Pareto-Optimality and Desirability Indices

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Abstract:
Pareto-Optimality and the Desirability Index are methods for multicriteria optimization in quality management. In this paper the pareto-optimality of the optimal influence factor settings of a process resulting from maximizing the DI is analyzed and is shown to be valid in most cases.

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1 Introduction

The Concept of Desirability, introduced by Harrington (1965), is a method for multicriteria optimization (MCO) in industrial quality management (Fig. 1). Via Desirability Functions (DFs), which allow for comparing different scales of process quality criteria (QCs) by mapping them to $[0, 1]$, and the Desirability Index (DI) the multivariate optimization problem is converted into a univariate one. Based on design of experiment methods optimal levels of process influencing factors can be determined that optimize all often competing QCs simultaneously.
Besides the DI several other MCO methods exist. One of them is an approach originally introduced in the context of microeconomics for determining pareto-optimal factor settings. As both concepts are competing it is therefore interesting to put the optimization results into one framework. This is done in section 4 by analyzing the pareto-optimality of the optimum factor settings resulting from MCO by means of the DI. Section 2 provides details regarding different types of DFs and DIs, and the Pareto Concept is outlined in section 3. A summary is given in section 5.

2 Desirability Functions and the Desirability Index

Harrington (1965) introduced two types of DFs which transform the QCs onto $(0, 1]$ (see Fig. 2). One aims at maximization of the QC (one-sided specification) whereas the other one reflects a target value problem (two-sided specification). Concerning the latter the transformation requires two specification limits (LSL, USL) for a QC Y symmetrically around the target value T, which

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Quality Optimization of a Process Using the Desirability Index:

1. Influence factors: $X_1, \ldots, X_n$
2. Quality Criteria: $Y_1, \ldots, Y_k$ with $Y_i = f_i(X_1, \ldots, X_n, \varepsilon_i)$
3. Which levels of the quality criteria are desired?

   $d_i(Y_i)(i = 1, \ldots, k)$, $d : \mathbb{R} \rightarrow [0, 1]$ bzw. $(0, 1]$ Desirability Function
4. Combination into a univariate quality measure:

   $D := f(d_1, \ldots, d_k)$, $D : [0, 1]^k/(0, 1]^k \rightarrow [0, 1]/(0, 1)$ Desirability Index
5. Determination of optimum factor settings X^{opt} by maximizing $\hat{D}$:

$$\max_{X_1,\ldots,X_n} \hat{D}(X_1, \ldots, X_n) = \sqrt[k]{\prod_{i=1}^{k} d_i(f_i(X_1, ..., X_n, 0))}$$

Figure 1: The Desirability Index in Multicriteria Optimization

are associated with a desirability of $1/e$. Then the DF d is defined as

$$d(Y') = e^{-|Y'|^n} \quad \text{with} \quad Y' = \frac{2Y - (USL + LSL)}{USL - LSL}. \tag{1}$$

The parameter $n > 0$ is to be chosen so that the resulting kurtosis of the function adequately meets the expert's preferences. The one-sided DF uses a special form of the Gompertz-Curve, where the kurtosis of the function is determined by the solution (b_0, b_1) of a system of two linear equations that require two values of Y and related values of d:

$$d(Y') = e^{-e^{-Y'}} \quad \text{with} \quad Y' = -[\ln(-\ln d)] = b_0 + b_1 Y. \tag{2}$$

The DI combines k individual desirability functions d_i into one overall quality value by

$$D := (\prod_{i=1}^{k} d_i)^{1/k}. \tag{3}$$

In the course of time modifications of Harrington's concept came up either in terms of different combination types of the DFs, e.g.

$$D := \min_{i=1,\ldots,k} d_i \quad \text{(Kim and Lin (2000))}, \tag{4}$$

or in terms of more flexible DFs. The approach introduced by Derringer and Suich (1980) became the most important one and is currently most frequently used in practice. In this case

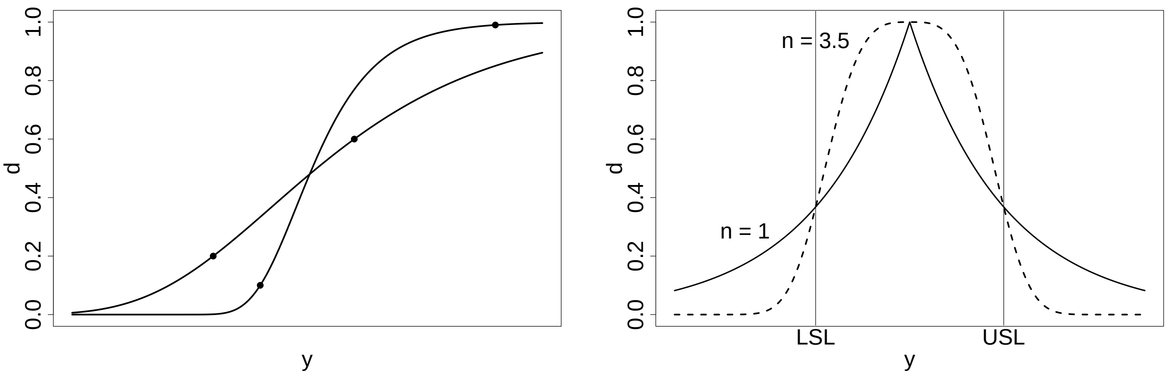

Figure 2: Harrington's one- and two-sided Desirability Functions

also asymmetric specifications become possible (see Fig. 3). In the two-sided case the DF is determined using

$$d_i(Y_i) = \begin{cases} 0, & Y_i < LSL_i \\ (\frac{Y_i - LSL_i}{T_i - LSL_i})^{l_i}, & LSL_i \leq Y_i \leq T_i \\ (\frac{Y_i - USL_i}{T_i - USL_i})^{r_i}, & T_i < Y_i \leq USL_i \\ 0, & Y_i > USL_i \end{cases}, \qquad i = 1, \ldots, k. \tag{5}$$

Thus values outside the specification limits result in an unacceptable process quality, i.e. the DF equals zero. The parameters l_i and r_i determine the shape of the DF. The one-sided specification is exemplary given for maximizing a QC, i.e.

$$d_i(Y_i) = \begin{cases} 0, & Y_i \leq LSL_i \\ (\frac{Y_i - LSL_i}{T_i - LSL_i})^{l_i}, & LSL_i < Y_i < T_i \\ 1, & Y_i \geq T_i \end{cases}, \qquad i = 1, \ldots, k. \tag{6}$$

In case the QC exceeds its target value no additional benefit is realized, therefore a value of 1 is constantly assigned. For minimizing a QC the part of the two sided DF exceeding T_i is used values falling below T_i are associated with a value of 1.

3 Pareto-Optimality

The concept of pareto-optimality was primarily introduced by Vilfredo Pareto (1896). He argued that an individual's preferences form the basis of economic analysis and developed the notion of a pareto optimal outcome in which no member of the society can be made better off without hurting, or decreasing the payoffs of someone else.

This approach can be transferred to multicriteria optimization problems in a straightforward manner. A realization of quality criteria (QC) $Y = (Y_1, \ldots, Y_k)'$ is said to be pareto-optimal if

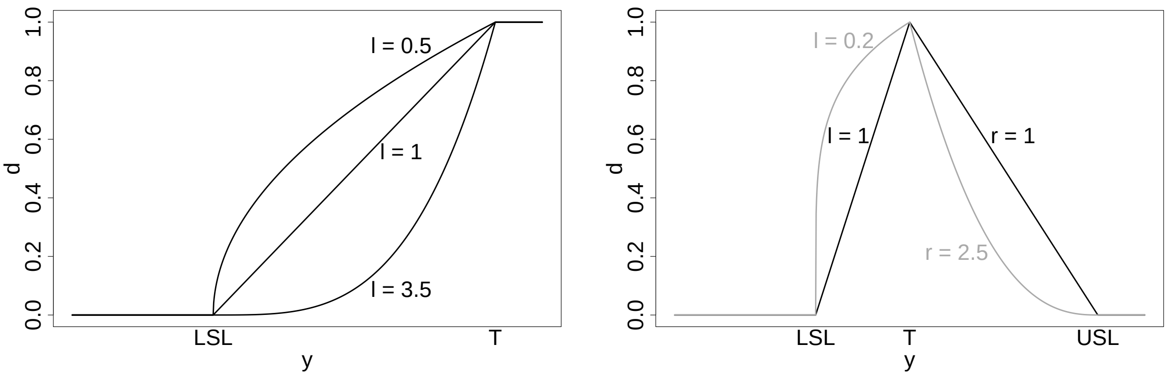

Figure 3: One- and two-sided Desirability Functions of Derringer-Suich type

there is no other realization that keeps up the process quality regarding all criteria and improves at least one criterion. Thus a pareto-optimal situation cannot be improved upon without deteriorating the process with respect to at least one quality criterion. A corresponding factor setting $X = (X_1, \dots, X_n)'$ then is pareto-optimal in factor space if the corresponding criteria vector Y is pareto-optimal in criteria space.

4 Pareto Optimality of optimal factor settings

In Steuer (1999) pareto-optimality is introduced using the concept of domination. A vector $Y = (Y_1, \dots, Y_k)'$ is defined as pareto-optimal if there exists no other realization Y^* which dominates Y, i.e. Y^* contains at least one value that exceeds the corresponding value of Y and simultaneously keeps up all the remaining ones. An improvement of a QC thus in this approach is only reflected by an increase of Y. Differing from this in the following an improvement of a QC is reflected by an increased value of the corresponding desirability function, which allows for including more specific requirements and preferences.

Theorem 1 Let a process characterized by quality criteria Y_i $(i = 1, ..., k)$ and influence factors X_j $(j = 1, ..., n)$ be given. Optimal influence factor levels $X^{opt} = (X_1^{opt}, \dots, X_n^{opt})'$ are assumed to have been determined based on DFs d_i $(i = 1, ..., k)$ and the DI (3) $D := (\prod_{i=1}^k d_i)^{1/k}$. Then it holds that X^{opt} is pareto-optimal.

Proof:
Assumption: X^{opt} is not pareto-optimal

$$\Rightarrow \exists\, X^* : \quad d_s(Y_s|X^*) > d_s(Y_s|X^{opt}) \text{ for } s \in \{1, \dots, k\} \tag{7}$$

$$\text{and} \quad d_j(Y_j|X^*) \geq d_j(Y_j|X^{opt}) \text{ for } j = 1, \dots, k; j \neq i. \tag{8}$$

$$\Rightarrow D^* = (\prod_{i=1}^{k} d_i(Y_i|X^*))^{1/k} > D^{opt} = (\prod_{i=1}^{k} d_i(Y_i|X^{opt}))^{1/k}.$$

That however is contradictory to the assumption of X^{opt} maximizing the DI. Thus X^{opt} must be pareto-optimal. □

If the minimum of the DFs (4) is used as a DI the pareto-optimality of X^{opt} is not guaranteed.

Theorem 2 Let a process be characterized by quality criteria Y_i $(i = 1, ..., k)$ and influence factors X_j $(j = 1, ..., n)$. Optimal influence factor levels $X^{opt} = (X_1^{opt}, \ldots, X_n^{opt})'$ are assumed to have been determined based on DFs d_i $(i = 1, ..., k)$ and the DI (4) $D := \min(d_1, \ldots, d_k)$. Let furthermore Y_p $(p \in \{1, \ldots, k\})$ be the criterion that takes the minimum value of the DI based on X^{opt}. Recalling conditions (7) and (8) for X^{opt} not being pareto-optimal it holds that:

1. $d_p = d_s \Rightarrow X^{opt}$ is pareto-optimal.
2. $d_p \neq d_s$, $(d_p|X^*) \neq (d_p|X^{opt}) \Rightarrow X^{opt}$ is pareto-optimal.
   Special cases:
   2a. $d_p \neq d_s$, $(Y_p|X^*) \neq (Y_p|X^{opt})$ and the DF d_p is strictly monotonic $\Rightarrow X^{opt}$ is pareto-optimal.
   2b. $d_p \neq d_s$, $(Y_p|X^*) \neq (Y_p|X^{opt})$ and the DF d_p is not strictly monotonic, but of one-sided Derringer-Suich-type (6) $\Rightarrow X^{opt}$ is pareto- optimal.
3. $d_p \neq d_s$ and $(Y_p|X^*) = (Y_p|X^{opt}) \Rightarrow X^{opt}$ is not pareto-optimal in general.

Proof:
For cases 1) and 2) X^{opt} is assumed to be not pareto-optimal. Thus conditions (7) and (8) must be fulfilled.

Referring to 1):
The criterion the process is improved upon is the one that yields the minimum value of the DFs:

$$\begin{aligned} d_p = d_s \quad &\Rightarrow \quad d_p(Y_p|X^*) > d_p(Y_p|X^{opt}) \\ &\Rightarrow \quad D^* = \min_{i=1,\ldots,k} d_i(Y_i|X^*) > D^{opt} = \min_{i=1,\ldots,k} d_i(Y_i|X^{opt}). \end{aligned}$$

This would imply that X^{opt} does not maximize the DI (Contradiction!). Therefore X^{opt} must be pareto-optimal.

Referring to 2-3):
The criterion the process is improved upon is not the one that yields the minimum value of the

DFs. For further analysis a distinction has to be made regarding the type of the desirability function d_p. The pareto-optimality of X^{opt} furthermore depends on the behaviour of $(Y_p|X^{opt})$ in case another criterion is improved by an altered X-vector X^*.

2:) $(d_p|X^*) \neq (d_p|X^{opt}) \overset{(7)}{\Rightarrow} d_p(Y_p|X^*) > d_p(Y_p|X^{opt})$

$\Rightarrow D^* > D^{opt}$ (Contradiction!)

$\Rightarrow X^{opt}$ is pareto-optimal.

2a:) $(Y_p|X^*) \neq (Y_p|X^{opt}) \Rightarrow d_p(Y_p|X^*) \neq d_p(Y_p|X^{opt})$ as d_p is strictly monotonic.

$\Rightarrow X^{opt}$ is pareto-optimal as shown in 2.).

2b:) A one-sided DF of Derringer-Suich-type only is not strictly monotonic outside the specification limits LSL and USL. Then it equals either 0 or 1. A value $d_p = 0$ does not make sense for optimal process quality. The situation $d_p = 1$ is also not possible as in this case $d_p(Y_p|X^{opt})] = d_s(Y_s|X^{opt})$ and an improvement of Y_s is not possible any more. Therefore these situations do not have to be taken into account and X^{opt} is pareto-optimal due to 2).

3:) If X^{opt} is pareto-optimal conditions (7) and (8) must not be fulfilled for any X^*. But for $(Y_p|X^*) = (Y_p|X^{opt})$ an improvement of the quality criterion Y_s (or addtionally even other quality criteria) would not change D^{opt}. Therefore in such situations X^{opt} is not pareto-optimal.

□

5 Summary and Conclusions

When using pareto-optimality as a means for multicriteria optimization the general problem is that the resulting factor settings are not unique. Usually the set of pareto-optimal factor settings ("Pareto-Set") on the one hand is not easy to determine and on the other hand there is no general guideline for selecting the appropriate solution. By showing that X^{opt} determined by maximizing the DI (3) is pareto-optimal the DI can be understood as a method for selecting a pareto-optimal solution from the Pareto-Set.

It may be argued that in praxis the whole Pareto-Set is of interest and that a single factor setting afterwards can be selected by expert knowledge. Following this approach therefore a compromise between often conflicting quality criteria has to be found a posteriori, i.e. after having provided the pareto-optimal solutions. An expert then has to cope with the responsibility of weighing different solutions which is a very complex task in general.

By using the desirability approach this step however is done a priori by specifiying DFs which reflect the preferences and requirements regarding the quality criteria. After the optimization of the DI a pareto-optimal solution is determined which automatically finds the "best" compromise. In addition in praxis one may also vary the DFs in order to get an idea how slight variations effect the optimal factor settings. The design of experiments step by which models $Y_i = f_i(X_1, \ldots, X_n, \varepsilon_i)$ are determined (see Figure 1) does not have to be repeated. Thus only the optimization step of the DI has to be rerun which is only a computational task. The desirability approach therefore facilitates the selection of a pareto-optimal solution out of the Pareto-Set.

In case the minimum of the DFs is applied as a DI situations may occur in which X^{opt} is not pareto-optimal. But it has to be kept in mind that the applying experts are chosing D^{min} being aware of this "non-pareto-optimality". In these cases only the minimum value of the quality criteria is of interest - the level of the remaining ones can be neglected. Therefore the "non-pareto-optimality" should not be viewed as a disadvantage as in those cases pareto-optimality of the optimal factor settings is not claimed on any account.

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