Pricing of options under different volatility models [1]

by

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Abstract

In this paper we compare the price of an option with one year maturity of the German stock index DAX for several volatility models including long memory and leverage effects. We compute the price by applying a present value scheme as well as the Black-Scholes and Hull-White formulas which includes stochastic volatility. We find that long memory as well as asymmetry affect the Black-Scholes price significantly whereas the Hull-White price is hardly affected by long memory but still by including asymmetries.

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1 Introduction

The evaluation of options is a problem of interest in econometrics in recent years. Beginning with the celebrated Black - Scholes formula (see Black and Scholes(1973) and Merton(1973)) the problem of evaluating options has been of more and more importance for researchers as well as practitioners. The Black - Scholes formula showed that the fair price of an option depends strongly on the volatility of the price process of the underlying financial asset. However, Black and Scholes assumed the volatility to be constant over time. Stock returns on the other hand have volatility clusters which show that the conditional volatilities are time dependent. Since the introduction of ARCH - models by Engle (1981) the application of models with stochastic conditional volatility to option pricing became important. Hull and White (1987) extended the Black - Scholes formula by allowing for stochastic volatilities. Unfortunately, the resulting formula cannot be given in an explicit form. For this reason the Black - Scholes formula is still very popular in practice.

In addition to stochastic volatility financial data show evidence of long memory in volatilities of returns. Long - range dependence allow for a better predictability of the volatilities and therefore affect the price of an option. However, the influence of different volatility models including long memory and asymmetries to option prices is hardly discovered so far. Also the differences in the several pricing formulas have hardly been considered. Bollerslev and Mikkelsen(1996) show that the option price is significantly different when standard models are applied as compared to models allowing for long memory. They examine option prices for the $S\&P500$ stock index by considering three different pricing formulas, namely the Present Value, the Black - Scholes and the Hull - White formula. They consider GARCH - models as well as EGARCH, FIEGARCH and IEGARCH and show that the price of an option increases with the degree of integration meaning that GARCH - models give the lowest price whereas the highest option price is obtained for the IGARCH - model.

This paper extends the work of Bollerslev and Mikkelsen(1996) by firstly using data of the German stock index DAX and by secondly extending the class of volatility models considered. The results are somewhat similar to those of Bollerslev and Mikkelsen meaning that we observe also an increasing price with an increasing degree of integration. Asymmetry does not have a large influence to the option price.

The rest of the paper is organized as follows. In the next section we introduce the Black - Scholes and the Hull - White option pricing formulas, section 3 describes the considered volatility models. Section 4 shows our results and section 5 concludes.

2 Pricing formulas

This paper considers a European call option with time to maturity T and exercise price K. We do not allow for arbitrage.

Let A_t be the price of the underlying stock at time t, let r denote a fixed interest rate and σ the volatility of the stock which is for the moment assumed to be constant over the time.

The Black - Scholes formula for the rational price of a European call option is given by

$$\begin{aligned} c^{BS}(A_t, t, K) &= A_t \Phi\left(\frac{\ln\left(\frac{A_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) \\ &- K\exp(-rt)\Phi\left(\frac{\ln\left(\frac{A_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right). \end{aligned} \tag{1}$$

Here, $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution. As it can be seen from (1) the option price depends on the volatility of the stock but not on the returns themselves.

The volatility σ is usually computed by using the market price of an option with time of maturity t and exercise price K which can be observed empirically.

However, the Black - Scholes formula assumes the volatility to be constant over time.

Therefore, Hull and White (1987) introduced an alternative model allowing for time varying volatilities. They also assume that the volatilities are independent of the price process. Unfortunately, the Hull - White formula has no closed representation. It can be interpreted as a Black - Scholes formula with an averaged variance. The Black - Scholes price of an option is above the Hull - White price for options at the money whereas the Hull - White price is higher for options out of the money or in the money.

3 Volatility models

Stock returns exhibit quite a lot of empirically observed stylized facts such as time dependent volatilities, long memory in volatilities and asymmetries. To take these into account as much as possible we consider a large variety of possible models, such as GARCH-, IGARCH, FIGARCH- and HYGARCH-models as well as a class of GARCH models introduced by Glosten, Jagannathan and Runkle (1993) which also allow for asymmetry.

For all models introduced in this section we assume that the demeaned returns ε_t follow the parametrization

$$\varepsilon_t = z_t \sigma_t,$$

where the z_t are iid $(0, 1)$ random variables. The demeaned returns y_t are defined by $\varepsilon_t = y_t - E(y_t)$. ARCH-models do now model the volatilities σ_t in the above equation.

The GARCH(p,q)-model was introduced by Bollerslev (1986). It is defined by

$$\sigma_t^2 = \omega + \alpha(B)\varepsilon_t^2 + \beta(B)\sigma_t^2. \qquad (2)$$

Here B denotes the lag operator and $\alpha(B)$ and $\beta(B)$ denote the AR- and MA-polynomials respectively. We assume that all roots of these polynomials are outside of the unit circle. Therefore, GARCH-models assume the volatilities to depend on past volatilities as well as on past innovations. Due to their similarity to ARMA-models GARCH-models are symmetric and have short memory.

IGARCH(p,q)-models are a kind of ARIMA-models for the volatilities. They are defined by

$$\phi(B)(1-B)\varepsilon_t^2 = \omega + [1-\beta(B)]\nu_t, \tag{3}$$

where $\phi(B) = (1-\alpha(B)-\beta(B))(1-B)^{-1}$. Here ν_t is an iid noise process with mean zero and finite variance. IGARCH-models imply that shocks to the series affect all future horizons. Although the assumption of short memory such as in GARCH-models is usually not fulfilled the implications of IGARCH-models are too strong compared to empirical findings. Sibbertsen(2004) among others shows that there is evidence for long memory in volatilities of stock returns.

Therefore, we consider fractionally integrated models. The first long-memory GARCH-model was the FIGARCH(p,d,q)-model introduced by Baillie et al. (1996). The FIGARCH-model is a generalization of the IGARCH-model (3) by replacing the operator $(1-B)$ by $(1-B)^d$, where d is the memory parameter. Thus, equation (3) becomes

$$\phi(B)(1-B)^d\varepsilon_t^2 = \omega + [1-\beta(B)]\nu_t, \tag{4}$$

where the polynomial $\phi(B)$ is as given above. FIGARCH-models exhibit long memory. They nest GARCH-models (for $d = 0$) as well as IGARCH-models (for $d = 1$). In contrast to ARFIMA-models where the memory parameter d is between zero and a half d is here between zero and one. Unfortunately, FIGARCH-processes are non-stationary like IGARCH-processes. This shows that the concept of unit roots can hardly be generalized from linear to non-linear processes. Furthermore, the interpretation of the memory parameter d is difficult in the FIGARCH set up.

For this reason Davidson(2004) extended the class of FIGARCH-models to HYGARCH(p,α,d,q)-models which stands for hyperbolic GARCH.

HYGARCH-models replace the operator $(1-B)^d$ in (4) by $[(1-\alpha)+\alpha(1-B)^d]$. The parametrization of HYGARCH-models is given by

$$\sigma_t^2 = \omega(1-\beta(B))^{-1}+\left[1-(\phi(B)(1+\alpha[(1-B)^d-1]))(1-\beta(B))^{-1}\right]\varepsilon_t^2. \quad (5)$$

The parameters α and d are assumed to be non-negative. HYGARCH-models nest GARCH-models (for $\alpha = 0$), FIGARCH-processes (for $\alpha = 1$) and IGARCH-models (for $\alpha = d = 1$).

All models introduced so far are symmetric. The most famous model allowing for asymmetry is the EGARCH-model by Nelson(1991). However, we want to consider a more flexible class of models introduced by Glosten, Jagannathan and Runkle(1993) and therefore referred to as GJR-GARCH-models. Denoting by 1_A the indicator function of the set A they have the form

$$\sigma_t^2 = \omega + \sum_{i=1}^{q}(\alpha_i\varepsilon_{t-i}^2 + \gamma_i\varepsilon_{t-i}^2 1_{]-\infty,0]}(\varepsilon_{t-i})) + \sum_{j=1}^{p}\beta_j\sigma_{t-j}^2. \quad (6)$$

Here, α_i and β_j are the coefficients of the polynomials $\alpha(B)$ and $\beta(B)$ respectively and γ_i are parameters describing the asymmetry of the model called leverage parameters because they model the leverage effect of the returns.

In the simulation study below we consider also combinations of the GJR-GARCH and the symmetric models above to allow for asymmetry as well as for long memory.

We estimate all models for the German stock index DAX by Maximum-Likelihood. Our data contains daily observations from 4. 1. 1960 to 28. 12. 2001. To the return process itself we fitted an $AR(3)$ process as it was done in Bollerslev and Mikkelsen(1996). The GARCH-order used for all volatility models was $p = 1$ and $q = 1$. We used the following parametrization which is given below for the $AR(3) - GJR - HYGARCH(1, \alpha, d, 1)$-model as this is the model having the most parameters. The equation for the returns is

$$y_t = \mu_0 + \mu_1 y_{t-1} + \mu_2 y_{t-2} + \mu_3 y_{t-3} + \varepsilon_t.$$

For the volatilities we have the parametrization

$$\sigma_t^2 = \frac{\omega}{1-\beta} + \left[1 - \frac{1}{1-\beta B}(1-\phi B)(\gamma 1_{]-\infty,0]}(\varepsilon_t)B)(1+\alpha[(1-B)^d - 1])\right]\varepsilon_t^2.$$

Here, the parameter ω is the mean of the volatilities, β and ϕ describe the GARCH-properties. γ is the asymmetry parameter and d is the memory parameter. In the tables below only those parameters are given which are included in the model meaning that the space for a parameter which is not included in the specific model is left empty in the table. The models are given in the columns, the parameters are in the rows.

The results are given in table I and II below. The numbers in brackets are the standard deviations. To compare the fit of the models we give the Akaike information criterion as well as the Schwarz information criterion. The model which maximizes these criteria has the best fit.

Table I *Parameter estimation for symmetric GARCH-models.*

	AR	AR-GARCH	AR-IGARCH	AR-FIGARCH	AR-HYIGARCH	AR-HYGARCH
μ_0	0.00023 (0.0001)	0.00028 (0.0001)	0.00027 (0.0001)	0.0003 (0.0000)	0.00028 (0.0001)	0.0003 (0.0000)
μ_1	0.0617 (0.0183)	0.12697 (0.0129)	0.12767 (0.0127)	0.12685 (0.0124)	0.12635 (0.0127)	0.12684 (0.0124)
μ_2	-0.05134 (0.0176)	-0.06072 (0.0133)	-0.06132 (0.0132)	-0.05985 (0.0126)	-0.06014 (0.0131)	-0.05985 (0.0127)
μ_3	-0.00491 (0.017)	-0.00013 (0.0121)	0.00045 (0.0121)	-0.00585 (0.0111)	-0.00073 (0.0120)	-0.00585 (0.011)
ω		0.00441 (0.0003)	0.00383 (0.0003)	0.00276 (0.0002)	0.00446 (0.0003)	0.00274 (0.0004)
β		0.83944 (0.0195)	0.83186 (0.0269)	0.3164 (0.0745)	0.82556 (0.0269)	0.31487 (0.0906)
ϕ		0.98127 (0.0067)	-0.02883 (0.0381)	0.03092 (0.0582)	-0.02924 (0.0337)	0.03038 (0.0602)
d			d = 1 fixed	0.4128 (0.062)	d = 1 fixed	0.41135 (0.0782)
α					0.97995 (0.0074)	1.00203 (0.0404)
AIC	32355.4	33879.7	33871.1	33941.8	33881.2	33941.8
SIC	32352.0	33876.1	33867.9	33938.2	33877.6	33942.8

Table II *Parameter estimation for asymmetric GARCH-models.*

	AR-GJR-GARCH	AR-GJR-IGARCH	AR-GJR-FIGARCH	AR-GJR-HYIGARCH	AR-GJR-HYGARCH
μ_0	0.00011 (0.0001)	0.00025 (0.0001)	0.00021 (0.0000)	0.00011 (0.0001)	0.00009 (0.0001)
μ_1	0.12889 (0.0125)	0.122793 (0.0128)	0.12654 (0.0118)	0.12844 (0.0124)	0.13103 (0.0123)
μ_2	-0.05587 (0.0125)	-0.06068 (0.0132)	-0.05826 (0.012)	-0.05566 (0.013)	-0.05416 (0.0123)
μ_3	0.00592 (0.0122)	0.00127 (0.0120)	-0.00491 (0.0106)	0.00563 (0.0122)	0.00029 (0.0109)
ω	0.00458 (0.00049)	0.00369 (0.004)	0.000 (0.000)	0.0046 (0.0004)	0.00181 (80.001)
β	0.84618 (0.0206)	0.83504 (0.0265)	0.12733 (0.1081)	0.83849 (0.0029)	0.83247 (0.0357)
ϕ	0.093886 (0.0122)	-0.02731 (0.0384)	-0.00741 (0.0705)	0.93731 (0.0132)	0.88235 (0.0333)
γ	0.86839 (0.02476)	0.03802 (0.086)	0.3088 (0.0846)	0.84156 (0.2563)	1.35538 (0.34959)
d		d = 1 fixed	0.31491 (0.0706)	d = 1 fixed	0.02444 (0.0495)
α				-0.01076 (0.0259)	2.24096 (4.0591)
AIC	33819.7	33781.8	33879.2	33820.1	33887.4
SIC	33816.1	33778.1	33875.6	33816.5	33883.8

As we can see from tables I and II for the DAX the symmetric models seem to give the better fit than the asymmetric models as they have the higher SIC in all cases. Comparing the SIC for all models the two long memory models are the chosen models. The SIC prefers the AR-HYGARCH-model to the AR-FIGARCH-model. The SIC for these two models is almost equal whereas it is much smaller for all the other models. The estimated memory parameter is

$d_{HYGARCH} = d_{FIGARCH} = 0.41$ and thus clearly positive and similar for both models.

4 Results

In this section we consider prices for European call options on the German stock index DAX. The options have a period of validity of $\tau = 260$ days which is one year. The options are sold on the 28th of December 2001 which is the last point of our data. At this day the DAX was at $A_0 = 5160.1$ points. To estimate the underlying price process of the DAX we use daily data from 4th January 1960 to 28th December 2001. Therefore, we have 10516 observations at hand to estimate the models. The estimation results are as given in tables I and II in the previous section. The simulations for the price process are based on $N = 1000$ replications. The risk free interest rate r is assumed to be $r = 0.07$. This is the rate used by Bollerslev and Mikkelsen(1996). We adopt the rate in this paper to obtain comparability of our results to those of Bollerslev and Mikkelsen.

In order to compute the option price we have at first to simulate the price process because we need the price at the time t. The logarithms of the returns y_t are therefore simulated by using the volatility models described in section 3. All simulations are carried out with the Ox package TSMod 4.03. The price process A_t is computed from the simulated returns

$$A_t = A_0 \exp\left(\sum_{i=1}^{t} y_i\right).$$

Denote with $A_{n,t}$ the n-th repetition of the simulated price process at time t.

In order to compute the Black - Scholes price of the option also in the presence of GARCH effects we have to replace the volatility by an average volatility during the period of validity of the option. It is obtained by

$$\sigma^{BS}(\tau)^2 = \frac{1}{\tau A_0^2} \frac{1}{N-1} \sum_{n=1}^{N} \left(A_{n,\tau} - \frac{1}{N} \sum_{i=1}^{N} A_{i,\tau}\right). \qquad (7)$$

This volatility estimator is the empirical variance of the marginal distribution of the price process at time τ weighted with the period of validity τ and the squared price A_0^2.

By substituting the volatility estimator (7) into the Black - Scholes formula (1) we obtain the Black - Scholes price for our option by

$$C^{BS}(\tau, K) = c^{BS}(\sigma^{BS}(\tau), \tau, K, A_0, r).$$

In order to compute the Hull - White price of the option we have the problem that the Hull - White pricing formula has no closed representation. However, as mentioned in section 2 the Hull - White price can be obtained as the expected Black - Scholes price integrated over the average variance during the period of validity of the option if the volatilities are independent of the price process. The volatilities for the Hull - White model are estimated by

$$\sigma^{HW}(\tau)_n^2 = \frac{1}{A_0^2}\frac{1}{\tau - 1}\sum_{t=1}^{\tau}\left(\Delta A_{n,t} - \frac{1}{\tau}\left[A_{n,t} - A_0\right]\right)^2. \tag{8}$$

Here, $\Delta A_{n,t} = A_{n,t} - A_{n,t-1}$. The Hull - White price is now obtained by substituting (8) into the Black - Scholes formula:

$$C^{HW}(\tau, K) = \frac{1}{N}c^{BS}(\sigma^{HW}(\tau)_n, \tau, K, A_0, r).$$

This formula is a discrete version of the Hull and White(1987) formula.

As a third pricing scheme we apply the present value scheme based on an idea by Engle and Mustafa (1992). It considers the mean of the possible profits of the option based on the simulated price process. As these profits have to compete with a risk free bond it is weighted with the interest rate r. The present value scheme is given by

$$C^{PV}(\tau, K) = \exp(-r\tau)\frac{1}{N}\sum_{n=1}^{N}\max(0, A_{n,\tau} - K).$$

We now consider the following situations. The period of validity for our option is one year which is $\tau = 260$. We consider options which are at the money ($K = A_0$), out of the money ($K = 1.05A_0$) and deep out of the money ($K = 1.1A_0$). For all these situations we compute the option price for all the models described in section 3. Similar to Bollerslev and Mikkelsen (1996) we model the returns with an $AR(3)$- process. Table III gives the results for the option at the money and tables IV and V has the other two scenarios respectively.

Table III *Price of the option at the money* ($K = A_0$).

Model	Present Value	Black - Scholes	Hull - White
AR	307.71	354.1	350.61
AR - GARCH	373.65	542.77	513.07
AR - IGARCH	352.68	571.89	513.07
AR-FIGARCH	392.76	554.87	513.29
AR - HY - IGARCH	371.95	572.24	509.89
AR - HYGARCH	392.82	561.21	513.38
AR - GJR - GARCH	162.95	506.35	512.58
AR - GJR - IGARCH	331.62	553.8	517.16
AR - GJR - FIGARCH	296.19	550.27	510.74
AR - GJR - HY - IGARCH	161.91	698.37	510.6
AR - GJR - HYGARCH	143.98	600	518.33

Table IV *Price of the option out of the money* ($K = 1.05A_0$).

Model	Present Value	Black - Scholes	Hull - White
AR	73.35	162.9	152.8
AR - GARCH	140.89	387.71	355.32
AR - IGARCH	125.13	417.99	358.85
AR-FIGARCH	158.64	400.33	355.67
AR - HY - IGARCH	141.41	418.34	351.94
AR - HYGARCH	159.02	406.92	355.65
AR - GJR - GARCH	17.05	349.31	354.8
AR - GJR - IGARCH	107.35	399.22	359.68
AR - GJR - FIGARCH	82.82	395.54	352.85
AR - GJR - HY - IGARCH	17.16	347.26	352.72
AR - GJR - HYGARCH	13.66	406.92	360.96

Table V *Price of the option deep out of the money* ($K = 1.1A_0$).

Model	Present Value	Black - Scholes	Hull - White
AR	0.15	55.11	46.2
AR - GARCH	13.64	270.38	239.07
AR - IGARCH	11.36	299.78	242.49
AR-FIGARCH	18.64	282.63	239.4
AR - HY - IGARCH	15.31	300.12	235.79
AR - HYGARCH	18.74	289.02	239.39
AR - GJR - GARCH	0.13	233.13	238.56
AR - GJR - IGARCH	8.42	281.55	243.29
AR - GJR - FIGARCH	3.81	277.97	236.67
AR - GJR - HY - IGARCH	0.17	231.15	236.55
AR - GJR - HYGARCH	0.14	289.02	244.52

As we can see from the tables the differences between the prices are enormous. Especially the difference between neglecting stochastic volatilities by just modelling the returns with an $AR(3)$-process and the models including stochastic volatilities is huge. The price can be more than five times lower with a constant volatility as it is for example for the Black - Scholes price of the option deep out of the money in table V. Obviously, the present value scheme takes the least account of stochastic volatility with the smallest price changes whereas the difference is biggest for the Black-Scholes price. For some reason including asymmetry collapses the present value price. However, the present value scheme does not seem to be a suitable way for pricing options.

For the Black-Scholes and Hull-White scheme we observe what we would expect. In the class of stochastic volatility models the price is the lowest for the GARCH-model and highest for the integrated IGARCH-model. The price of the long-memory alternatives is somewhere in between. The price for the HYGARCH-model is slightly higher than the FIGARCH-price although both of them are at the same range. Including long memory into the model can change the prices for about 7% as it can be seen for the price of the option deep out of the money in table V. We can see that the difference becomes larger if the exercise price increases. Therefore, for options being deep out of the money it becomes more and more important to specify the model correctly. Including long memory seriously affects the price of an option.

Furthermore, the differences are bigger for the Black-Scholes price than for the Hull-White price. The Hull-White price is less affected by using different models and including long memory. Long memory changes the prices only by about 0.1%.

Including asymmetries also drops the Black-Scholes price by quite a bit. All the other findings are similar to those described above. Again the Hull-White price is affected the least by introducing asymmetry. It seems to be most robust against model changes as long as stochastic volatility is taken into account.

The price differences found in our paper are of a similar magnitude as those in Bollerslev and Mikkelsen (1996). They also find that the Hull-White scheme is the most robust to model changes.

5 Conclusion

In this paper we simulate the fair price of a European call option for the DAX starting December 2001 with a period of validity of one year. We consider options which are at the money, out of the money and deep out of the money. The present value price, Black-Scholes price and Hull-White price are computed. It can be seen that the present value scheme is not suitable for pricing options. Neglecting stochastic volatility results in a by far too low price. Including effects such as long memory and asymmetry changes the price significantly and has to be taken into account when pricing options. The effect increases for options deep out of the money. The Black-Scholes price is more affected than the Hull-White price which seems to be quite robust against changes of the model. Asymmetry affects the Hull-White price more than long memory.

The superior model for the DAX was a symmetric long memory GARCH model. Having the above results in mind long memory should therefore be taken into account when pricing DAX options.

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