Logics which allow Degrees of Truth and Degrees of Validity

A way of handling Graded Truth Assessment and Graded Trust Assessment within a single framework

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Preface

In this dissertation, the semantics of logical systems which are able to express *vagueness* and *graded truth assessment* as well as *doubt* and *graded trust assessment* are investigated from the point of view of mathematical logic.

Traditionally, logics for modelling **graded truth** have been many-valued logics which allow truth values *between* 0 (false) and 1 (true). In applications, sometimes truth values are attached to formulae to **assess** the truth of the formula.

In logics for modelling **graded trust**, usually *trust* (or plausibility, or possibility, or belief) degrees are attached to formulae from classical two-valued logic to **assess** the trust in the *knowledge* expressed by this formula.

Several logical systems using *labelled formulae* (i. e. formulae to which some label is attached) have been described in the literature, with varying interpretations concerning structure and semantics of labels. In many cases, however, the *meaning* of a label is not precisely specified, casting doubt on what, from a semantic point of view, is really *formalised* by labelled formulae or a corresponding inference mechanism.

Without a specific background theory for the *meaning* of labels (as is given, for instance, by probability theory), of course no *canonical* paradigm for specifying the structure and processing of labels exists. Consequently, several different such paradigms have been developed. Differences between these systems combined with the lack of a precisely defined semantics for labels have led to critique of such logical systems as a whole, because it must seem suspicious if from one and the same knowledge base of labelled formulae, it is possible to infer totally different results, without a clear semantic theory which can explain the differences.

There have been attempts to clarify this situation, especially by distinguishing whether a system of labelled logical formulae is used for the representation of *graded truth assessment* or *graded trust* (or *possibility*, *necessity*, *plausibility*, *uncertainty*, *belief*) *assessment* with respect to the states of affairs being modelled. Logical systems which can accomplish one or the other task have been defined, studied and compared.

In this dissertation, a very general approach to the definition of labels for expressing graded truth *and* graded trust is described. This definition gives rise to a canonical definition of the concepts of model and semantic consequence for the resulting logic of labelled formulae.

The expressive power of such logics is very high. A label can express uncertainty about truth or trust or any combination of both. A systematic study of the semantics of these logical systems is given here, as well as a discussion and comparison of special cases.

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1 Introduction

> **uncertain** /ʌnˈsəːt(ə)n, -tɪn/ *a.* ME. **1** Not determined or fixed; liable to change, variable, erratic; (of a person) changeable, capricious. ME. **2** About which one cannot be certain, unreliable; (of a path etc.) not clearly leading to a certain goal or destination. ME. **3a** Not known with certainty; not established beyond doubt. ME. **b** Without clear meaning; ambiguous. LME. **c** Not clearly identified, located, or determined; (of something seen) not clearly defined or outlined. E17. [...]
>
> *The New Shorter Oxford English Dictionary.*

Most disciplines of artificial intelligence, for instance *knowledge representation*, *machine learning*, and *planning*, profit from research on the theoretical foundations of 'classical' two-valued logic. This is done by employing the logical language for the formulation of *knowledge bases* and (via *logic programming languages*) for the implementation of *inference engines*. This way, an extensive theoretical background can be exploited for establishing consistency or independence of knowledge bases or for the design of automated inference mechanisms.

Whenever problem definitions, input data, or expert knowledge involve **uncertainty**, classical two-valued logic alone is often insufficient as a tool for knowledge representation. From the possible meanings of uncertainty cited above from the *New Shorter Oxford English Dictionary*, the following two are focused on in this dissertation.

1. Uncertainty in the sense of ambiguity, vagueness or impreciseness (items **3b** and **3c** of the above quote). The concept that this type of uncertainty is given by *degree* in systems for knowledge representation is referred to as **graded truth assessment**[1].

2. Uncertainty in the sense that something is ill-known or doubtful (item **3a** of the above quote). The concept that this type of uncertainty is given by *degree* in systems for knowledge representation is referred to as **graded trust assessment**.

In both cases, it is assumed that uncertainty can be given by *degree*. This dissertation is devoted to studying logics which allow to represent both types of graded uncertainty, strictly from the perspective of mathematical logic. For representing the distinct types of uncertainty in logical systems, two distinct concepts which are two-valued in classical logic are allowed to become *many-valued*:

1. **Graded truth assessment** is achieved by making the classical concept of **truth** many-valued and making *degrees of truth* available within the logical language (in the form of **labels**).

[1]The author has abstained from calling this type of uncertainty *uncertainty about truth* (and the second type *uncertainty about knowledge*) because the term *uncertainty* is too much overloaded in the literature, especially by measure-theoretic investigations like Dempster-Shafer *theory*. Since this dissertation deals exclusively with the *linguistic* aspects of uncertainty, i. e. the meaning of this term in natural language and the formalisation of this meaning as the semantics of suitable logical systems, the chance of misunderstandings is reduced by avoiding the term *uncertainty* as much as possible.

2. **Graded trust assessment** is achieved by making the classical concept of **validity** many-valued and making *degrees of validity* available within the logical language (in the form of **labels**). Because of the purpose of many-valued validity for modelling *graded trust assessment*, degrees of validity used within the logical language for modelling purposes are also called *degrees of trust*.

All this will presently be explained in detail; but first, a short survey of the state of the art.

These two aspects of uncertainty are nowadays discussed and distinguished in most books on **fuzzy logic**, usually under the names **vagueness** and **uncertainty**. See for instance the preface to [41] by G. Gerla or [84, section 1.1] by V. Novák, I. Perfilieva, and J. Močkoř for another discussion of the two concepts (although only vagueness, and hence **graded truth assessment**, is studied further in [84]). See also [53, chapter eight] by P. Hájek.

For the representation of **graded truth**, it seems natural to allow that logical formulae assume *truth values* between 0 (for 'false') and 1 (for 'true'). This assures that there is again a strong theoretical basis, namely the theory of *many-valued logic* (see for instance S. Gottwald [45] or P. Hájek [53]).

However, the 'classical' approaches to many-valued logic (see S. Gottwald [45] and also P. Hájek [53]) rely on employing the language of two-valued logic, varying *only* the set of *truth values* and the interpretation of the propositional logical operators (e. g. *implication*, *and*, *or*, *not*), of the quantifiers and finally of the concepts of *model* and *semantic consequence*.

It has to be stressed again that in most classical approaches to many-valued logic, the 'outer appearance' of logical formulae is retained while all changes happen 'behind the scenes'.

Obviously, for knowledge representation involving *graded truth assessment*, the mere change of interpretation from two-valued to many-valued logic while retaining the logical language is not sufficient; there has to be a means for *assessing* the truth of formulae in a knowledge base. One 'standard technique' for the representation of graded truth in the language of logical formulae which has been investigated also from the theoretical perspective lies in 'attaching' truth values to formulae (see for instance J. Pavelka [85], V. Novák et al [84]).

The representation of **graded trust** has been addressed, for instance, in *possibilistic logic* (see D. Dubois, J. Lang, and H. Prade [19]). There, the classical two-valued logic is employed (i. e. no graded truth is present) and degrees of possibility are attached to formulae.

Logics of both kinds have been studied and compared, for instance, by G. Gerla [41].

Labelled formulae, i. e. formulae from a classical logical language to which some *label* is attached, are widely used for the representation of uncertainty in knowledge. In the literature, various interpretations of the *semantics* of labels exist, and also variations regarding the *structure* of labels (see for instance L. A. Zadeh [105]; J. Baldwin [2]; D. Dubois, J. Lang, and H. Prade [18]; R. Hähnle [46]; J. J. Lu, N. V. Murray, and E. Rosenthal [74]; E. Y. Shapiro [90]).

In many cases, however, the *meaning* of a label is not precisely specified in such systems, casting doubt on what, from a semantic point of view, is really *formalised* by the labelled formulae or a corresponding inference mechanism.

There have been attempts to clarify this situation, especially by distinguishing whether a system of labelled logical formulae is used for *graded truth assessment* or *graded trust assessment* with respect to the states of affairs being modelled (see D. Dubois, H. Prade and others [17, 18, 24, 28, 30]; compare also P. Hájek and others [54, 56] and G. Gerla [41]). Logical systems which can accomplish one or the other task have been defined, studied and compared. It has been remarked that

> "The frequent confusion pervading the relationship between truth and (un)certainty in the approximate reasoning literature is apparently due to the lack of a dedicated paradigm for interpreting graded truth and degrees of uncertainty in a single framework." [18, p. 210]; see also [19]

Since this comment was formulated, some logical systems have been provided which allow a combination of *many-valued truth* with *degrees of necessity* [1, 29] and *degrees of belief* [49, 51, 55, 56], respectively (see section 5.7 for a comparison with the approach developed there).

In this dissertation, labelled logics for the representation of graded truth assessment and graded trust assessment are defined, taking a very general approach which supports a wide range of possible definitions of labels. In particular, it is possible to define logics which allow the *simultaneous* use of graded truth assessment and graded trust assessment in labels, with a precise definition of the semantic meaning of the labels. Suitable definitions for the *fundamental* concepts of **model** and **semantic consequence** are given.

Thus, the modelling power of a logic of labelled formulae can be raised *without* changing the underlying many-valued logic, simply by employing more expressive concepts of **label**. The theoretical apparatus of many-valued logic can still be applied to the underlying logic.

The remainder of the current chapter provides some more motivational remarks and concept clarifications, as well as some preliminaries from mathematics and fuzzy set theory which are needed for the further development.

Chapter 2 contains a 'tool-box' of results about fuzzy filters in lattices, a notion on which the definition of *label* shall be based.

Chapter 3 gives a first introduction of the notions *logical formula* and *labelled formula*. In accordance with the presentation of J. PAVELKA [85], the concepts are introduced in a rather abstract form which is largely independent of the concrete logical system employed.

The central semantic concepts of *model* and *semantic consequence* are defined in chapter 4. A systematic study of their properties is given.

In chapter 5, the logical definitions and theorems are illustrated by giving some special cases of logical systems definable by the means developed in chapters 3 and 4. It is demonstrated that some of the most popular logics for the representation of *graded truth assessment* (for instance, PAVELKA-style logics [85]) and *graded trust assessment* (for instance, possibilistic logic [19]) can be derived as special cases. The common framework of definition is exploited for giving a systematic comparison of these two types of logics. Furthermore, some particular cases of logics of *graded truth and graded trust assessment* which are generalisations of both PAVELKA-style logics and possibilistic logics are studied. The unified treatment of graded truth and graded trust in a single framework allows it to shed some light on the issue of *compositionality* which has been discussed at length in the literature under the keyword *truth-functionality.* Chapter 5 concludes with a comparison of existing paradigms for representing *graded truth and graded trust assessment* with the system presented in this dissertation.

Chapter 6 is devoted to summarising the results of this dissertation, describing extensions of the logical systems presented and possibilities for future developments. Among the subjects covered are some preliminary steps towards *automated deduction* in the logics described here (concerning *refutation*, *normal forms* and *derivation rules*) and alternatives to *lattice-based measures* for the representation of *graded trust.*

1.1 Degrees of Truth vs. Degrees of Validity

When a logical system for the formalisation of *fuzzy knowledge*, i. e. knowledge of which certain aspects are given by degree, is to be defined, it is natural to start with a system of classical two-valued logic and make certain parts of it, which may assume bivalent states only in the classic definition, many-valued.

The most well-known method of defining such logical systems is known as *many-valued logic* (see for instance [88]). This leads to the concept of *truth value*. While truth is two-valued in classical logic, i. e. a formula is either true or false (under an interpretation), it can assume any one from a fixed set of *truth values* in many-valued logic.

There is, however, another concept in classical logic which could be made many-valued, namely *validity*. This leads to the concept of *validity degree*. Validity degrees are relevant for such concepts as *satisfaction*, *modelness*, *validity* (of a formula), all of which are bivalent in classical (and also in many-valued) logics.

There are logics in which truth is two-valued and validity many-valued (such as *possibilistic logic*, see [19]), but obviously the most interesting case with most expressive power for fuzziness is the case in which *both* truth and validity are many-valued.

In this dissertation, a formal methodology is provided for defining logical systems in which truth and validity form two dimensions for making a logic ‘fuzzy’. This means many-valued truth and many-valued validity can be studied and applied independently, with a precise specification of their meaning and impact on the expressive power of the resulting logic.

The crucial idea for the systematic investigation of these logics is the use of *labelled formulae*. For precise definitions see chapter 3. For now, it should suffice to state that the definition of *labelled logic* is based on an *underlying logic* which is a classical two-valued or many-valued logic. To formulae from the underlying logic, *labels* are attached, and higher-level logical concepts like *model* or *semantic consequence* are defined for labelled formulae only.

While the underlying logic is still subject to all the well-known laws and tools of many-valued logic, the additional expressive power needed for representing fuzziness is put into the labels, which are able to express both dimensions of fuzziness studied here.

In the following considerations, truth values will be employed to express **graded truth**. If a formula from the underlying logic attains a certain truth value under a given interpretation, this means it is true to a certain degree. When a truth value t is attached to a formula F as a *label*, this shall express a *constraint* on the truth of the formula: To be valid under an interpretation, it is sufficient for F to attain or exceed the truth value t. This is in fact a relaxation of the strong constraint of classical many-valued logic [76], where a formula has to attain the truth value 1 to be considered valid. Clearly this constraint is too strong in the presence of fuzziness.

Validity degrees will be employed to express **graded trust**. When attached to formulae as labels, validity degrees shall also be called *degrees of trust*, because they express an assessment of the trust that the formula is valid. The more trust in the validity of the formula, the higher the degree attached.

All the concepts motivated here shall be discussed more deeply in the sequel. They are defined precisely and studied in chapters 3–5.

After having introduced the general framework in chapters 3–4, concrete logical systems are discussed in chapter 5. There, three types of logics are distinguished:

Logics of graded truth assessment: In this class, all logics are collected for which validity is two-valued. In this case, it is not possible to express graded trust in a label, and thus such logics are suited mainly for the expression of knowledge pertaining to *graded truth assessment*.

Logics of graded trust assessment: In this class, all logics are collected for which truth is two-valued. In this case, it is not possible to express graded truth in a label, and thus such logics are suited mainly for the expression of knowledge pertaining to *graded trust assessment*.

Logics of graded truth and graded trust assessment: In this class, all logics are collected for which truth and validity are *both* many-valued. In this case, a label can express graded truth as well as graded trust, yielding a logic of very high expressive power. However, this brings about also a high complexity of the resulting logics, and thus there has not been much of a formal study of such logics so far. From an informal, semantically oriented point of view, such logics have been proposed, applied and studied under several names (see L. A. Zadeh [106] and the survey in [18]).

Note that the *logics of graded truth assessment* correspond to the case "(b) Fuzzy statement; complete information" in the survey [18] of D. Dubois, J. Lang and H. Prade. *Logics of graded trust assessment* correspond to the case "(c) Crisp statement; incomplete information" and *logics of graded truth and graded trust assessment* correspond to the case "(d) Fuzzy statement; incomplete information".

There are some subtle differences between the cases distinguished in [18] and the different classes of logics studied here; for instance, to represent knowledge in a logic of graded truth assessment, it is not necessary to be *completely* (i. e. unambiguously) informed about the truth value of a formula which is to be part of a knowledge base; it is just not possible to express existing incomplete information in the form of *graded trust*. Some more explanations and illustrations of the exact modelling power of the classes of logics studied here will be given in the sequel; see in particular sections 3.4, 5.4, and 5.5.

The case "(a) Crisp statement; complete information" from [18] corresponds to classical two-valued logic, which is a special case (i. e. making truth *and* validity two-valued) of both *logics of graded truth assessment* and *logics of graded trust assessment*.

In the following chapters, the different logical systems are investigated in depth. After having laid the foundations in chapters 2 and 3, in chapter 4 properties shared by all the logics from all classes are investigated. Finally, in chapter 5, the different classes of logics distinguished above are studied separately and compared. To motivate the apparatus developed in chapters 2–4, some remarks on the fundamental differences between *degrees of truth* and *degrees of validity* are given in the following two subsections. Further motivations and explanations are given in sections 3.4, 3.5, and 4.1.

1.1.1 Truth Values

In classical two-valued and many-valued logic, the concept of truth value is well-known and well-understood for a long time. A couple of basic, well-known facts about truth values are listed below.

1. A truth value is induced in a formula by an interpretation of the symbols from the logical language.

   It is obvious that the concept of a "truth value of a formula" does not make sense without a corresponding interpretation, so "truth" is not a property of a formula in itself, but only of a formula together with an interpretation.

2. It is not a custom in logic to make interpretations 'available'. Instead, when defining higher level concepts like *validity*, *semantic equivalence* or *semantic consequence*, interpretations are usually 'quantified over': The definitions are obtained by quantifying over all interpretations and processing the resulting *set* of truth values, without regard as to which interpretation induced which truth value.

3. From the two previous items, a striking fact can be concluded: Truth values, though one of the most basic concepts of many-valued logics, are for *internal* use only, **not** on the 'user level'. The person defining and using systems of many-valued logics is **not** concerned with interpretations or truth values, but only with validity, semantic equivalence or semantic consequences of formulae.

   In fact, when looking at publications concerned *only* with many-valued logics (for instance [5, 8, 78, 100]), one may note that truth values play almost no role at all (unless constants for truth values are present in the logical language).

4. To summarise: A truth value is a property of a formula together with an interpretation; it is not available to the 'user' of a logical system, but is quantified over when defining user-level concepts like validity, semantic equivalence or semantic consequence.

1.1.2 Degrees of Validity

Using degrees of validity is much less common in investigations of logical systems. Classically, validity (and, correspondingly, the model relation) is a bivalent notion, even in many-valued logics. In many-valued logics, it is common to define a set of *designated truth values*. If the truth value of a formula under an interpretation falls into the set of designated truth values, the interpretation is considered to be a model for the formula (to satisfy the formula), otherwise it is not considered to be a model for the formula. If a formula is satisfied by all interpretations, it is considered to be *valid*.

Obviously, there is no notion of degree associated with this concept of model, and hence, validity. Even Pavelka's logic (see [85]) and "fuzzy logic in the narrow sense" (see [84], for instance), where the set of designated truth values is localised to formulae by attaching a truth value as a label to every formula, the notions of model and validity are two-valued.

The only logics investigated so far where the model relation and the concept of validity are given by degrees are based on *two-valued* logic. These logics are, for instance, possibilistic [19] and probabilistic [37, 77] logics, where formulae of two-valued logic are labelled by gradual assessments of the trust, possibility, necessity, or probability of the formula to be valid. The degree of validity of a labelled formula under an interpretation is then calculated depending on the truth value induced by the interpretation, based on the 'trust' in the formula expressed by the label.

As there are virtually no investigations so far of many-valued logics in which validity is given by degree, in the following, the basic intentions behind the forthcoming definitions are summarised:

1. Degrees of validity are properties of *labelled formulae*. The degree of validity of a labelled formula under an interpretation (degree of satisfaction of the formula by the interpretation) can only be determined by considering the truth value of the formula under the interpretation *and* the label.

   The label expresses the *trust* in the validity of the statement represented by the formula. If the formula is completely true, it should be considered completely valid. But if it is not completely true, it might still be considered somewhat valid (if the statement represented by the formula cannot be trusted to be always completely true).

   The validity of a labelled formula is then calculated by quantifying over the degrees of satisfaction by all possible interpretations.

2. As the degree of validity of a labelled formula depends essentially on the label, the 'user', i. e. the person using the logical system, has a strong influence on the resulting validity

degree. If they select a very strong label, the formula will be valid only if it is almost completely true under all interpretations. If they select a very weak label, the formula might attain a high degree of validity even if it has a very low truth value under certain interpretations.

3. Degrees of truth are, from an algebraic point of view, obviously truth-theoretic in nature, and thus will obey algebraic laws of e. g. MV-algebras, residuated or boolean lattices. In this thesis, the generic form of a complete lattice (see chapter 3) has been selected as the most general superstructure of all the possible truth-theoretic algebras.

   In contrast with this, degrees of validity seem to be basically *measure-theoretic* in nature. By choosing (again) a complete lattice (see chapter 3) as the algebraic structure for validity degrees, this dissertation is committed to possibility measures (see [12,13]). This is not the only choice, however. By choosing a HAUSDORFF space with an appropriate definition of integral, it would pose no principal problem to consider degrees of validity as probability degrees, as it has already been investigated for two-valued logics in the field of probabilistic logics. The adaption of the definitions and results from this dissertation to the case of probabilistic validity measures is an interesting subject for future investigations (see chapter 6).

1.2 Notation

The sets of all **natural numbers** and **real numbers** are denoted by $\mathbb{N}(=_{\text{def}} \{0, 1, \dots\})$ and $\mathbb{R}$, respectively. The notation $\langle r, s\rangle$ is used for the **closed interval** of all real numbers $t \in \mathbb{R}$ with $r \leqq t \leqq s$. The notation (r, s) is used for the **open interval** of all real numbers $t \in \mathbb{R}$ with $r < t < s$. The **half-open intervals** $(r, s\rangle$ and $\langle r, s)$ are defined accordingly.

The symbol $\mathfrak{P}$ denotes the classical concept of **power set**, that is, for an arbitrary set S, the set of all subsets of S is written $\mathfrak{P}S$. The **empty set** is denoted by $\varnothing$.

For two sets S, T, the **set of all mappings** from S into T is written T^S.

The **range** of a mapping $f : S \to T$ is denoted $\operatorname{rg} f =_{\text{def}} \left\{t \middle| t \in T \text{ and } \exists s \in S : t = f(s)\right\}$.

Ordered pairs are denoted by using square brackets, i. e. the **ordered pair** of a and b is written $[a, b]$. **Ordered tuples** of arbitrary length are defined canonically by iterating ordered pairs. For $n \in \mathbb{N}$ and a set S, the notation S^n denotes the **n-fold Cartesian product** of S, i. e. the set of all n-tuples of elements from S, with the special cases $S^0 =_{\text{def}} \{\varnothing\}$ and $S^1 =_{\text{def}} S$.

1.3 Lattices

Following the approaches of J. A. GOGUEN [43] and J. PAVELKA [85–87], the set of truth values for the many-valued logic underlying the labelled formulae (the concepts of *truth value*, *formula* and *labelled formula* are introduced in chapter 3) is assumed to possess a *complete lattice structure* $[L, \sqcap, \sqcup]$ such that L contains at least two distinct elements. This approach makes it possible to apply the following results to a variety of logical systems, including finitely and infinitely many-valued logics and (by Observation 1.4.1) even logics the truth values of which are *fuzzy sets*. This makes this approach compatible with current trends in applications, for instance approximate reasoning systems which use *fuzzy sets* as truth values (see for instance H. THIELE [94]).

Remark

All lattices considered shall (explicitly or implicitly) be assumed to be *complete* (see below); the **unit element** 1 and the **zero element** 0 of a complete lattice are assumed to be distinct,

i. e. $1 \neq 0$ (unless stated otherwise). These properties are not always needed, but for logical considerations this assures compatibility with the classical case. •

Definition 1.3.1 (Lattice)
Given a non-empty set L and two binary operations $\sqcap$, $\sqcup$ on L, the triple $\mathfrak{L} = [L, \sqcap, \sqcup]$ is said to be a ***Lattice***
$=_{\text{def}}$ *1. $\sqcap, \sqcup$ are* ***commutative****, i. e. for all $a, b \in L$,*

$$a \sqcap b = b \sqcap a \qquad\qquad a \sqcup b = b \sqcup a \tag{1.1}$$

2. $\sqcap, \sqcup$ are ***associative****, i. e. for all $a, b, c \in L$,*

$$a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c \qquad\qquad a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c \tag{1.2}$$

3. $\sqcap, \sqcup$ fulfil the ***absorption laws****, i. e. for all $a, b \in L$,*

$$a \sqcap (a \sqcup b) = a \qquad\qquad a \sqcup (a \sqcap b) = a \tag{1.3}$$

L is said to be the ***domain*** *of $\mathfrak{L}$ and $\sqcap, \sqcup$ are said to be the* ***meet*** *and* ***join*** *of $\mathfrak{L}$, respectively.*

The fundamentals of lattice theory, which may be found for instance in [4] by G. Birkhoff, shall not be introduced in detail. It should however be mentioned that the lattice meet $\sqcap$ and join $\sqcup$ induce on L a **partial order relation** $\sqsubseteq$ by

$$a \sqsubseteq b =_{\text{def}} a \sqcap b = a \qquad \text{(equivalent with } a \sqcup b = b\text{).} \qquad (a, b \in L) \tag{1.4}$$

Vice versa, for every **partially ordered set** $[L, \sqsubseteq]$ for which every two-element subset $M = \{a, b\} \subseteqq L$ has a **greatest lower bound** $\bigsqcap M$ and a **least upper bound** $\bigsqcup M$ with respect to $\sqsubseteq$, one can define a lattice structure $[L, \sqcap, \sqcup]$ by

$$a \sqcap b =_{\text{def}} \bigsqcap\{a, b\}, \qquad (a, b \in L) \tag{1.5}$$

$$a \sqcup b =_{\text{def}} \bigsqcup\{a, b\}. \tag{1.6}$$

Furthermore, if $\bigsqcap$ and $\bigsqcup$ are defined with respect to the partial order $\sqsubseteq$ induced by a lattice structure $[L, \sqcap, \sqcup]$ via definition (1.4), then for every $a, b \in L$, $a \sqcap b$ and $a \sqcup b$ coincide with $\bigsqcap\{a, b\}$ and $\bigsqcup\{a, b\}$, respectively.

The **completeness** of the lattice $[L, \sqcap, \sqcup]$ is equivalent with the statement that for *every* subset $M \subseteqq L$, the **greatest lower bound** $\bigsqcap M$ and the **least upper bound** $\bigsqcup M$ with respect to $\sqsubseteq$ exist and lie in L. In particular, $[L, \sqcap, \sqcup]$ has a **unit element** 1 and a **zero element** 0 defined by

$$1 =_{\text{def}} \bigsqcup L \tag{1.7}$$

$$0 =_{\text{def}} \bigsqcap L. \tag{1.8}$$

In the following, it is assumed that always $0 \neq 1$ holds.

Given a lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$, its **dual** lattice is defined by

$$\mathscr{D}(\mathfrak{L}) =_{\text{def}} [L, \sqcup, \sqcap].$$

$\mathfrak{L}$ is said to be **distributive** iff the following equations hold for all $a, b, c \in L$:

$$a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c), \tag{1.9}$$

$$(1.10) \qquad a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c).$$

$\mathfrak{L}$ is said to be **completely distributive** wrt. $\bigsqcap$ iff the following equation holds for all $a \in L$ and $M \subseteqq L$:

$$(1.11) \qquad a \sqcup \bigsqcap M = \bigsqcap \{a \sqcup b | b \in M\}.$$

Analogously, $\mathfrak{L}$ is said to be **completely distributive** wrt. $\bigsqcup$ iff the following equation holds for all $a \in L$ and $M \subseteqq L$:

$$(1.12) \qquad a \sqcap \bigsqcup M = \bigsqcup \{a \sqcap b | b \in M\}.$$

$\mathfrak{L} = [L, \sqcap, \sqcup]$ is said to be a **chain** iff for all $a, b \in L$,

$$(1.13) \qquad a \sqsubseteq b \quad \text{or} \quad b \sqsubseteq a.$$

A unary mapping $\nu : L \to L$ is said to be a **complementation** on a lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$ (with zero 0 and unit 1) iff for all $a \in L$,

$$(1.14) \qquad a \sqcap \nu(a) = 0 \text{ and } a \sqcup \nu(a) = 1.$$

$\mathfrak{L}$ is said to be **complementary** iff there exists a complementation on $\mathfrak{L}$. If $\mathfrak{L}$ is distributive, then a complementation (if existent) is unique. A complementary and distributive lattice is said to be a **Boolean algebra**.

Note that on a Boolean algebra, the complementation is **bijective** and **involutive**, i. e. for all $a \in L$,

$$\nu(\nu(a)) = a.$$

Furthermore, in a Boolean algebra, the complementation is **order-reversing**, i. e. for all $a, b \in L$,

$$a \sqsubseteq b \text{ iff } \nu(b) \sqsubseteq \nu(a).$$

None of these properties, however, is necessary for a complementation if $\mathfrak{L}$ is not distributive.

Vice versa, being an involutive, order-reversing bijection is not sufficient for ν to be a complementation on $\mathfrak{L}$.

Given two lattices $\mathfrak{L} = [L, \sqcap, \sqcup]$ and $\mathfrak{L}' = [L', \sqcap', \sqcup']$, then $\mathfrak{L}$ is said to be a **sublattice** of $\mathfrak{L}'$ ($\mathfrak{L} \Subset \mathfrak{L}'$) iff $L \subseteqq L'$ and $\sqcap$ coincides with $\sqcap'$ and $\sqcup$ coincides with $\sqcup'$ on L.

Definition 1.3.2 (Filters of a lattice)
Let $\mathfrak{L} = [L, \sqcap, \sqcup]$ be a lattice.

A nonempty subset F of L is said to be a ***filter*** *of $[L, \sqcap, \sqcup]$*
$=_{\text{def}}$
1. *If $a, b \in F$, then $a \sqcap b \in F$.*
2. *If $a \in F$ and $b \in L$, then $a \sqcup b \in F$.*

The set of all filters of $\mathfrak{L}$ is denoted by $\mathrm{Fl}(\mathfrak{L})$.

To each lattice element $a \in L$ its ***principal filter*** *$\overline{a}$ is associated by*

$$(1.15) \qquad \overline{a} =_{\text{def}} \{b | b \in L \text{ and } a \sqsubseteq b\}.$$

The set of all principal filters of $\mathfrak{L}$ is denoted by $\mathrm{PFl}(\mathfrak{L}) =_{\text{def}} \{\overline{a} | a \in L\}$.

Observations 1.3.1 (Properties of filters)
The following observations are cited from the literature (see for instance [57]):

1. Requirement 2 of Definition 1.3.2 is equivalent with any one of the following statements:

   2a If $a \in F$ and $b \in L$ and $a \sqsubseteq b$, then $b \in F$.

   2b If $a, b \in L$ and $a \sqcap b \in F$, then $a \in F$.

2. Item 2b above implies

   A nonempty subset F of L is a **filter** of $[L, \sqcap, \sqcup]$ if and only if for all $a, b \in L$,

   $$a, b \in F \;\text{ iff }\; a \sqcap b \in F.$$

3. If $\mathfrak{L} = [L, \sqcap, \sqcup]$ is a lattice with 1, then the ordinary *subset relation* $\subseteqq$ induces on $\mathrm{Fl}(\mathfrak{L})$ a complete lattice structure, the meet and greatest lower bound of which coincide with the intersection $\cap$ and the greatest lower bound $\bigcap$ in the complete lattice of all subsets of L.

   This lattice is denoted $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$.

   The join $\uplus$ of $\mathrm{Fl}(\mathfrak{L})$ is uniquely determined by definition (1.6), which can be formulated as follows, for $F, G \in \mathrm{Fl}(\mathfrak{L})$.

   $$F \uplus G =_{\text{def}} \bigcap \left\{ H \,\middle|\, H \in \mathrm{Fl}(\mathfrak{L}) \;\text{ and }\; F \cup G \subseteqq H \right\}$$

   This definition is equivalent with

   (1.16) $\qquad F \uplus G =_{\text{def}} \{c | c \in L \;\text{ and there are }\; a \in F,\, b \in G \;\text{ such that }\; a \sqcap b \sqsubseteq c\}\,.$

4. The *unit element* of $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$ is L. The *zero element* of $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$ is $\{1\}$. If $\mathfrak{L}$ is a *chain*, then so is $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$.

5. For every lattice element $a \in L$, the *principal filter* $\overline{a}$ is a filter of $\mathfrak{L}$. $\overline{0}$ coincides with the unit element L of $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$. $\overline{1}$ coincides with the zero element $\{1\}$ of $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$.

6. In the lattice $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$, the following holds for $a, b \in L$:

   $$\overline{a} \cap \overline{b} = \overline{a \sqcup b}$$
   $$\overline{a} \uplus \overline{b} = \overline{a \sqcap b}$$

   Thus $\left[\mathrm{PFl}(\mathfrak{L}), \cap, \uplus\right]$is a *sublattice* of the complete lattice $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\right]$.

   Furthermore,

   $$\overline{a} = \overline{b} \;\text{ iff }\; a = b,$$

   thus $\bar{\cdot}$ is a *lattice isomorphism* from $\mathfrak{L}$ onto the *dual* lattice $\left[\mathrm{PFl}(\mathfrak{L}), \uplus, \cap\right]$.

7. Let $a \in L$ and $F \in \mathrm{Fl}(\mathfrak{L})$.

   Then $\overline{a} \subseteqq F$ if and only if $a \in F$. •

Remark
A variety of lattice structures shall be employed in the following, for all of which the operations and induced partial orders *may* be different, unless stated otherwise. This is made clear by using different symbols for different operations wherever possible, but at some places, overloading cannot be avoided. The meaning of overloaded operator symbols will always be clear from the context. •

Examples 1.3.1 (Lattices)

1. The classical **two-valued Boolean lattice** $\mathfrak{B} =_{\text{def}} \left[\{0,1\}, \text{and}, \text{or}\right]$, where and and or are characterised by the 'truth table'

a	b	and(a,b)	or(a,b)
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

   is the basis of two-valued logic. It is finite and thus trivially complete. The induced partial order coincides on $\{0,1\}$ with the usual order $\leqq$ of real numbers.

2. The **real unit interval**, i. e. the closed interval $\langle 0,1\rangle$ of real numbers, which plays a fundamental role in fuzzy logic as the most common *set of truth values*, is a lattice with respect to the operations min and max for meet and join, respectively. The induced partial order is the usual order $\leqq$ of real numbers. The lattice $\mathfrak{F} =_{\text{def}} \left[\langle 0,1\rangle, \min, \max\right]$ is complete.

   Unit and zero element turn out to be 1 and 0, respectively.

   The Filters of $\mathfrak{F}$ are all nonempty intervals from $\langle 0,1\rangle$ which are closed above with 1, i. e. if $I \in \mathrm{Fl}(\mathfrak{F})$, then

   - there is a real number $r \in \langle 0,1\rangle$ such that $I = \langle r,1\rangle$
   - or there is a real number $r \in \langle 0,1)$ such that $I = (r,1\rangle$. •

1.4 $\mathfrak{L}$-Fuzzy Sets

Fix a non-empty set U called **universe** and a complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$. In this section, some simple facts about $\mathfrak{L}$-fuzzy set are recalled. The two simple observations at the end of the section can be found in most textbooks on the subject.

An **$\mathfrak{L}$-fuzzy set** on U (see J. A. Goguen [42]) is defined to be a mapping

$$\mathcal{F} : U \to L.$$

Remark
In the remainder of this dissertation, different types of sets will play the role of *universe*. •

For $u \in U$, the value $\mathcal{F}(u)$ is said to be the **degree of membership** of u in $\mathcal{F}$.

The lattice operations $\sqcap, \sqcup$ and the partial ordering $\sqsubseteq$ induced by the lattice structure on L are used for defining on L^U a **meet** $\cap$, a **join** $\cup$ and a **subset relation** $\subseteqq$. Let $\mathcal{F}, \mathcal{G}$ be $\mathfrak{L}$-fuzzy sets on U.

$$(\mathcal{F} \cap \mathcal{G})(u) =_{\text{def}} \mathcal{F}(u) \sqcap \mathcal{G}(u) \qquad (u \in U) \tag{1.17}$$

$$(\mathcal{F} \cup \mathcal{G})(u) =_{\text{def}} \mathcal{F}(u) \sqcup \mathcal{G}(u) \tag{1.18}$$

$$\mathcal{F} \subseteqq \mathcal{G} =_{\text{def}} \mathcal{F}(u) \sqsubseteq \mathcal{G}(u) \text{ for every } u \in U \tag{1.19}$$

For the unit interval $\mathfrak{J}$, these definitions correspond to L. A. Zadeh's original fuzzy set theory [104].

Observation 1.4.1 (Complete lattice of $\mathfrak{L}$-fuzzy sets)
If $[L, \sqcap, \sqcup]$ is a complete lattice, then $\left[L^U, \cap, \cup\right]$ as defined by (1.17) and (1.18) is a complete lattice with induced partial order $\subseteqq$ as defined in (1.19). •

Proof
(see also J. Pavelka [85, p. 46])

By the pointwise definitions of $\cap, \cup, \subseteqq$ for $\mathfrak{L}$-fuzzy sets, the lattice properties for $\left[L^U, \cap, \cup\right]$ follow immediately. Furthermore, for every $\Phi \subseteqq L^U$ and $u \in U$,

$$\left(\bigcap \Phi\right)(u) = \bigsqcap \left\{\mathcal{F}(u) \middle| \mathcal{F} \in \Phi\right\}, \tag{1.20}$$

$$\left(\bigcup \Phi\right)(u) = \bigsqcup \left\{\mathcal{F}(u) \middle| \mathcal{F} \in \Phi\right\}, \tag{1.21}$$

hence the completeness of $\left[L^U, \cap, \cup\right]$ follows directly from the completeness of $[L, \sqcap, \sqcup]$. □

The **empty $\mathfrak{L}$-fuzzy set** $\not{\emptyset}$ on U is defined for every $u \in U$ by

$$\not{\emptyset}(u) =_{\text{def}} 0. \qquad \text{(where 0 is defined in (1.8))} \tag{1.22}$$

As a possibility of translating between fuzzy sets and classical sets, for $\mathcal{F} \in L^U$ and every $a \in L$ the **a-cut** of $\mathcal{F}$ is defined by

$$\mathrm{CUT}_a(\mathcal{F}) =_{\text{def}} \left\{u \middle| u \in U \text{ and } a \sqsubseteq \mathcal{F}(u)\right\}. \tag{1.23}$$

By the completeness of the lattice $\mathfrak{L}$, the following simple characterisation of $\mathfrak{L}$-fuzzy sets by their a-cuts is obtained.

Observation 1.4.2 (Constructing a fuzzy set from its cuts)
For every $\mathcal{F} \in L^U$ and every $u \in U$,

$$\mathcal{F}(u) = \bigsqcup \left\{a \middle| a \in L \text{ and } u \in \mathrm{CUT}_a(\mathcal{F})\right\}.$$

•

Proof
The proof is very simple, carried out in two steps.

1. $\mathcal{F}(u) \sqsubseteq \bigsqcup \left\{a \middle| a \in L \text{ and } u \in \mathrm{CUT}_a(\mathcal{F})\right\}$.
   It is sufficient to prove that $\mathcal{F}(u) \in \left\{a \middle| a \in L \text{ and } u \in \mathrm{CUT}_a(\mathcal{F})\right\}$, which is obtained from $u \in \mathrm{CUT}_{\mathcal{F}(u)}(\mathcal{F})$, which in turn follows by (1.23) from $\mathcal{F}(u) \sqsubseteq \mathcal{F}(u)$.

2. $\bigsqcup \left\{ a \middle| a \in L \text{ and } u \in \mathrm{CUT}_a(\mathcal{F}) \right\} \sqsubseteq \mathcal{F}(u)$.

   It is sufficient to prove that for every $a \in L$,

$$\text{if } u \in \mathrm{CUT}_a(\mathcal{F})\,, \text{ then } a \sqsubseteq \mathcal{F}(u).$$

   But this is just the definition of CUT_a (see (1.23)). □

Finally, the **support** $\operatorname{supp} \mathcal{F}$ of an $\mathfrak{L}$-fuzzy set $\mathcal{F} \in L^U$ is defined by

$$\operatorname{supp} \mathcal{F} =_{\mathrm{def}} \left\{ u \middle| u \in U \text{ and } \mathcal{F}(u) \neq 0 \right\}. \tag{1.24}$$

$\mathcal{F}$ is said to be **finite** iff $\operatorname{supp} \mathcal{F}$ is finite.

2 Fuzzy Filters in Lattices

In this chapter, a class of *special* $\mathfrak{L}$-fuzzy sets is defined which form a 'fuzzy' counterpart to *filters* of a lattice and at the same time form a 'link' between two lattice structures. In the sequel, **fuzzy filters** of the truth-value lattice are used as **labels** for formulae, yielding a considerable gain in expressive power of the resulting **labelled logic**.

The concept of **fuzzy filter** on a lattice is well-known in the literature (see for instance M. A. de Prada Vicente and M. Saralegui Aranguren [16], B. Yuan and W. Wu [103], P. Eklund and W. Gähler [32], W. Gähler [39,40], Y. J. Lee [65]). Some of the approaches referred to make strong assumptions on the lattices involved (of being distributive, chains, or function spaces; sometimes only the unit interval is considered as the lattice of membership values).

One of the more general and comprehensive studies of fuzzy filters on a lattice has been reported by B. Yuan and W. Wu [103], for the special case that the lattice is *distributive* and that the *membership degrees* of the fuzzy sets which are used to model fuzzy filters are taken from the *unit interval* $\mathfrak{F}$.

In the following, a more general definition is given, leaving out the distributivity condition and considering membership degrees from an arbitrary complete lattice. Furthermore, the thrust of the investigations presented here is slightly different. While B. Yuan and W. Wu study the relationship between fuzzy filters and fuzzy congruences on a distributive lattice, in the following the complete lattice of fuzzy filters of an arbitrary complete lattice is investigated. Selected results from this chapter have been reported by the author in [71].

In further chapters, the semantics of logics in which formulae are labelled by fuzzy filters of the truth-value lattice are studied, which is impossible without a fairly complete theory of fuzzy filters on a lattice. The lattice-theoretic foundations for theoretical investigations of the semantics of fuzzy filter-based logics are laid in this chapter.

2.1 Basic Definitions and Propositions

Let $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$ be **complete lattices** with induced partial orders $\sqsubseteq, \preccurlyeq$, respectively. The following definition of a **fuzzy filter** is well-known in the literature, with slight differences stemming from the fact that in most publications, the lattices $\mathfrak{L}$ and/or $\mathfrak{L}'$ have a special form. In particular, $\mathfrak{L}$ is often a lattice of *mappings*, and $\mathfrak{L}'$ is frequently the *real unit interval*. The definition given here is the most general one. The definition (2.1) of a **principal fuzzy filter** is also found in [39, section 4.1].

Definition 2.1.1 (Fuzzy filters of a lattice)
*An $\mathfrak{L}'$-fuzzy set $\mathcal{F}$ on L is said to be an **$\mathfrak{L}'$-fuzzy filter** of $\mathfrak{L}$*
$=_{\text{def}}$

1. *For all $a, b \in L$, $\mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcap b)$.*
2. *For all $a, b \in L$, $\mathcal{F}(a) \preccurlyeq \mathcal{F}(a \sqcup b)$.*
3. *$\mathcal{F}(1) = 1$.*

The set of all $\mathfrak{L}'$-fuzzy filters of $\mathfrak{L}$ is denoted by $\mathfrak{L}'\text{-Fl}(\mathfrak{L})$.

For $d \in L'$, associate with each lattice element $a \in L$ its **principal fuzzy d-filter** ${}^{d}\overline{a}$ *by*

$$ {}^{d}\overline{a}\,(b) =_{\text{def}} \begin{cases} 1, & \text{if } b = 1 \\ d, & \text{if } b \neq 1 \text{ and } a \sqsubseteq b \\ 0, & \text{if } \textbf{not } a \sqsubseteq b \end{cases} \qquad (b \in L) \tag{2.1}$$

The set of all principal fuzzy d-filters of $\mathfrak{L}$ is denoted by $d\text{-PFl}\,(\mathfrak{L}) =_{\text{def}} \left\{ {}^{d}\overline{a} \,\middle|\, a \in L \right\}$. *The set of* ***all*** *principal fuzzy filters of $\mathfrak{L}$ is denoted by* $\mathfrak{L}'\text{-PFl}\,(\mathfrak{L}) =_{\text{def}} \bigcup \left\{ d\text{-PFl}\,(\mathfrak{L}) \,\middle|\, d \in L' \right\}$.

Note that the concept of *fuzzy filter* is meant to be a *generalisation* of the concept of *filter* with respect to the lattice $\mathfrak{L}$. This should be distinguished carefully from the concept of *filter* with respect to the lattice $\left[L^U, \cap, \cup\right]$, which is *not* related.

First of all, some compatibility results with the classical case.

Proposition 2.1.1 (Cuts of fuzzy filters are filters)

1. *$\mathcal{F} \in \mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$ if and only if for every $d \in L'$, the* ***d-cut*** *$\mathrm{CUT}_d(\mathcal{F})$ of $\mathcal{F}$ is a Filter of $\mathfrak{L}$.*
2. *For every $d, d' \in L'$, $\mathrm{CUT}_d\left({}^{d'}\overline{a}\right) \in \mathrm{Fl}(\mathfrak{L})$. In particular, if $d \neq 0$, then $\mathrm{CUT}_d\left({}^{d}\overline{a}\right)$ is the principal filter $\overline{a}$.*

Proof

ad 1. Both implications are proved separately.

1. If $\mathcal{F} \in \mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$, then for every $d \in L'$, $\mathrm{CUT}_d(\mathcal{F}) \in \mathrm{Fl}(\mathfrak{L})$.
   That for every $d \in L'$, $\mathrm{CUT}_d(\mathcal{F})$ is nonempty is guaranteed by condition 3 of Definition 2.1.1. Conditions 1 and 2 of Definition 1.3.2 remain to be checked.
   ad 1. Let $a, b \in \mathrm{CUT}_d(\mathcal{F})$. By definition of CUT_d, $d \preccurlyeq \mathcal{F}(a)$ and $d \preccurlyeq \mathcal{F}(b)$. Thus, d is a lower bound of $\mathcal{F}(a)$ and $\mathcal{F}(b)$, hence
   $$d \preccurlyeq \mathcal{F}(a) \curlywedge \mathcal{F}(b)$$
   and because $\mathcal{F}$ is an $\mathfrak{L}'$-fuzzy filter of $\mathfrak{L}$,
   $$\mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcap b),$$
   thus
   $$d \preccurlyeq \mathcal{F}(a \sqcap b).$$
   This means $a \sqcap b \in \mathrm{CUT}_d(\mathcal{F})$, which establishes condition 1.
   ad 2. Let $a \in \mathrm{CUT}_d(\mathcal{F})$ and $b \in L$. By definition of CUT_d, $d \preccurlyeq \mathcal{F}(a)$. By condition 2 of Definition 2.1.1,
   $$d \preccurlyeq \mathcal{F}(a \sqcup b).$$
   This means $a \sqcup b \in \mathrm{CUT}_d(\mathcal{F})$, which establishes condition 2.
2. If for every $d \in L'$, $\mathrm{CUT}_d(\mathcal{F}) \in \mathrm{Fl}(\mathfrak{L})$, then $\mathcal{F} \in \mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$.
   Conditions 1, 2, and 3 of Definition 2.1.1 are to be checked.

ad 1. Let $a, b \in L$. Then

$$\mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a)$$
$$\mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(b)$$

and thus by (1.23),

$$a \in \mathrm{CUT}_{\mathcal{F}(a) \curlywedge \mathcal{F}(b)}(\mathcal{F})$$
$$b \in \mathrm{CUT}_{\mathcal{F}(a) \curlywedge \mathcal{F}(b)}(\mathcal{F}),$$

hence, because $\mathrm{CUT}_{\mathcal{F}(a) \curlywedge \mathcal{F}(b)}(\mathcal{F})$ is a filter of $\mathfrak{L}$,

$$a \sqcap b \in \mathrm{CUT}_{\mathcal{F}(a) \curlywedge \mathcal{F}(b)}(\mathcal{F}),$$

thus, again by (1.23),

$$\mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcap b),$$

which had to be proved.

ad 2. Let $a, b \in L$. Obviously,

$$a \in \mathrm{CUT}_{\mathcal{F}(a)}(\mathcal{F})$$

and thus, because $\mathrm{CUT}_{\mathcal{F}(a)}(\mathcal{F})$ is a filter of $\mathfrak{L}$,

$$a \sqcup b \in \mathrm{CUT}_{\mathcal{F}(a)}(\mathcal{F}).$$

By definition (1.23) of CUT, this means

$$\mathcal{F}(a) \preccurlyeq \mathcal{F}(a \sqcup b),$$

which had to be proved.

ad 3. Trivial, because 1 is an element of every filter of $\mathfrak{L}$, thus also of $\mathrm{CUT}_1(\mathcal{F})$.

ad 2. To prove that for every $d, d' \in L'$, $\mathrm{CUT}_d\left({}^{d'}\overline{a}\right) \in \mathrm{Fl}(\mathfrak{L})$, three cases are distinguished.

Case 1. $d = 0$.
By definition (1.23), $\mathrm{CUT}_d\left({}^{d'}\overline{a}\right) = L$, which is obviously a filter of $\mathfrak{L}$.

Case 2. $d \neq 0$ and $d \preccurlyeq d'$.
By definition (1.23),

$$\mathrm{CUT}_d\left({}^{d'}\overline{a}\right) = \left\{ b \,\middle|\, b \in L \text{ and } d \preccurlyeq {}^{d'}\overline{a}\,(b) \right\}. \tag{2.2}$$

Excluding the case ${}^{d'}\overline{a}\,(b) = 0$ from definition (2.1) because $d \neq 0$, and considering that $d \preccurlyeq d'$, this yields

$$= \{ b | b \in L \text{ and } a \sqsubseteq b \},$$

which is just the definition of $\overline{a}$ by (1.15), thus establishing the result by Observation 1.3.1.5.

Case 3. **Not** $d \preccurlyeq d'$.
Considering (2.2) and (2.1), in this case $\mathrm{CUT}_d\left({}^{d'}\overline{a}\right) = \{1\}$, which is a filter of $\mathfrak{L}$.

The claim that if $d \neq 0$, then $\mathrm{CUT}_d\left({}^{d}\overline{a}\right) = \overline{a}$ has been proved in case 2 above. □

Corollary 2.1.2 (Principal fuzzy filters are fuzzy filters)
For every lattice element $a \in L$ and $d \in L'$, the ***principal fuzzy d-filter*** ${}^{d}\overline{a}$ *is an* $\mathfrak{L}'$***-fuzzy filter*** *of* $\mathfrak{L}$.

Proof
By combining item 2 and item 1 of Proposition 2.1.1. □

The following lemma corresponds to [39, proposition 4.1].

Lemma 2.1.3 (Monotonicity properties of principal fuzzy filters)

1. *Let $a, b \in L$ and $d, d' \in L'$. If $b \sqsubseteq a$ and $d \preccurlyeq d'$, then*

$$ {}^{d}\overline{a} \subseteqq {}^{d'}\overline{b} \tag{2.3}$$

2. *Let $a, b \in L$ and $d, d' \in L'$. If $a \neq 1$ and $d \neq 0$ and*

$$ {}^{d}\overline{a} \subseteqq {}^{d'}\overline{b}, $$

then $b \sqsubseteq a$ and $d \preccurlyeq d'$.

Proof
Follows immediately from the definitions (2.1) and (1.19). □

Observation 2.1.4 (Fuzzy filters of a chain)
If $\mathfrak{L}$ is a chain, then for every $\mathcal{F} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$ and all $a, b \in L$,

$$ \mathcal{F}(a) \curlyvee \mathcal{F}(b) = \mathcal{F}(a \sqcup b). \tag{2.4}$$ •

Proof
$\mathcal{F}(a) \curlyvee \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcup b)$ follows immediately from Definition 2.1.1.2.

For establishing $\mathcal{F}(a \sqcup b) \preccurlyeq \mathcal{F}(a) \curlyvee \mathcal{F}(b)$, observe that from the fact that $\mathfrak{L}$ is a chain, it follows that $a \sqcup b = a$ or $a \sqcup b = b$. But $\mathcal{F}(a) \preccurlyeq \mathcal{F}(a) \curlyvee \mathcal{F}(b)$ and $\mathcal{F}(b) \preccurlyeq \mathcal{F}(a) \curlyvee \mathcal{F}(b)$ hold trivially. □

The following proposition is meant to tighten the compatibility with the paper of B. Yuan and W. Wu [103].

Observation 2.1.5 (Fuzzy filters are fuzzy sublattices)
Every $\mathfrak{L}'$-fuzzy filter of $\mathfrak{L}$ is a $\mathfrak{L}'$**-fuzzy sublattice** of $\mathfrak{L}$. This means that for every $\mathcal{F} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$ and all $a, b \in L$,

$$ \mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcup b) \curlywedge \mathcal{F}(a \sqcap b). \tag{2.5}$$ •

Proof
By Definition 2.1.1, $\mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcap b)$ and $\mathcal{F}(a) \preccurlyeq \mathcal{F}(a \sqcup b)$.

The result then follows immediately from the fact that $\mathfrak{L}'$ is a lattice. □

The results collected in Observations 1.3.1 are now obtained in a generalised form as theorems. Some of the easier observations are stated in the remainder of this section, while the more involved results concerning complete lattices of fuzzy filters are presented in section 2.2.

Proposition 2.1.6 (Equivalent definitions of fuzzy filter)
Requirement 2 of Definition 2.1.1 *is equivalent with any one of the following two equivalent statements:*

2a *For all $a, b \in L$, if $a \sqsubseteq b$, then $\mathcal{F}(a) \preccurlyeq \mathcal{F}(b)$.*

2b *For all $a, b \in L$, $\mathcal{F}(a \sqcap b) \preccurlyeq \mathcal{F}(a) \curlywedge \mathcal{F}(b)$.*

Proof
2 ⇒ 2a. Let $a, b \in L$ such that $a \sqsubseteq b$. By (1.4), this means $a \sqcup b = b$, thus $\mathcal{F}(a) \preccurlyeq \mathcal{F}(b)$ by condition 2.

2a ⇒ 2b. Let $a, b \in L$. $a \sqcap b$ is a lower bound of a and b, thus $a \sqcap b \sqsubseteq a$ and $a \sqcap b \sqsubseteq b$. By 2a, $\mathcal{F}(a \sqcap b) \preccurlyeq \mathcal{F}(a)$ and $\mathcal{F}(a \sqcap b) \preccurlyeq \mathcal{F}(b)$. Thus $\mathcal{F}(a \sqcap b)$ is a lower bound of $\mathcal{F}(a)$ and $\mathcal{F}(b)$, hence $\mathcal{F}(a \sqcap b) \preccurlyeq \mathcal{F}(a) \curlywedge \mathcal{F}(b)$.

2b ⇒ 2. Let $a, b \in L$. Then

$$\begin{aligned}\mathcal{F}(a) &= \mathcal{F}(a \sqcap (a \sqcap b)) && \text{(by the absorption law of lattices)}\\ &\preccurlyeq \mathcal{F}(a) \curlywedge \mathcal{F}(a \sqcup b) && \text{(by 2b)}\end{aligned}$$

and, because $\mathcal{F}(a) \curlywedge \mathcal{F}(a \sqcup b)$ is a lower bound of $\mathcal{F}(a)$ and $\mathcal{F}(a \sqcup b)$,

$$\preccurlyeq \mathcal{F}(a \sqcup b)$$ □

Remark
B. Yuan and W. Wu [103] characterise fuzzy filters by (2.5) and 2a.

Thus an $\mathfrak{F}$-fuzzy filter of $\mathfrak{L}$ is a *fuzzy filter* on $\mathfrak{L}$ in the sense of [103]. The reverse direction does not hold, in general, because of the additional condition 3 in Definition 2.1.1.

In fact, B. Yuan and W. Wu do not state *any* non-emptiness condition for fuzzy filters, thereby sacrificing the compatibility with the classical case.

For this definition, condition 3 cannot be weakened without sacrificing either the fact that every filter must in some sense be non-empty or the completeness of the lattice of fuzzy filters established in theorem 2.2.1. •

Corollary 2.1.7 (Short definition of fuzzy filter)
An $\mathfrak{L}'$-fuzzy set $\mathcal{F}$ on L is an **$\mathfrak{L}'$-fuzzy filter** *of $\mathfrak{L}$ if and only if $\mathcal{F}(1) = 1$ and for all $a, b \in L$, $\mathcal{F}(a) \curlywedge \mathcal{F}(b) = \mathcal{F}(a \sqcap b)$.*

Proof
Follows immediately from Definition 2.1.1 and item 2b above. □

Remark
As remarked in [103], Corollary 2.1.7 implies that $\mathcal{F}$ is an **$\mathfrak{L}'$-fuzzy filter** of $\mathfrak{L}$ if and only if $\mathcal{F}$ is a **homomorphism** from the structure $[L, \sqcap, 1]$ into the structure $\left[L', \curlywedge, 1\right]$. •

The following lemma corresponds to [39, proposition 4.1].

Lemma 2.1.8 (Degree of membership vs. containment of principal filter in a fuzzy filter)
Let $a \in L$, let $d \in L'$ and $\mathcal{F} \in \mathfrak{L}'\text{-}\mathrm{Fl}(\mathfrak{L})$. Then ${}^{d}\overline{a} \subseteq \mathcal{F}$ if and only if $d \preccurlyeq \mathcal{F}(a)$.

Proof
The result is proved in two steps.

Step 1: „$\Rightarrow$“. Assume ${}^{d}\overline{a} \subseteqq \mathcal{F}$.

By the assumption and definition (1.19),

$$ {}^{d}\overline{a}\,(a) \preccurlyeq \mathcal{F}(a) \tag{2.6}$$

and by definition (2.1),

$$ d \preccurlyeq {}^{d}\overline{a}\,(a). \tag{2.7}$$

From (2.6) and (2.7) it follows by the transitivity of $\preccurlyeq$ that

$$d \preccurlyeq \mathcal{F}(a),$$

which had to be proved.

Step 2: „$\Leftarrow$“. Assume $d \preccurlyeq \mathcal{F}(a)$. By definition (1.19) it suffices to prove that for every $b \in L$,

$$ {}^{d}\overline{a}\,(b) \preccurlyeq \mathcal{F}(b).$$

Case 1. $b = 1$.

By definitions (2.1) and 2.1.1.3,

$$ {}^{d}\overline{a}\,(b) = 1 = \mathcal{F}(b).$$

Case 2. $b \neq 1$ and $a \sqsubseteq b$.

This means

$$\begin{aligned} {}^{d}\overline{a}\,(b) &= d && \text{(by (2.1))}\\ &\preccurlyeq \mathcal{F}(a) && \text{(by assumption)}\\ &\preccurlyeq \mathcal{F}(b). && \text{(by Proposition 2.1.6.2a)} \end{aligned}$$

Case 3. Not $a \sqsubseteq b$.

In this case,

$$\begin{aligned} {}^{d}\overline{a}\,(b) &= 0 && \text{(by (2.1))}\\ &\preccurlyeq \mathcal{F}(b). && \text{(by definition (1.8) of 0)} \ \square \end{aligned}$$

Examples 2.1.1 (Fuzzy filters)

1. A $\mathfrak{B}$-fuzzy filter of $\mathfrak{L}$ is just the **characteristic function** of a filter of $\mathfrak{L}$. This follows from Proposition 2.1.1.1 and the two-valuedness of $\mathfrak{B}$.

2. The $\mathfrak{F}$-fuzzy filters of the lattice $\mathfrak{F}$ (see example 1.3.1.2) are all functions $f : \langle 0, 1\rangle \to \langle 0, 1\rangle$ which are *nondecreasing* with respect to the usual order $\leqq$ of the reals and fulfil $f(1) = 1$.

   This class includes the 'positive' examples of *truth value restrictions* given by J. F. Baldwin [2], i. e. *true*, *very true*, *fairly true*, *absolutely true*, and *unrestricted.* •

2.2 Complete Lattices of Fuzzy Filters of a Lattice

The statement of the following theorem is mentioned in [103] and also in [69], without proof.

Theorem 2.2.1 (Complete lattice of fuzzy filters of a lattice)
Let $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$ be complete lattices with induced partial orders $\sqsubseteq, \preccurlyeq$, respectively.

*The ordinary **fuzzy subset relation** $\subseteqq$ from (1.19) induces on $\mathfrak{L}'\text{-Fl}(\mathfrak{L})$ a complete lattice structure, the meet and greatest lower bound of which coincide with the meet $\cap$ and the greatest lower bound $\bigcap$ in the complete lattice of all $\mathfrak{L}'$-fuzzy sets on L (see Observation 1.4.1).*

This lattice is denoted by $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \Cup\right]$.

The join $\Cup$ of $\mathfrak{L}'\text{-Fl}(\mathfrak{L})$ is uniquely determined by definition (1.6), which can be formulated as follows, for $\mathcal{F}, \mathcal{G} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$.

$$\mathcal{F} \Cup \mathcal{G} =_{\text{def}} \bigcap \left\{\mathcal{H} \middle| \mathcal{H} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L}) \text{ and } \mathcal{F} \cup \mathcal{G} \subseteqq \mathcal{H}\right\}. \tag{2.8}$$

Proof
First of all, prove that for every $\Phi \subseteqq \mathfrak{L}'\text{-Fl}(\mathfrak{L})$, the element $\bigcap \Phi$ of L'^L (see Observation 1.4.1) is an element of $\mathfrak{L}'\text{-Fl}(\mathfrak{L})$. For this, the conditions from Corollary 2.1.7 are verified separately.

1. $\left(\bigcap \Phi\right)(1) = 1$.

$$\begin{aligned}
\left(\bigcap \Phi\right)(1) &= \bigcurlywedge \left\{\mathcal{F}(1) \middle| \mathcal{F} \in \Phi\right\} && \text{(by (1.20))}\\
&= \bigcurlywedge \{1\} && \text{(by Definition 2.1.1)}\\
&= 1
\end{aligned}$$

2. For all $a, b \in L$, $\left(\bigcap \Phi\right)(a) \curlywedge \left(\bigcap \Phi\right)(b) = \left(\bigcap \Phi\right)(a \sqcap b)$.
   Let $a, b \in L$. Then

$$\begin{aligned}
&\left(\bigcap \Phi\right)(a) \curlywedge \left(\bigcap \Phi\right)(b)\\
&\quad= \bigcurlywedge \left\{\mathcal{F}(a) \middle| \mathcal{F} \in \Phi\right\} \curlywedge \bigcurlywedge \left\{\mathcal{F}(b) \middle| \mathcal{F} \in \Phi\right\} && \text{(by (1.20))}\\
&\quad= \bigcurlywedge \left(\left\{\mathcal{F}(a) \middle| \mathcal{F} \in \Phi\right\} \cup \left\{\mathcal{F}(b) \middle| \mathcal{F} \in \Phi\right\}\right) && \text{(by (1.5))}\\
&\quad= \bigcurlywedge \left\{\mathcal{F}(a) \curlywedge \mathcal{F}(b) \middle| \mathcal{F} \in \Phi\right\} && \text{(by (1.5))}\\
&\quad= \bigcurlywedge \left\{\mathcal{F}(a \sqcap b) \middle| \mathcal{F} \in \Phi\right\} && \text{(by Corollary 2.1.7)}\\
&\quad= \left(\bigcap \Phi\right)(a \sqcap b) && \text{(by (1.20))}
\end{aligned}$$

As $\bigcap \Phi$ is the greatest lower bound of Φ wrt. $\subseteqq$ in L'^L, of course by $\bigcap \Phi \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$, it is also the greatest lower bound of Φ wrt. $\subseteqq$ in the subset $\mathfrak{L}'\text{-Fl}(\mathfrak{L})$ of L'^L.

The meet induced by $\subseteqq$ is obtained from definition (1.5); of course it is identical with the meet $\cap$ from (1.17).

It is well known from lattice theory that from the previous part of this proof, it already follows that $\subseteqq$ induces a complete lattice structure on $\mathfrak{L}'\text{-Fl}(\mathfrak{L})$, and that the least upper bound of a set $\Phi \subseteqq \mathfrak{L}'\text{-Fl}(\mathfrak{L})$ (which shall be denoted by $\dot\bigcup \Phi$) is given by the greatest lower bound of the set of upper bounds of Φ:

$$\dot\bigcup \Phi = \bigcap \left\{ \mathcal{H} \middle| \mathcal{H} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L}) \text{ and } \bigcup \Phi \subseteqq \mathcal{H} \right\}.$$

Equation (2.8) then immediately follows from (1.5). □

Theorem 2.2.2 (Alternative definition of join in the lattice of fuzzy filters)
Let $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$ be complete lattices with induced partial orders $\sqsubseteq, \preccurlyeq$, respectively. Furthermore, let $\mathfrak{L}'$ be ***completely distributive*** *wrt. $\curlyvee$ (see (1.12)).*

The join of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \dot\cup\right]$ is also given by the following equation, for $\mathcal{F}, \mathcal{G} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$:

$$(\mathcal{F} \dot\cup \mathcal{G})(c) = \bigcurlyvee \left\{ \mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \right\}. \quad (c \in L) \tag{2.9}$$

Proof
Define $\mathcal{U} \in {L'}^L$ by

$$\mathcal{U}(c) =_{\text{def}} \bigcurlyvee \left\{ \mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \right\}. \qquad (c \in L)$$

It is to be proved that $\mathcal{F} \dot\cup \mathcal{G} = \mathcal{U}$.

This is done in three steps:

1. $\mathcal{U}$ is an upper bound of $\mathcal{F}, \mathcal{G}$ wrt. $\subseteqq$.

   Let $c \in L$. It is sufficient to prove $\mathcal{F}(c) \preccurlyeq \mathcal{U}(c)$ and $\mathcal{G}(c) \preccurlyeq \mathcal{U}(c)$.

   In fact,

   $$c \sqcap 1 = c \sqsubseteq c$$

   and thus

   $$\mathcal{F}(c) \curlywedge \mathcal{G}(1) = \mathcal{F}(c) \in \left\{ \mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \right\},$$

   hence

   $$\mathcal{F}(c) \preccurlyeq \bigcurlyvee \left\{ \mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \right\} = \mathcal{U}(c).$$

   $\mathcal{G}(c) \preccurlyeq \mathcal{U}(c)$ is proved analogously.

2. $\mathcal{U} \subseteqq \mathcal{F} \dot\cup \mathcal{G}$.

   Let $c \in L$. It is sufficient to prove that for all $a, b \in L$ with $a \sqcap b \sqsubseteq c$,

   $$\mathcal{F}(a) \curlywedge \mathcal{G}(b) \preccurlyeq (\mathcal{F} \dot\cup \mathcal{G})(c).$$

   First of all,

   $$\begin{aligned} \mathcal{F}(a) &\preccurlyeq (\mathcal{F} \dot\cup \mathcal{G})(a) \\ \mathcal{G}(b) &\preccurlyeq (\mathcal{F} \dot\cup \mathcal{G})(b) \end{aligned}$$

   and thus

   $$\begin{aligned} \mathcal{F}(a) \curlywedge \mathcal{G}(b) &\preccurlyeq (\mathcal{F} \dot\cup \mathcal{G})(a) \curlywedge (\mathcal{F} \dot\cup \mathcal{G})(b) \\ &\preccurlyeq (\mathcal{F} \dot\cup \mathcal{G})(a \sqcap b) && \text{(by condition 1 of Definition 2.1.1)} \\ &\preccurlyeq (\mathcal{F} \dot\cup \mathcal{G})(c) && \text{(by condition 2a of Proposition 2.1.6)} \end{aligned}$$

3. $\mathcal{U} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$.

   This is proved by verifying the conditions from Corollary 2.1.7.

   3.1. $\mathcal{U}(1) = 1$.
   Trivially, $1 \sqcap 1 \sqsubseteq 1$ and thus

$$1 = \mathcal{F}(1) \curlywedge \mathcal{G}(1) \in \left\{\mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq 1\right\},$$

   hence

$$\mathcal{U}(1) = \bigvee \left\{\mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq 1\right\} = 1.$$

   3.2. For all $c, d \in L$, $\mathcal{U}(c) \curlywedge \mathcal{U}(d) = \mathcal{U}(c \sqcap d)$.
   By expanding definitions,

$$\begin{aligned}\mathcal{U}(c) \curlywedge \mathcal{U}(d) = &\bigvee \left\{\mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c\right\}\\ &\curlywedge \bigvee \left\{\mathcal{F}(a') \curlywedge \mathcal{G}(b') \middle| a', b' \in L \text{ and } a' \sqcap b' \sqsubseteq d\right\},\end{aligned}$$

   from which it follows by (1.12) that

$$= \bigvee \left\{ \mathcal{F}(a) \curlywedge \mathcal{G}(b) \curlywedge \mathcal{F}(a') \curlywedge \mathcal{G}(b') \,\middle|\, \begin{array}{l} a, b, a', b' \in L \\ \text{and } a \sqcap b \sqsubseteq c \\ \text{and } a' \sqcap b' \sqsubseteq d \end{array} \right\}$$

   and by Corollary 2.1.7

$$= \bigvee \left\{ \mathcal{F}(a \sqcap a') \curlywedge \mathcal{G}(b \sqcap b') \,\middle|\, \begin{array}{l} a, b, a', b' \in L \\ \text{and } a \sqcap b \sqsubseteq c \text{ and } a' \sqcap b' \sqsubseteq d \end{array} \right\}.$$

   Furthermore,

$$\mathcal{U}(c \sqcap d) = \bigvee \left\{\mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \sqcap d\right\}.$$

   To establish $\mathcal{U}(c) \curlywedge \mathcal{U}(d) = \mathcal{U}(c \sqcap d)$, it is sufficient to prove that

$$\begin{aligned}&\left\{ \mathcal{F}(a \sqcap a') \curlywedge \mathcal{G}(b \sqcap b') \,\middle|\, \begin{array}{l} a, b, a', b' \in L \\ \text{and } a \sqcap b \sqsubseteq c \text{ and } a' \sqcap b' \sqsubseteq d \end{array} \right\}\\ &\qquad = \left\{\mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \sqcap d\right\}.\end{aligned}$$

   This is demonstrated in two steps.
   First, show that for all $a, b, a', b' \in L$ such that $a \sqcap b \sqsubseteq c$ and $a' \sqcap b' \sqsubseteq d$,

$$\mathcal{F}(a \sqcap a') \curlywedge \mathcal{G}(b \sqcap b') \in \left\{\mathcal{F}(a) \curlywedge \mathcal{G}(b) \middle| a, b \in L \text{ and } a \sqcap b \sqsubseteq c \sqcap d\right\}.$$

   For this, it is sufficient to prove that for all $a, b, a', b' \in L$ such that $a \sqcap b \sqsubseteq c$ and $a' \sqcap b' \sqsubseteq d$,

$$(a \sqcap a') \sqcap (b \sqcap b') \sqsubseteq c \sqcap d.$$

   But this follows trivially from the properties of a lattice.

Secondly, show that for all $a, b \in L$ such that $a \sqcap b \sqsubseteq c \sqcap d$,

$$(2.10) \quad \mathcal{F}(a) \curlywedge \mathcal{G}(b) \in \left\{ \mathcal{F}(a \sqcap a') \curlywedge \mathcal{G}(b \sqcap b') \;\middle|\; \begin{array}{l} a, b, a', b' \in L \\ \text{and } a \sqcap b \sqsubseteq c \text{ and } a' \sqcap b' \sqsubseteq d \end{array} \right\}.$$

For this, two simple observations are sufficient. First of all, it is obvious that under the precondition $a \sqcap b \sqsubseteq c \sqcap d$,

$$a \sqcap b \sqsubseteq c \text{ and } a \sqcap b \sqsubseteq d.$$

From this it follows that

$$\mathcal{F}(a \sqcap a) \curlywedge \mathcal{G}(b \sqcap b) \in \left\{ \mathcal{F}(a \sqcap a') \curlywedge \mathcal{G}(b \sqcap b') \;\middle|\; \begin{array}{l} a, b, a', b' \in L \\ \text{and } a \sqcap b \sqsubseteq c \text{ and } a' \sqcap b' \sqsubseteq d \end{array} \right\},$$

and finally the claim (2.10) by the fact that $a \sqcap a = a$ and $b \sqcap b = b$.

From step 1 and step 3, it follows that $\mathcal{F} \uplus \mathcal{G} \subseteqq \mathcal{U}$, and together with step 2, $\mathcal{U} = \mathcal{F} \uplus \mathcal{G}$ is obtained, which concludes the proof. □

Observation 2.2.3 ($\cup$ vs. $\uplus$)
Let $\mathcal{F}, \mathcal{G} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$. If $\mathcal{F} \cup \mathcal{G} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$, then $\mathcal{F} \cup \mathcal{G} = \mathcal{F} \uplus \mathcal{G}$. •

Proof
Trivial by (2.8). □

The following observation follows from the previous one by verifying the conditions from Corollary 2.1.7 for $\mathcal{F} \cup \mathcal{G}$.

Observation 2.2.4 (Join in the lattice of fuzzy filters of a chain)
Let $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = [L', \curlywedge, \curlyvee]$ be complete lattices. Furthermore, let $\mathfrak{L}$ be a *chain*.
Then in the complete lattice $[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus]$, $\mathcal{F} \uplus \mathcal{G} = \mathcal{F} \cup \mathcal{G}$. •

Remark
In Theorem 2.2.2 and Observation 2.2.4, for the first time, additional assumptions have been placed on $\mathfrak{L}$ or $\mathfrak{L}'$, apart from being complete lattices. It should be observed, however, that the examples $\mathfrak{B}$, $\mathfrak{F}$ from Example 1.3.1 fulfil *all* the assumptions made in Theorem 2.2.2 and Observation 2.2.4, so each could play the role of $\mathfrak{L}$ as well as the role of $\mathfrak{L}'$ in each of Theorem 2.2.2 and Observation 2.2.4. •

In the following, the remaining items of Observation 1.3.1 are translated to fuzzy filters. Let $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = [L', \curlywedge, \curlyvee]$ be complete lattices with induced partial orders $\sqsubseteq$, $\preccurlyeq$, respectively.

Observation 2.2.5 (Zero and unit in the lattice of fuzzy filters)

1. The **zero element** of the complete lattice $[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus]$ is the mapping $\mathbb{0} : L \to L'$ defined for $a \in L$ by

$$(2.11) \qquad \mathbb{0}(a) = \begin{cases} 0, & \text{if } a \neq 1 \\ 1, & \text{if } a = 1 \end{cases}$$

2. The **unit element** of the complete lattice $[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus]$ is the mapping $\mathbb{1} : L \to L'$ defined for $a \in L$ by

$$(2.12) \qquad \mathbb{1}(a) = 1.$$

•

Proof
Trivial by $\mathbb{0}, \mathbb{1} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})$, by the fact that the induced partial order of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \Cup\right]$ is the usual subset relation $\subseteqq$ for fuzzy sets, and by condition 3 of Definition 2.1.1. □

Observation 2.2.6 (Special fuzzy principal filters)
The principal fuzzy 1-filter ${}^{1}\overline{0}$ of 0 coincides with the **unit element** $\mathbb{1}$ of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \Cup\right]$.

For every lattice element $d \in L'$, ${}^{d}\overline{1}$ coincides with the **zero element** $\mathbb{0}$ of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \Cup\right]$.
For every lattice element $a \in L$, ${}^{0}\overline{a}$ coincides with the **zero element** $\mathbb{0}$ of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \Cup\right]$.
•

Proof
Follows trivially from definition (2.1) and Observation 2.2.5. □

Next, lattice structures based on principal fuzzy filters are studied. Some preparations are necessary for this. First, it is demonstrated that from every element of $\mathfrak{L}'\text{-PFl}(\mathfrak{L})$, the parameters d and a can be 'reconstructed' if $d \neq 0$ or $a \neq 1$, using newly defined functions δ, α to be applied to principal fuzzy filters.

Definition 2.2.1 (Operators for extracting the parameters of a fuzzy principal filter)
Let $\mathcal{F} \in L'^{L}$. Define $\delta : L'^{L} \to L'$, $\alpha : L'^{L} \to L$ by

$$\delta(\mathcal{F}) =_{\text{def}} \curlyvee \left\{\mathcal{F}(b) \middle| b \in L \setminus \{1\}\right\}$$
$$\alpha(\mathcal{F}) =_{\text{def}} \bigsqcap \left\{b \middle| b \in L \text{ and } \mathcal{F}(b) \neq 0\right\}$$

Lemma 2.2.7 (Conditions for extracting the parameters of fuzzy principal filters)

1. *Let $a \in L$ and $d \in L'$. Then*
   1.1. *if $d \neq 0$, then $\alpha\left({}^{d}\overline{a}\right) = a$,*
   1.2. *if $a \neq 1$, then $\delta\left({}^{d}\overline{a}\right) = d$,*
   1.3. *if $d = 0$ or $a = 1$, then $\alpha\left({}^{d}\overline{a}\right) = 1$ and $\delta\left({}^{d}\overline{a}\right) = 0$.*
2. *Let $\mathcal{P} \in \mathfrak{L}'\text{-PFl}(\mathfrak{L})$. Then*

$$\mathcal{P} = {}^{\delta(\mathcal{P})}\overline{\alpha(\mathcal{P})}\,.$$

Proof
ad 1.1. In the case $d \neq 0$, Proposition 2.1.1.2 yields

$$\left\{b \middle| b \in L \text{ and } {}^{d}\overline{a}(b) \neq 0\right\} = \overline{a} = \{b | b \in L \text{ and } a \sqsubseteq b\}\,,$$

thus $\alpha\left({}^{d}\overline{a}\right) = \bigsqcap \{b | b \in L \text{ and } a \sqsubseteq b\} = a$ is obvious.

ad 1.2. In the case $a \neq 1$, it holds by definition of ${}^{d}\overline{a}$ that

$$\left\{{}^{d}\overline{a}(b) \middle| b \in L \setminus \{1\}\right\} = \{0, d\},$$

thus $\delta\left({}^{d}\overline{a}\right) = \curlyvee\{0, d\} = d$ is obvious.

ad 1.3. Observation 2.2.6 yields that in the case $d = 0$ or $a = 1$, ${}^{d}\overline{a} = \mathbb{0}$ and thus

$$\left\{ {}^{d}\overline{a}\,(b) \middle| b \in L \setminus \{1\} \right\} = \{0\}$$
$$\left\{ b \middle| b \in L \text{ and } {}^{d}\overline{a}\,(b) \neq 0 \right\} = \{1\},$$

hence trivially $\alpha\left({}^{d}\overline{a}\right) = \bigsqcap\{1\} = 1$ and $\delta\left({}^{d}\overline{a}\right) = \curlyvee\{0\} = 0$.

ad 2. From $\mathcal{P} \in \mathfrak{L}'\text{-PFl}\,(\mathfrak{L})$, it follows that there exist $a \in L$ and $d \in L'$ such that $\mathcal{P} = {}^{d}\overline{a}$. Two cases are distinguished:

Case 1. $d = 0$ or $a = 1$.
By Observation 2.2.6, $\mathcal{P} = \mathbb{0}$. Furthermore, $\alpha(\mathcal{P}) = 1$ and $\delta(\mathcal{P}) = 0$ by item 1.3. Hence

$${}^{\delta(\mathcal{P})}\overline{\alpha(\mathcal{P})} = {}^{0}\overline{1} = \mathbb{0} = \mathcal{P}.$$

Case 2. $d \neq 0$ and $a \neq 1$.
By items 1.1 and 1.2,

$$\alpha(\mathcal{P}) = a,$$
$$\delta(\mathcal{P}) = d,$$

thus

$${}^{\delta(\mathcal{P})}\overline{\alpha(\mathcal{P})} = {}^{d}\overline{a} = \mathcal{P}.$$

□

Secondly, a new operation for combining principal fuzzy filters is defined.

Definition 2.2.2 (Alternative join for principal fuzzy filters)
Let $\mathcal{P}, \mathcal{P}' \in \mathfrak{L}'\text{-PFl}\,(\mathfrak{L})$.

$$\mathcal{P} \uplus \mathcal{P}' =_{\text{def}} {}^{\left(\delta(\mathcal{P}) \curlyvee \delta(\mathcal{P}')\right)}\overline{\alpha(\mathcal{P}) \sqcap \alpha(\mathcal{P}')}\,. \tag{2.13}$$

Observation 2.2.8 (Joining principal fuzzy filters)
Let $a, b \in L$ and $d, d' \in L'$. If $a, b \neq 1$ and $d, d' \neq 0$, then

$${}^{d}\overline{a} \uplus {}^{d'}\overline{b} = {}^{(d \curlyvee d')}\overline{a \sqcap b}\,.$$

•

Proof
Trivial from the definition and Lemma 2.2.7. □

Remark
Note that this result can not be extended to arbitrary $a, b \in L$ and $d, d' \in L'$, in general, because for instance if $d = 0$, then the value of a cannot be obtained from ${}^{d}\overline{a}$. •

Theorem 2.2.9 (Lattice of principal fuzzy filters)
For all $a, b \in L$, and $d, d' \in L'$,

$${}^{d}\overline{a} \cap {}^{d'}\overline{b} = {}^{(d \curlywedge d')}\overline{a \sqcup b} \tag{2.14}$$

Furthermore, $\left[\mathfrak{L}'\text{-PFl}\,(\mathfrak{L}), \cap, \uplus\right]$ is a lattice.

Proof

First, (2.14) is proved.

Let $c \in L$. By definition (1.18), it suffices to prove

$$^{(d \curlywedge d')}\overline{a \sqcup b}\,(c) = {}^{d}\overline{a}\,(c) \curlywedge {}^{d'}\overline{b}\,(c) \tag{2.15}$$

Case 1 $c = 1$.

By definition (2.1),

$$^{(d \curlywedge d')}\overline{a \sqcup b}\,(c) = 1 = 1 \curlywedge 1 = {}^{d}\overline{a}\,(c) \curlywedge {}^{d'}\overline{b}\,(c).$$

Case 2. $c \neq 1$ and $a \sqcup b \sqsubseteq c$.

By definition (2.1),

$$^{(d \curlywedge d')}\overline{a \sqcup b}\,(c) = d \curlywedge d'.$$

Furthermore, $a \sqcup b$ is an upper bound of a, b, so

$$\begin{aligned} a &\sqsubseteq c \\ b &\sqsubseteq c, \end{aligned}$$

thus by (2.1),

$$\begin{aligned} {}^{d}\overline{a}\,(c) &= d \\ {}^{d'}\overline{b}\,(c) &= d' \end{aligned}$$

thus

$$^{d}\overline{a}\,(c) \curlywedge {}^{d'}\overline{b}\,(c) = d \curlywedge d'$$

which establishes (2.15) in this case.

Case 3. not $a \sqcup b \sqsubseteq c$.

By definition (2.1),

$$^{(d \curlywedge d')}\overline{a \sqcup b}\,(c) = 0.$$

Furthermore, $a \sqcup b$ is the *least* upper bound of a, b, so

$$\text{not } a \sqsubseteq c \;\text{ or }\; \text{not } b \sqsubseteq c,$$

which means by definition (2.1)

$$^{d}\overline{a}\,(c) = 0 \;\text{ or }\; {}^{d'}\overline{b}\,(c) = 0$$

thus

$$^{d}\overline{a}\,(c) \curlywedge {}^{d'}\overline{b}\,(c) = 0 = {}^{(d \curlywedge d')}\overline{a \sqcup b}\,(c),$$

which establishes (2.15) in this case.

For the demonstration that $\left[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \uplus\right]$ is a lattice, so far it has been established that for all $\mathcal{P}, \mathcal{P}' \in \mathfrak{L}'\text{-PFl}(\mathfrak{L})$, $\mathcal{P} \cap \mathcal{P}' \in \mathfrak{L}'\text{-PFl}(\mathfrak{L})$. This means by (1.4) that if $\left[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \uplus\right]$ is a lattice, then its induced partial order is the fuzzy subset relation $\subseteqq$.

To establish that $\left[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \uplus\right]$ is indeed a lattice, it is now sufficient to prove the following.

1. For all $\mathcal{P}, \mathcal{P}' \in \mathfrak{L}'\text{-PFl}(\mathfrak{L})$, $\mathcal{P} \uplus \mathcal{P}' \in \mathfrak{L}'\text{-PFl}(\mathfrak{L})$.

   This is obvious by definition (2.13).

2. For $d, d' \in L'$ and $a, b \in L$, ${}^{d}\overline{a} \subseteqq {}^{d}\overline{a} \uplus {}^{d'}\overline{b}$ and ${}^{d'}\overline{b} \subseteqq {}^{d}\overline{a} \uplus {}^{d'}\overline{b}$.

   It is proved that ${}^{d}\overline{a} \subseteqq {}^{d}\overline{a} \uplus {}^{d'}\overline{b}$. The statement ${}^{d'}\overline{b} \subseteqq {}^{d}\overline{a} \uplus {}^{d'}\overline{b}$ is proved analogously.

   Case 1. $d = 0$ or $a = 1$.
   In this case, ${}^{d}\overline{a} = \mathbb{0}$, thus trivially ${}^{d}\overline{a} \subseteqq {}^{d}\overline{a} \uplus {}^{d'}\overline{b}$.

   Case 2. $d \neq 0$ and $a \neq 1$.
   By Lemma 2.2.7,

$${}^{d}\overline{a} \uplus {}^{d'}\overline{b} = {}^{\left(d \curlyvee \delta\left({}^{d'}\overline{b}\right)\right)}\overline{a \sqcap \alpha\left({}^{d'}\overline{b}\right)}.$$

   Furthermore,

$$d \preccurlyeq d \curlyvee \delta\left({}^{d'}\overline{b}\right),$$
$$a \sqcap \alpha\left({}^{d'}\overline{b}\right) \sqsubseteq a.$$

   From this,

$${}^{d}\overline{a} \subseteqq {}^{d}\overline{a} \uplus {}^{d'}\overline{b}$$

   follows by (2.3).

3. For $d, d', d'' \in L'$ and $a, b, c \in L$, if ${}^{d}\overline{a} \subseteqq {}^{d''}\overline{c}$ and ${}^{d'}\overline{b} \subseteqq {}^{d''}\overline{c}$, then ${}^{d}\overline{a} \uplus {}^{d'}\overline{b} \subseteqq {}^{d''}\overline{c}$.

   Case 1. $d = 0$ or $a = 1$.
   By Lemma 2.2.7, $\alpha\left({}^{d}\overline{a}\right) = 1$ and $\delta\left({}^{d}\overline{a}\right) = 0$, thus by definition (2.13),

$${}^{d}\overline{a} \uplus {}^{d'}\overline{b} = {}^{d'}\overline{b},$$

   and ${}^{d'}\overline{b} \subseteqq {}^{d''}\overline{c}$ holds by assumption.

   Case 2. $d' = 0$ or $b = 1$.
   Is handled exactly as the previous case.

   Case 3. $d \neq 0$ and $a \neq 1$ and $d' \neq 0$ and $b \neq 1$.
   By Lemma 2.1.3, $d \sqsubseteq d''$ and $d' \sqsubseteq d''$ and $c \sqsubseteq a$ and $c \sqsubseteq b$. Thus

$$d \curlyvee d' \preccurlyeq d'',$$
$$c \sqsubseteq a \sqcap b.$$

and by (2.3), ${}^{(d \curlyvee d')}\overline{a \sqcap b} \subseteqq {}^{d''}\overline{c}$. By Observation 2.2.8,

$$ {}^{(d \curlyvee d')}\overline{a \sqcap b} = {}^{d}\overline{a} \circledast {}^{d'}\overline{b}, $$

hence the result.

Thus, the theorem is proved. □

Theorem 2.2.10 ('Horizontally' embedding lattices of principal fuzzy filters)
For all $a, b \in L$, and $d \in L'$,

$$ {}^{d}\overline{a} \cap {}^{d}\overline{b} = {}^{d}\overline{a \sqcup b} \tag{2.16} $$

$$ {}^{d}\overline{a} \cup {}^{d}\overline{b} = {}^{d}\overline{a \sqcap b} \tag{2.17} $$

Thus, $[d\text{-PFl}(\mathfrak{L}), \cap, \cup]$ is a ***sublattice*** *of the complete lattice $[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \cup]$.*
Furthermore, $[d\text{-PFl}(\mathfrak{L}), \cap, \cup]$ is a ***sublattice*** *of the lattice $[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \circledast]$.*

Proof
(2.16) holds by (2.14) and the fact that $d \curlywedge d = d$.
(2.17) is proved in two steps.

Step 1. To prove ${}^{d}\overline{a \sqcap b} \subseteqq {}^{d}\overline{a} \cup {}^{d}\overline{b}$, by (2.8) it is to be proved that

$$ {}^{d}\overline{a \sqcap b} \subseteqq \bigcap \left\{ \mathcal{F} \middle| \mathcal{F} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L}) \text{ and } {}^{d}\overline{a} \cup {}^{d}\overline{b} \subseteqq \mathcal{F} \right\}, $$

thus it suffices to show that for every $\mathfrak{L}'$-fuzzy filter $\mathcal{F}$ of $\mathfrak{L}$ such that

$$ {}^{d}\overline{a} \cup {}^{d}\overline{b} \subseteqq \mathcal{F}, \tag{2.18} $$

it holds that

$$ {}^{d}\overline{a \sqcap b} \subseteqq \mathcal{F}. $$

By Lemma 2.1.8 it suffices to show

$$ d \preccurlyeq \mathcal{F}(a \sqcap b). \tag{2.19} $$

By (2.18),

$$ {}^{d}\overline{a} \subseteqq \mathcal{F} \text{ and } {}^{d}\overline{b} \subseteqq \mathcal{F}, $$

thus again by Lemma 2.1.8

$$ d \preccurlyeq \mathcal{F}(a) \text{ and } d \preccurlyeq \mathcal{F}(b), $$

so, because $\mathcal{F}(a) \curlywedge \mathcal{F}(b)$ is the *greatest* lower bound of $\mathcal{F}(a), \mathcal{F}(b)$,

$$ d \preccurlyeq \mathcal{F}(a) \curlywedge \mathcal{F}(b) \tag{2.20} $$

and, because $\mathcal{F}$ is an $\mathfrak{L}'$-fuzzy filter of $\mathfrak{L}$, it follows from Definition 2.1.1.1 that

$$ \mathcal{F}(a) \curlywedge \mathcal{F}(b) \preccurlyeq \mathcal{F}(a \sqcap b), \tag{2.21} $$

so (2.19) follows from (2.20) and (2.21) by the transitivity of $\preccurlyeq$.

Step 2. To prove ${}^{d}\overline{a} \Cup {}^{d}\overline{b} \subseteqq {}^{d}\overline{a \sqcap b}$, by (2.8) it is sufficient to prove that

(2.22) $$ {}^{d}\overline{a \sqcap b} \in \left\{ \mathcal{F} \middle| \mathcal{F} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L}) \text{ and } {}^{d}\overline{a} \cup {}^{d}\overline{b} \subseteqq \mathcal{F} \right\}, $$

thus it suffices to show that

(2.23) $$ {}^{d}\overline{a \sqcap b} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L}) $$

and

(2.24) $$ {}^{d}\overline{a} \cup {}^{d}\overline{b} \subseteqq {}^{d}\overline{a \sqcap b}. $$

(2.23) follows from Corollary 2.1.2. It remains to be proved that (2.24) holds. By (1.18) and (1.19), it suffices to show that for every $c \in L$,

(2.25) $$ {}^{d}\overline{a}(c) \curlyvee {}^{d}\overline{b}(c) \preccurlyeq {}^{d}\overline{a \sqcap b}(c) $$

Case 1. $c = 1$.

By definition (2.1),

$$ {}^{d}\overline{a}(c) \curlyvee {}^{d}\overline{b}(c) = 1 \curlyvee 1 = 1 = {}^{d}\overline{a \sqcap b}(c). $$

Case 2. $c \neq 1$ and ($a \sqsubseteq c$ or $b \sqsubseteq c$).

Because $a \sqcap b$ is a lower bound of a and b,

$$ a \sqcap b \sqsubseteq c, $$

thus by definition (2.1)

$$ {}^{d}\overline{a \sqcap b}(c) = d. $$

By assumption and definition (2.1),

$$ {}^{d}\overline{a}(c) \preccurlyeq d $$
$$ {}^{d}\overline{b}(c) \preccurlyeq d, $$

thus

$$ {}^{d}\overline{a}(c) \curlyvee {}^{d}\overline{b}(c) \preccurlyeq d \curlyvee d = d = {}^{d}\overline{a \sqcap b}(c) $$

Case 3. **not** $a \sqsubseteq c$ and **not** $b \sqsubseteq c$.

This case is trivial because by definition (2.1),

$$ {}^{d}\overline{a}(c) \curlyvee {}^{d}\overline{b}(c) = 0 \curlyvee 0 = 0. $$

Thus (2.24) and consequently (2.22) is proved.

This completes the proof of (2.17).

The result that $\left[d\text{-PFl}(\mathfrak{L}), \cap, \Cup\right]$ is a sublattice of the complete lattice $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \Cup\right]$ follows immediately.

To show that $\left[d\text{-PFl}(\mathfrak{L}), \cap, \Cup\right]$ is a sublattice of the lattice $\left[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \circledast\right]$, by (2.16), it suffices to prove that for all $a, b \in L$, and $d \in L'$,

(2.26) $$ {}^{d}\overline{a} \Cup {}^{d}\overline{b} = {}^{d}\overline{a} \circledast {}^{d}\overline{b}. $$

In the case $d = 0$, $d\text{-PFl}(\mathfrak{L}) = \{\mathbb{0}\}$, so this case is trivial. The case $a = 1$ or $b = 1$ is also trivial. Otherwise, (2.26) follows from Observation 2.2.8 and the fact that $d \curlyvee d = d$. □

Remark
Note that $[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \Cup]$ is **not** a sublattice of $[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus]$, in general, because for arbitrary principal fuzzy filters $\mathcal{P}, \mathcal{P}' \in \mathfrak{L}'\text{-PFl}(\mathfrak{L})$, $\mathcal{P} \Cup \mathcal{P}'$ will not be equal with $\mathcal{P} \uplus \mathcal{P}'$, in general ($\mathcal{P} \uplus \mathcal{P}'$ is a fuzzy filter, but not necessarily a principal fuzzy filter). •

Theorem 2.2.11 ($\mathfrak{L}$ is isomorphic with $\mathscr{D}\left(\left[d\text{-}\mathbf{PFl}(\mathfrak{L}), \cap, \uplus\right]\right)$)
For all $a, b \in L$, and $d \in L' \setminus \{0\}$,

$$ {}^{d}\overline{a} = {}^{d}\overline{b} \text{ iff } a = b \tag{2.27}$$

*Thus, ${}^{d}\overline{\cdot}$ is a **lattice isomorphism** from $\mathfrak{L}$ onto the **dual** lattice*

$$\mathscr{D}\left(\left[d\text{-PFl}(\mathfrak{L}), \cap, \uplus\right]\right) = \left[d\text{-PFl}(\mathfrak{L}), \uplus, \cap\right].$$

Proof
(2.27) follows immediately from the definition of ${}^{d}\overline{\cdot}$, and follows also by Lemma 2.1.3.

That ${}^{d}\overline{\cdot}$ is a lattice isomorphism from $\mathfrak{L}$ onto $[d\text{-PFl}(\mathfrak{L}), \cap, \uplus]$ is then obvious by Theorem 2.2.10. □

Fixing the 'horizontal direction' $\mathfrak{L}$ and varying the 'vertical direction' $\mathfrak{L}'$ yields a result analogous with Theorems 2.2.10 and 2.2.11 as simple conclusions of the definitions.

For this result, however, an additional definition is necessary.

Definition 2.2.3 (Set of all principal fuzzy filters of a lattice element)
The set of all principal fuzzy filters of a lattice element $a \in \mathfrak{L}$ is denoted by $\mathfrak{L}'\text{-PFl}(a) =_{\text{def}} \left\{ {}^{d}\overline{a} \,\middle|\, d \in L' \right\}$.

Observation 2.2.12 ('Vertically' embedding lattices of principal fuzzy filters)
For all $a \in L$ and $d, d' \in L'$,

$$ {}^{d}\overline{a} \cap {}^{d'}\overline{a} = {}^{(d \curlywedge d')}\overline{a} \tag{2.28}$$

$$ {}^{d}\overline{a} \uplus {}^{d'}\overline{a} = {}^{(d \curlyvee d')}\overline{a} = {}^{d}\overline{a} \cup {}^{d'}\overline{a} \tag{2.29}$$

Thus, $[\mathfrak{L}'\text{-PFl}(a), \cap, \uplus] = [\mathfrak{L}'\text{-PFl}(a), \cap, \cup]$ is a **sublattice** of the complete lattice $[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus]$.

Furthermore, $[\mathfrak{L}'\text{-PFl}(a), \cap, \uplus]$ is a **sublattice** of the lattice $[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \Cup]$. •

Proof
Trivial from the definitions, especially definition (1.18). □

Observation 2.2.13 ($\mathfrak{L}'$ is isomorphic with $[\mathfrak{L}'\text{-}\mathbf{PFl}(a), \cap, \cup]$)
For all $a \in L \setminus \{1\}$ and $d, d' \in L'$,

$$ {}^{d}\overline{a} = {}^{d'}\overline{a} \text{ iff } d = d' \tag{2.30}$$

Thus, ${}^{\cdot}\overline{a}$ is a **lattice isomorphism** from $\mathfrak{L}'$ onto $[\mathfrak{L}'\text{-PFl}(a), \cap, \uplus] = [\mathfrak{L}'\text{-PFl}(a), \cap, \cup]$. •

Proof
Is also trivial. □

Remark
Note that both lattices $\left[d\text{-PFl}(\mathfrak{L}), \uplus, \cap\right]$ and $\left[\mathfrak{L}'\text{-PFl}(a), \cap, \cup\right]$ are **complete** lattices by the isomorphisms established in Theorem 2.2.11 and Observation 2.2.13.

But $\left[d\text{-PFl}(\mathfrak{L}), \uplus, \cap\right]$ is not necessarily a **complete sublattice** of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \uplus, \cap\right]$, because the greatest lower bound and least upper bound, respectively, of a subset of d-PFl$(\mathfrak{L})$ may be different when taken wrt $\left[d\text{-PFl}(\mathfrak{L}), \uplus, \cap\right]$ and $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \uplus, \cap\right]$, respectively.

In contrast with this, $\left[\mathfrak{L}'\text{-PFl}(a), \cap, \cup\right]$ is (trivially) indeed a **complete sublattice** of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$. •

Observation 2.2.14 ('Vertically' embedding lattices of principal fuzzy filters is complete)
For all $a \in L$ and $M \subseteqq L'$,

$$\bigcap_{d \in M} {}^{d}\overline{a} = {}^{\left(\bigwedge_{d \in M} d\right)}\overline{a} \tag{2.31}$$

$$\dot{\bigcup_{d \in M}} {}^{d}\overline{a} = {}^{\left(\bigvee_{d \in M} d\right)}\overline{a} \tag{2.32}$$

Thus, $\left[\mathfrak{L}'\text{-PFl}(a), \cap, \uplus\right] = \left[\mathfrak{L}'\text{-PFl}(a), \cap, \cup\right]$ is a **complete sublattice** of the complete lattice $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$. •

Proof
Trivial from the definitions. □

It has been remarked above that $\left[\mathfrak{L}'\text{-PFl}(\mathfrak{L}), \cap, \uplus\right]$ is **not** a sublattice of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$, and that applying $\uplus$ to fuzzy principal filters will lead out of $\mathfrak{L}'$-PFl$(\mathfrak{L})$, in general. In spite of this, sublattices of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$ which contain $\mathfrak{L}'$-PFl$(\mathfrak{L})$ will be studied in the next section and subsequent chapters. To ease formulations, an explicit definition of the sublattice of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$ *generated* by $\mathfrak{L}'$-PFl$(\mathfrak{L})$ is given.

Definition 2.2.4 (Sublattice of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$ generated by $\mathfrak{L}'$-PFl$(\mathfrak{L})$)
*The **smallest** subset of $\mathfrak{L}'$-Fl$(\mathfrak{L})$ which contains $\mathfrak{L}'$-PFl$(\mathfrak{L})$ and is closed wrt. the lattice operations of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$ is defined to be*

$$\mathscr{P}(\mathfrak{L}', \mathfrak{L}) =_{\text{def}} \bigcap \left\{ \Phi \,\middle|\, \mathfrak{L}'\text{-PFl}(\mathfrak{L}) \subseteqq \Phi \text{ and } [\Phi, \cap, \uplus] \Subset \left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right] \right\} \tag{2.33}$$

*The sublattice of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$ **generated** by $\mathfrak{L}'$-PFl$(\mathfrak{L})$ is $\left[\mathscr{P}(\mathfrak{L}', \mathfrak{L}), \cap, \uplus\right]$.*

Proposition 2.2.15 (Characterisation of $\mathscr{P}(\mathfrak{L}', \mathfrak{L})$)
*$\mathcal{F} \in \mathscr{P}(\mathfrak{L}', \mathfrak{L})$ if and only if there exists $n \in \mathbb{N}, n \geqq 1$ and $\mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathfrak{L}'$-PFl$(\mathfrak{L})$ such that $\mathcal{F}$ is obtained from $\mathcal{P}_1, \ldots, \mathcal{P}_n$ by **finitely many** applications of $\cap, \uplus$ (within $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \uplus\right]$).*

Proof
Trivial. □

Remarks

1. Obviously, $\left[\mathscr{P}(\mathfrak{L}', \mathfrak{L}), \cap, \uplus\right]$ is a lattice, and hence a sublattice of $\left[\mathfrak{L}'\text{-Fl}\,(\mathfrak{L}), \cap, \uplus\right]$.
2. The structure of $\left[\mathscr{P}(\mathfrak{L}', \mathfrak{L}), \cap, \uplus\right]$ is not studied any further here. Obviously, $\mathscr{P}(\mathfrak{L}', \mathfrak{L})$ is very large, much larger than $\mathfrak{L}'\text{-PFl}\,(\mathfrak{L})$.

   If $\mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$ is *finite*, then trivially, $\mathscr{P}(\mathfrak{L}', \mathfrak{L}) = \mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$. But if $\mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$ is *infinite*, then $\mathscr{P}(\mathfrak{L}', \mathfrak{L})$ is a proper subset of $\mathfrak{L}'\text{-Fl}\,(\mathfrak{L})$, in general. •

2.3 Expanding a Complete Lattice by Another Complete Lattice

In this section, a method is defined which is needed in section 3.5 for *expanding* a **truth value lattice** $\mathfrak{T}$ into a **label lattice** $\mathfrak{L}$ by means of a lattice $\mathfrak{D}$ of **degrees of trust**. In this way, the **expressive power** of the logic can be raised in a controlled way and a *canonical* method is established for defining a *model relation* graded by elements from $\mathfrak{D}$.

Definition 2.3.1 (Expansion)
Given three complete lattices $\mathfrak{L}_1 = [L_1, \sqcap, \sqcup]$, $\mathfrak{L}_2 = [L_2, \curlywedge, \curlyvee]$, $\mathfrak{L}_3 = [L_3, \Cap, \Cup]$ and a mapping $\iota : L_3 \to \mathfrak{L}_2\text{-Fl}\,(\mathfrak{L}_1)$, $\mathfrak{L}_3$ ***is said to be an expansion of** $\mathfrak{L}_1$ **by** $\mathfrak{L}_2$**, by means of** ι*
$=_{\text{def}}$ *There exists a lattice $\mathfrak{L}_3'$ such that*

1. $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \uplus, \cap\right] \Subset \mathfrak{L}_3' \Subset \left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{L}_1), \uplus, \cap\right]$.
2. *ι is a* ***lattice isomorphism*** *from $\mathfrak{L}_3$ onto $\mathfrak{L}_3'$.*

Remarks

1. Note that by the isomorphism between $\mathfrak{L}_3$ and $\mathfrak{L}_3'$, the lattice $\mathfrak{L}_3'$ is necessarily complete, but **not** necessarily a complete sublattice of $\left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{L}_1), \uplus, \cap\right]$.

   This means that for every subset L_3'' of L_3', a greatest lower bound and least upper bound **in** $\mathfrak{L}_3'$ exist, but are not necessarily identical with the greatest lower bound and least upper bound of L_3'' **in** $\left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{L}_1), \uplus, \cap\right]$.
2. As the choice of ι plays an important role the later definition of the **model relation** and thus influences the **semantics** of **fuzzy filter-based** logics (see section 4.1), ι is incorporated into the definition of *expansion*.

   It should be noted that if $\mathfrak{L}_3$ is a **finite chain**, then ι is uniquely determined. In particular, in this case a **successor relation** can be defined which will cover the range from 0 to 1, via all elements of L_3, and which has to be respected by ι. From this it follows that ι is unique.
3. In case ι can be disregarded, it does not need to be mentioned, i. e. "$\mathfrak{L}_3$ is an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$" means there exists a mapping $\iota : L_3 \to \mathfrak{L}_2\text{-Fl}\,(\mathfrak{L}_1)$ such that $\mathfrak{L}_3$ is an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$, by means of ι. •

The following result gives a first justification of the above definition by establishing that an *expansion* of a lattice $\mathfrak{L}_1$ can be regarded to be a 'generalisation' of $\mathfrak{L}_1$.

Proposition 2.3.1 (If $\mathfrak{L}_3$ is an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$, then $\mathfrak{L}_1$ is embeddable into $\mathfrak{L}_3$)
Given three complete lattices $\mathfrak{L}_1$, $\mathfrak{L}_2$, $\mathfrak{L}_3$, such that $\mathfrak{L}_3$ is an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$.

$\mathfrak{L}_1$ is ***embeddable*** *into $\mathfrak{L}_3$, i. e. there exists a* ***lattice monomorphism*** *(an injective lattice homomorphism) from $\mathfrak{L}_1$ into $\mathfrak{L}_3$ (see* P. M. Cohn *[10]).*

Proof
Let $\mathfrak{L}_1 = [L_1, \sqcap, \sqcup]$ and $\mathfrak{L}'_3 = \left[L'_3, \cup\!\!\cdot, \cap\right]$ be the sublattice of $\left[\mathfrak{L}_2\text{-Fl}\left(\mathfrak{L}_1\right), \cup\!\!\cdot, \cap\right]$ from Definition 2.3.1; let ι be the isomorphism between $\mathfrak{L}_3$ and $\mathfrak{L}'_3$ by means of which $\mathfrak{L}_1$ is expanded to $\mathfrak{L}_3$. It is sufficient to prove that for the unit element 1 of $\mathfrak{L}_2$,

$$h =_{\text{def}} \iota^{-1} \circ {}^{1}\!\cdot \tag{2.34}$$

is a lattice monomorphism from $\mathfrak{L}_1$ into $\mathfrak{L}_3$. By the fact that ι is an isomorphism, it suffices to show that ${}^{1}\!\cdot$ is a lattice monomorphism from $\mathfrak{L}_1$ into $\mathfrak{L}'_3$.

By Theorem 2.2.10, $\left[1\text{-PFl}\left(L_1\right), \cup\!\!\cdot, \cap\right]$ is a sublattice of $\left[\mathfrak{L}_2\text{-Fl}\left(\mathfrak{L}_1\right), \cup\!\!\cdot, \cap\right]$. Furthermore, $1\text{-PFl}\left(L_1\right) \subseteqq L'_3$ by item 1 of Definition 2.3.1. So $\left[1\text{-PFl}\left(L_1\right), \cup\!\!\cdot, \cap\right]$ is a sublattice of $\mathfrak{L}'_3$.

By Theorem 2.2.11 and the fact that $0 \neq 1$ (see the introduction), $\mathfrak{L}_1$ is isomorphic with $\left[1\text{-PFl}\left(L_1\right), \cup\!\!\cdot, \cap\right]$ and thus, because $\left[1\text{-PFl}\left(L_1\right), \cup\!\!\cdot, \cap\right]$ is a sublattice of $\mathfrak{L}'_3$, trivially monomorphic with $\mathfrak{L}'_3$. □

For illustrating the concept of *expansion*, some examples of lattices are given in the following.

Examples 2.3.1 (Lattices for expansions)

1. Denote the classical BOOLEan lattice of two-valued logic by $\mathfrak{B} =_{\text{def}} \left[\{0,1\}, \text{and}, \text{or}\right]$ (see Example 1.3.1.1). The induced partial order of this lattice coincides on $\{0,1\}$ with the standard ordering relation $\leqq$ of real numbers.

2. For a fixed complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$, denote the complete lattice structure for filters of $\mathfrak{L}$ described in Observation 1.3.1 by $\left[\text{Fl}(\mathfrak{L}), \cap, \cup\!\!\cdot\right]$. The *dual lattice* $\mathscr{D}\left(\left[\text{Fl}(\mathfrak{L}), \cap, \cup\!\!\cdot\right]\right) = \left[\text{Fl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right]$ is denoted by $\mathscr{F}(\mathfrak{L})$. The induced partial order of this lattice is the *superset relation* $\supseteqq$.

3. For a fixed complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$, denote the complete lattice structure for **$\mathfrak{L}$-fuzzy** filters of $\mathfrak{L}$ established in Theorem 2.2.1 by $\left[\mathfrak{L}\text{-Fl}\left(\mathfrak{L}\right), \cap, \cup\!\!\cdot\right]$. The *dual lattice* $\mathscr{D}\left(\left[\mathfrak{L}\text{-Fl}\left(\mathfrak{L}\right), \cap, \cup\!\!\cdot\right]\right) = \left[\mathfrak{L}\text{-Fl}\left(\mathfrak{L}\right), \cup\!\!\cdot, \cap\right]$ is denoted by $F\mathscr{F}(\mathfrak{L})$. The induced partial order of this lattice is the *superset relation* $\supseteqq$ for *$\mathfrak{L}$-fuzzy sets* (the inversion of the relation $\subseteqq$ defined in (1.19)). •

The following observations are to illustrate the concept of *expansion* (using the examples above). Let $\mathfrak{L}, \mathfrak{L}'$ be complete lattices.

Proposition 2.3.2 (Lattices expanded by the two-valued Boolean lattice)
$\mathfrak{L}$ is expanded to $\mathfrak{L}'$ by $\mathfrak{B}$ if and only if there exists a lattice $\mathfrak{L}''$ isomorphic with $\mathfrak{L}'$ such that

$$\left[\text{PFl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right] \Subset \mathfrak{L}'' \Subset \left[\text{Fl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right].$$

Proof
By Definition 2.3.1, it is sufficient to prove that $\left[\text{PFl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right]$ is isomorphic with $\left[\mathscr{P}(\mathfrak{B}, \mathfrak{L}), \cup\!\!\cdot, \cap\right]$ and $\left[\text{Fl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right]$ is isomorphic with $\left[\mathfrak{B}\text{-Fl}\left(\mathfrak{L}\right), \cup\!\!\cdot, \cap\right]$.

These results are proved separately.

1. $\left[\text{PFl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right]$ is isomorphic with $\left[\mathscr{P}(\mathfrak{B}, \mathfrak{L}), \cup\!\!\cdot, \cap\right]$.

   First of all, $\left[\text{PFl}(\mathfrak{L}), \cup\!\!\cdot, \cap\right]$ is isomorphic with $\mathfrak{L}$ by Observation 1.3.1.6.

   Secondly, $\mathfrak{L}$ is isomorphic with $\left[1\text{-PFl}\left(\mathfrak{L}\right), \cup\!\!\cdot, \cap\right]$ by Theorem 2.2.11.

It remains to prove that $\left[1\text{-PFl}(\mathfrak{L}), \uplus, \cap\right]$ is isomorphic with $\left[\mathscr{P}(\mathfrak{B}, \mathfrak{L}), \uplus, \cap\right]$. For this, it is sufficient to prove

$$\mathfrak{B}\text{-PFl}(\mathfrak{L}) = 1\text{-PFl}(\mathfrak{L}). \tag{2.35}$$

Obviously,

$$\mathfrak{B}\text{-PFl}(\mathfrak{L}) = 0\text{-PFl}(\mathfrak{L}) \cup 1\text{-PFl}(\mathfrak{L}).$$

By Observation 2.2.6,

$$0\text{-PFl}(\mathfrak{L}) = \left\{ {}^{1}\overline{1} \right\},$$

hence

$$0\text{-PFl}(\mathfrak{L}) \subsetneqq 1\text{-PFl}(\mathfrak{L}),$$

establishing (2.35).

2. $\left[\text{Fl}(\mathfrak{L}), \uplus, \cap\right]$ is isomorphic with $\left[\mathfrak{B}\text{-Fl}(\mathfrak{L}), \uplus, \cap\right]$.
Consider the mapping

$$h : \text{Fl}(\mathfrak{L}) \to \{0, 1\}^{L}$$
$$h(F)(a) =_{\text{def}} \begin{cases} 0, & \text{if } a \notin F \\ 1, & \text{if } a \in F \end{cases} \qquad F \in \text{Fl}(\mathfrak{L}), a \in L \tag{2.36}$$

That h is a bijection between $\text{Fl}(\mathfrak{L})$ and $\mathfrak{B}\text{-Fl}(\mathfrak{L})$ follows from Proposition 2.1.1.1, by observing that $\text{CUT}_1\left(h(F)\right) = F$ and $\text{CUT}_0(\mathcal{F}) = L \in \text{Fl}(\mathfrak{L})$ for all $\mathcal{F} \in \{0, 1\}^{L}$.

That h is a lattice homomorphism is assured by definition (1.17), taking into account that the lattice operation involved is the BOOLEan operation and which is also the basis for defining the set intersection, and that $\uplus$ is uniquely determined by $\cap$.

Hence, h is a lattice isomorphism between $\left[\text{Fl}(\mathfrak{L}), \uplus, \cap\right]$ and $\left[\mathfrak{B}\text{-Fl}(\mathfrak{L}), \uplus, \cap\right]$, establishing the claim. □

Corollary 2.3.3 (Lattices expanded by the two-valued Boolean lattice)

1. *If $\mathfrak{L}$ is* ***isomorphic*** *with $\mathfrak{L}'$, then $\mathfrak{L}$ is expanded to $\mathfrak{L}'$ by the* BOOLE*an lattice $\mathfrak{B}$.*
2. *$\mathfrak{L}$ is expanded to $\mathfrak{L}$ by $\mathfrak{B}$.*
3. *$\mathfrak{L}$ is expanded to $\mathscr{F}(\mathfrak{L})$ by $\mathfrak{B}$.*

Proof
Follows immediately from Proposition 2.3.2. □

Proposition 2.3.4 (Expanding the two-valued Boolean lattice)
Given lattices $\mathfrak{L}, \mathfrak{L}'$, $\mathfrak{L}$ is an expansion of $\mathfrak{B}$ by $\mathfrak{L}'$ if and only if $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{L}')$.

Proof
Let $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$. First, show that $\mathfrak{L}'\text{-PFl}\,(\mathfrak{B}) = \mathfrak{L}'\text{-Fl}\,(\mathfrak{B})$. It is a trivial observation that $\mathfrak{B}$ has only two filters, namely

$$\begin{aligned}\{0,1\} &= \overline{0},\\ \{1\} &= \overline{1}.\end{aligned}$$

It is easy to conclude from definition (2.1) and the fact that by Proposition 2.1.1, every cut of every fuzzy filter of $\mathfrak{B}$ is one of $\overline{0}$, $\overline{1}$ that all fuzzy filters of $\mathfrak{B}$ are fuzzy principal filters, i. e. every $\mathcal{F} \in \mathfrak{L}'\text{-Fl}\,(\mathfrak{B})$ is either ${}^{\ell}\overline{0}$ or ${}^{\ell}\overline{1}$ for $\ell \in L'$.

Furthermore, Observation 2.2.6 yields

$$\mathfrak{L}'\text{-PFl}\,(1) = \left\{ {}^{0}\overline{0} \right\},$$

hence

$$\mathfrak{L}'\text{-PFl}\,(0) = \mathfrak{L}'\text{-PFl}\,(\mathfrak{B}) = \mathfrak{L}'\text{-Fl}\,(\mathfrak{B})\,.$$

By item 1 of Definition 2.3.1, thus every expansion of $\mathfrak{B}$ by $\mathfrak{L}'$ is isomorphic with $\left[\mathfrak{L}'\text{-PFl}\,(0)\,, \uplus, \cap\right]$.

By Observation 2.2.13, $\mathscr{D}(\mathfrak{L}')$ is isomorphic with $\left[\mathfrak{L}'\text{-PFl}\,(0)\,, \uplus, \cap\right]$, hence the result. □

Observation 2.3.5 (Expanding a lattice by itself)
$\mathfrak{L}$ is expanded to $F\mathscr{F}(\mathfrak{L})$ by $\mathfrak{L}$. •

Proof
Obvious by the definition of $F\mathscr{F}(\mathfrak{L})$. □

In chapter 5, logics based on the examples of expansions given above shall be discussed in detail. There, a **truth value lattice** $\mathfrak{T}$ is expanded into a **label lattice** $\mathfrak{L}$ by a lattice $\mathfrak{D}$ of **degrees of trust**. The logical implications of basing the definition of *label* on the concept of *expansion* are discussed at the end of the following chapter.

This chapter is closed with a discussion of criteria for an expansion to be a *chain*.

Proposition 2.3.6 (Expanding chains)
Let complete lattices $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$ be given, such that $\mathfrak{L}_3$ is an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$.

Then $\mathfrak{L}_3$ is a chain if and only if $\mathfrak{L}_1$ is a chain and $\mathfrak{L}_2 = \mathfrak{B}$ or $\mathfrak{L}_1 = \mathfrak{B}$ and $\mathfrak{L}_2$ is a chain.

Proof
The 'if' direction is proved first, i. e. if $\mathfrak{L}_1$ is a chain and $\mathfrak{L}_2 = \mathfrak{B}$ or $\mathfrak{L}_1 = \mathfrak{B}$ and $\mathfrak{L}_2$ is a chain, then $\mathfrak{L}_3$ is a chain. Two cases are distinguished:

Case 1. Assume $\mathfrak{L}_1$ is a chain and $\mathfrak{L}_2 = \mathfrak{B}$.

It is sufficient to prove that $\left[\mathfrak{B}\text{-Fl}\,(\mathfrak{L}_1)\,, \uplus, \cap\right]$ is a chain, from which the same follows for every lattice isomorphic with a sublattice of $\left[\mathfrak{B}\text{-Fl}\,(\mathfrak{L}_1)\,, \uplus, \cap\right]$.

In the proof of Proposition 2.3.2, it has been shown that $\left[\mathfrak{B}\text{-Fl}\,(\mathfrak{L}_1)\,, \uplus, \cap\right]$ is isomorphic with $\mathscr{F}(\mathfrak{L}_1) = \left[\text{Fl}(\mathfrak{L}_1), \uplus, \cap\right]$. By Observation 1.3.1.4, $\left[\text{Fl}(\mathfrak{L}_1), \cap, \uplus\right]$ is a chain, so $\mathscr{F}(\mathfrak{L}_1)$ and hence $\left[\mathfrak{B}\text{-Fl}\,(\mathfrak{L}_1)\,, \uplus, \cap\right]$ are chains, too.

Case 2. Assume $\mathfrak{L}_1 = \mathfrak{B}$ and $\mathfrak{L}_2$ is a chain. It is sufficient to prove that $\left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{B})\,, \uplus, \cap\right]$ is a chain, from which the same follows for every lattice isomorphic with a sublattice of $\left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{B})\,, \uplus, \cap\right]$.

In the proof of Proposition 2.3.4, it has been shown that $\left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{B})\,, \uplus, \cap\right]$ is isomorphic with $\mathscr{D}\,(\mathfrak{L}_2)$, which is just the dual lattice of $\mathfrak{L}_2$ and thus a chain. Hence $\left[\mathfrak{L}_2\text{-Fl}\,(\mathfrak{B})\,, \uplus, \cap\right]$ is a chain, too.

Next, the 'only if' direction is proved, i. e. if $\mathfrak{L}_1$ is a chain, then $\mathfrak{L}_1$ is a chain and $\mathfrak{L}_2 = \mathfrak{B}$ or $\mathfrak{L}_1 = \mathfrak{B}$ and $\mathfrak{L}_2$ is a chain. For the proof by contradiction, two cases are distinguished:

Case 1. $\mathfrak{L}_1 \neq \mathfrak{B}$ and $\mathfrak{L}_2 \neq \mathfrak{B}$.

This means that there exists $\ell_1 \in L_1$ such that $\ell_1 \neq 0$ and $\ell_1 \neq 1$ holds and there exists $\ell_2 \in L_2$ such that $\ell_2 \neq 0$ and $\ell_2 \neq 1$ holds.

It is sufficient to prove that $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \uplus, \cap\right]$ is not a chain. From this it trivially follows that every expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$ is not a chain (see Definition 2.3.1).

Define the following two $\mathfrak{L}_2$-fuzzy sets $\mathcal{F}, \mathcal{G}$ on $\mathfrak{L}_1$, for $\ell \in L_1$:

$$\mathcal{F}(\ell) =_{\text{def}} \begin{cases} \ell_2, & \text{if } \ell \neq 1 \\ 1, & \text{if } \ell = 1 \end{cases}$$

$$\mathcal{G}(\ell) =_{\text{def}} \begin{cases} 1, & \text{if } \ell_1 \sqsubseteq \ell \\ 0, & \text{if not } \ell_1 \sqsubseteq \ell \end{cases}$$

i. e. $\mathcal{F} = {}^{\ell_2}\overline{0}$ and $\mathcal{G} = {}^{1}\overline{\ell_1}$.

It is obvious that $\mathcal{F}, \mathcal{G}$ are both in $\mathfrak{L}_2$-PFl$\,(\mathfrak{L}_1)$, and $\mathcal{F} \neq \mathcal{G}$. But it is also simple to check that neither $\mathcal{F} \subseteqq \mathcal{G}$ nor $\mathcal{G} \subseteqq \mathcal{F}$, because $\mathcal{G}(0) = 0 \sqsubseteq \ell_2 = \mathcal{F}(0)$, where $0 \neq \ell_2$ and $\mathcal{F}(\ell_1) = \ell_2 \sqsubseteq 1 = \mathcal{G}(\ell_1)$, where $\ell_2 \neq 1$.

It follows that $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \uplus, \cap\right]$ is not a chain.

Case 2. $\mathfrak{L}_1$ is not a chain or $\mathfrak{L}_2$ is not a chain.

By Definition 2.3.1, $\mathfrak{L}_3$ contains a sublattice isomorphic with $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \uplus, \cap\right]$.

By Theorem 2.2.11, $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \uplus, \cap\right]$ contains a sublattice isomorphic with $\mathfrak{L}_1$.

By Observation 2.2.13, $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \uplus, \cap\right]$ contains a sublattice isomorphic with the *dual* of $\mathfrak{L}_2$.

If either one of $\mathfrak{L}_1$ or $\mathfrak{L}_2$ were not a chain, then trivially, $\mathfrak{L}_3$ wouldn't be a chain, either.

As all possible cases are covered, the proof is complete. □

3 Fuzzy Filter-Based Logics

In this chapter, the basic concepts of the logical systems considered in this dissertation are defined. Features of the *underlying logic* for the labelled formulae are not referred to in any depth, apart from the fact that truth values of formulae are taken from a given lattice $\mathfrak{T}$.

After having defined the underlying **$\mathfrak{T}$-valued logical formulae**, the concept of **labelled formula** is established. The remainder of this dissertation is devoted to the study of the **fuzzy filter-based logics** defined in this chapter. The concepts of **model** and **semantic consequence** are studied in chapter 4. Chapter 5 contains a study of different variants of fuzzy filter-based logics and their expressive power and gives examples of some particular logics.

In the following, three distinct lattices shall frequently be referred to, assumed to be given when defining a particular logic:

- A complete lattice $\mathfrak{T}$ of **truth values**.
- A complete lattice $\mathfrak{D}$ of **degrees of trust** (or **validity**).
- A complete lattice $\mathfrak{L}$ of **labels**, to be attached to formulae as an *assessment* of their *validity*.

These lattices are illustrated by (few) examples, and there are scattered remarks in this and the following chapters, trying to explain the *meaning* of the values from $\mathfrak{T}$, $\mathfrak{D}$, and $\mathfrak{L}$, but basically, in this chapter and chapter 4, the lattices are treated as abstract algebraic entities without a specific fixed meaning. This allows to formulate properties of the logics under consideration from a very abstract point of view, neglecting superficial details of *semantics* which depend on the applications.

For a better understanding of the expressive power of the logics which are studied from an abstract point of view in chapter 4, in chapter 5 particular classes of logics resulting from the choice of particular lattices for $\mathfrak{T}$, $\mathfrak{D}$, and $\mathfrak{L}$ are presented and compared.

3.1 Syntax

Following Pavelka [85], an arbitrary nonempty set Frm of **logical formulae** is assumed to be given. For the time being, no assumptions whatsoever are made about the structure of Frm.

Example 3.1.1 (Propositional logic)
The most simple example of a *logical language* is that of **propositional logic**. It is determined by

1. A non-empty set PV of **propositional variables**.
2. A non-empty set Ω of **operator symbols** or **connectives**.
3. A mapping $\mathrm{Ar} : \Omega \to \mathbb{N}$ giving the **arity** of each operator.

Definition 3.1.1 (Propositional formula)
The set $\mathrm{PFrm}(\mathrm{PV}, \Omega, \mathrm{Ar})$ *of all* ***propositional formulae*** *with respect to the sets* PV, Ω *and the mapping* Ar *as defined above is the smallest set such that*

1. $\mathrm{PV} \subseteqq \mathrm{PFrm}(\mathrm{PV}, \Omega, \mathrm{Ar})$,
2. *For each* $\omega \in \Omega$ *and formulae* $x_1, \ldots, x_{\mathrm{Ar}(\omega)} \in \mathrm{PFrm}(\mathrm{PV}, \Omega, \mathrm{Ar})$, *the symbol sequence*

$$\omega\, x_1 \ldots x_{\mathrm{Ar}(\omega)}$$

is contained in $\mathrm{PFrm}(\mathrm{PV}, \Omega, \mathrm{Ar})$.

Remark
When an operator symbol $\omega \in \Omega$ is *binary*, i. e. $\mathrm{Ar}(\omega) = 2$, then the *infix notation*

$$(x_1\, \omega\, x_2) \qquad \text{instead of} \qquad \omega\, x_1 x_2$$

is also allowed. •

Example 3.1.2 (First order predicate logic)
The most common *logical language* is that of **first order predicate logic**. It is determined by

1. A non-empty set IV of **individual variables**.
2. A set Func of **function symbols**.
3. A non-empty set Pred of **predicate symbols**.
4. Mappings $\mathrm{Ar}_{\mathrm{Func}} : \mathrm{Func} \to \mathbb{N}$ giving the **arity** of each function and $\mathrm{Ar}_{\mathrm{Pred}} : \mathrm{Pred} \to \mathbb{N}$ giving the **arity** of each predicate.

   0-ary function symbols are called **individual constants** and 0-ary predicate symbols are called **propositional constants**.
5. A non-empty set Ω of **operator symbols** or **connectives**.
6. A mapping $\mathrm{Ar} : \Omega \to \mathbb{N}$ giving the **arity** of each operator.

Definition 3.1.2 (Terms and formulae in first order logic)

1. *The set* $\mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$ *of all* ***terms*** *with respect to the sets* IV, Func *and the mapping* $\mathrm{Ar}_{\mathrm{Func}}$ *as defined above is the smallest set such that*

   1.1. $\mathrm{IV} \subseteqq \mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$,

   1.2. *for each* $f \in \mathrm{Func}$ *and terms* $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Func}}(f)} \in \mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$, *the symbol sequence*

$$f\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Func}}(f)}$$

   is contained in $\mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$.

2. *The set* $\mathrm{FOFrm}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega, \mathrm{Ar})$ *of all* ***first order formulae*** *with respect to the sets* IV, Func, Pred, Ω *and the mappings* Ar, $\mathrm{Ar}_{\mathrm{Func}}$, $\mathrm{Ar}_{\mathrm{Pred}}$ *as defined above is the smallest set such that*

2.1. *For each* $p \in \text{Pred}$ *and terms* $t_1, \ldots, t_{\text{Ar}_{\text{Pred}}(p)} \in \text{Term}(\text{IV}, \text{Func}, \text{Ar}_{\text{Func}})$, *the symbol sequence*

$$p\, t_1 \ldots t_{\text{Ar}_{\text{Pred}}(p)}$$

is contained in $\text{FOFrm}(\text{IV}, \text{Func}, \text{Ar}_{\text{Func}}, \text{Pred}, \text{Ar}_{\text{Pred}}, \Omega, \text{Ar})$.
This type of formula is called ***atomic formula****.*

2.2. *For each* $\omega \in \Omega$ *and formulae*

$$x_1, \ldots, x_{\text{Ar}(\omega)} \in \text{FOFrm}(\text{IV}, \text{Func}, \text{Ar}_{\text{Func}}, \text{Pred}, \text{Ar}_{\text{Pred}}, \Omega, \text{Ar}),$$

the symbol sequence

$$\omega\, x_1 \ldots x_{\text{Ar}(\omega)}$$

is contained in $\text{FOFrm}(\text{IV}, \text{Func}, \text{Ar}_{\text{Func}}, \text{Pred}, \text{Ar}_{\text{Pred}}, \Omega, \text{Ar})$.

2.3. *For each* $v \in \text{IV}$ *and each formula*

$$x \in \text{FOFrm}(\text{IV}, \text{Func}, \text{Ar}_{\text{Func}}, \text{Pred}, \text{Ar}_{\text{Pred}}, \Omega, \text{Ar}),$$

the symbol sequences

$$\forall v\, x \qquad \textit{(universal quantifier)}$$
$$\textit{and} \quad \exists v\, x \qquad \textit{(existential quantifier)}$$

are contained in $\text{FOFrm}(\text{IV}, \text{Func}, \text{Ar}_{\text{Func}}, \text{Pred}, \text{Ar}_{\text{Pred}}, \Omega, \text{Ar})$.
Formulae of this type are called ***quantified formulae****.*

Note that many other variants for defining formulae of first order predicate logic exist in the literature.

For instance, in many variants function symbols are not present or a special equality predicate = is required to be in Pred.

Typical *extensions* of the variant presented here include the **many-sorted** variant and a variant where instead of fixing quantifiers to universal and existential one, an arbitrary set of *quantifier symbols* is employed. The latter variant is especially interesting for many-valued logics, but is neglected here to avoid cluttering up the presentation. •

The remark above allowing infix notation for binary operator symbols is used for first order formulae as well.

Furthermore, for better readability, a term of the form

$$f\, t_1 \ldots t_n$$

will sometimes be written $f(t_1, \ldots, t_n)$ or even $(t_1\ f\ t_2)$ if $n = 2$ and a first order formula

$$p\, t_1 \ldots t_m$$

will sometimes be written $p(t_1, \ldots, t_m)$ or even $(t_1\, p\, t_2)$ if $m = 2$.

Example 3.1.3 (Classical Logical Operators)
The standard set of **logical operator symbols** is defined by

$$\begin{aligned}
\Omega_{\mathrm{S}} &=_{\mathrm{def}} \{\neg, \wedge, \vee, \rightarrow\} && \\
\mathrm{Ar}_{\mathrm{S}}(\neg) &=_{\mathrm{def}} 1 && \text{(Negation)} \\
\mathrm{Ar}_{\mathrm{S}}(\wedge) &=_{\mathrm{def}} 2 && \text{(Conjunction)} \\
\mathrm{Ar}_{\mathrm{S}}(\vee) &=_{\mathrm{def}} 2 && \text{(Disjunction)} \\
\mathrm{Ar}_{\mathrm{S}}(\rightarrow) &=_{\mathrm{def}} 2 && \text{(Implication)}
\end{aligned}$$

For a fixed set PV of propositional variables, a shorthand for the standard *propositional language* is defined by $\mathrm{PFrm}_{\mathrm{S}} =_{\mathrm{def}} \mathrm{PFrm}(\mathrm{PV}, \Omega_{\mathrm{S}}, \mathrm{Ar}_{\mathrm{S}})$.

Assuming that $p, q \in \mathrm{PV}$, examples for *propositional formulae* from $\mathrm{PFrm}_{\mathrm{S}}$ include

$$\begin{gathered}
p \\
\neg q \\
(p \vee q) \\
\neg(\neg p \wedge \neg q)
\end{gathered}$$

For a fixed set IV of individual variables, a fixed set Func of function symbols with corresponding arity mapping $\mathrm{Ar}_{\mathrm{Func}} : \mathrm{Func} \rightarrow \mathbb{N}$, and a fixed set Pred of predicate symbols with corresponding arity mapping $\mathrm{Ar}_{\mathrm{Pred}} : \mathrm{Pred} \rightarrow \mathbb{N}$, a shorthand for the standard *first order language* is defined by

$$\mathrm{FOFrm}_{\mathrm{S}} =_{\mathrm{def}} \mathrm{FOFrm}\left(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega_{\mathrm{S}}, \mathrm{Ar}_{\mathrm{S}}\right).$$

Assuming that $v, w \in \mathrm{IV}$, $\xi, f, * \in \mathrm{Func}$ such that $\mathrm{Ar}_{\mathrm{Func}}(\xi) = 0$, $\mathrm{Ar}_{\mathrm{Func}}(f) = 1$, and $\mathrm{Ar}_{\mathrm{Func}}(*) = 2$, and furthermore $p, = \in \mathrm{Pred}$ such that $\mathrm{Ar}_{\mathrm{Pred}}(p) = 1$ and $\mathrm{Ar}_{\mathrm{Pred}}(=) = 2$, then the following are examples for *first order formulae* from $\mathrm{FOFrm}_{\mathrm{S}}$:

$$\begin{gathered}
p(f(\xi)) \\
\neg(v = f(v)) \\
\exists v \exists w((v * f(w)) = \xi) \\
\forall v(p(v) \vee p(f(v)))
\end{gathered}$$

•

There are logical languages with even more expressive power than first order predicate logic, for instance second order predicate logics or modal logics. As the semantic considerations in the following chapters are very general in nature, and are potentially applicable to a wide range of logical systems, the logical language is restricted as little as possible, and the examples of concrete languages given above are used only where examples of formulae are needed. In this case, *propositional logic* is mostly sufficient for demonstrating certain effects; *first order logic* is only used in some examples of *knowledge representation* at the very end of this dissertation (see section 5.5.2).

In the major part of this dissertation, however, Frm is completely arbitrary.

It will sometimes be interesting to study the behaviour of **semantic operators** (like *semantic entailment*) wrt. certain logical connectives, without having to restrict the set of formulae to a specific type of logical language. To this end, in the following a logical language Frm shall be said to **contain** a connective ω of arity $n \in \mathbb{N}$ if and only if for all $x_1, \ldots, x_n \in \mathrm{Frm}$,

$$\omega\, x_1 \ldots x_n \in \mathrm{Frm}, \tag{3.1}$$

and furthermore, for every $x \in \mathrm{Frm}$, it can be uniquely determined whether it is of the form (3.1). This way, it is possible to speak in a sensible manner about formulae constructed by using ω without overly restricting the range of concrete structures of logical formulae. If $n = 2$, the notation

$$(x_1\, \omega\, x_2) \qquad \text{instead of} \qquad \omega\, x_1 x_2$$

is also allowed.

3.2 Semantics

The basis for the definition of many-valued **semantics** is a complete lattice $\mathfrak{T} =_{\mathrm{def}} [T, \boxed{\sqcap}, \boxed{\sqcup}]$. The members of T are called **truth values**.

Because the structure of logical formulae has not been fixed, the semantics of a logic are defined to be an arbitrary collection of **truth valuation functions**, i. e. a **semantics** for Frm is a set

$$\mathfrak{S} \subseteqq T^{\mathrm{Frm}}$$

of **truth valuation functions** $\mathrm{Val} \in \mathfrak{S}$, $\mathrm{Val} : \mathrm{Frm} \to T$.

For some characterisation theorems, the following property will be necessary:

(3.2) For every $t \in T$ there exists $\mathrm{Val}_t \in \mathfrak{S}$ and $x_t \in \mathrm{Frm}$ such that $\mathrm{Val}_t(x_t) = t$.

This fundamental property will be assumed in the following without further mention. It is not really a very severe restriction on the admissible logical systems. In proofs, it shall be indicated where it is essential.

Example 3.2.1 (Semantics for Propositional Logics)
The semantics for a **propositional language** $\mathrm{PFrm}(\mathrm{PV}, \Omega, \mathrm{Ar})$ is a set of valuation functions induced by *assignments* of truth values to the propositional variables.

An **assignment** of truth values to the propositional variables is a mapping of the form

$$\mathcal{A} : \mathrm{PV} \to T.$$

Following the *principle of extensionality*, it is assumed that an $\mathrm{Ar}(\omega)$-ary truth value function

$$\varphi_\omega : T^{\mathrm{Ar}(\omega)} \to T$$

is associated with each operator symbol $\omega \in \Omega$.

With every assignment $\mathcal{A} \in T^{\mathrm{PV}}$, a valuation function $\mathrm{Val}_{\mathcal{A}}$ is associated inductively as follows. Let $x \in \mathrm{Frm}$.

Definition 3.2.1 (Valuation function in propositional logic)

1. *If* $x \in \mathrm{PV}$, *then* $\mathrm{Val}_{\mathcal{A}}(x) =_{\mathrm{def}} \mathcal{A}(x)$.
2. *If there are* $\omega \in \Omega$ *and propositional formulae* $x_1, \ldots, x_{\mathrm{Ar}(\omega)}$ *such that*

$$x = \omega\, x_1 \ldots x_{\mathrm{Ar}(\omega)}$$

then

$$\mathrm{Val}_{\mathcal{A}}(x) =_{\mathrm{def}} \varphi_\omega \left(\mathrm{Val}_{\mathcal{A}}(x_1), \ldots, \mathrm{Val}_{\mathcal{A}}\left(x_{\mathrm{Ar}(\omega)}\right)\right).$$

Assuming $\mathfrak{T}$, PV, Ω, Ar and the mappings φ_ω for all $\omega \in \Omega$ to be given, the **semantics** $\mathfrak{S}$ for $\mathrm{PFrm}(\mathrm{PV}, \Omega, \mathrm{Ar})$ is defined to be

$$\mathfrak{S} =_{\mathrm{def}} \{\mathrm{Val}_{\mathcal{A}} | \mathcal{A} : \mathrm{PV} \to T\}. \tag{3.3}$$

For the example of propositional logic, assumption (3.2) is trivially fulfilled. For an arbitrary propositional variable $p \in \mathrm{PV}$, define $x_t =_{\mathrm{def}} p$ for every $t \in T$, and as Val_t, employ any valuation function $\mathrm{Val}_{\mathcal{A}}$ such that $\mathcal{A}(p) = t$. •

Example 3.2.2 (Semantics for First Order Logics)
The semantics for a **first order language** $\mathrm{FOFrm}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega, \mathrm{Ar})$ is a set of valuation functions induced by *interpretations* which specify a *domain* containing all *individuals* under consideration and assign *fuzzy relations* (on the domain) to *predicate symbols*, and *functions* (on the domain) to *function symbols*.

Definition 3.2.2 (Interpretations in First Order Logic)
Given a ***first order language*** $\mathrm{Frm} = \mathrm{FOFrm}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega, \mathrm{Ar})$ *(see* Example 3.1.2 *for a definition of* IV*,* Func*,* $\mathrm{Ar}_{\mathrm{Func}}$*,* Pred*,* $\mathrm{Ar}_{\mathrm{Pred}}$*,* Ω*, and* Ar*), an* ***interpretation*** *for* Frm *is given by a tuple*

$$\mathfrak{I} = [U, \Pi, \Phi]$$

where

1. *U is an arbitrary non-empty set called* ***domain*** *or* ***universe****.*
2. $\Pi : \mathrm{Pred} \to \bigcup \left\{ T^{U^n} \middle| n \in \mathbb{N} \right\}$ *such that for every* $p \in \mathrm{Pred}$, $\Pi(p) \in T^{U^{\mathrm{Ar}_{\mathrm{Pred}}(p)}}$.
3. $\Phi : \mathrm{Func} \to \bigcup \left\{ U^{U^n} \middle| n \in \mathbb{N} \right\}$ *such that for every* $f \in \mathrm{Func}$, $\Phi(f) \in U^{U^{\mathrm{Ar}_{\mathrm{Func}}(f)}}$.

Logical operator symbols are interpreted as for *propositional logic* (see Example 3.2.1), i. e. it is assumed that an $\mathrm{Ar}(\omega)$-ary truth value function

$$\varphi_\omega : T^{\mathrm{Ar}(\omega)} \to T$$

is associated with each operator symbol $\omega \in \Omega$.

With every *interpretation* $\mathfrak{I} = [U, \Pi, \Phi]$ as specified above, a valuation function $\mathrm{Val}_{\mathfrak{I}}$ is associated inductively as follows.

Definition 3.2.3 (Valuation of terms and formulae in first order logic)
Let a ***first order language*** $\mathrm{Frm} = \mathrm{FOFrm}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega, \mathrm{Ar})$ *and an* ***interpretation*** $\mathfrak{I} = [U, \Pi, \Phi]$ *for* Frm *be given.*

For this definition, ***assignments***

$$\sigma : \mathrm{IV} \to U$$

are used. Given an assignment $\sigma : \mathrm{IV} \to U$*, an individual variable* $v \in \mathrm{IV}$ *and an element* $u \in U$ *of the domain, the notation* $\sigma_{v:=u}$ *denotes the assignment given for* $w \in \mathrm{IV}$ *by*

$$\sigma_{v:=u}(w) =_{\mathrm{def}} \begin{cases} u, & \text{if } w = v \\ \sigma(w), & \text{if } w \neq v. \end{cases}$$

Next, the interpretation of terms and formulae of (multiple-valued) first order logic is defined.

1. *Given an assignment* $\sigma : \mathrm{IV} \to U$ *and a* ***Term*** $t \in \mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$, *the* ***individual*** *associated with* t *by* $\mathfrak{I}$ *and* σ *is denoted by* $\mathrm{Ind}(t, \mathfrak{I}, \sigma) \in U$ *and defined inductively by*

   1.1. $\mathrm{Ind}(t, \mathfrak{I}, \sigma) =_{\mathrm{def}} \sigma(t)$ *if* $t \in \mathrm{IV}$;

   1.2. *for* $f \in \mathrm{Func}$ *and* $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Func}}(f)} \in \mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$ *such that* $t = f\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Func}}(f)}$,

   $$\mathrm{Ind}(t, \mathfrak{I}, \sigma) =_{\mathrm{def}} \Phi(f) \left(\mathrm{Ind}\,(t_1, \mathfrak{I}, \sigma), \ldots, \mathrm{Ind}\left(t_{\mathrm{Ar}_{\mathrm{Func}}(f)}, \mathfrak{I}, \sigma\right) \right).$$

2. *Given an assignment* $\sigma : \mathrm{IV} \to U$ *and a* ***Formula*** $x \in \mathrm{Frm}$, *the* ***truth value*** *associated with* x *by* $\mathfrak{I}$ *and* σ *is denoted by* $\mathrm{Val}(x, \mathfrak{I}, \sigma) \in T$ *and defined inductively as follows.*

   2.1. *For* $p \in \mathrm{Pred}$ *and* $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \in \mathrm{Term}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}})$ *such that* $x = p\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}$,

   $$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}} \Pi(p) \left(\mathrm{Ind}\,(t_1, \mathfrak{I}, \sigma), \ldots, \mathrm{Ind}\left(t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}, \mathfrak{I}, \sigma\right) \right).$$

   2.2. *For* $\omega \in \Omega$ *and* $x_1, \ldots, x_{\mathrm{Ar}(\omega)} \in \mathrm{Frm}$ *such that* $x = \omega\, x_1 \ldots x_{\mathrm{Ar}(\omega)}$,

   $$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}} \varphi_\omega \left(\mathrm{Val}(x_1, \mathfrak{I}, \sigma), \ldots, \mathrm{Val}\left(x_{\mathrm{Ar}(\omega)}, \mathfrak{I}, \sigma\right) \right).$$

   2.3. *For* $v \in \mathrm{IV}$ *and* $y \in \mathrm{Frm}$ *such that* $x = \forall v\, y$,

   $$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}} \overline{\underline{|\mathrm{T}}|} \left\{ \mathrm{Val}\,(y, \mathfrak{I}, \sigma_{v:=u}) \,\middle|\, u \in U \right\}. \tag{3.4}$$

   2.4. *For* $v \in \mathrm{IV}$ *and* $y \in \mathrm{Frm}$ *such that* $x = \exists v\, y$,

   $$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}} \underline{|\mathrm{T}|} \left\{ \mathrm{Val}\,(y, \mathfrak{I}, \sigma_{v:=u}) \,\middle|\, u \in U \right\}.$$

3. *The valuation function* $\mathrm{Val}_{\mathfrak{I}} : \mathrm{Frm} \to T$ *induced by* $\mathfrak{I}$ *is now defined as follows. Let* $x \in \mathrm{Frm}$ *be given. Then*

   $$\mathrm{Val}_{\mathfrak{I}}(x) =_{\mathrm{def}} \overline{|\mathrm{T}|} \left\{ \mathrm{Val}\,(x, \mathfrak{I}, \sigma) \,\middle|\, \sigma : \mathrm{IV} \to U \right\} \tag{3.5}$$

Assuming $\mathfrak{T}$, IV, Func, $\mathrm{Ar}_{\mathrm{Func}}$, Pred, $\mathrm{Ar}_{\mathrm{Pred}}$, Ω, Ar and all the mappings φ_ω to be given, the **semantics** $\mathfrak{S}$ for $\mathrm{FOFrm}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega, \mathrm{Ar})$ is defined to be

$$\mathfrak{S} =_{\mathrm{def}} \left\{ \mathrm{Val}_{\mathfrak{I}} \,\middle|\, \mathfrak{I} = [U, \Pi, \Phi] \text{ as defined in Definition 3.2.2} \right\}. \tag{3.6}$$

For the example of first order logic, assumption (3.2) is trivially fulfilled. As Pred is required to be non-empty, there exists $p \in \mathrm{Pred}$. Given $t \in T$, define $x_t =_{\mathrm{def}} p\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}$ for arbitrary terms $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}$ and define Val_t as $\mathrm{Val}_{[U,\Pi,\Phi]}$ such that $\Pi(p)$ is constantly t. •

Remarks

1. Note that several variants for defining the semantics of *fuzzy first order logic* exist.

   Some are driven by variants of **syntax**. For instance, in many-sorted logics, an interpretation has to contain several *domains* associated with *sorts*. When a specialised *equality predicate* = exists, then there has to be a fixed *fuzzy equality relation* (see [58, 59]) interpreting it.

Other variants result from different ways of handling **free individual variables**. The variant presented here is **generalisation invariant**, i. e. for an arbitrary individual variable v, formula x and interpretation $\mathfrak{I}$, it holds that

$$\mathrm{Val}_{\mathfrak{I}}(x) = \mathrm{Val}_{\mathfrak{I}}(\forall v\, x).$$

Exchanging $\lceil \mathrm{T} \rceil$ with $\lfloor \mathrm{T} \rfloor$ in (3.5) leads to **specialisation invariance**, i. e. for an arbitrary individual variable v, formula x and interpretation $\mathfrak{I}$, it holds that

$$\mathrm{Val}_{\mathfrak{I}}(x) = \mathrm{Val}_{\mathfrak{I}}(\exists v\, x).$$

A third variant, namely including an assignment $\sigma : \mathrm{IV} \to U$ into every interpretation and defining $\mathrm{Val}_{\mathfrak{I}}(x)$ to be $\mathrm{Val}(x, \mathfrak{I}, \sigma)$ for this specific assignment leads to an equivalence between free individual variables and **individual constants**, i. e. 0-ary function symbols.

Another possible variation of semantics could be wrt **quantifiers**. Here, the *canonical quantifiers* given by the infinitary lattice connectives are used. An alternative is to use **fuzzy quantifiers** as described in [95,96] as a replacement for or addition to the canonical ones.

Finally, the definition of **terms** given here is in a certain sense *two-valued*. The terms themselves are interpreted by individuals in a crisp way. Fuzziness is introduced at the level of predicate symbols acting on terms as *fuzzy predicates*. In contrast with this definition, in the domain of fuzzy logic, constants like "high temperature" are often interpreted by fuzzy sets over the domain (also called *linguistic terms*). The way *first order many-valued logic* is introduced here, it would be very hard to incorporate terms which are interpreted by fuzzy sets. The logic PLFC (see section 5.7.2), for instance, allows *fuzzy constants*, but uses *two-valued predicates*. Still, a complicated definition is needed for evaluating the truth value of an atomic formula involving fuzzy constants. With fuzzy predicates on top of that, the semantics of a simple atomic formula would get even more complicated. Among other problems, there is in fact no canonical method for calculating the truth value of an atomic formula in this case, which involves several applications of operators for combining truth values. These operators are by no means unique, creating a lot of case distinctions to be considered.

Note that fuzzy constants can be simulated in the first order many-valued logic defined here by defining an *indicator predicate* p_f for the fuzzy constant f such that for an individual variable v, the atomic formula $p_f(v)$ gives the membership degree of the individual assigned to v in the fuzzy constant f.

2. For examples of *knowledge modelling* to be given in the sequel, it will sometimes be convenient to *fix* several properties of an *interpretation*, i. e. to *restrict* the range of interpretations forming the basis of semantics.

   This will be done here in a completely naive way, i. e. the statement "fix the domain to be the set of all natural numbers and the function symbol + to be interpreted by addition" means that for the example at hand, $\mathfrak{S}$ consists only of those valuations $\mathrm{Val}_{[U,\Pi,\Phi]}$ for which $U = \mathbb{N}$ and $\Phi(+)$ is the usual addition of natural numbers.

   From a model-theoretic point of view, and in particular for the axiomatisation of the respective logic, this kind of restriction can lead to severe problems, but as it doesn't compromise the property (3.2), and as first order logic is used here only for simple examples of knowledge bases anyway, this doesn't lead to problems in this dissertation.

   For a proper handling of this kind of restriction in first order logic, the respective *constraints* should be expressed by *axioms* to be added to the knowledge base, effecting a restriction of the set of admissible interpretations from *within* the knowledge base. This can lead to problems because of the limited expressive power of first order logic (for instance, the natural numbers cannot be characterised up to isomorphism by axioms in first order predicate logic), but it is neutral from a model theoretic point of view. •

Example 3.2.3 (Logical operators based on the lattice connectives)
Given a truth value lattice $\mathfrak{T} = [T, \sqcap_{\mathfrak{T}}, \sqcup_{\mathfrak{T}}]$, the simplest interpretation of the classical logical operator symbols from Ω_S is the one which is defined using the **lattice connectives**. To this end, $\wedge$ and $\vee$ are interpreted by the lattice connectives $\sqcap_{\mathfrak{T}}$ and $\sqcup_{\mathfrak{T}}$ of $\mathfrak{T}$, respectively, i. e.

$$\varphi_\wedge =_{\text{def}} \sqcap_{\mathfrak{T}}$$
$$\varphi_\vee =_{\text{def}} \sqcup_{\mathfrak{T}}$$

Additionally, an interpretation for the **negation connective** $\neg$ is needed. For this, assume that a bijective unary function $\varphi_\neg : T \to T$ is given which is *order-reversing*, i. e. for $s, t \in T$,

$$s \sqsubseteq_{\mathfrak{T}} t \text{ iff } \varphi_\neg(t) \sqsubseteq_{\mathfrak{T}} \varphi_\neg(s).$$

Such functions on a lattice have been studied, for instance, by G. DE COOMAN and E. E. KERRE [15] under the name **negation operators**.

For the **implication connective**, basically two choices for defining the corresponding truth value function $\varphi_\to$ exist.

Choosing the **s-implication** of $\sqcup_{\mathfrak{T}}$ wrt. $\varphi_\neg$ means that $\varphi_\to$ is expressed directly in terms of $\sqcup_{\mathfrak{T}}$ and $\varphi_\neg$ as follows, for $s, t \in T$:

$$\varphi_\to(s,t) =_{\text{def}} \varphi_\neg(s) \sqcup_{\mathfrak{T}} t. \tag{3.7}$$

This choice has the disadvantage of not adding to the expressive power of the logical operators, but on the other hand, it means that implication is easily eliminated from formulae, for instance when constructing *normal forms*. The situation is exactly opposite for the next definition.

Choosing the **r-implication** of $\sqcap_{\mathfrak{T}}$ means that $\varphi_\to$ is expressed indirectly in terms of $\sqcap_{\mathfrak{T}}$ as follows, for $s, t \in T$:

$$\varphi_\to(s,t) =_{\text{def}} \bigsqcup_{\mathfrak{T}} \{r \,|\, r \in T \text{ and } s \sqcap_{\mathfrak{T}} r \sqsubseteq_{\mathfrak{T}} t\}\,. \tag{3.8}$$

Implications will not be used very much in this dissertation, so the issue is not discussed further here. See [61] for a deep study of this subject. •

Example 3.2.4 (Lattice logics on the two-valued lattice and the unit interval)
The previous example is illustrated further by taking a look at the resulting logics for two particular lattices (see Examples 1.3.1).

1. For the classical **two-valued Boolean lattice** $\mathfrak{B} =_{\text{def}} \big[\{0,1\}, \text{and}, \text{or}\big]$, the only negation operator is the classical two-valued negation $\varphi_\neg : \{0,1\} \to \{0,1\}$, defined by

   $$\varphi_\neg(0) =_{\text{def}} 1, \quad \varphi_\neg(1) =_{\text{def}} 0. \tag{3.9}$$

   S-implication as well as r-implication both yield the same truth value function in this case, namely classical two-valued implication, given by the following truth table:

s	t	$\varphi_\to(s,t)$
0	0	1
0	1	1
1	0	0
1	1	1

Hence, the resulting lattice-based logic on the two-valued lattice is uniquely determined to be the classical BOOLEan propositional logic, with the following interpretations of the logical operators:

t	$\varphi_\neg(t)$
0	1
1	0

s	t	$\varphi_\wedge(s,t)$	$\varphi_\vee(s,t)$	$\varphi_\rightarrow(s,t)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	1

The **semantics** for **propositional logic** defined (wrt. Ω_{S}, Ar_{S}, and a fixed set PV of propositional variables) by (3.3) is denoted by $\mathfrak{S}^{\mathrm{P}}_{\mathrm{B}}$.

The **semantics** for **first order predicate logic** defined (wrt. Ω_{S}, Ar_{S}, and fixed IV, Func, $\mathrm{Ar}_{\mathrm{Func}}$, Pred, $\mathrm{Ar}_{\mathrm{Pred}}$, see Example 3.1.2) by (3.6) is denoted by $\mathfrak{S}^{\mathrm{F}}_{\mathrm{B}}$.

2. For the **real unit interval** $\mathfrak{F} =_{\mathrm{def}} \left[\langle 0,1\rangle, \min, \max\right]$, the canonical choice for a negation operator is the classical ŁUKASIEWICZ negation $\varphi_\neg : \langle 0,1\rangle \to \langle 0,1\rangle$, defined by

$$\varphi_\neg(t) =_{\mathrm{def}} 1 - t. \qquad (t \in \langle 0,1\rangle) \tag{3.10}$$

Concerning implication, the s-implication and r-implication in this setting are different. For the s-implication, (3.7) yields the KLEENE-DIENES implication $\mathrm{imp}_{\mathrm{KD}}$, given for $s, t \in \langle 0,1\rangle$ by

$$\mathrm{imp}_{\mathrm{KD}}(s,t) = \max(1-s, t). \tag{3.11}$$

For the r-implication, (3.8) yields the GÖDEL implication $\mathrm{imp}_{\mathrm{G}}$, given for $s, t \in \langle 0,1\rangle$ by

$$\mathrm{imp}_{\mathrm{G}}(s,t) = \begin{cases} 1, & \text{if } s \leqq t \\ t, & \text{if } s > t \end{cases} \tag{3.12}$$

Here, the s-implication $\mathrm{imp}_{\mathrm{KD}}$ is chosen as the 'standard' interpretation of the implication operator $\rightarrow$ for lattice-based logic on the unit interval. This means that $\rightarrow$ is a *defined operator*; in particular, all logical operators from Ω_{S} can be expressed by means of $\vee, \neg$, which is important for considerations involving **clausal form** (compare Example 4.2.1, Example 4.3.1, Observation 5.2.6, and section 5.4.1).

This way, the well-known min-max fuzzy logic on the unit interval is obtained, which was studied for instance by LEE and CHANG [63, 64]. For convenience, the interpretations of the logical operators from Ω_{S} are repeated here, for $s, t \in \langle 0,1\rangle$:

$$\begin{aligned} \varphi_\neg(t) &= 1 - t \\ \varphi_\wedge(s,t) &= \min(s,t) \\ \varphi_\vee(s,t) &= \max(s,t) \\ \varphi_\rightarrow(s,t) &= \max(1-s,t). \end{aligned}$$

The **semantics** for **propositional logic** defined (wrt. Ω_{S}, Ar_{S}, and a fixed set PV of propositional variables) by (3.3) is denoted by $\mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$.

The **semantics** for **first order predicate logic** defined (wrt. Ω_{S}, Ar_{S}, and fixed IV, Func, $\mathrm{Ar}_{\mathrm{Func}}$, Pred, $\mathrm{Ar}_{\mathrm{Pred}}$, see Example 3.1.2) by (3.6) is denoted by $\mathfrak{S}^{\mathrm{F}}_{\mathrm{F}}$. •

Remark

Note that the standard interpretation of logical operators chosen here for *two-valued logic* is in fact the only sensible one, while for many-valued logic there are many possible choices, none of which is popular enough to have become a 'standard'.

In fact, the interpretation chosen in Example 3.2.4.2 isn't even a very popular one as it has very limited expressive power. Its advantage wrt the considerations in this dissertation are mainly that all connectives are based purely on the *lattice structure* of the truth values and a simple order-reversing bijection interpreting the negation operator, and hence no more complicated algebraic structures, like **residuated lattices** or **MV-algebras** have to be considered.

When more expressive power is required, then the implication operator should be interpreted by $\mathrm{imp}_{\mathrm{G}}$ (leading to a *residuated lattice structure*) or by the Łukasiewicz implication $\mathrm{imp}_{\mathrm{L}}$, given for $s, t \in \langle 0, 1\rangle$ by

$$\mathrm{imp}_{\mathrm{L}}(s,t) = \min(1, 1 - s + t).$$

This leads to the well-known **Łukasiewicz infinitely many-valued logic** (compare [76]) and equips the truth value lattice with an **MV algebra** structure (compare [9]). •

For investigating properties of logical operators without having to completely fix the structure of logical formulae, a logic defined by Frm, $\mathfrak{T}$ and $\mathfrak{S}$ is said to **contain** an n-ary connective ω **interpreted by** the n-ary function $\varphi_\omega : T^n \to T$ if and only if Frm contains ω and for every formula $x \in \mathrm{Frm}$ of the form (3.1), and for every $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Val}(x) = \varphi_\omega(\mathrm{Val}(x_1), \ldots, \mathrm{Val}(x_n)). \tag{3.13}$$

From this point of view, every logic of the form described in Example 3.2.3 contains a binary operator $\wedge$ interpreted by $\overline{|\mathrm{T}|}$.

3.3 Properties of and Relations Between Formulae

Neither properties of nor relations between formulae are studied in depth in this dissertation, as considerations are focused on properties of and relations between **labelled formulae** (see section 4.2). In the following, only those concepts are defined which are needed later for establishing properties of and relations between labelled formulae.

For two logical formulae the meaning of their being **semantically equivalent** is obvious. Later, it will become apparent that the model relation is needed to define the same for labelled formulae.

Definition 3.3.1 (Semantic equivalence of formulae and sets of formulae)

1. *Two formulae* $x, y \in \mathrm{Frm}$ *are said to be* ***semantically equivalent*** $(x \equiv y)$ $=_{\mathrm{def}}$ *for every* $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Val}(x) = \mathrm{Val}(y).$$

2. *Let* $X, Y \subseteqq \mathrm{Frm}$. *Then* X *and* Y *are said to be* ***semantically equivalent*** $(X \equiv Y)$ $=_{\mathrm{def}}$ *for every* $\mathrm{Val} \in \mathfrak{S}$,

$$\underset{x\in X}{\overline{|\mathrm{T}|}}\,\mathrm{Val}(x) = \underset{y\in Y}{\overline{|\mathrm{T}|}}\,\mathrm{Val}(y).$$

Remark
Obviously, for $x, y \in \mathrm{Frm}$, $x \equiv y$ is equivalent with $\{x\} \equiv \{y\}$. •

Observation 3.3.1 ($\equiv$ is an equivalence relation)
$\equiv$ is an **equivalence relation** on Frm and also on $\mathfrak{P}\mathrm{Frm}$. •

Proof
Follows immediately from the definitions. □

Definition 3.3.2 (Equivalence classes of formulae)

1. *Given $x \in \mathrm{Frm}$, the **equivalence class** $[x]$ of x wrt. $\equiv$ is defined by*

$$[x] =_{\mathrm{def}} \{y \,|\, y \in \mathrm{Frm} \text{ and } x \equiv y\}\,. \tag{3.14}$$

2. *For $X \subseteqq \mathrm{Frm}$, the **quotient set** $X_{\equiv}$ of X wrt. $\equiv$ is defined by*

$$X_{\equiv} =_{\mathrm{def}} \big\{[x] \cap X \,\big|\, x \in X\big\}\,. \tag{3.15}$$

Definition 3.3.3 (Semantic covering, syntax transformation)

1. *Given $X, Y \subseteqq \mathrm{Frm}$, X is said to be a **semantic covering** of Y (denoted $Y \leqq X$)*
   $=_{\mathrm{def}}$ for every $y \in Y$, there exists $x \in X$ such that $y \equiv x$.

2. *Given $X \subseteqq \mathrm{Frm}$, an operator $\mathscr{T} : \mathrm{Frm} \to \mathfrak{P}\mathrm{Frm}$ is said to be a **semantic-preserving syntax transformation operator** wrt X*
   *$=_{\mathrm{def}}$ for every $x \in X$, $\mathscr{T}(x)$ is **finite** and $\{x\} \equiv \mathscr{T}(x)$.*

Remarks

1. The notion of *semantic covering* is very strong. In particular, it is **not** implied by semantic equivalence, in general (but $X \leqq Y$ and $Y \leqq X$ obviously implies $X \equiv Y$). It is useful mainly for *normal form* theorems, where Y is an arbitrary set of formulae and X is a set of formulae in normal form.

2. A *syntax transformation operator* is obviously also useful mainly for transformation into *normal form*. In contrast to a normal form given by a *semantic covering*, a *syntax transformation operator* can be used for *weak normal forms* where a *set of formulae* is associated with every formula, e. g. **clausal form**.

3. Obviously, if $X \leqq Y$, then there exists a semantic-preserving syntax transformation operator $\mathscr{T}$ wrt X such that $\bigcup \mathrm{rg}\,\mathscr{T} \subseteqq Y$. Simply let $\mathscr{T}(x) = \{y\}$ for $y \in Y$ such that $x \equiv y$ (such an y exists by $X \leqq Y$).

   But for arbitrary $X, Y \subseteqq \mathrm{Frm}$, the existence of a semantic-preserving syntax transformation operator $\mathscr{T}$ wrt X such that $\bigcup \mathrm{rg}\,\mathscr{T} \subseteqq Y$ does not imply $X \leqq Y$, in general.

4. Given $X, Y \subseteqq \mathrm{Frm}$, if there exists a semantic-preserving syntax transformation operator $\mathscr{T}_X$ wrt X such that $\bigcup \mathrm{rg}\,\mathscr{T}_X \subseteqq Y$ and vice versa there exists a semantic-preserving syntax transformation operator $\mathscr{T}_Y$ wrt Y such that $\bigcup \mathrm{rg}\,\mathscr{T}_Y \subseteqq X$, this implies $X \equiv Y$.

   But $X \equiv Y$ does not imply the existence of semantic-preserving syntax transformation operators $\mathscr{T}_X$ or $\mathscr{T}_Y$ as described above, in general. •

Definition 3.3.4 (Tautology and satisfiability index)

1. *For a formula $x \in \mathrm{Frm}$, its* ***inherent truth*** *(or* ***tautology index****)* $\mathrm{taut}(x)$ *is given by*

$$\mathrm{taut}(x) =_{\mathrm{def}} \lceil \mathrm{T} \rceil \left\{ \mathrm{Val}(x) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\}.$$

A formula x is said to be a ***tautology*** *iff* $\mathrm{taut}(x) = 1$. *The set of all tautologies is denoted* Taut.

2. *For a formula $x \in \mathrm{Frm}$, its* ***satisfiability index*** $\mathrm{sat}(x)$ *is given by*

$$\mathrm{sat}(x) =_{\mathrm{def}} \lfloor \mathrm{T} \rfloor \left\{ \mathrm{Val}(x) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\}.$$

A formula x is said to be ***insatisfiable*** *iff* $\mathrm{sat}(x) = 0$.

Remark

Obviously, for $x \in \mathrm{Frm}$,

$$x \in \mathrm{Taut} \text{ iff } \forall\, \mathrm{Val} \in \mathfrak{S}, \mathrm{Val}(x) = 1. \tag{3.16}$$

•

3.4 Expressing Uncertainty in Many-Valued Logics

Before in the next section, the **labelled logics** to be used in the remainder of this dissertation are introduced and justified, in this section a short discussion is given concerning means of expressing **graded truth assessment** and/or **graded trust assessment** in logical systems like those defined in the preceding part of this chapter.

The conclusion of this section is that graded truth assessment and graded trust assessment is best expressed in *labelled logics* the labels of which are (essentially) fuzzy filters, as will be done in the remainder of this dissertation. For alternative approaches, see section 5.7.

The power of a multiple-valued logic to express uncertainty is illustrated using the **model relation** $\models$ between valuations and formulae defined in the respective logic. Several *levels* of definitions for this concept with increasing expressive power are presented in the next two subsections. Note that higher logical concepts like *semantic consequence*, *semantic equivalence* and *validity* (of a formula) are all based on the model relation. In chapter 4, the interrelations between these concepts are presented in detail.

Note that the presentation in this section is still very much from the intuitive point of view. The respective logics are described only briefly, just enough to compare their expressive power. Only labelled logics with fuzzy filters as labels are studied more deeply in the sequel. However, most of the other levels can be embedded in this system, because of the very high expressive power (see section 5.5).

3.4.1 Expressing Graded Truth Assessment

As explained in chapter 1, the expression *graded truth assessment* is used in this dissertation to describe the modelling of *vagueness* or *ambiguity* using a given many-valued logic. For carrying out this modelling, there has to be a possibility of *assessing* the truth of formulae, and the definition of the model relation should reflect this assessment.

Level 1: Classical Many-valued Logic. In this case, $\mathrm{Val} \in \mathfrak{S}$ is said to be a **model** for a formula $x \in \mathrm{Frm}$ ($\mathrm{Val} \models x$) iff $\mathrm{Val}(x) = 1$.

In this level, a priori there is no means of expressing graded truth assessment wrt a formula. Val has to make x completely true to be considered a model of x.

If Frm contains *truth constants* and a *residual implication*, then graded truth assessment can be expressed *inside* the logical language (compare [53]), but this aspect will not be investigated any further here (see, however, section 5.7).

Level 2: Using a set of Designated Truth Values. Slightly more expressive power than in level 1 is gained by defining a set $D \subsetneqq T$ of **designated truth values** and considering $\mathrm{Val} \in \mathfrak{S}$ to be a **model** for $x \in \mathrm{Frm}$ ($\mathrm{Val} \models x$) iff $\mathrm{Val}(x) \in D$.

Obviously, level 1 is reduced to level 2 by defining $D =_{\mathrm{def}} \{1\}$.

This level leaves at least a little room for *graded truth assessment* because D can be chosen to reflect an *application*-specific assessment of the truth of the formulae in a given knowledge base; the expressive power is very limited, however, because one and the same set D of designated truth values is employed for *all* formulae, making it impossible to assess the truth of individual formulae differently.

Note that for modelling *uncertainty* about the truth of formulae, it can be expected that D is a **filter** of $\mathfrak{T}$. Property 2a from Observation 1.3.1.1 is essential in this case, because it guarantees the **monotonicity** of $\models$. Assume that for fixed $\mathrm{Val} \in \mathfrak{S}, x \in \mathrm{Frm}$, it holds that $\mathrm{Val} \models x$ and there exists a formula y such that $\mathrm{Val}(x) \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(y)$, i. e. y is 'more true' under the interpretation Val than x. It is to be expected that $\mathrm{Val} \models y$ should also hold, which is exactly the property guaranteed by property 2a from Observation 1.3.1.1.

For more information concerning logics on this level compare [45] by S. Gottwald.

Level 3: Using Truth Value-Labelled Formulae. This level addresses the problem of *independently* assessing the truth of different formulae. This is effected by appending a *truth value* to every formula, i. e. a **truth value-labelled formula** is an ordered pair $[x, t]$ for $x \in \mathrm{Frm}$ and $t \in T$. $\mathrm{Val} \in \mathfrak{S}$ is considered to be a **model** for $[x, t]$ ($\mathrm{Val} \models [x, t]$) iff $t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x)$.

Obviously, this corresponds to an *individualisation* of the set D of *designated truth values* if D is the **principal filter** $\overline{t}$ (see (1.15)), i. e. every labelled formula $[x, t]$ has its own set of designated truth values $\overline{t}$.

Hence, level 2 can be reduced to level 3 iff there exists $t \in T$ such that $D = \overline{t}$ by choosing t as the label for every formula.

The resulting labelled logic corresponds to the one defined and studied by J. Pavelka [85–87]; compare section 5.2.1.

Level 4: Labelling formulae with Sets of Truth Values. The next level consists of assigning an individual set of **designated truth values** to every formula, i. e. a **set-labelled formula** is an ordered pair $[x, D]$ for $x \in \mathrm{Frm}$ and $D \subsetneqq T$. $\mathrm{Val} \in \mathfrak{S}$ is considered to be a **model** for $[x, D]$ ($\mathrm{Val} \models [x, D]$) iff $\mathrm{Val}(x) \in D$.

Level 2 can be reduced to level 4 by choosing the set D of designated truth values as the label for every formula. Level 3 can be reduced to level 4 by restricting labels to principal filters $\overline{t}$ for $t \in T$.

By the reasoning given in the description of level 2, in this dissertation only **filters** of $\mathfrak{T}$ will be employed as labels for set-labelled formulae. Labels of this kind have been used in [68, 72] for resolution-based automated reasoning. Compare also [47].

3.4.2 Expressing Graded Trust Assessment

As explained in chapter 1, the expression *graded trust assessment* is used in this dissertation to describe the modelling of (graded) *ill-knowledge* or *doubt* using formulae of a given many-valued logic. For carrying out this modelling, there has to be a possibility of *assessing* the trust in the information expressed by a formula, and the definition of the model relation should reflect this assessment.

A completely *novel* approach taken in this dissertation lies in assuming to be given a (complete) lattice $\mathfrak{D} = [D, \boxed{\sqcap}, \boxed{\sqcup}]$ of **degrees of validity** using which the graded trust assessment is achieved.

The lattice $\mathfrak{D}$ can be chosen freely and independently from $\mathfrak{T}$. In particular, it is possible to choose $\mathfrak{D} = \mathfrak{B}$, leading to two-valued validity and hence to a *logic of graded truth assessment* (see section 5.2), or to choose $\mathfrak{T} = \mathfrak{B}$, leading to two-valued truth and hence to a *logic of graded trust assessment* (see section 5.3).

That graded trust assessment and graded truth assessment should be distinguished is illustrated by the following example. Consider the statement

> "The door is locked."

Leaving time- and context-dependency of this statement aside, it makes no sense to express *uncertainty* about this statement using graded truth assessment. The door can only be locked or not locked, not anything in-between. In particular, no paradox arises from evaluating this statement in *two-valued logic*.

If the *knowledge* about this statement stems from second-hand information, however, it makes sense to model the resulting uncertainty using graded trust assessment, even if the underlying logic is two-valued. Arbitrarily casting this statement in *many-valued logic* and employing graded truth assessment for expressing uncertainty caused by a doubtful source is bound to lead to paradoxical and uninterpretable results. See section 5.4 for a demonstration that even the simplest examples of logics of graded truth assessment and logics of graded trust assessment show significant semantical differences.

This justifies considering a second *dimension of many-valuedness* corresponding to **trust**. In the following discussion of the different levels of logics for expressing graded trust assessment, it is explained how this second dimension is provided by the lattice $\mathfrak{D}$ of **degrees of validity**. See also section 5.5 for an illustration of how the two dimensions are employed for uncertainty modelling.

Next, consider the statement

> "The highway is jammed."

Here, "jammed" obviously is a *vague* predicate, so two scenarios are conceivable:

1. The information is completely certain, so *graded truth assessment* is sufficient for expressing the vague knowledge.

2. The source of the information is doubtful, so *graded truth assessment* and *graded trust assessment* have to be combined for expressing the vague and uncertain knowledge.

In this subsection, the expression of *graded trust assessment* is illustrated independently of the question whether the underlying logic is two-valued or many-valued, so no assumptions are made about the nature of $\mathfrak{T}$. A systematic study of the most general case is given in chapter 4; the special case of an underlying two-valued logic is studied in section 5.3; see also sections 5.4 and 5.5 for further illustrations of the similarities and differences between graded truth assessment and graded trust assessment.

That *validity degrees* are used for expressing graded trust assessment naturally leads to a *graded model relation* (graded with values from $\mathfrak{D}$), for two reasons. First, as remarked above, the model relation as the most basic semantic concept should reflect the graded trust assessment placed on a formula, and there is no other way for this, in particular when $\mathfrak{T}$ is two-valued. Secondly, in the absence of validity degrees, the validity of a formula is defined by applying a universal quantifier to the set of all models of the formula. In the case that validity is many-valued, it sees natural to define the *degree of validity* of a formula by applying a fuzzy universal quantifier (in this case, the infimum of the complete lattice $\mathfrak{D}$) to the fuzzy set of models (induced by the graded model relation) of the formula.

Consequently, the model relation is assumed to be graded by values from $\mathfrak{D}$ in the following. Next, several levels of definitions for this concept, with increasing expressive power, are presented.

Level 1′: Identifying truth values and validity degrees. This level requires $\mathfrak{T} = \mathfrak{D}$. $\mathrm{Val} \in \mathfrak{S}$ is said to be a **model** for a formula $x \in \mathrm{Frm}$ to degree $d \in D$ ($\mathrm{Val} \models_{d} x$) iff $\mathrm{Val}(x) = d$.

This level lacks several of the advantages of using separate lattices for truth values and validity degrees, respectively, underlined in the beginning of this subsection. For instance, the strict distinction between truth values and validity degrees is obscured and the range of all definable logics is radically reduced when no two different lattices may be chosen.

Defining the *validity* of a formula on this basis leads to the concept called **tautology index** in Definition 3.3.4.1; in this dissertation, it is classified as a *truth-theoretic* concept and not used for expressing graded trust assessment.

The above definition of graded model relation (or rather, the *semantic consequence operator* based on this definition) has been used seldomly in the literature on many-valued logics (compare [64]), but not in the context of graded trust assessment.

One more disadvantage of this approach is that there is no straightforward way of combining it with labelled formulae for *graded truth assessment*.

Level 2′: Using a Fuzzy Set of Designated Truth Values. For this level, a $\mathfrak{D}$-fuzzy set $\mathcal{D} \in D^T$ representing a **degree of designation** of truth values is assumed to be given. Then $\mathrm{Val} \in \mathfrak{S}$ is a **model** for $x \in \mathrm{Frm}$ to degree $d \in D$ ($\mathrm{Val} \models_{d} x$) iff $\mathcal{D}\big(\mathrm{Val}(x)\big) = d$.

Obviously, in the case $\mathfrak{T} = \mathfrak{D}$ level 1′ is reduced to level 2′ by defining $\mathcal{D}(t) =_{\mathrm{def}} t$.

This level allows ‘real’ *graded trust assessment* by expressing the trust in the source of the information represented by formulae through the definition of $\mathcal{D}$. If the information is *completely trusted*, then the *characteristic function* of an appropriate set D of designated truth values in the sense of level 2 should be used for $\mathcal{D}$. If the information is *completely mistrusted*, then $\mathcal{D}$ should be chosen to be always 1 because in the absence of information, no *constraint* should be placed on the truth values a formula may assume. Usually, the trust placed in the given information will be somewhere between *complete trust* and *complete mistrust*, so there will be some truth values which are completely excluded by the constraint placed by $\mathcal{D}$ (i. e. the value of $\mathcal{D}$ is 0), some truth values which lead to full validity even under the constraint placed by $\mathcal{D}$ (i. e. the value of $\mathcal{D}$ is 1), and further values which are not fully constrained, but also don’t lead to full validity (i. e. the value of $\mathcal{D}$ is strictly between 0 and 1).

How this indeed leads to an adequate modelling of graded trust assessment is illustrated further by remarks in the next section, at the beginning of section 4.1 and by the examples in chapter 5.

By an analogous reasoning as for level 2, for modelling *uncertainty* it is assumed that the class of possible definitions for $\mathcal{D}$ is essentially the class of all **$\mathfrak{D}$-fuzzy filters** of $\mathfrak{T}$ (in particular, if a formula gets more true, it has to be considered to be more valid).

The flaw of this level lies in the fact that $\mathcal{D}$ is applied to *all* formulae, making it impossible to assess the trust in individual formulae differently. This level, as the previous one, furthermore offers no natural way of combining graded trust assessment with labelled formulae for *graded truth assessment*.

To the author's knowledge, logics of this level have not been studied in the literature. They are not considered any further here, but this level forms a special case of level $4'$ (see below), so the considerations there can also be applied to this level.

Level $4'$: Labelling formulae with Fuzzy Sets of Truth Values. This level consists of assigning an individual fuzzy set to every formula, i. e. a **fuzzy set-labelled formula** is an ordered pair $[x, \mathcal{D}]$ for $x \in \mathrm{Frm}$ and $\mathcal{D} \in D^T$. $\mathrm{Val} \in \mathfrak{S}$ is considered to be a **model** for $[x, \mathcal{D}]$ to degree $d \in D$ ($\mathrm{Val} \models_{\overline{d}} [x, \mathcal{D}]$) iff $\mathcal{D}\big(\mathrm{Val}(x)\big) = d$.

This corresponds to an *individualisation* of the fuzzy set $\mathcal{D}$, i. e. every labelled formula $[x, \mathcal{D}]$ has its own graded assessment of the trust placed in the information represented by the formula. This way, the expression of graded trust assessment is effectively possible. Note further that every label indeed represents a **combination** of graded truth assessment and graded trust assessment.

Level $2'$ can be reduced to level $4'$ by choosing the same fuzzy set $\mathcal{D}$ as the label for every formula. Note that there is no level $3'$ because it makes no sense to create a special level just for principal fuzzy filters.

By the reasoning given for levels 2 and $2'$, in this dissertation only **$\mathfrak{D}$-fuzzy filters** of $\mathfrak{T}$ will be employed as labels for fuzzy set-labelled formulae. Labels of this kind have not been used in the literature before. The rest of this dissertation is devoted to their study.

Remark
Of course, other approaches for defining a **graded model relation** than those sketched above exist in the literature. Some of them are described in section 5.7.

One variant which is not comparable with the approaches described here is *similarity-based logic* [36]. Here, the underlying logic is two-valued, and a **fuzzy equivalence relation** (similarity relation) is declared on the set of all interpretations. The *degree of modelness* of an interpretation wrt. a *formula* is defined to be the supremum of the degrees of similarity of this interpretation with all models of the formula.

This variant is too far away from the approach of this dissertation to be considered here. It will not be mentioned any more in the following. •

In the following section, the syntax used here for logics of level $4'$ is formally defined. In chapter 4, the semantics (in particular, the model relation and semantic consequence relation) are formally defined and studied. Chapter 5 contains a discussion of special cases, including some logics on levels 3 and 4.

3.5 Labels and Labelled Formulae

As already stated in the introduction, classically (see J. Łukasiewicz and A. Tarski [75] or C. C. Chang [7]), concepts of **model** or **semantic consequence** are defined with respect to *sets* of logical formulae.

In chapter 1, it has been argued that this representation is insufficient for the modelling of *uncertain knowledge*. This is remedied here by developing means to *assess* the validity of a formula which is part of the knowledge base. This means that to every formula present in a knowledge base, a **label** is attached for the purpose of assessing its validity, thereby expressing the knowledge engineer's **uncertainty** about the validity of each individual formula. This expression of uncertainty may take several forms:

1. It may be known exactly that a formula does not need to take always the highest **truth value** to be valid. If a **range** of truth values can be given for which the formula can still be considered valid, then this range should be expressible in the form of a **label**. This is the case of **graded truth assessment** because the formula can take 'suboptimal' truth values and still be valid.

2. One may be sure that a formula has to be **absolutely true** to be absolutely valid, but be uncertain about the *reliability* of the source of this formula. If conclusions are drawn from this formula, it should be ascertained that these conclusions are also not completely reliable. This is the case of **graded trust**, and it should be possible to express this type of uncertainty in a **label** in the form of a **degree of trust**.

3. Combinations of the above should be possible, in particular a label should be able to express different degrees of trust for different truth values a formula can take.

In applications, this could mean that a language of 'names' for labels is provided (like TRUE, FALSE, AMBIGUOUS) and 'modifiers' (like VERY, FAIRLY) in the sense that an **assessment** of its validity is attached to each formula in the knowledge base. This leads to formulae named "Type IV" in L. A. ZADEH's paper [105] (in particular, truth and possibility qualifications). An example from [105]:

"Abe is young is NOT VERY TRUE."

Here, "Abe is young" could be translated into a formula of the underlying logic (as defined in section 3.1) while "is NOT VERY TRUE" could be translated into the **label** for the formula (see for instance ZADEH [105, Sec. 6] or BALDWIN [2]).

In the following, an 'algebraic' definition for the concept of **label** is given which is based on the results of chapter 2 and which shall be justified by the definition of the **model** and **semantic consequence** relations in chapter 4.

In addition to the truth value lattice $\mathfrak{T}$, let a lattice $\mathfrak{D}$ of **degrees of trust** (or validity) be given. The lattices $\mathfrak{T}$ and $\mathfrak{D}$ need not be equal; in fact, the distinction between **logics of graded truth assessment** and **logics of graded trust assessment** (see chapter 5 for some examples), which is possible precisely because $\mathfrak{T}$ and $\mathfrak{D}$ *can* be chosen to be different, is one of the most interesting features of the logics thus defined.

The degrees of validity given by the lattice $\mathfrak{D}$ are used to measure 'how valid' a formula can be assumed to be, given that its truth value (from $\mathfrak{T}$) is known. The correspondence between the truth values and the degrees of validity is given by the **label** associated with a formula. To this end, a lattice $\mathfrak{L}$ of **labels** is required, which is an **expansion** (see Definition 2.3.1) of $\mathfrak{T}$ by $\mathfrak{D}$. Hence, every label ℓ from $\mathfrak{L}$ is associated with a $\mathfrak{D}$-fuzzy filter of $\mathfrak{T}$, i. e. a mapping from $\mathfrak{T}$ into $\mathfrak{D}$. Given a formula $x \in \mathrm{Frm}$, a valuation $\mathrm{Val} \in \mathfrak{S}$ and the truth value $\mathrm{Val}(x)$ of x under the valuation Val, the degree to which x, labelled by ℓ, is valid under Val, is the value from $\mathfrak{D}$ associated with $\mathrm{Val}(x)$ by ℓ. This is basically the definition of the model relation of this logical system. Before a mathematical definition and a deeper study of this model relation is given in chapter 4, the concept of **labelled formula** is defined.

Let a fixed set Frm of logical formulae and a fixed complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$ with induced partial order $\sqsubseteq$ be given.[1] For the definition of a **labelled formula**, it is not necessary to postulate anything further about the structure of $\mathfrak{L}$. As a matter of fact, different alternatives for characterising $\mathfrak{L}$ (algebraically by fixing $\mathfrak{L}$ to be an expansion or logically by axioms about the model relation) are discussed in section 4.1.

The elements of L shall be called **labels**. In the following, $\mathfrak{L}$-fuzzy sets of formulae are considered.

Definition 3.5.1 (Labelled formula)
*An $\mathfrak{L}$-fuzzy set $\mathfrak{x} \in L^{\mathrm{Frm}}$ is said to be an **$\mathfrak{L}$-labelled formula** if and only if $\mathfrak{x} = \emptyset$ or there exists $x \in \mathrm{Frm}$ such that $\operatorname{supp} \mathfrak{x} = \{x\}$. If $\operatorname{supp} \mathfrak{x} = \{x\}$, then $\mathfrak{x}$ is identified with the ordered pair $\big[x, \mathfrak{x}(x)\big]$.*

To avoid special cases, given $x \in \mathrm{Frm}$ the notation $[x, 0]$ is allowed for the $\mathfrak{L}$-labelled formula $\emptyset$, although it is no longer possible to unambiguously identify $\mathfrak{x}$ with the ordered pair $[x, 0]$.

For a fixed $\mathfrak{L}$, the set of all $\mathfrak{L}$-labelled formulae is denoted by LFrm.

Remarks

1. The idea of using fuzzy sets of formulae goes back to J. Pavelka [85]. Alternatively, it would be possible to start from the definition of *labelled formula* (as an ordered pair $[x, \ell]$) and then use sets of labelled formulae. There are subtle differences between both approaches which are inconsequential for considerations concerned only with *semantics*, but can lead to problems when *syntactic derivations* are concerned. This issue is discussed further in the sequel.

2. The crucial idea in **this** definition of labelled formulae is the **separation** of truth value and label structures. This gives additional flexibility for knowledge modelling; on the other hand, strong connections between truth values and labels are established by Definition 2.3.1 and Conclusion 2.3.1 (in the case of an algebraic definition of labels) or by the logical axioms which are given in Definition 4.1.3.

   Some possible combinations of truth value and label structures are illustrated in Examples 2.3.1, Proposition 2.3.2, Corollary 2.3.3, Proposition 2.3.4, Observation 2.3.5. In chapter 5, an extensive overview of logics representable as fuzzy filter-based logics is given.

3. The idea of using (practically) $\mathfrak{D}$-fuzzy sets on $\mathfrak{T}$ as labels for logical formulae and using this as the basis for logics of graded truth assessment *and* graded trust goes back to the **fuzzy truth values** of L. A. Zadeh [106]. Such logics have been applied in **fuzzy expert systems** and also studied from a theoretical point of view, under various names (see [18] for an overview), but so far, a complete theory of such logics from the perspective of *mathematical logic* seems to be lacking.

   Furthermore, the name **fuzzy truth value** invites misunderstandings in the sense that somehow the truth value $\mathrm{Val}(x)$ of a formula x under a valuation Val is being ‘fuzzified’. To avoid such misunderstandings, a careful distinction shall be made in the sequel between the following concepts:

   - The **truth value** $\mathrm{Val}(x) \in T$ of a formula x under a valuation Val.

[1] Note that by being an **expansion** of $\mathfrak{T}$ by $\mathfrak{D}$, $\mathfrak{L}$ is essentially isomorphic with the *dual* of a lattice of fuzzy sets. $\sqcup$ corresponds to the fuzzy set *intersection* $\cap$ and $\sqcap$ corresponds to the *fuzzy filter join* $\uplus$. The *induced partial order* $\sqsubseteq$ of $\mathfrak{L}$ corresponds to the *inversion* of the fuzzy subset relation.

- The **label** $\ell \in L$ associated with a formula by its membership degree in a fuzzy set $\mathcal{X} \in L^{\mathrm{Frm}}$, which is by the fact that $\mathfrak{T}$ is expanded to $\mathfrak{L}$ by $\mathfrak{D}$ essentially a mapping from T into D.

  See the remark following Definition 4.1.1 for a "logical" interpretation of labels and their ordering.
- The **degree of validity** of a labelled formula $[x, \ell]$ under a valuation Val, which coincides with the degree to which Val is a *model* of $[x, \ell]$ (see chapter 4).

By strictly distinguishing between the lattices $\mathfrak{T}$, $\mathfrak{D}$ and $\mathfrak{L}$ and the different meanings of their elements, some common misconceptions can be avoided which seem to pervade discussions about logics of graded truth assessment and logics of graded trust assessment (for instance, the mystical *truth-functionality*).

By introducing the isomorphism ι explicitly into the definition of *expansion*, it is not necessary to use fuzzy sets of truth values explicitly as labels, unless the great expressive power provided by choosing neither $\mathfrak{T}$ nor $\mathfrak{D}$ to be the two-valued lattice $\mathfrak{B}$ is needed. This way, it can easily be verified that most commonly used logics of graded truth assessment and logics of graded trust assessment are indeed special cases of this definition. See Examples 2.3.1, Proposition 2.3.2, Corollary 2.3.3, Proposition 2.3.4, Observation 2.3.5 for a first illustration and chapter 5 for an extensive survey. •

4 Models and Semantic Consequence

In chapters 2 and 3, the foundation has been laid on which a very general class of logical systems based on *labelled* logical formulae can be defined.

Let a fixed set Frm of logical formulae and a semantics $\mathfrak{S}$ of valuation functions for Frm be given, based on a complete lattice $\mathfrak{T} = [T, \sqcap_T, \sqcup_T]$ with induced partial order $\sqsubseteq_T$. Furthermore, let fixed complete lattices $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ with induced partial order $\sqsubseteq_L$ and $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$ with induced partial order $\sqsubseteq_D$ be given.

4.1 The Model Relation

First, a model relation for labelled formulae is defined in a purely algebraic fashion by assuming $\mathfrak{L}$ to be an *expansion* of $\mathfrak{T}$ by $\mathfrak{D}$.

Definition 4.1.1 (Model relation for labelled formulae)
*Assume that $\mathfrak{L}$ is an **expansion** of $\mathfrak{T}$ by $\mathfrak{D}$ (see Definition 2.3.1). Furthermore, fix the isomorphism ι by means of which $\mathfrak{T}$ is expanded to $\mathfrak{L}$ by $\mathfrak{D}$.*

*A ternary **model relation** $\models$ is defined as follows:*

*Given a valuation $\mathrm{Val} \in \mathfrak{S}$, an $\mathfrak{L}$-labelled formula $[x, \ell] \in \mathrm{LFrm}$ and a validity degree $d \in D$, Val **is a model for** $[x, \ell]$ **to the degree** d,*

$$\mathrm{Val} \models_{d} [x, \ell] =_{\mathrm{def}} d = \iota(\ell)(\mathrm{Val}(x)). \tag{4.1}$$

Remark
Considering the definition of the *model relation* based on the algebraic characterisation of *fuzzy filters* on the truth value lattice, a first explanation of the *meaning* of the different lattice structures from a logical point of view can be given.

1. The *truth value lattice* $\mathfrak{T}$ provides a set of truth values for logical formulae. The induced partial order $\sqsubseteq_T$ of this lattice is to be interpreted as meaning *less true than or equally true*, i. e. if two formulae $x, y \in \mathrm{Frm}$ assume truth values $s =_{\mathrm{def}} \mathrm{Val}(x)$ and $t =_{\mathrm{def}} \mathrm{Val}(y)$ under some valuation $\mathrm{Val} \in \mathfrak{S}$, and furthermore,

   $$s \sqsubseteq_T t, \quad t \neq s,$$

   then y can be assumed to be *more true* than x under Val.

   In this context, the truth value 1 means *completely true* and 0 means *completely false*. Observe that if the logic contains a 'decent' unary negation operator $\neg$, then $\mathrm{Val}(x) = 0$ will mean $\mathrm{Val}(\neg x) = 1$.

2. The lattice $\mathfrak{D}$ of *degrees of validity* makes it possible to specify in Definition 4.1.1 a *graded* model relation between valuation functions and *labelled formulae* which gives the degree of validity of a labelled formula under a given valuation.

The induced partial order $\sqsubseteq_{\mathfrak{D}}$ of the lattice $\mathfrak{D}$ is to be interpreted as meaning *less valid than or equally valid*, i. e. if for two labelled formulae $\mathfrak{x}, \mathfrak{y} \in \mathrm{LFrm}$, it holds that $\mathrm{Val} \models_{c} \mathfrak{x}$ and $\mathrm{Val} \models_{d} \mathfrak{y}$, for some valuation $\mathrm{Val} \in \mathfrak{S}$, and furthermore,

$$c \sqsubseteq_{\mathfrak{D}} d, \quad c \neq d,$$

then $\mathfrak{y}$ can be assumed to be *more valid* than $\mathfrak{x}$ under Val.

In this context, the validity degree 1 means *completely valid* and 0 means *completely invalid*. Observe that no logical connectives which operate on labelled formulae have been defined, so given $\mathfrak{x} \in \mathrm{LFrm}$, there is (yet) no way of deriving from $\mathfrak{x}$ some other labelled formula $\mathfrak{y}$ such that $\mathrm{Val} \models_{1} \mathfrak{x}$ if and only if $\mathrm{Val} \models_{0} \mathfrak{y}$.

3. A *label* ℓ from the lattice $\mathfrak{L}$ corresponds to a $\mathfrak{D}$-fuzzy set $\iota(\ell)$ on $\mathfrak{T}$ by the isomorphism ι.

   Thus from the logical point of view, a label associates with every truth value a degree of validity. The interpretation of this fact is as follows:

   - If a formula x is *completely true* under a valuation Val, then $[x, \ell]$ is *completely valid* under Val for every label ℓ (which is assured by the axioms of *fuzzy filters*, see Definition 2.1.1).
   - Whenever x is *not completely true* under Val, the *trust* in the validity of the formula expressed by the label has to be considered. When the formula is trusted to be always necessarily completely true, then the labelled formula has to be considered *invalid* under Val. Otherwise, a certain degree of validity can be attained because the formula (as part of a knowledge base) is not completely trusted.

   In fact this is a *relaxation* of the laws of classical logic: In classical many-valued logics like BOOLEan or ŁUKASIEWICZ's logic, a formula is said to be valid under an interpretation **if and only if** its truth value is exactly 1; otherwise it is considered *completely invalid*.

   If the knowledge about the validity of formulae which are to be put into a knowledge base is *uncertain*, this is too hard a constraint: For a valuation to be a model of the knowledge base, it would be necessary for every formula in the knowledge base to be *completely true*, in contradiction with the uncertainty. So the label attached to a formula can be interpreted as the expression of a *soft constraint* on the validity of this formula.

   The induced partial order $\sqsubseteq_{\mathfrak{L}}$ of the label lattice is an order of *strength*, i. e. if for two labels ℓ, ℓ',

$$\ell' \sqsubseteq_{\mathfrak{L}} \ell, \quad \ell' \neq \ell,$$

   then ℓ can be assumed to be a *stronger* constraint on the validity of a logical formula than ℓ'. This is consistent with the fact that $\sqsubseteq_{\mathfrak{L}}$ corresponds via ι to the *inverse* of the partial order $\subseteq$ of fuzzy sets: If a fuzzy set associating validity degrees with truth values gets smaller, it expresses a *stronger* constraint.

   Thus the label 1, corresponding to the function $\mathbb{0}$ (see (2.11)), is the *strongest* constraint, expressing *complete certainty* about the complete truth of a formula and thus equivalent with the classical concept of validity, while the label 0, corresponding to the function $\mathbb{1}$ (see (2.12)), is the *weakest* constraint (in fact, no constraint at all; every truth value, even 0 is considered equally completely valid), expressing complete uncertainty. •

Proposition 4.1.1 (Properties of the graded model relation)

1. *For **every** valuation* $\mathrm{Val} \in \mathfrak{S}$ *and **every*** $\mathfrak{L}$*-labelled formula* $[x, \ell] \in \mathrm{LFrm}$*, there exists a **unique*** $d \in D$ *such that* $\mathrm{Val} \models_{d} [x, \ell]$.

   Thus $\models$ *can be regarded as a mapping from* $\mathfrak{S} \times \mathrm{LFrm}$ *to* D.

2. *For all* $\mathrm{Val} \in \mathfrak{S}$ *and* $x \in \mathrm{Frm}$,

$$\mathrm{Val} \models_{1} [x, 0]\,. \tag{4.2}$$

3. *For all* $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, *and* $\ell \in L$,

$$\textit{if } \mathrm{Val}(x) = 1\,,\ \textit{then } \mathrm{Val} \models_{1} [x, \ell]\,. \tag{4.3}$$

4. *For all* $\mathrm{Val}_1, \mathrm{Val}_2, \mathrm{Val}_3 \in \mathfrak{S}$, $x, y, z \in \mathrm{Frm}$, $\ell \in L$, *and* $d_1, d_2, d_3 \in D$ *such that*

$$\mathrm{Val}_1 \models_{d_1} [x, \ell] \ \textit{and}\ \mathrm{Val}_2 \models_{d_2} [y, \ell] \ \textit{and}\ \mathrm{Val}_3 \models_{d_3} [z, \ell]\,,$$

   the following holds:

$$\textit{if } \mathrm{Val}_1(x) \sqcap_{\mathrm{T}} \mathrm{Val}_2(y) = \mathrm{Val}_3(z)\,,\ \textit{then } d_1 \sqcap_{\mathrm{D}} d_2 = d_3. \tag{4.4}$$

5. *For all* $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, $\ell, \ell' \in L$, *and* $c, d, e \in D$,

$$\textit{if } \mathrm{Val} \models_{c} \left[x, \ell'\right] \ \textit{and}\ \mathrm{Val} \models_{d} [x, \ell]\,,\ \textit{then } \mathrm{Val} \models_{c \sqcap_{\mathrm{D}} d} \left[x, \ell' \sqcup_{\mathrm{L}} \ell\right]. \tag{4.5}$$

6. *Let* $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$ *and* $M \subseteqq L$*. Furthermore, let*

$$C =_{\mathrm{def}} \left\{ d \,\middle|\, d \in D \textit{ and there exists } \ell \in M \textit{ such that } \mathrm{Val} \models_{d} [x, \ell] \right\}.$$

   Then

$$\mathrm{Val} \models_{\bigsqcup_{\mathrm{D}} C} \left[x, \bigsqcup_{\mathrm{L}} M\right]. \tag{4.6}$$

7. *For all* $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, $\ell, \ell' \in L$, *and* $d \in D$,

$$\textit{if } \mathrm{Val} \models_{d} \left[x, \ell' \sqcap_{\mathrm{L}} \ell\right], \tag{4.7}$$

$$\textit{then } \bigsqcup_{\mathrm{D}} \left\{ d_1 \sqcap_{\mathrm{D}} d_2 \,\middle|\, \begin{array}{l} \textit{There exist } \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm},\ \textit{and } d_1, d_2 \in D \\ \textit{such that } \mathrm{Val}_1 \models_{d_1} \left[y, \ell'\right] \ \textit{and}\ \mathrm{Val}_2 \models_{d_2} [z, \ell] \\ \textit{and } \mathrm{Val}_1(y) \sqcap_{\mathrm{T}} \mathrm{Val}_2(z) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \end{array} \right\} \sqsubseteq_{\mathrm{D}} d.$$

8. *In each of the following cases,*

   8.1. $\mathfrak{T}$ *is a **chain** or*

   8.2. $\mathfrak{D}$ *is **completely distributive** wrt.* $\bigsqcup_{\mathrm{D}}$,

   the following holds.

   For all $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, $\ell, \ell' \in L$, *and* $d \in D$,

$$\textit{if } \mathrm{Val} \models_{d} \left[x, \ell' \sqcap_{\mathrm{L}} \ell\right],$$

$$\textit{then } d = \bigsqcup_{\mathrm{D}} \left\{ d_1 \sqcap_{\mathrm{D}} d_2 \,\middle|\, \begin{array}{l} \textit{There exist } \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm},\ \textit{and } d_1, d_2 \in D \\ \textit{such that } \mathrm{Val}_1 \models_{d_1} \left[y, \ell'\right] \ \textit{and}\ \mathrm{Val}_2 \models_{d_2} [z, \ell] \\ \textit{and } \mathrm{Val}_1(y) \sqcap_{\mathrm{T}} \mathrm{Val}_2(z) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \end{array} \right\}.$$

   (For the proof of this item, assumption (3.2) is needed.)

9. $\models$ *has the following* ***monotonicity*** *property, for* $\mathrm{Val}, \mathrm{Val}' \in \mathfrak{S}$, $\ell, \ell' \in L$, $c, d \in D$, *and* $x, y \in \mathrm{Frm}$*:*

$$\text{If } \mathrm{Val}(x) \sqsubseteq_{\mathrm{T}} \mathrm{Val}'(y) \ \text{ and } \ \ell' \sqsubseteq_{\mathrm{L}} \ell \ \text{ and } \ \mathrm{Val} \models_{d} [x, \ell] \ \text{ and } \ \mathrm{Val}' \models_{c} \left[y, \ell'\right], \text{ then } d \sqsubseteq_{\mathrm{D}} c.$$

Proof

ad 1. Follows immediately from the definition of $\models$.

ad 2. It is sufficient to prove

$$\iota(0)(\mathrm{Val}(x)) = 1$$

for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$.

Being an isomorphism, ι maps the zero element 0 of $\mathfrak{L}$ to the zero element of a sublattice $\mathfrak{L}' = \left[L', \cup, \cap\right]$ of the dual lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \cup, \cap\right]$ of the complete lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \cap, \cup\right]$ (see Definition 2.3.1 and Theorem 2.2.1).

By Observation 2.2.5, the zero element $\mathbb{1}$ of $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \cup, \cap\right]$ is a principal fuzzy filter, contained in $\mathfrak{L}'$ by Definition 2.3.1, hence $\iota(0) = \mathbb{1}$. Thus,

$$\iota(0)(\mathrm{Val}(x)) = \mathbb{1}(\mathrm{Val}(x)) = 1$$

by definition (2.12).

ad 3. It is sufficient to prove that for every $\ell \in L$,

$$\iota(\ell)(\mathrm{Val}(x)) = 1$$

if $\mathrm{Val}(x) = 1$.

By definition, ι maps ℓ to a $\mathfrak{D}$-fuzzy filter $\mathcal{F}$ of $\mathfrak{T}$. By item 3 of Definition 2.1.1,

$$\iota(\ell)(\mathrm{Val}(x)) = \mathcal{F}(1) = 1.$$

ad 4. Let $\mathrm{Val}_1, \mathrm{Val}_2, \mathrm{Val}_3, x, y, z, \ell, d_1, d_2, d_3$ be given as assumed. It is sufficient to prove that

$$\text{if } \mathrm{Val}_1(x) \sqcap_{\mathrm{T}} \mathrm{Val}_2(y) = \mathrm{Val}_3(z)\,, \text{ then } \ \iota(\ell)(\mathrm{Val}_1(x)) \sqcap_{\mathrm{D}} \iota(\ell)(\mathrm{Val}_2(y)) = \iota(\ell)(\mathrm{Val}_3(z)).$$

This follows trivially from the fact that $\iota(\ell)$ is a $\mathfrak{D}$-fuzzy filter of $\mathfrak{T}$, and Corollary 2.1.7.

ad 5. Let $\mathrm{Val}, x, \ell, \ell', c, d$ be given as assumed and let

$$\mathrm{Val} \models_{c} \left[x, \ell'\right] \ \text{ and } \ \mathrm{Val} \models_{d} [x, \ell]\,.$$

It is sufficient to prove that

$$\iota(\ell' \sqcup_{\mathrm{L}} \ell)(\mathrm{Val}(x)) = \iota(\ell')(\mathrm{Val}(x)) \sqcap_{\mathrm{D}} \iota(\ell)(\mathrm{Val}(x)). \tag{4.8}$$

Let $\mathfrak{L}' = \left[L', \cup, \cap\right]$ be the sublattice of the dual lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \cup, \cap\right]$ of the complete lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \cap, \cup\right]$ such that ι is an isomorphism between $\mathfrak{L}$ and $\mathfrak{L}'$ (see Definition 2.3.1 and Theorem 2.2.1).

By the fact that ι is an isomorphism,

$$\iota(\ell' \sqcup_{\mathrm{L}} \ell) = \iota(\ell') \cap \iota(\ell),$$

thus (4.8) follows trivially from the definition (1.17) of $\cap$.

ad 6. Is proved exactly as item 5, taking into account that $\mathfrak{L}$ is a complete lattice and thus the isomorphism ι admits arbitrary joins.

ad 7. Let $\mathrm{Val}, x, \ell, \ell', d$ be given such that

$$\mathrm{Val} \models_{d} \left[x, \ell' \sqcup_L \ell\right]. \tag{4.9}$$

Let $\mathfrak{L}' = \left[L', \uplus, \cap\right]$ be the sublattice of the dual lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \uplus, \cap\right]$ of the complete lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \cap, \uplus\right]$ such that ι is an isomorphism between $\mathfrak{L}$ and $\mathfrak{L}'$ (see Definition 2.3.1 and Theorem 2.2.1).

(4.9) means that

$$\iota(\ell' \sqcup_L \ell)(\mathrm{Val}(x)) = d.$$

By the fact that ι is an isomorphism,

$$\iota(\ell' \sqcup_L \ell) = \iota(\ell') \uplus \iota(\ell).$$

Define

$$d' =_{\mathrm{def}} \bigsqcup_D \left\{ \iota(\ell')(\mathrm{Val}_1(y)) \sqcap_D \iota(\ell)(\mathrm{Val}_2(z)) \,\middle|\, \begin{array}{l} \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm}, \\ \text{and } \mathrm{Val}_1(y) \sqcap_T \mathrm{Val}_2(z) \sqsubseteq_T \mathrm{Val}(x) \end{array} \right\}. \tag{4.10}$$

It is sufficient to prove that

$$\begin{aligned} d' &\sqsubseteq_D d \\ &= (\iota(\ell') \uplus \iota(\ell))(\mathrm{Val}(x)). \end{aligned} \tag{4.11}$$

Obviously,

$$\begin{aligned} &\left\{ \iota(\ell')(\mathrm{Val}_1(y)) \sqcap_D \iota(\ell)(\mathrm{Val}_2(z)) \,\middle|\, \begin{array}{l} \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm}, \\ \text{and } \mathrm{Val}_1(y) \sqcap_T \mathrm{Val}_2(z) \sqsubseteq_T \mathrm{Val}(x) \end{array} \right\} \\ &\qquad \subseteq \left\{ \iota(\ell')(s) \sqcap_D \iota(\ell)(t) \,\middle|\, s, t \in T \text{ and } s \sqcap_T t \sqsubseteq_T \mathrm{Val}(x) \right\}. \end{aligned}$$

On page 22 (proof of Theorem 2.2.2, ad (2.9), item 2), it has been proved that in general,

$$\bigsqcup_D \left\{ \iota(\ell')(s) \sqcap_D \iota(\ell)(t) \,\middle|\, s, t \in T \text{ and } s \sqcap_T t \sqsubseteq_T \mathrm{Val}(x) \right\} \sqsubseteq_D d,$$

hence

$$\begin{aligned} d' &= \bigsqcup_D \left\{ \iota(\ell')(\mathrm{Val}_1(y)) \sqcap_D \iota(\ell)(\mathrm{Val}_2(z)) \,\middle|\, \begin{array}{l} \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm}, \\ \text{and } \mathrm{Val}_1(y) \sqcap_T \mathrm{Val}_2(z) \sqsubseteq_T \mathrm{Val}(x) \end{array} \right\} \\ &\sqsubseteq_D \bigsqcup_D \left\{ \iota(\ell')(s) \sqcap_D \iota(\ell)(t) \,\middle|\, s, t \in T \text{ and } s \sqcap_T t \sqsubseteq_T \mathrm{Val}(x) \right\} \\ &\sqsubseteq_D d. \end{aligned}$$

ad 8. Let everything be given and defined as in item 7.

Bearing in mind this item, it is sufficient to prove that

$$d \sqsubseteq_D d'. \tag{4.12}$$

Distinguish two cases:

Case 1. Assumption 8.1 holds.

By Observation 2.2.4,

$$\iota(\ell') \mathbin{\dot\cup} \iota(\ell) = \iota(\ell') \cup \iota(\ell),$$

thus

$$d = (\iota(\ell') \cup \iota(\ell))(\mathrm{Val}(x)) = \iota(\ell')(\mathrm{Val}(x)) \sqcup_{\mathrm{D}} \iota(\ell)(\mathrm{Val}(x)).$$

To establish the claim of this item, it is thus sufficient to prove

$$\begin{aligned}(4.13)\quad & \iota(\ell')(\mathrm{Val}(x)) \\ & \in \left\{ \iota(\ell')(\mathrm{Val}_1(y)) \sqcap_{\mathrm{D}} \iota(\ell)(\mathrm{Val}_2(z)) \;\middle|\; \begin{array}{l} \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm}, \\ \text{and } \mathrm{Val}_1(y) \sqcap_{\mathrm{T}} \mathrm{Val}_2(z) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \end{array} \right\}\end{aligned}$$

and

$$\begin{aligned}(4.14)\quad & \iota(\ell)(\mathrm{Val}(x)) \\ & \in \left\{ \iota(\ell')(\mathrm{Val}_1(y)) \sqcap_{\mathrm{D}} \iota(\ell)(\mathrm{Val}_2(z)) \;\middle|\; \begin{array}{l} \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm}, \\ \text{and } \mathrm{Val}_1(y) \sqcap_{\mathrm{T}} \mathrm{Val}_2(z) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \end{array} \right\}.\end{aligned}$$

By assumption (3.2), there are $\mathrm{Val}_2 \in \mathfrak{S}$ and $z \in \mathrm{Frm}$ such that $\mathrm{Val}_2(z) = 1$. Then

$$\mathrm{Val}(x) \sqcap_{\mathrm{T}} \mathrm{Val}_2(z) = \mathrm{Val}(x) \sqcap_{\mathrm{T}} 1 = \mathrm{Val}(x) \tag{4.15}$$

and, because $\iota(\ell)$ is a $\mathfrak{D}$-fuzzy filter of $\mathfrak{T}$,

$$\begin{aligned}\iota(\ell)(\mathrm{Val}_2(z)) &= \iota(\ell)(1) \\ &= 1,\end{aligned}$$

hence because of (4.15),

$$\begin{aligned} & \iota(\ell')(\mathrm{Val}(x)) \\ & \quad = \iota(\ell')(\mathrm{Val}(x)) \sqcap_{\mathrm{D}} 1 \\ & \quad = \iota(\ell')(\mathrm{Val}(x)) \sqcap_{\mathrm{D}} \iota(\ell)(\mathrm{Val}_2(z)) \\ & \quad \in \left\{ \iota(\ell')(\mathrm{Val}_1(y)) \sqcap_{\mathrm{D}} \iota(\ell)(\mathrm{Val}_2(z)) \;\middle|\; \begin{array}{l} \mathrm{Val}_1, \mathrm{Val}_2 \in \mathfrak{S},\ y, z \in \mathrm{Frm}, \\ \text{and } \mathrm{Val}_1(y) \sqcap_{\mathrm{T}} \mathrm{Val}_2(z) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \end{array} \right\}.\end{aligned}$$

(4.14) is proved analogously.

Case 2. Assumption 8.2 holds.

By assumption (3.2), (4.12) is equivalent with

$$\bigsqcup_{\mathrm{D}} \left\{ \iota(\ell')(s) \sqcap_{\mathrm{D}} \iota(\ell)(t) \,\middle|\, s, t \in T \text{ and } s \sqcap_{\mathrm{T}} t \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \right\} \sqsubseteq_{\mathrm{D}} d'.$$

But under this interpretation, (4.11) has been proved in Theorem 2.2.2, equation (2.9).

ad 9. It has to be proved that under the given conditions,

$$\iota(\ell)(\mathrm{Val}(x)) \sqsubseteq_{\mathfrak{D}} \iota\left(\ell'\right)\left(\mathrm{Val}'(y)\right).$$

Let $\mathfrak{L}' = \left[L', \Cup, \Cap\right]$ be the sublattice of the dual lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \Cup, \Cap\right]$ of the complete lattice $\left[\mathfrak{D}\text{-Fl}\left(\mathfrak{T}\right), \Cap, \Cup\right]$ such that ι is an isomorphism between $\mathfrak{L}$ and $\mathfrak{L}'$ (see Definition 2.3.1 and Theorem 2.2.1).

By the fact that the induced partial order of $\mathfrak{L}'$ is the usual superset relation $\supseteqq$ of fuzzy sets, by the fact that ι is an isomorphism and by $\ell' \sqsubseteq_{\mathfrak{L}} \ell$,

$$\iota(\ell) \subseteqq \iota(\ell'). \tag{4.16}$$

Now, because $\iota(\ell)$ and $\iota(\ell')$ are both $\mathfrak{D}$-fuzzy filters of $\mathfrak{T}$, condition 2a of Proposition 2.1.6 can be applied to the fact that $\mathrm{Val}(x) \sqsubseteq_{\mathfrak{T}} \mathrm{Val}'(y)$, yielding

$$\iota(\ell)(\mathrm{Val}(x)) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(\mathrm{Val}'(y)). \tag{4.17}$$

Summing up,

$$\begin{aligned}
d &= \iota(\ell)(\mathrm{Val}(x)) && \text{(by definition)}\\
&\sqsubseteq_{\mathfrak{D}} \iota(\ell)(\mathrm{Val}'(y)) && \text{(by (4.17))}\\
&\sqsubseteq_{\mathfrak{D}} \iota(\ell')(\mathrm{Val}'(y)) && \text{(by (4.16))}\\
&= c. && \square
\end{aligned}$$

From Proposition 4.1.1, one can already get the impression that Definition 4.1.1 indeed characterises a large class of 'sensible' labelled logics. The remainder of this dissertation is devoted to the study of the resulting logics. In the remainder of this chapter, these logics are investigated from an abstract point of view, defining logical concepts like **semantic consequence**, **semantic equivalence** etc, and studying their interrelationship. In the subsequent chapter, the expressive power of the definitions made here is illustrated, giving examples of concrete logics from this class.

To facilitate discussions about the labelled logics studied here, a formal definition of the class of logics generated by Definition 4.1.1 is given.

Definition 4.1.2 (Fuzzy filter-based logic)
A tuple $\Lambda =_{\mathrm{def}} [\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \iota]$ *shall be called a* ***fuzzy filter-based logic***

- *with* ***logical language*** Frm,
- *with* ***truth value lattice*** $\mathfrak{T}$,
- *with* ***semantics*** $\mathfrak{S}$,
- *with* ***validity degree lattice*** $\mathfrak{D}$,
- *and with* ***label lattice*** $\mathfrak{L}$,

$=_{\mathrm{def}}$
1. Frm *is a nonempty set,*
2. $\mathfrak{T} = [T, \sqcap_{\mathfrak{T}}, \sqcup_{\mathfrak{T}}]$, $\mathfrak{D} = [D, \sqcap_{\mathfrak{D}}, \sqcup_{\mathfrak{D}}]$, $\mathfrak{L} = [L, \sqcap_{\mathfrak{L}}, \sqcup_{\mathfrak{L}}]$ *are complete lattices with at least two elements each, with induced partial orders* $\sqsubseteq_{\mathfrak{T}}$, $\sqsubseteq_{\mathfrak{D}}$, $\sqsubseteq_{\mathfrak{L}}$, *respectively,*
3. $\mathfrak{S} \subseteqq T^{\mathrm{Frm}}$,

4. $\iota : L \to \mathfrak{D}\text{-Fl}\,(\mathfrak{T})$,
5. $\mathfrak{L}$ *is an expansion of* $\mathfrak{T}$ *by* $\mathfrak{D}$, *by means of* ι.

The relation $\models$ *defined in* Definition 4.1.1 *shall be called the* ***model relation induced by*** Λ.

Observe that claim 5 of Definition 4.1.2 is not really *logically* justified. It is demonstrated in Proposition 4.1.1 that the 'logical' consequences of this claim are reasonable, but it would be more satisfying to replace claim 5 by assumptions on the 'logical' properties of the model relation, thus avoiding the detour of employing fuzzy filters and the mapping ι. Indeed, it is possible to characterise a large subclass of all fuzzy filter-based logics by those properties which were presented in Proposition 4.1.1.

Definition 4.1.3 (Logic of graded truth and graded trust assessment)
(Excessive use of assumption (3.2) *is made in the following definition, especially of the valuation* Val_t *and the formula* x_t *with* $\mathrm{Val}_t(x_t) = t$, *for* $t \in T$.*)*

A tuple $\Lambda =_{\text{def}} \left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\right]$ *shall be called a* ***logic of graded truth and graded trust assessment***

- *with* ***logical language*** Frm,
- *with* ***truth value lattice*** $\mathfrak{T}$,
- *with* ***semantics*** $\mathfrak{S}$,
- *with* ***validity degree lattice*** $\mathfrak{D}$,
- *with* ***label lattice*** $\mathfrak{L}$,
- *and with* ***model relation*** $\models$,

$=_{\text{def}}$

1. Frm *is a nonempty set,*
2. $\mathfrak{T} = [T, \sqcap_T, \sqcup_T]$, $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$, $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ *are complete lattices with at least two elements each, with induced partial orders* $\sqsubseteq_T$, $\sqsubseteq_D$, $\sqsubseteq_L$, *respectively,*
3. $\mathfrak{S} \subseteqq T^{\mathrm{Frm}}$,
4. $\models$ *is a ternary relation on* $\mathfrak{S} \times \mathrm{LFrm} \times D$ *such that for every* $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, *and* $\ell \in L$ *there exists a* ***unique*** $d \in D$ *such that* $\mathrm{Val} \models_d [x, \ell]$,
5. *if* $x, y \in \mathrm{Frm}$ *and* $\mathrm{Val}, \mathrm{Val}' \in \mathfrak{S}$ *such that* $\mathrm{Val}(x) = \mathrm{Val}'(y)$, *then for all* $\ell \in L$ *and* $d \in D$,
$$\mathrm{Val} \models_d [x, \ell] \text{ iff } \mathrm{Val}' \models_d [y, \ell]\,, \tag{4.18}$$
6. *if* $\ell, \ell' \in L$ *such that* $\ell' \neq \ell$, *then there exists* $t \in T$ *such that for* $d, d' \in D$,
$$\text{if } \mathrm{Val}_t \models_d \left[x_t, \ell'\right] \text{ and } \mathrm{Val}_t \models_{d'} [x_t, \ell]\,, \text{ then } d \neq d', \tag{4.19}$$
7. *for all* $\ell \in L$,
$$\mathrm{Val}_1 \models_1 [x_1, \ell]\,, \tag{4.20}$$
8. *for every* $t \in T$ *and* $d \in D$, *there exists* $\ell^t_d \in L$ *such that for* $t' \in T$ *and* $d' \in D$,
$$\text{if } \mathrm{Val}_{t'} \models_{d'} \left[x_{t'}, \ell^t_d\right], \text{ then } d' = \begin{cases} 1, & \text{if } t' = 1 \\ d, & \text{if } t' \neq 1 \text{ and } t \sqsubseteq_T t' \\ 0, & \text{if } \textbf{not } t \sqsubseteq_T t' \end{cases} \tag{4.21}$$

9. *for $s, t \in T$, $\ell \in L$, and $c, d \in D$ such that*

$$\mathrm{Val}_s \models_{c} [x_s, \ell] \text{ and } \mathrm{Val}_t \models_{d} [x_t, \ell],$$

it holds that

$$\mathrm{Val}_{s \sqcap t} \models_{c \sqcap d} [x_{s \sqcap t}, \ell], \tag{4.22}$$

10. *for $t \in T$, $\ell, \ell' \in L$, and $c, d \in D$ such that*

$$\mathrm{Val}_t \models_{c} [x_t, \ell'] \text{ and } \mathrm{Val}_t \models_{d} [x_t, \ell],$$

it holds that

$$\mathrm{Val}_t \models_{c \sqcap d} [x_t, \ell' \sqcup \ell], \tag{4.23}$$

11. *for $t \in T$, $\ell, \ell' \in L$, and $d \in D$ such that*

$$\mathrm{Val}_t \models_{d} [x_t, \ell' \sqcup \ell],$$

it holds that

$$d = \bigsqcup^{D} \left\{ d_1 \sqcup d_2 \;\middle|\; \begin{array}{l} \text{There exist } t_1, t_2 \in T \text{ and } d_1, d_2 \in D \\ \text{such that } \mathrm{Val}_{t_1} \models_{d_1} [x_{t_1}, \ell'] \text{ and } \mathrm{Val}_{t_2} \models_{d_2} [x_{t_2}, \ell] \\ \text{and } t_1 \sqcap t_2 \sqsubseteq t \end{array} \right\}. \tag{4.24}$$

First of all, it is a simple observation that most 'sensible' fuzzy filter-based logics are also logics of graded truth and graded trust assessment.

Observation 4.1.2 (From fuzzy filter-based logics to logics of graded truth and graded trust assessment)
If $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \iota]$ is a fuzzy filter-based logic with induced model relation $\models$ and furthermore

1. $\mathfrak{T}$ is a **chain** or
2. $\mathfrak{D}$ is **completely distributive** wrt. $\bigsqcup^{D}$,

then $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models]$ is a logic of graded truth and graded trust assessment, where (see Definition 4.1.3.8) $\iota\left(\ell^t_d\right) = {}^{d}\bar{t}$. •

Proof
Items 1–3 of Definition 4.1.3 are identical with the respective items of Definition 4.1.2.

Items 4, 7, and 9–11 of Definition 4.1.3 have been proved, under the given assumptions, in Proposition 4.1.1.

Item 5 of Definition 4.1.3 follows immediately from definition (4.1).

Item 6 of Definition 4.1.3 follows from definition (4.1) by assumption (3.2) and the injectivity of ι.

Item 8 of Definition 4.1.3 and the condition $\iota\left(\ell^t_d\right) = {}^{d}\bar{t}$ follow from definition (4.1) by item 1 of Definition 2.3.1. □

The proof of the fact that the class of all logics of graded truth and graded trust assessment is a subclass of the class of all fuzzy filter-based logics requires a little more effort, because a suitable lattice of fuzzy filters and an isomorphism have to be provided.

Theorem 4.1.3 (From logics of graded truth and graded trust assessment to fuzzy filter-based logics)
If $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models]$ *is a logic of graded truth and graded trust assessment, then there exists a mapping* $\iota : L \to \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$ *such that* $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \iota]$ *is a fuzzy filter-based logic with induced model relation* $\models$ *and (see* Definition 4.1.3.*8)* $\iota\left(\ell_d^t\right) = {}^d\overline{t}$.

Proof
Define a mapping $\iota : L \to D^T$ for $\ell \in L$ and $t \in T$ by

$$\iota(\ell)(t) =_{\mathrm{def}} d \text{ such that } \mathrm{Val}_t \models_{d} [x_t, \ell]. \tag{4.25}$$

That by (4.25), ι is indeed uniquely defined to be a mapping from L into the set of all $\mathfrak{D}$-fuzzy sets on T is guaranteed by assumption (3.2) and item 4 of Definition 4.1.3.

For convenience, in the remainder of this proof the $\mathfrak{D}$-fuzzy set $\iota(\ell)$ on T (given $\ell \in L$) shall be denoted by $\mathcal{F}_\ell$.

Next, define a set

$$L' =_{\mathrm{def}} \{\mathcal{F}_\ell | \ell \in L\}.$$

It is to be proved that $[L', \uplus, \cap]$ is a sublattice of the dual lattice $[\mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T}), \uplus, \cap]$ of the complete lattice of $\mathfrak{D}$-fuzzy filters of $\mathfrak{T}$ and that ι is an isomorphism from $\mathfrak{L}$ onto $[L', \uplus, \cap]$ such that the conditions of Definition 2.3.1 are fulfilled.

First of all, prove that every $\mathcal{F} \in L'$ is a $\mathfrak{D}$-fuzzy filter of $\mathfrak{T}$. To this end, the conditions from Corollary 2.1.7 are verified.

Let $\ell \in L$ such that $\mathcal{F} = \mathcal{F}_\ell$.

1. $\mathcal{F}_\ell(1) = 1$.

   It is sufficient to show

   $$\mathrm{Val}_1 \models_{1} [x_1, \ell].$$

   But this is equivalent with assumption 7 of Definition 4.1.3.

2. For all $s, t \in T$, $\mathcal{F}_\ell(s) \boxdot \mathcal{F}_\ell(t) = \mathcal{F}_\ell(s \sqcap t)$.

   Let $c, d, e \in D$ be such that

   $$\mathrm{Val}_s \models_{c} [x_s, \ell] \text{ and } \mathrm{Val}_t \models_{d} [x_t, \ell] \text{ and } \mathrm{Val}_{s \sqcap t} \models_{e} [x_{s \sqcap t}, \ell].$$

   By definition (4.25), it suffices to show

   $$c \boxdot d = e,$$

   which follows from assumption 9 of Definition 4.1.3.

So far, it is proved that $L' \subseteqq \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$. To establish that $[L', \uplus, \cap]$ is a sublattice of $[\mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T}), \uplus, \cap]$, it suffices to prove that for all $\mathcal{F}, \mathcal{G} \in L'$,

$$\mathcal{F} \cap \mathcal{G} \in L' \tag{4.26}$$

$$\text{and} \quad \mathcal{F} \uplus \mathcal{G} \in L'. \tag{4.27}$$

Let $\ell, \ell' \in L$ such that $\mathcal{F} = \mathcal{F}_{\ell'}$ and $\mathcal{G} = \mathcal{F}_\ell$.

ad (4.26)**.** It is sufficient to prove

$$\mathcal{F}_{\ell'} \cap \mathcal{F}_{\ell} = \mathcal{F}_{\ell' \sqcup_{\mathbb{L}} \ell}. \tag{4.28}$$

For $t \in T$,

$$(\mathcal{F}_{\ell'} \cap \mathcal{F}_{\ell})(t) = \mathcal{F}_{\ell'}(t) \sqcap_{\mathbb{D}} \mathcal{F}_{\ell}(t).$$

Let $c, d \in D$ be such that $\mathrm{Val}_t \models_{c} \left[x_t, \ell'\right]$ and $\mathrm{Val}_t \models_{d} [x_t, \ell]$. By definition, this means

$$(\mathcal{F}_{\ell'} \cap \mathcal{F}_{\ell})(t) = c \sqcap_{\mathbb{D}} d.$$

To obtain

$$(\mathcal{F}_{\ell'} \cap \mathcal{F}_{\ell})(t) = (\mathcal{F}_{\ell' \sqcup_{\mathbb{L}} \ell})(t),$$

It is thus sufficient to prove

$$\mathrm{Val}_t \models_{c \sqcap_{\mathbb{D}} d} \left[x_t, \ell' \sqcup_{\mathbb{L}} \ell\right].$$

But this is true by assumption 10 of Definition 4.1.3.

ad (4.27)**.** The equation

$$\mathcal{F}_{\ell'} \dot{\cup} \mathcal{F}_{\ell} = \mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell} \tag{4.29}$$

is proved in two steps.

1. $\mathcal{F}_{\ell'} \dot{\cup} \mathcal{F}_{\ell} \subseteqq \mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell}$.
   By equation (2.8), it is sufficient to prove

   $$\mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell} \in \left\{\mathcal{H} \middle| \mathcal{H} \in \mathfrak{D}\text{-Fl}\,(\mathfrak{T}) \text{ and } \mathcal{F}_{\ell'} \cup \mathcal{F}_{\ell} \subseteqq \mathcal{H}\right\}.$$

   Thus it suffices to show that

   $$\mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell} \in \mathfrak{D}\text{-Fl}\,(\mathfrak{T}) \tag{4.30}$$

   and

   $$\mathcal{F}_{\ell'} \cup \mathcal{F}_{\ell} \subseteqq \mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell}. \tag{4.31}$$

   (4.30) has been proved already. For demonstrating (4.31), it is sufficient to show that for every $t \in T$,

   $$(\mathcal{F}_{\ell'} \cup \mathcal{F}_{\ell})(t) \sqsubseteq_{\mathbb{D}} \mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell}(t),$$

   i. e.

   $$\mathcal{F}_{\ell'}(t) \sqcup_{\mathbb{D}} \mathcal{F}_{\ell}(t) \sqsubseteq_{\mathbb{D}} \mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell}(t).$$

   In the following, it is established that

   $$\mathcal{F}_{\ell'}(t) \sqsubseteq_{\mathbb{D}} \mathcal{F}_{\ell' \sqcap_{\mathbb{L}} \ell}(t). \tag{4.32}$$

$\mathcal{F}_\ell(t) \sqsubseteq_D \mathcal{F}_{\ell' \sqcup_L \ell}(t)$ is proved analogously, and the result then follows from the fact that $\mathcal{F}_{\ell'}(t) \sqcup_D \mathcal{F}_\ell(t)$ is the **least** upper bound of $\mathcal{F}_{\ell'}(t)$ and $\mathcal{F}_\ell(t)$.

Proving (4.32) boils down to showing that for $c, d \in D$ such that $\mathrm{Val}_t \models_c [x_t, \ell']$ and $\mathrm{Val}_t \models_d [x_t, \ell' \sqcap_L \ell]$, it holds that $c \sqsubseteq_D d$.

By the absorption law of the lattice $\mathfrak{L}$,

$$\ell' \sqcup_L (\ell' \sqcap_L \ell) = \ell',$$

so by assumption 10 of Definition 4.1.3,

$$c = c \sqcap_D d.$$

But this is equivalent with $c \sqsubseteq_D d$ by equation (1.4).

2. $\mathcal{F}_{\ell' \sqcap_L \ell} \subseteq \mathcal{F}_{\ell'} \dot\cup \mathcal{F}_\ell$.
   It is sufficient to prove that for every $t \in T$,

$$\mathcal{F}_{\ell' \sqcap_L \ell}(t) \sqsubseteq_D (\mathcal{F}_{\ell'} \dot\cup \mathcal{F}_\ell)(t).$$

Let

$$d =_{\mathrm{def}} \bigsqcup_D \left\{ \mathcal{F}_{\ell'}(t_1) \sqcap_D \mathcal{F}_\ell(t_2) \,\middle|\, t_1, t_2 \in T \ \text{ and } \ t_1 \sqcap_T t_2 \sqsubseteq_T t \right\}.$$

On page 22 (proof of Theorem 2.2.2, ad (2.9), item 2), it has been proved that

$$d \sqsubseteq_D (\mathcal{F}_{\ell'} \dot\cup \mathcal{F}_\ell)(t).$$

Thus it is sufficient to prove that

$$\mathcal{F}_{\ell' \sqcap_L \ell}(t) \sqsubseteq_D d. \tag{4.33}$$

By definition (4.25), for $d' =_{\mathrm{def}} \mathcal{F}_{\ell' \sqcap_L \ell}(t)$ it holds that

$$\mathrm{Val}_t \models_{d'} [x_t, \ell' \sqcap_L \ell].$$

By assumption 11 of Definition 4.1.3, this means

$$d' = \bigsqcup_D \left\{ d_1 \sqcap_D d_2 \,\middle|\, \begin{array}{l} \text{There exist } t_1, t_2 \in T \text{ and } d_1, d_2 \in D \\ \text{such that } \mathrm{Val}_{t_1} \models_{d_1} [x_{t_1}, \ell'] \text{ and } \mathrm{Val}_{t_2} \models_{d_2} [x_{t_2}, \ell] \\ \text{and } t_1 \sqcap_T t_2 \sqsubseteq_T t \end{array} \right\}.$$

Again by definition (4.25),

$$\begin{aligned} d' &= \bigsqcup_D \left\{ \mathcal{F}_{\ell'}(t_1) \sqcap_D \mathcal{F}_\ell(t_2) \,\middle|\, t_1, t_2 \in T \ \text{ and } \ t_1 \sqcap_T t_2 \sqsubseteq_T t \right\} \\ &= d, \end{aligned}$$

from which (4.33) follows.

So far, it has been established that $[L', \dot\cup, \cap]$ is a sublattice of $[\mathfrak{D}\text{-Fl}(\mathfrak{T}), \dot\cup, \cap]$. Next, prove that

$$\mathfrak{D}\text{-PFl}(\mathfrak{T}) \subseteq L'.$$

Let $d \in D$ and $t \in T$. By assumption 8 of Definition 4.1.3, there exists $\ell_d^t \in L$ defined by equation (4.21). It is sufficient to prove that

$$\mathcal{F}_{\ell_d^t} = {}^{d}\overline{t}\,,$$

additionally establishing the condition $\iota\left(\ell_d^t\right) = {}^{d}\overline{t}$ of the theorem.

Prove that for all $t' \in T$,

$$\mathcal{F}_{\ell_d^t}(t') = {}^{d}\overline{t}\,(t'). \tag{4.34}$$

Let $\mathcal{F}_{\ell_d^t}(t') = d'$. By definition (4.25), this is equivalent with

$$\mathrm{Val}_{\,t'} \models_{d'} \left[x_{t'}, \ell_d^t\right].$$

The claim (4.34) now follows immediately from the definitions (2.1) and (4.21).

To complete the proof, it only remains to present an isomorphism from $[L, \sqcap_{\mathrm{L}}, \sqcup_{\mathrm{L}}]$ onto $\left[L', \dot\cup, \cap\right]$ and to prove that $\models$ coincides with the relation defined in (4.1).

By the definition (4.25) and by the equations (4.28) and (4.29) which have already been established, it is obvious that the mapping ι is a surjective lattice homomorphism from $[L, \sqcap_{\mathrm{L}}, \sqcup_{\mathrm{L}}]$ onto $\left[L', \dot\cup, \cap\right]$. To prove that ι is an isomorphism, it remains to show that ι is injective.

Let $\ell, \ell' \in L$ be given such that

$$\mathcal{F}_{\ell'} = \mathcal{F}_{\ell}.$$

This means that for all $t \in T$ and $d \in D$,

$$\mathrm{Val}_{\,t} \models_{d} \left[x_t, \ell'\right] \text{ iff } \mathrm{Val}_{\,t} \models_{d} \left[x_t, \ell\right].$$

But from this it follows by assumption 6 of Definition 4.1.3 that $\ell' = \ell$, thus the injectivity of ι and hence the fact that ι is the desired isomorphism from $[L, \sqcap_{\mathrm{L}}, \sqcup_{\mathrm{L}}]$ onto $\left[L', \dot\cup, \cap\right]$ is proved.

That $\models$ coincides with the relation defined in Definition 4.1.1 follows from (4.1) and (4.25) by assumption 5 of Definition 4.1.3. □

Remarks

From the proof of Theorem 4.1.3, it is obvious that the defining properties of a fuzzy filter-based logic have been 'translated' into logical notation. It is nevertheless interesting to note which 'logical' form these properties take and how natural they appear.

Of course it can be argued that some of the items of Definition 4.1.3 could be dispensed with, from a logical point of view. It would be interesting to study what exactly the logical consequences of leaving out one or the other of the assumptions from Definition 4.1.3 would be. For the time being, however, all the defining properties are exploited and the resulting logical systems are characterised.

The theorems to be proved in the remainder of this chapter and the examples of logics of graded truth and graded trust assessment in the following chapter shall justify this proceeding by demonstrating that logics of graded truth and graded trust assessment possess all the properties one would expect from a logic capable of representing graded truth and graded trust assessment and furthermore illustrating that a large variety of well-known valuable logical systems fall into this class.

Next, a discussion of the defining properties of logics of graded truth and graded trust assessment is given from an intuitive point of view.

- Items 1, 2, and 3 of Definition 4.1.3 are trivial assumptions and appear identically in Definition 4.1.2.
- Item 4 basically means that $\models$ is a **graded model relation**, i. e. for every valuation $\mathrm{Val} \in \mathfrak{S}$ and every labelled formula $\mathfrak{x}$, there exists a degree $d \in D$ to which Val models $\mathfrak{x}$.

  The only thing that should be noted is the fact that $\models$ really is a **mapping** from $\mathfrak{S} \times \mathrm{LFrm}$ into D, in contrast with for instance [35], where the graded consequence relation has a certain *monotonicity* property, meaning that if $\mathrm{Val} \models_{d} \mathfrak{x}$ and $d' \sqsubseteq_{\mathbb{D}} d$, then $\mathrm{Val} \models_{d'} \mathfrak{x}$.

  To explain the philosophy behind this approach, consider the case that $d = 0$. For the approach taken here, $\mathrm{Val} \models_{0} \mathfrak{x}$ means that Val is not a model of $\mathfrak{x}$ at all, i. e. the labelled formula $\mathfrak{x}$ is not satisfied by the valuation Val. If $\models$ was monotone, $\mathrm{Val} \models_{0} \mathfrak{x}$ would hold for all Val and $\mathfrak{x}$, in particular, if $\mathrm{Val} \models_{1} \mathfrak{x}$, then it would still be the case that $\mathrm{Val} \models_{0} \mathfrak{x}$, i. e. $\mathfrak{x}$ would at the same time be completely satisfied by Val and not satisfied by Val. But this would make this system incompatible with the classical case.
- Item 5 states that if two formulae are **indistinguishable** by their truth values, then neither can they be distinguished by their validity degrees, for any label. This means a certain *extensionality* property of the model relation: it only depends on the *truth value* of a formula under a valuation (and the label, of course).
- Item 6 means that if two labels are **indistinguishable** by logical means, then they should be equal.

  While it is obvious where this claim is needed in the proof of Theorem 4.1.3 to establish the injectivity of ι, it is also justified from a logical point of view. There is no more basic notion in the logical system than the model relation, and as labels are for logical purposes only, different labels should be distinguishable by means of $\models$.
- Item 7 means that a formula which is **absolutely true** under a valuation, should also be **absolutely valid**, regardless of the label. This condition already makes clear that a label must always be a *positive constraint* on the validity of a formula; it is not possible to express by a label that a formula must be *false* (to achieve this, a label must be placed on the negation of the formula, provided a suitable negation operator is contained in the logic).
- Item 8 is mainly needed for technical purposes (it corresponds to item 1 of Definition 2.3.1). It can be interpreted as demanding a minimal level of **expressive power** from the label lattice, namely that a basic set of simple constraints (for every t and d, there must be a label ℓ_d^t such that, whenever x takes a truth value greater or equal to t, then $\left[x, \ell_d^t\right]$ is d-valid, taking into account of course that if x is absolutely true then $\left[x, \ell_d^t\right]$ has to be absolutely valid) can be expressed.
- Item 9 claims a **compatibility** between the *meet* of truth values and the meet of validity degrees wrt the graded model relation. In particular, from this property the following **monotonicity** property of $\models$ follows:

  For $s, t \in T$, $\ell \in L$, and $c, d \in D$ such that

  $$\mathrm{Val}_s \models_{c} [x_s, \ell] \text{ and } \mathrm{Val}_t \models_{d} [x_t, \ell]\,,$$

  if $s \sqsubseteq_{\mathbb{T}} t$ then $c \sqsubseteq_{\mathbb{D}} d$, i. e. if a formula gets more true, the corresponding labelled formula has to get more valid.

This kind of monotonicity is important for *uncertainty modelling*, because if one is uncertain about the validity of a formula, then of course ones belief in the validity of the formula should increase if the formula gets more true.

- Item 10 claims a **compatibility** between the *join* of labels and the meet of validity degrees wrt. the graded model relation. In particular, from this property the following **comonotonicity** property of $\models$ follows:

  For $t \in T$, $\ell, \ell' \in L$, and $c, d \in D$ such that

  $$\mathrm{Val}_t \underset{c}{\models} \left[x_t, \ell'\right] \text{ and } \mathrm{Val}_t \underset{d}{\models} \left[x_t, \ell\right],$$

  if $\ell \sqsubseteq_{\mathbb{L}} \ell'$ then $c \sqsupseteq_{\mathbb{D}} d$, i. e. if the same formula is endowed with a weaker label, the corresponding labelled formula has to get more valid.

  Again, this property matches the intuition for *uncertainty modelling*. By the explanations earlier in this chapter, labels are to be ordered by *strength* of the associated constraint on the validity of the labelled formula.

  So it is natural to assume that the same formula, endowed with a stronger label, will lose validity.

- Item 11 is again of a technical nature, stating that (and in which way) the **validity distribution** created by a compound label $\ell' \sqcup_{\mathbb{L}} \ell$ is completely determined by the validity distributions created by the labels ℓ' and ℓ.

Summing up, items 1, 2, 3, and 4 provide the basis for the further definitions, without really contributing to the logical properties of the model relation.

Items 5, 6, 8, and 11 provide technical properties of labels which are needed to establish the isomorphism with a lattice of fuzzy filters.

Items 7, 9, and 10, finally state important logical properties which link the notions of truth, validity and label, and make precise the intended meaning of these concepts in uncertainty modelling.

It would of course be desirable to have more 'logical' explanations for items 6, 8, and 11, but this is a subject for future research. •

Observations 4.1.4 (Special cases of logics of graded truth and graded trust assessment)
The *logical* characterisation of fuzzy filter-based logics is illustrated further by looking at some special cases.

1. In the case that $\mathfrak{T}$ is a **chain**, item 9 of Definition 4.1.3 is equivalent with the following **monotonicity** condition:

   9* for $s, t \in T$, $\ell \in L$, and $c, d \in D$ such that

   $$\mathrm{Val}_s \underset{c}{\models} [x_s, \ell] \text{ and } \mathrm{Val}_t \underset{d}{\models} [x_t, \ell],$$

   it holds that

   $$\text{if } s \sqsubseteq_{\mathbb{T}} t \text{, then } c \sqsubseteq_{\mathbb{D}} d. \tag{4.35}$$

   Furthermore, item 11 is equivalent with

11* for $t \in T$, $\ell, \ell' \in L$, and $c, d \in D$ such that

$$\mathrm{Val}_t \underset{c}{\models} [x_t, \ell'] \text{ and } \mathrm{Val}_t \underset{d}{\models} [x_t, \ell],$$

it holds that

$$\mathrm{Val}_t \underset{c \sqcup_{\mathbb{D}} d}{\models} [x_t, \ell' \sqcap_{\mathbb{L}} \ell], \tag{4.36}$$

By Observation 4.1.2 and Theorem 4.1.3, in the case that $\mathfrak{T}$ is a chain, thus the class of all fuzzy filter-based logics is completely characterised by the axioms 1, 2, 3, 4, 5, 6, 7, 8, 9*, 10, and 11*.

2. In the case that $\mathfrak{L}$ is a **chain** (which implies by Proposition 2.3.6 that $\mathfrak{T}$ is a chain and $\mathfrak{D} = \mathfrak{B}$ or $\mathfrak{T} = \mathfrak{B}$ and $\mathfrak{D}$ is a chain), item 10 of Definition 4.1.3 is equivalent with the following **monotonicity** condition:

   10* for $t \in T$, $\ell, \ell' \in L$, and $c, d \in D$ such that

   $$\mathrm{Val}_t \underset{c}{\models} [x_t, \ell'] \text{ and } \mathrm{Val}_t \underset{d}{\models} [x_t, \ell],$$

   it holds that

   $$\text{if } \ell \sqsubseteq_{\mathbb{L}} \ell', \text{ then } c \sqsupseteq_{\mathbb{D}} d. \tag{4.37}$$

   Furthermore, item 11* is also equivalent with 10*, thus in this case, the class of all fuzzy filter-based logics is completely characterised by the axioms 1, 2, 3, 4, 5, 6, 7, 8, 9*, 10*.

   This means that the axioms reduce to a couple of technical trivialities (axioms 1–8) and two simple monotonicity conditions (axioms 9* and 10*) — a very simple and intuitively pleasing axiom system which nonetheless characterises a lot of well-known logics (see chapter 5). •

Proof
By applying (1.4) and (1.13) to Definition 4.1.3, and applying Observation 4.1.2 and Theorem 4.1.3. □

Next, the semantic theory of logics of graded truth and graded trust assessment is developed further.

Definition 4.1.4 (Model relation for $\mathfrak{L}$-fuzzy sets of formulae)
The relation $\models$ *from* Definition 4.1.1 *is extended to $\mathfrak{L}$-fuzzy sets of formulae as follows. Given an $\mathfrak{L}$-fuzzy set $\mathcal{X} : \mathrm{Frm} \to L$ and a valuation* $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Val} \underset{d}{\models} \mathcal{X} =_{\mathrm{def}} d = \bigsqcap_{\mathbb{D}} \left\{ d' \,\middle|\, x \in \mathrm{Frm} \textit{ and } \mathrm{Val} \underset{d'}{\models} [x, \mathcal{X}(x)] \right\} \tag{4.38}$$

Remarks

1. Because by Proposition 4.1.1.1, $\mathrm{Val} \underset{1}{\models} [x, \mathcal{X}(x)]$ whenever $\mathcal{X}(x) = 0$, equation (4.38) is equivalent with

   $$\mathrm{Val} \underset{d}{\models} \mathcal{X} =_{\mathrm{def}} d = \bigsqcap_{\mathbb{D}} \left\{ d' \,\middle|\, x \in \mathrm{Frm} \text{ and } \mathcal{X}(x) \neq 0 \text{ and } \mathrm{Val} \underset{d'}{\models} [x, \mathcal{X}(x)] \right\}. \tag{4.39}$$

   This is important when $\mathcal{X}$ is to be characterised by a (possibly finite) **knowledge base** consisting of a set $X =_{\mathrm{def}} \{[x_1, \ell_1], [x_2, \ell_2], \dots\}$ of labelled formulae, such that $x_i \neq x_j$ for $i \neq j$. X can then be identified with an $\mathfrak{L}$-fuzzy set $\mathcal{X}$ by setting $\mathcal{X}(x_i) =_{\mathrm{def}} \ell_i$ for every i and $\mathcal{X}(x) =_{\mathrm{def}} 0$ for every $x \in \mathrm{Frm}$ which does not appear as x_i for any i. The calculation of $\mathrm{Val} \underset{d}{\models} \mathcal{X}$ can then be reduced to the calculation of $\mathrm{Val} \underset{d'}{\models} [x_i, \ell_i]$, ignoring those $x \in \mathrm{Frm}$ which do not appear as x_i for any i.

2. The formulation "the relation $\models$ is extended to $\mathfrak{L}$-fuzzy sets of formulae" has to be justified by proving that the model relations defined in Definition 4.1.1 and Definition 4.1.4 coincide on labelled formulae.

   But this follows immediately from the previous item, because $[x, \ell]$ is just the $\mathfrak{D}$-fuzzy set of formulae associated with $\{[x, \ell]\}$.

3. Proposition 4.1.1.2 means that for all $\mathrm{Val} \in \mathfrak{S}$,

   (4.40) $$\mathrm{Val} \underset{1}{\models} \emptyset.$$

   (Compare (1.22).) •

The primary idea behind the above definition of $\models$ is to provide a basis for a 'sensible' definition of **semantic entailment**.

The second goal in this definition is to be general enough to allow for interesting interpretations of **labels** which endow the resulting logic with sufficient expressive power to be suitable for modelling graded truth assessment *and* graded trust assessment in *knowledge representation*. The flexibility of this approach is illustrated in chapter 5. In particular, section 5.5 contains examples and a discussion of special labels and their logical meaning.

4.2 Some Logical Concepts Based on the Model Relation

Before defining and studying the **semantic consequence** operator in section 4.3, in this section some simpler semantic concepts based on the **graded model relation** like **model fuzzy sets**, the properties **validity** and **consistency** for labelled formulae, and the **semantic equivalence** relation are defined and studied.

First of all, for every $\mathfrak{L}$-fuzzy set of formulae, its **$\mathfrak{D}$-fuzzy set of models** can be defined.

Definition 4.2.1 (Model fuzzy set of a fuzzy set of formulae)
For $\mathcal{X} \in L^{\mathrm{Frm}}$, define the ***$\mathfrak{D}$-fuzzy set*** $\mathrm{Mod}(\mathcal{X}) \in D^{\mathfrak{S}}$ ***of models of*** $\mathcal{X}$ *for* $\mathrm{Val} \in \mathfrak{S}$ *and* $d \in D$ *by*

(4.41) $$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = d =_{\mathrm{def}} \mathrm{Val} \underset{d}{\models} \mathcal{X}.$$

(This is possible because $d \in D$ such that $\mathrm{Val} \underset{d}{\models} \mathcal{X}$ *is uniquely defined.)*

Proposition 4.2.1 (Compatibility of model fuzzy sets with operations on fuzzy sets)
Let $\mathcal{X}, \mathcal{Y} \in L^{\mathrm{Frm}}$. Then

(4.42) $$\mathrm{Mod}\,(\mathcal{X}) \cap \mathrm{Mod}\,(\mathcal{Y}) = \mathrm{Mod}\,(\mathcal{X} \cup \mathcal{Y})\,,$$

(4.43) $$\textit{if } \mathcal{X} \subseteqq \mathcal{Y}\,,\ \textit{then } \mathrm{Mod}(\mathcal{Y}) \subseteqq \mathrm{Mod}(\mathcal{X}).$$

(4.44) $$\mathrm{Mod}\,(\mathcal{X}) \cup \mathrm{Mod}\,(\mathcal{Y}) \subseteqq \mathrm{Mod}\,(\mathcal{X} \cap \mathcal{Y})\,,$$

Proof
ad (4.42). Let $\mathrm{Val} \in \mathfrak{S}$. Then

$$\begin{aligned} &\big(\mathrm{Mod}\,(\mathcal{X}) \cap \mathrm{Mod}\,(\mathcal{Y})\big)\,(\mathrm{Val}) \\ &= \bigsqcap_{\mathbb{D}} \Big\{ d' \,\Big|\, \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d'}{\models} \big[x, \mathcal{X}(x)\big] \Big\} && \text{By (4.41), (4.38), (1.17)} \\ &\quad \sqcap_{\mathbb{D}} \bigsqcap_{\mathbb{D}} \Big\{ d' \,\Big|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \underset{d'}{\models} \big[y, \mathcal{Y}(y)\big] \Big\} \end{aligned}$$

$$= \boxed{\mathrm{D}}\left(\begin{array}{l}\left\{d' \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d'}{\models} \left[x, \mathcal{X}(x)\right]\right\} \\ \cup \left\{d' \middle| \exists y \in \mathrm{Frm} : \mathrm{Val} \underset{d'}{\models} \left[y, \mathcal{Y}(y)\right]\right\}\end{array}\right)$$

$$= \boxed{\mathrm{D}}\left\{d \boxdot d' \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d}{\models} \left[x, \mathcal{X}(x)\right] \text{ and } \mathrm{Val} \underset{d'}{\models} \left[x, \mathcal{Y}(x)\right]\right\}$$

$$= \boxed{\mathrm{D}}\left\{d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d}{\models} \left[x, \mathcal{X}(x) \sqcup \mathcal{Y}(x)\right]\right\} \qquad \text{By (4.5)}$$

$$= \boxed{\mathrm{D}}\left\{d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d}{\models} \left[x, \left(\mathcal{X} \cup \mathcal{Y}\right)(x)\right]\right\}$$

$$= \left(\mathrm{Mod}\left(\mathcal{X} \cup \mathcal{Y}\right)\right)(\mathrm{Val})$$

ad (4.43)**.** If $\mathcal{X} \subseteqq \mathcal{Y}$, then $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$, thus (4.42) is just the definition of

$$\mathrm{Mod}(\mathcal{Y}) \subseteqq \mathrm{Mod}(\mathcal{X})$$

in the lattice of $\mathfrak{D}$-fuzzy sets on $\mathfrak{S}$ (compare (1.4)).

ad (4.44)**.** $\mathcal{X} \cap \mathcal{Y} \subseteqq \mathcal{X}$ and $\mathcal{X} \cap \mathcal{Y} \subseteqq \mathcal{Y}$ hold trivially, hence by (4.43),

$$\mathrm{Mod}\left(\mathcal{X}\right) \subseteqq \mathrm{Mod}\left(\mathcal{X} \cap \mathcal{Y}\right)$$

and

$$\mathrm{Mod}\left(\mathcal{Y}\right) \subseteqq \mathrm{Mod}\left(\mathcal{X} \cap \mathcal{Y}\right),$$

from which (4.44) follows immediately. □

Proposition 4.2.2 (Compatibility of model fuzzy sets with infinitary join)
Let $\mathfrak{X} \subseteqq L^{\mathrm{Frm}}$. Then

$$\bigcap \left\{\mathrm{Mod}(\mathcal{X}) \middle| \mathcal{X} \in \mathfrak{X}\right\} = \mathrm{Mod}\left(\bigcup \mathfrak{X}\right). \tag{4.45}$$

Proof
The result is proved analogously to (4.42). Let $\mathrm{Val} \in \mathfrak{S}$. Then

$$\left(\bigcap \left\{\mathrm{Mod}(\mathcal{X}) \middle| \mathcal{X} \in \mathfrak{X}\right\}\right)(\mathrm{Val})$$

$$= \boxed{\mathrm{D}}\left\{\boxed{\mathrm{D}}\left\{d' \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d'}{\models} \left[x, \mathcal{X}(x)\right]\right\} \middle| \mathcal{X} \in \mathfrak{X}\right\}$$

$$= \boxed{\mathrm{D}}\bigcup\left\{\left\{d' \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d'}{\models} \left[x, \mathcal{X}(x)\right]\right\} \middle| \mathcal{X} \in \mathfrak{X}\right\}$$

$$= \boxed{\mathrm{D}}\left\{\boxed{\mathrm{D}}\, C \middle| \begin{array}{l}\exists x \in \mathrm{Frm} : \\ C = \left\{d \middle| d \in D \text{ and there exists } \mathcal{X} \in \mathfrak{X} \text{ such that } \mathrm{Val} \underset{d}{\models} \left[x, \mathcal{X}(x)\right]\right\}\end{array}\right\}$$

$$= \boxed{\mathrm{D}}\left\{d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d}{\models} \left[x, \left(\boxed{\mathrm{L}}\left\{\mathcal{X}(x) \middle| \mathcal{X} \in \mathfrak{X}\right\}\right)\right]\right\} \qquad \text{By (4.6)}$$

$$= \boxed{\mathrm{D}}\left\{d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \underset{d}{\models} \left[x, \left(\bigcup \mathfrak{X}\right)(x)\right]\right\}$$

$$= \left(\mathrm{Mod}\left(\bigcup \mathfrak{X}\right)\right)(\mathrm{Val})$$

□

Remark
Note that from the observation above, it follows that the set of all **model fuzzy sets** for an arbitrary fuzzy filter-based logic again forms a complete lattice, the infimum of which coincides with the canonical one on the set of all $\mathfrak{D}$-fuzzy sets on $\mathfrak{S}$. •

Observation 4.2.3 (Monotonicity of Mod wrt truth values)

1. Let $x, y \in \mathrm{Frm}$. If for every $\mathrm{Val} \in \mathfrak{S}$,

   $$\mathrm{Val}(x) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(y) \tag{4.46}$$

   holds, then for every $\ell \in L$, $\mathrm{Mod}\big([x,\ell]\big) \subseteqq \mathrm{Mod}\big([y,\ell]\big)$.

2. If the logic constituted by Frm, $\mathfrak{T}$ and $\mathfrak{S}$ contains a binary operator $\vee$ interpreted by $\sqcup_{\mathrm{T}}$, then for all $x, y \in \mathrm{Frm}$ and all $\ell \in L$,

   $$\mathrm{Mod}\big([x,\ell]\big) \subseteqq \mathrm{Mod}\big([x \vee y,\ell]\big) \;\text{ and }\; \mathrm{Mod}\big([y,\ell]\big) \subseteqq \mathrm{Mod}\big([x \vee y,\ell]\big)\,. \tag{4.47}$$ •

Proof

ad 1. Let x, y be given as required. By definition of Mod, it is to be proved that for every $\ell \in L$, every $\mathrm{Val} \in \mathfrak{S}$, and all $d, d' \in D$,

$$\text{if } \mathrm{Val} \models_{d} [x,\ell] \text{ and } \mathrm{Val} \models_{d'} [y,\ell]\,, \text{ then } d \sqsubseteq_{\mathrm{D}} d',$$

which follows from Proposition 4.1.1.9.

ad 2. Follows from the previous item by the fact that for every $\mathrm{Val} \in \mathfrak{S}$,

$$\begin{aligned} \mathrm{Val}(x) &\sqsubseteq_{\mathrm{T}} \mathrm{Val}(x) \sqcup_{\mathrm{T}} \mathrm{Val}(y) && \text{(by the fact that } \mathfrak{T} \text{ is a lattice)}\\ &= \mathrm{Val}(x \vee y). && \text{(by definition)} \end{aligned}$$

Analogously for $\mathrm{Val}(y) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(x \vee y)$. □

An interesting property of a fuzzy set of formulae is the degree to which it *must* be valid (which should be distinguished from the *inherent truth* taut which is a truth value defined for a formula; see Definition 3.3.4), and the degree to which it *can* be valid.

Definition 4.2.2 (Validity and consistency index)
Let $\mathcal{X} \in L^{\mathrm{Frm}}$.

1. *Define the* ***validity index*** *(or inherent validity) of $\mathcal{X}$,*

   $$\mathrm{valid}(\mathcal{X}) =_{\mathrm{def}} \boxed{\sqcap_{\mathrm{D}}} \big\{\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \,\big|\, \mathrm{Val} \in \mathfrak{S}\big\}\,. \tag{4.48}$$

   If $\mathrm{valid}(\mathcal{X}) = 1$, *$\mathcal{X}$ is said to be* ***valid****. The set of all valid $\mathfrak{L}$-fuzzy sets of formulae is denoted* Valid.

2. *Define the* ***consistency index*** *of $\mathcal{X}$,*

   $$\mathrm{cst}(\mathcal{X}) =_{\mathrm{def}} \boxed{\sqcup_{\mathrm{D}}} \big\{\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \,\big|\, \mathrm{Val} \in \mathfrak{S}\big\}\,. \tag{4.49}$$

   If $\mathrm{cst}(\mathcal{X}) = 1$, *$\mathcal{X}$ is said to be* ***consistent****. $\mathcal{X}$ is said to be* ***strictly consistent*** *iff there exists* $\mathrm{Val} \in \mathfrak{S}$ *such that* $\mathrm{Val} \models_{1} \mathcal{X}$. *$\mathcal{X}$ is said to be* ***inconsistent*** *iff* $\mathrm{cst}(\mathcal{X}) = 0$. *The set of all inconsistent $\mathfrak{L}$-fuzzy sets of formulae is denoted* Incons.

Remark
The definition of cst above corresponds to the definition of the **consistency degree** Cons in [19, section 3.3]. •

Observations 4.2.4 (Properties of valid and cst)

1. $\not\!\emptyset \in \mathrm{Valid}$.
2. For $[x,\ell] \in \mathrm{LFrm}$,
$$\iota(\ell)(\mathrm{taut}(x)) \sqsubseteq_{\mathbb{D}} \mathrm{valid}([x,\ell]).$$
3. If $x \in \mathrm{Frm}$ is a **tautology**, then for every $\ell \in L$, $[x,\ell]$ is **valid**.
4. For every $x \in \mathrm{Frm}$, $[x,0]$ is **valid**.
5. For $x \in \mathrm{Frm}$,
$$\iota(1)(\mathrm{taut}(x)) = \mathrm{valid}([x,1]).$$
6. If $[x,1] \in \mathrm{LFrm}$ is **valid**, then x is a **tautology**.
7. For every $\mathcal{X} \in \mathrm{LFrm}$, $\mathrm{valid}(\mathcal{X}) \sqsubseteq_{\mathbb{D}} \mathrm{cst}(\mathcal{X})$.
8. If $\mathcal{X}$ is **strictly consistent**, then $\mathcal{X}$ is **consistent**.
9. If $[x,1] \in \mathrm{LFrm}$ is **consistent**, then $[x,1]$ is **strictly consistent**.
10. For $[x,\ell] \in \mathrm{LFrm}$, if there exists $\mathrm{Val} \in \mathfrak{S}$ such that $\mathrm{Val}(x) = 1$, then $[x,\ell]$ is **strictly consistent**.
11. For $x \in \mathrm{Frm}$, if $[x,1]$ is **consistent**, then there exists $\mathrm{Val} \in \mathfrak{S}$ such that $\mathrm{Val}(x) = 1$. •

Proof

ad 1. Trivial because by (4.40), $\mathrm{Mod}(\not\!\emptyset)(\mathrm{Val}) = 1$ for all $\mathrm{Val} \in \mathfrak{S}$.

ad 2. It is sufficient to prove that for every $\mathrm{Val} \in \mathfrak{S}$ and $d \in D$ such that $\mathrm{Val} \models_{d} [x,\ell]$,
$$\iota(\ell)(\mathrm{taut}(x)) \sqsubseteq_{\mathbb{D}} d.$$
By the definition of taut,
$$\mathrm{taut}(x) \sqsubseteq_{\mathbb{T}} \mathrm{Val}(x),$$
thus, because $\iota(\ell)$ is a $\mathfrak{D}$-fuzzy filter of $\mathfrak{T}$,
$$\iota(\ell)(\mathrm{taut}(x)) \sqsubseteq_{\mathbb{D}} \iota(\ell)(\mathrm{Val}(x)).$$
But by the definition of $\models$,
$$d = \iota(\ell)(\mathrm{Val}(x)),$$
so the proof of this item is complete.

Please note that the reverse direction (i. e. $\mathrm{valid}([x,\ell]) \sqsubseteq_{\mathbb{D}} \iota(\ell)(\mathrm{taut}(x))$) does not hold, in general. To establish this inequation, some sort of *continuity* of the mapping $\iota(\ell)$ would be needed, which cannot even be formulated so far. But in item 5, a weak form of the reverse inequation is proved for the special case $\ell = 1$.

ad 3. Follows immediately from the previous item, because $\iota(\ell)(1) = 1$ by the definition of a fuzzy filter.

ad 4. Follows immediately from Proposition 4.1.1, item 2.

ad 5. Taking into account item 2, it remains to prove

$$\text{valid}([x,1]) \sqsubseteq \iota(1)(\text{taut}(x)), \tag{4.50}$$

where

$$\begin{aligned}\text{valid}([x,1]) &= \boxed{\mathrm{D}}\left\{d \,\middle|\, \text{Val} \in \mathfrak{S} \text{ and } \text{Val} \models_{d} [x,1]\right\} \\ &= \boxed{\mathrm{D}}\left\{\iota(1)(\text{Val}(x)) \,\middle|\, \text{Val} \in \mathfrak{S}\right\}.\end{aligned}$$

By an analogous argumentation as in the proof of item 2 of Proposition 4.1.1, it follows that $\iota(1)$ is the zero element $\mathbb{0}$ of the lattice $\left[\mathfrak{D}\text{-Fl}\,(\mathfrak{T}), \cap, \Cup\right]$. By the characterisation of this mapping in item 1 of Observation 2.2.5, it follows that for every $t \in T$,

$$\mathbb{0}(t) \in \{0,1\} \tag{4.51}$$

and furthermore

$$\mathbb{0}(t) = 1 \text{ iff } t = 1. \tag{4.52}$$

Hence for (4.50), it is sufficient to show that

$$\text{if } \iota(1)(\text{taut}(x)) = 0\,, \text{ then } 0 \in \left\{\iota(1)(\text{Val}(x)) \,\middle|\, \text{Val} \in \mathfrak{S}\right\}. \tag{4.53}$$

If $\iota(1)(\text{taut}(x)) = 0$, then $\text{taut}(x) \neq 1$, i.e. there exists $\text{Val} \in \mathfrak{S}$ such that $\text{Val}(x) \neq 1$. But in this case $\iota(1)(\text{Val}(x)) = 0$, which establishes the claim.

ad 6. Follows immediately from the contraposition of (4.53) and from (4.52).

ad 7 and 8. Trivial by the definition and the fact that $\mathfrak{D}$ is a complete lattice.

ad 9. Follows immediately from the fact that for every $\text{Val} \in \mathfrak{S}$, $\text{Mod}\big([x,1]\big)(\text{Val}) \in \{0,1\}$ (compare (4.51)).

ad 10. Follows immediately from the definition by Proposition 4.1.1.3.

ad 11. Is proved analogously to item 5. □

Next, the relation of **semantic equivalence** is defined for labelled formulae resp. $\mathfrak{L}$-fuzzy sets of formulae.

Definition 4.2.3 (Semantic equivalence)

1. *$\mathcal{X}, \mathcal{Y} \in L^{\text{Frm}}$ are said to be **semantically equivalent** ($\mathcal{X} \equiv \mathcal{Y}$) $=_{\text{def}}$ $\text{Mod}(\mathcal{X}) = \text{Mod}(\mathcal{Y})$.*

   (The overloading of $\equiv$ to denote semantic equivalence both for formulae and $\mathfrak{L}$-fuzzy sets of formulae should pose no problems because it will always be clear from the context which interpretation is meant.)

2. *$\mathcal{X}, \mathcal{Y} \in L^{\text{Frm}}$ are said to be **consistency-equivalent** ($\mathcal{X} \cong \mathcal{Y}$) $=_{\text{def}}$ $\text{cst}(\mathcal{X}) = \text{cst}(\mathcal{Y})$.*

Observations 4.2.5 (Properties of $\equiv$)

1. Both $\equiv$ and $\cong$ are **equivalence relations** on L^{Frm}.

2. $\mathcal{X} \in \mathrm{Valid}$ iff $\mathcal{X} \equiv \emptyset$.

   If $\mathcal{X} \in \mathrm{Incons}$, then $\mathcal{Y} \in \mathrm{Incons}$ iff $\mathcal{X} \equiv \mathcal{Y}$.

   This means that Valid, Incons (if non-empty) are **equivalence classes** wrt $\equiv$.

3. For $x, y \in \mathrm{Frm}$, if $x \equiv y$, then for every $\ell \in L$,

$$[x, \ell] \equiv [y, \ell]\,, \tag{4.54}$$
$$[x, \ell] \cong [y, \ell]\,. \tag{4.55}$$

4. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in L^{\mathrm{Frm}}$. If $\mathcal{X} \equiv \mathcal{Y}$, then $\mathcal{X} \cup \mathcal{Z} \equiv \mathcal{Y} \cup \mathcal{Z}$.

5. Let $\mathfrak{X}, \mathfrak{Y} \subseteqq L^{\mathrm{Frm}}$. If

$$\begin{array}{rl} & \text{for every } \mathcal{X} \in \mathfrak{X} \text{ there exists } \mathcal{Y} \in \mathfrak{Y} \text{ such that } \mathcal{X} \equiv \mathcal{Y} \\ \text{and} & \text{for every } \mathcal{Y} \in \mathfrak{Y} \text{ there exists } \mathcal{X} \in \mathfrak{X} \text{ such that } \mathcal{X} \equiv \mathcal{Y}, \end{array} \tag{4.56}$$

   then $\bigcup \mathfrak{X} \equiv \bigcup \mathfrak{Y}$. •

Proof

ad 1. Obvious by the definitions.

ad 2. Trivial because $\mathcal{X} \in \mathrm{Valid}$ iff $\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = 1$ for all $\mathrm{Val} \in \mathfrak{S}$ (compare Observation 4.2.4.1) and $\mathcal{X} \in \mathrm{Incons}$ iff $\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = 0$ for all $\mathrm{Val} \in \mathfrak{S}$.

ad 3. Follows immediately from definition (4.1).

ad 4. Obvious by Proposition 4.2.1.

ad 5. Let $\mathfrak{X}, \mathfrak{Y}$ be given as assumed. It is to be proved that

$$\mathrm{Mod}\left(\bigcup \mathfrak{X}\right) = \mathrm{Mod}\left(\bigcup \mathfrak{Y}\right).$$

By (4.45), it is sufficient to show that

$$\bigcap \left\{\mathrm{Mod}(\mathcal{X}) \middle| \mathcal{X} \in \mathfrak{X}\right\} = \bigcap \left\{\mathrm{Mod}(\mathcal{Y}) \middle| \mathcal{Y} \in \mathfrak{Y}\right\}.$$

It suffices to prove

$$\left\{\mathrm{Mod}(\mathcal{X}) \middle| \mathcal{X} \in \mathfrak{X}\right\} = \left\{\mathrm{Mod}(\mathcal{Y}) \middle| \mathcal{Y} \in \mathfrak{Y}\right\}.$$

ad "$\subseteqq$". Let $\mathcal{X} \in \mathfrak{X}$. It is sufficient to show that $\mathrm{Mod}(\mathcal{X}) \in \left\{\mathrm{Mod}(\mathcal{Y}) \middle| \mathcal{Y} \in \mathfrak{Y}\right\}$. By assumption (4.56), there exists $\mathcal{Y} \in \mathfrak{Y}$ such that $\mathcal{X} \equiv \mathcal{Y}$, i.e. $\mathrm{Mod}(\mathcal{X}) = \mathrm{Mod}(\mathcal{Y})$, which establishes the claim.

ad "$\supseteqq$". Is proved analogously.

This concludes the proof. □

The following theorems are very important because they allow to transfer results about semantic equivalence of formulae (from the theory of normal forms, for instance) immediately to labelled formulae by simply *replacing* semantically equivalent formulae in a fuzzy set. To formulate the results more elegantly, an auxiliary definition is introduced.

Definition 4.2.4 (Modifying fuzzy sets by crisp sets)
Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $X \subseteqq \mathrm{Frm}$ be given.

1. *Denote by $\mathcal{X} \setminus X$ the fuzzy set such that for all $x \in \mathrm{Frm}$*

$$\big(\mathcal{X} \setminus X\big)(x) =_{\mathrm{def}} \begin{cases} 0, & \text{if } x \in X \\ \mathcal{X}(x), & \text{if } x \notin X \end{cases} \tag{4.57}$$

2. *$\mathcal{X} \cap X$ denotes the fuzzy set $\mathcal{X} \setminus \big(\mathrm{Frm} \setminus X\big)$, i. e. for all $x \in \mathrm{Frm}$,*

$$(\mathcal{X} \cap X)(x) =_{\mathrm{def}} \begin{cases} 0, & \text{if } x \notin X \\ \mathcal{X}(x), & \text{if } x \in X \end{cases} \tag{4.58}$$

Theorem 4.2.6 (Replacement)

1. *Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x, y \in \mathrm{Frm}$ with $x \equiv y$. Then*

$$\mathcal{X} \equiv \mathcal{X} \cup \big[x, \mathcal{X}(y)\big]. \tag{4.59}$$
$$\equiv (\mathcal{X} \setminus \{y\}) \cup \big[x, \mathcal{X}(y)\big]. \tag{4.60}$$

2. *The result can be extended a little, to the case where one formula is semantically equivalent with a finite set of formulae.*

   Let $\mathcal{X} \in L^{\mathrm{Frm}}$, a formula $y \in \mathrm{Frm}$ and a finite set $Y = \{y_1, \ldots, y_n\} \subseteqq \mathrm{Frm}$, for $n \in \mathbb{N}$, be given, such that $\{y\} \equiv Y$.

   Furthermore, let

$$\mathcal{Y} =_{\mathrm{def}} \big[y_1, \mathcal{X}(y)\big] \cup \cdots \cup \big[y_n, \mathcal{X}(y)\big]$$

   Then

$$\mathcal{X} \equiv \mathcal{X} \cup \mathcal{Y}. \tag{4.61}$$
$$\equiv (\mathcal{X} \setminus \{y\}) \cup \mathcal{Y}. \tag{4.62}$$

Proof
ad (4.59)**.** Let $\mathrm{Val} \in \mathfrak{S}$ and $d' \in D$ be such that

$$\mathrm{Val} \models_{d'} \big[y, \mathcal{X}(y)\big]. \tag{4.63}$$

It is obvious by $x \equiv y$ and definition (4.1) that also

$$\mathrm{Val} \models_{d'} \big[x, \mathcal{X}(y)\big].$$

By (4.42),

$$\begin{aligned} \mathrm{Mod}\Big(\mathcal{X} \cup \big[x, \mathcal{X}(y)\big]\Big)(\mathrm{Val}) &= \Big(\mathrm{Mod}(\mathcal{X}) \cap \mathrm{Mod}\Big(\big[x, \mathcal{X}(y)\big]\Big)\Big)(\mathrm{Val}) \\ &= \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqcap_{\mathrm{D}} d' \\ &= \bigsqcap_{\mathrm{D}} \Big\{ d \,\Big|\, z \in \mathrm{Frm} \text{ and } \mathrm{Val} \models_{d} \big[z, \mathcal{X}(z)\big] \Big\} \sqcap_{\mathrm{D}} d'. \end{aligned}$$

But $d' \in \Big\{ d \,\Big|\, z \in \mathrm{Frm} \text{ and } \mathrm{Val} \models_{d} \big[z, \mathcal{X}(z)\big] \Big\}$ by (4.63), hence

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqcap_{\mathrm{D}} d' = \mathrm{Mod}\Big(\mathcal{X} \cup \big[x, \mathcal{X}(y)\big]\Big)(\mathrm{Val}).$$

Thus (4.59) is proved.

ad (4.60)**.** Follows exactly like (4.59), taking into account that

$$\mathrm{Mod}\big(\mathcal{X} \setminus \{y\}\big)\,(\mathrm{Val}) \sqcap_{\mathrm{D}} d' = \mathrm{Mod}(\mathcal{X})(\mathrm{Val}).$$

ad (4.61)**.** The proof of (4.59) is adapted to this case. Let $\mathrm{Val} \in \mathfrak{S}$ and $d' \in D$ be such that

$$\mathrm{Val} \models_{d'} \mathcal{Y}.$$

Furthermore, let $d_1, \ldots, d_n$ be such that

$$\begin{aligned} \mathrm{Val} &\models_{d_1} \big[y_1, \mathcal{X}(y)\big]\,, \\ &\vdots \\ \mathrm{Val} &\models_{d_n} \big[y_n, \mathcal{X}(y)\big]\,. \end{aligned}$$

From (4.38) it follows that $d' = d_1 \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} d_n$. Furthermore, $\{y\} \equiv Y$ means that $\mathrm{Val}(y) = \mathrm{Val}(y_1) \sqcap_{\mathrm{T}} \ldots \sqcap_{\mathrm{T}} \mathrm{Val}(y_n)$. Thus a simple induction using (4.4) yields

$$\mathrm{Val} \models_{d'} \big[y, \mathcal{X}(y)\big]\,. \tag{4.64}$$

By (4.42),

$$\begin{aligned} \mathrm{Mod}\,(\mathcal{X} \cup \mathcal{Y})\,(\mathrm{Val}) &= \big(\mathrm{Mod}(\mathcal{X}) \cap \mathrm{Mod}\,(\mathcal{Y})\big)\,(\mathrm{Val}) \\ &= \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqcap_{\mathrm{D}} d' \\ &= \bigsqcap_{\mathrm{D}} \Big\{ d \,\Big|\, z \in \mathrm{Frm} \ \text{ and } \ \mathrm{Val} \models_{d} \big[z, \mathcal{X}(z)\big] \Big\} \sqcap_{\mathrm{D}} d'. \end{aligned}$$

But $d' \in \Big\{ d \,\Big|\, z \in \mathrm{Frm} \ \text{ and } \ \mathrm{Val} \models_{d} \big[z, \mathcal{X}(z)\big] \Big\}$ by (4.64), hence

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqcap_{\mathrm{D}} d' = \mathrm{Mod}\,(\mathcal{X} \cup \mathcal{Y})\,(\mathrm{Val}).$$

Thus (4.61) is proved.

ad (4.62)**.** Follows exactly like (4.61), taking into account that

$$\mathrm{Mod}\big(\mathcal{X} \setminus \{y\}\big)\,(\mathrm{Val}) \sqcap_{\mathrm{D}} d' = \mathrm{Mod}(\mathcal{X})(\mathrm{Val}). \qquad \square$$

From (4.62), an interesting corollary is obtained which allows to *dissolve* conjunctive formulae in a logic where conjunction is interpreted by the lattice meet.

Corollary 4.2.7 (Dissolving conjunctions)
If the underlying logic constituted by Frm, $\mathfrak{T}$ *and* $\mathfrak{S}$ *contains a binary operator* $\wedge$ *interpreted by* $\sqcap_{\mathrm{T}}$*, then for every* $\mathcal{X} : \mathrm{Frm} \to L$ *and all* $x, y \in \mathrm{Frm}$*,*

$$\mathcal{X} \equiv \big(\mathcal{X} \setminus \{x \wedge y\}\big) \cup \big[x, \mathcal{X}(x \wedge y)\big] \cup \big[y, \mathcal{X}(x \wedge y)\big]\,.$$

Proof
By (4.62), it suffices to prove $\{x \wedge y\} \equiv \{x, y\}$. This follows by definition. $\square$

Next, more 'substantial' replacements are considered.

Theorem 4.2.8 (Transforming fuzzy sets of formulae into normal form)
Let $\mathcal{X} \in L^{\mathrm{Frm}}$.

1. *Let* $x \in \mathrm{Frm}$ *be given. The labels of all elements from* x*'s* ***equivalence class*** $[x]$ *(see* Definition 3.3.2*) can be made equal, as follows.*

   Define

   $$\ell_x =_{\mathrm{def}} \boxed{\mathrm{L}} \left\{ \mathcal{X}(y) \,\middle|\, y \in [x] \right\}, \tag{4.65}$$

   $$\mathcal{X}_x =_{\mathrm{def}} \bigcup \left\{ [y, \ell_x] \,\middle|\, y \in [x] \right\}. \tag{4.66}$$

   Then the following holds:

   $$\mathcal{X} \equiv \mathcal{X} \cup \mathcal{X}_x \tag{4.67}$$
   $$\equiv \big(\mathcal{X} \setminus [x]\big) \cup \mathcal{X}_x \tag{4.68}$$

2. $\mathcal{X}$ *can even be* ***factorised*** *this way (using the definitions for* ℓ_x *and* $\mathcal{X}_x$ *given above):*

   $$\mathcal{X} \equiv \bigcup \left\{ \mathcal{X}_x \,\middle|\, x \in \operatorname{supp} \mathcal{X} \right\} \tag{4.69}$$

3. *If* $N \subseteqq \mathrm{Frm}$ *is a* ***semantic covering*** *(see* Definition 3.3.3*.1) of* $\operatorname{supp} \mathcal{X}$*, then*

   $$\mathcal{X} \equiv \bigcup \left\{ [x, \ell_x] \,\middle|\, x \in N \right\} \tag{4.70}$$

4. *If* $\mathscr{T}$ *is a* ***semantic-preserving syntax transformation operator*** *(see* Definition 3.3.3*.2) wrt* $\operatorname{supp} \mathcal{X}$*, then*

   $$\mathcal{X} \equiv \bigcup \left\{ \bigcup \left\{ [y, \ell_x] \,\middle|\, y \in \mathscr{T}(x) \right\} \,\middle|\, x \in \operatorname{supp} \mathcal{X} \right\} \tag{4.71}$$

Proof
ad (4.67)**.** It is sufficient to prove

$$\mathcal{X}_x \equiv \mathcal{X} \cap [x], \tag{4.72}$$

because obviously, $\mathcal{X} = \mathcal{X} \cup \big(\mathcal{X} \cap [x]\big)$, hence (4.67) follows from Observation 4.2.5.4.

For establishing (4.72), by Definition 4.2.3 and (4.41) it is sufficient to prove that for every $\mathrm{Val} \in \mathfrak{S}$ and $d \in D$,

$$\mathrm{Val} \models_{d} \mathcal{X}_x \;\text{ iff }\; \mathrm{Val} \models_{d} \mathcal{X} \cap [x], \tag{4.73}$$

which, by (4.38), is equivalent with

$$\boxed{\mathrm{D}} \left\{ d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \models_{d'} [y, \mathcal{X}_x(y)] \right\} = \boxed{\mathrm{D}} \left\{ d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \models_{d'} \Big[y, \big(\mathcal{X} \cap [x]\big)(y)\Big] \right\} \tag{4.74}$$

By definition, $\mathcal{X}_x(y) = \big(\mathcal{X} \cap [x]\big)(y) = 0$ for every $y \in \mathrm{Frm}$ such that $y \notin [x]$. Furthermore, for every $y \in [x]$, $\mathcal{X}_x(y) = \ell_x$, so defining d'' such that $\mathrm{Val} \models_{d''} [y, \ell_x]$, it is sufficient to prove

$$d'' = \boxed{\mathrm{D}} \left\{ d' \,\middle|\, y \in [x] \text{ and } \mathrm{Val} \models_{d'} [y, \mathcal{X}(y)] \right\} \tag{4.75}$$

This equality follows immediately from Proposition 4.1.1.7 and the definition (4.65) of ℓ_x.

ad (4.68)**.** Is proved exactly as (4.67) after observing that $\mathcal{X} = \big(\mathcal{X} \setminus [x]\big) \cup \big(\mathcal{X} \cap [x]\big)$.

ad (4.69)**.** First of all, observe that

$$\mathcal{X} = \bigcup \big\{ \mathcal{X} \cap [x] \,\big|\, x \in \operatorname{supp} \mathcal{X} \big\} . \tag{4.76}$$

That $\mathcal{X} \cap [x] \equiv \mathcal{X}_x$ for every $x \in \operatorname{supp} \mathcal{X}$ has been established in the proof of equation (4.67) (compare (4.72)).

Hence, for $\big\{ \mathcal{X} \cap [x] \,\big|\, x \in \operatorname{supp} \mathcal{X} \big\}$ and $\{ \mathcal{X}_x \,|\, x \in \operatorname{supp} \mathcal{X} \}$, assumption (4.56) of Observation 4.2.5.5 is fulfilled. Thus,

$$\bigcup \big\{ \mathcal{X} \cap [x] \,\big|\, x \in \operatorname{supp} \mathcal{X} \big\} \equiv \bigcup \{ \mathcal{X}_x \,|\, x \in \operatorname{supp} \mathcal{X} \}$$

follows from Observation 4.2.5.5, establishing the result.

ad (4.70)**.** By the definition of semantic covering, N contains at least one representative from each equivalence class in $(\operatorname{supp} \mathcal{X})_{\equiv}$. It shall be established that for every $x \in \mathrm{Frm}$,

$$[x, \ell_x] \equiv \mathcal{X}_x . \tag{4.77}$$

This means that assumption (4.56) of Observation 4.2.5.5 is fulfilled for $\big\{ [x, \ell_x] \,\big|\, x \in N \big\}$ and $\{ \mathcal{X}_x \,|\, x \in \operatorname{supp} \mathcal{X} \}$, hence the claim follows from (4.69) and Observation 4.2.5.5.

For establishing (4.77), it is sufficient to prove

$$\mathrm{Mod}\big([x, \ell_x]\big) = \mathrm{Mod}\,(\mathcal{X}_x) ,$$

i. e. for every $\mathrm{Val} \in \mathfrak{S}$ and $d \in D$,

$$\mathrm{Val} \models_{d} [x, \ell_x] \;\text{ iff }\; d = \boxed{\mathbb{D}} \Big\{ d' \,\Big|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \models_{d'} \big[y, \mathcal{X}_x(y)\big] \Big\} .$$

Let $d \in D$ be given such that $\mathrm{Val} \models_{d} [x, \ell_x]$. The equality is proved in two steps.

1. $d \sqsubseteq_{\mathbb{D}} \boxed{\mathbb{D}} \Big\{ d' \,\Big|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \models_{d'} \big[y, \mathcal{X}_x(y)\big] \Big\}$.

   It is sufficient to prove that for every $y \in \mathrm{Frm}$ and $d' \in D$ such that $\mathrm{Val} \models_{d'} \big[y, \mathcal{X}_x(y)\big]$, $d \sqsubseteq_{\mathbb{D}} d'$. Distinguish two cases.

   Case 1. $y \notin [x]$.
   In this case, $\mathcal{X}_x(y) = 0$ by the definition (4.66) of $\mathcal{X}_x$. From (4.2), it follows that $d' = 1$, which implies $d \sqsubseteq_{\mathbb{D}} d'$.

   Case 2. $y \in [x]$, i. e. $y \equiv x$.
   In this case, $\mathcal{X}_x(y) = \ell_x$ by the definition (4.66) of $\mathcal{X}_x$, hence $[x, \ell_x] \equiv \big[y, \mathcal{X}_x(y)\big]$ by (4.54). It follows that $d' = d$, which implies $d \sqsubseteq_{\mathbb{D}} d'$.

2. $\boxed{\mathbb{D}} \Big\{ d' \,\Big|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \models_{d'} \big[y, \mathcal{X}_x(y)\big] \Big\} \sqsubseteq_{\mathbb{D}} d$.

   It is sufficient to prove $d \in \Big\{ d' \,\Big|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \models_{d'} \big[y, \mathcal{X}_x(y)\big] \Big\}$, which follows immediately from the fact that $x \in [x]$ and the definition (4.66) of $\mathcal{X}_x$.

This concludes the proof of equation (4.70).

ad (4.71)**.** Obviously, by (4.69) and (4.77),

$$\mathcal{X} \equiv \bigcup \left\{ [x, \ell_x] \,\middle|\, x \in \operatorname{supp} \mathcal{X} \right\} . \tag{4.78}$$

Thus, by Observation 4.2.5.5 it is sufficient to prove

$$[x, \ell_x] \equiv \bigcup \left\{ [y, \ell_x] \,\middle|\, y \in \mathscr{T}(x) \right\} , \tag{4.79}$$

which follows immediately from Theorem 4.2.6.2, taking into account that the finiteness of $\mathscr{T}(x)$ and $x \equiv \mathscr{T}(x)$ follow from the definition of $\mathscr{T}$. □

Observations 4.2.9 (Replacements and normal forms)

1. In fact, (4.59) is a much weakened version of (4.67).
2. With respect to (4.67) and (4.68), it is easily observed that for every $y \in [x]$, $\mathcal{X}(y) \sqsubseteq \ell_x$, thus even $\mathcal{X} \cup \mathcal{X}_x = \big(\mathcal{X} \setminus [x]\big) \cup \mathcal{X}_x$.
3. Observe that the definitions of ℓ_x and $\mathcal{X}_x$ are independent from the representative x of the equivalence class $[x]$, so it would be justified to denote ℓ_x by $\ell_{[x]}$ and $\mathcal{X}_x$ by $\mathcal{X}_{[x]}$.

   With respect to (4.69), it is easily observed that

   $$\{\mathcal{X}_x \,|\, x \in \mathrm{Frm}\} = \left\{ \mathcal{X}_{[x]} \,\middle|\, [x] \in \mathrm{Frm}_{\equiv} \right\} \tag{4.80}$$

4. It follows from (4.70) that it is sufficient to select *one* representative x from each equivalence class $[x] \in \mathrm{Frm}_{\equiv}$. N would then be the set of *all* these representatives, preferably in some normal form. $\mathcal{X}$ can be *compressed* by attaching the label ℓ_x to every $x \in N$ and 0 to every formula not in N.

   This can mean a drastic reduction in the *size* of $\mathcal{X}$, i. e. the cardinality of $\operatorname{supp} \mathcal{X}$. •

Theorem 4.2.6, Corollary 4.2.7, and Theorem 4.2.8 are very convenient for all kinds of semantically equivalent transformations and manipulations on an $\mathfrak{L}$-fuzzy set of formulae. The result of Theorem 4.2.8.3 in particular allows an $\mathfrak{L}$-fuzzy set of formulae to be transformed into some kind of **normal form**.

As an application of these results, it is demonstrated that an $\mathfrak{L}$-fuzzy set of formulae from a propositional logic based on the lattice connectives (compare Example 3.2.3) can be transformed into **clausal form**.

Example 4.2.1 (Conjunctive normal form and clausal form)
For the extent of this example, fix a set PV of propositional variables and $\mathrm{Frm} =_{\mathrm{def}} \mathrm{PFrm}_{\mathrm{S}}$ (see Example 3.1.3).

For the interpretation of the logical operators, let $\varphi_\wedge =_{\mathrm{def}} \sqcap$ and $\varphi_\vee =_{\mathrm{def}} \sqcup$ as described in Example 3.2.3.

Fix $\mathfrak{T}$ to be a **De Morgan algebra** wrt. $\varphi_\neg$. This means that $\mathfrak{T}$ is a lattice which is **distributive**, i. e. for all $t_1, t_2, t_3 \in T$,

$$t_1 \sqcap (t_2 \sqcup t_3) \equiv (t_1 \sqcap t_2) \sqcup (t_1 \sqcap t_3) \, , \tag{4.81}$$

$$t_1 \sqcup (t_2 \sqcap t_3) \equiv (t_1 \sqcup t_2) \sqcap (t_1 \sqcup t_3) \, , \tag{4.82}$$

and furthermore, De Morgan's laws hold for all $t_1, t_2 \in T$:

$$\varphi_\neg\,(t_1 \sqcup t_2) = \varphi_\neg(t_1) \sqcap \varphi_\neg(t_2), \tag{4.83}$$

$$\varphi_\neg\,(t_1 \sqcap t_2) = \varphi_\neg(t_1) \sqcup \varphi_\neg(t_2). \tag{4.84}$$

In addition, assume that $\varphi_\neg$ is **involutive**, i. e. for all $t \in T$,

$$\varphi_\neg\,\big(\varphi_\neg\,(t)\big) = t. \tag{4.85}$$

Remark
Note that from (4.83) and (4.85) it follows that $\varphi_\neg$ is an order-reversing bijection as demanded in Example 3.2.3.

On the other hand, if $\varphi_\neg$ is an order-reversing bijection, it fulfils (4.83) and (4.84), but not necessarily (4.85). •

Assume $\varphi_\rightarrow$ to be the **s-implication** of $\boxed{\scriptstyle\top}$ wrt. $\varphi_\neg$, i.e. for $s, t \in T$:

(4.86) $$\varphi_\rightarrow(s,t) =_{\text{def}} \varphi_\neg(s) \boxed{\scriptstyle\top}\, t.$$

Let $\mathfrak{S}$ be the semantics given by (3.3) for this interpretation of the logical operators.

$\mathfrak{D}$ may be chosen arbitrarily; let $\mathfrak{L}, \models$ be chosen such that $\left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\right]$ is a **logic of graded truth and graded trust assessment**.

To simplify notation in the following, some notational conventions are introduced.

Define the set Lit of all **literals** to be the set of all propositional variables and negations of propositional variables:

(4.87) $$\mathrm{Lit} =_{\text{def}} \mathrm{PV} \cup \{\neg p | p \in \mathrm{PV}\}\,.$$

For later considerations concerning **refutation**, the **complement** of a literal $l \in \mathrm{Lit}$ (denoted $\bar{l}$) is defined by

(4.88) $$\bar{l} =_{\text{def}} \begin{cases} \neg l & \text{if } l \in \mathrm{PV} \\ p & \text{if } l = \neg p \text{ and } p \in \mathrm{PV} \end{cases}$$

Because of the associativity of the functions $\varphi_\wedge$, $\varphi_\vee$, parentheses are left out when denoting multiple conjunctions or disjunctions. That is, given $n \in \mathbb{N}, n \geqq 1$ and $x_1, \ldots, x_n \in \mathrm{PFrm}_{\mathrm{S}}$,

(4.89) $$\text{write } (x_1 \wedge x_2 \wedge \cdots \wedge x_n) \text{ or } \bigwedge_{i=1}^{n} x_i \text{ instead of } \Big(\big(\ldots(x_1 \wedge x_2) \wedge \ldots\big) \wedge x_n\Big)$$

(4.90) $$\text{write } (x_1 \vee x_2 \vee \cdots \vee x_n) \text{ or } \bigvee_{i=1}^{n} x_i \text{ instead of } \Big(\big(\ldots(x_1 \vee x_2) \vee \ldots\big) \vee x_n\Big)$$

Definition 4.2.5 (Clauses, conjunctive normal form)
Let $x \in \mathrm{Frm}$.

1. *x is said to be a* ***clause***
   $=_{\text{def}}$ *x is a disjunction of literals, i.e. there exists $n \in \mathbb{N}, n \geqq 1$ and there exist literals $l_1, \ldots, l_n \in \mathrm{Lit}$ such that*
   $$x = \bigvee_{i=1}^{n} l_i.$$
   The set of all clauses in Frm *is denoted by* Cls.

2. *x is said to be in* ***conjunctive normal form***
   $=_{\text{def}}$ *x is a conjunction of clauses, i.e. there exists $n \in \mathbb{N}, n \geqq 1$ and there exist clauses $c_1, \ldots, c_n \in \mathrm{Cls}$ such that*
   $$x = \bigwedge_{i=1}^{n} c_i.$$
   The set of all formulae in Frm *which are in conjunctive normal form is denoted by* Cnf.

Remark
As $\lor$ is commutative and **idempotent** (i. e. $(x \lor x) \equiv x$), it is convenient (see especially section 5.4) to identify a **clause** with the *set of all literals* occurring in it.

Consequently, in the sequel a clause

$$\bigvee_{i=1}^{n} l_i \quad \text{is identified with the set} \quad \{l_1, \ldots, l_n\},$$

with the obvious extension of valuations from $\mathfrak{S}$ to sets of literals. •

Proposition 4.2.10 (Transforming into clausal form)

1. *Let* $x \in \mathrm{Frm}$. *Then there exists* $x_{\mathrm{Cnf}} \in \mathrm{Cnf}$ *such that* $x \equiv x_{\mathrm{Cnf}}$.
2. Cnf *is a* ***semantic covering*** *of* Frm.
3. *Given* $x \in \mathrm{Frm}$, *let* $n_x \in \mathbb{N}$ *and* $c_{x,1}, \ldots, c_{x,n_x} \in \mathrm{Cls}$ *such that*
$$x_{\mathrm{Cnf}} = \bigwedge_{i=1}^{n_x} c_{x,i}.$$
Then defining $\mathscr{T}_{\mathrm{Cls}} : \mathrm{Frm} \to \mathfrak{P}\mathrm{Frm}$ *by*
$$\mathscr{T}_{\mathrm{Cls}}(x) =_{\mathrm{def}} \{c_{x,1}, \ldots, c_{x,n_x}\},$$
$\mathscr{T}_{\mathrm{Cls}}$ *is a* ***semantic-preserving syntax transformation operator*** *wrt* Frm.
4. *For every* $\mathcal{X} \in L^{\mathrm{Frm}}$, *there exists* $\mathcal{X}_{\mathrm{Cnf}} \in L^{\mathrm{Cnf}}$ *such that* $\mathcal{X} \equiv \mathcal{X}_{\mathrm{Cnf}}$.
5. *For every* $\mathcal{X} \in L^{\mathrm{Frm}}$, *there exists* $\mathcal{X}_{\mathrm{Cls}} \in L^{\mathrm{Cls}}$ *such that* $\mathcal{X} \equiv \mathcal{X}_{\mathrm{Cls}}$.

Proof

ad 1. Straightforward induction on the structure of x, using the following equivalences implied by (4.81)–(4.86) (for $x_1, x_2, x_3 \in \mathrm{Frm}$)

$$(x_1 \to x_2) \equiv (\neg x_1 \lor x_2) \qquad \text{(elimination of } \to) \tag{4.91}$$
$$\big(x_1 \land (x_2 \lor x_3)\big) \equiv \big((x_1 \land x_2) \lor (x_1 \land x_3)\big), \tag{4.92}$$
$$\big(x_1 \lor (x_2 \land x_3)\big) \equiv \big((x_1 \lor x_2) \land (x_1 \lor x_3)\big), \qquad \text{(distributive laws)} \tag{4.93}$$
$$\neg(x_1 \lor x_2) \equiv (\neg x_1 \land \neg x_2) \tag{4.94}$$
$$\neg(x_1 \land x_2) \equiv (\neg x_1 \lor \neg x_2) \qquad \text{(De Morgan's laws)} \tag{4.95}$$
$$\neg\neg x_1 \equiv x_1 \qquad \text{(involution)} \tag{4.96}$$

ad 2. Obvious by the previous item and the definition of semantic covering.

ad 3. Follows from the previous item and the fact that by definition,

$$\{c_1, \ldots, c_n\} \equiv \left\{\bigwedge_{i=1}^{n} c_i\right\}.$$

ad 4. Follows from item 2 and Theorem 4.2.8.3.

ad 5. Follows from item 3 and Theorem 4.2.8.4. □

This concludes the example. It has been demonstrated that a **normal form** which exists for the underlying many-valued logic can be *transferred* to the corresponding labelled logic. Observe that this result can be reproduced for (almost) arbitrary normal form results on the underlying logic by means of the strong results in Theorem 4.2.8. More general results will be presented in section 6.2.1.2. •

4.3 Semantic Consequences

In this section, a **semantic consequence relation** (or **semantic entailment relation**) $\Vdash$ is defined, with the intention is that this relation allows to determine whether a labelled formula *follows* from an $\mathfrak{L}$-fuzzy set of formulae. Furthermore, an $\mathfrak{L}$-fuzzy set $\mathrm{Cons}(\mathcal{X})$ of **consequences** of an $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae is defined. In this fuzzy set, every formula assumes the *supremum* of all labels with which it is a semantic consequence of $\mathcal{X}$. Taking into account that labels are ordered by strength, this means that the label of a formula x in $\mathrm{Cons}(\mathcal{X})$ allows to estimate the *greatest strength* with which x is a consequence of $\mathcal{X}$. Cons is a **fuzzy logical operator** on Frm. In the remainder of this section, properties and applications of semantic consequence are studied.

4.3.1 Basic Definitions and Properties

Definition 4.3.1 (Semantic consequence)
Let an $\mathfrak{L}$-fuzzy set $\mathcal{X} : \mathrm{Frm} \to L$ be given.

1. *Given an $\mathfrak{L}$-labelled formula $\mathfrak{x}$, it is said that $\mathcal{X}$* ***entails*** *$\mathfrak{x}$,*

   (4.97) $$\mathcal{X} \Vdash \mathfrak{x} \;=_{\mathrm{def}}\; \mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}(\mathfrak{x}).$$

2. *The* ***$\mathfrak{L}$-fuzzy set of consequences of*** *$\mathcal{X}$ is defined by*

   (4.98) $$\mathrm{Cons}(\mathcal{X}) =_{\mathrm{def}} \bigcup \left\{\mathfrak{x} \middle| \mathfrak{x} \in \mathrm{LFrm} \text{ and } \mathcal{X} \Vdash \mathfrak{x}\right\}.$$

From this definition and the properties of Mod established in section 4.2, some properties of $\Vdash$ and Cons can be derived.

Propositions 4.3.1 (Properties of $\Vdash$ and Cons)
Let $\mathcal{X}, \mathcal{Y} : \mathrm{Frm} \to L$, $x, y \in \mathrm{Frm}$ and $\ell, \ell' \in L$.

1. *If $\mathcal{X} \Vdash [x, \ell]$ and $[x, \ell] \Vdash [y, \ell']$, then $\mathcal{X} \Vdash [y, \ell']$.*
2. *If $x \equiv y$, then*

   (4.99) $$\forall \ell \in L : \mathcal{X} \Vdash [x, \ell] \text{ iff } \mathcal{X} \Vdash [y, \ell],$$

   (4.100) $$\mathrm{Cons}(\mathcal{X})(x) = \mathrm{Cons}(\mathcal{X})(y).$$

3. $\mathcal{X} \equiv \mathrm{Cons}(\mathcal{X})$.
4. *$\mathcal{X} \equiv \mathcal{Y}$ if and only if $\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}(\mathcal{Y})$.*
5. *If $\mathcal{X} \subseteqq \mathcal{Y}$ and $\mathcal{X} \Vdash [x, \ell]$, then $\mathcal{Y} \Vdash [x, \ell]$.*
6. *If $\ell' \sqsubseteq \ell$ and $\mathcal{X} \Vdash [x, \ell]$, then $\mathcal{X} \Vdash [x, \ell']$.*
7. *If for every $\mathrm{Val} \in \mathfrak{S}$,*

   (4.101) $$\mathrm{Val}(x) \sqsubseteq \mathrm{Val}(y)$$

   holds, then

   7.1. *if $\mathcal{X} \Vdash [x, \ell]$, then $\mathcal{X} \Vdash [y, \ell]$;*
   7.2. $\mathrm{Cons}(\mathcal{X})(x) \sqsubseteq \mathrm{Cons}(\mathcal{X})(y)$.

8. $\mathcal{X} \Vdash [x, \mathrm{Cons}(\mathcal{X})(x)]$.
9. *$\mathcal{X} \Vdash [x, \ell]$ iff $\ell \sqsubseteq \mathrm{Cons}(\mathcal{X})(x)$.*

Proof

ad 1. Immediate by the definition of $\Vdash$ and the fact that $\subseteqq$ is *transitive*.

ad (4.99)**.** Follows immediately from the definition of $\Vdash$ by applying (4.54).

ad (4.100)**.** Follows from the previous item by definition of Cons.

ad 3. It is to be proved that

$$\text{Mod}(\mathcal{X}) = \text{Mod}\left(\text{Cons}(\mathcal{X})\right). \tag{4.102}$$

By definition,

$$\text{Mod}\left(\text{Cons}(\mathcal{X})\right) = \text{Mod}\left(\bigcup\left\{\mathfrak{x}\,\middle|\,\mathfrak{x} \in \text{LFrm and } \mathcal{X} \Vdash \mathfrak{x}\right\}\right),$$

hence, by (4.45), for establishing (4.102) it is sufficient to prove

$$\text{Mod}(\mathcal{X}) = \bigcap\left\{\text{Mod}(\mathfrak{x})\,\middle|\,\mathfrak{x} \in \text{LFrm and } \mathcal{X} \Vdash \mathfrak{x}\right\},$$

i. e., by definition of $\Vdash$,

$$\text{Mod}(\mathcal{X}) = \bigcap\left\{\text{Mod}(\mathfrak{x})\,\middle|\,\mathfrak{x} \in \text{LFrm and } \text{Mod}(\mathcal{X}) \subseteqq \text{Mod}(\mathfrak{x})\right\}. \tag{4.103}$$

Let

$$M =_{\text{def}} \left\{\text{Mod}(\mathfrak{x})\,\middle|\,\mathfrak{x} \in \text{LFrm and } \text{Mod}(\mathcal{X}) \subseteqq \text{Mod}(\mathfrak{x})\right\} \tag{4.104}$$

(4.103) is proved in two steps.

(i) $\text{Mod}(\mathcal{X}) \subseteqq \bigcap M$.
Follows immediately from the fact that by definition of M, $\text{Mod}(\mathcal{X}) \subseteqq \text{Mod}(\mathfrak{x})$ holds for every $\mathfrak{x} \in \text{LFrm}$ such that $\text{Mod}(\mathfrak{x}) \in M$.

(ii) $\bigcap M \subseteqq \text{Mod}(\mathcal{X})$.
Start out with the (trivial) observation that

$$\mathcal{X} = \bigcup\left\{\left[x, \mathcal{X}(x)\right]\,\middle|\,x \in \text{Frm}\right\},$$

hence, by (4.45),

$$\text{Mod}(\mathcal{X}) = \bigcap\left\{\text{Mod}\left(\left[x, \mathcal{X}(x)\right]\right)\,\middle|\,x \in \text{Frm}\right\}.$$

It is thus sufficient to prove

$$\bigcap M \subseteqq \bigcap\left\{\text{Mod}\left(\left[x, \mathcal{X}(x)\right]\right)\,\middle|\,x \in \text{Frm}\right\},$$

which follows if for every $x \in \text{Frm}$,

$$\text{Mod}\left(\left[x, \mathcal{X}(x)\right]\right) \in M. \tag{4.105}$$

For this, it is by definition (4.104) of M sufficient to prove that

$$\text{Mod}(\mathcal{X}) \subseteqq \text{Mod}\left(\left[x, \mathcal{X}(x)\right]\right), \tag{4.106}$$

which follows by (4.43) from the fact that $\left[x, \mathcal{X}(x)\right] \subseteqq \mathcal{X}$.

This concludes the proof of this item.

ad 4. For the "if" direction, assume $\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}(\mathcal{Y})$. From the previous item, it follows that

$$\mathcal{X} \equiv \mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}(\mathcal{Y}) \equiv \mathcal{Y}.$$

The "only if" direction follows by applying the definition of $\equiv$ to (4.98) via (4.97).

ad 5. Assume

$$\mathcal{X} \subseteqq \mathcal{Y} \tag{4.107}$$

and

$$\mathcal{X} \Vdash [x, \ell] \tag{4.108}$$

hold.

To prove $\mathcal{Y} \Vdash [x, \ell]$ means to establish

$$\mathrm{Mod}(\mathcal{Y}) \subseteqq \mathrm{Mod}\left([x, \ell]\right). \tag{4.109}$$

(4.108) means by definition

$$\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}\left([x, \ell]\right). \tag{4.110}$$

From (4.107) and (4.43) it follows that $\mathrm{Mod}(\mathcal{Y}) \subseteqq \mathrm{Mod}(\mathcal{X})$, hence (4.109) follows immediately.

ad 6. From $\ell' \sqsubseteq \ell$, it follows that $[x, \ell] \subseteqq \left[x, \ell'\right]$, hence this item can be proved analogously to the previous one.

ad 7.1. From assumption (4.101), it follows by Observation 4.2.3.1 that

$$\mathrm{Mod}\left([x, \ell]\right) \subseteqq \mathrm{Mod}\left([y, \ell]\right)$$

and $\mathcal{X} \Vdash [x, \ell]$ means $\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}\left([x, \ell]\right)$, hence $\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}\left([y, \ell]\right)$ and thus $\mathcal{X} \Vdash [y, \ell]$ follows immediately.

ad 7.2. From the previous item, it follows that

$$\left\{[x, \ell] \,\middle|\, \ell \in L \text{ and } \mathcal{X} \Vdash [x, \ell]\right\} \subseteqq \left\{[y, \ell] \,\middle|\, \ell \in L \text{ and } \mathcal{X} \Vdash [y, \ell]\right\},$$

from which $\mathrm{Cons}(\mathcal{X})(x) \sqsubseteq \mathrm{Cons}(\mathcal{X})(y)$ follows immediately by (4.98).

ad 8. It is to be proved that

$$\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}\Big(\big[x, \mathrm{Cons}(\mathcal{X})(x)\big]\Big).$$

By item 3, $\mathrm{Mod}(\mathcal{X}) = \mathrm{Mod}\left(\mathrm{Cons}(\mathcal{X})\right)$, hence it is sufficient to prove

$$\mathrm{Mod}\left(\mathrm{Cons}(\mathcal{X})\right) \subseteqq \mathrm{Mod}\Big(\big[x, \mathrm{Cons}(\mathcal{X})(x)\big]\Big),$$

which follows by (4.43) from the fact that $\big[x, \mathrm{Cons}(\mathcal{X})(x)\big] \subseteqq \mathrm{Cons}(\mathcal{X})$.

ad 9. Two implications are proved separately:

(i) If $\mathcal{X} \Vdash [x, \ell]$, then $\ell \sqsubseteq \mathrm{Cons}(\mathcal{X})(x)$.
From $\mathcal{X} \Vdash [x, \ell]$ it follows that $\ell \in \left\{\ell'' \middle| \ell'' \in L \text{ and } \mathcal{X} \Vdash \left[x, \ell''\right]\right\}$, and thus trivially $\ell \sqsubseteq \bigsqcup \left\{\ell'' \middle| \ell'' \in L \text{ and } \mathcal{X} \Vdash \left[x, \ell''\right]\right\} = \mathrm{Cons}(\mathcal{X})(x)$.

(ii) If $\ell \sqsubseteq \mathrm{Cons}(\mathcal{X})(x)$, then $\mathcal{X} \Vdash [x, \ell]$.
Follows immediately from item 6, taking into account that $\mathcal{X} \Vdash \left[x, \mathrm{Cons}(\mathcal{X})(x)\right]$ by item 8. □

To justify the above definitions of **semantic consequence**, it is proved that Cons has the important property of being a **fuzzy closure operator** on Frm.

Theorem 4.3.2 (Cons is a fuzzy closure operator on Frm)
Cons *is an* ***$\mathfrak{L}$-fuzzy closure operator*** *on* Frm, *i. e. for all* $\mathcal{X}, \mathcal{Y} : \mathrm{Frm} \to L$,

1. Cons *is* ***embedding***:

$$\mathcal{X} \subseteqq \mathrm{Cons}(\mathcal{X})$$

2. Cons *is* ***closed***:

$$\mathrm{Cons}(\mathrm{Cons}(\mathcal{X})) \subseteqq \mathrm{Cons}(\mathcal{X})$$

3. Cons *is* ***monotone***:

$$\textit{If } \mathcal{X} \subseteqq \mathcal{Y}\,,\ \textit{then } \mathrm{Cons}(\mathcal{X}) \subseteqq \mathrm{Cons}(\mathcal{Y})$$

Proof
ad 1. It suffices to show that for every $x \in \mathrm{Frm}$,

$$\mathcal{X}(x) \sqsubseteq \mathrm{Cons}(\mathcal{X})(x).$$

By Proposition 4.3.1.9, it suffices to show that

$$\mathcal{X} \Vdash \left[x, \mathcal{X}(x)\right].$$

By definition, this means that

$$\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}\left(\left[x, \mathcal{X}(x)\right]\right),$$

which follows immediately from (4.43) by the fact that $\left[x, \mathcal{X}(x)\right] \subseteqq \mathcal{X}$.

ad 2. Follows immediately from Proposition 4.3.1.4 and the fact that $\mathcal{X} \equiv \mathrm{Cons}(\mathcal{X})$ by Proposition 4.3.1.3.

ad 3. Let $\mathcal{X}, \mathcal{Y} : \mathrm{Frm} \to L$ such that $\mathcal{X} \subseteqq \mathcal{Y}$. For establishing $\mathrm{Cons}(\mathcal{X}) \subseteqq \mathrm{Cons}(\mathcal{Y})$, i. e.

$$\bigcup \left\{\mathfrak{x} \middle| \mathfrak{x} \in \mathrm{LFrm} \text{ and } \mathcal{X} \Vdash \mathfrak{x}\right\} \subseteqq \bigcup \left\{\mathfrak{y} \middle| \mathfrak{y} \in \mathrm{LFrm} \text{ and } \mathcal{Y} \Vdash \mathfrak{y}\right\},$$

it is sufficient to prove that for every $\mathfrak{x} \in \mathrm{LFrm}$ such that $\mathcal{X} \Vdash \mathfrak{x}$, it holds that $\mathcal{Y} \Vdash \mathfrak{x}$.
But this follows from Proposition 4.3.1.6. □

There is another characterisation of Cons which is analogous to PAVELKA's [85] *definition* of Cons (definition (4.98) is analogous to PAVELKA's observation 5).

Theorem 4.3.3 (Alternative definition of Cons)
For every $\mathfrak{L}$-fuzzy set $\mathcal{X} : \mathrm{Frm} \to L$ and every $x \in \mathrm{Frm}$,

$$\mathrm{Cons}(\mathcal{X})(x) = \bigsqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\} \tag{4.111}$$

Proof
Let $\mathfrak{L}' = \left[L', \dot\cup, \cap\right]$ be the sublattice of the dual lattice $\left[\mathfrak{D}\text{-}\mathrm{Fl}\left(\mathfrak{T}\right), \dot\cup, \cap\right]$ of the complete lattice $\left[\mathfrak{D}\text{-}\mathrm{Fl}\left(\mathfrak{T}\right), \cap, \dot\cup\right]$ such that ι is an isomorphism between $\mathfrak{L}$ and $\mathfrak{L}'$ (see Definition 2.3.1 and Theorem 2.2.1).

First of all, observe that

$$\bigsqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\} \in L$$

because by item 1 of Definition 2.3.1, for all $d \in D$ and $t \in T$, ${}^{d}\overline{t} \in L'$, and because of the completeness of $\mathfrak{L}$.

The claimed equation is proved in two steps.

1. $\mathrm{Cons}(\mathcal{X})(x) \sqsubseteq_{\mathfrak{L}} \bigsqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\}$.

   By the definition of Cons, it is to be proved that

$$\bigsqcup_{\mathfrak{L}} \left\{ \ell \middle| \ell \in L \text{ and } \mathcal{X} \Vdash [x, \ell] \right\} \sqsubseteq_{\mathfrak{L}} \bigsqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\}.$$

   It suffices to show that for every $\ell \in L$ such that $\mathcal{X} \Vdash [x, \ell]$ and every $\mathrm{Val} \in \mathfrak{S}$,

$$\ell \sqsubseteq_{\mathfrak{L}} \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right).$$

   Because ι is an isomorphism onto $\mathfrak{L}'$, this claim is equivalent with

$${}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \subseteq \iota(\ell).$$

   (The order is reversed because $\mathfrak{L}'$ is a sublattice of the *dual* of $\mathfrak{D}\text{-}\mathrm{Fl}\left(\mathfrak{T}\right)$.)

   By Lemma 2.1.8, this is equivalent with

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(\mathrm{Val}(x)). \tag{4.112}$$

   Now, let

$$d =_{\mathrm{def}} \iota(\ell)(\mathrm{Val}(x)).$$

   By (4.1), this means

$$\mathrm{Val} \models_{d} [x, \ell],$$

   thus from $\mathcal{X} \Vdash [x, \ell]$ and Definition 4.3.1.1 it follows that

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} d,$$

   which proves (4.112).

2. $\sqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\} \sqsubseteq_{\mathfrak{L}} \mathrm{Cons}(\mathcal{X})(x)$.

By the definition of Cons, it is to be proved that

$$\sqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\} \sqsubseteq_{\mathfrak{L}} \sqcup_{\mathfrak{L}} \left\{ \ell \middle| \ell \in L \text{ and } \mathcal{X} \Vdash [x, \ell] \right\}. \tag{4.113}$$

Let

$$\ell' =_{\mathrm{def}} \sqcap_{\mathfrak{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\} \tag{4.114}$$

It follows that for every $\mathrm{Val} \in \mathfrak{S}$,

$$\ell' \sqsubseteq_{\mathfrak{L}} \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right)$$

and because ι is an isomorphism onto $\mathfrak{L}'$, it follows that

$${}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \subseteq \iota(\ell').$$

By Lemma 2.1.8, this is equivalent with

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} \iota(\ell')(\mathrm{Val}(x)).$$

Now, let

$$d' =_{\mathrm{def}} \iota(\ell')(\mathrm{Val}(x)).$$

By (4.1), this means $\mathrm{Val} \models_{d'} [x, \ell']$, hence $\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} \mathrm{Mod}\left([x, \ell']\right)(\mathrm{Val})$.

As the above holds for every $\mathrm{Val} \in \mathfrak{S}$, Definition 4.3.1.1 yields

$$\mathcal{X} \Vdash [x, \ell'],$$

thus

$$\ell' \in \left\{ \ell \middle| \ell \in L \text{ and } \mathcal{X} \Vdash [x, \ell] \right\},$$

and it follows

$$\ell' \sqsubseteq_{\mathfrak{L}} \sqcup_{\mathfrak{L}} \left\{ \ell \middle| \ell \in L \text{ and } \mathcal{X} \Vdash [x, \ell] \right\},$$

which yields (4.113) by (4.114). □

In both characterisations of Cons, i. e. (4.98) (via (4.97)) and (4.111), the value of $\mathrm{Cons}(\mathcal{X})$ is completely determined by the **model fuzzy set** $\mathrm{Mod}(\mathcal{X})$ of $\mathcal{X}$. By an analogous definition, of course *every* $\mathfrak{D}$-fuzzy set on $\mathfrak{S}$ (not only $\mathrm{Mod}(\mathcal{X})$) induces an $\mathfrak{L}$-fuzzy set on Frm. Prompted by this observation, semantic consequences of a $\mathfrak{D}$-fuzzy set $\mathcal{S}$ on $\mathfrak{S}$ are defined, a more general definition than that of semantic consequences of a fuzzy set of formulae.

Definition 4.3.2 (Semantic consequences of a fuzzy set of valuations)
Let a $\mathfrak{D}$-fuzzy set $\mathcal{S} : \mathfrak{S} \to D$ be given.

1. *Given an $\mathfrak{L}$-labelled formula $\mathfrak{x}$, it is said that $\mathcal{S}$ **entails** $\mathfrak{x}$,*

$$\mathcal{S} \Vdash \mathfrak{x} =_{\mathrm{def}} \mathcal{S} \subseteqq \mathrm{Mod}(\mathfrak{x}). \tag{4.115}$$

2. *The **$\mathfrak{L}$-fuzzy set of consequences of** $\mathcal{S}$ is defined by*

$$\mathrm{Cons}(\mathcal{S}) =_{\mathrm{def}} \bigcup \left\{ \mathfrak{x} \middle| \mathfrak{x} \in \mathrm{LFrm} \ \textit{and} \ \mathcal{S} \Vdash \mathfrak{x} \right\}. \tag{4.116}$$

Remarks

1. Of course, by the above definition, for $\mathcal{X} : \mathrm{Frm} \to L$ and $\mathfrak{x} \in \mathrm{LFrm}$,

$$\mathcal{X} \Vdash \mathfrak{x} \text{ iff } \mathrm{Mod}(\mathcal{X}) \Vdash \mathfrak{x} \tag{4.117}$$

and

$$\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}(\mathrm{Mod}(\mathcal{X})) \tag{4.118}$$

holds.

(4.117) corresponds to Corollary 3.2.3 in [19].

2. As in Proposition 4.3.1,

$$\text{If } \mathcal{S} \Vdash [x, \ell] \text{ and } [x, \ell] \Vdash [y, \ell'] \text{, then } \mathcal{S} \Vdash [y, \ell']. \tag{4.119}$$
$$\text{If } \ell' \sqsubseteq_{\mathbb{L}} \ell \text{ and } \mathcal{S} \Vdash [x, \ell] \text{, then } \mathcal{S} \Vdash [x, \ell']. \tag{4.120}$$
$$\mathcal{S} \Vdash [x, \mathrm{Cons}(\mathcal{S})(x)]. \tag{4.121}$$
$$\mathcal{S} \Vdash [x, \ell] \text{ iff } \ell \sqsubseteq_{\mathbb{L}} \mathrm{Cons}(\mathcal{S})(x). \tag{4.122}$$

3. The proof of Theorem 4.3.3 can easily be adapted to establish

$$\mathrm{Cons}(\mathcal{S})(x) = \overline{\underline{\mathbb{L}}|} \left\{ \iota^{-1} \left({}^{\mathcal{S}(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\}. \tag{4.123}$$

•

4.3.2 Characterising Validity and Inconsistency using Cons

Some more properties of Cons are studied in the following; in particular, differences and similarities with the classical semantic consequence operator are pointed out.

First of all, it is demonstrated how to use the semantic consequence operator to characterise certain properties of a (labelled) formula.

Propositions 4.3.4 (Characterising validity and inconsistency using Cons)

1. *For a labelled formula $\mathfrak{x} \in \mathrm{LFrm}$, the following statements are equivalent:*
   - *(i)* $\mathfrak{x} \in \mathrm{Valid}$ *(see* Definition 4.2.2*)*
   - *(ii) for every* $\mathcal{S} \in D^{\mathfrak{S}}$, $\mathcal{S} \Vdash \mathfrak{x}$ *(see* (4.123)*)*
   - *(iii) for every* $\mathcal{X} \in L^{\mathrm{Frm}}$, $\mathcal{X} \Vdash \mathfrak{x}$
   - *(iv)* $\emptyset \Vdash \mathfrak{x}$

2. *For a formula $x \in \mathrm{Frm}$, the following statements are equivalent:*

   *(i) x is a **tautology** (see* Definition 3.3.4*)*

   (ii) for all $\mathcal{S} \in D^{\mathfrak{S}}$: $\mathrm{Cons}(\mathcal{S})(x) = 1$ (see (4.123))

   (iii) for all $\mathcal{X} \in L^{\mathrm{Frm}}$: $\mathrm{Cons}(\mathcal{X})(x) = 1$

3. *If $\mathfrak{x} \in \mathrm{LFrm}$ is **inconsistent** (see* Definition 4.2.2*), then $\mathcal{X} \Vdash \mathfrak{x}$ if and only if $\mathcal{X} \in \mathrm{Incons}$.*

4. *If $\mathrm{Incons} \neq \varnothing$, then for every $\mathcal{X} \in L^{\mathrm{Frm}}$, the following statements are equivalent:*

   (i) $\mathcal{X} \in \mathrm{Incons}$ (see Definition 4.2.2*)*

   (ii) for every $\mathfrak{x} \in \mathrm{LFrm}$, $\mathcal{X} \Vdash \mathfrak{x}$

   (iii) for all $x \in \mathrm{Frm}$, $\mathrm{Cons}(\mathcal{X})(x) = 1$

   If $\mathrm{Incons} = \varnothing$, then for every $\mathcal{X} \in L^{\mathrm{Frm}}$, 4.ii and 4.iii are equivalent, but 4.i does not follow from 4.ii, in general.

Proof
ad 1. The following implications are proved:

1.i ⇒ 1.ii. Assume that $\mathfrak{x} \in \mathrm{Valid}$, i. e.

$$\boxed{\mathrm{D}}\left\{\mathrm{Mod}(\mathfrak{x})(\mathrm{Val}) \,\middle|\, \mathrm{Val} \in \mathfrak{S}\right\} = 1.$$

This means that for every $\mathrm{Val} \in \mathfrak{S}$, $\mathrm{Mod}(\mathfrak{x})(\mathrm{Val}) = 1$. By (4.115), it is to be proved that $\mathcal{S} \subseteqq \mathrm{Mod}(\mathfrak{x})$, which holds trivially.

1.ii ⇒ 1.iii. Trivial by (4.117).

1.iii ⇒ 1.iv. As $\not{\mathcal{O}} \in L^{\mathrm{Frm}}$, this is also trivial.

1.iv ⇒ 1.i. Assume $\not{\mathcal{O}} \Vdash \mathfrak{x}$.
Let $\mathrm{Val} \in \mathfrak{S}$. By (4.40),

$$\mathrm{Mod}(\not{\mathcal{O}})(\mathrm{Val}) = 1,$$

so $\not{\mathcal{O}} \Vdash \mathfrak{x}$ implies by (4.97) that $\mathrm{Mod}(\mathfrak{x})(\mathrm{Val}) = 1$ holds for all $\mathrm{Val} \in \mathfrak{S}$, which yields the absolute validity of $\mathfrak{x}$ by (4.48).

ad 2. The following implications are proved:

2.i ⇒ 2.ii. From the fact that x is a tautology, it follows by Observation 4.2.4.3 that $[x, \ell]$ is valid for every $\ell \in L$.
By item 1.ii, this means $\mathcal{S} \Vdash [x, \ell]$ for every $\ell \in L$. By (4.98), hence

$$\mathrm{Cons}(\mathcal{S})(x) = \boxed{\mathrm{L}}\{\ell | \ell \in L\} = 1.$$

2.ii ⇒ 2.iii. As $\mathrm{Cons}(\mathcal{X})$ is induced by $\mathrm{Mod}(\mathcal{X})$ according to (4.111), this is obvious.

2.iii ⇒ 2.i. In particular, $\mathrm{Cons}(\not{\mathcal{O}})(x) = 1$ is obtained. From Proposition 4.3.1.8, it follows that $\not{\mathcal{O}} \Vdash [x, 1]$.
By item 1.iv, this means that $[x, 1]$ is valid, from which it follows by Observation 4.2.4.6 that x is a tautology.

ad 3. Let $\mathfrak{x} \in \text{Incons}$, i. e. $\text{Mod}(\mathfrak{x})(\text{Val}) = 0$ for all $\text{Val} \in \mathfrak{S}$.

Obviously, $\text{Mod}(\mathcal{X}) \subseteqq \text{Mod}(\mathfrak{x})$ (and hence $\mathcal{X} \Vdash \mathfrak{x}$) is equivalent with $\text{Mod}(\mathcal{X})(\text{Val}) = 0$ for all $\text{Val} \in \mathfrak{S}$ (and hence $\mathcal{X} \in \text{Incons}$).

ad 4. The following implications are proved:

4.i ⇒ 4.ii. If $\mathcal{X} \in \text{Incons}$, then $\text{Mod}(\mathcal{X})(\text{Val}) = 0$ for all $\text{Val} \in \mathfrak{S}$, from which $\text{Mod}(\mathcal{X}) \subseteqq \text{Mod}(\mathfrak{x})$ and hence $\mathcal{X} \Vdash \mathfrak{x}$ follows trivially for all $\mathfrak{x} \in \text{LFrm}$.

4.ii ⇒ 4.iii. Follows immediately by (4.98).

4.iii ⇒ 4.ii. Follows from Proposition 4.3.1.9.

4.ii ⇒ 4.i. Assume that for every $\mathfrak{x} \in \text{LFrm}$, $\mathcal{X} \Vdash \mathfrak{x}$. By assumption, $\text{Incons} \neq \varnothing$, so there exists $\mathcal{X}_{\text{Incons}} \in \text{Incons}$, which means $\text{Mod}\left(\mathcal{X}_{\text{Incons}}\right)(\text{Val}) = 0$ for all $\text{Val} \in \mathfrak{S}$.

Obviously, $\mathcal{X}_{\text{Incons}} = \bigcup\left\{ \left[x, \mathcal{X}_{\text{Incons}}(x)\right] \,\middle|\, x \in \text{Frm}\right\}$. From Proposition 4.2.2, it follows that

$$\bigcap\left\{\text{Mod}\left(\left[x, \mathcal{X}_{\text{Incons}}(x)\right]\right) \,\middle|\, x \in \text{Frm}\right\} = \text{Mod}\left(\mathcal{X}_{\text{Incons}}\right).$$

On the other hand, it holds by assumption that for every $x \in \text{Frm}$, $\mathcal{X} \Vdash \left[x, \mathcal{X}_{\text{Incons}}(x)\right]$, hence $\text{Mod}(\mathcal{X}) \subseteqq \text{Mod}\left(\left[x, \mathcal{X}_{\text{Incons}}(x)\right]\right)$, from which it follows that

$$\text{Mod}(\mathcal{X}) \subseteqq \bigcap\left\{\text{Mod}\left(\left[x, \mathcal{X}_{\text{Incons}}(x)\right]\right) \,\middle|\, x \in \text{Frm}\right\} = \text{Mod}\left(\mathcal{X}_{\text{Incons}}\right),$$

from which $\mathcal{X} \in \text{Incons}$ follows by the fact that $\mathcal{X}_{\text{Incons}} \in \text{Incons}$ by assumption.

From the above proofs, it is clear that the equivalence 4.ii ⇔ 4.iii (and in fact the implication 4.i ⇒ 4.ii) does not depend on the assumption $\text{Incons} \neq \varnothing$.

On the other hand, if $\text{Incons} = \varnothing$, it is easy to see that the implication 4.ii ⇒ 4.i does not hold, in general. □

4.3.3 Inconsistency and Refutation

Definition 4.3.3 (Inconsistency distribution)
For this definition, assume that Frm *contains a formula* $\perp$ *such that for all* $\text{Val} \in \mathfrak{S}$, $\text{Val}(\perp) = 0$.
Let $\mathcal{X} \in L^{\text{Frm}}$. *The* ***inconsistency distribution*** *of* $\mathcal{X}$ *is defined by*

$$\text{inc}(\mathcal{X}) =_{\text{def}} \text{Cons}(\mathcal{X})(\perp). \tag{4.124}$$

Remarks

1. The definition (4.124) of inc corresponds to the result of Proposition 3.3.2 in [19] about the **inconsistency degree** Incons. The definition given in [19] for Incons corresponds to equation (5.100) of this dissertation which makes sense only in the special case of *possibilistic logic* as presented in [19].

2. The *meaning* of the preceding definition is clear: $\mathcal{X}$ is inconsistent to the extent in which an **insatisfiable** formula (compare Definition 3.3.4.2) follows from $\mathcal{X}$. This definition has to be compared with the definition (4.49) of the **consistency degree** cst.

First of all, the value of cst is a **degree of validity** from $\mathfrak{D}$, while the value of inc is a **label** from $\mathfrak{L}$. As labels are implicitly $\mathfrak{D}$-fuzzy filters of $\mathfrak{T}$, inc has been named **inconsistency distribution**.

Secondly, it has to be expected that inc is somehow *complementary* to cst.

The exact nature of the fuzzy filter characterised by $\mathrm{inc}(\mathcal{X})$ and the relationship between inc and cst is made precise by the following observation. •

Proposition 4.3.5 (Properties of inc)
For every $\mathcal{X} \in L^{\mathrm{Frm}}$,

$$\mathrm{inc}(\mathcal{X}) = \iota^{-1}\left({}^{\mathrm{cst}(\mathcal{X})}\overline{0}\right) \tag{4.125}$$

$$= \ell^{0}_{\mathrm{cst}(\mathcal{X})} \text{ for a logic of graded truth and graded trust assessment.} \tag{4.126}$$

$$\mathrm{Incons} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \mathrm{inc}(\mathcal{X}) = 1\right\}. \tag{4.127}$$

$$\mathcal{X} \Vdash \left[x, \mathrm{inc}(\mathcal{X})\right] \text{ for all } x \in \mathrm{Frm}. \tag{4.128}$$

Proof
ad (4.125)**,** (4.126)**.** Let $\mathcal{X} \in L^{\mathrm{Frm}}$.

$$\begin{aligned}
\mathrm{inc}(\mathcal{X}) &= \mathrm{Cons}(\mathcal{X})(\bot) && \text{(by (4.124))}\\
&= \boxed{\mathrm{L}}\left\{\iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{0}\right) \,\middle|\, \mathrm{Val} \in \mathfrak{S}\right\} && \text{(by (4.111) and the definition of } \bot)\\
&= \iota^{-1}\left({}^{\left(\bigsqcup^{\mathrm{D}}_{\mathrm{Val}\in\mathfrak{S}} \mathrm{Mod}(\mathcal{X})(\mathrm{Val})\right)}\overline{0}\right) && \text{(by (2.32))}\\
&= \iota^{-1}\left({}^{\mathrm{cst}(\mathcal{X})}\overline{0}\right) && \text{(by (4.49))}\\
&= \ell^{0}_{\mathrm{cst}(\mathcal{X})} && \text{(by Theorem 4.1.3)}
\end{aligned}$$

ad (4.127)**.** Follows immediately from the previous item and the fact that $\iota^{-1}\left({}^{0}\overline{0}\right) = \ell^{0}_{0}$ is the **unit element** of $\mathfrak{L}$.

ad (4.128)**.** By definition (4.97), $\mathcal{X} \Vdash \left[x, \mathrm{inc}(\mathcal{X})\right]$ iff

$$\mathrm{Mod}(\mathcal{X}) \sqsubseteq \mathrm{Mod}\left(\left[x, \mathrm{inc}(\mathcal{X})\right]\right),$$

which means that for every $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathrm{D}} \mathrm{Mod}\left(\left[x, \mathrm{inc}(\mathcal{X})\right]\right)(\mathrm{Val}),$$

which means by definition (4.41) of Mod and definition (4.1) of $\models$ that for all $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathrm{D}} \iota\left(\mathrm{inc}(\mathcal{X})\right)\left(\mathrm{Val}(x)\right),$$

which is by (4.125) equivalent with

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathrm{D}} {}^{\mathrm{cst}(\mathcal{X})}\overline{0}\left(\mathrm{Val}(x)\right).$$

In the case $\mathrm{Val}(x) = 1$, it follows by definition (2.1) that ${}^{\mathrm{cst}(\mathcal{X})}\overline{0}\,\big(\mathrm{Val}(x)\big) = 1$, in which case the inequation holds trivially. Otherwise, ${}^{\mathrm{cst}(\mathcal{X})}\overline{0}\,\big(\mathrm{Val}(x)\big) = \mathrm{cst}(\mathcal{X})$, and

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathbb{D}} \mathrm{cst}(\mathcal{X})$$

holds by the definition (4.49) of cst. $\square$

Remarks
Proposition 4.3.5 clarifies the nature of inc:

1. inc is not really a **distribution**, it is uniquely determined by the value of cst.
2. The **complementation** between inc and cst is effected indirectly by the fact that ι implicitly contains a dualisation.
3. Still, the relationship between inc and cst is not completely trivial because of the influence of ι. This will be illustrated by examples in chapter 5.
4. Because of Proposition 4.3.5, in the following (4.125) (resp. (4.126)) will be used as the definition of inc even in logics where no appropriate formula $\perp$ exists. •

Propositions 4.3.6 (inc without $\perp$)
For the following observations, assume inc *to be defined by* (4.125) *(resp.* (4.126)*), hence the existence of a formula* $\perp$ *as required by* Definition 4.3.3 *is not necessary.*

1. *Observations* (4.127) *and* (4.128) *hold even in the case that* inc *is defined by* (4.125) *(resp.* (4.126)*).*
2. *For every* $\mathcal{X} \in L^{\mathrm{Frm}}$,
$$\mathrm{inc}(\mathcal{X}) \sqsubseteq_{\mathbb{L}} \boxed{\mathbb{L}}\left\{\mathrm{Cons}(\mathcal{X})(x) \,\middle|\, x \in \mathrm{Frm}\right\}.$$
3. *If for every* $\mathrm{Val} \in \mathfrak{S}$*, there exists* $x \in \mathrm{Frm}$ *such that* $\mathrm{Val}(x) = 0$*, then for every* $\mathcal{X} \in L^{\mathrm{Frm}}$,
$$\mathrm{inc}(\mathcal{X}) = \boxed{\mathbb{L}}\left\{\mathrm{Cons}(\mathcal{X})(x) \,\middle|\, x \in \mathrm{Frm}\right\}.$$

Proof
ad 1. It suffices to note that in the proofs of (4.127) and (4.128), only (4.125) has been used.

ad 2. It is sufficient to prove that for every $x \in \mathrm{Frm}$,

$$\mathrm{inc}(\mathcal{X}) \sqsubseteq_{\mathbb{L}} \mathrm{Cons}(\mathcal{X})(x),$$

i. e. by (4.111)

$$\sqsubseteq_{\mathbb{L}} \boxed{\mathbb{L}}\left\{\iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)}\right) \,\middle|\, \mathrm{Val} \in \mathfrak{S}\right\},$$

which means it is sufficient to prove that for every $x \in \mathrm{Frm}$ and every $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{inc}(\mathcal{X}) \sqsubseteq_{\mathbb{L}} \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)}\right),$$

which means by (4.125) that for every $x \in \mathrm{Frm}$ and every $\mathrm{Val} \in \mathfrak{S}$,

$$ {}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \subseteqq {}^{\mathrm{cst}(\mathcal{X})}\overline{0}\,. $$

For establishing this inequation, by the definition (4.49) of cst and the definition (2.1) of fuzzy principal filters it is sufficient to prove that for every $\mathrm{Val} \in \mathfrak{S}$,

$$ \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathbb{D}} \bigsqcup_{\mathbb{D}} \left\{ \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\}, $$

which holds trivially.

ad 3. Taking into account the previous item, it is sufficient to prove that

$$ \bigsqcap_{\mathbb{L}} \left\{ \mathrm{Cons}(\mathcal{X})(x) \,\middle|\, x \in \mathrm{Frm} \right\} \sqsubseteq_{\mathbb{L}} \mathrm{inc}(\mathcal{X}), $$

i.e. by (4.111),

$$ \bigsqcap_{\mathbb{L}} \left\{ \bigsqcap_{\mathbb{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\} \,\middle|\, x \in \mathrm{Frm} \right\} \sqsubseteq_{\mathbb{L}} \mathrm{inc}(\mathcal{X}), $$

which is equivalent with

$$ \bigsqcap_{\mathbb{L}} \left\{ \bigsqcap_{\mathbb{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \,\middle|\, x \in \mathrm{Frm} \right\} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\} \sqsubseteq_{\mathbb{L}} \mathrm{inc}(\mathcal{X}). \tag{4.129} $$

Obviously, for every $x \in \mathrm{Frm}$, $\iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{0} \right) \sqsubseteq_{\mathbb{L}} \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right)$, hence by the assumption that for every $\mathrm{Val} \in \mathfrak{S}$, there exists $x \in \mathrm{Frm}$ such that $\mathrm{Val}(x) = 0$, (4.129) is equivalent with

$$ \bigsqcap_{\mathbb{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{0} \right) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\} \sqsubseteq_{\mathbb{L}} \mathrm{inc}(\mathcal{X}). \tag{4.130} $$

Now, $\mathfrak{L}$ is not necessarily isomorphic with a complete sublattice of $\left[\mathfrak{D}\text{-Fl}\,(\mathfrak{T})\,, \dot\cup, \cap\right]$ (see the first remark following Definition 2.3.1), but from the fact that ι is a lattice isomorphism, in any case it follows that

$$ \dot\bigcup \left\{ {}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{0} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\} \subseteqq \iota\left(\bigsqcap_{\mathbb{L}} \left\{ \iota^{-1}\left({}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{0} \right) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\} \right), $$

hence for establishing (4.130) it is sufficient to prove that

$$ \iota(\mathrm{inc}(\mathcal{X})) \subseteqq \dot\bigcup \left\{ {}^{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})}\overline{0} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\}, $$

which means by (4.125) and (2.32) that

$$ {}^{\mathrm{cst}(\mathcal{X})}\overline{0} \subseteqq {}^{\left(\bigsqcup_{\mathbb{D}\,\mathrm{Val} \in \mathfrak{S}} \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \right)}\overline{0}\,, $$

which holds by definition (4.49) of cst. $\square$

Remark
Proposition 4.3.6.2 is a *fuzzification* of the classical statement „if a set of formulae is inconsistent, then it entails every formula".

For replacing implication with equivalence in the above statement (Proposition 4.3.6.3), an additional assumption is necessary about the semantics of the underlying logic.

This assumption is weaker than assuming the existence of $\perp$, and is trivially fulfilled in classical two-valued logic, but does not hold in some many-valued logics (for instance, it does not hold in LEE's fuzzy logic; confirm Example 3.2.4.2).

In logics where the assumption of Proposition 4.3.6.3 does not hold, inc (as defined by (4.125)) represents a genuinely stronger concept than the one represented by $\boxed{\mathrm{L}}\left\{\mathrm{Cons}(\mathcal{X})(x) \,\middle|\, x \in \mathrm{Frm}\right\}$. •

To illustrate uses of the inconsistency distribution further, some general observations regarding **refutation systems** are made in the following.

Definition 4.3.4 (Refutation)
Assume to be given two unary mappings $\nu_{\mathfrak{D}} : D \to D$, $\nu_{\mathfrak{T}} : T \to T$ with the following properties, for $c, d \in D$ and $s, t \in T$:

$$(4.131) \qquad s \mathrel{\underline{\boxed{\scriptstyle \mathrm{T}}}} t \text{ iff } \nu_{\mathfrak{T}}(t) \mathrel{\underline{\boxed{\scriptstyle \mathrm{T}}}} \nu_{\mathfrak{T}}(s) \qquad c \mathrel{\underline{\boxed{\scriptstyle \mathrm{D}}}} d \text{ iff } \nu_{\mathfrak{D}}(d) \mathrel{\underline{\boxed{\scriptstyle \mathrm{D}}}} \nu_{\mathfrak{D}}(c) \qquad \textit{(order reversion)}$$

$$(4.132) \qquad \nu_{\mathfrak{T}}(\nu_{\mathfrak{T}}(t)) = t \qquad \nu_{\mathfrak{D}}(\nu_{\mathfrak{D}}(d)) = d \qquad \textit{(involution)}$$

and assume further that Frm *contains a unary operator symbol $\neg$ interpreted by $\nu_{\mathfrak{T}}$.*

Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $[x, \ell] \in \mathrm{LFrm}$ be given. ℓ is said to ***admit refutation***
$=_{\mathrm{def}}$ *the mapping $\mathcal{F}_\ell : T \to D$ defined for $t \in T$ by*

$$(4.133) \qquad \mathcal{F}_\ell(t) =_{\mathrm{def}} \begin{cases} 1 & \text{if } t = 1 \\ \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\nu_{\mathfrak{T}}(t)\right)\right) & \text{if } t \neq 1 \end{cases}$$

is in $\mathrm{rg}\,\iota$.

If ℓ admits refutation, then $\iota^{-1}(\mathcal{F}_\ell)$ is denoted by $\widetilde{\ell}$.

If ℓ admits refutation, then $\mathcal{X} \Vdash [x, \ell]$ is said to be ***characterised by refutation***
$=_{\mathrm{def}}$ *$\mathcal{X} \Vdash [x, \ell]$ iff $\ell \mathrel{\underline{\boxed{\scriptstyle \mathrm{L}}}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$.*

Remarks

1. The concept of refutation allows to reduce the task of finding a label ℓ such that $\mathcal{X} \Vdash [x, \ell]$ to the task of finding $\mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$.

   This is especially important for **automated theorem proving**, as all classical methods of automated theorem proving which allow at least some degree of efficiency (most of them stemming from **tableau-** or **resolution**-based methods) are based on refutation systems. That is, the method itself only allows to automatically find whether a set of formulae is (classically) inconsistent.

   It can be expected that the methods themselves can be adapted to labelled formulae (using **labelled deductive systems**, compare [38]) for finding the inconsistency distribution of a fuzzy set of formulae, but to characterise entailment this way, the refutation system sketched above has to be applied.

2. Note that from (4.132) it follows that $\nu_{\mathfrak{D}}, \nu_{\mathfrak{T}}$ are bijections on D, T, respectively.

   The mere existence of order-reversing bijections on $\mathfrak{D}, \mathfrak{T}$, respectively, is an effective restriction of the validity degree lattices $\mathfrak{D}$ and truth value lattices $\mathfrak{T}$ for which the concept of **refutation** defined above is applicable.

   A further restriction is effected by the necessity of a label ℓ to **admit refutation** before the concept that $\mathcal{X} \Vdash [x, \ell]$ is **characterised by refutation** can even be formulated.

   The following theorem sheds some light on the range of labels which admit refutation. •

Theorem 4.3.7 (Label lattices admitting refutation)

1. *If $\mathfrak{T}$ is a chain and $\operatorname{rg} \iota = \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$, then every $\ell \in L$ admits refutation.*

2. *If every $\ell \in L$ admits refutation, then $\mathfrak{T}$ is a chain.*

Proof

ad 1. Assume both premises are fulfilled.

It is sufficient to prove that for every $\mathcal{F} \in \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$, the mapping $\mathcal{G} : T \to D$ defined for $t \in T$ by

$$\mathcal{G}(t) =_{\mathrm{def}} \begin{cases} 1 & \text{if } t = 1 \\ \nu_{\mathfrak{D}}\left(\mathcal{F}\left(\nu_{\mathfrak{T}}(t)\right)\right) & \text{if } t \neq 1 \end{cases} \tag{4.134}$$

is in $\mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$.

This is established by verifying the claims from Corollary 2.1.7. $\mathcal{G}(1) = 1$ is assured by the definition of $\mathcal{G}$. It remains to prove that for all $s, t \in T$,

$$\mathcal{G}(s) \sqcap_{\mathfrak{D}} \mathcal{G}(t) = \mathcal{G}\left(s \sqcap_{\mathfrak{T}} t\right).$$

First, consider the case that $s = 1$ or $t = 1$, wlg $s = 1$. By definition of $\mathcal{G}$, it follows that

$$\mathcal{G}(s) \sqcap_{\mathfrak{D}} \mathcal{G}(t) = \mathcal{G}(1) \sqcap_{\mathfrak{D}} \mathcal{G}(t) = 1 \sqcap_{\mathfrak{D}} \mathcal{G}(t) = \mathcal{G}(t) = \mathcal{G}\left(1 \sqcap_{\mathfrak{T}} t\right) = \mathcal{G}\left(s \sqcap_{\mathfrak{T}} t\right).$$

Now, assume $s \neq 1$ and $t \neq 1$. It follows that $s \sqcap_{\mathfrak{T}} t \neq 1$ and

$$\begin{aligned}
\mathcal{G}\left(s \sqcap_{\mathfrak{T}} t\right) &= \nu_{\mathfrak{D}}\left(\mathcal{F}\left(\nu_{\mathfrak{T}}(s \sqcap_{\mathfrak{T}} t)\right)\right) && \text{(by definition of } \mathcal{G}) \\
&= \nu_{\mathfrak{D}}\left(\mathcal{F}\left(\nu_{\mathfrak{T}}(s) \sqcup_{\mathfrak{T}} \nu_{\mathfrak{T}}(t)\right)\right) && \text{(by definition of } \nu_{\mathfrak{T}}) \\
&= \nu_{\mathfrak{D}}\left(\mathcal{F}\left(\nu_{\mathfrak{T}}(s)\right) \sqcup_{\mathfrak{D}} \mathcal{F}\left(\nu_{\mathfrak{T}}(t)\right)\right) && \text{(by (2.4), because } \mathfrak{T} \text{ is a chain)} \\
&= \nu_{\mathfrak{D}}\left(\mathcal{F}\left(\nu_{\mathfrak{T}}(s)\right)\right) \sqcap_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\mathcal{F}\left(\nu_{\mathfrak{T}}(t)\right)\right) && \text{(by definition of } \nu_{\mathfrak{D}}) \\
&= \mathcal{G}(s) \sqcap_{\mathfrak{D}} \mathcal{G}(t) && \text{(by definition of } \mathcal{G})
\end{aligned}$$

ad 2. For proving this claim by contraposition, assume that $\mathfrak{T}$ is not a chain and establish that there exists $\ell \in L$ which does not admit refutation.

If $\mathfrak{T}$ is not a chain, then there are $s, t \in T$ such that neither $s \sqsubseteq_{\mathfrak{T}} t$ nor $t \sqsubseteq_{\mathfrak{T}} s$. Consequently, $\{s, t, s \sqcap_{\mathfrak{T}} t, s \sqcup_{\mathfrak{T}} t\}$ forms a 4-element sublattice of $\mathfrak{T}$ as sketched in Figure 4.1.

Consider $\ell =_{\mathrm{def}} \ell_1^{s \sqcup_{\mathfrak{T}} t} = \iota^{-1}\left({}^{1}\overline{s \sqcup_{\mathfrak{T}} t}\right)$ (compare Observation 4.1.2). By Definition 2.3.1.1, $\ell \in L$. It is to be proved that ℓ does not admit refutation.

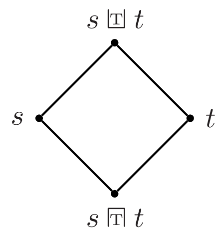

Figure 4.1: The smallest non-chain

By definition of ℓ,

$$\iota(\ell)(s) = 0, \qquad \iota(\ell)(t) = 0,$$
$$\iota(\ell)(s \sqcap_{\mathrm{T}} t) = 0, \qquad \iota(\ell)(s \sqcup_{\mathrm{T}} t) = 1. \tag{4.135}$$

From the fact that $\nu_{\mathfrak{T}}$ is an order-reversing bijection, it follows that

$$\nu_{\mathfrak{T}}(s \sqcap_{\mathrm{T}} t) = \nu_{\mathfrak{T}}(s) \sqcup_{\mathrm{T}} \nu_{\mathfrak{T}}(t), \qquad \nu_{\mathfrak{T}}(s \sqcup_{\mathrm{T}} t) = \nu_{\mathfrak{T}}(s) \sqcap_{\mathrm{T}} \nu_{\mathfrak{T}}(t),$$

hence by the definition of $\mathcal{F}_\ell$,

$$\begin{aligned}
\mathcal{F}_\ell\left(\nu_{\mathfrak{T}}^{-1}(s) \sqcap_{\mathrm{T}} \nu_{\mathfrak{T}}^{-1}(t)\right) &= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\nu_{\mathfrak{T}}\left(\nu_{\mathfrak{T}}^{-1}(s) \sqcap_{\mathrm{T}} \nu_{\mathfrak{T}}^{-1}(t)\right)\right)\right) \\
&= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\nu_{\mathfrak{T}}\left(\nu_{\mathfrak{T}}^{-1}(s)\right) \sqcup_{\mathrm{T}} \nu_{\mathfrak{T}}\left(\nu_{\mathfrak{T}}^{-1}(t)\right)\right)\right) \\
&= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(s \sqcup_{\mathrm{T}} t\right)\right) \\
&= \nu_{\mathfrak{D}}(1) \\
&= 0.
\end{aligned} \tag{4.136}$$

(note that obviously, $s \neq 0$ and $t \neq 0$, so $\nu_{\mathfrak{T}}^{-1}(s) \sqcap_{\mathrm{T}} \nu_{\mathfrak{T}}^{-1}(t) \neq 1$)

From (4.135) it follows immediately that

$$\mathcal{F}_\ell\left(\nu_{\mathfrak{T}}^{-1}(s)\right) = 1 \qquad \mathcal{F}_\ell\left(\nu_{\mathfrak{T}}^{-1}(t)\right) = 1,$$

thus $\mathcal{F}_\ell\left(\nu_{\mathfrak{T}}^{-1}(s)\right) \sqcap_{\mathrm{D}} \mathcal{F}_\ell\left(\nu_{\mathfrak{T}}^{-1}(t)\right) = 1$, which in combination with (4.136) establishes by Corollary 2.1.7 that $\mathcal{F}_\ell$ is not in $\mathfrak{D}$-$\mathrm{Fl}(\mathfrak{T})$. Obviously, this means $\mathcal{F}_\ell$ is not in $\mathrm{rg}\,\iota$, hence ℓ does **not** admit refutation.

This completes the proof of this item. $\square$

Remark

The question what the fact that every $\ell \in L$ admits refutation implies for $\mathrm{rg}\,\iota$ (which could lead to the reverse implication of Theorem 4.3.7.1) is left open.

It leads to the study of sublattices of $\left[\mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T}), \cup, \cap\right]$ which contain $\mathfrak{D}$-$\mathrm{PFl}(\mathfrak{T})$ and are **closed** wrt the operation defined by equation (4.134). This subject, which is of great importance for the study of automated reasoning in fuzzy filter-based logics, is left for future investigations.

The following proposition gives some first results in this direction, concerning **principal fuzzy filters**. •

Proposition 4.3.8 (Principal fuzzy filters admitting refutation)

1. *Given $t \in T$ and $d \in D$,*

$$\mathcal{F}_{\ell_d^t}(s) = \begin{cases} 1, & \text{if } s = 1 \text{ or } \textbf{not } s \sqsubseteq_{\mathfrak{T}} \nu_{\mathfrak{T}}(t) \\ \nu_{\mathfrak{D}}(d), & \text{if } s \neq 0 \text{ and } s \neq 1 \text{ and } s \sqsubseteq_{\mathfrak{T}} \nu_{\mathfrak{T}}(t) \\ 0, & \text{if } s = 0 \end{cases} \tag{4.137}$$

2. *Given $t \in T$ and $d \in D$, if ℓ_d^t admits refutation, then*

$$\mathcal{F}_{\widetilde{\ell_d^t}}(s) = \begin{cases} 1, & \text{if } s = 1 \\ d, & \text{if } s \neq 1 \text{ and } s \neq 0 \text{ and } t \sqsubseteq_{\mathfrak{T}} s \\ 0, & \text{if } s = 0 \text{ or } \textbf{not } t \sqsubseteq_{\mathfrak{T}} s \end{cases} \tag{4.138}$$

3. *Given $t \in T$ and $d \in D$, if ℓ_d^t and $\widetilde{\ell_d^t}$ admit refutation, then $\mathcal{F}_{\widetilde{\widetilde{\ell_d^t}}} = \mathcal{F}_{\ell_d^t}$.*

4. *That for all $t \in T$, $d \in D$, $\mathcal{F}_{\ell_d^t} \in \mathfrak{D}\text{-PFl}(\mathfrak{T})$ holds if and only if*
   - *(i) T contains at most three elements or*
   - *(ii) $\mathfrak{D}$ is two-valued and for every $t \in T$ with $t \neq 1$, there exists $t' \in T$ such that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$.*

5. *If for every $t \in T$ with $t \neq 1$, there exists $t' \in T$ such that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$, then for all $t \in T$, $d \in D$, $\mathcal{F}_{\ell_d^t} \in \mathscr{P}(\mathfrak{D}, \mathfrak{T})$.*

6. *$\left[\mathfrak{D}\text{-PFl}(\mathfrak{T}), \Cup, \Cap\right]$ is a **minimal** sublattice of $\left[\mathfrak{D}\text{-Fl}(\mathfrak{T}), \Cup, \Cap\right]$ which contains $\mathfrak{D}\text{-PFl}(\mathfrak{T})$ and is **closed** wrt the operation defined by equation (4.134) if and only if*
   - *(i) $\mathfrak{T}$ is two-valued or*
   - *(ii) $\mathfrak{D}$ is two-valued and for every $t \in T$ with $t \neq 1$ there exists $t' \in T$ such that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$.*

Proof

ad 1–3. Follow immediately from the definitions.

ad 4. Both implications are proved separately.

"if". It is to be proved that in each of the cases 4.i and 4.ii, for all $t \in T$ and $d \in D$, it holds that $\mathcal{F}_{\ell_d^t} \in \mathfrak{D}\text{-PFl}(\mathfrak{T})$. Let $t \in T$, $d \in D$ be given. Both cases are considered separately.

ad 4.i. Wlg assume that $T = \{0, \tau, 1\}$ such that $\tau \neq 0$ and $\tau \neq 1$.

Distinguish two cases:

1. $t = 0$ or $t = \tau$.

   As $\nu_{\mathfrak{T}}$ is a bijection, obviously $\nu_{\mathfrak{T}}(\tau) = \tau$. Hence, it follows from (4.137) that for all $s \in T$,

$$\mathcal{F}_{\ell_d^t}(s) = \begin{cases} 1, & \text{if } s = 1 \\ \nu_{\mathfrak{D}}(d), & \text{if } s = \tau \\ 0, & \text{if } s = 0 \end{cases}$$

   i. e. $\mathcal{F}_{\ell_d^t} = {}^{\nu_{\mathfrak{D}}(d)}\overline{\tau} \in \mathfrak{D}\text{-PFl}(\mathfrak{T})$.

2. $t = 1$.
   It follows from (4.137) that for all $s \in T$,

$$\mathcal{F}_{\ell_d^1}(s) = \begin{cases} 1, & \text{if } s \in \{\tau, 1\} \\ 0, & \text{if } s = 0 \end{cases}$$

   i. e. $\mathcal{F}_{\ell_d^1} = {}^{1}\overline{\tau} \in \mathfrak{D}\text{-PFl}\,(\mathfrak{T})$.

ad 4.ii. Assume that $\mathfrak{D}$ is two-valued and for every $t \in T$ with $t \neq 1$ there exists $t' \in T$ such that for every $s \in T$, $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$.
Distinguish two cases:

1. $d = 0$.
   It follows from (4.137) that for all $s \in T$,

$$\mathcal{F}_{\ell_0^t}(s) = \begin{cases} 1, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0 \end{cases}$$

   Applying the assumption to 0 yields

$$\mathcal{F}_{\ell_0^t}(s) = \begin{cases} 1, & \text{if } 0' \sqsubseteq_{\mathfrak{T}} s \\ 0, & \text{if } \textbf{not } 0' \sqsubseteq_{\mathfrak{T}} s \end{cases}$$

   i. e. $\mathcal{F}_{\ell_0^t} = {}^{1}\overline{0'}$.

2. $d = 1$.
   It follows from (4.137) that for all $s \in T$,

$$\mathcal{F}_{\ell_1^t}(s) = \begin{cases} 1, & \text{if } s = 1 \text{ or } \textbf{not } s \sqsubseteq_{\mathfrak{T}} \nu_{\mathfrak{T}}(t) \\ 0, & \text{if } s \neq 1 \text{ and } s \sqsubseteq_{\mathfrak{T}} \nu_{\mathfrak{T}}(t) \end{cases}$$

   In the case $\nu_{\mathfrak{T}}(t) = 1$, obviously $\mathcal{F}_{\ell_0^t} = \mathbb{0} \in \mathfrak{D}\text{-PFl}\,(\mathfrak{T})$. Otherwise, the assumption of this item can be applied to $\nu_{\mathfrak{T}}(t)$, yielding

$$\mathcal{F}_{\ell_0^t}(s) = \begin{cases} 1, & \text{if } s = 1 \text{ or } \nu_{\mathfrak{T}}(t)' \sqsubseteq_{\mathfrak{T}} s \\ 0, & \text{if } \textbf{not } \nu_{\mathfrak{T}}(t)' \sqsubseteq_{\mathfrak{T}} s \end{cases}$$

   i. e. $\mathcal{F}_{\ell_0^t} = {}^{1}\overline{\nu_{\mathfrak{T}}(t)'}$.

"only if". The contraposition is proved.
Assume that T contains at least four elements and either D contains at least three elements or there exists $t \in T$ with $t \neq 1$ such that there is **no** $t' \in T$ with the property that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$.
It is to be proved that there exist $t \in T$ and $d \in D$ such that $\mathcal{F}_{\ell_d^t} \notin \mathfrak{D}\text{-PFl}\,(\mathfrak{T})$.
In Theorem 4.3.7.2 it has been proved that if $\mathfrak{T}$ is not a chain, then there exist $t \in T$ and $d \in D$ such that $\mathcal{F}_{\ell_d^t} \notin \mathfrak{D}\text{-PFl}\,(\mathfrak{T})$. Hence, in the following, wlg it is assumed that $\mathfrak{T}$ is a chain of at least four elements.
Two cases are distinguished:

1. D contains at least three elements.
   As $\mathfrak{T}$ is assumed to be a chain with at least four elements, obviously there exist $\tau, \tau' \in T$ such that $\tau \neq 0$ and $\tau \neq \tau'$ and $\tau \sqsubseteq_{\mathfrak{T}} \tau'$ and $\tau' \neq 1$. Let $S =_{\text{def}} \{s \,|\, \text{not } s \sqsubseteq_{\mathfrak{T}} \tau\}$. Note that $\{\tau', 1\} \subseteq S$.

As D is assumed to contain at least three elements there exists $\delta \in D$ with $\delta \neq 0$ and $\delta \neq 1$.
Consider $t =_{\text{def}} \nu_{\mathfrak{T}}^{-1}(\tau)$ and $d =_{\text{def}} \delta$. It follows from (4.137) that for all $s \in T$,

$$\mathcal{F}_{\ell_{\delta}^{\nu_{\mathfrak{T}}^{-1}(\tau)}}(s) = \begin{cases} 1, & \text{if } s \in S \\ \nu_{\mathfrak{D}}(\delta), & \text{if } s \neq 0 \text{ and } s \notin S \\ 0, & \text{if } s = 0 \end{cases}$$

As S contains more than two elements and $\mathcal{F}_{\ell_{\delta}^{\nu_{\mathfrak{T}}^{-1}(\tau)}}(\tau) = \nu_{\mathfrak{D}}(\delta) \notin \{0,1\}$, $\mathcal{F}_{\ell_{\delta}^{\nu_{\mathfrak{T}}^{-1}(\tau)}}$ is clearly not in $\mathfrak{D}\text{-PFl}(\mathfrak{T})$ (compare (2.1)).

2. There exists $t \in T$ with $t \neq 1$ such that there is **no** $t' \in T$ with the property that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$.
Let $t'' =_{\text{def}} \nu_{\mathfrak{T}}^{-1}(t)$. By (4.137), for all $s \in T$ (considering $t \neq 1$),

$$\mathcal{F}_{\ell_1^{t''}}(s) = \begin{cases} 1, & \text{if } \textbf{not } s \sqsubseteq_{\mathfrak{T}} t \\ 0, & \text{if } s \sqsubseteq_{\mathfrak{T}} t \end{cases} \qquad (4.139)$$

That $\mathcal{F}_{\ell_1^{t''}} \notin \mathfrak{D}\text{-PFl}(\mathfrak{T})$ is proved by contradiction. The assumption $\mathcal{F}_{\ell_1^{t''}} \in \mathfrak{D}\text{-PFl}(\mathfrak{T})$ leads to a contradiction with the assumption of this item.
Obviously, $\mathcal{F}_{\ell_1^{t''}} \in \mathfrak{D}\text{-PFl}(\mathfrak{T})$ iff there exists $t' \in T$ such that $\mathcal{F}_{\ell_1^{t''}} = {}^{1}\overline{t'}$ (note that by $t \neq 1$, $t'' \neq 0$). By (2.1), ${}^{1}\overline{t'}$ is defined as

$${}^{1}\overline{t'}(s) = \begin{cases} 1, & \text{if } t' \sqsubseteq_{\mathfrak{T}} s \\ 0, & \text{if } \textbf{not } t' \sqsubseteq_{\mathfrak{T}} s \end{cases} \qquad (4.140)$$

But comparing (4.139) and (4.140), the existence of $t' \in T$ such that $\mathcal{F}_{\ell_1^{t''}} = {}^{1}\overline{t'}$ leads to a contradiction with the assumption that there exists **no** $t' \in T$ with the property that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$.
This contradiction completes the proof of this item.

Combining both cases establishes the claim of this item.

ad 5. Assume that for every $t \in T$ with $t \neq 1$ there exists $t' \in T$ such that for every $s \in T$, $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$. It is to be proved that for all $t \in T$ and $d \in D$, it holds that $\mathcal{F}_{\ell_d^t} \in \mathscr{P}(\mathfrak{D},\mathfrak{T})$. Let $t \in T$, $d \in D$ be given.

Applying the assumption to t and 0, by (4.137) obviously

$$\mathcal{F}_{\ell_d^t} = {}^{\nu_{\mathfrak{D}}(d)}\overline{0'} \uplus {}^{1}\overline{t'} \in \mathscr{P}(\mathfrak{D},\mathfrak{T}).$$

ad 6. Both implications are proved separately.

"if". That in both cases, $\mathfrak{D}\text{-PFl}(\mathfrak{T})$ is closed wrt the operation defined by equation (4.134) follows immediately from item 4.
It remains to be proved that $\left[\mathfrak{D}\text{-PFl}(\mathfrak{T}), \uplus, \cap\right]$ is a sublattice of $\left[\mathfrak{D}\text{-Fl}(\mathfrak{T}), \uplus, \cap\right]$ (from which the minimality follows by the requirement to contain $\mathfrak{D}\text{-PFl}(\mathfrak{T})$).
That in each of the cases 6.i and 6.ii, $\left[\mathfrak{D}\text{-PFl}(\mathfrak{T}), \uplus, \cap\right]$ is a sublattice of $\left[\mathfrak{D}\text{-Fl}(\mathfrak{T}), \uplus, \cap\right]$, follows from Proposition 2.3.2 and Proposition 2.3.4.

"only if". The contraposition follows immediately from the respective direction of item 4, taking into account the trivial observation that if neither $\mathfrak{T}$ not $\mathfrak{D}$ is two-valued, then $\mathfrak{D}\text{-PFl}(\mathfrak{T}) \neq \mathscr{P}(\mathfrak{D},\mathfrak{T})$, hence $\left[\mathfrak{D}\text{-PFl}(\mathfrak{T}), \uplus, \cap\right]$ is not a sublattice of $\left[\mathfrak{D}\text{-Fl}(\mathfrak{T}), \uplus, \cap\right]$. □

Remarks

1. Note that $\mathcal{F}_{\widetilde{\ell^t_d}}$ and ${}^d\overline{t}$ differ only in the additional condition $s = 0$ for the case $\mathcal{F}_{\widetilde{\ell^t_d}}(s) = 0$.

2. Item 6 gives a first result concerning a *minimal* label lattice all labels of which admit refutation, under severe assumptions about the lattices of truth values and degrees of validity, respectively.

   It is left open how this result can be extended.

   Considering items 1–3 of the above observation, it is clear that if every ℓ^t_d admits refutation and every $\widetilde{\ell^t_d}$ admits refutation, then the set of all fuzzy filters represented by ℓ^t_d, $\widetilde{\ell^t_d}$, $\widetilde{\widetilde{\ell^t_d}}$ is a minimal *subset* of $\mathfrak{D}\text{-Fl}\,(\mathfrak{T})$ containing $\mathfrak{D}\text{-PFl}\,(\mathfrak{T})$ which is **closed** wrt the operation defined by equation (4.134).

   Unfortunately, this still leaves open the question of a minimal *sublattice* of $\mathfrak{D}\text{-Fl}\,(\mathfrak{T})$ which contains $\mathfrak{D}\text{-PFl}\,(\mathfrak{T})$ and is **closed** wrt the operation defined by equation (4.134) because there is no straightforward way of equipping this set with a (complete) lattice structure compatible with that of $\mathfrak{D}\text{-Fl}\,(\mathfrak{T})$.

   Another open question in this context is under which conditions $\mathscr{P}(\mathfrak{D},\mathfrak{T})$ is closed wrt the operation defined by equation (4.134) when $\mathfrak{D}\text{-PFl}\,(\mathfrak{T}) \neq \mathscr{P}(\mathfrak{D},\mathfrak{T})$ (in which case obviously $\left[\mathscr{P}(\mathfrak{D},\mathfrak{T}), \Cup, \Cap\right]$ is a *minimal* label lattice all labels of which admit refutation).

3. For proving the reverse implication of item 5, it would be necessary to show that the counterexample constructed in the proof of the "only if" direction of item 4 can not be in $\mathscr{P}(\mathfrak{D},\mathfrak{T})$. While this seems obvious, for a proof more information about the structure of $\mathscr{P}(\mathfrak{D},\mathfrak{T})$ is needed than has been provided so far. •

Next, it is investigated under which conditions semantic entailment is characterised by refutation.

For simplifying the following proofs, the conditions involved are expanded in the next lemma.

Lemma 4.3.9 (Expanding the definitions of $\boldsymbol{\mathcal{X}} \Vdash [\boldsymbol{x}, \boldsymbol{\ell}]$ and $\boldsymbol{\ell} \sqsubseteq \textbf{inc}\left(\boldsymbol{\mathcal{X}} \cup \left[\neg \boldsymbol{x}, \widetilde{\boldsymbol{\ell}}\right]\right)$)

Given $\mathcal{X} \in L^{\mathrm{Frm}}$ *and* $[x,\ell] \in \mathrm{LFrm}$,

1. $\mathcal{X} \Vdash [x,\ell]$ *iff for all* $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqsubseteq \iota(\ell)\left(\mathrm{Val}(x)\right).$$

2. *If* ℓ *admits refutation, then* $\ell \sqsubseteq \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$ *iff for all* $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqcap \mathcal{F}_{\ell}\left(\mathrm{Val}(\neg x)\right) \sqsubseteq \iota(\ell)(0).$$

Proof

ad 1. By definition (4.97), $\mathcal{X} \Vdash [x,\ell]$ means

$$\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}\left([x,\ell]\right),$$

i. e. for all $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq \mathrm{Mod}\left([x,\ell]\right)(\mathrm{Val}),$$

which means by definition (4.41) of Mod and definition (4.1) of $\models\!\!=$ that for all $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} \iota(\ell)\left(\mathrm{Val}(x)\right),$$

which had to be established.

ad 2. $\ell \sqsubseteq_{\mathfrak{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$ means by (4.125) that

$$\left(\mathrm{cst}\left(\mathcal{X}\cup\left[\neg x,\widetilde{\ell}\right]\right)\overline{0}\right) \subseteqq \iota(\ell),$$

which means by (2.1) that for every $t \in T$ with $t \neq 1$,

$$\mathrm{cst}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t). \tag{4.141}$$

Now, by Proposition 2.1.6.2a, for every $t \in T$, $\iota(\ell)(0) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$, hence (4.141) is equivalent with

$$\mathrm{cst}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0),$$

which means by Definition 4.2.2.2 that for every $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0). \tag{4.142}$$

By (4.42),

$$\mathrm{Mod}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)(\mathrm{Val}) = \mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqcap_{\mathfrak{D}} \mathrm{Mod}\left(\left[\neg x, \widetilde{\ell}\right]\right)(\mathrm{Val}),$$

which means by definition (4.41) of Mod and definition (4.1) of $\models\!\!=$

$$= \mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqcap_{\mathfrak{D}} \iota\left(\widetilde{\ell}\right)\left(\mathrm{Val}(\neg x)\right)$$

and by definition of $\widetilde{\ell}$,

$$= \mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqcap_{\mathfrak{D}} \mathcal{F}_{\ell}\left(\mathrm{Val}(\neg x)\right),$$

which, combined with (4.142), establishes the claim of this item. □

Theorem 4.3.10 (From entailment to refutation)
Let $\ell \in \mathfrak{L}$ be given such that ℓ admits refutation.
Then the statement

$$\text{For all } \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } x \in \mathrm{Frm}, \text{ if } \mathcal{X} \Vdash [x,\ell], \text{ then } \ell \sqsubseteq_{\mathfrak{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right) \tag{4.143}$$

holds if and only if for all $t \in T$,

$$\iota(\ell)(t) \sqcap_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0). \tag{4.144}$$

Proof

Let $\ell \in \mathfrak{L}$ be given such that ℓ admits refutation. Both directions of the claimed equivalence are proved separately.

"if". Assume that (4.144) holds. For proving (4.143), let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ be given. It is proved that under the assumption $\mathcal{X} \Vdash [x, \ell]$, it holds that $\ell \sqsubseteq_{\mathbb{L}} \operatorname{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$.

By Lemma 4.3.9.2, it is sufficient to prove that for every $\mathrm{Val} \in \mathfrak{S}$,

$$\operatorname{Mod}(\mathcal{X})(\mathrm{Val}) \sqcap_{\mathbb{D}} \mathcal{F}_\ell\left(\mathrm{Val}(\neg x)\right) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0).$$

From the assumption $\mathcal{X} \Vdash [x, \ell]$, by Lemma 4.3.9.1 it is sufficient to prove that for every $\mathrm{Val} \in \mathfrak{S}$,

$$\iota(\ell)\left(\mathrm{Val}(x)\right) \sqcap_{\mathbb{D}} \mathcal{F}_\ell\left(\mathrm{Val}(\neg x)\right) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0). \tag{4.145}$$

By (4.133), two cases are distinguished.

1. $\mathrm{Val}(\neg x) = 1$.
   In this case,

   $$\mathcal{F}_\ell\left(\mathrm{Val}(\neg x)\right) = 1$$

   hence by (4.145), it is sufficient to prove

   $$\iota(\ell)\left(\mathrm{Val}(x)\right) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0).$$

   From $\mathrm{Val}(\neg x) = 1$, it follows by the fact that $\neg$ is interpreted by $\nu_{\mathfrak{T}}$ and $\nu_{\mathfrak{T}}$ is an order-reversing bijection that $\mathrm{Val}(x) = 0$, which establishes the claim.
2. $\mathrm{Val}(\neg x) \neq 1$.
   In this case,

   $$\begin{aligned}\mathcal{F}_\ell\left(\mathrm{Val}(\neg x)\right) &= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\nu_{\mathfrak{T}}\left(\mathrm{Val}(\neg x)\right)\right)\right)\\ &= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\mathrm{Val}(x)\right)\right)\end{aligned}$$

   as $\neg$ is interpreted by $\nu_{\mathfrak{T}}$ and $\nu_{\mathfrak{T}}$ is involutive, hence by (4.145), it is to be proved that

   $$\iota(\ell)\left(\mathrm{Val}(x)\right) \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\mathrm{Val}(x)\right)\right) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0),$$

   which holds by assumption (4.144).

"only if". Property (3.2) is needed for proving this item.

The contraposition of the "only if" direction is proved, i. e. it is proved that from the assumption that (4.144) does **not** hold, it follows that (4.143) does not hold.

For disproving (4.143), it is established that there exist $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ such that $\mathcal{X} \Vdash [x, \ell]$, but $\ell \sqsubseteq_{\mathbb{L}} \operatorname{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$ does not hold.

From the assumption that (4.144) does not hold, it follows that there exists $t \in T$ such that

$$\iota(\ell)(t) \sqcap_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0). \tag{4.146}$$

does not hold.

By assumption (3.2), there exist $x_t \in \mathrm{Frm}$ and $\mathrm{Val}_t \in \mathfrak{S}$ such that $\mathrm{Val}_t(x_t) = t$. Consider $x =_{\mathrm{def}} x_t$ and $\mathcal{X} =_{\mathrm{def}} [x_t, \ell]$.

Obviously, $\mathcal{X} \Vdash [x, \ell]$. For disproving $\ell \sqsubseteq_{\mathfrak{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$, by Lemma 4.3.9.2 it is sufficient to show that there exists $\mathrm{Val} \in \mathfrak{S}$ such that

$$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqcap_{\mathfrak{D}} \mathcal{F}_\ell\left(\mathrm{Val}(\neg x)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$$

does not hold. Choose $\mathrm{Val} =_{\mathrm{def}} \mathrm{Val}_t$, i.e. it is to be proved that

$$\mathrm{Mod}\left([x_t, \ell]\right)(\mathrm{Val}_t) \sqcap_{\mathfrak{D}} \mathcal{F}_\ell\left(\mathrm{Val}_t\,(\neg x_t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0) \tag{4.147}$$

does not hold.

By definition (4.41) of Mod and definition (4.1) of $\models\!=$,

$$\mathrm{Mod}\left([x_t, \ell]\right)(\mathrm{Val}_t) = \iota(\ell)(t). \tag{4.148}$$

Furthermore, clearly $t \neq 0$, because otherwise, a contradiction to the assumption that (4.146) does not hold would occur. Hence,

$$\begin{aligned}\mathcal{F}_\ell\left(\mathrm{Val}_t\,(\neg x_t)\right) &= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\nu_{\mathfrak{T}}\left(\mathrm{Val}_t\,(\neg x_t)\right)\right)\right)\\ &= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\mathrm{Val}_t\,(x_t)\right)\right)\end{aligned}$$

as $\neg$ is interpreted by $\nu_{\mathfrak{T}}$ and $\nu_{\mathfrak{T}}$ is involutive

$$= \nu_{\mathfrak{D}}\left(\iota(\ell)\,(t)\right). \tag{4.149}$$

That (4.147) does not hold now follows immediately from (4.146) by inserting (4.148) and (4.149) into (4.147). □

The following observations illustrate the criterion (4.144) by exhibiting some cases when it holds. First, some cases which assure (4.144) independently of the lattices $\mathfrak{D}$, $\mathfrak{T}$.

Observation 4.3.11 (Criteria for going from entailment to refutation)

1. (4.144) holds for all $\ell \in \mathfrak{L}$ such that for every $t \in T$,

$$\iota(\ell)(t) = \iota(\ell)(0) \qquad \text{or} \qquad \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t).$$

2. (4.144) holds for all $\ell \in \mathfrak{L}$ such that

$$\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0).$$

3. If $\nu_{\mathfrak{D}}$ has a **fixed point** $e \in D$ (i.e. $\nu_{\mathfrak{D}}(e) = e$), then (4.144) holds for all $\ell \in \mathfrak{L}$ such that $e \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$. •

Proof

ad 1. Let $t \in T$. If $\iota(\ell)(t) = \iota(\ell)(0)$, then (4.144) follows immediately.

If $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq \iota(\ell)(t)$, then $\nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq \iota(\ell)(0)$ follows by the fact that $\nu_{\mathfrak{D}}$ is order-reversing and involutive, and from this (4.144) follows immediately.

ad 2. Let $t \in T$. By Proposition 2.1.6.2a, $\iota(\ell)(0) \sqsubseteq \iota(\ell)(t)$. From this, $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq \iota(\ell)(t)$ follows by assumption, hence the claim follows from the previous item.

ad 3. Follows immediately from the previous item because obviously, if $e \sqsubseteq \iota(\ell)(0)$, then $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq \iota(\ell)(0)$. □

Remarks

1. In Observation 4.3.11.1, the requirement $\iota(\ell)(t) = \iota(\ell)(0)$ is equivalent with $\iota(\ell)(t) \sqsubseteq \iota(\ell)(0)$, as $\iota(\ell)(0) \sqsubseteq \iota(\ell)(t)$ follows from Proposition 2.1.6.2a.

   If $\iota(\ell)(0) \sqsubseteq \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$, it is easily proved that in the case $\iota(\ell)(0) \sqsubseteq \iota(\ell)(t) \sqsubseteq \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$, (4.144) does **not** hold, hence in this case, the range of validity degrees between $\iota(\ell)(0)$ and $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$ is *taboo* for labels for which entailment should imply refutation.

2. Observation 4.3.11.2 requires labels to have a high level of uncertainty. Even the truth value 0 (standing for "absolutely false") needs to be assigned a validity degree high enough to allow $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq \iota(\ell)(0)$.

   Observation 4.3.11.1 allows these and additionally labels which express "uncertainty about truth".

   In this class of labels, which contains ℓ_1^t for all $t \in T$, every truth value has to be assigned a validity degree which is either very "low" (equal to $\iota(\ell)(0)$) or very "high" (allowing $\iota(\ell)(0) \sqcup \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq \iota(\ell)(t)$), hence such labels are useful mainly for characterising a set of 'designated truth values' (being assigned a "high" validity degree), between which little variation of the assigned validity degree is possible. Non-designated truth values are all assigned validity degree $\iota(\ell)(0)$. •

The next proposition clarifies under which circumstances (4.144) holds for all labels.

Proposition 4.3.12 (When do all labels allow to go from entailment to refutation?)
(4.144) *holds for all* $\ell \in \mathfrak{L}$ *if and only if*

(i) $\mathfrak{D}$ *is a* ***complementary lattice*** *a complementation of which is represented by* $\nu_{\mathfrak{D}}$ *or*

(ii) $\mathfrak{T}$ *is two-valued.*

Proof

Both implications are proved separately.

"if". For both conditions (i) and (ii), it is proved separately that for all $\ell \in \mathfrak{L}$, (4.144) holds. Let $\ell \in \mathfrak{L}$ be given.

ad (i). If $\nu_{\mathfrak{D}}$ represents a **complement** in $\mathfrak{D}$, then for all $t \in T$,

$$\iota(\ell)(t) \sqcap \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) = 0,$$

from which (4.144) follows immediately.

ad (ii). Let $t \in T$. As $\mathfrak{T}$ is assumed to be two-valued, it is sufficient to distinguish two cases:

1. $t = 0$.

$$\iota(\ell)(0) \sqcap_{\mathrm{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq_{\mathrm{D}} \iota(\ell)(0)$$

   holds trivially, establishing (4.144) in this case.

2. $t = 1$.
   From Definition 2.1.1.3, it follows that $\iota(\ell)(t) = 1$, from which it follows that $\nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) = 0$, hence

$$\iota(\ell)(t) \sqcap_{\mathrm{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) = 0 \sqsubseteq_{\mathrm{D}} \iota(\ell)(0),$$

   establishing (4.144) in this case.

"only if". For proving the contraposition, it is proved that under the assumption that (i) and (ii) do **not** hold, there exists $\ell \in \mathfrak{L}$ such that (4.144) does not hold.

Assume T contains an element τ such that $\tau \neq 0$ and $\tau \neq 1$. Assume furthermore that $\nu_{\mathfrak{D}}$ does not represent a **complement** in $\mathfrak{D}$. This means there exists $d \in D$ such that

$$d \sqcap_{\mathrm{D}} \nu_{\mathfrak{D}}(d) \neq 0.$$

It is sufficient to present a label ℓ for which (4.144) does not hold.

Consider $\ell_d^\tau = \iota\left({}^{d}\overline{\tau}\right)$. From the fact that $\tau \neq 0$, it follows by definition (2.1) that $\iota\left(\ell_d^\tau\right)(0) = 0$. From the fact that $\tau \neq 1$, it follows by definition (2.1) that $\iota\left(\ell_d^\tau\right)(\tau) = d$. Hence

$$\iota\left(\ell_d^\tau\right)(\tau) \sqcap_{\mathrm{D}} \nu_{\mathfrak{D}}\left(\iota\left(\ell_d^\tau\right)(\tau)\right) = d \sqcap_{\mathrm{D}} \nu_{\mathfrak{D}}(d) \neq 0,$$

from which it follows immediately that (4.144) does not hold. □

Theorem 4.3.13 (From refutation to entailment)
Let $\ell \in \mathfrak{L}$ be given such that ℓ admits refutation.
Then the statement

(4.150) *For all $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$, if $\ell \sqsubseteq_{\mathrm{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$, then $\mathcal{X} \Vdash [x, \ell]$*

holds if and only if for all $t \in T \setminus \{0\}$ and all $d \in D$,

(4.151) *if $d \sqcap_{\mathrm{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathrm{D}} \iota(\ell)(0)$, then $d \sqsubseteq_{\mathrm{D}} \iota(\ell)(t)$.*

Proof
Let $\ell \in \mathfrak{L}$ be given such that ℓ admits refutation. Both directions of the claimed equivalence are proved separately.

"if". Assume that (4.151) holds. For proving (4.150), let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ be given. It is proved that under the assumption $\ell \sqsubseteq_{\mathrm{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$, it holds that $\mathcal{X} \Vdash [x, \ell]$.

By Lemma 4.3.9.2, $\ell \sqsubseteq_{\mathrm{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$ means that for every $\mathrm{Val} \in \mathfrak{S}$,

(4.152) $$\mathrm{Mod}\left(\mathcal{X}\right)\left(\mathrm{Val}\right) \sqcap_{\mathrm{D}} \mathcal{F}_\ell\left(\mathrm{Val}(\neg x)\right) \sqsubseteq_{\mathrm{D}} \iota(\ell)(0).$$

It is to be proved that $\mathcal{X} \Vdash [x, \ell]$, which means by Lemma 4.3.9.1 that for all $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqsubseteq_{\mathbb{D}} \iota(\ell)\,\big(\mathrm{Val}(x)\big)\,.$$

By (4.133), two cases are distinguished.

1. $\mathrm{Val}(\neg x) = 1$.
   In this case,

   $$\mathcal{F}_\ell\,\big(\mathrm{Val}(\neg x)\big) = 1,$$

   hence by (4.152),

   $$\begin{aligned}\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) &\sqsubseteq_{\mathbb{D}} \iota(\ell)(0)\\ &\sqsubseteq_{\mathbb{D}} \iota(\ell)\,\big(\mathrm{Val}(x)\big)\,, && \text{(by Proposition 2.1.6.2a)}\end{aligned}$$

   which had to be proved.
2. $\mathrm{Val}(\neg x) \neq 1$.
   In this case,

   $$\begin{aligned}\mathcal{F}_\ell\,\big(\mathrm{Val}(\neg x)\big) &= \nu_{\mathfrak{D}}\left(\iota(\ell)\,\Big(\nu_{\mathfrak{T}}\,\big(\mathrm{Val}(\neg x)\big)\Big)\right)\\ &= \nu_{\mathfrak{D}}\,\Big(\iota(\ell)\,\big(\mathrm{Val}(x)\big)\Big)\end{aligned}$$

   as $\neg$ is interpreted by $\nu_{\mathfrak{T}}$ and $\nu_{\mathfrak{T}}$ is involutive, hence by (4.152),

   $$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}}\,\Big(\iota(\ell)\,\big(\mathrm{Val}(x)\big)\Big) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0).$$

   As $\mathrm{Val}(x) \neq 0$ by the assumption of this case and the fact that $\neg$ is interpreted by $\nu_{\mathfrak{T}}$ and $\nu_{\mathfrak{T}}$ is order-reversing, (4.152) can be applied to yield

   $$\mathrm{Mod}\,(\mathcal{X})\,(\mathrm{Val}) \sqsubseteq_{\mathbb{D}} \iota(\ell)\,\big(\mathrm{Val}(x)\big)\,,$$

   which had to be proved.

"only if". Property (3.2) is needed for proving this item.

The contraposition of the "only if" direction is proved, i. e. it is proved that from the assumption that (4.151) does **not** hold, it follows that (4.150) does not hold.

For disproving (4.150), it is established that there exist $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ such that $\ell \sqsubseteq_{\mathbb{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$, but $\mathcal{X} \Vdash [x, \ell]$ does not hold.

From the assumption that (4.151) does not hold, it follows that there exist $t \in T \setminus \{0\}$ and $d \in D$ such that

$$d \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}}\,\big(\iota(\ell)(t)\big) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0) \tag{4.153}$$

holds and

$$d \sqsubseteq_{\mathbb{D}} \iota(\ell)(t) \tag{4.154}$$

does **not** hold.

By assumption (3.2), there exist $x_t \in \mathrm{Frm}$ and $\mathrm{Val}_t \in \mathfrak{S}$ such that $\mathrm{Val}_t(x_t) = t$. Consider $x =_{\mathrm{def}} x_t$ and $\mathcal{X} =_{\mathrm{def}} \left[x_t, \ell_d^t\right]$.

First, it is proved that $\ell \sqsubseteq_{\mathbb{L}} \mathrm{inc}\left(\left[x_t, \ell_d^t\right] \cup \left[\neg x_t, \widetilde{\ell}\right]\right)$. By Lemma 4.3.9.2 it is sufficient to show that for all $\mathrm{Val} \in \mathfrak{S}$,

$$\mathrm{Mod}\left(\left[x_t, \ell_d^t\right]\right)(\mathrm{Val}) \sqsupset_{\mathbb{D}} \mathcal{F}_\ell\left(\mathrm{Val}(\neg x_t)\right) \sqsubseteq_{\mathbb{D}} \iota(\ell)(0). \tag{4.155}$$

Let $\mathrm{Val} \in \mathfrak{S}$. By definition (4.41) of Mod and definition (4.1) of $\models$,

$$\mathrm{Mod}\left(\left[x_t, \ell_d^t\right]\right)(\mathrm{Val}) = \iota\left(\ell_d^t\right)\left(\mathrm{Val}(x_t)\right) = {}^d\overline{t}\left(\mathrm{Val}(x_t)\right).$$

By (2.1), for establishing (4.155), three cases are distinguished:

1. $\mathrm{Val}(x_t) = 1$.
   In this case, $\mathrm{Val}(\neg x_t) = 0$, hence by (4.133),

   $$\mathcal{F}_\ell\left(\mathrm{Val}(\neg x_t)\right) = \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\nu_{\mathfrak{T}}\left(\mathrm{Val}(\neg x_t)\right)\right)\right),$$

   and as $\neg$ is interpreted by $\nu_{\mathfrak{T}}$ and $\nu_{\mathfrak{T}}$ is involutive,

   $$\begin{aligned} &= \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\mathrm{Val}(x_t)\right)\right), \\ &= \nu_{\mathfrak{D}}\left(\iota(\ell)(1)\right) \\ &= \nu_{\mathfrak{D}}(1) \\ &= 0, \end{aligned}$$

   from which (4.155) follows immediately.
2. $\mathrm{Val}(x_t) \neq 1$ and $t \sqsubseteq_{\mathbb{T}} \mathrm{Val}(x_t)$.
   In this case,

   $$\mathrm{Mod}\left(\left[x_t, \ell_d^t\right]\right)(\mathrm{Val}) = {}^d\overline{t}\left(\mathrm{Val}(x_t)\right) = d. \tag{4.156}$$

   Furthermore, as $t \neq 0$ by assumption, from $t \sqsubseteq_{\mathbb{T}} \mathrm{Val}(x_t)$ it follows immediately that $\mathrm{Val}(x_t) \neq 0$, hence $\mathrm{Val}(\neg x_t) \neq 1$ and from (4.133),

   $$\mathcal{F}_\ell\left(\mathrm{Val}(\neg x_t)\right) = \nu_{\mathfrak{D}}\left(\iota(\ell)\left(\mathrm{Val}(x_t)\right)\right) \tag{4.157}$$

   follows as in the previous item.
   Finally, from

   $$t \sqsubseteq_{\mathbb{T}} \mathrm{Val}(x_t)$$

   it follows by Proposition 2.1.6.2a that

   $$\iota(\ell)(t) \sqsubseteq_{\mathbb{D}} \iota(\ell)\left(\mathrm{Val}(x_t)\right),$$

   from which it follows by the fact that $\nu_{\mathfrak{D}}$ is order-reversing that

   $$\nu_{\mathfrak{D}}\left(\iota(\ell)\left(\mathrm{Val}(x_t)\right)\right) \sqsubseteq_{\mathbb{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right), \tag{4.158}$$

   hence (4.155) follows by combining (4.153) with (4.156), (4.157) and (4.158).

3. **not** $t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x_t)$.

   In this case, $\mathrm{Mod}\left(\left[x_t, \ell^t_d\right]\right)(\mathrm{Val}) = {}^d\overline{t}\left(\mathrm{Val}(x_t)\right) = 0$, form which (4.155) follows trivially.

Next, it is proved that $\left[x_t, \ell^t_d\right] \Vdash [x_t, \ell]$ does not hold. By Lemma 4.3.9.1, it is sufficient to prove that there exists $\mathrm{Val} \in \mathfrak{S}$ such that

$$\mathrm{Mod}\left(\left[x_t, \ell^t_d\right]\right)(\mathrm{Val}) \sqsubseteq_{\mathfrak{D}} \iota(\ell)\left(\mathrm{Val}(x_t)\right)$$

does not hold. Choose $\mathrm{Val} =_{\mathrm{def}} \mathrm{Val}_t$, i. e. it is to be proved that

$$(4.159) \qquad {}^d\overline{t}\left(\mathrm{Val}_t(x_t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$$

does not hold.

By assumption (3.2), $\mathrm{Val}_t(x_t) = t$. Furthermore, obviously $t \neq 1$, because otherwise $\iota(\ell)(t) = 1$ and a contradiction to the assumption that (4.154) does not hold would occur. By (2.1), ${}^d\overline{t}\left(\mathrm{Val}_t(x_t)\right) = {}^d\overline{t}\,(t) = d$, from which the fact that (4.159) does not hold follows immediately by assumption (4.154). □

The following propositions illustrate criterion (4.151). First, some necessary and some sufficient conditions for (4.151) to hold for a single label.

Proposition 4.3.14 (Criteria for going from refutation to entailment)

1. *Given $\ell \in \mathfrak{L}$,* (4.151) *holds for ℓ only if for all $t \in T \setminus \{0\}$, it holds that*

$$\text{if } \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)\,, \text{ then } \iota(\ell)(t) = 1.$$

2. (4.151) *holds for all $\ell \in \mathfrak{L}$ such that for all $t \in T \setminus \{0\}$,*

$$\iota(\ell)(t) \in \{0, 1\}\,.$$

3. *If $\mathfrak{D}$ is a chain, then* (4.151) *holds for $\ell \in \mathfrak{L}$ if and only if for all $t \in T \setminus \{0\}$,*

$$\iota(\ell)(t) \sqsubseteq_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \qquad \text{or} \qquad \iota(\ell)(t) = 1.$$

Proof

ad 1. For proving the contraposition, assume that there exists $t \in T \setminus \{0\}$ such that $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ but $\iota(\ell)(t) \neq 1$.

Then defining $d =_{\mathrm{def}} 1$, $d \sqcap_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$ holds, but $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ does **not** hold, as $\iota(\ell)(t) \neq 1$ by assumption. Hence, (4.151) is disproved and the contraposition is proved.

ad 2. Let $\ell \in \mathfrak{L}$ and $t \in T \setminus \{0\}$ be given. Two cases have to be distinguished.

1. $\iota(\ell)(t) = 1$.

   In this case, $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ and hence (4.151) holds trivially.

2. $\iota(\ell)(t) = 0$.

   In this case, $\nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) = 1$, hence $d \sqcap_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$ means $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$. $\iota(\ell)(0) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ follows from Proposition 2.1.6.2a, hence $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ and thus (4.151) follows by the transitivity of $\sqsubseteq_{\mathfrak{D}}$.

ad 3. Assume that $\mathfrak{D}$ is a chain. Let $\ell \in \mathfrak{L}$.

Both implications are proved separately.

"if". Let $t \in T \setminus \{0\}$ be given. Two cases have to be distinguished.

1. $\iota(\ell)(t) = 1$.
   In this case, $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ and hence (4.151) holds trivially.
2. $\iota(\ell)(t) \sqsubseteq_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$.
   This means $\iota(\ell)(0) \sqsubseteq_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right)$, and as $\mathfrak{D}$ is assumed to be a chain, $d \sqsubset_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$ implies $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$. $\iota(\ell)(0) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ follows from Proposition 2.1.6.2a, hence $d \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ and thus (4.151) follows by the transitivity of $\sqsubseteq_{\mathfrak{D}}$.

"only if". For proving the contraposition, assume there exists $t \in T \setminus \{0\}$ such that neither $\iota(\ell)(t) \sqsubseteq_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$ nor $\iota(\ell)(t) = 1$ hold.

As $\mathfrak{D}$ is assumed to be a chain, if $\iota(\ell)(t) \sqsubseteq_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$ does not hold then $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(t)$ holds, hence it follows from item 1 (by the fact that $\iota(\ell)(t) \neq 1$) that (4.151) does not hold, proving the contraposition. □

Remarks

1. Proposition 4.3.14.1 requires a certain level of uncertainty: if $\iota(\ell)(t)$ is high enough to be above $\nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$, then it has to be equal to 1.

2. The "if" part of condition 3 of Proposition 4.3.14 is a relaxation of condition 2 for the special case that $\mathfrak{D}$ is a chain (relaxing the condition $\iota(\ell)(t) = 0$ to $\iota(\ell)(t) \sqsubseteq_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$).

   The sufficient condition 2 cannot easily be made more general for arbitrary lattices, because if neither $\iota(\ell)(t)$ nor $\nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right)$ is equal to 1, (if $\mathfrak{D}$ is not a chain) it is possible that there exists $d \in D$ such that $d \sqsubset_{\mathfrak{D}} \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq_{\mathfrak{D}} \iota(\ell)(0)$, but d is not comparable with $\iota(\ell)(t)$.

   Note, however, that $\iota(\ell)(0)$ is not restricted by this condition. •

The next proposition clarifies under which circumstances (4.151) holds for all labels.

Proposition 4.3.15 (When do all labels allow to go from refutation to entailment?)

1. *If*

   (i) $\mathfrak{D}$ *is a* ***Boolean algebra*** *the complement of which is represented by* $\nu_{\mathfrak{D}}$ *or*

   (ii) $\mathfrak{T}$ *is two-valued,*

   then (4.151) *holds for all* $\ell \in \mathfrak{L}$.

2. *If* (4.151) *holds for all* $\ell \in \mathfrak{L}$*, then*

   (i) $\mathfrak{D}$ *is a* ***complementary*** *lattice the complement of which is uniquely defined and represented by* $\nu_{\mathfrak{D}}$ *or*

   (ii) $\mathfrak{T}$ *is two-valued.*

Proof

ad 1. For both conditions 1.i and 1.ii, it is proved separately that for all $\ell \in \mathfrak{L}$, (4.151) holds. Let $\ell \in \mathfrak{L}$ be given.

ad 1.i. Let $t \in T$ and $d \in D$ be given such that

$$\begin{aligned} d \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}} \big(\iota(\ell)(t)\big) &\sqsubseteq_{\mathbb{D}} \iota(\ell)(0) \\ &\sqsubseteq_{\mathbb{D}} \iota(\ell)\,(t)\,. \qquad \text{(by Proposition 2.1.6.2a)} \end{aligned}$$

This means by (1.4) that

$$d \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}} \big(\iota(\ell)(t)\big) \sqcap_{\mathbb{D}} \iota(\ell)(t) = d \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}} \big(\iota(\ell)(t)\big)$$

from which it follows by the assumption that $\nu_{\mathfrak{D}}$ represents the **complement** in $\mathfrak{D}$, that

$$d \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}} \big(\iota(\ell)(t)\big) = 0 \tag{4.160}$$

from which it follows by the fact that $\mathfrak{D}$ is a BOOLEan algebra that

$$d \sqsubseteq_{\mathbb{D}} \iota(\ell)(t), \tag{4.161}$$

which had to be proved.

ad 1.i. As T is assumed to be two-valued and (4.151) is required to hold only for $t \in T \setminus \{0\}$, it is sufficient to prove (4.151) for $t = 1$. But by Definition 2.1.1.3, $\iota(\ell)(1) = 1$, hence

$$d \sqsubseteq_{\mathbb{D}} \iota(\ell)(1)$$

holds unconditionally for all $d \in D$.

ad 2. For proving the contraposition, it is proved that under the assumption that 2.i and 2.ii do **not** hold, (4.151) does not hold.

Assume T contains an element τ such that $\tau \neq 0$ and $\tau \neq 1$. Assume furthermore that $\mathfrak{D}$ is not a complementary lattice the complement of which is uniquely defined and represented by $\nu_{\mathfrak{D}}$.

The negation of the claim for $\mathfrak{D}$ leads to two cases, in each of which it has to be proved that there exists $\ell \in \mathfrak{L}$ such that (4.151) does not hold.

1. $\nu_{\mathfrak{D}}$ does not represent a **complement** in $\mathfrak{D}$.
   This means there exists $\delta \in D$ such that

   $$\delta \sqcap_{\mathbb{D}} \nu_{\mathfrak{D}}(\delta) \neq 0. \tag{4.162}$$ [1]

   It is sufficient to present a label ℓ for which (4.151) does not hold.
   Consider

   $$\ell =_{\text{def}} \ell_{\delta}^{0} \sqcap_{\mathbb{L}} \ell^{\tau}_{\delta \sqcup_{\mathbb{D}} \nu_{\mathfrak{D}}(\delta)}.$$

   From the fact that $\tau \neq 0$, it follows by definition (2.1) that

   $$\iota\,(\ell)\,(0) = \iota\left(\ell_{\delta}^{0}\right)(0) = {}^{\delta}\overline{0}\,(0) = \delta. \tag{4.163}$$

[1] Note that the case that there exists $\delta \in D$ such that $\delta \sqcup_{\mathbb{D}} \nu_{\mathfrak{D}}(\delta) \neq 1$ is equivalent with (4.162) by the fact that $\nu_{\mathfrak{D}}$ is order-reversing and involutive (just apply $\nu_{\mathfrak{D}}$ to both sides of (4.162)).

Furthermore, $\tau \neq 1$ and from the fact that $\delta \sqsubseteq \delta \sqcup \nu_{\mathfrak{D}}(\delta)$, it follows by definition (2.1) and Observation 2.2.3 that

$$\iota\left(\ell\right)\left(\tau\right) = \iota\left(\ell^{\tau}_{\delta \sqcup \nu_{\mathfrak{D}}(\delta)}\right)\left(\tau\right) = {}^{\delta \sqcup \nu_{\mathfrak{D}}(\delta)}\overline{\tau}\left(\tau\right) = \delta \sqcup \nu_{\mathfrak{D}}(\delta). \tag{4.164}$$

Let $d =_{\mathrm{def}} 1$ and $t =_{\mathrm{def}} \tau$ in (4.151). From (4.163) and (4.164), it follows that

$$\begin{aligned} d \sqcap \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) &= 1 \sqcap \nu_{\mathfrak{D}}\left(\delta \sqcup \nu_{\mathfrak{D}}(\delta)\right) \\ &= \delta \sqcap \nu_{\mathfrak{D}}(\delta) \\ &\sqsubseteq \delta = \iota(\ell)(0). \end{aligned}$$

On the other hand, from (4.162) it follows that

$$\begin{aligned} \iota\left(\ell\right)\left(\tau\right) &= \delta \sqcup \nu_{\mathfrak{D}}(\delta) \\ &= \nu_{\mathfrak{D}}\left(\delta \sqcap \nu_{\mathfrak{D}}(\delta)\right) \\ &\neq 1, \end{aligned}$$

from which it follows immediately that

$$d \sqsubseteq \iota\left(\ell\right)\left(\tau\right),$$

which is equivalent with

$$1 \sqsubseteq \nu_{\mathfrak{D}}\left(\delta \sqcap \nu_{\mathfrak{D}}(\delta)\right),$$

does **not** hold. Consequently, (4.151) does not hold.

2. Complements in $\mathfrak{D}$ are not **unique**.
   By the previous item, it can be assumed that $\nu_{\mathfrak{D}}$ represents a complement in $\mathfrak{D}$.
   The assumption that complements are not unique in $\mathfrak{D}$ means there exist $c, d \in D$ such that

$$c \sqcap d = 0 \ \text{ and } \ c \sqcup d = 1 \ \text{ and } \ d \neq \nu_{\mathfrak{D}}(c). \tag{4.165}$$

Wlg assume that **not** $d \sqsubseteq \nu_{\mathfrak{D}}(c)$ (otherwise, just define $c' =_{\mathrm{def}} \nu_{\mathfrak{D}}(c)$ and $d' =_{\mathrm{def}} \nu_{\mathfrak{D}}(d)$; from the fact that $\nu_{\mathfrak{D}}$ represents a complement in $\mathfrak{D}$, it follows that (4.165) still holds for c', d' and if $d \sqsubseteq \nu_{\mathfrak{D}}(c)$, then $c \sqsubseteq \nu_{\mathfrak{D}}(d)$, hence $d' = \nu_{\mathfrak{D}}(d) \sqsubseteq c = \nu_{\mathfrak{D}}(c')$ does not hold because otherwise, $d' = \nu_{\mathfrak{D}}(c')$).
Consider the label $\ell =_{\mathrm{def}} \ell^{\tau}_{\nu_{\mathfrak{D}}(c)}$. It follows that

$$\begin{aligned} d \sqcap \nu_{\mathfrak{D}}\left(\iota(\ell)(\tau)\right) &= d \sqcap c && \text{(as } \tau \neq 1 \text{ and } \nu_{\mathfrak{D}} \text{ is involutive)} \\ &= 0 && \text{(by (4.165))} \\ &\sqsubseteq \iota(\ell)(0). \end{aligned}$$

On the other hand, $\iota(\ell)(\tau) = \nu_{\mathfrak{D}}(c)$, and $d \sqsubseteq \nu_{\mathfrak{D}}(c)$ does **not** hold by assumption, hence (4.151) is disproved in this case. $\square$

Remark
Note that the requirements 1.i and 2.i from Proposition 4.3.15 are equivalent in the case that $\mathfrak{D}$ is atomic.

Otherwise, requirement 1.i is genuinely stronger than 2.i because complements are unique in every BOOLEan algebra while there exist (non-atomic) non-distributive lattices with unique complements which are thus not BOOLEan algebras.

It seems that 2.i is too weak for proving Proposition 4.3.15.1 (as distributivity is needed for going from (4.160) to (4.161)).

On the other hand, 1.i seems too strong for proving Proposition 4.3.15.2 because distributivity doesn't seem to follow from (4.151).

Whether there is a condition *between* 1.i and 2.i which provides a necessary *and* sufficient condition for (4.151) is left open for future investigations. •

Corollary 4.3.16
If (4.151) *holds for all $\ell \in \mathfrak{L}$, then* (4.144) *holds for all $\ell \in \mathfrak{L}$.*

Proof
Follows by combining Proposition 4.3.15 with Proposition 4.3.12. □

The following series of corollaries combines the results of Theorem 4.3.10, Observation 4.3.11, Proposition 4.3.12, Theorem 4.3.13, Proposition 4.3.14, Proposition 4.3.15, giving criteria for $\mathcal{X} \Vdash [x, \ell]$ to be **characterised by refutation**.

Corollary 4.3.17 (Characterising entailment by refutation)
Let $\ell \in \mathfrak{L}$ be given such that ℓ admits refutation.

Then for all $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$, $\mathcal{X} \Vdash [x, \ell]$ is ***characterised by refutation*** *if and only if*

(i) $\iota(\ell)(t) \sqcap \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq \iota(\ell)(0)$

(ii) *and for all $t \in T \setminus \{0\}$ and all $d \in D$, if $d \sqcap \nu_{\mathfrak{D}}\left(\iota(\ell)(t)\right) \sqsubseteq \iota(\ell)(0)$, then $d \sqsubseteq \iota(\ell)(t)$.*

Corollary 4.3.18 (Criteria for characterising entailment by refutation)
Let $\ell \in \mathfrak{L}$ be given such that ℓ admits refutation.

Then for all $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$, $\mathcal{X} \Vdash [x, \ell]$ is ***characterised by refutation*** *in each of the following cases:*

1. *for all $t \in T \setminus \{0\}$, $\iota(\ell)(t) = 1$;*
2. *$\iota(\ell)(0) = 0$ and for all $t \in T \setminus \{0\}$, $\iota(\ell)(t) \in \{0, 1\}$;*
3. *$\mathfrak{D}$ is a chain and $\iota(\ell)(0) \sqsubseteq \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right)$ and for all $t \in T \setminus \{0\}$, $\iota(\ell)(t) \in \left\{\iota(\ell)(0), \nu_{\mathfrak{D}}\left(\iota(\ell)(0)\right), 1\right\}$.*

Proof
Item 1 follows by combining Observation 4.3.11.1 with Proposition 4.3.14.2.

Item 2 also follows by combining Observation 4.3.11.1 with Proposition 4.3.14.2, taking into account that $\iota(\ell)(0) \sqsubseteq \iota(\ell)(t)$ for every fuzzy filter, hence the case $\iota(\ell)(t) = 0$ is possible only if $\iota(\ell)(0) = 0$.

Item 3 follows by combining Observation 4.3.11.1 with Proposition 4.3.14.3.

Note that all other combinations of items of Observation 4.3.11 and Proposition 4.3.14 are either meaningless or reduce to one of the above cases. □

Corollary 4.3.19 (When do all labels allow to characterise entailment by refutation?)

1. *$\mathcal{X} \Vdash [x, \ell]$ is characterised by refutation for all $\ell \in \mathfrak{L}$ which admit refutation, all $\mathcal{X} \in L^{\mathrm{Frm}}$ and all $x \in \mathrm{Frm}$ if*

   (i) $\mathfrak{D}$ is a **Boolean algebra** *the complement of which is represented by $\nu_{\mathfrak{D}}$ or*

   (ii) $\mathfrak{T}$ is two-valued.

2. *If all $\ell \in \mathfrak{L}$ admit refutation[2], then if for all $\mathcal{X} \in L^{\mathrm{Frm}}$ and all $x \in \mathrm{Frm}$, it holds that*

$$\text{if } \ell \sqsubseteq_{\mathbb{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right), \text{ then } \mathcal{X} \Vdash [x, \ell],$$

   then for all $\mathcal{X} \in L^{\mathrm{Frm}}$ and all $x \in \mathrm{Frm}$, it holds that

$$\text{if } \mathcal{X} \Vdash [x, \ell], \text{ then } \ell \sqsubseteq_{\mathbb{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right).$$

3. *If all $\ell \in \mathfrak{L}$ admit refutation, then $\mathcal{X} \Vdash [x, \ell]$ is characterised by refutation for all $\ell \in \mathfrak{L}$, all $\mathcal{X} \in L^{\mathrm{Frm}}$ and all $x \in \mathrm{Frm}$ only if*

   (i) $\mathfrak{D}$ is a ***complementary*** *lattice the complement of which is uniquely defined and represented by $\nu_{\mathfrak{D}}$ or*

   (ii) $\mathfrak{T}$ is two-valued.

Proof

ad 1. By combining the "if" directions of Proposition 4.3.12 and Proposition 4.3.15.

ad 2. If all labels admit refutation, then the premise of this item implies by Theorem 4.3.13 that (4.151) holds for all $\ell \in \mathfrak{L}$. By Proposition 4.3.15 this means that one of the conditions 2.i or 2.ii of Proposition 4.3.15 holds. This in turn means by Proposition 4.3.12 that (4.144) holds for all $\ell \in \mathfrak{L}$, which implies the conclusion of this item by Theorem 4.3.10.

ad 3. By combining the "only if" directions of Proposition 4.3.12 and Proposition 4.3.15. □

This summary closes subsection 4.3.3. Unfortunately, some of the results, especially the compound results in Corollary 4.3.19, are quite discouraging. It seems that for an effective **refutation system** to exist, strong conditions have to be placed on the lattices $\mathfrak{T}$ and $\mathfrak{D}$.

Note, however, that by Theorem 4.3.13, whenever (4.151) and $\ell \sqsubseteq_{\mathbb{L}} \mathrm{inc}\left(\mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right]\right)$ are fulfilled (for given $\ell, \mathcal{X}, x$ such that ℓ admits refutation), then by Proposition 4.3.1.9, $\ell \sqsubseteq_{\mathbb{L}} \mathrm{Cons}(\mathcal{X})(x)$. Hence, it is possible to **approximate** $\mathrm{Cons}(\mathcal{X})(x)$ (from below) even if not all labels admit refutation or allow $\mathcal{X} \Vdash [x, \ell]$ to be characterised by refutation.

The subject of estimating the **error** made by approximating $\mathrm{Cons}(\mathcal{X})(x)$ using only labels which admit refutation and for which (4.151) holds, is left for future investigations.

4.3.4 Compatibility wrt Logical Operator Symbols; Normal Forms

Next, some properties of the semantic consequence operator are studied which are important for *normal form generation.*

[2] A condition under which all labels admit refutation is given in Theorem 4.3.7.

Proposition 4.3.20 (Compatibility of Cons wrt lattice connectives)

1. *If the logic constituted by* Frm, $\mathfrak{T}$ *and* $\mathfrak{S}$ *contains a binary operator* $\wedge$ *interpreted by* $\sqcap_{\mathrm{T}}$, *then for every* $\mathcal{S} : \mathfrak{S} \to D$ *and all* $x, y \in \mathrm{Frm}$,

$$\text{(4.166)} \qquad \textit{for every } \ell \in L : \mathcal{S} \Vdash [x \wedge y, \ell] \; \textit{ iff } \; \mathcal{S} \Vdash [x, \ell] \; \textit{ and } \; \mathcal{S} \Vdash [y, \ell] \,,$$

$$\text{(4.167)} \qquad \mathrm{Cons}(\mathcal{S})(x \wedge y) = \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y).$$

2. *If the logic constituted by* Frm, $\mathfrak{T}$ *and* $\mathfrak{S}$ *contains a binary operator* $\vee$ *interpreted by* $\sqcup_{\mathrm{T}}$, *then for every* $\mathcal{S} : \mathfrak{S} \to D$ *and all* $x, y \in \mathrm{Frm}$,

$$\text{(4.168)} \qquad \textit{for every } \ell \in L : \textit{if } \mathcal{S} \Vdash [x, \ell] \; \textit{ or } \; \mathcal{S} \Vdash [y, \ell] \,, \textit{ then } \; \mathcal{S} \Vdash [x \vee y, \ell] \,,$$

$$\text{(4.169)} \qquad \mathrm{Cons}(\mathcal{S})(x) \sqcup_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(x \vee y).$$

Proof

ad (4.166)**.** Let $\mathcal{S} : \mathfrak{S} \to D$, $x, y \in \mathrm{Frm}$, and $\ell \in L$ be given. By definition (4.115) of $\Vdash$, (4.166) is equivalent with

$$\mathcal{S} \subseteqq \mathrm{Mod}\big([x \wedge y, \ell]\big) \; \text{ iff } \; \mathcal{S} \subseteqq \mathrm{Mod}\big([x, \ell]\big) \; \text{ and } \; \mathcal{S} \subseteqq \mathrm{Mod}\big([y, \ell]\big) \,,$$

which, by Corollary 4.2.7 and (4.42), is equivalent with

$$\mathcal{S} \subseteqq \mathrm{Mod}\big([x, \ell]\big) \cap \mathrm{Mod}\big([y, \ell]\big) \; \text{ iff } \; \mathcal{S} \subseteqq \mathrm{Mod}\big([x, \ell]\big) \; \text{ and } \; \mathcal{S} \subseteqq \mathrm{Mod}\big([y, \ell]\big) \,,$$

which holds trivially.

ad (4.167)**.** The result is proved in two steps:

1. $\mathrm{Cons}(\mathcal{S})(x \wedge y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y)$.
   It is sufficient to prove

$$\mathrm{Cons}(\mathcal{S})(x \wedge y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(x) \; \text{ and } \; \mathrm{Cons}(\mathcal{S})(x \wedge y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y).$$

   It suffices to prove

$$\text{(4.170)} \qquad \mathrm{Cons}(\mathcal{S})(x \wedge y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(x),$$

   $\mathrm{Cons}(\mathcal{S})(x \wedge y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y)$ follows by symmetry.
   By definition (4.116) of Cons, (4.170) means

$$\bigsqcup_{\mathrm{L}} \big\{ \ell \,\big|\, \ell \in L \; \text{ and } \; \mathcal{S} \Vdash [x \wedge y, \ell] \big\} \sqsubseteq_{\mathrm{L}} \bigsqcup_{\mathrm{L}} \big\{ \ell \,\big|\, \ell \in L \; \text{ and } \; \mathcal{S} \Vdash [x, \ell] \big\} \,,$$

   which follows immediately from (4.166) which implies

$$\big\{ \ell \,\big|\, \ell \in L \; \text{ and } \; \mathcal{S} \Vdash [x \wedge y, \ell] \big\} \subseteqq \big\{ \ell \,\big|\, \ell \in L \; \text{ and } \; \mathcal{S} \Vdash [x, \ell] \big\} \,.$$

2. $\mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(x \wedge y)$.
   By (4.122), this is equivalent with

$$\mathcal{S} \Vdash \big[x \wedge y, \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y) \big] \,.$$

   By (4.166), it is sufficient to prove that

$$\mathcal{S} \Vdash \big[x, \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y) \big] \; \text{ and } \; \mathcal{S} \Vdash \big[y, \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y) \big] \,.$$

From (4.121), it follows that

$$\mathcal{S} \Vdash \left[x, \mathrm{Cons}(\mathcal{S})(x)\right],$$

so $\mathcal{S} \Vdash \left[x, \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y)\right]$ follows from (4.120) by the fact that $\mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(x)$.
$\mathcal{S} \Vdash \left[y, \mathrm{Cons}(\mathcal{S})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{S})(y)\right]$ is proved analogously.

This concludes the proof of this item.

ad (4.168). Let $\mathcal{S} : \mathfrak{S} \to D$, $x, y \in \mathrm{Frm}$, and $\ell \in L$ be given. The claim of this item follows from (4.119) because $[x, \ell] \Vdash [x \vee y, \ell]$ by Observation 4.2.3.2.

ad (4.169). Let $\mathcal{S} : \mathfrak{S} \to D$ and $x, y \in \mathrm{Frm}$ be given. Expanding the definition of Cons, it is sufficient to prove

$$\begin{aligned}(4.171)\quad \bigsqcup_{\mathrm{L}} \left\{\ell \middle| \ell \in L \text{ and } \mathcal{S} \Vdash [x, \ell]\right\} \sqcup_{\mathrm{L}} \bigsqcup_{\mathrm{L}} \left\{\ell \middle| \ell \in L \text{ and } \mathcal{S} \Vdash [y, \ell]\right\} \\ \sqsubseteq_{\mathrm{L}} \bigsqcup_{\mathrm{L}} \left\{\ell \middle| \ell \in L \text{ and } \mathcal{S} \Vdash [x \vee y, \ell]\right\}.\end{aligned}$$

But obviously,

$$\begin{aligned}\bigsqcup_{\mathrm{L}} \left\{\ell \middle| \ell \in L \text{ and } \mathcal{S} \Vdash [x, \ell]\right\} \sqcup_{\mathrm{L}} \bigsqcup_{\mathrm{L}} \left\{\ell \middle| \ell \in L \text{ and } \mathcal{S} \Vdash [y, \ell]\right\} \\ = \bigsqcup_{\mathrm{L}} \left\{\ell \middle| \ell \in L \text{ and } \left(\mathcal{S} \Vdash [x, \ell] \text{ or } \mathcal{S} \Vdash [y, \ell]\right)\right\}.\end{aligned}$$

From (4.168), it follows that

$$\left\{\ell \middle| \ell \in L \text{ and } \left(\mathcal{S} \Vdash [x, \ell] \text{ or } \mathcal{S} \Vdash [y, \ell]\right)\right\} \subseteq \left\{\ell \middle| \ell \in L \text{ and } \mathcal{S} \Vdash [x \vee y, \ell]\right\},$$

from which (4.171) follows immediately. □

The following Corollary is easily established using (4.118).

Corollary 4.3.21 (Compatibility of Cons wrt lattice connectives)

1. *If the logic constituted by* Frm, $\mathfrak{T}$ *and* $\mathfrak{S}$ *contains a binary operator symbol* $\wedge$ *interpreted by* $\sqcap_{\mathrm{T}}$, *then for every* $\mathcal{X} : \mathrm{Frm} \to L$ *and all* $x, y \in \mathrm{Frm}$,

$$(4.172)\qquad \textit{for every } \ell \in L : \mathcal{X} \Vdash [x \wedge y, \ell] \textit{ iff } \mathcal{X} \Vdash [x, \ell] \textit{ and } \mathcal{X} \Vdash [y, \ell],$$

$$(4.173)\qquad \mathrm{Cons}(\mathcal{X})(x \wedge y) = \mathrm{Cons}(\mathcal{X})(x) \sqcap_{\mathrm{L}} \mathrm{Cons}(\mathcal{X})(y).$$

2. *If the logic constituted by* Frm, $\mathfrak{T}$ *and* $\mathfrak{S}$ *contains a binary operator symbol* $\vee$ *interpreted by* $\sqcup_{\mathrm{T}}$, *then for every* $\mathcal{X} : \mathrm{Frm} \to L$ *and all* $x, y \in \mathrm{Frm}$,

$$(4.174)\qquad \textit{for every } \ell \in L : \textit{if } \mathcal{X} \Vdash [x, \ell] \textit{ or } \mathcal{X} \Vdash [y, \ell], \textit{ then } \mathcal{X} \Vdash [x \vee y, \ell],$$

$$(4.175)\qquad \mathrm{Cons}(\mathcal{X})(x) \sqcup_{\mathrm{L}} \mathrm{Cons}(\mathcal{X})(y) \sqsubseteq_{\mathrm{L}} \mathrm{Cons}(\mathcal{X})(x \vee y).$$

By Proposition 4.3.1.4, the results of the replacement theorems Theorem 4.2.6 and Theorem 4.2.8 are transferred to Cons.

Corollary 4.3.22 (to Theorem 4.2.6 **and** Theorem 4.2.8**)**
Let $\mathcal{X} \in L^{\mathrm{Frm}}$.

1. *Let* $x, y \in \mathrm{Frm}$ *with* $x \equiv y$*. Then*

$$\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}\Big((\mathcal{X} \setminus \{y\}) \cup \big[x, \mathcal{X}(y)\big]\Big). \tag{4.176}$$

2. *Let a formula* $y \in \mathrm{Frm}$ *and a finite set* $Y = \{y_1, \ldots, y_n\} \subseteqq \mathrm{Frm}$*, for* $n \in \mathbb{N}$*, be given, such that* $\{y\} \equiv Y$*.*

   Then

$$\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}\Big((\mathcal{X} \setminus \{y\}) \cup \big[y_1, \mathcal{X}(y)\big] \cup \cdots \cup \big[y_n, \mathcal{X}(y)\big]\Big). \tag{4.177}$$

3. *If the logic constituted by* Frm, $\mathfrak{T}$ *and* $\mathfrak{S}$ *contains a binary operator symbol* $\wedge$ *interpreted by* $\overline{\sqcap}$*, then for all* $x, y \in \mathrm{Frm}$*,*

$$\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}\Big((\mathcal{X} \setminus \{x \wedge y\}) \cup \big[x, \mathcal{X}(x \wedge y)\big] \cup \big[y, \mathcal{X}(x \wedge y)\big]\Big). \tag{4.178}$$

4. *If* $N \subseteqq \mathrm{Frm}$ *is a* ***semantic covering*** *of* $\mathrm{supp}\, \mathcal{X}$*, then*

$$\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}\Big(\bigcup \big\{[x, \ell_x] \,\big|\, x \in N\big\}\Big). \tag{4.179}$$

   (Where ℓ_x *is defined by* (4.65)*.)*

5. *If* $\mathscr{T}$ *is a* ***semantic-preserving syntax transformation operator*** *(see* Definition 3.3.3.2*) wrt* $\mathrm{supp}\, \mathcal{X}$*, then*

$$\mathrm{Cons}(\mathcal{X}) = \mathrm{Cons}\left(\bigcup \left\{\bigcup \big\{[y, \ell_x] \,\big|\, y \in \mathscr{T}(x)\big\} \,\middle|\, x \in \mathrm{supp}\, \mathcal{X}\right\}\right). \tag{4.180}$$

Similarly, Example 4.2.1 can be extended to Cons:

Example 4.3.1 (Semantic consequence and clausal form)
Let a **logic of graded truth and graded trust assessment** $\big[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=\big]$ be given exactly as in Example 4.2.1.

Furthermore, let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $[x, \ell] \in \mathrm{LFrm}$ be given.

By Proposition 4.2.10, there exist $\mathcal{X}_{\mathsf{Cls}} \in L^{\mathrm{Cls}}$ such that $\mathcal{X} \equiv \mathcal{X}_{\mathsf{Cls}}$ and $x_{\mathrm{Cnf}} \in \mathrm{Cnf}$ such that $x \equiv x_{\mathrm{Cnf}}$.

Let $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathrm{Cls}$ be such that

$$x_{\mathrm{Cnf}} = \bigwedge_{i=1}^{n} c_i.$$

By Proposition 4.3.1.4,

$$\mathcal{X} \Vdash [x, \ell] \text{ iff } \mathcal{X}_{\mathsf{Cls}} \Vdash [x, \ell].$$

By (4.172),

$$\mathcal{X} \Vdash [x, \ell] \text{ iff for every } i \in \{1, \ldots, n\} : \mathcal{X} \Vdash [c_i, \ell],$$

hence

$$\mathcal{X} \Vdash [x, \ell] \text{ iff for every } i \in \{1, \ldots, n\} : \mathcal{X}_{\mathrm{Cls}} \Vdash [c_i, \ell] \,.$$

This example demonstrates that for lattice-based logics of graded truth and graded trust assessment, it is sufficient to consider **clauses** when studying semantic consequences.

In chapter 5, it is demonstrated how this facilitates the comparison of such logics as well as the construction of **resolution**-based automated deduction systems.

Obviously, the case that the *underlying many-valued logic* contains only the lattice connectives of $\mathfrak{T}$ (and an appropriate negation) is the simplest possible case, and it is not surprising that as soon as an underlying many-valued logic with more expressive power is chosen, the simple clausal form construction presented in this example is no longer suitable. In chapter 6.2.1, a more sophisticated clause-based normal form is presented which is suitable for a wider range of logics of graded truth and graded trust assessment. •

5 On the Expressive Power of Fuzzy Filter-Based Logics

To justify the definitions so far, in this chapter the range of logical systems which can be represented in the form of **fuzzy filter-based logics** is investigated.

Some examples of well-known logics for the representation of graded truth and graded trust assessment are given and it is demonstrated in how far they can be represented in the form of fuzzy filter-based logics. In particular, similarities and differences between the respective logical systems and their interrelationships are pointed out. This type of investigation is often difficult to carry out for logics which have been developed by different people for different purposes, because of differences in terminology and presentation. It is demonstrated how this comparison is facilitated by first casting these different logics in the common framework of fuzzy filter-based logics. Parts of this chapter have been published by the author in [73]

At this point, the reader is encouraged to step back and reread the motivations in section 1.1. Hopefully, the reader will be able to make connections between the general concepts which are introduced in section 1.1 from an intuitive point of view, and the concrete mathematical interpretations of these concepts provided in chapters 2–4.

As a reminder, the classification of logics from page 4 is repeated in the following, augmented by the concepts from chapters 2–4 which take the place of the intuitive concepts from section 1.1.

Logics of graded truth assessment: In this class, all logics are collected for which, when presented as a **fuzzy filter-based logic**, the lattice $\mathfrak{D}$ of degrees of validity (or trust) is the two-valued lattice $\mathfrak{B}$. In this case, it is not possible to express graded trust in a label, and thus such logics are suited mainly for the expression of knowledge pertaining to *graded truth assessment*.

This class of logics is studied in section 5.2. In how far uncertainty can be expressed in these logics is investigated in section 5.4.

Logics of graded trust assessment: In this class, all logics are collected for which, when presented as a **fuzzy filter-based logic**, the lattice $\mathfrak{T}$ of truth values is the two-valued lattice $\mathfrak{B}$. In this case, it is not possible to express graded truth in a label, and thus such logics are suited mainly for the expression of uncertainty with respect to *graded trust assessment*.

This class of logics is studied in section 5.3. A comparison between logics of graded trust assessment and logics of graded truth assessment is given in section 5.4.

Logics of graded truth and graded trust assessment: In this class, all logics are collected for which, when presented as a **fuzzy filter-based logic**, neither the lattice $\mathfrak{T}$ of truth values nor the lattice $\mathfrak{D}$ of degrees of validity is the two-valued lattice $\mathfrak{B}$. In this case, a label can express graded truth assessment as well as graded trust assessment, yielding a logic of very high expressive power.

As this is the most complex case which can be represented by fuzzy filter-based logics, in fact chapter 4 provides a detailed survey of the properties of logics from this class. Specific examples and a discussion of the expressive power of such logics are given in section 5.5.

Apart from discussing and comparing specific examples of **fuzzy filter-based logics**, there are further aspects in connection with the expressive power of fuzzy filter-based logics worth mentioning.

In section 5.6, the issue of **compositionality** is addressed which has been discussed at length in the literature on **uncertainty logics** (see for instance [26, 27]). The unifying framework of **fuzzy filter-based logics** makes it possible to define and discuss this matter much more precisely than usual.

A comparison of fuzzy filter-based logics with other logical paradigms of similar expressive power is provided in section 5.7.

5.1 Degrees of Truth vs. Degrees of Validity

Before the different logical systems are investigated, a more explicit version of sections 1.1.1 and 1.1.2 is presented. Most of the contents of these sections is repeated here, augmented with details on the concrete representation of the concepts within the framework of **fuzzy filter-based logics**.

These concepts were already used — in the form of the lattices $\mathfrak{T}$ and $\mathfrak{D}$ — in the two preceding chapters, and it should have become clear that values from these lattices were *employed* for very different purposes. The following should make clear that the *meanings* of the values from these lattices are completely different, as well.

5.1.1 Truth Values

1. A **truth value** is induced in a formula by an interpretation of the symbols from the logical language. In the presentation in chapter 3, this fact has been obscured a little by not fixing what exactly the logical language *is*. This led to the definition of **semantics** as a set of truth value assignments for formulae. However, it has to be kept in mind that a **valuation function** Val from the given semantics always reflects an *interpretation* of the logical formulae.

   It is obvious that the concept of a "truth value of a formula" does not make sense without a corresponding interpretation, so "truth" is not a property of a formula in itself, but only of a formula together with an interpretation.

2. It is not a custom in logic to make interpretations 'available'. Instead, when defining higher level concepts like **validity**, **semantic equivalence** or **semantic consequence** (compare definitions 4.2.2, 4.2.3, and 4.3.1), interpretations are usually 'quantified over': The definitions are obtained by quantifying over all interpretations (in this case: all valuation functions from $\mathfrak{S}$) and processing the resulting *set* of truth values, without regard as to which interpretation induced which truth value.

3. From the two previous items, a striking fact can be concluded: Truth values, though one of the most basic concepts of many-valued logics, are for *internal* use only, **not** on the 'user level'. The person defining and using systems of many-valued logics is **not** concerned with interpretations or truth values, but only with validity, semantic equivalence or semantic consequence of formulae.

In fact, when looking at publications concerned *only* with many-valued logics (for instance [5, 8, 78, 100]), it may be observed that truth values play almost no role at all (unless constants for truth values are present in the logical language).

4. To summarise: A **truth value** is a property of a formula together with an **interpretation**; it is not available to the 'user' of a logical system, but is quantified over when defining user-level concepts like **validity**, **semantic equivalence** or **semantic consequence**.

5.1.2 Degrees of Validity

1. **Degrees of validity** are properties of **labelled formulae**. The degree of validity of a labelled formula under an interpretation (degree of satisfaction of the labelled formula by the interpretation) can only be determined by considering the truth value of the formula under the interpretation *and* the label.

   The label expresses the *trust* in the validity of the statement represented by the formula. If the formula is completely true, it should be considered completely valid. But if it is not completely true, it might still be considered somewhat valid (if the statement represented by the formula cannot be trusted to be always completely true).

   The validity of a labelled formula is then calculated by quantifying over the degrees of satisfaction by all possible interpretations.

2. As the degree of validity of a labelled formula depends essentially on the label, the 'user', i. e. the person using the logical system has a strong influence on the resulting validity degree. If they select a very strong label, the formula will be valid only if it is almost completely true under all interpretations. If they select a very weak label, the formula might attain a high degree of validity even if it has a very low truth value under certain interpretations.

3. Degrees of truth are, from an algebraic point of view, obviously **truth-theoretic** in nature, and thus will obey algebraic laws of e. g. MV-algebras, residuated or boolean lattices. In this thesis, the generic form of a complete lattice has been selected as the most general superstructure of all the possible truth-theoretic algebras.

   In contrast with this, degrees of validity seem to be basically **measure-theoretic** in nature. By choosing (again) a complete lattice as the algebraic structure for $\mathfrak{D}$, this dissertation is committed to possibility measures (see [12, 13]). This is not the only choice, however. By choosing $\mathfrak{D}$ to be a HAUSDORFF space with an appropriate definition of integral, it would pose no principal problem to consider degrees of validity as probability degrees, as it has already been investigated for two-valued logics in the field of *probabilistic logics* [37, 77]. The adaption of the definitions and results from this dissertation to the case of probabilistic validity measures is an interesting subject for future investigations.

5.2 Logics of Graded Truth Assessment

In this section, logical systems are discussed which are obtained as **fuzzy filter-based logics** by setting $\mathfrak{D} =_{\text{def}} \mathfrak{B}$. The most well-known examples correspond to the case presented in Corollary 2.3.3.2, but there are also examples of logics corresponding to the case presented in Corollary 2.3.3.3.

If $\mathfrak{D} = \mathfrak{B}$, degrees of validity may be neglected altogether. In this case a **binary** relation $\models$ is defined for $\mathrm{Val} \in \mathfrak{S}$ and $[x, \ell] \in \mathrm{LFrm}$ by

$$\mathrm{Val} \models [x, \ell] =_{\mathrm{def}} \mathrm{Val} \models_{\overline{1}} [x, \ell] \tag{5.1}$$

and analogously for $\mathfrak{L}$-fuzzy sets of formulae. The case that **not** $\mathrm{Val} \models [x, \ell]$ (i. e. $\mathrm{Val} \models_{\overline{0}} [x, \ell]$) is written $\mathrm{Val} \not\models [x, \ell]$.

Examining Definition 4.1.3, a definition for **logic of graded truth assessment** can be given in the special case that $\mathfrak{D} = \mathfrak{B}$, taking into account the definition (5.1) of $\models$.

Definition 5.2.1 (Logic of graded truth assessment)
(In the following definition, excessive use is made of assumption (3.2)*, especially of the valuation* Val_t *and the formula* x_t *with* $\mathrm{Val}_t(x_t) = t$*, for* $t \in T$*.)*

A tuple $\Lambda =_{\mathrm{def}} \left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models\right]$ *is said to be a* ***logic of graded truth assessment***

- *with* ***logical language*** Frm,
- *with* ***truth value lattice*** $\mathfrak{T}$,
- *with* ***semantics*** $\mathfrak{S}$,
- *with* ***label lattice*** $\mathfrak{L}$,
- *and with* ***model relation*** $\models$,

$=_{\mathrm{def}}$

1. Frm *is a nonempty set,*
2. $\mathfrak{T} = [T, \sqcap_T, \sqcup_T]$ *and* $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ *are complete lattices with at least two elements each, with induced partial orders* $\sqsubseteq_T$, $\sqsubseteq_L$*, respectively*
3. $\mathfrak{S} \subseteqq T^{\mathrm{Frm}}$,
4. $\models$ *is a binary relation between* $\mathfrak{S}$ *and* LFrm*,*
5. *if* $x, y \in \mathrm{Frm}$ *and* $\mathrm{Val}, \mathrm{Val}' \in \mathfrak{S}$ *such that* $\mathrm{Val}(x) = \mathrm{Val}'(y)$*, then for all* $\ell \in L$*,*
$$\mathrm{Val} \models [x, \ell] \text{ iff } \mathrm{Val}' \models [y, \ell]\,, \tag{5.2}$$
6. *if* $\ell, \ell' \in L$ *such that* $\ell' \neq \ell$*, then there exists* $t \in T$ *such that*
$$\mathrm{Val}_t \models \left[x_t, \ell'\right] \text{ and } \mathrm{Val}_t \not\models [x_t, \ell] \text{ or } \mathrm{Val}_t \not\models \left[x_t, \ell'\right] \text{ and } \mathrm{Val}_t \models [x_t, \ell]\,, \tag{5.3}$$
7. *for all* $\ell \in L$*,*
$$\mathrm{Val}_1 \models [x_1, \ell]\,, \tag{5.4}$$
8. *for every* $t \in T$*, there exists* $\ell^t \in L$ *such that for* $t' \in T$*,*
$$\mathrm{Val}_{t'} \models \left[x_{t'}, \ell^t\right] \text{ iff } t \sqsubseteq_T t', \tag{5.5}$$
9. *for* $s, t \in T$ *and* $\ell \in L$*,*
$$\mathrm{Val}_s \models [x_s, \ell] \text{ and } \mathrm{Val}_t \models [x_t, \ell] \text{ iff } \mathrm{Val}_{s \sqcap_T t} \models [x_{s \sqcap_T t}, \ell]\,, \tag{5.6}$$
10. *for* $t \in T$ *and* $\ell, \ell' \in L$*,*
$$\mathrm{Val}_t \models \left[x_t, \ell'\right] \text{ and } \mathrm{Val}_t \models [x_t, \ell] \text{ iff } \mathrm{Val}_t \models \left[x_t, \ell' \sqcup_L \ell\right], \tag{5.7}$$
11. *for* $t \in T$ *and* $\ell, \ell' \in L$*,*
$\mathrm{Val}_t \models \left[x_t, \ell' \sqcap_L \ell\right]$ *iff there exist* $t_1, t_2 \in T$ *such that* $\mathrm{Val}_{t_1} \models \left[x_{t_1}, \ell'\right]$ *and* $\mathrm{Val}_{t_2} \models \left[x_{t_2}, \ell\right]$ *and* $t_1 \sqcap_T t_2 \sqsubseteq_T t$.

Remark
The remarks on pages 71–73 accompanying Definition 4.1.3 hold in a more special form (disregarding $\mathfrak{D}$) also for **logics of graded truth assessment**. •

Observations 5.2.1 (Logics of graded truth assessment vs. fuzzy filter-based logics)

1. $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{B}, \mathfrak{L}, \models\!\!=]$ is a **logic of graded truth and graded trust assessment** if and only if $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models]$ is a **logic of graded truth assessment** (where $\models$ is defined by (5.1)) and (see Definition 4.1.3.8 and Definition 5.2.1.8) $\ell^t = \ell_1^t$.

2. $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{B}, \mathfrak{L}, \iota]$ is a **fuzzy filter-based logic** with induced model relation $\models\!\!=$ if and only if $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models]$ is a **logic of graded truth assessment** (where $\models$ is defined by (5.1)) and (see Definition 5.2.1.8) $\iota\left(\ell^t\right) = {}^1\overline{t}$.

3. In the case that $\mathfrak{T}$ is a **chain**, by Proposition 2.3.6 $\mathfrak{L}$ is also a chain, and thus Observation 4.1.4 can be employed to simplify the axioms.

   Item 9 of Definition 5.2.1 is then equivalent with the following **monotonicity** condition:

   9* for $s, t \in T$ and $\ell \in L$,

   $$\text{if } \mathrm{Val}_s \models [x_s, \ell] \text{ and } s \sqsubseteq_{\mathfrak{T}} t \text{, then } \mathrm{Val}_t \models [x_t, \ell]\,. \tag{5.8}$$

   Furthermore, each one of items 10 and 11 is equivalent with

   10* for $t \in T$ and $\ell, \ell' \in L$,

   $$\text{if } \mathrm{Val}_t \models \left[x_t, \ell'\right] \text{ and } \ell' \sqsubseteq_{\mathfrak{L}} \ell \text{, then } \mathrm{Val}_t \models [x_t, \ell]\,. \tag{5.9}$$

   Hence, the class of all fuzzy filter-based logics for a chain $\mathfrak{T}$ and $\mathfrak{D} = \mathfrak{B}$ is completely characterised by the axioms 1, 2, 3, 4, 5, 6, 7, 8, 9*, 10*.

   The examples which are discussed in the remainder of this section shall demonstrate that this simple axiom system characterises an interesting class of logics. •

Proof

ad 1. It is sufficient to observe that the axioms of Definition 5.2.1 are special cases of those given in Definition 4.1.3, taking into account that $\mathfrak{D} = \mathfrak{B}$ and the definition (5.1) of $\models$.

ad 2. Follows from the previous item by applying Observation 4.1.2 and Theorem 4.1.3, taking into account that $\mathfrak{B}$ is completely distributive wrt. its least upper bound.

ad 3. The proof is analogous to that for Observation 4.1.4. □

Corollary 5.2.2 (Admissible label lattices for logics of graded truth assessment)
Given sets $\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}$*, the three following statements are equivalent:*

(i) *there exists* $\models\!\!=$ *such that* $\left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{B}, \mathfrak{L}, \models\!\!=\right]$ *is a* ***logic of graded truth and graded trust assessment***

(ii) *there exists* $\models$ *such that* $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models]$ *is a* ***logic of graded truth assessment***

(iii) $\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}$ *fulfil the axioms 1 to 3 of* Definition 4.1.3 *and there exists a lattice* $\mathfrak{L}'$ *isomorphic with* $\mathfrak{L}$ *such that* $\left[\mathrm{PFl}(\mathfrak{T}), \Cup, \cap\right] \Subset \mathfrak{L}' \Subset \left[\mathrm{Fl}(\mathfrak{T}), \Cup, \cap\right]$.

Proof

The equivalence of items (i) and (ii) follows from Observation 5.2.1.1.

The equivalence of items (i) and (iii) follows from Proposition 2.3.2 (taking into account Observation 5.2.1.2). □

In the light of Corollary 5.2.2, it can safely be assumed that

$$\left[\mathrm{PFl}(\mathfrak{T}), \uplus, \cap\right] \Subset \mathfrak{L} \Subset \left[\mathrm{Fl}(\mathfrak{T}), \uplus, \cap\right],$$

i.e. $\mathfrak{L}$ is a sublattice (containing all principal filters) of the lattice $\mathscr{F}(\mathfrak{T})$ defined in Example 2.3.1.2.

Furthermore, wlg let ι be the isomorphism defined in (2.36). Consequently, in this section, ℓ is identified with the filter for which $\iota(\ell)$ is the characteristic function. For $\models$, this means

$$\begin{aligned} \mathrm{Val} \models [x, \ell] \;&\text{iff}\; \iota(\ell)(\mathrm{Val}(x)) = 1 && \text{(by (5.1))} \\ &\text{iff}\; \mathrm{Val}(x) \in \ell. && \text{(by (2.36))} \end{aligned} \tag{5.10}$$

Remarks

1. The identification of labels with filters means that, as $\mathscr{F}(\mathfrak{T})$ is the **dual** lattice of the lattice of all filters of $\mathfrak{T}$, the induced partial order $\sqsubseteq$ of $\mathfrak{L}$ is the **superset** relation $\supseteq$, the *join* $\sqcup$ is the **set intersection** $\cap$ and the *meet* $\sqcap$ is the **set union** $\cup$.

   This corresponds to the understanding that $\mathfrak{L}$ is ordered by **strength** (a smaller set poses a stronger constraint).

   Moreover, since all labels are filters, the label ℓ of the labelled formula $[x, \ell]$ essentially specifies a **range** of truth values x may assume such that $[x, \ell]$ is still considered valid. Every filter specifies some sort of **interval** of truth values which is closed above with 1. If $\mathfrak{T}$ is a chain or ℓ is a principal filter (see section 5.2.1), this is strictly true.

   So the difference to classical many-valued logic (where a formula x is said to be valid iff the truth value of x is 1) is that some **uncertainty** wrt. the truth of x is allowed, expressed by allowing a larger range of truth values to be assumed by x without challenging the validity of x. In particular, in a labelled formula this uncertainty can be expressed *local* to the formula, by adapting the label of each formula to the exact uncertainty one wishes to express about its truth value.

   The notion of **validity**, however, is still two-valued, so uncertainty cannot be expressed by giving a **degree** of validity to be associated with a labelled formula $[x, \ell]$, depending on the truth value x assumes. $[x, \ell]$ has to be considered valid or not valid at all.

   The issue of many-valued validity is tackled in section 5.3 and ultimately in section 5.5.

2. Note that wrt. the level scheme described in section 3.4, logics of graded truth assessment are located on level 4.

   In particular, *classical many-valued logics* from level 1 can be reduced to the respective logic of graded truth assessment by choosing all labels equal to $\{1\}$ (which is a principal filter of $\mathfrak{T}$ and thus guaranteed to be in L).

   A *many-valued logic* from level 2 (i.e. using a set $D \subsetneqq T$ of **designated truth values**) can be reduced to a corresponding logic of graded truth assessment iff $D \in L$, i.e. D is a *filter* of $\mathfrak{T}$ included in L. The reduction is then done by choosing all labels equal to D. •

The extension (4.38) of the model relation to fuzzy sets of formulae can be considered to be a binary relation analogously to (5.1), yielding

$$\begin{aligned} \mathrm{Val} \models \mathcal{X} \ &\text{iff}\ \forall x \in \mathrm{Frm} : \mathrm{Val} \models \big[x, \mathcal{X}(x)\big] \\ &\text{iff}\ \forall x \in \mathrm{Frm} : \mathrm{Val}(x) \in \mathcal{X}(x). \end{aligned} \tag{5.11}$$

As defined in equation (4.41), an $\mathfrak{L}$-fuzzy set $\mathcal{X}$ on Frm induces on $\mathfrak{S}$ a $\mathfrak{B}$-fuzzy set $\mathrm{Mod}(\mathcal{X})$ of **models** of $\mathcal{X}$. By the two-valuedness of $\mathfrak{B}$, $\mathrm{Mod}(\mathcal{X})$ can be identified with a (classical) set, given by

$$\mathrm{Mod}(\mathcal{X}) = \big\{\mathrm{Val} \big| \ \mathrm{Val} \in \mathfrak{S} \ \text{ and } \ \mathrm{Val} \models \mathcal{X}\big\} \tag{5.12}$$

$$= \big\{\mathrm{Val} \big| \ \mathrm{Val} \in \mathfrak{S} \ \text{ and } \ \forall x \in \mathrm{Frm} : \mathrm{Val}(x) \in \mathcal{X}(x)\big\}\,. \tag{5.13}$$

The set Valid of all **valid** $\mathfrak{L}$-fuzzy sets of formulae is

$$\mathrm{Valid} = \Big\{\mathcal{X} \Big| \mathcal{X} \in L^{\mathrm{Frm}} \ \text{ and } \ \forall\, \mathrm{Val} \in \mathfrak{S} : \mathrm{Val} \models \mathcal{X}\Big\}\,, \tag{5.14}$$

the set Incons of all **inconsistent** $\mathfrak{L}$-fuzzy sets of formulae is

$$\mathrm{Incons} = \Big\{\mathcal{X} \Big| \mathcal{X} \in L^{\mathrm{Frm}} \ \text{ and } \ \forall\, \mathrm{Val} \in \mathfrak{S} : \mathrm{Val} \not\models \mathcal{X}\Big\}\,. \tag{5.15}$$

Remark

Note that by the two-valuedness of $\mathfrak{B}$, the **validity index** valid and the **consistency index** cst add no further information; they just indicate which $\mathfrak{L}$-fuzzy sets of formulae are contained in Valid, Incons, respectively. For instance, for $\mathcal{X} \in L^{\mathrm{Frm}}$,

$$\mathrm{cst}(\mathcal{X}) = \begin{cases} 0, & \text{if } \mathcal{X} \in \mathrm{Incons} \\ 1, & \text{if } \mathcal{X} \notin \mathrm{Incons} \end{cases} \tag{5.16}$$

Furthermore, the **inconsistency distribution** inc also just characterises Incons because by (4.125), for every $\mathcal{X} \in L^{\mathrm{Frm}}$,

$$\begin{aligned} \mathrm{inc}(\mathcal{X}) &= \iota^{-1}\left({}^{\mathrm{cst}(\mathcal{X})}\overline{0}\right) \\ &= \begin{cases} \iota^{-1}\left({}^{0}\overline{0}\right), & \text{if } \mathcal{X} \in \mathrm{Incons} \\ \iota^{-1}\left({}^{1}\overline{0}\right), & \text{if } \mathcal{X} \notin \mathrm{Incons} \end{cases} && \text{(by (5.16))} \\ &= \begin{cases} \ell^{1}, & \text{if } \mathcal{X} \in \mathrm{Incons} \\ \ell^{0}, & \text{if } \mathcal{X} \notin \mathrm{Incons} \end{cases} && \text{(by Observation 5.2.1.2 and Observation 2.2.6)} \\ &= \begin{cases} 1, & \text{if } \mathcal{X} \in \mathrm{Incons} \\ 0, & \text{if } \mathcal{X} \notin \mathrm{Incons} \end{cases} \end{aligned} \tag{5.17}$$

By (4.124), this means that from every $\mathcal{X} \in L^{\mathrm{Frm}}$, $\perp$ either follows completely (which means $\mathcal{X} \in \mathrm{Incons}$) or not at all (which means $\mathcal{X} \notin \mathrm{Incons}$). •

For the **semantic consequence relation**, (5.13) implies

$$\mathcal{X} \Vdash \mathfrak{x} \ \text{ iff } \ \mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}(\mathfrak{x}) \qquad \text{(by Definition 4.3.1)} \tag{5.18}$$

$$\text{iff } \ \text{for all } \mathrm{Val} \in \mathfrak{S}, \text{ if } \mathrm{Val} \models \mathcal{X} \text{ then } \mathrm{Val} \models \mathfrak{x}. \qquad \text{(by (5.12))} \tag{5.19}$$

$$\text{(5.20)} \qquad \operatorname{Cons}(\mathcal{X}) = \bigcup \left\{ \mathfrak{x} \,\middle|\, \mathfrak{x} \in \text{LFrm and } \mathcal{X} \Vdash \mathfrak{x} \right\}$$

$$\text{(5.21)} \qquad \operatorname{Cons}(\mathcal{X})(x) = \bigcup\!\!\!\cdot\; \left\{ \overline{\operatorname{Val}(x)} \,\middle|\, \operatorname{Val} \models \mathcal{X} \right\}. \qquad \text{(by Theorem 4.3.3)}$$

Note that although elements of L have been identified with **sets of truth values**, the greatest lower bound $\bigcup\!\!\cdot$ of $\mathfrak{L}$ does not need to coincide with the set-theoretical least upper bound. $\mathfrak{L}$ is regarded as a **sublattice** of the (dual) lattice of all filters of $\mathfrak{T}$, but although $\mathfrak{L}$ must be complete, it is possible that $\mathfrak{L}$ is **not** a **complete sublattice** of the (dual) lattice of all filters of $\mathfrak{T}$.

Concerning **refutation**, applying Corollary 4.3.19 in this case yields the following Observation.

Observation 5.2.3 (Refutation system for logics of graded truth assessment)
Let the mapping $\nu_{\mathfrak{D}} : \{0,1\} \to \{0,1\}$ from Definition 4.3.4 be given by the **negation operator** $\varphi_\neg$ defined in (3.9). Let a mapping $\nu_{\mathfrak{T}} : T \to T$ be given as specified in (4.131),(4.132) and assume that Frm contains a unary operator symbol $\neg$ interpreted by $\nu_{\mathfrak{T}}$.

Then for all $\mathcal{X} \in L^{\text{Frm}}$ and $[x, \ell] \in \text{LFrm}$ such that ℓ **admits refutation**, $\mathcal{X} \Vdash [x, \ell]$ is **characterised by refutation**, i. e.

$$\text{(5.22)} \qquad \mathcal{X} \Vdash [x, \ell] \text{ iff } \ell = 0 \text{ or } \mathcal{X} \cup \left[\neg x, \widetilde{\ell}\right] \in \text{Incons}.$$

Considering $\widetilde{\ell}$ to be a set of truth values leads to the equation

$$\text{(5.23)} \qquad \widetilde{\ell} = \left\{ \nu_{\mathfrak{T}}(t) \,\middle|\, t \in T \text{ and } t \notin \ell \right\}. \qquad \bullet$$

Proof
Taking into account Corollary 4.3.19, it is sufficient to observe that $\varphi_\neg$ is the (unique) **complementation** on the BOOLEan algebra $\mathfrak{B}$.

The definition (5.22) of being characterised by refutation (in a logic of graded truth assessment) is derived from Definition 4.3.4 by expanding the characterisation (5.17) of inc.

Equation (5.23) follows by simply expanding the definition of $\nu_{\mathfrak{D}}$ in (4.133). □

Remark
Some observations concerning the set of all labels which admit refutation are given in the next two subsections, where special label lattices are considered. •

In Corollary 5.2.2, the range of possible label lattices for logics of graded truth assessment has been specified, in particular, $\mathfrak{L}$ has to lie (up to isomorphism) between the (dual) lattice of all **principal filters** of $\mathfrak{T}$ and the (dual) lattice of all **filters** of $\mathfrak{T}$. In the following two subsections, the two extreme cases $\mathfrak{L} = \left[\text{PFl}(\mathfrak{T}), \uplus, \cap\right]$ and $\mathfrak{L} = \left[\text{Fl}(\mathfrak{T}), \uplus, \cap\right]$ are investigated.

5.2.1 Using Truth Values as Labels

In this subsection, logics of graded truth assessment are studied for which $\mathfrak{L}$ is isomorphic with the (dual) lattice $\left[\text{PFl}(\mathfrak{T}), \uplus, \cap\right]$ of all **principal filters** of $\mathfrak{T}$ (compare Corollary 5.2.2). As $\left[\text{PFl}(\mathfrak{T}), \uplus, \cap\right]$ is isomorphic with $\mathfrak{T}$ by Observation 1.3.1.6, it can safely be assumed that $\mathfrak{L} = \mathfrak{T}$ (compare Corollary 2.3.3.2).

Observation 5.2.4 (Logics of graded truth assessment using principal filters as labels) Given sets Frm, $\mathfrak{S}$ and lattices $\mathfrak{T}, \mathfrak{L}$, the following statements are equivalent:

(i) there exists $\models$ such that $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models]$ is a **logic of graded truth assessment** and $L = \left\{\ell^t \middle| t \in T\right\}$ (compare Definition 5.2.1.8)

(ii) there exists $\models$ such that $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models]$ is a **logic of graded truth assessment** and additionally fulfils the following axiom:

$8'$ for every $\ell \in L$, there exists $t^\ell \in T$ such that for $t' \in T$,

$$\mathrm{Val}_{t'} \models [x_{t'}, \ell] \text{ iff } t^\ell \sqsubseteq_{\mathfrak{T}} t',$$

(iii) $\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}$ fulfil the axioms 1 to 3 of Definition 4.1.3 and $\mathfrak{L}$ is isomorphic with $\left[\mathrm{PFl}(\mathfrak{T}), \cup, \cap\right]$.

(iv) $\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}$ fulfil the axioms 1 to 3 of Definition 4.1.3 and $\mathfrak{L}$ is isomorphic with $\mathfrak{T}$. •

Proof
The equivalence of (i) with (ii) is obvious.

The equivalence of (i) with (iii) follows from Corollary 5.2.2.(iii) and Observation 5.2.1.2.

The equivalence of (iii) with (iv) follows from Observation 1.3.1.6. □

Choosing $\mathfrak{L} = \mathfrak{T}$ leads to a logic where *formulae are labelled by truth values*, an approach well-known in theory and applications of **fuzzy logic**. It seems to have originated in research on *standard* expert systems, where the need for a possibility to deal with uncertain knowledge was felt, but where corresponding reasoning mechanisms were mostly implemented in an *ad hoc* manner (see, for instance, E. Y. Shapiro [90]).

Jan Pavelka gave a systematic study of fuzzy model-theoretic concepts based on this idea and issues of their axiomatisation in 1979 [85–87]. His ideas were taken up and investigated by several researchers (see for instance works by V. Novák and others [79–82,84], by J. L. Castro and E. Trillas [6], by E. Turunen [98, 99], by G. Gerla [41]).

The author of the presented dissertation has investigated this approach from the perspective of **automated reasoning** [66, 67, 72].

The class of logics presented in this subsection corresponds to level 3 in the scheme described in section 3.4.

It is interesting to study the meaning of the **model relation** in this special case.

Let $[x, t] \in \mathrm{LFrm}$ be given. By the identification of $\left[\mathrm{PFl}(\mathfrak{T}), \cup, \cap\right]$ with $\mathfrak{T}$ (wlg it is assumed that $t \in T$ is associated with $\ell^t = \overline{t}$ and vice versa) made in the introduction of this subsection, combining definition (5.10) with the definition (1.15) of the principal filter $\overline{t}$ yields

$$\begin{aligned} \mathrm{Val} \models [x, t] &\text{ iff } \mathrm{Val}(x) \in \overline{t} && \text{(by (5.10))} \\ &\text{ iff } t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x). && \text{(by (1.15))} \end{aligned} \tag{5.24}$$

Analogously, for $\mathcal{X} : \mathrm{Frm} \to T$,

$$\mathrm{Val} \models \mathcal{X} \text{ iff } \forall x \in \mathrm{Frm} : \mathcal{X}(x) \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x). \qquad \text{(combining (5.24) with (5.11))}$$

Remark
This model relation coincides with the one which can be derived from the work of J. PAVELKA [85], so PAVELKA-style logics have been obtained as a special case of fuzzy filter-based logics in the case $\mathfrak{L} = \mathfrak{T}$. •

The equations (5.13), (5.19) and (5.20) for Mod, $\Vdash$ and Cons do not need to be adapted in this special case, but in (5.21), the nature of the infimum can now be specified:

$$\text{(5.25)} \qquad \mathrm{Cons}(\mathcal{X})(x) = \boxed{\top}\left\{\mathrm{Val}(x) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \ \text{ and } \ \mathrm{Val} \models \mathcal{X}\right\},$$

where $\boxed{\top}$ is the infimum in the lattice of truth values.

Remark
This equation is identical with definition (3) of semantic consequence in [85]. •

Concerning **refutation**, applying Proposition 4.3.8 in this case yields the following Observation.

Observation 5.2.5 (When do truth values admit refutation?)
Let the mapping $\nu_{\mathfrak{D}} : \{0,1\} \to \{0,1\}$ from Definition 4.3.4 be given by the **negation operator** $\varphi_\neg$ defined in (3.9). Let a mapping $\nu_{\mathfrak{T}} : T \to T$ be given as specified in (4.131),(4.132).

Then all $\ell \in L$ admit refutation if and only if

> for every $t \in T$ with $t \neq 1$, there exists $t' \in T$ such that for every $s \in T$, it holds that $t' \sqsubseteq_{\mathfrak{T}} s$ iff **not** $s \sqsubseteq_{\mathfrak{T}} t$. •

Proof
Follows immediately from Proposition 4.3.8.4, taking into account that $\mathfrak{D} = \mathfrak{B}$ and $\mathfrak{L}$ is isomorphic with $\left[\mathrm{PFl}(\mathfrak{T}), \uplus, \cap\right]$. □

Remark
Note that the condition of Observation 5.2.5 is fulfilled if $\mathfrak{T}$ is a finite chain. •

In the following two subsections, two special cases of logics of graded truth assessment for which $\mathfrak{L} = \mathfrak{T}$ are discussed.

5.2.1.1 Pavelka-Style Lattice-Based Propositional Logic

In the beginning of the first part of his series on *fuzzy logic* [85–87], J. PAVELKA allows arbitrary sets of formulae and arbitrary semantics. It has been proved above that the whole range of logics allowed by PAVELKA's scheme of definition is identical (up to isomorphism) with logics of graded truth assessment for which $\mathfrak{L} = \mathfrak{T}$.

In the remainder of his series on *fuzzy logic* [85–87], J. PAVELKA studies mainly the special case where the *underlying many-valued logic* is **Łukasiewiczs infinitely many-valued propositional logic**. In this subsection (and the next one), another, simpler underlying many-valued logic shall be studied: **lattice-based propositional logic**.

Let the set of **formulae** be given by Frm $=_{\mathrm{def}}$ $\mathrm{PFrm}_{\mathrm{S}}$, i. e. classical propositional syntax (see Example 3.1.3). Furthermore, let the semantics be based on the lattice connectives, i. e. $\wedge$ and $\vee$ are interpreted by the lattice connectives $\boxed{\sqcap}$ and $\boxed{\sqcup}$ of $\mathfrak{T}$, respectively, and the negation connective $\neg$ is interpreted by a bijective unary function $\varphi_\neg : T \to T$ which is *order-reversing* (see Example 3.2.3). Let $\varphi_\to$ be the *s-implication* of $\boxed{\sqcup}$ wrt. $\varphi_\neg$ (see (3.7)).

In subsection 5.2.1.2, $\mathfrak{T}$ and $\varphi_\neg$ are interpreted by a concrete lattice and a concrete negation function, respectively.

5.2.1.2 Lee's Fuzzy Logic with Truth Value-Labelled Formulae

In this subsection, the case discussed in the previous subsection is made even more concrete. Let $\mathrm{Frm} =_{\mathrm{def}} \mathrm{PFrms_S}$ (as above) and furthermore, $\mathfrak{T} =_{\mathrm{def}} \mathfrak{F} = \left[\langle 0,1\rangle, \min, \max\right]$ and choose the semantics $\mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$ (see Example 3.2.4.2) of LEE's fuzzy logic.

The underlying many-valued logic is in fact one of the first logical systems ever to be called **fuzzy logic**. It was introduced by R. C. T. LEE and C. L. CHANG in 1971 [63, 64].

The labelled version (with truth values as labels) was studied by G. ESCALADA-IMAZ and F. MANYÀ in [33] and forms a special case of R. HÄHNLES *regular logics* [46, 47].

The model and semantic entailment relation and semantic consequence operator defined in (5.24), (5.19), and (5.20), respectively, are denoted by $\models^{\mathrm{L}}, \Vdash_{\mathrm{L}}, \mathrm{Cons_L}$ in this case and the resulting logic of graded truth assessment is denoted by

$$\Lambda_{\mathrm{L}} =_{\mathrm{def}} \left[\mathrm{PFrms_S}, \mathfrak{F}, \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}, \mathfrak{F}, \models^{\mathrm{L}}\right]. \tag{5.26}$$

For convenience (in particular with respect to the comparison in section 5.4), definitions for the basic logical concepts (see equations (5.19), (5.20), (5.24), (5.25)) are repeated here for this special case.

For $x \in \mathrm{PFrms_S}$, $\mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$, and $t \in \langle 0,1\rangle$,

$$\mathrm{Val} \models^{\mathrm{L}} [x,t] \text{ iff } t \leqq \mathrm{Val}(x). \tag{5.27}$$

For $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{PFrms_S}}$,

$$\mathrm{Val} \models^{\mathrm{L}} \mathcal{X} \text{ iff } \forall x \in \mathrm{PFrms_S} : \mathcal{X}(x) \leqq \mathrm{Val}(x). \tag{5.28}$$

For $[x,t] \in \mathrm{LFrm}$,

$$\mathcal{X} \Vdash_{\mathrm{L}} [x,t] \text{ iff } \forall\, \mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}, \text{ if } \mathrm{Val} \models^{\mathrm{L}} \mathcal{X}, \text{ then } \mathrm{Val} \models^{\mathrm{L}} [x,t]. \tag{5.29}$$

$$\begin{aligned}\mathrm{Cons_L}(\mathcal{X})(x) &= \sup\left\{t \,\middle|\, t \in \langle 0,1\rangle \text{ and } \mathcal{X} \Vdash_{\mathrm{L}} [x,t]\right\}\\ &= \inf\left\{\mathrm{Val}(x) \,\middle|\, \mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}} \text{ and } \mathrm{Val} \models^{\mathrm{L}} \mathcal{X}\right\}.\end{aligned} \tag{5.30}$$

In subsequent investigations (see section 5.4), the following trivial observation will be useful.

Observation 5.2.6 (Clausal form in Lee's fuzzy logic with truth value-labelled formulae)
Λ_{L} is a **lattice logic** as defined in Example 3.2.3, and furthermore, $\mathfrak{F}$ is a DE MORGAN algebra wrt. the negation operator $\varphi_\neg$ defined in (3.10). Hence, all conditions given in Example 4.2.1 for the applicability of **conjunctive normal form** are fulfilled.

By Proposition 4.2.10 and Example 4.3.1, it can safely be assumed that all formulae of Λ_{L} are **clauses**. •

For the simplified logical language Cls, the sets Taut of all **tautologies** wrt. $\mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$ (denoted $\mathrm{Taut_F}$) and the sets Valid, Incons of all **valid** and **inconsistent** $\mathfrak{F}$-fuzzy sets of clauses wrt. Λ_{L} (denoted $\mathrm{Valid_L}$, $\mathrm{Incons_L}$) can be given explicitly.

Observation 5.2.7 (Tautologies, validity, and inconsistency in Lee's fuzzy logic)

$$\mathrm{Taut_F} = \emptyset. \tag{5.31}$$

$$\mathrm{Valid_L} = \left\{\mathcal{X} \,\middle|\, \begin{array}{l}\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \forall c \in \mathrm{Cls}:\\ \text{if } \mathcal{X}(c) > 0, \text{ then } \mathcal{X}(c) \leqq \frac{1}{2} \text{ and } \exists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c\end{array}\right\}. \tag{5.32}$$

$$\mathrm{Incons_L} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \forall\, \mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}} : \mathrm{Val} \not\models \mathcal{X}\right\}. \tag{5.33}$$

•

Proof

ad (5.31)**.** Trivial by Definition 3.3.4 and the fact that for every $c \in \mathrm{Cls}$,

$$\exists\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} : \mathrm{Val}(c) \leqq \frac{1}{2}. \tag{5.34}$$

ad (5.32)**.** Start from (5.14):

$$\begin{aligned} \mathrm{Valid}_{\mathrm{L}} &= \left\{ \mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \forall\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} : \mathrm{Val} \mathrel{|\!\!\overset{\mathrm{L}}{=}} \mathcal{X} \right\} \\ &= \left\{ \mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \forall\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}, c \in \mathrm{Cls} : \mathcal{X}(c) \leqq \mathrm{Val}(c) \right\} \quad (\text{by } (5.28)) \end{aligned} \tag{5.35}$$

Let $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}}$ and $c \in \mathrm{Cls}$ such that $\mathcal{X}(c) > 0$ (the case $\mathcal{X}(c) = 0$ is trivial). For establishing that the sets given in (5.32) and (5.35) are equal, it is sufficient to prove that

$$\forall\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} : \mathcal{X}(c) \leqq \mathrm{Val}(c) \text{ iff } \mathcal{X}(c) \leqq \frac{1}{2} \text{ and } \exists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c. \tag{5.36}$$

For proving this, the following trivial observations wrt $\mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ are employed which hold for every $c \in \mathrm{Cls}$:

$$\text{if } \exists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c\,, \text{ then } \forall\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} : \mathrm{Val}(c) \geqq \frac{1}{2} \tag{5.37}$$

$$\text{if } \nexists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c\,, \text{ then } \exists\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} : \mathrm{Val}(c) = 0 \tag{5.38}$$

The "if" direction of (5.36) now follows from (5.34) and (5.38) (taking into account that $\mathcal{X}(c) > 0$) while the "only if" direction of (5.36) follows from (5.37).

ad (5.33)**.** Is identical with (5.15). □

Concerning **refutation**, the following Observation gives a quite negative result for Lee's fuzzy logic with truth value-labelled formulae.

Observation 5.2.8 (Labels from $\langle 0, 1\rangle$ don't admit refutation)

Let the mapping $\nu_{\mathfrak{D}} : \{0,1\} \to \{0,1\}$ from Definition 4.3.4 be given by the **negation operator** $\varphi_\neg$ defined in (3.9). Let a mapping $\nu_{\mathfrak{T}} : \langle 0,1\rangle \to \langle 0,1\rangle$ be given as specified in (4.131),(4.132).

Then $t \in \langle 0,1\rangle$ admits refutation if and only if $t = 0$. •

Proof

It is easily derived from the proof of the "only if" direction of Proposition 4.3.8.4, case 2 on page 105 that in the special case of Lee's fuzzy logic with truth value-labelled formulae, the following holds for every $t \in \langle 0,1\rangle$ with $t \neq 1$:

if there is **no** $t' \in T$ with the property that
for every $s \in T$, it holds that $t' \leqq s$ iff **not** $s \leqq t$,
then $\nu_{\mathfrak{T}}^{-1}(t)$ does not admit refutation.

Obviously, **no** $t \in \langle 0,1)$ has the property that there exists $t' \in T$ such that for every $s \in T$, it holds that $t' \leqq s$ iff **not** $s \sqsubseteq_{\mathrm{T}} t$.

As $\nu_{\mathfrak{T}}^{-1}(1) = 0$, this implies that **no** $t \in (0,1\rangle$ admits refutation.

An easy calculation yields $\mathcal{F}_0 = {}^{1}\overline{1}$ (compare (4.133)), hence 0 admits refutation and $\widetilde{0} = 1$.

This concludes the proof. □

5.2.2 Using Sets of Truth Values as Labels

In this subsection, logics of graded truth assessment are studied for which $\mathfrak{L}$ is isomorphic with the (dual) lattice $\left[\mathrm{Fl}(\mathfrak{T}), \uplus, \cap\right]$ of all **filters** of $\mathfrak{T}$ (compare Corollary 5.2.2). Wlg, let $\mathfrak{L} = \mathscr{F}(\mathfrak{T})$ (see Example 2.3.1.2). This type of logic is very similar to Hähnle's *regular logics* [47].

In recent years, the technique of using set-labelled formulae has been emerging as a theoretical tool for automated deduction in many-valued logics. Recent references (where truth-value sets are used as **signs** in reasoning systems for **finitely many-valued logics**) are R. Hähnle's book [46] and [74] by J. J. Lu, N. V. Murray, and E. Rosenthal.

For **continuously many-valued logics**, the set of *all* sets of truth values is too 'large' to be manageable as a class of labels for formulae.

In [68, 72], however, it has been demonstrated how $\mathscr{F}(\mathfrak{T})$ can be used as a label class for a resolution-based **automated reasoning system** for a labelled extension to Łukasiewicz's continuously many-valued propositional logic, with the model relation (5.10).

Logics of graded truth assessment where $\mathfrak{L} = \mathscr{F}(\mathfrak{T})$ have the advantage (over Pavelka's logic) that the range of labels admitting refutation and hence the range of labelled formulae for which entailment can be characterised by refutation is much larger, and hence they are better suited for resolution-based reasoning.

Applying Theorem 4.3.7 in this case yields the following Observation.

Observation 5.2.9 (When do filters admit refutation?)
Let the mapping $\nu_{\mathfrak{D}} : \{0,1\} \to \{0,1\}$ from Definition 4.3.4 be given by the **negation operator** $\varphi_\neg$ defined in (3.9). Let a mapping $\nu_{\mathfrak{T}} : T \to T$ be given as specified in (4.131),(4.132).

Then all $\ell \in L$ **admit refutation** if and only if $\mathfrak{T}$ is a **chain**. •

Remark
Note that the condition from Observation 5.2.9 is strictly weaker than that from Observation 5.2.5, in fact much weaker. For instance, it allows to choose $\mathfrak{T} = \mathfrak{F}$ (as exploited in [68,72]), while the condition from Observation 5.2.5 does not (compare Observation 5.2.8). •

Combining the above result with Observation 5.2.3 yields the following Corollary.

Corollary 5.2.10 (Refutation in logics of graded truth assessment with filters as labels)
Assume that $\mathfrak{T}$ is a ***chain****.*

Let the mapping $\nu_{\mathfrak{D}} : \{0,1\} \to \{0,1\}$ from Definition 4.3.4 *be given by the* ***negation operator*** *$\varphi_\neg$ defined in* (3.9). *Let a mapping $\nu_{\mathfrak{T}} : T \to T$ be given as specified in* (4.131),(4.132) *and assume that* Frm *contains a unary operator symbol $\neg$ interpreted by $\nu_{\mathfrak{T}}$.*

Then for all $\mathcal{X} \in L^{\mathrm{Frm}}$ and $[x,\ell] \in \mathrm{LFrm}$, ℓ ***admits refutation*** *and $\mathcal{X} \Vdash [x,\ell]$ is* ***characterised by refutation****.*

5.3 Logics of Graded Trust Assessment

In this section, logical systems are discussed which are obtained in the form of **fuzzy filter-based logics** by setting $\mathfrak{T} =_{\mathrm{def}} \mathfrak{B}$. These logical systems correspond to the case presented in Proposition 2.3.4.

Examining Definition 4.1.3 and Observation 4.1.4, a definition for a **logic of graded trust assessment** can be given in the special case that $\mathfrak{T} = \mathfrak{B}$.

Definition 5.3.1 (Logic of graded trust assessment)
A tuple $\Lambda =_{\text{def}} \left[\text{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\right]$ *shall be called a* ***logic of graded trust assessment***

- *with* ***logical language*** Frm,
- *with* ***semantics*** $\mathfrak{S}$,
- *with* ***validity degree lattice*** $\mathfrak{D}$,
- *with* ***label lattice*** $\mathfrak{L}$,
- *and with* ***model relation*** $\models$,

$=_{\text{def}}$

1. Frm *is a nonempty set,*
2. $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$ *and* $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ *are complete lattices with at least two elements each, with induced partial orders* $\sqsubseteq_D$, $\sqsubseteq_L$, *respectively*
3. $\mathfrak{S} \subseteqq \{0, 1\}^{\text{Frm}}$,
4. $\models$ *is a ternary relation on* $\mathfrak{S} \times \text{LFrm} \times D$ *such that for every* $\text{Val} \in \mathfrak{S}$, $x \in \text{Frm}$, *and* $\ell \in L$ *there exists a* ***unique*** $d \in D$ *such that* $\text{Val} \models_{d} [x, \ell]$,
5. *if* $x, y \in \text{Frm}$ *and* $\text{Val}, \text{Val}' \in \mathfrak{S}$ *such that* $\text{Val}(x) = \text{Val}'(y)$, *then for all* $\ell \in L$ *and* $d \in D$,

$$\text{Val} \models_{d} [x, \ell] \ \textit{iff} \ \text{Val}' \models_{d} [y, \ell] \,, \tag{5.39}$$

6. *if* $\ell, \ell' \in L$ *such that* $\ell \neq \ell'$, *then for* $d, d' \in D$,

$$\textit{if } \text{Val}_0 \models_{d} [x_0, \ell] \ \textit{ and } \ \text{Val}_0 \models_{d'} \left[x_0, \ell'\right] \,, \textit{ then } \ d \neq d', \tag{5.40}$$

7. *for all* $\ell \in L$,

$$\text{Val}_1 \models_{1} [x_1, \ell] \,, \tag{5.41}$$

8. *for every* $d \in D$, *there exists* $\ell_d \in L$ *such that*

$$\text{Val}_0 \models_{d} [x_0, \ell_d] \tag{5.42}$$

10. *for* $\ell, \ell' \in L$, *and* $c, d \in D$ *such that*

$$\begin{aligned} & \text{Val}_0 \models_{c} \left[x_0, \ell'\right] \\ \textit{and} \quad & \text{Val}_0 \models_{d} [x_0, \ell] \,, \end{aligned}$$

it holds that

$$\text{Val}_0 \models_{c \sqcap_D d} \left[x_0, \ell' \sqcup_L \ell\right] \,, \tag{5.43}$$

11. *for* $\ell, \ell' \in L$, *and* $c, d \in D$ *such that*

$$\begin{aligned} & \text{Val}_0 \models_{c} \left[x_0, \ell'\right] \\ \textit{and} \quad & \text{Val}_0 \models_{d} [x_0, \ell] \,, \end{aligned}$$

it holds that

$$\text{Val}_0 \models_{c \sqcup_D d} \left[x_0, \ell' \sqcap_L \ell\right] \,. \tag{5.44}$$

Remark
The remarks on pages 71–73 accompanying Definition 4.1.3 hold in a more special form (disregarding $\mathfrak{T}$) also for **logics of graded trust assessment**. •

Observations 5.3.1 (Logics of graded trust assessment vs. fuzzy filter-based logics)

1. In a logic of graded trust assessment $[\mathrm{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models]$ (where $\mathfrak{L} = [L, \sqcap, \sqcup]$), it holds that $L = \{\ell_d | d \in D\}$ (compare Definition 5.3.1.8).

2. $[\mathrm{Frm}, \mathfrak{B}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models]$ is a **logic of graded truth and graded trust assessment** if and only if $[\mathrm{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models]$ is a logic of graded trust assessment and (see Definition 4.1.3.8 and Definition 5.3.1.8) $\ell_d = \ell_d^0$.

3. $[\mathrm{Frm}, \mathfrak{B}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \iota]$ is a **fuzzy filter-based logic** with induced model relation $\models$ if and only if $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models]$ is a logic of graded trust assessment and (see Definition 5.3.1.8) $\iota(\ell_d) = {}^{d}\overline{0}$.

4. $\mathfrak{D}$ is a **chain** iff $\mathfrak{L}$ is a chain, and in this case axioms 10 and 11 can be replaced by the following monotonicity condition:

   10* for $\ell, \ell' \in L$ and $c, d \in D$ such that

$$\begin{aligned} &\mathrm{Val}_0 \models_{c} [x_0, \ell'] \\ \text{and} \quad &\mathrm{Val}_0 \not\models_{d} [x_0, \ell], \end{aligned}$$

   it holds that

$$\text{if } \ell \sqsubseteq_{\mathfrak{L}} \ell', \text{ then } c \sqsubseteq_{\mathfrak{D}} d. \tag{5.45}$$

   Hence, the class of all fuzzy filter-based logics for which $\mathfrak{T} = \mathfrak{B}$ and $\mathfrak{L}$ is a chain, is completely characterised by axioms 1, 2, 3, 4, 5, 6, 7, 8, and 10*. •

Proof

ad 1. Follows immediately from items 6 and 8 of Definition 5.3.1.

ad 2. It is sufficient to observe that the axioms given in Definition 5.3.1 are equivalent to those given in Definition 4.1.3 in the case $\mathfrak{T} = \mathfrak{B}$.

As $\mathfrak{B}$ is a chain, the axioms from Observation 4.1.4.1 can be used. The most significant change from the general case is that $\mathfrak{T}$ has only two values, and as $\mathrm{Val} \models_{1} [x, \ell]$ is fixed for all ℓ and all Val, x such that $\mathrm{Val}(x) = 1$ by axioms 7 and 5, it is sufficient to consider Val_0 in all the places where Val_t features in Definition 4.1.3 and Observation 4.1.4.1. Axiom 9* is redundant for exactly this reason.

Establishing that the axioms given in Definition 5.3.1 are indeed equivalent to those given in Definition 4.1.3 and Observation 4.1.4.1 in the case $\mathfrak{T} = \mathfrak{B}$ is as simple as examining the axioms in turn and discussing the (two) possible cases $t = 0$ and $t = 1$.

ad 3. Follows form the previous item by applying Observation 4.1.2 and Theorem 4.1.3, taking into account that $\mathfrak{B}$ is a chain.

ad 4. That $\mathfrak{D}$ is a chain iff $\mathfrak{L}$ is a chain follows from the previous item and Proposition 2.3.6.

The rest follows from item 2 and Observation 4.1.4.2. □

Corollary 5.3.2 (Admissible label lattices for logics of graded trust assessment)
Given sets $\mathrm{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}$,

(i) *there exists* $\models\!\!=$ *such that* $[\mathrm{Frm}, \mathfrak{B}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=]$ *is a* ***logic of graded truth and graded trust assessment***

(ii) *iff there exists* $\models\!\!=$ *such that* $[\mathrm{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=]$ *is a* ***logic of graded trust assessment***

(iii) *iff* $\mathrm{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}$ *fulfil the axioms 1 to 3 of* Definition 4.1.3 *and* $\mathfrak{L}$ *is isomorphic with* $\mathscr{D}(\mathfrak{D})$.

Proof
The equivalence of items (i) and (ii) follows from Observation 5.3.1.1.

The equivalence of items (i) and (iii) follows from Proposition 2.3.4.

As a matter of fact, it is easy to see how the axioms in Definition 5.3.1 force $\mathfrak{L}$ to be isomorphic with $\mathscr{D}(\mathfrak{D})$. The existence and all the characterising features of an isomorphism between the two lattices (injectivity, surjectivity, and the lattice homomorphism conditions) are coded in the axioms (compare Observation 5.3.1.1 which essentially means that there is a bijection between D and L). □

It is interesting to discuss the meaning of the model and semantic consequence relation in the special case of a logic of graded trust assessment. To this end, L is identified with the set $\{\ell_d \,|\, d \in D\}$ (compare Observation 5.3.1.1 and Definition 5.3.1.8).

Because in the following, some semantic concepts of logics of graded trust assessment are reduced to semantic concepts of two-valued logic, some semantic concepts of *classical* two-valued logic are defined.

Definition 5.3.2 (Semantic concepts of two-valued logic)
Let $\mathrm{Val} \in \mathfrak{S}$.

Val *is said to be a* ***model*** *of* $x \in \mathrm{Frm}$ *classically (denoted* $\mathrm{Val} \models x$*)* $=_{\mathrm{def}} \mathrm{Val}(x) = 1$.

The case that ***not*** $\mathrm{Val} \models x$ *is denoted by* $\mathrm{Val} \not\models x$.

For $X \subseteqq \mathrm{Frm}$, $\mathrm{Val} \models X =_{\mathrm{def}}$ *for all* $x \in X$, $\mathrm{Val} \models x$.

Define the classical ***entailment*** *of* $x \in \mathrm{Frm}$ *by* $X \subseteqq \mathrm{Frm}$ *(denoted* $X \Vdash x$*)* $=_{\mathrm{def}}$ *for every* $\mathrm{Val} \in \mathfrak{S}$, *if* $\mathrm{Val} \models X$, *then* $\mathrm{Val} \models x$.

Finally, $X \subseteqq \mathrm{Frm}$ *is said to be (classically)* ***satisfiable*** $=_{\mathrm{def}}$ *there exists* $\mathrm{Val} \in \mathfrak{S}$ *such that* $\mathrm{Val} \models X$.

The set of all satisfiable $X \subseteqq \mathrm{Frm}$ *is denoted* Sat.

Axiom 7 of Definition 5.3.1 yields that for all $x \in \mathrm{Frm}$ and $d \in D$,

$$\text{if } \mathrm{Val} \models x \text{ , then } \mathrm{Val} \mathrel{\models\!\!\!=_{\mathsf{T}}} [x, \ell_d] \,.$$

This means that ℓ_d is only significant if $\mathrm{Val} \not\models x$. Indeed, if $\mathrm{Val} \mathrel{\models\!\!\!=_{d'}} [x, \ell_d]$, then

$$d' = \begin{cases} 1, & \text{if } \mathrm{Val} \models x \\ d, & \text{if } \mathrm{Val} \not\models x \end{cases} \qquad \text{(By axioms 7 and 8)} \tag{5.46}$$

The degree $d \in D$ to which a valuation Val is a model of an $\mathfrak{L}$-fuzzy set $\mathscr{X}$ on Frm (written $\mathrm{Val} \mathrel{\models\!\!\!=_{d}} \mathscr{X}$) is defined in equation (4.38) to be

$$d = \boxed{\mathrm{D}} \left\{ d' \,\middle|\, \exists x \in \mathrm{Frm} : \mathrm{Val} \mathrel{\models\!\!\!=_{d'}} [x, \mathscr{X}(x)] \right\} .$$

By equation (5.46), this is equivalent with

$$d = \overline{\mathbb{D}} \left\{ d' \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \mathcal{X}(x) = \ell_{d'} \right\}. \tag{5.47}$$

As defined in equation (4.41), an $\mathfrak{L}$-fuzzy set $\mathcal{X}$ on Frm induces on $\mathfrak{S}$ a $\mathfrak{D}$-fuzzy set $\mathrm{Mod}(\mathcal{X})$ of **models** of $\mathcal{X}$ by

$$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = \overline{\mathbb{D}} \left\{ d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \mathcal{X}(x) = \ell_d \right\}. \tag{5.48}$$

This definition of Mod corresponds to the **possibility distribution** $\pi_{\mathcal{X}}$ defined in [19, Proposition 3.2.2]. See also the discussion concerning *fuzzy sets of models* in [19, section 3.4].

For $\mathcal{X} \in L^{\mathrm{Frm}}$, the **validity index** $\mathrm{valid}(\mathcal{X})$, **consistency index** $\mathrm{cst}(\mathcal{X})$, and **inconsistency distribution** inc are given as

$$\mathrm{valid}(\mathcal{X}) = \overline{\mathbb{D}} \left\{ \overline{\mathbb{D}} \left\{ d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \mathcal{X}(x) = \ell_d \right\} \middle| \mathrm{Val} \in \mathfrak{S} \right\} \tag{5.49}$$

$$\mathrm{cst}(\mathcal{X}) = \underline{\mathbb{D}} \left\{ \overline{\mathbb{D}} \left\{ d \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \mathcal{X}(x) = \ell_d \right\} \middle| \mathrm{Val} \in \mathfrak{S} \right\} \tag{5.50}$$

$$\mathrm{inc}(\mathcal{X}) = \overline{\mathbb{L}} \left\{ \underline{\mathbb{L}} \left\{ \mathcal{X}(x) \middle| \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \right\} \middle| \mathrm{Val} \in \mathfrak{S} \right\} \tag{5.51}$$

(5.49) and (5.50) are obtained by just inserting (5.48) into (4.48) and (4.49), respectively.

(5.51) is easily established using (4.126), Observation 2.2.14 and Observation 5.3.1.3, but (assuming an appropriate formula $\perp$ exists in Frm) it also follows immediately from (4.124) using the definition (5.57) of Cons for logics of graded trust assessment and the fact that $\mathrm{Val} \not\models \perp$ for all $\mathrm{Val} \in \mathfrak{S}$ by assumption.

Proposition 5.3.9 contains some more characterisations (depending on observations about Cons) of valid, cst, inc and also the sets Valid, Incons of all **valid** and all **inconsistent** $\mathfrak{L}$-fuzzy sets of formulae.

Next, the **semantic consequence relation** is studied in this special case. Having already analysed the meaning of the model relation, it is convenient to start with the semantic consequence relation for $\mathfrak{D}$-fuzzy sets of valuations (see Definition 4.3.2). Let $\mathcal{S} : \mathfrak{S} \to D$ be given.

For $[x, \ell_d] \in \mathrm{LFrm}$, Definition 4.3.2.1 yields

$$\mathcal{S} \Vdash [x, \ell_d] \text{ iff For every Val} \in \mathfrak{S} \text{ and all } d' \in D \text{, if Val} \models_{\overline{d'}} [x, \ell_d] \text{, then } \mathcal{S}(\mathrm{Val}) \sqsubseteq_{\mathbb{D}} d'.$$

By (5.46), this is equivalent with

$$\mathcal{S} \Vdash [x, \ell_d] \text{ iff For every Val} \in \mathfrak{S} \text{, if Val} \not\models x \text{, then } \mathcal{S}(\mathrm{Val}) \sqsubseteq_{\mathbb{D}} d. \tag{5.52}$$

The $\mathfrak{L}$-fuzzy set of semantic consequences of a $\mathfrak{D}$-fuzzy set $\mathcal{S} : \mathfrak{S} \to D$ of valuations is given by equation (4.123) to be, for $x \in \mathrm{Frm}$,

$$\mathrm{Cons}(\mathcal{S})(x) = \overline{\mathbb{L}} \left\{ \iota^{-1} \left({}^{\mathcal{S}(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) \middle| \mathrm{Val} \in \mathfrak{S} \right\}. \tag{5.53}$$

Let $\mathrm{Val} \in \mathcal{S}$ and $x \in \mathrm{Frm}$ be given. If $\mathrm{Val}(x) = 0$, then by Observation 5.3.1.3,

$$\iota^{-1} \left({}^{\mathcal{S}(\mathrm{Val})}\overline{\mathrm{Val}(x)} \right) = \ell_{\mathcal{S}(\mathrm{Val})}.$$

If $\mathrm{Val}(x) = 1$, then by Observation 2.2.6, ${}^{\mathcal{S}(\mathrm{Val})}\overline{\mathrm{Val}(x)} = {}^{0}\overline{0}$, so

$$\iota^{-1}\left({}^{\mathcal{S}(\mathrm{Val})}\overline{\mathrm{Val}(x)}\right) = \ell_0.$$

By the fact that $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{D})$, obviously ℓ_0 is the unit element of $\mathfrak{L}$, hence equation (5.53) is equivalent with

$$\mathrm{Cons}(\mathcal{S})(x) = \boxed{\mathrm{L}}\left\{\ell_{\mathcal{S}(\mathrm{Val})}\,\middle|\, \mathrm{Val} \in \mathfrak{S} \text{ and } \mathrm{Val} \not\models x\right\}. \tag{5.54}$$

Next, consider semantic consequences of an $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae.
(5.52), together with (4.117) yields

$$\begin{aligned}
&\mathcal{X} \Vdash [x, \ell_d] \\
&\Leftrightarrow \text{For every } \mathrm{Val} \in \mathfrak{S}, \text{ if } \mathrm{Val} \not\models x, \text{ then } \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \sqsubseteq_{\mathrm{D}} d. \\
&\Leftrightarrow \text{For every } \mathrm{Val} \in \mathfrak{S}, && \text{(by (5.48))} \\
&\quad\ \text{if } \mathrm{Val} \not\models x, \text{ then } \boxed{\mathrm{D}}\left\{d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'}\right\} \sqsubseteq_{\mathrm{D}} d.
\end{aligned}$$

By (4.98),

$$\begin{aligned}
&\mathrm{Cons}(\mathcal{X})(x) \\
&= \boxed{\mathrm{L}}\left\{\ell_d \,\middle|\, d \in D \text{ and } \mathcal{X} \Vdash [x, \ell_d]\right\} \\
&= \boxed{\mathrm{L}}\left\{\ell_d \,\middle|\, \begin{array}{l} d \in D \text{ and for every } \mathrm{Val} \in \mathfrak{S}, \\ \text{if } \mathrm{Val} \not\models x, \text{ then } \boxed{\mathrm{D}}\left\{d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'}\right\} \sqsubseteq_{\mathrm{D}} d. \end{array}\right\} && (5.55)
\end{aligned}$$

By (5.54), considering (4.118) and (5.48),

$$\begin{aligned}
&\mathrm{Cons}(\mathcal{X})(x) \\
&= \boxed{\mathrm{L}}\left\{\ell_{\mathrm{Mod}(\mathcal{X})(\mathrm{Val})} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \text{ and } \mathrm{Val} \not\models x\right\} \\
&= \boxed{\mathrm{L}}\left\{\ell_{\boxed{\mathrm{D}}\left\{d' \,\middle|\, \exists y \in \mathrm{Frm}: \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'}\right\}} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \text{ and } \mathrm{Val} \not\models x\right\}. && (5.56)
\end{aligned}$$

Taking into account Observation 5.3.1.3 and Observation 2.2.14, it follows

$$= \boxed{\mathrm{L}}\left\{\boxed{\mathrm{L}}\left\{\mathcal{X}(y) \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y\right\} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \text{ and } \mathrm{Val} \not\models x\right\}. \tag{5.57}$$

Remark
It is trivial that (5.55) and (5.57) are equal in this case. As $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{D})$,

$$\left\{\ell_d \,\middle|\, \begin{array}{l} d \in D \text{ and for every } \mathrm{Val} \in \mathfrak{S}, \\ \text{if } \mathrm{Val} \not\models x, \text{ then } \boxed{\mathrm{D}}\left\{d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'}\right\} \sqsubseteq_{\mathrm{D}} d. \end{array}\right\}$$

is simply the set of all lower bounds of

$$\left\{\boxed{\mathrm{L}}\left\{\mathcal{X}(y) \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y\right\} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \text{ and } \mathrm{Val} \not\models x\right\}, \tag{5.58}$$

so (5.55) implements the definition of **greatest lower bound** of (5.58). •

Theorem 5.3.3 (Characterising semantic consequence by two-valued entailment)
Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ be fixed.

For $\mathrm{Val} \in \mathfrak{S}$, define

$$D_{\mathrm{Val}} =_{\mathrm{def}} \left\{ d \,\middle|\, \textit{There exists } y \in \mathrm{Frm} \textit{ such that } \mathrm{Val} \not\models y \textit{ and } \ell_d \sqsubseteq_{\mathrm{L}} \mathcal{X}(y) \right\}. \tag{5.59}$$

Furthermore, let

$$\Delta =_{\mathrm{def}} \left\{ D_{\mathrm{Val}} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \textit{ and } \mathrm{Val} \not\models x \right\}. \tag{5.60}$$

Then

$$\textstyle\sqcap_{\mathrm{D}} \bigcap \Delta = \bigsqcup_{\mathrm{D}\; D_{\mathrm{Val}} \in \Delta} \sqcap_{\mathrm{D}} D_{\mathrm{Val}}, \tag{5.61}$$

if and only if

$$\mathrm{Cons}(\mathcal{X})(x) = \bigsqcup_{\mathrm{L}} \left\{ \ell_d \,\middle|\, d \in D \textit{ and } \mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x \right\}. \tag{5.62}$$

Proof
The definition of $\mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x$ can be expanded by using Definition 5.3.2 and (1.23), yielding

$$\begin{aligned}\mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x \Leftrightarrow\ & \text{For all } \mathrm{Val} \in \mathfrak{S}, \\ & \text{if for all } y \in \mathrm{Frm} \text{ such that } \ell_d \sqsubseteq_{\mathrm{L}} \mathcal{X}(y),\ \mathrm{Val} \models y \text{ holds,} \\ & \text{then } \mathrm{Val} \models x.\end{aligned}$$

and by contraposition

$$\begin{aligned}\mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x \Leftrightarrow\ & \text{For all } \mathrm{Val} \in \mathfrak{S}, \text{ if } \mathrm{Val} \not\models x, \\ & \text{then there exists } y \in \mathrm{Frm} \text{ such that } \ell_d \sqsubseteq_{\mathrm{L}} \mathcal{X}(y) \text{ and } \mathrm{Val} \not\models y. \\ \Leftrightarrow\ & d \in \bigcap \Delta \end{aligned} \tag{5.63}$$

$\mathrm{Cons}(\mathcal{X})(x)$ is expanded using equation (5.56).
First of all, observe that

$$\sqcap_{\mathrm{D}} \left\{ d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'} \right\} = \sqcap_{\mathrm{D}} D_{\mathrm{Val}}. \tag{5.64}$$

Obviously, $\left\{ d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'} \right\} \subseteqq D_{\mathrm{Val}}$, but furthermore, because $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{D})$, for every $d \in D_{\mathrm{Val}}$, there exists $d' \in \left\{ d' \,\middle|\, \exists y \in \mathrm{Frm} : \mathrm{Val} \not\models y \text{ and } \mathcal{X}(y) = \ell_{d'} \right\}$ such that $d' \sqsubseteq_{\mathrm{D}} d$. Hence, the additional elements in D_{Val} do not influence the value of the infimum.

From (5.64) and the definition of Δ it follows that

$$\mathrm{Cons}(\mathcal{X})(x) = \bigsqcap_{\mathrm{L}\; D_{\mathrm{Val}} \in \Delta} \ell_{\sqcap_{\mathrm{D}} D_{\mathrm{Val}}}. \tag{5.65}$$

Hence by inserting (5.63) and (5.65) into (5.62), considering that $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{D})$, it follows that (5.61) and (5.62) are indeed equivalent. □

Corollary 5.3.4 (Semantic consequence in logics of graded trust is a matter of threshold)
Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ and $d \in D$ be fixed.

Define Δ as in (5.60) and assume that (5.61) holds.

Then

1. $\mathcal{X} \Vdash [x, \ell_d]$ *iff* $\ell_d \sqsubseteq_{\mathbb{L}} \bigsqcup_{\mathbb{L}} \{\ell_d | d \in D \text{ and } \mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x\}$*;*

2. *if* $\mathrm{rg}\,\mathcal{X}$ *is finite, then*

$$\mathcal{X} \Vdash [x, \ell_d] \text{ iff } \mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x. \tag{5.66}$$

Proof
ad 1. Follows from (5.62) and Proposition 4.3.1.9.

ad 2. From the assumption it follows that $\{\mathrm{CUT}_{\ell_d}(\mathcal{X}) | d \in D\}$ is finite, hence

$$\ell_d \sqsubseteq_{\mathbb{L}} \bigsqcup_{\mathbb{L}} \{\ell_d | d \in D \text{ and } \mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x\} \tag{5.67}$$

is equivalent with the existence of $d' \in D$ such that $\mathrm{CUT}_{\ell_{d'}}(\mathcal{X}) \Vdash x$ and

$$\ell_d \sqsubseteq_{\mathbb{L}} \ell_{d'}, \tag{5.68}$$

and as from $\mathrm{CUT}_{\ell_{d'}}(\mathcal{X}) \Vdash x$ and $\ell_d \sqsubseteq_{\mathbb{L}} \ell_{d'}$ it follows that $\mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x$, (5.67) is equivalent with

$$\ell_d \in \{\ell_d | d \in D \text{ and } \mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x\} \tag{5.69}$$

which is equivalent with

$$\mathrm{CUT}_{\ell_d}(\mathcal{X}) \Vdash x. \tag{5.70}$$

□

Remarks
(5.66) corresponds to the result of [19, Proposition 3.5.6].

Note that the assumption of Corollary 5.3.4.2 is fulfilled if $\mathcal{X}$ is finite.

The importance of Theorem 5.3.3 and Corollary 5.3.4 cannot be overestimated, as (5.62) (compare in particular (5.66)) is fundamental in two respects:

1. When (5.62) holds, it allows to reduce considerations with respect to semantic consequence in logics of graded trust assessment (especially, for instance, concerning **deduction**) to considerations with respect to semantic consequence in **classical logic**.

2. (5.62) (and especially (5.66)) demonstrates that semantic consequence in logics of graded trust assessment is, by nature, a matter of **threshold**. To put it simply, if one wishes to ascertain that a statement x follows from $\mathcal{X}$ with a certain degree of trust, it is sufficient to use the label associated with this degree of trust as a threshold such that x follows classically from all evidence in $\mathcal{X}$ which is trusted at least to this threshold.

   Statements which belong to $\mathcal{X}$ to a degree below this threshold (i. e. are not sufficiently trustworthy) may not be considered when trying to establish x.

The above reasoning shows that labels in logics of graded trust assessment indeed express uncertainty about the **knowledge** of the formulae in $\mathcal{X}$. The more certain one is about some statement x, the higher the label of x in $\mathcal{X}$, and the higher are the thresholds at which x will still be considered when reasoning from the knowledge in $\mathcal{X}$. If some piece x of knowledge is completely certain, its label is 1 and hence x will be considered in every inference drawn from the knowledge represented by $\mathcal{X}$, regardless of the threshold. •

For being able to fully profit from (5.62) whenever it holds, next some criteria for (5.61) to hold are given.

Definition 5.3.3 (Infinite distributive law)
Let $\mathfrak{D} = [D, \sqcap_{\mathfrak{D}}, \sqsubseteq_{\mathfrak{D}}]$ be a ***complete lattice*** *and $\Delta \subseteqq \mathfrak{P}D$ a set of subsets of D.*

Furthermore, let Φ_Δ be the set of all ***choice functions*** *for Δ, i.e. the set of all mappings $\varphi : \Delta \to \bigcup \Delta$ such that for every $D' \in \Delta$, $\varphi(D') \in D'$.*

Then the ***infinite distributive law*** *wrt. Δ holds in $\mathfrak{D}$*
$=_{\text{def}}$

$$\bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \bigsqcap^{\mathfrak{D}} D' = \bigsqcap^{\mathfrak{D}}_{\varphi \in \Phi_\Delta} \bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \varphi(D'). \tag{5.71}$$

The ***infinite distributive law*** *holds in $\mathfrak{D}$ (compare [57])*
$=_{\text{def}}$ (5.71) *holds for every set $\Delta \subseteqq \mathfrak{P}D$.*

(Note that of course there exists a **dual** *law to* (5.71)*, which is not considered here.)*

Lemma 5.3.5 (Connection between infinite distributive law and semantic entailment)
For every set $\Delta \subseteqq \mathfrak{P}D$ of ***ascending*** *subsets of D (where ascending means that for every $D' \in \Delta$ and $d \in D'$, if $d' \in D$ such that $d \sqsubseteq_{\mathfrak{D}} d'$, then $d' \in D'$),*

$$\bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \bigsqcap^{\mathfrak{D}} D' = \bigsqcap^{\mathfrak{D}}_{\varphi \in \Phi_\Delta} \bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \varphi(D')$$

is equivalent with

$$\bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \bigsqcap^{\mathfrak{D}} D' = \bigsqcap^{\mathfrak{D}} \bigcap \Delta.$$

Proof
It suffices to prove that

$$\bigsqcap^{\mathfrak{D}}_{\varphi \in \Phi_\Delta} \bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \varphi(D') = \bigsqcap^{\mathfrak{D}} \bigcap \Delta$$

holds in this case. It is sufficient to prove

$$\left\{ \bigsqcup^{\mathfrak{D}}_{D' \in \Delta} \varphi(D') \,\middle|\, \varphi \in \Phi_\Delta \right\} = \bigcap \Delta,$$

which is carried out in two steps:

1. $\left\{ \bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D') \middle| \varphi \in \Phi_\Delta \right\} \subseteqq \bigcap \Delta$.

   Let $\varphi \in \Phi_\Delta$. Then for every $D' \in \Delta$,
   $$\varphi(D') \sqsubseteq^{\mathfrak{D}} \bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D'),$$
   and from the fact that $\varphi(D') \in D'$ and D' is ascending, it follows that for every $D' \in \Delta$,
   $$\bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D') \in D'.$$
   It follows immediately that
   $$\bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D') \in \bigcap \Delta.$$

2. $\bigcap \Delta \subseteqq \left\{ \bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D') \middle| \varphi \in \Phi_\Delta \right\}$.

   Let $d \in \bigcap \Delta$. There exists a choice function $\varphi : \Delta \to \bigcup \Delta$ such that for every $D' \in \Delta$, $\varphi(D') = d$. Obviously, $\bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D') = d$, hence
   $$d \in \left\{ \bigsqcup^{\mathfrak{D}}_{D'\in\Delta} \varphi(D') \middle| \varphi \in \Phi_\Delta \right\}.$$
   □

Next, some criteria are exhibited under which equation (5.62) holds.

Corollary 5.3.6 (Criteria for semantic entailment to be characterised by threshold I) *Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ be fixed, and let Δ be given by (5.60).*

(5.62) holds in each of the following cases:

1. *The **infinite distributive law** wrt. Δ holds in $\mathfrak{D}$. (This is even equivalent with the validity of (5.62).)*
2. *$\mathfrak{D}$ is a **chain** on $\bigcup \Delta$.*
3. *The complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is an **atomic boolean algebra**.*
4. *The complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is a **boolean algebra** isomorphic with a **power set lattice**.*
5. *Δ is **finite** and the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is **completely distributive** wrt. $\bigsqcap^{\mathfrak{D}}$.*
6. *$\mathrm{rg}\,\mathcal{X}$ is **finite** and the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is **completely distributive** wrt. $\bigsqcap^{\mathfrak{D}}$.*
7. *$\mathcal{X}$ is **finite** and the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is **completely distributive** wrt. $\bigsqcap^{\mathfrak{D}}$.*
8. *$\bigcup \Delta$ is **finite** and the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is **distributive**.*

Proof

ad 1. Follows from Lemma 5.3.5 and the fact that every $D_{\mathrm{Val}} \in \Delta$ is an ascending set.

ad 2. Obviously, the infinite distributive law wrt. Δ holds in $\mathfrak{D}$ in this case.

ad 3. Let $\mathfrak{D}'$ be the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$. That $\mathfrak{D}'$ exists is trivial by the completeness of $\mathfrak{D}$. It is a theorem of lattice theory (see [57, Satz 24.3]) that the infinite distributive law wrt. Δ holds in $\mathfrak{D}'$ when $\mathfrak{D}'$ is an atomic boolean algebra. Obviously it then also holds (wrt. Δ) in $\mathfrak{D}$.

ad 4. Same as the previous item.

ad 5. The infinite distributive law wrt. Δ is equivalent with the complete distributivity of the sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ wrt. $\boxed{\mathrm{D}}$ in this case.

ad 6. Obviously, if the range of $\mathcal{X}$ is finite then Δ is also finite, so this item follows from the previous one.

ad 7. Follows from the previous item.

ad 8. In the case that $\bigcup \Delta$ is finite, the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$ is also finite, hence the infinite distributive law wrt. Δ is equivalent with the distributivity of the complete sublattice of $\mathfrak{D}$ generated by $\bigcup \Delta$. □

Corollary 5.3.7 (Criteria for semantic entailment to be characterised by threshold II)
(5.62) *holds for every* ***finite*** $\mathcal{X} \in L^{\mathrm{Frm}}$ *and* $x \in \mathrm{Frm}$ *if* $\mathfrak{D}$ *is* ***completely distributive*** *wrt.* $\boxed{\mathrm{D}}$.

Proof
Follows immediately from Corollary 5.3.6.7. □

Corollary 5.3.8 (Criteria for semantic entailment to be characterised by threshold III)
(5.62) *holds for every* $\mathcal{X} \in L^{\mathrm{Frm}}$ *and* $x \in \mathrm{Frm}$ *in each of the following cases:*

1. *The* ***infinite distributive law*** *holds in* $\mathfrak{D}$.
2. $\mathfrak{D}$ *is a* ***chain***.
3. $\mathfrak{D}$ *is an* ***atomic boolean algebra***.
4. $\mathfrak{D}$ *is a* ***boolean algebra*** *isomorphic with a* ***power set lattice***.
5. $\mathfrak{D}$ *is* ***finite*** *and* ***distributive***.

Proof
Follows immediately from the corresponding items of Corollary 5.3.6. □

Remark
Corollary 5.3.8 should make it clear that there are a lot of 'plausible' instances of $\mathfrak{D}$ for which the equations from Theorem 5.3.3 and Corollary 5.3.4 hold.

So, in accordance with the remark on page 144, it is henceforth assumed that semantic consequence in logics of graded trust assessment is mainly a matter of threshold. •

Propositions 5.3.9 (Characterising validity and consistency by cuts)
In a logic of graded trust assessment, the following hold:

1. *For all* $\mathcal{X} \in L^{\mathrm{Frm}}$,

$$\text{valid}(\mathcal{X}) = \overline{\underline{\mathrm{D}}}\left\{d \,\middle|\, \mathrm{CUT}_{\ell_d}(\mathcal{X}) \nsubseteq \mathrm{Taut}\right\}. \tag{5.72}$$

2. *Let* $\mathcal{X} \in L^{\mathrm{Frm}}$. *For* $\mathrm{Val} \in \mathfrak{S}$, *define* D_{Val} *as in* (5.59). *Let*

$$\Delta =_{\mathrm{def}} \left\{D_{\mathrm{Val}} \,\middle|\, \mathrm{Val} \in \mathfrak{S}\right\}. \tag{5.73}$$

Then (5.61) *holds (for this definition of* Δ*) if and only if*

$$\text{cst}(\mathcal{X}) = \overline{\underline{\mathrm{D}}}\left\{d \,\middle|\, \mathrm{CUT}_{\ell_d}(\mathcal{X}) \notin \mathrm{Sat}\right\}. \tag{5.74}$$

3. *Let* $\mathcal{X} \in L^{\mathrm{Frm}}$.
(5.61) *holds for* Δ *as defined in* (5.73) *if and only if*

$$\text{inc}(\mathcal{X}) = \underline{\mathrm{L}}\left\{\ell_d \,\middle|\, d \in D \text{ and } \mathrm{CUT}_{\ell_d}(\mathcal{X}) \notin \mathrm{Sat}\right\}. \tag{5.75}$$

4. *The sets of all* ***valid*** *and all* ***inconsistent*** $\mathfrak{L}$*-fuzzy sets of formulae are given by*

$$\text{Valid} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \operatorname{supp}\mathcal{X} \subseteqq \mathrm{Taut}\right\} \tag{5.76}$$

$$\text{Incons} = \left\{\mathcal{X} \,\middle|\, \begin{array}{l} \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S}: \\ \overline{\underline{\mathrm{D}}}\left\{d \,\middle|\, \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \mathcal{X}(x) = \ell_d\right\} = 0 \end{array}\right\}. \tag{5.77}$$

Under the condition given in item 2,

$$\text{Incons} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \overline{\underline{\mathrm{D}}}\left\{d \,\middle|\, \mathrm{CUT}_{\ell_d}(\mathcal{X}) \notin \mathrm{Sat}\right\} = 0\right\} \tag{5.78}$$

Proof
ad 1. Let $\mathcal{X} \in L^{\mathrm{Frm}}$. Combining both occurrences of $\overline{\underline{\mathrm{D}}}$ in (5.49) yields

$$\text{valid}(\mathcal{X}) = \overline{\underline{\mathrm{D}}}\left\{d \,\middle|\, \exists\,\mathrm{Val} \in \mathfrak{S}, x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \mathcal{X}(x) = \ell_d\right\}$$

Obviously, the value of the greatest lower bound does not change when $\mathcal{X}(x) = \ell_d$ is replaced by $\ell_d \sqsubseteq_{\mathfrak{L}} \mathcal{X}(x)$ (remember that $\mathfrak{L}$ is isomorphic with the *dual* of $\mathfrak{D}$), so

$$= \overline{\underline{\mathrm{D}}}\left\{d \,\middle|\, \exists\,\mathrm{Val} \in \mathfrak{S}, x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \ell_d \sqsubseteq_{\mathfrak{L}} \mathcal{X}(x)\right\}.$$

Considering the definition (3.16) of Taut, it follows that

$$\exists\,\mathrm{Val} \in \mathfrak{S}, x \in \mathrm{Frm} : \mathrm{Val} \not\models x \text{ and } \ell_d \sqsubseteq_{\mathfrak{L}} \mathcal{X}(x) \quad \text{iff} \quad \mathrm{CUT}_{\ell_d}(\mathcal{X}) \nsubseteq \mathrm{Taut},$$

establishing the claim.

ad 2. Let $\mathcal{X} \in L^{\mathrm{Frm}}$ and let Δ be defined by (5.73). The proof that (5.61) holds iff (5.74) holds is analogous to that of Theorem 5.3.3, but much simpler.

It is sufficient to observe that

$$
\begin{aligned}
d \in \bigcap \Delta \ &\text{iff } \forall\, \mathrm{Val} \in \mathfrak{S} : d \in D_{\mathrm{Val}} && \text{by (5.73)}\\
&\text{iff } \forall\, \mathrm{Val} \in \mathfrak{S}\ \exists y \in \mathrm{Frm} : \ell_d \sqsubseteq \mathcal{X}(y) \ \text{ and } \ \mathrm{Val} \not\models y && \text{by (5.59)}\\
&\text{iff } \forall\, \mathrm{Val} \in \mathfrak{S}\ \exists y \in \mathrm{CUT}_{\ell_d} : \mathrm{Val} \not\models y && \text{by (1.23)}\\
&\text{iff } \mathrm{CUT}_{\ell_d} \notin \mathrm{Sat} && \text{by Definition 3.3.4.2}
\end{aligned}
$$

and by definition (5.50),

$$
\mathrm{cst}(\mathcal{X}) = \bigsqcup_{\mathbb{D}} \left\{ \bigsqcap_{\mathbb{D}} \left\{ d \,\middle|\, \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \ \text{ and } \ \mathcal{X}(x) = \ell_d \right\} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\},
$$

where $\mathcal{X}(x) = \ell_d$ can be replaced by $\ell_d \sqsubseteq \mathcal{X}(x)$ without changing the value of the (inner) greatest lower bound, yielding

$$
\begin{aligned}
&= \bigsqcup_{\mathbb{D}} \left\{ \bigsqcap_{\mathbb{D}} \left\{ d \,\middle|\, \exists x \in \mathrm{Frm} : \mathrm{Val} \not\models x \ \text{ and } \ \ell_d \sqsubseteq \mathcal{X}(x) \right\} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\}\\
&= \bigsqcup_{\mathbb{D}} \left\{ \bigsqcap_{\mathbb{D}} D_{\mathrm{Val}} \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\} && \text{by (5.59)}\\
&= \bigsqcup_{\mathbb{D}\ D_{\mathrm{Val}} \in \Delta} \bigsqcap_{\mathbb{D}} D_{\mathrm{Val}}. && \text{by (5.73)}
\end{aligned}
$$

ad 3. Follows from (5.74) and (4.126), taking into account that $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{D})$.

ad (5.76)**.** From the definition of Valid in Definition 4.2.2.1 and item 1, it follows that

$$
\mathcal{X} \in \mathrm{Valid} \quad \text{iff} \quad \mathrm{valid}(\mathcal{X}) = 1 \quad \text{iff} \quad \left\{ d \,\middle|\, \mathrm{CUT}_{\ell_d}(\mathcal{X}) \not\subseteq \mathrm{Taut} \right\} \subseteq \{1\}.
$$

It follows that

$$
\mathcal{X} \in \mathrm{Valid} \quad \text{iff} \quad \text{for every } \ell_d \in L \text{ with } \ell_d \neq \ell_1,\ \mathrm{CUT}_{\ell_d}(\mathcal{X}) \subseteq \mathrm{Taut}. \tag{5.79}
$$

But ℓ_1 is the zero element of $\mathfrak{L}$ by the fact that $\mathfrak{L}$ is isomorphic with $\mathscr{D}(\mathfrak{D})$, hence

$$
\mathcal{X} \in \mathrm{Valid} \quad \text{iff} \quad \mathrm{supp}\,\mathcal{X} \subseteq \mathrm{Taut}
$$

follows from (5.79) by (1.24).

ad (5.77)**.** Follows immediately from the definition of Incons.

ad (5.78)**.** Follows by combining Definition 4.2.2.2 with (5.74). □

Remark

1. The conditions for the validity of (5.62) exhibited in Corollaries 5.3.6–5.3.8 also guarantee the validity of the characterisations in items 2 and 3 and equation (5.78) of Proposition 5.3.9.

2. If for $\mathcal{X} \in L^{\mathrm{Frm}}$, $\mathrm{rg}\,\mathcal{X}$ is finite, then by a proof analogous to that of Corollary 5.3.4.2,

$$
\mathrm{CUT}_{\mathrm{inc}(\mathcal{X})}(\mathcal{X}) \notin \mathrm{Sat}.
$$

Note that $\mathrm{rg}\,\mathcal{X}$ is finite if $\mathcal{X}$ is finite.

3. Under the conditions for (5.78), obviously

$$\left\{\mathcal{X}\middle|\mathcal{X}\in L^{\mathrm{Frm}} \text{ and } \mathrm{CUT}_{\ell_0}(\mathcal{X})\notin\mathrm{Sat}\right\}\subseteqq\mathrm{Incons}.$$

The equation

$$\mathrm{Incons}=\left\{\mathcal{X}\middle|\mathcal{X}\in L^{\mathrm{Frm}} \text{ and } \mathrm{CUT}_{\ell_0}(\mathcal{X})\notin\mathrm{Sat}\right\} \tag{5.80}$$

holds if and only if for all $\mathcal{X}\in L^{\mathrm{Frm}}$,

$$\boxed{\mathrm{D}}\left\{d\middle|\ \mathrm{CUT}_{\ell_d}(\mathcal{X})\notin\mathrm{Sat}\right\}=0 \iff 0\in\left\{d\middle|\ \mathrm{CUT}_{\ell_d}(\mathcal{X})\notin\mathrm{Sat}\right\}.$$

It should be clear that this will hold only under very special circumstances, for instance if $\mathfrak{D}$ is an atomic chain or an atomic Boolean algebra.

With a proof analogous to that of Corollary 5.3.4.2 it can be established that if for $\mathcal{X}\in L^{\mathrm{Frm}}$, $\mathrm{rg}\,\mathcal{X}$ is finite, it holds that

$$\mathcal{X}\in\mathrm{Incons} \text{ iff } \mathrm{CUT}_{\ell_0}(\mathcal{X})\notin\mathrm{Sat}.$$

•

Concerning **refutation**, applying Theorem 4.3.7 and Corollary 4.3.19 in this case yields the following encouraging Observation.

Observation 5.3.10 (Refutation system for logics of graded trust assessment)
Let the mapping $\nu_{\mathfrak{T}}:\{0,1\}\to\{0,1\}$ from Definition 4.3.4 be given by the **negation operator** $\varphi_{\neg}$ defined in (3.9) and assume that Frm contains a unary operator symbol $\neg$ interpreted by $\nu_{\mathfrak{T}}$. Let a mapping $\nu_{\mathfrak{D}}:D\to D$ be given as specified in (4.131),(4.132).

Then for all $\mathcal{X}\in L^{\mathrm{Frm}}$ and $[x,\ell]\in\mathrm{LFrm}$, ℓ **admits refutation** and $\mathcal{X}\Vdash[x,\ell]$ is **characterised by refutation**, i. e.

$$\mathcal{X}\Vdash[x,\ell] \text{ iff } \ell\sqsubseteq_{\mathfrak{L}}\mathrm{inc}\left(\mathcal{X}\cup[\neg x,1]\right). \tag{5.81}$$

•

Proof
$\mathfrak{T}$ is two-valued and thus obviously a chain, so the fact that every ℓ admits refutation and $\mathcal{X}\Vdash[x,\ell]$ is characterised by refutation follows by simply applying Theorem 4.3.7 and Corollary 4.3.19.

For justifying equation (5.81), it is sufficient to observe that in equation (4.137), due to the two-valuedness of $\mathfrak{T}$, only the cases $s=1$ and $s=0$ occur. Taking into account items 1 and 3 of Observation 5.3.1, it is clear that $\mathcal{F}_\ell$ is the principal fuzzy filter ${}^{0}\overline{0}$ for every $\ell\in L$, hence $\widetilde{\ell}=\ell_0$ by Observation 5.3.1.4, which is the unit element of $\mathfrak{L}$. □

Observation 5.3.10 means that refutation is applicable without restriction in all logics of graded trust assessment, the only condition being the existence of a suitable order-reversing involution $\nu_{\mathfrak{D}}$ on $\mathfrak{D}$.

5.3.1 Possibilistic Logic

Possibilistic logic is by far the most well-known representative of logics of graded trust assessment, and it is in fact the only logical system so far where the difference between graded truth and graded trust is made explicit and forms an integral part of the definition of logical concepts.

Possibilistic logic is studied in great detail by D. Dubois, J. Lang and H. Prade [19] (see also Gert de Cooman [13] and G. Gerla [41, chapter 6]).

In the following subsection, it is demonstrated how the most simple case of *possibilistic logic* (possibilistic logic with necessity-valued formulae; see [19]) can be derived as a special case of a logic of graded trust assessment.

Possibilistic logic with necessity-valued and possibility-valued formulae has more expressive power (compare [19, section 4.1]). Its study and comparison with logics of graded trust assessment is left for future research.

Note that in [19, section 4.3], even possibilistic logic with 'necessity degrees' taken from an arbitrary complete and distributive lattice is mentioned, which is almost equivalent with the concept of logics of graded trust assessment. It seems, however, that no detailed study of this type of logic from the perspective of mathematical logic has taken place yet.

5.3.1.1 Possibilistic Logic with Necessity-Valued Formulae

For the special case (the most intensively studied one) that all formulae of possibilistic logic are valuated with a **necessity degree**, let $\mathfrak{D} =_{\text{def}} \mathfrak{F} = \left[\langle 0,1\rangle, \min, \max\right]$.

Furthermore, define the set of **formulae** to be given by Frm $=_{\text{def}}$ PFrm$_{\text{S}}$, i.e. classical propositional syntax (see Example 3.1.3) and the semantics to be given by $\mathfrak{S}^{\text{P}}_{\text{B}}$ (see Example 3.2.4.1).

By Corollary 5.3.2.(iii), to obtain a **logic of graded trust assessment** according to Definition 5.3.1, the label lattice $\mathfrak{L}$ has to be isomorphic with the dual lattice $\mathscr{D}(\mathfrak{F}) = \left[\langle 0,1\rangle, \max, \min\right]$ of $\mathfrak{F}$. For compatibility with the original definition of possibilistic logic with necessity-valued formulae [19], let $\mathfrak{L} =_{\text{def}} \mathfrak{F}$ and choose the isomorphism

$$\iota(r) =_{\text{def}} 1 - r \qquad r \in \langle 0,1\rangle \tag{5.82}$$

between $\mathfrak{L}$ and $\mathscr{D}(\mathfrak{F})$, i.e. ℓ_d from Definition 5.3.1.8 is $1-d$ for all $d \in \langle 0,1\rangle$. Note that the greatest lower bound $\boxed{\text{L}}$ of the label lattice $\mathfrak{F}$ is the infimum inf of the complete lattice of all real numbers and the least upper bound $\boxed{\text{L}}$ of the label lattice $\mathfrak{F}$ is the supremum sup.

The model relation and model fuzzy set, the semantic entailment relation, and the semantic consequence operator defined in (5.46), (5.48), (5.55), and (5.57), respectively, are denoted by $\models^{\text{P}}$, Mod$_{\text{P}}$, $\Vdash_{\text{P}}$, Cons$_{\text{P}}$ in this case and the resulting logic of graded trust assessment is denoted by

$$\Lambda_{\text{P}} =_{\text{def}} \left[\text{PFrm}_{\text{S}}, \mathfrak{S}^{\text{P}}_{\text{B}}, \mathfrak{F}, \mathfrak{F}, \models^{\text{P}}\right]. \tag{5.83}$$

According to [19], "a necessity-valued formula is a pair $(\varphi\ \alpha)$ where φ is a classical first-order, closed formula and $\alpha \in (0,1\rangle$ is a positive number."

In this section, syntax is restricted to propositional logic, but otherwise, a "necessity-valued formula" is identical with a labelled formula of the logic of graded trust assessment Λ_{P}.

For convenience (in particular with respect to the comparison in section 5.4), definitions for the basic logical concepts (see equations (5.46)–(5.62)) are repeated here for this special case.

For $x \in \text{PFrm}_{\text{S}}$, Val $\in \mathfrak{S}^{\text{P}}_{\text{B}}$, and $d, d' \in \langle 0,1\rangle$,

$$\text{Val} \models^{\text{P}}_{d'} [x,d] \text{ iff } d' = \begin{cases} 1, & \text{if Val} \models x \\ 1-d, & \text{if Val} \not\models x \end{cases} \tag{5.84}$$

For $\mathcal{X} \in \langle 0,1\rangle^{\text{PFrm}_{\text{S}}}$,

$$\text{Val} \models^{\text{P}}_{d} \mathcal{X} \text{ iff } d = \inf\left\{1 - \mathcal{X}(x) \,\middle|\, x \in \text{Frm and Val} \not\models x\right\}, \tag{5.85}$$

$$\text{Mod}_{\text{P}}(\mathcal{X})(\text{Val}) = \inf\left\{1 - \mathcal{X}(x) \,\middle|\, x \in \text{Frm and Val} \not\models x\right\}. \tag{5.86}$$

This definition of Mod_P corresponds to the **possibility distribution** $\pi_{\mathcal{X}}$ defined in [19, Proposition 3.2.2].
For $[x,d] \in \mathrm{LFrm}$ and $\mathcal{S} : \mathfrak{S}^\mathrm{P}_\mathrm{B} \to \langle 0,1\rangle$,

$$\mathcal{S} \Vdash_{\overline{\mathrm{P}}} [x,d] \text{ iff for every } \mathrm{Val} \in \mathfrak{S}^\mathrm{P}_\mathrm{B}, \text{ if } \mathrm{Val} \not\models x, \text{ then } \mathcal{S}(\mathrm{Val}) \leqq 1-d,$$

$$\mathrm{Cons}_\mathrm{P}(\mathcal{S})(x) = \inf\left\{1-\mathcal{S}(\mathrm{Val}) \,\middle|\, \mathrm{Val} \in \mathfrak{S}^\mathrm{P}_\mathrm{B} \text{ and } \mathrm{Val} \not\models x\right\}. \tag{5.87}$$

For $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{PFrm_S}}$ and $[x,d] \in \mathrm{LFrm}$,

$$\begin{aligned}\mathcal{X} \Vdash_{\overline{\mathrm{P}}} [x,d] \text{ iff } & \text{For every } \mathrm{Val} \in \mathfrak{S}^\mathrm{P}_\mathrm{B}, \\ & \text{if } \mathrm{Val} \not\models x, \text{ then } d \leqq \sup\left\{\mathcal{X}(y) \,\middle|\, y \in \mathrm{PFrm_S} \text{ and } \mathrm{Val} \not\models y\right\}.\end{aligned} \tag{5.88}$$

As $\mathfrak{F}$ is a chain, Corollary 5.3.8.2 and hence Corollary 5.3.4.1 can be applied, yielding

$$\text{iff } d \leqq \sup\left\{d \,\middle|\, d \in \langle 0,1\rangle \text{ and } \mathrm{CUT}_d(\mathcal{X}) \Vdash x\right\}. \tag{5.89}$$

$$\mathrm{Cons}_\mathrm{P}(\mathcal{X})(x) = \inf\left\{\sup\left\{\mathcal{X}(y) \,\middle|\, y \in \mathrm{PFrm_S} \text{ and } \mathrm{Val} \not\models y\right\} \,\middle|\, \mathrm{Val} \in \mathfrak{S}^\mathrm{P}_\mathrm{B} \text{ and } \mathrm{Val} \not\models x\right\}.$$

As above, Theorem 5.3.3 can be applied, yielding

$$= \sup\left\{d \,\middle|\, d \in \langle 0,1\rangle \text{ and } \mathrm{CUT}_d(\mathcal{X}) \Vdash x\right\}. \tag{5.90}$$

In possibilistic logic [19], "possibility distributions" on the set $\mathfrak{S}^\mathrm{P}_\mathrm{B}$ of all Boolean valuation functions are studied. These correspond to the $\mathfrak{F}$-fuzzy sets $\mathcal{S} : \mathfrak{S}^\mathrm{P}_\mathrm{B} \to \langle 0,1\rangle$ studied here.

In [19, section 3.2], a **necessity measure** on the set of all formulae is induced by a possibility distribution $\mathcal{S} : \mathfrak{S}^\mathrm{P}_\mathrm{B} \to \langle 0,1\rangle$ for $x \in \mathrm{PFrm_S}$ by

$$N(x) =_{\mathrm{def}} \inf\left\{1-\mathcal{S}(\mathrm{Val}) \,\middle|\, \mathrm{Val} \in \mathfrak{S}^\mathrm{P}_\mathrm{B} \text{ and } \mathrm{Val} \not\models x\right\}. \tag{5.91}$$

Observe that (5.91) is identical with (5.87), yielding that the necessity measure N is just $\mathrm{Cons}(\mathcal{S})$:

$$N = \mathrm{Cons}(\mathcal{S}). \tag{5.92}$$

[19] reports for N the following properties, for $x, y \in \mathrm{PFrm_S}$:

$$\text{if } \mathrm{Val} \models x \text{ for every } \mathrm{Val} \in \mathfrak{S}^\mathrm{P}_\mathrm{B}, \text{ then } N(x) = 1 \tag{5.93}$$

$$N(x \wedge y) = \min(N(x), N(y)) \tag{5.94}$$

$$N(x \vee y) \geqq \max(N(x), N(y)) \tag{5.95}$$

It is not surprising that the same properties have been established for $\mathrm{Cons}(\mathcal{S})$, in a more general form, in Proposition 4.3.4 and Proposition 4.3.20.

In [19], a "possibility distribution" $\mathcal{S} : \mathfrak{S}^\mathrm{P}_\mathrm{B} \to \langle 0,1\rangle$ is said to **satisfy** a labelled formula $[x,d]$ iff $N(x) \geqq d$, where N is induced by $\mathcal{S}$. In the context of logics of graded trust assessment, it can be said that this is really a notion of **semantic consequence**, in the sense that

$$\begin{aligned} & d \leqq N(x) & \\ \Leftrightarrow\ & d \leqq \mathrm{Cons}(\mathcal{S})(x) & \text{(by (5.92))} \\ \Leftrightarrow\ & \mathcal{S} \Vdash [x,d] & \text{(by (4.122))}\end{aligned}$$

Furthermore, in [19], a labelled formula $[x, d]$ is said to be a *logical consequence* of a set of labelled formulae (here denoted as a fuzzy set $\mathcal{X} \in \langle 0, 1\rangle^{\mathrm{PFrms}}$) iff

$$\text{for every } \mathcal{S} \in \langle 0, 1\rangle^{\mathfrak{S}^{\mathrm{P}}_{\mathrm{B}}} : \text{if } \mathcal{S} \Vdash \mathcal{X} \text{ , then } \mathcal{S} \Vdash [x, d] \,.$$

By [19, Proposition 3.2.2], this is equivalent with $\mathcal{X} \Vdash [x, d]$ because of the fact that the **possibility distribution** $\pi_{\mathcal{X}}$ mentioned there is equal with $\mathrm{Mod}(\mathcal{X})$.

In subsequent investigations (see section 5.4), the following trivial observation will be useful.

Observation 5.3.11 (Clausal form in possibilistic logic with necessity-valued formulae)
Λ_{P} is a lattice logic as defined in Example 3.2.3, and furthermore, $\mathfrak{B}$ is a De Morgan algebra wrt. the negation operator $\varphi_{\neg}$ defined in (3.9). Hence, all conditions given in Example 4.2.1 for the applicability of **conjunctive normal form** are fulfilled.

By Proposition 4.2.10 and Example 4.3.1, it can safely be assumed that all formulae of Λ_{P} are **clauses**. •

For the simplified logical language Cls, the set Taut of all **tautologies** wrt. $\mathfrak{S}^{\mathrm{P}}_{\mathrm{B}}$ (denoted $\mathrm{Taut}_{\mathrm{B}}$) can be given explicitly.

Observation 5.3.12 (Tautologies in two-valued propositional logic)

$$\mathrm{Taut}_{\mathrm{B}} = \left\{c \,\middle|\, \exists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c\right\} . \tag{5.96}$$ •

Proof
Trivial. □

The set of all satisfiable sets of clauses is denoted $\mathrm{Sat}_{\mathrm{B}}$ (but setting Frm = Cls does not allow a simpler definition than that given in Definition 5.3.2).

As $\mathfrak{F}$ is a chain, Proposition 5.3.9 yields the following characterisations of the **validity index** valid (denoted $\mathrm{valid}_{\mathrm{P}}(\mathcal{X})$), **consistency index** cst (denoted $\mathrm{cst}_{\mathrm{P}}(\mathcal{X})$), **inconsistency distribution** inc (denoted $\mathrm{inc}_{\mathrm{P}}(\mathcal{X})$; note that by the fact that $\mathfrak{L} = \mathfrak{F}$, inc is really also an index) and the sets Valid, Incons of all **valid** and **inconsistent** $\mathfrak{F}$-fuzzy sets of clauses wrt. Λ_{P} (denoted $\mathrm{Valid}_{\mathrm{P}}$, $\mathrm{Incons}_{\mathrm{P}}$).

Observation 5.3.13 (Validity, consistency, and inconsistency in possibilistic logic)
For all $\mathcal{X} \in \langle 0, 1\rangle^{\mathrm{Cls}}$,

$$\mathrm{valid}_{\mathrm{P}}(\mathcal{X}) = \inf\left\{d \,\middle|\, \mathrm{CUT}_{1-d}(\mathcal{X}) \not\subseteqq \mathrm{Taut}_{\mathrm{B}}\right\} . \tag{5.97}$$

$$\mathrm{cst}_{\mathrm{P}}(\mathcal{X}) = \inf\left\{d \,\middle|\, \mathrm{CUT}_{1-d}(\mathcal{X}) \notin \mathrm{Sat}_{\mathrm{B}}\right\} . \tag{5.98}$$

$$\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = \sup\left\{d \,\middle|\, \mathrm{CUT}_{d}(\mathcal{X}) \notin \mathrm{Sat}_{\mathrm{B}}\right\} . \tag{5.99}$$

$$= 1 - \mathrm{cst}_{\mathrm{P}}(\mathcal{X}) \tag{5.100}$$

and furthermore

$$\mathrm{Valid}_{\mathrm{P}} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0, 1\rangle^{\mathrm{Cls}} \text{ and } \mathrm{supp}\,\mathcal{X} \subseteqq \mathrm{Taut}_{\mathrm{B}}\right\} . \tag{5.101}$$

$$\mathrm{Incons}_{\mathrm{P}} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0, 1\rangle^{\mathrm{Cls}} \text{ and } \sup\left\{d \,\middle|\, \mathrm{CUT}_{d}(\mathcal{X}) \notin \mathrm{Sat}_{\mathrm{B}}\right\} = 1\right\} . \tag{5.102}$$ •

Proof
All equations are obtained immediately from Proposition 5.3.9. □

Remark
Equation (5.99) corresponds to the result of Proposition 3.5.2 in [19].

Equation (5.100) is the definition of the **inconsistency degree** Incons in [19, section 3.3]. •

Concerning refutation, the following Observation holds in the special case of possibilistic logic with necessity-valued formulae.

Observation 5.3.14 (Refutation system for possibilistic logic with necessity-valued formulae)
Let the mapping $\nu_{\mathfrak{T}} : \{0,1\} \to \{0,1\}$ from Definition 4.3.4 be given by the **negation operator** $\varphi_{\neg}$ defined in (3.9) and assume that Frm contains a unary operator symbol $\neg$ interpreted by $\nu_{\mathfrak{T}}$. Let a mapping $\nu_{\mathfrak{D}} : \langle 0,1\rangle \to \langle 0,1\rangle$ be given as specified in (4.131),(4.132).

Let $\mathcal{X} \in L^{\mathrm{Cls}}$ and $[c,d] \in \mathrm{LFrm}$ such that $c = \{l_1,\ldots,l_n\}$ for $n \in \mathbb{N}, n \geqq 1$ and $l_1,\ldots,l_n \in \mathrm{Lit}$.

Then d **admits refutation** and $\mathcal{X} \Vdash [c,d]$ is **characterised by refutation**, i. e.

$$\mathcal{X} \Vdash [c,d] \text{ iff } d \leqq \operatorname{inc}\left(\mathcal{X} \cup \left[\overline{l_1}, 1\right] \cup \ldots \cup \left[\overline{l_n}, 1\right]\right). \tag{5.103}$$ •

Proof
Follows from Observation 5.3.10 and Corollary 4.3.22.3, taking into account the definition (4.124) of inc (note that $\perp$ exists in two-valued propositional logic) and the fact that by equation (4.83) (which holds in two-valued propositional logic),

$$\neg c \equiv \overline{l_1} \wedge \cdots \wedge \overline{l_n}.$$ □

Remark
Observation 5.3.14 corresponds to Proposition 3.5.5 in [19]. •

5.4 Comparison between logics of graded truth assessment and logics of graded trust assessment

In the two previous sections, two aspects of logical systems have been studied which can be made many-valued, namely **truth** and **validity**. While the role of these concepts in the definition of logical systems is very different, it has been made sure that the general principles along which **logics of graded truth assessment** (in which truth is many-valued and validity is two-valued) and **logics of graded trust assessment** (in which truth is two-valued and validity is many-valued) are defined are equal.

This general definition principle now allows for a comparison of both types of logics. In section 1.1, it has been argued that the two notions of **truth** and **validity** are **orthogonal**.

And indeed, the situation described in the preceding two sections can be visualised as in Figure 5.1.

Visualising the lattice of truth values as one axis (i. e. assuming $\mathfrak{T}$ is a chain), labels in a logic of graded truth assessment represent **ranges** of admissible truth values *on* this axis while labels in a logic of graded trust assessment represent **thresholds** on an axis of degrees of validity which is **orthogonal** to the truth value axis and in fact **stratifies** any fuzzy set of formulae into layers according to the thresholds expressed by the labels.

This concept of visualising labels in a coordinate system spanned by the two orthogonal axes **truth** and **validity** is elaborated further in section 5.5 where logics are studied in which

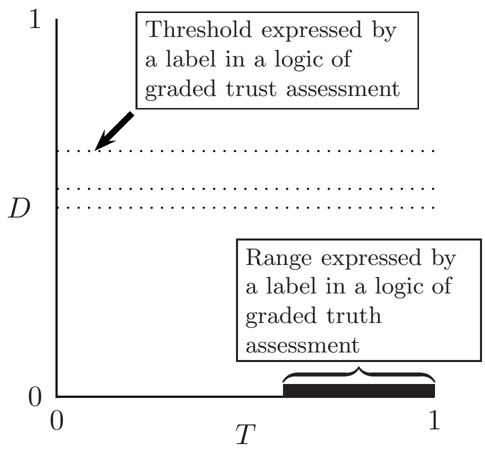

Figure 5.1: Dimensions of many-valuedness.

both truth and validity are many-valued. There, a couple of examples for 'two-dimensional' labels and their intuitive meaning are given.

In this section, a comparison is given between two particular logical systems, one a logic of graded truth assessment and the other a logic of graded trust assessment.

5.4.1 Definitions of the Logics to be Compared

For this comparison, one particular representative of each class is selected, namely **possibilistic logic with necessity-labelled formulae** (see section 5.3.1.1) and **Lee's fuzzy logic with truth values as labels** (see section 5.2.1.2). The relationships between these representatives are studied in the following.

In both examples, the logical language is given by $\mathrm{Frm} =_{\mathrm{def}} \mathrm{PFrm}_{\mathrm{S}}$, i. e. classical propositional syntax (see Example 3.1.3). For the subsequent considerations, formulae are assumed to be in **clausal form**, so the additional syntactic concepts like **literal** and **clause** defined in Example 4.2.1 are used freely in the sequel.

For this section, fix $\mathfrak{L} =_{\mathrm{def}} \mathfrak{F} = \left[\langle 0,1\rangle, \min, \max\right]$, i. e. the lattice of labels is fixed to be the unit interval together with the usual minimum and maximum of real numbers.

Lee's fuzzy logic with truth values as labels is defined to be

$$\Lambda_{\mathrm{L}} =_{\mathrm{def}} \left[\mathrm{PFrm}_{\mathrm{S}}, \mathfrak{F}, \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}, \mathfrak{F}, \overset{\mathrm{L}}{\models}\right]$$

in section 5.2.1.2. Validity is two-valued, as this is a logic of graded truth assessment.

Possibilistic logic with necessity-labelled formulae is defined to be

$$\Lambda_{\mathrm{P}} =_{\mathrm{def}} \left[\mathrm{PFrm}_{\mathrm{S}}, \mathfrak{S}^{\mathrm{P}}_{\mathrm{B}}, \mathfrak{F}, \mathfrak{F}, \overset{\mathrm{P}}{\models}\right]$$

in section 5.3.1.1. Truth is two-valued, as this is a logic of graded trust assessment.

The comparison of the two logics given above should be based on the most basic logical concept, the **semantic consequence operators** $\mathrm{Cons}_{\mathrm{L}}, \mathrm{Cons}_{\mathrm{P}}$ defined by them, which both operate on the *same* set $\langle 0,1\rangle^{\mathrm{PFrms}}$ of $\mathfrak{F}$-fuzzy sets of propositional formulae. Both operators are meant to model semantic consequence in special logics of propositional formulae labelled by numbers from the unit interval, both are **closure operators** by Theorem 4.3.2, and both are defined using only the **lattice structure** of $\langle 0,1\rangle$, i. e. using $\leqq, \min, \max$.

It is now obviously tempting to compare the logical systems characterised by these operators, in particular because in both systems, labels have almost identical semantics from the intuitive point of view: The higher the label, the more trust is placed in the truth resp. validity of the labelled formula. Both systems differ only in the 'implementation' of this intuitive notion, and both have shown their merits in applications. Still, it has so far not been attempted to characterise exactly the relationship between these two logics.

Comparing both logics is done by comparing some selected characteristics of these logics. Most of the results of this section have been published by the author in [73]; some preliminary results can be found in [20, 21] (coauthored between Didier Dubois[1], the author of this dissertation, and Henri Prade[1]).

In the following, the similarities and differences between the entailment relations $\Vdash_{\overline{\mathrm{L}}}$, $\Vdash_{\overline{\mathrm{P}}}$ and consequence operators $\mathrm{Cons}_\mathrm{L}, \mathrm{Cons}_\mathrm{P}$ are studied.

As both logics considered here allow formulae to be represented in **clausal form** (see Observation 5.2.6, Observation 5.3.11), in the following formulae are assumed to be **clauses**, i.e. non-empty, finite sets of **literals** (see Definition 4.2.5 and the remark following it); as a reminder the denotation Cls_S is used instead of PFrm_S in the sequel, defined by

$$\mathrm{Cls}_\mathrm{S} =_{\mathrm{def}} \left\{c \middle| c \subseteqq \mathrm{Lit}_\mathrm{S} \text{ and } c \neq \varnothing \text{ and } c \text{ is finite}\right\}, \tag{5.104}$$

where $\mathrm{Lit}_\mathrm{S} =_{\mathrm{def}} \mathrm{PV} \cup \{\neg p | p \in \mathrm{PV}\}$ like in (4.87). The set LCls_S of all **labelled clauses** is defined accordingly.

The comparison between the two logics is now carried out in a series of propositions.

5.4.2 Compactness

The first result established concerns a further similarity between both entailment operators. For this, an auxiliary definition is needed.

Definition 5.4.1 (Compactness)
*An entailment relation $\Vdash$ is said to be **compact** iff for every $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}_\mathrm{S}}$ and every $[c,r] \in \mathrm{LCls}_\mathrm{S}$ such that $\mathcal{X} \Vdash [c,r]$, there exists a **finite** $\mathcal{X}_{\mathrm{fin}} \subseteqq \mathcal{X}$ such that $\mathcal{X}_{\mathrm{fin}} \Vdash [c,r]$.*

As compactness is trivial if Cls_S is finite, for this subsection assume that PV is infinite.

Theorem 5.4.1 (Compactness)

1. *Neither $\Vdash_{\overline{\mathrm{L}}}$ nor $\Vdash_{\overline{\mathrm{P}}}$ is compact.*
2. *$\Vdash_{\overline{\mathrm{L}}}, \Vdash_{\overline{\mathrm{P}}}$ are however **weakly compact** in the following sense: For every $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}_\mathrm{S}}$ and every $[c,r] \in \mathrm{LCls}_\mathrm{S}$ such that $\mathcal{X} \Vdash [c,r]$ and $\mathrm{rg}\,\mathcal{X}$ is **finite**, there exists a finite $\mathcal{X}_{\mathrm{fin}} \subseteqq \mathcal{X}$ such that $\mathcal{X}_{\mathrm{fin}} \Vdash [c,r]$.*

Proof

ad 1. A counterexample is constructed which works for *both* entailment relations. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real numbers from the half-open real interval $\langle 0,1)$ such that $\sup\limits_{n\in\mathbb{N}} r_n = 1$. Furthermore, let pairwise different propositional variables $p, p_1, p_2, \ldots \in \mathrm{PV}$ be given. Define

$$\mathcal{X} =_{\mathrm{def}} \left[\{\neg p_1\}, 1\right] \cup \left[\{\neg p_2\}, 1\right] \cup \cdots \cup \left[\{p, p_1\}, r_1\right] \cup \left[\{p, p_2\}, r_2\right] \cup \cdots$$

[1] Institut de Recherche en Informatique de Toulouse (I.R.I.T) — C.N.R.S, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France.

By expanding the definitions of $\Vdash_{\mathrm{L}}$, $\Vdash_{\mathrm{P}}$, it is proved that

$$\mathcal{X} \Vdash_{\mathrm{L}} [\{p\}, 1]\,,$$
$$\mathcal{X} \Vdash_{\mathrm{P}} [\{p\}, 1]\,.$$

ad $\Vdash_{\mathrm{L}}$. By (5.29), (5.27), and (5.28),

$$\mathcal{X} \Vdash_{\mathrm{L}} [\{p\}, 1] \quad \text{iff } \forall\,\mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}, \text{ if } \forall x \in \mathrm{PFrm_S} : \mathcal{X}(x) \leqq \mathrm{Val}(x)\,, \text{ then } \mathrm{Val}(p) = 1.$$

Let $\mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$ be given such that $\forall x \in \mathrm{PFrm_S} : \mathcal{X}(x) \leqq \mathrm{Val}(x)$. This means by the definition of $\mathcal{X}$ that

$$\begin{aligned} \mathrm{Val}\,(\neg p_1) &= 1, \\ \mathrm{Val}\,(\neg p_2) &= 1, \\ &\vdots \\ \mathrm{Val}\,(p \vee p_1) &\geqq r_1, \\ \mathrm{Val}\,(p \vee p_2) &\geqq r_2, \\ &\vdots \end{aligned}$$

Considering that $\mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$, this means that there exists an assignment $\mathcal{A} \in \langle 0, 1\rangle^{\mathrm{PV}}$ such that

$$\begin{aligned} \mathcal{A}(p_1) &= 0, \\ \mathcal{A}(p_2) &= 0, \\ &\vdots \\ \max\big(\mathcal{A}(p), \mathcal{A}(p_1)\big) = \mathcal{A}(p) &\geqq r_1, \\ \max\big(\mathcal{A}(p), \mathcal{A}(p_2)\big) = \mathcal{A}(p) &\geqq r_2, \\ &\vdots \end{aligned}$$

and from the fact that $\sup\limits_{n\in\mathbb{N}} r_n = 1$, it follows that $\mathcal{A}(p) = 1$ and hence $\mathrm{Val}(p) = 1$, establishing $\mathcal{X} \Vdash_{\mathrm{L}} [\{p\}, 1]$.

ad $\Vdash_{\mathrm{P}}$. By (5.89),

$$\mathcal{X} \Vdash_{\mathrm{P}} [\{p\}, 1] \text{ iff } \sup\big\{d \,\big|\, d \in \langle 0, 1\rangle \text{ and } \mathrm{CUT}_d(\mathcal{X}) \Vdash p\big\} = 1.$$

By the fact that $\sup\limits_{n\in\mathbb{N}} r_n = 1$, it is sufficient to prove that for every $n \in \mathbb{N}$, $\mathrm{CUT}_{r_n}(\mathcal{X}) \Vdash p$.

By the definition of $\mathcal{X}$, $\{\neg p_n, p \vee p_n\} \subseteqq \mathrm{CUT}_{r_n}(\mathcal{X})$, hence $\mathrm{CUT}_{r_n}(\mathcal{X}) \Vdash p$ by the fact that $\Vdash$ here stands for the classical entailment of two-valued propositional logic.

That there exists **no** finite $\mathcal{X}_{\mathrm{fin}} \subseteqq \mathcal{X}$ such that any one of $\mathcal{X}_{\mathrm{fin}} \Vdash_{\mathrm{L}} [\{p\}, 1]$ or $\mathcal{X}_{\mathrm{fin}} \Vdash_{\mathrm{P}} [\{p\}, 1]$ holds is trivial by the reasoning above.

ad 2. Let $\mathcal{X} \in \langle 0, 1\rangle^{\mathrm{Clss}}$ and $[c, r] \in \mathrm{LClss_S}$ be given such that $\mathrm{rg}\,\mathcal{X}$ is finite, i.e. only *finitely many* different real numbers occur in $\mathcal{X}$ as labels.

ad $\Vdash_{\overline{\mathrm{L}}}$. Assume $\mathcal{X} \Vdash_{\overline{\mathrm{L}}} [c, r]$. It will be established that there exists a finite $\mathcal{X}_{\text{fin}} \subseteqq \mathcal{X}$ such that $\mathcal{X}_{\text{fin}} \Vdash_{\overline{\mathrm{L}}} [c, r]$.

Let $c = \{\ell_1, \dots, \ell_n\}$, for $\ell_1, \dots, \ell_n \in \mathrm{Lit}$. By the assumption $\mathcal{X} \Vdash_{\overline{\mathrm{L}}} [c, r]$ and the refutation system established in Theorem 5.4.8.1, there exists $r' > 1 - r$ such that $\mathcal{X} \cup \left[\overline{\ell_1}, r'\right] \cup \dots \cup \left[\overline{\ell_n}, r'\right] \in \mathrm{Incons}_{\mathrm{L}}$.

Hence, again by Theorem 5.4.8.1, it is sufficient to prove that there exists a finite $\mathcal{X}_{\text{fin}} \subseteqq \mathcal{X}$ such that $\mathcal{X}_{\text{fin}} \cup \left[\overline{\ell_1}, r'\right] \cup \dots \cup \left[\overline{\ell_n}, r'\right]$ has no model.

This result follows immediately from the next proposition, which is an adapted form of an analogous result proved by S. Gottwald in [44, Theorem 6.7]. For convenience, the reasoning is sketched here, with notation adapted to that used here.

Proposition 5.4.2 (Compactness wrt models in Lee's fuzzy logic)
*Given $\mathcal{X} \in \langle 0, 1\rangle^{\mathrm{Clss}}$, if every **finite** $\mathcal{X}_{\text{fin}} \subseteqq \mathcal{X}$ has a model, then $\mathcal{X}$ has a model (wrt $\overset{\mathrm{L}}{\models}$).*

For proving Proposition 5.4.2, consider the metric spaces $\left[\langle 0, 1\rangle, d\right]$ and $\left[\langle 0, 1\rangle^{\mathrm{PV}}, \rho\right]$ where d is the usual metric of the real line and ρ is the usual uniform metric, defined for $\mathcal{A}, \mathcal{B} \in \langle 0, 1\rangle^{\mathrm{PV}}$ by

$$\rho(\mathcal{A}, \mathcal{B}) =_{\text{def}} \mathrm{Sup}\left\{ d\left(\mathcal{A}(p), \mathcal{B}(p)\right) \,\middle|\, p \in \mathrm{PV}\right\}.$$

Then the metric spaces $\left[\langle 0, 1\rangle, d\right]$ and $\left[\langle 0, 1\rangle^{\mathrm{PV}}, \rho\right]$ are **compact**.

Lemma 5.4.3 ([44, Lemma 6.5])
*For every $c \in \mathrm{Clss}_{\mathrm{S}}$ the function $\mathrm{Val}_{\cdot}(c)$ is a **continuous** one from $\left[\langle 0, 1\rangle^{\mathrm{PV}}, \rho\right]$ into $\left[\langle 0, 1\rangle, d\right]$.*

Proof
Follows immediately from the continuity of max and $1 - x$. □

Lemma 5.4.4 (analogous with [44, Lemma 6.6])
For every $[c, r] \in \mathrm{LClss}_{\mathrm{S}}$, the set

$$M_{[c,r]} =_{\text{def}} \left\{\mathcal{A} \,\middle|\, \mathcal{A} \in \langle 0, 1\rangle^{\mathrm{PV}} \text{ and } \mathrm{Val}_{\mathcal{A}} \overset{\mathrm{L}}{\models} [c, r]\right\}$$

*is a **closed** subset of $\left[\langle 0, 1\rangle^{\mathrm{PV}}, \rho\right]$.*

Proof
In analogy with the proof of [44, Lemma 6.6], it is sufficient to prove that the set

$$\langle 0, 1\rangle^{\mathrm{PV}} \setminus M_{[c,r]} = \left\{\mathcal{A} \,\middle|\, \mathcal{A} \in \langle 0, 1\rangle^{\mathrm{PV}} \text{ and } \mathrm{Val}_{\mathcal{A}}(c) < r\right\}$$

is an open subset of $\left[\langle 0, 1\rangle^{\mathrm{PV}}, \rho\right]$.

The straightforward proof is carried out exactly as the proof of [44, Lemma 6.6] (see there or [68, Lemma 4.2.2]). □

Proof (of Proposition 5.4.2)
Compare the proof of [44, Theorem 6.7].
Let $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls_S}}$ be given such that every *finite* $\mathcal{X}_{\mathrm{fin}} \subseteqq \mathcal{X}$ has a model.
It follows that every finite intersection of sets from

$$\left\{ M_{[c,\mathcal{X}(c)]} \middle| c \in \mathrm{Cls_S} \right\}$$

is nonempty.
Hence by the compactness of $\left[\langle 0,1\rangle^{\mathrm{PV}}, \rho\right]$,

$$\bigcap \left\{ M_{[c,\mathcal{X}(c)]} \middle| c \in \mathrm{Cls_S} \right\}$$

is also nonempty. But obviously, every $\mathcal{A} \in \bigcap \left\{ M_{[c,\mathcal{X}(c)]} \middle| c \in \mathrm{Cls_S} \right\}$ induces a model $\mathrm{Val}_{\mathcal{A}}$ of $\mathcal{X}$. □

ad $\Vdash_{\overline{\mathrm{P}}}$. Assume $\mathcal{X} \Vdash_{\overline{\mathrm{P}}} [c,r]$. It will be established that there exists a *finite* $\mathcal{X}_{\mathrm{fin}} \subseteqq \mathcal{X}$ such that $\mathcal{X}_{\mathrm{fin}} \Vdash_{\overline{\mathrm{P}}} [c,r]$.
By the assumption that $\operatorname{rg}\mathcal{X}$ is *finite*, it follows from (5.89) (compare (5.66)) that $\mathcal{X} \Vdash_{\overline{\mathrm{P}}} [c,r]$ is equivalent with

$$\mathrm{CUT}_r(\mathcal{X}) \Vdash c.$$

As the classical entailment relation in two-valued propositional logic is compact it follows that there exists a *finite* $X_{\mathrm{fin}} \subseteqq \mathrm{CUT}_r(\mathcal{X})$ such that $X_{\mathrm{fin}} \Vdash c$.
That there exists a finite $\mathcal{X}_{\mathrm{fin}} \subseteqq \mathcal{X}$ such that $\mathrm{CUT}_r(\mathcal{X}_{\mathrm{fin}}) = X_{\mathrm{fin}}$ and hence $\mathcal{X}_{\mathrm{fin}} \Vdash_{\overline{\mathrm{P}}} [c,r]$ is obvious. □

To make the further comparison more feasible, in the remainder of this section it shall be implicitly assumed that the condition for **weak compactness** (i.e. that $\operatorname{rg}\mathcal{X}$ is *finite*) is fulfilled; it shall be pointed out where it is indispensable.

5.4.3 Validity Indices and Valid $\mathfrak{F}$-Fuzzy Sets of Clauses

The first significant difference between the logics of graded truth assessment and graded trust assessment being compared in this section is that in LEE's labelled fuzzy logic, the **validity index** valid is *two-valued* while in possibilistic logic with necessity-labelled formulae, valid takes values from $\langle 0,1\rangle$, hence it can be expected that in possibilistic logic with necessity-labelled formulae, a finer characterisation of the concept of validity is possible.

Before concentrating on graded validity, it is interesting to compare valid formulae in both logics. In sections 5.2.1.2 and 5.3.1.1, the following characterisations of the sets $\mathrm{Valid_L}, \mathrm{Valid_P}$ of all **valid** $\mathfrak{F}$-fuzzy sets of clauses wrt. Λ_{L}, Λ_{P}, respectively, were given:

$$\mathrm{Valid_L} = \left\{ \mathcal{X} \middle| \begin{array}{l} \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls_S}} \text{ and } \operatorname{supp}\mathcal{X} \subseteqq \mathrm{Taut_B} \\ \text{and } \forall c \in \mathrm{Cls_S} : \mathcal{X}(c) \leqq \frac{1}{2} \end{array} \right\}, \qquad \text{(by (5.32) and (5.96))}$$

$$\mathrm{Valid_P} = \left\{ \mathcal{X} \middle| \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls_S}} \text{ and } \operatorname{supp}\mathcal{X} \subseteqq \mathrm{Taut_B} \right\}, \qquad \text{(by (5.101))}$$

i.e. $\mathrm{Valid_L}$ is a proper subset of $\mathrm{Valid_P}$.

While any fuzzy set of tautologies of two-valued propositional logic is valid in necessity-labelled possibilistic logic, a fuzzy set of classical tautologies is valid in LEE's labelled fuzzy logic if and only if none of the tautologies is labelled with a value above $\frac{1}{2}$.

With respect to semantic entailment, valid fuzzy sets of formulae are not very significant: From Proposition 4.3.1.4 and Observation 4.2.5.2 it follows that

$$\mathcal{X} \in \text{Valid iff } \text{Cons}(\mathcal{X}) = \text{Cons}(\emptyset),$$

hence wrt. semantic consequence, an valid fuzzy set of formulae carries no more information than the *empty* fuzzy set of formulae. Furthermore, by Proposition 4.3.4.1 a labelled formula is valid iff it is entailed by *all* fuzzy sets of formulae, hence entailment of an valid labelled formula does not allow to distinguish between fuzzy sets of formulae.

From the above reasoning, it can be concluded that in LEE's labelled fuzzy logic, there are *more* fuzzy sets of clauses which are significant for semantic consequence: If $\mathcal{X} \in \text{Valid}_\text{P} \setminus \text{Valid}_\text{L}$, then $\text{Cons}_\text{L}(\mathcal{X}) = \text{Cons}_\text{L}(\emptyset)$ does not hold any more, and if $\mathcal{X}$ is a labelled clause $\mathfrak{x}$, then $\mathfrak{x}$ is not entailed by all fuzzy sets of clauses.

Next, consider the **validity index** valid. In possibilistic logic with necessity-labelled formulae, it is possible to consider fuzzy sets of formulae which are **not** *valid*, and determine their validity index with an uncountably infinite number of degrees. In Observation 5.3.13, the following characterisation was given:

$$\text{valid}_\text{P}(\mathcal{X}) = \inf \left\{ d \middle| \ \text{CUT}_{1-d}(\mathcal{X}) \not\subseteqq \text{Taut}_\text{B} \right\}.$$

By definition, Valid_P is the set of all $\mathcal{X}$ for which $\text{valid}_\text{P}(\mathcal{X}) = 1$, so valid_P allows to study fuzzy sets of formulae which are *less* than valid. It is possible to define a set of fuzzy sets of clauses which are *at least d-valid*, given $d \in \langle 0, 1 \rangle$:

$$\text{Valid}_\text{P}^d =_\text{def} \left\{ \mathcal{X} \middle| \mathcal{X} \in \langle 0, 1 \rangle^{\text{Cls}_\text{S}} \text{ and } \text{valid}_\text{P}(\mathcal{X}) \geqq d \right\}.$$

Observations 5.4.5 (Properties of Valid_P^d)

1. $\text{Valid}_\text{P}^1 = \text{Valid}_\text{P}$, $\text{Valid}_\text{P}^0 = \langle 0, 1 \rangle^{\text{Cls}_\text{S}}$.
2. If $d \geqq d'$, then $\text{Valid}_\text{P}^d \subseteqq \text{Valid}_\text{P}^{d'}$.
3. $\text{Valid}_\text{P}^d = \left\{ \mathcal{X} \middle| \mathcal{X} \in \langle 0, 1 \rangle^{\text{Cls}_\text{S}} \text{ and } \forall d' > 1 - d : \text{CUT}_{d'}(\mathcal{X}) \subseteqq \text{Taut}_\text{B} \right\}$.
4. For all $d \in \langle 0, 1 \rangle$, Valid_L is a proper subset of Valid_P^d. •

Proof

Obvious from the definitions. □

As a conclusion to this subsection, it can be stated that

- in LEE's labelled fuzzy logic, only the distinction between *valid* and **not** valid fuzzy sets of formulae can be made;
- in possibilistic logic with necessity-labelled formulae, there is a hierarchy of sets of fuzzy sets of formulae which are at least d-valid (indexed by $\langle 0, 1 \rangle$, ordered by the superset relation);

- every formula which is *valid* in LEE's labelled fuzzy logic is valid in possibilistic logic with necessity-labelled formulae (and of course at least d-valid in possibilistic logic with necessity-labelled formulae to any degree d), so on the whole, validity in possibilistic logic with necessity-labelled formulae is a **weaker** concept that in LEE's labelled fuzzy logic;

- in both logics considered here, the concept of validity naturally reduces to validity in classical two-valued logic (denoted *tautology* here).

5.4.4 Inconsistency

Concerning inconsistency, the situation is similar as with validity. In LEE's labelled fuzzy logic, the **consistency index** cst is *two-valued* while in possibilistic logic with necessity-labelled formulae, cst takes values from $\langle 0, 1\rangle$, hence it can be expected that in possibilistic logic with necessity-labelled formulae, a finer characterisation of the concept of (in)consistency is possible.

Before concentrating on graded inconsistency, it is interesting to compare inconsistent formulae in both logics. In sections 5.2.1.2 and 5.3.1.1, the following characterisations of the sets $\mathrm{Incons}_{\mathrm{L}}, \mathrm{Incons}_{\mathrm{P}}$ of all **inconsistent** $\mathfrak{F}$-fuzzy sets of clauses wrt. Λ_{L}, Λ_{P}, respectively, were given:

$$\mathrm{Incons}_{\mathrm{L}} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}_{\mathrm{S}}} \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}} : \mathrm{Val} \not\models \mathcal{X}\right\}, \qquad \text{(by (5.33))}$$

$$\mathrm{Incons}_{\mathrm{P}} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}_{\mathrm{S}}} \text{ and } \sup\left\{d \,\middle|\, \mathrm{CUT}_d(\mathcal{X}) \notin \mathrm{Sat}_{\mathrm{B}}\right\} = 1\right\}. \qquad \text{(by (5.102))}$$

It can be observed that in contrast with absolute validity, a natural reduction of absolute inconsistency to two-valued logic exists only in possibilistic logic with necessity-labelled formulae, while in LEE's labelled fuzzy logic, absolute inconsistency appears to depend on the many-valued interpretation of logical formulae. This is supported by the fact that by the **refutation systems** given in subsection 5.4.5, entailment in LEE's labelled fuzzy logic is completely characterised by absolute inconsistency.

By Proposition 4.3.4.4, a fuzzy set of formulae is inconsistent if and only if entailment from it is trivial (in the sense that every labelled formula is entailed). In this sense, the inconsistency degree $\mathrm{inc}_{\mathrm{P}}$ can be considered to be a way of dealing *gracefully* with inconsistencies, in the sense that meaningful consequences can be drawn from a fuzzy set $\mathcal{X}$ of formulae even if some inconsistency is present in $\mathcal{X}$, as long as it is not inconsistent (i. e. $0 < \mathrm{inc}_{\mathrm{P}}(\mathcal{X}) < 1$). In LEE's labelled fuzzy logic, this is impossible because inconsistency is two-valued: either a fuzzy set of formulae is inconsistent, trivialising semantic consequences from it, or it is not inconsistent at all.

The treatment of inconsistency in possibilistic logic with necessity-labelled formulae and LEE's labelled fuzzy logic, respectively, is illustrated further in the following, employing the inconsistency degree $\mathrm{inc}_{\mathrm{P}}$ from possibilistic logic with necessity-labelled formulae. In Observation 5.3.13, the following characterisation was given:

$$\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = \sup\left\{d \,\middle|\, \mathrm{CUT}_d(\mathcal{X}) \notin \mathrm{Sat}_{\mathrm{B}}\right\}. \tag{5.105}$$

Proposition 5.4.6 ($\mathrm{inc}_{\mathrm{P}}$ vs. $\mathrm{Incons}_{\mathrm{L}}$)

1. *If* $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) > \frac{1}{2}$, *then* $\mathcal{X} \in \mathrm{Incons}_{\mathrm{L}}$.

2. *If* $\mathcal{X} \in \mathrm{Incons}_{\mathrm{L}}$, *then* $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) > 0$.

Proof

ad 1. For the proof by contradiction, assume that $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) > \frac{1}{2}$ and furthermore, $\mathcal{X} \in \mathrm{Incons}_{\mathrm{L}}$ does **not** hold, i.e. there is $\mathcal{A} : \mathrm{PV} \to \langle 0,1\rangle$ such that $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}$ in the sense of Example 3.2.4.2 and section 5.2.1.2.

Now, by the assumption $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) > \frac{1}{2}$, there is a real number $r > \frac{1}{2}$ such that $\mathrm{CUT}_r(\mathcal{X})$ is classically inconsistent. For every $c \in \mathrm{CUT}_r(\mathcal{X})$, it holds by definition that $\mathcal{X}(c) > \frac{1}{2}$. Let $\mathcal{X}'$ be the union of all these $\big[c, \mathcal{X}(c)\big]$. Obviously, if $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}$, then $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}'$.

Recalling that $\mathrm{Cls}_{\mathrm{S}}$ is assumed to consist of sets of literals, it is clear that for each $c \in \mathrm{CUT}_r(\mathcal{X})$ there exists a literal $l_c \in c$ such that $\mathrm{Val}_{\mathcal{A}}(l_c) > \frac{1}{2}$ (otherwise $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}'$ would not hold).

Fix $\mathcal{A}' : \mathrm{PV} \to \{0,1\}$ by

$$\mathcal{A}'(l) =_{\mathrm{def}} 1 \text{ for every } l \in \mathrm{Lit} \text{ such that } \mathrm{Val}_{\mathcal{A}}(l) > \frac{1}{2}$$

and choosing arbitrary (matching) values for all literals not fixed by this. This always effectively determines some $\mathcal{A}' : \mathrm{PV} \to \{0,1\}$ because a literal and its complement cannot simultaneously have a value exceeding $\frac{1}{2}$ under $\mathcal{A}$.

The claim that $\mathcal{A}' \models \mathrm{CUT}_r(\mathcal{X})$ classically is trivial because for every $c \in \mathrm{CUT}_r(\mathcal{X})$, $l_c \in c$ (existing by the argument above) is made true by $\mathcal{A}'$.

Thus, a contradiction has been derived from the assumption $\mathcal{X} \notin \mathrm{Incons}_{\mathrm{L}}$, establishing the claim of the proposition.

ad 2. For the proof by contraposition, assume that $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = 0$ and establish that there exists $\mathcal{A} : \mathrm{PV} \to \langle 0,1\rangle$ such that $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}$.

By (5.105), from $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = 0$ it follows that $\mathrm{supp}\,\mathcal{X} \in \mathrm{Sat}_{\mathrm{B}}$, hence there exists $\mathcal{A} : \mathrm{PV} \to \{0,1\}$ such that $\mathrm{Val}_{\mathcal{A}} \models \mathrm{supp}\,\mathcal{X}$ in the sense of Example 3.2.4.1 and section 5.3.1.1.

Obviously, $\mathcal{A}$ can be considered to be a mapping from PV to $\langle 0,1\rangle$. Establish that $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}$ in the sense of Example 3.2.4.2 and section 5.2.1.2. From the definitions in Example 3.2.4.2 and the assumption $\mathrm{Val}_{\mathcal{A}} \models \mathrm{supp}\,\mathcal{X}$, it follows that $\mathrm{Val}_{\mathcal{A}}(c) = 1$ for every $c \in \mathrm{supp}\,\mathcal{X}$. This means that $\mathrm{Val}_{\mathcal{A}}(c) \geqq \mathcal{X}(c)$ for all $c \in \mathrm{Cls}_{\mathrm{S}}$, from which $\mathrm{Val}_{\mathcal{A}} \models^{\mathrm{L}} \mathcal{X}$ follows immediately, establishing the claim. □

By (5.100), $\mathrm{Incons}_{\mathrm{P}}$ is the set of all $\mathcal{X}$ for which $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = 1$, so $\mathrm{inc}_{\mathrm{P}}$ allows to study fuzzy sets of formulae which are *less* than inconsistent. It is possible to define a set of formulae which are *at least d-inconsistent*, given $d \in \langle 0,1\rangle$:

$$\mathrm{Incons}^d_{\mathrm{P}} =_{\mathrm{def}} \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}_{\mathrm{S}}} \text{ and } \mathrm{inc}_{\mathrm{P}}(\mathcal{X}) \geqq d\right\}.$$

Observations 5.4.7 (Properties of $\mathrm{Incons}^d_{\mathrm{P}}$)

1. $\mathrm{Incons}^1_{\mathrm{P}} = \mathrm{Incons}_{\mathrm{P}}$, $\mathrm{Incons}^0_{\mathrm{P}} = \langle 0,1\rangle^{\mathrm{Cls}_{\mathrm{S}}}$.
2. If $d \geqq d'$, then $\mathrm{Incons}^d_{\mathrm{P}} \subseteqq \mathrm{Incons}^{d'}_{\mathrm{P}}$.
3. $\mathrm{Incons}^d_{\mathrm{P}} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}_{\mathrm{S}}} \text{ and } \forall d' > d : \mathrm{CUT}_{d'}(\mathcal{X}) \in \mathrm{Sat}_{\mathrm{B}}\right\}$.
4. $\bigcup_{d>\frac{1}{2}} \mathrm{Incons}^d_{\mathrm{P}} \subseteqq \mathrm{Incons}_{\mathrm{L}} \subseteqq \bigcup_{d>0} \mathrm{Incons}^d_{\mathrm{P}}$.
5. $\mathrm{Incons}_{\mathrm{P}}$ is a proper subset of $\mathrm{Incons}_{\mathrm{L}}$. •

Proof
Obvious from the definitions and Proposition 5.4.6. Item 5 follows from the fact that the monotonicity claimed in item 2 is strict really. □

Observation 5.4.7.4 justifies the claim that the variant of possibilistic logic studied here handles inconsistencies better in some sense, because for all fuzzy sets $\mathcal{X}$ from $\bigcup_{1>d>\frac{1}{2}} \mathrm{Incons}_{\mathrm{P}}^{d}$, $\mathcal{X} \Vdash_{\mathrm{P}} [c, r]$ is still meaningful for $r > \mathrm{inc}_{\mathrm{P}}(\mathcal{X})$ (compare (4.128)), while $\mathcal{X} \Vdash_{\mathrm{L}} [c, r]$ holds trivially for all $[c, r] \in \mathrm{LCls}_{\mathrm{S}}$.

The case $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) \leqq \frac{1}{2}$ is illustrated by two simple examples. Let $p \in \mathrm{PV}$ and

$$\mathcal{X} =_{\mathrm{def}} [p, 0.4] \cup [\neg p, 0.5]\,,$$
$$\mathcal{Y} =_{\mathrm{def}} [p, 0.4] \cup [\neg p, 0.7]\,.$$

(here, the clause notation has been simplified by leaving out the set braces for singleton clauses.)

Obviously, $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = \mathrm{inc}_{\mathrm{P}}(\mathcal{Y}) = 0.4$, but $\mathcal{X} \notin \mathrm{Incons}_{\mathrm{L}}$ while $\mathcal{Y} \in \mathrm{Incons}_{\mathrm{L}}$. The maximal labels with which labelled clauses are entailed by $\mathcal{X}, \mathcal{Y}$ in both systems are given as follows:

$$\begin{array}{ll} \mathcal{X} \Vdash_{\mathrm{P}} [p, 0.4] & \mathcal{X} \Vdash_{\mathrm{L}} [p, 0.4] \\ \mathcal{X} \Vdash_{\mathrm{P}} [\neg p, 0.5] & \mathcal{X} \Vdash_{\mathrm{L}} [\neg p, 0.5] \\ \mathcal{Y} \Vdash_{\mathrm{P}} [p, 0.4] & \mathcal{Y} \Vdash_{\mathrm{L}} [p, 1] \\ \mathcal{Y} \Vdash_{\mathrm{P}} [\neg p, 0.7] & \mathcal{Y} \Vdash_{\mathrm{L}} [\neg p, 1] \end{array}$$

To these, of course all sorts of trivial entailments have to be added, like $\mathcal{X} \Vdash_{\mathrm{P}} [c, 0.4]$ and $\mathcal{Y} \Vdash_{\mathrm{P}} [c, 0.4]$ and $\mathcal{Y} \Vdash_{\mathrm{L}} [c, 1]$ for all $c \in \mathrm{Cls}_{\mathrm{S}}$ because of the inconsistencies, and the valid labelled clauses.

Because of inconsistencies, the entailments $\mathcal{X} \Vdash_{\mathrm{P}} [p, 0.4]$, $\mathcal{Y} \Vdash_{\mathrm{P}} [p, 0.4]$, $\mathcal{Y} \Vdash_{\mathrm{L}} [p, 1]$, $\mathcal{Y} \Vdash_{\mathrm{L}} [\neg p, 1]$ have to be considered trivial, so from these simple examples, it can not easily be decided which system handles inconsistencies better if $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) \leqq 0.5$.

As a conclusion to this subsection, it can be stated that

- in Lee's labelled fuzzy logic, only the distinction between *inconsistent* and **not** inconsistent fuzzy sets of formulae can be made;
- in possibilistic logic with necessity-labelled formulae, there is a hierarchy of sets $\mathrm{Incons}_{\mathrm{P}}^{d}$ of formulae which are at least d-inconsistent (indexed by $\langle 0, 1\rangle$, ordered by the superset relation);
- the set of formulae which are *inconsistent* in Lee's labelled fuzzy logic is 'completely covered' by the hierarchy of sets of d-inconsistent formulae of possibilistic logic, in the sense that $\mathrm{Incons}_{\mathrm{P}}$ is a proper subset of $\mathrm{Incons}_{\mathrm{L}}$ and $\mathrm{Incons}_{\mathrm{L}}$ is a subset of $\bigcup_{d>0} \mathrm{Incons}_{\mathrm{P}}^{d}$;
- in possibilistic logic with necessity-labelled formulae, the concept of inconsistency naturally reduces to inconsistency in classical two-valued logic (denoted *insatisfiability* here), while inconsistency in Lee's labelled fuzzy logic depends on the many-valued interpretation of logical formulae.

5.4.5 Refutation

Refutation means a characterisation of entailment by inconsistency, effected by adding the 'negation' of the labelled clause to be entailed to the set of labelled clauses to be entailed from, to achieve inconsistency.

The concept of refutation has been studied, from a general point of view, in section 4.3.3.

It is obvious that a system like LEE's labelled logic, where the model relation is **compact** (see [44, 68]), does not admit a general refutation system because this would contradict the non-compactness of entailment.

Furthermore, Observation 5.2.8 states that the refutation system established in section 4.3.3 is not applicable here because the labels used in LEE's labelled logic do not admit refutation (in the sense of Definition 4.3.4), in general, i. e. the label set has too little expressive power. In the 'weakly compact case', where it is assumed that only finitely many different labels occur, there is a refutation system though, which is based on a different method of calculating the labels for the labelled clauses to be added.

In the case of necessity-labelled possibilistic logic, Observation 5.3.14 gives a complete refutation system.

The results are summarised in the following Theorem.

Theorem 5.4.8 (Refutation)
Let $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Clss}}$ *and* $[c,r] \in \mathrm{LClss_S}$ *with* $c = \{l_1, \ldots, l_n\}$.

1. *Assume that* $\operatorname{rg}\mathcal{X}$ *is* ***finite*** *and* $r > 0$. *Then there exists* $r' \in \langle 0,1\rangle$ *with* $r' > 1 - r$ *such that* $\mathcal{X} \Vdash_{\mathrm{L}} [c,r]$ *if and only if* $\mathcal{X} \cup \left[\overline{l_1}, r'\right] \cup \cdots \cup \left[\overline{l_n}, r'\right] \in \mathrm{Incons_L}$.
2. $\mathcal{X} \Vdash_{\mathrm{P}} [c,r]$ *if and only if* $\mathrm{inc_P}\left(\mathcal{X} \cup \left[\overline{l_1}, 1\right] \cup \cdots \cup \left[\overline{l_n}, 1\right]\right) \geqq r$.

Proof

ad 1. $\mathcal{X} \Vdash_{\mathrm{L}} [c,r]$ is equivalent with $\mathrm{Val}(c) \geqq r$ for every $\mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ such that $\mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X}$. $\mathrm{Val}(c) \geqq r$ in turn is equivalent with $\mathrm{Val}(\neg c) \leqq 1 - r$. Considering the semantics $\mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ defined in Example 3.2.4.2, it is immediately observed that

$$\mathcal{X} \Vdash_{\mathrm{L}} [c,r] \text{ iff for all } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that } \mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X}, \quad \mathrm{Val}\left(\overline{l_1}\right) \leqq 1 - r \text{ or } \ldots \text{ or } \mathrm{Val}\left(\overline{l_n}\right) \leqq 1 - r. \tag{5.106}$$

Furthermore, for every $r' \in \langle 0,1\rangle$ it holds that

$$\mathcal{X} \cup \left[\overline{l_1}, r'\right] \cup \cdots \cup \left[\overline{l_n}, r'\right] \in \mathrm{Incons_L} \text{ iff for all } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that } \mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X}, \quad \mathrm{Val}\left(\overline{l_1}\right) < r' \text{ or } \ldots \text{ or } \mathrm{Val}\left(\overline{l_n}\right) < r'. \tag{5.107}$$

Combining (5.106) with (5.107) yields that it is sufficient to prove that there exists $r' > 1 - r$ such that

$$\text{for all } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that } \mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X},\ \mathrm{Val}\left(\overline{l_1}\right) \leqq 1 - r \text{ or } \ldots \text{ or } \mathrm{Val}\left(\overline{l_n}\right) \leqq 1 - r$$

if and only if

$$\text{for all } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that } \mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X},\ \mathrm{Val}\left(\overline{l_1}\right) < r' \text{ or } \ldots \text{ or } \mathrm{Val}\left(\overline{l_n}\right) < r'.$$

Eliminating negation and complementation for ease of notation, it is sufficient to prove the following:

Given $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{Clss}}$ such that $\mathrm{rg}\,\mathcal{X}$ is *finite*, $l_1, \ldots, l_n \in \mathrm{Lit}$ and $r < 1$, there exists $r' > r$ such that

$$\text{(5.108)} \qquad \text{for all Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that Val} \models^{\mathrm{L}} \mathcal{X},\ \mathrm{Val}\,(l_1) \leqq r \text{ or } \ldots \text{ or } \mathrm{Val}\,(l_n) \leqq r$$

if and only if

$$\text{for all Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that Val} \models^{\mathrm{L}} \mathcal{X},\ \mathrm{Val}\,(l_1) < r' \text{ or } \ldots \text{ or } \mathrm{Val}\,(l_n) < r'.$$

Define

$$\text{(5.109)} \qquad r' =_{\mathrm{def}} \inf \{ s \,|\, 1 - s \in \mathrm{rg}\,\mathcal{X} \text{ and } s > r \}\,.$$

As $\mathrm{rg}\,\mathcal{X}$ is finite, the infimum is reached and thus $r' > r$ (note that this even holds if the set is empty, as the infimum of the empty set wrt. the real unit interval yields $r' = 1$, and $r < 1$ by assumption).

It remains to prove the equivalence (5.108) for this r'.

The "only if" direction obviously holds for every $r' > r$. It remains to prove the "if" direction.

Assume that

$$\text{(5.110)} \qquad \text{for all Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that Val} \models^{\mathrm{L}} \mathcal{X},\ \mathrm{Val}\,(l_1) < r' \text{ or } \ldots \text{ or } \mathrm{Val}\,(l_n) < r'.$$

Let $\mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ be given such that $\mathrm{Val} \models^{\mathrm{L}} \mathcal{X}$. It remains to prove

$$\text{(5.111)} \qquad \mathrm{Val}\,(l_1) \leqq r \text{ or } \ldots \text{ or } \mathrm{Val}\,(l_n) \leqq r.$$

This holds always if $\{l_1, \ldots, l_n\}$ contains two complementary literals and $r \geqq \frac{1}{2}$, so it is assumed in the following that this case doesn't occur.

From (5.110), it follows that

$$\mathrm{Val}\,(l_1) < r' \text{ or } \ldots \text{ or } \mathrm{Val}\,(l_n) < r'.$$

For simplicity, for the remainder of this proof it is assumed that $\mathrm{Val}\,(l_1) < r'$ and for all $k > 1$, $\mathrm{Val}\,(l_k) \geqq r'$.

Extending the proof to the general case that an arbitrary non-empty selection from $l_1, \ldots, l_n$ assumes a value strictly below r' is straightforward and involves only some organisational overhead. It is neglected here as it provides no further insights.

Under the assumption made above, it is sufficient to prove

$$\text{(5.112)} \qquad \mathrm{Val}\,(l_1) \leqq r.$$

This is achieved with a proof by contradiction, i.e. the assumption $\mathrm{Val}\,(l_1) > r$ is led to a contradiction, establishing (5.112).

As $\mathrm{Val}\,(l_1) < r'$ holds by assumption, it remains to derive a contradiction from the assumption

$$\text{(5.113)} \qquad r < \mathrm{Val}(l_1) < r'.$$

By the definition of the semantics $\mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}$ (compare Examples 3.2.1–3.2.4), there exists an assignment $\mathcal{A} : \mathrm{PV} \to \langle 0, 1 \rangle$ such that $\mathrm{Val} = \mathrm{Val}_{\mathcal{A}}$. Wlg. assume that

$$l_1 = p \text{ for } p \in \mathrm{PV}. \tag{5.114}$$

Consider the assignment $\mathcal{A}' : \mathrm{PV} \to \langle 0, 1 \rangle$ defined for all $q \in \mathrm{PV}$ by

$$\mathcal{A}'(q) =_{\mathrm{def}} \begin{cases} r', & \text{if } q = p \\ \mathcal{A}(q), & \text{if } q \neq p. \end{cases} \tag{5.115}$$

For completing the proof, two cases are distinguished.

Case 1. None of $l_2, \ldots, l_n$ is the *complementary literal* to l_1.

In this case, by the fact that only the value of p has been changed and the assumption that for all $k > 1$, $\mathrm{Val}_{\mathcal{A}}(l_k) \geqq r'$, it follows that for all $k \geqq 1$, $\mathrm{Val}_{\mathcal{A}'}(l_k) \geqq r'$, hence by (5.110), $\mathrm{Val}_{\mathcal{A}'} \not\models \mathcal{X}$.

By definition, this means there exists $c \in \mathrm{Cls}$ such that

$$\mathrm{Val}_{\mathcal{A}'}(c) < \mathcal{X}(c). \tag{5.116}$$

Furthermore,

$$\mathrm{Val}_{\mathcal{A}}(c) \geqq \mathcal{X}(c) \tag{5.117}$$

by the assumption $\mathrm{Val}_{\mathcal{A}} \models \mathcal{X}$.

From the preconditions

(i) $\mathcal{A}$ and $\mathcal{A}'$ differ only in the assignment of p, (by (5.115))

(ii) $\mathcal{A}(p) < \mathcal{A}'(p)$, (by (5.113), (5.114), (5.115))

(iii) $\mathrm{Val}_{\mathcal{A}'}(c) < \mathrm{Val}_{\mathcal{A}}(c)$, (by (5.116), (5.117))

the following conclusions can be drawn:

$$\neg p \in c \qquad \text{(by (ii) and (iii))} \tag{5.118}$$

$$\mathrm{Val}_{\mathcal{A}}(c) = \mathcal{A}(\neg p) \qquad \text{(by (i), (5.118), and (iii))} \tag{5.119}$$

$$\mathrm{Val}_{\mathcal{A}'}(c) \geqq \mathcal{A}'(\neg p) \qquad \text{(by (5.118) and the semantics of clauses)} \tag{5.120}$$

$$\mathcal{X}(c) > 1 - r'. \qquad \text{(by (5.116), (5.120), and (5.115))} \tag{5.121}$$

$$\mathcal{X}(c) < 1 - r. \qquad \text{(by (5.113), (5.114), (5.119), (5.117))} \tag{5.122}$$

The inequation (5.122) means that $1 - \mathcal{X}(c) \in \{s \,|\, 1 - s \in \mathrm{rg}\,\mathcal{X} \text{ and } s > r\}$, hence by (5.109),

$$r' \leqq 1 - \mathcal{X}(c),$$

a contradiction to (5.121).

This contradiction concludes the proof of (5.112) in this case.

Case 2. There exists $k \in \{2, \ldots, n\}$ such that $l_k = \overline{l_1}$.

This case is slightly more complicated than the previous one because $\mathrm{Val}_{\mathcal{A}}(l_k) \neq \mathrm{Val}_{\mathcal{A}'}(l_k)$, so for achieving the requirement $\mathrm{Val}_{\mathcal{A}'} \not\models \mathcal{X}$ of the previous case, some additional measures have to be taken. In particular, it has to be made sure that $\mathrm{Val}_{\mathcal{A}'}(l_k) \geqq r'$ holds even though $l_k = \overline{l_1}$.

In this case, by the assumption on page 165 (below (5.111)), $r < \frac{1}{2}$.

It is easily verified that by choosing $r' =_{\mathrm{def}} \frac{1}{2}$ in case (5.109) yields a value *above* $\frac{1}{2}$, the proof for the previous case can be used otherwise unchanged. In particular, obviously $r < r'$ and if $\mathrm{Val}_{\mathcal{A}'}(l_1) = r' = \frac{1}{2}$, then $\mathrm{Val}_{\mathcal{A}'}(l_k) = \mathrm{Val}_{\mathcal{A}'}\left(\overline{l_1}\right) = \frac{1}{2} \geqq r'$.

This concludes the proof of item 1.

ad 2. Is identical with Observation 5.3.14. □

5.4.6 Concluding Remarks

As a conclusion to this section, the relation between $\Vdash_{\mathrm{L}}$ and $\Vdash_{\mathrm{P}}$ is illustrated further by two additional results.

The following proposition demonstrates that in the most simple case, i. e. where only finitely many singleton clauses occur in $\mathcal{X}$, $\Vdash_{\mathrm{L}}$ and $\Vdash_{\mathrm{P}}$ are equal and can easily be determined.

Proposition 5.4.9 (Comparison of $\Vdash_{\mathbf{L}}$ and $\Vdash_{\mathbf{P}}$ wrt fuzzy sets of literals)
Let $n \in \mathbb{N}, n \geqq 1$ and $\mathcal{X} =_{\mathrm{def}} [l_1, r_1] \cup \cdots \cup [l_n, r_n]$ for $l_i \in \mathrm{Lit}$ and $r_i \in \langle 0, 1\rangle$ $(i \in \{1, \ldots, n\})$; let $[c, r] \in \mathrm{LCls}_{\mathrm{S}}$. Assume $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = 0$ and $\mathrm{valid}_{\mathrm{P}}\big([c, r]\big) = 0$.

Then the following statements are equivalent:

(i) $\mathcal{X} \Vdash_{\mathrm{L}} [c, r]$,

(ii) $\mathcal{X} \Vdash_{\mathrm{P}} [c, r]$,

(iii) $c \cap \{l_1, \ldots, l_n\} \neq \varnothing$ *and* $r \leqq \max\big\{r_i \big| i \in \{1, \ldots, n\}$ *and* $l_i \in c\big\}$.

Proof
From $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = 0$ holds if and only if there are no two complementary literals in $\{l_1, \ldots, l_n\}$. $\mathrm{valid}_{\mathrm{P}}\big([c, r]\big) = 0$ holds if and only if there are no two complementary literals in c.

The equivalence of $\mathcal{X} \Vdash_{\mathrm{L}} [c, r]$ and $\mathcal{X} \Vdash_{\mathrm{P}} [c, r]$ with the explicit term is checked separately.

ad (i)⇔(iii). $\mathcal{X} \Vdash_{\mathrm{L}} [c, r]$ means that for every $\mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ such that $\mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X}$, $\mathrm{Val}(c) \geqq r$.

$\mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X}$ means that for every $i \in \{1, \ldots, n\}$, $\mathrm{Val}(l_i) \geqq r_i$.

$\mathrm{Val}(c) \geqq r$ means that there exists $l \in c$ such that $\mathrm{Val}(l) \geqq r$.

Hence, $\mathcal{X} \Vdash_{\mathrm{L}} [c, r]$ is equivalent with

$$\begin{aligned}&\text{for every } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that for every } i \in \{1, \ldots, n\}, \mathrm{Val}(l_i) \geqq r_i,\\ &\qquad\qquad\qquad\qquad \text{there exists } l \in c \text{ such that } \mathrm{Val}(l) \geqq r.\end{aligned}$$

Because of the structure of $\mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ (see Example 3.2.4.2), this statement is equivalent with

$$\begin{aligned}&\text{for every } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that for every } i \in \{1, \ldots, n\}, \mathrm{Val}(l_i) \geqq r_i,\\ &\qquad\qquad\qquad\qquad \text{there exists } l \in c \cap \{l_1, \ldots, l_n\} \text{ such that } \mathrm{Val}(l) \geqq r,\end{aligned}$$

which in turn is equivalent with

$$\begin{aligned}&\text{for every } \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} \text{ such that for every } i \in \{1, \ldots, n\}, \mathrm{Val}(l_i) \geqq r_i,\\ &\qquad\qquad \text{there exists } i \in \{1, \ldots, n\} \text{ such that } l_i \in c \text{ and } \mathrm{Val}(l_i) \geqq r,\end{aligned}$$

which obviously is equivalent with the statement (iii).

ad (ii)⇔(iii). As $\mathrm{rg}\,\mathcal{X}$ is finite, $\mathcal{X} \Vdash_{\mathrm{P}} [c, r]$ is equivalent with

$$\mathrm{CUT}_r(\mathcal{X}) \Vdash c,$$

which, by definition of $\Vdash$ in two-valued propositional logic, is equivalent with

$$\mathrm{CUT}_r(\mathcal{X}) \cap c \neq \varnothing. \tag{5.123}$$

$\mathrm{CUT}_r(\mathcal{X})$ is $\left\{l_i \middle| i \in \{1,\ldots,n\} \text{ and } r \leqq r_i\right\}$, hence (5.123) is equivalent with

$$\text{there exists } i \in \{1,\ldots,n\} \text{ such that } l_i \in c \text{ and } r \leqq r_i,$$

which obviously is equivalent with the statement (iii). □

The following example illustrates that even for very simple cases (which do not meet the requirements of the previous proposition), $\Vdash_{\mathrm{L}}$ and $\Vdash_{\mathrm{P}}$ are different.

Example 5.4.1 Let $\mathcal{X} =_{\mathrm{def}} \left[\{p, q\}, 0.7\right] \cup [\neg p, 0.4]$.
Then

$$X \Vdash_{\mathrm{L}} [q, 0.7]\,,$$
$$X \Vdash_{\mathrm{P}} [q, 0.4]\,.$$
•

This concludes the comparison of $\Vdash_{\mathrm{L}}$ and $\Vdash_{\mathrm{P}}$. The purpose of this section is to shed some light on differences and similarities between logics of graded truth assessment and logics of graded trust assessment.

It should have become clear that even the simple examples of such logics selected in this section, which were as a matter of fact selected to achieve maximal similarity between the logics, clearly show significantly *different* characteristics, justifying the claim that **truth** and **validity** are *orthogonal dimensions* of many-valuedness in logics.

The differences are certain to become much more significant as soon as in the case of many-valued truth, a more complicated algebraic structure is induced on the lattice of truth values by the logical operators than just the lattice structure itself (an *MV-algebra*, for instance).

The comparison of logics of graded truth assessment and logics of graded trust assessment is a matter of ongoing investigation and cooperation with the *Institut de Recherche en Informatique de Toulouse (I.R.I.T)*. Most of the results of this section have been published by the author in [73]; some preliminary results can be found in [20, 21] (coauthored between Didier Dubois[2], the author of this dissertation, and Henri Prade[2]); further joint publications are forthcoming.

5.5 Logics of Graded Truth and Graded Trust Assessment

As a matter of fact, a systematic study of the foundations of fuzzy filter-based logics for which *neither* the lattice $\mathfrak{T}$ of truth values *nor* the lattice $\mathfrak{D}$ of degrees of validity is the two-valued lattice $\mathfrak{B}$ has been given in chapter 4.

As the theoretical investigations there might seem abstract and sometimes overly general, in this section some aspects are cast in a more concrete form. The expressive power of *logics of graded truth and graded trust assessment* is illustrated by a series of examples.

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5.5.1 Examples of Labels

Here, some examples of labels are given. Their meaning and usefulness for the modelling of uncertain knowledge is explained.

For this subsection, both the lattice $\mathfrak{T}$ of truth values and the lattice $\mathfrak{D}$ of degrees of validity are assumed to be **chains**. This restriction is valuable for illustration purposes: it makes it possible to draw labels like *function graphs* along two linear axes.

Note that fuzzy filter-based logics for which $\mathfrak{T}$ is a chain have been called *simple logics of graded truth and graded trust assessment* in [73]; these logics are characterised by a very convenient and intuitively pleasing set of axioms (Axioms 1, 2, 3, 4, 5, 6, 7, 8, 10 from Definition 4.1.3 and axioms 9*, 11* from Observation 4.1.4).

For the extent of this subsection, every label is identified with the $\mathfrak{D}$-fuzzy filter on $\mathfrak{T}$ associated with it by the isomorphism ι. Hence, labels are considered to be monotone mappings from $\mathfrak{T}$ into $\mathfrak{D}$ (mapping 1 to 1), so it is justified to visualise them in the form of a diagram, using $\mathfrak{T}$ and $\mathfrak{D}$ as x and y axes in a coordinate system (see Figure 5.1).

5.5.1.1 'Simple' Labels

In Figure 5.2, some simple labels are sketched. Their meaning is discussed in the following. Note that it is assumed that T and D have 'enough' elements to make the labels which are illustrated effectively definable. The diagrams are drawn under the assumption that T and D contain a *continuity* of elements, but it should be easy to adapt the issues discussed below to a 'smaller' number of elements (larger than 2). In the following, several *classes* of labels are distinguished, as visualised in Figure 5.2.

Note that most of the simple labels discussed below are necessarily elements of L, because they correspond to principal $\mathfrak{D}$-fuzzy filters of $\mathfrak{T}$. It is noted below for which examples this is not the case.

Unknown. The **weakest** label, denoted by ℓ^{U}, stands for total lack of knowledge. It is defined by

$$\ell^{\mathrm{U}}(t) =_{\mathrm{def}} 1. \qquad (t \in T)$$

It is the **zero element** of the label lattice $\mathfrak{L}$ (which is assumed to be ordered by the *superset relation* of fuzzy sets). Obviously, by definition, for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$,

$$\mathrm{Val} \models_{\overline{\mathrm{T}}} \left[x, \ell^{\mathrm{U}}\right].$$

This means that Unknown does not restrict the possible validity of a formula at all: lacking any knowledge of constraints on the validity of x, it has to be assumed to be equally completely valid, regardless of its *truth value*.

Note that for a fuzzy set $\mathcal{X}$ of formulae, ℓ^{U}, the zero element of $\mathfrak{L}$, is the membership value assumed by all formulae which are "not in $\mathcal{X}$".

Absolutely True. The **strongest** label, denoted by ℓ^{AT}, stands for complete knowledge of total truth. It is defined by

$$\ell^{\mathrm{AT}}(t) =_{\mathrm{def}} \begin{cases} 1, & \text{if } t = 1 \\ 0, & \text{if } t \neq 1 \end{cases} \qquad (t \in T)$$

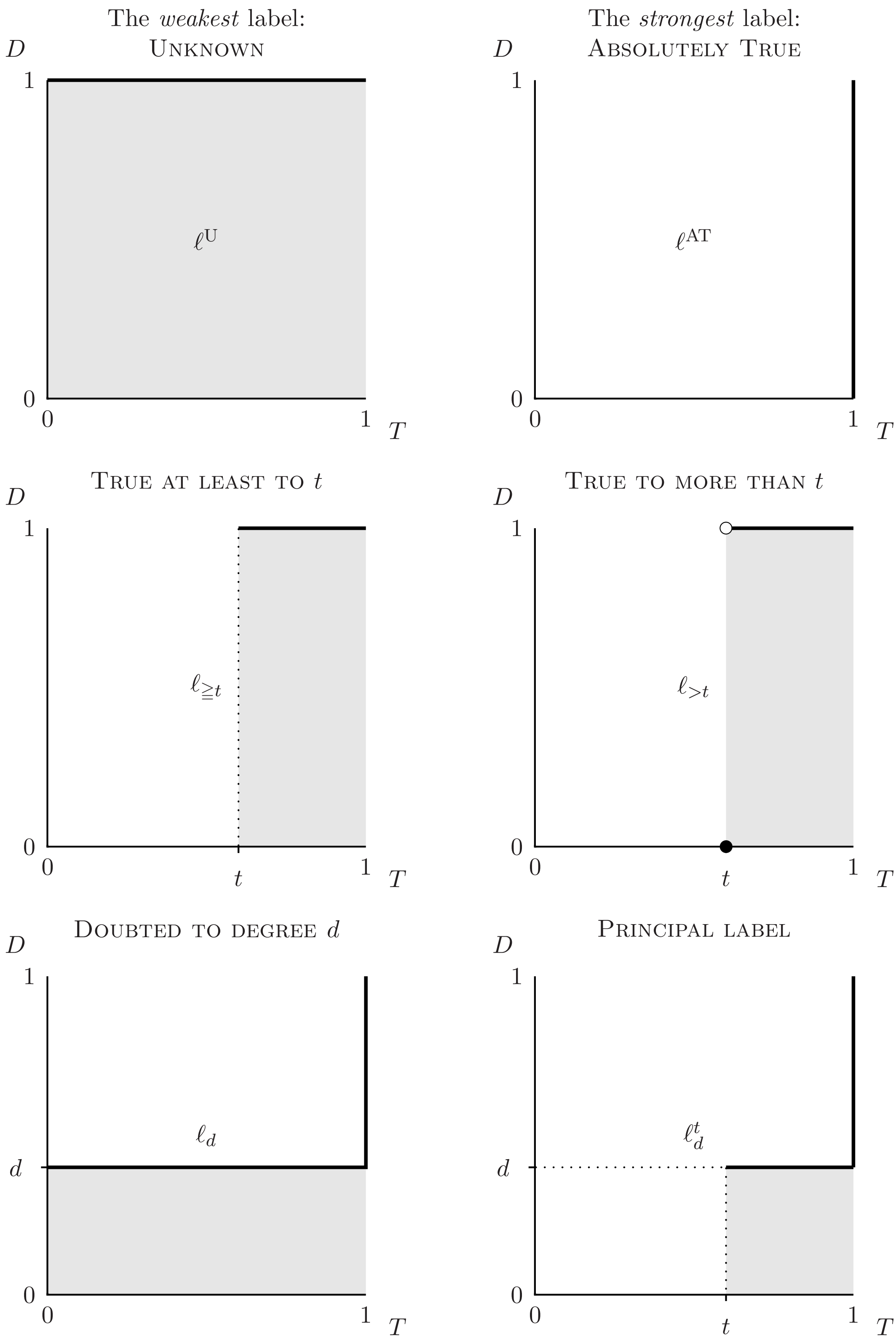


Figure 5.2: Some simple labels.

It is the **unit element** of the label lattice $\mathfrak{L}$ (which consists only of fuzzy sets assuming the value 1 for the truth value 1). By definition, for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$,

$$\mathrm{Val} \models_{\overline{1}} \left[x, \ell^{\mathrm{AT}}\right] \text{ if } \mathrm{Val}(x) = 1$$
$$\text{and } \mathrm{Val} \models_{\overline{0}} \left[x, \ell^{\mathrm{AT}}\right] \text{ if } \mathrm{Val}(x) \neq 1$$

This means that Absolutely True forces the formula x to be completely true if it is to be considered valid at all.

Remark

Let $\mathcal{P}\mathrm{Frm} =_{\mathrm{def}} \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \mathrm{rg}\,\mathcal{X} \subseteqq \left\{\ell^{\mathrm{U}}, \ell^{\mathrm{AT}}\right\}\right\}$. Obviously, the mapping $f : \mathcal{P}\mathrm{Frm} \to \mathfrak{P}\mathrm{Frm}$ which is defined for $\mathcal{X} \in \mathcal{P}\mathrm{Frm}$ by

$$f(\mathcal{X}) =_{\mathrm{def}} \left\{x \,\middle|\, x \in \mathrm{Frm} \text{ and } \mathcal{X}(x) = \ell^{\mathrm{AT}}\right\}$$

is a bijection between $\mathcal{P}\mathrm{Frm}$ and $\mathfrak{P}\mathrm{Frm}$.

Let $\models$ denote the classical model relation of many-valued logic wrt. Frm and $\mathfrak{S}$, i.e. for $\mathrm{Val} \in \mathfrak{S}$ and $X \in \mathfrak{P}\mathrm{Frm}$,

$$\mathrm{Val} \models X =_{\mathrm{def}} \forall x \in X : \mathrm{Val}(x) = 1.$$

Then for every $\mathrm{Val} \in \mathfrak{S}$ and $X \in \mathfrak{P}\mathrm{Frm}$,

$$\mathrm{Val} \models X \text{ iff } \mathrm{Val} \models_{\overline{1}} f^{-1}(X),$$

i.e. classical many-valued logic (denoted level 1 in section 3.4.1) can be embedded into the framework of fuzzy filter-based logic by appropriately restricting the range of admissible labels.

•

$\boxed{\textbf{True at least to } t\textbf{.}}$ A class of labels representing **truth values** is defined as follows. For each $t \in T$, the label True at least to t is denoted by $\ell_{\geqq t}$. It is defined by

$$\ell_{\geqq t}\left(t'\right) =_{\mathrm{def}} \begin{cases} 1, & \text{if } t \sqsubseteq_T t' \\ 0, & \text{if not } t \sqsubseteq_T t' \end{cases} \qquad (t' \in T)$$

By definition, for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$,

$$\mathrm{Val} \models_{\overline{1}} \left[x, \ell_{\geqq t}\right] \text{ if } t \sqsubseteq_T \mathrm{Val}(x)$$
$$\text{and } \mathrm{Val} \models_{\overline{0}} \left[x, \ell_{\geqq t}\right] \text{ if not } t \sqsubseteq_T \mathrm{Val}(x)$$

This means that the label $\ell_{\geqq t}$ forces the formula x to be true at least to the truth value t if it is to be considered valid at all.

The labels Unknown and Absolutely True are obtained as special cases: $\ell^{\mathrm{U}} = \ell_{\geqq 0}$ and $\ell^{\mathrm{AT}} = \ell_{\geqq 1}$.

Remark

Let $\mathcal{P}\mathrm{Frm} =_{\mathrm{def}} \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \mathrm{rg}\,\mathcal{X} \subseteqq \left\{\ell_{\geqq t} \,\middle|\, t \in T\right\}\right\}$. The mapping $f : T^{\mathrm{Frm}} \to \mathcal{P}\mathrm{Frm}$ which is defined for $\mathcal{X} \in T^{\mathrm{Frm}}$ and $x \in \mathrm{Frm}$ by

$$f(\mathcal{X})(x) =_{\mathrm{def}} \ell_{\geqq \mathcal{X}(x)}$$

is a bijection between T^{Frm} and $\mathcal{P}\mathrm{Frm}$.

Let $\models$ denote the model relation for the logic $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{T}, \models]$ of graded truth assessment where truth values are used as labels (compare section 5.2.1), i. e. for $\mathrm{Val} \in \mathfrak{S}$ and $\mathcal{X} \in T^{\mathrm{Frm}}$,

$$\mathrm{Val} \models \mathcal{X} =_{\mathrm{def}} \forall x \in \mathrm{Frm} : \mathcal{X}(x) \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x).$$

Then for every $\mathrm{Val} \in \mathfrak{S}$ and $\mathcal{X} \in T^{\mathrm{Frm}}$,

$$\mathrm{Val} \models \mathcal{X} \text{ iff } \mathrm{Val} \models_{\overline{1}} f(\mathcal{X}), \tag{5.124}$$

i. e. logics of graded truth assessment with truth value-labelled formulae (and hence PAVELKA's logic) can be embedded into the framework of fuzzy filter-based logic by appropriately restricting the range of admissible labels.

Note that if $\mathfrak{L}$ contains labels of the form $\ell_{>t}$ (see below), then it might happen that $\operatorname{rg}\mathrm{Cons}(\mathcal{X}) \subseteqq \left\{\ell_{\geqq t} \,\middle|\, t \in T\right\}$ does **not** hold even though $\operatorname{rg}\mathcal{X} \subseteqq \left\{\ell_{\geqq t} \,\middle|\, t \in T\right\}$ holds, because the definition of Cons involves an infinitary operation on $\mathfrak{L}$. •

True to more than t. Another class of labels based on truth values is defined as follows. For each $t \in T \setminus \{1\}$, the label TRUE TO MORE THAN t is denoted by $\ell_{>t}$. It is defined by

$$\ell_{>t}(t') =_{\mathrm{def}} \begin{cases} 1, & \text{if } t \sqsubseteq_{\mathfrak{T}} t' \text{ and } t \neq t' \\ 0, & \text{if } t = t' \text{ or not } t \sqsubseteq_{\mathfrak{T}} t' \end{cases} \qquad (t' \in T)$$

Remarks

1. Depending on the nature of $\mathfrak{T}$, it may be that $\left\{\ell_{>t} \,\middle|\, t \in T \setminus \{1\}\right\} \subseteqq \left\{\ell_{\geqq t} \,\middle|\, t \in T\right\}$, for instance if T is *finite*, because all filters are principal filters in this case.

2. While all labels of the form $\ell_{\geqq t}$ *have* to be in L because they correspond to principal $\mathfrak{D}$-fuzzy filters of $\mathfrak{T}$, this is not the case for labels of the form $\ell_{>t}$, which may be left out (in case there exist $\ell_{>t}$ which are not equal to $\ell_{\geqq t'}$ for some $t' \in T$). •

By definition, for every $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, and $t \in T \setminus \{1\}$, if $\ell_{>t} \in L$, then

$$\begin{aligned} \mathrm{Val} &\models_{\overline{1}} [x, \ell_{>t}] \text{ if } t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x) \text{ and } t \neq \mathrm{Val}(x) \\ \text{and } \mathrm{Val} &\models_{\overline{0}} [x, \ell_{>t}] \text{ if } t = \mathrm{Val}(x) \text{ or not } t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x) \end{aligned}$$

This means that the label $\ell_{>t}$ forces the formula x to be *more* true than the truth value t if it is to be considered valid at all.

Remarks

Assume $\left\{\ell_{>t} \,\middle|\, t \in T \setminus \{1\}\right\} \subseteqq L$.

1. Let $\mathcal{P}\mathrm{Frm} =_{\mathrm{def}} \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \operatorname{rg}\mathcal{X} \subseteqq \left\{\ell_{\geqq t} \,\middle|\, t \in T\right\} \cup \left\{\ell_{>t} \,\middle|\, t \in T \setminus \{1\}\right\}\right\}$. Define $f : \mathcal{P}\mathrm{Frm} \to (\mathfrak{P}(T))^{\mathrm{Frm}}$ as follows, for $\mathcal{X} \in \mathcal{P}\mathrm{Frm}$ and $x \in \mathrm{Frm}$:

$$f(\mathcal{X})(x) =_{\mathrm{def}} \mathrm{CUT}_1(\mathcal{X}(x)).$$

Lemma 5.5.1
*f is a **bijection** between $\mathcal{P}\mathrm{Frm}$ and $\mathrm{Fl}(\mathfrak{T})^{\mathrm{Frm}}$.*

Proof
Only $\mathrm{Fl}(\mathfrak{T}) = \bigcup\limits_{\mathcal{X}\in\mathcal{P}\mathrm{Frm}} \mathrm{rg}\, f(\mathcal{X})$ is proved; the rest is trivial.

For $\mathcal{X} \in \mathcal{P}\mathrm{Frm}$, by Proposition 2.1.1, $\mathrm{rg}\, f(\mathcal{X}) \subseteqq \mathrm{Fl}(\mathfrak{T})$.

It remains to establish that for every $F \in \mathrm{Fl}(\mathfrak{T})$, there exists $t \in T$ such that $F = \mathrm{CUT}_1\left(\ell_{\geqq t}\right)$ or $t \in T \setminus \{1\}$ such that $F = \mathrm{CUT}_1\left(\ell_{>t}\right)$.

Consider $\lfloor\mathrm{T}\rfloor F$. As $\mathfrak{T}$ is a chain, it holds that

$$F = \begin{cases} \left\{t \middle| t \in T \text{ and } \lfloor\mathrm{T}\rfloor F \sqsubseteq_{\mathfrak{T}} t\right\}, & \text{if } \lfloor\mathrm{T}\rfloor F \in F \\ \left\{t \middle| t \in T \text{ and } \lfloor\mathrm{T}\rfloor F \sqsubseteq_{\mathfrak{T}} t \text{ and } t \neq \lfloor\mathrm{T}\rfloor F\right\}, & \text{if } \lfloor\mathrm{T}\rfloor F \notin F \end{cases}$$

and in the latter case, obviously $\lfloor\mathrm{T}\rfloor F \neq 1$.

But

$$\begin{aligned} \left\{t \middle| t \in T \text{ and } \lfloor\mathrm{T}\rfloor F \sqsubseteq_{\mathfrak{T}} t\right\} &= \mathrm{CUT}_1\left(\ell_{\geqq\lfloor\mathrm{T}\rfloor F}\right) \\ \text{and } \left\{t \middle| t \in T \text{ and } \lfloor\mathrm{T}\rfloor F \sqsubseteq_{\mathfrak{T}} t \text{ and } t \neq \lfloor\mathrm{T}\rfloor F\right\} &= \mathrm{CUT}_1\left(\ell_{>\lfloor\mathrm{T}\rfloor F}\right), \end{aligned}$$

concluding the proof. □

Let $\models$ denote the model relation for the logic $\left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathrm{Fl}(\mathfrak{T}), \models\right]$ of graded truth assessment where *filters* are used as labels (compare section 5.2.2), i. e. for $\mathrm{Val} \in \mathfrak{S}$ and $\mathcal{X} \in \mathrm{Fl}(\mathfrak{T})^{\mathrm{Frm}}$,

$$\mathrm{Val} \models \mathcal{X} =_{\mathrm{def}} \forall x \in \mathrm{Frm} : \mathrm{Val}(x) \in \mathcal{X}(x).$$

Then for every $\mathrm{Val} \in \mathfrak{S}$ and $\mathcal{X} \in \mathcal{P}\mathrm{Frm}$,

$$\mathrm{Val} \models_{\mathrm{T}} \mathcal{X} \text{ iff } \mathrm{Val} \models f(\mathcal{X}),$$

i. e. logics of graded truth assessment with filter-labelled formulae can be embedded into the framework of fuzzy filter-based logic by appropriately restricting the range of admissible labels.

2. Note that if $\mathcal{X} \in \mathcal{P}\mathrm{Frm}$, then $\mathrm{Cons}(\mathcal{X}) \in \mathcal{P}\mathrm{Frm}$.

3. Let a mapping $\nu_{\mathfrak{D}} : D \to D$ and a mapping $\nu_{\mathfrak{T}} : T \to T$ be given as specified in (4.131),(4.132) and assume that Frm contains a unary operator symbol $\neg$ interpreted by $\nu_{\mathfrak{T}}$.

   Then for all $\mathcal{X} \in \mathcal{P}\mathrm{Frm}$ and $[x, \ell] \in \mathrm{Frm} \times \left\{\ell_{\geqq t} \middle| t \in T\right\} \cup \left\{\ell_{>t} \middle| t \in T \setminus \{1\}\right\}$, ℓ **admits refutation** and $\mathcal{X} \Vdash [x, \ell]$ is **characterised by refutation**.

   This result follows immediately from the two previous items by Corollary 5.2.10 and the fact that $\nu_{\mathfrak{D}}$ has to coincide with $\varphi_{\neg}$ defined in (3.9) on the set $\{0, 1\}$. •

Doubted to degree *d*. Analogously as for truth values, a class of labels representing **degrees of validity** is defined. For each $d \in D$, the label DOUBTED TO DEGREE d is denoted by ℓ_d. It is defined by

$$\ell_d(t) =_{\mathrm{def}} \begin{cases} 1, & \text{if } t = 1 \\ d, & \text{if } t \neq 1 \end{cases} \qquad (t \in T)$$

By definition, for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$,

$$\begin{aligned} \mathrm{Val} &\models_{1} [x, \ell_d] \text{ if } \mathrm{Val}(x) = 1 \\ \text{and } \mathrm{Val} &\models_{d} [x, \ell_d] \text{ if } \mathrm{Val}(x) \neq 1 \end{aligned}$$

This means that only "full truth" is assumed to lead to full validity. All other truth values, however, instead of considering them to lead to non-validity (as in classical many-valued logic), are given the 'benefit of the doubt', i. e. because the knowledge of x is not fully trusted, a certain degree of validity is assigned even if x is not completely true.

The weakest and strongest labels are obtained as special cases: $\ell^{\mathrm{U}} = \ell_1$ (doubted to degree 1) and $\ell^{\mathrm{AT}} = \ell_0$ (completely known).

Remark

It is easy to 'embed' *logics of graded truth assessment* into *logics of graded truth and graded trust assessment*. As demonstrated above, all that is necessary is to 'emulate' the two-valuedness of $\mathfrak{D}$ by appropriately restricting the set of admissible labels.

The situation is more difficult for *logics of graded trust assessment* because there is no straightforward way of 'emulating' the two-valuedness of $\mathfrak{T}$ just by choosing the right set of admissible labels. In particular, it is impossible to formulate a statement of equivalence like (5.124) if the valuations on both sides of the biimplication have to be taken from different semantics. •

Principal labels. The last class of 'simple' labels considered here has the most expressive power. They are called "primitive" or "principal" labels because they correspond to $\mathfrak{D}$-*fuzzy principal filters on* $\mathfrak{T}$. Each of these labels depends on a truth value t and a degree of validity d, is denoted by ℓ_d^t and defined by

$$\ell_d^t(t') =_{\mathrm{def}} \begin{cases} 1, & \text{if } t' = 1 \\ d, & \text{if } t' \neq 1 \text{ and } t \sqsubseteq_{\mathfrak{T}} t' \\ 0, & \text{if not } t \sqsubseteq_{\mathfrak{T}} t' \end{cases} \qquad (t' \in T)$$

(Compare (2.1) and (4.21).)

By definition, for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$,

$$\begin{aligned} \mathrm{Val} &\models_{1} \left[x, \ell_d^t\right] \text{ if } \mathrm{Val}(x) = 1 \\ \text{and } \mathrm{Val} &\models_{d} \left[x, \ell_d^t\right] \text{ if } \mathrm{Val}(x) \neq 1 \text{ and } t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x) \\ \text{and } \mathrm{Val} &\models_{0} \left[x, \ell_d^t\right] \text{ if not } t \sqsubseteq_{\mathfrak{T}} \mathrm{Val}(x) \end{aligned}$$

This means that only "full truth" is assumed to lead to full validity. The range of truth values above t is given the 'benefit of the doubt' and assigned the validity degree d. Truth values below t lead to total non-validity.

The labels from the classes $\ell_{\geqq t}$ and ℓ_d are obtained as special cases: $\ell_{\geqq t} = \ell_1^t$ and $\ell_d = \ell_d^0$.

Note that the class of principal labels forms a **base** of $\mathfrak{L}$, i. e. every label can be represented as an (infinitary) intersection[3] of principal labels (compare [39, Corollary 4.2]).

Furthermore, $\ell_d^t = \ell_d \sqcup \ell_{\geqq t}$, so $\left\{\ell_{\geqq t} \middle| t \in T\right\} \cup \{\ell_d \,|\, d \in D\}$ forms a **subbase** of $\mathfrak{L}$.

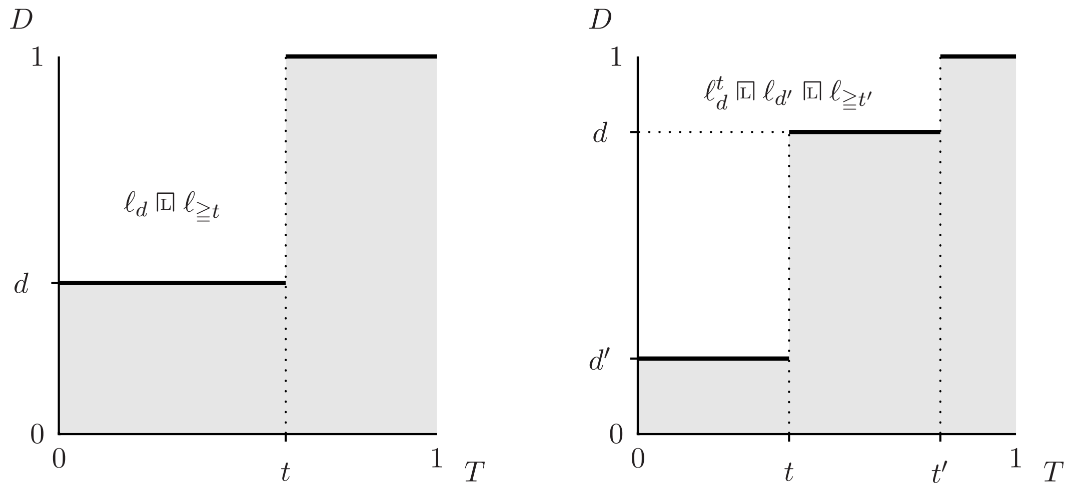


Figure 5.3: Composition of simple labels.

Composite Labels. Definition 2.3.1.1 guarantees that every *principal label* is contained in $\mathfrak{L}$, which extends immediately to those classes of labels from Figure 5.2 which are special cases of principal labels, i. e. $\ell^{\mathrm{U}}, \ell^{\mathrm{AT}}, \ell_{\geqq t}, \ell_d$. The only class which is not guaranteed to be covered by $\mathfrak{L}$ is $\left\{\ell_{>t} \middle| t \in T \setminus \{1\}\right\}$ (see above for a discussion).

In fact, Definition 2.3.1.1 guarantees the existence of a lot more labels, namely all labels corresponding to elements of $\mathscr{P}(\mathfrak{D}, \mathfrak{T})$. By Proposition 2.2.15, these are all labels stemming from *finitely many* superpositions of union and intersection of **principal labels**. Two examples of such **composite labels** are illustrated in Figure 5.3.

5.5.1.2 Labels Based on True

The considerations in this subsection require $\mathfrak{T} = \mathfrak{D}$. This makes it possible to meaningfully use the *identity mapping* id from T into D as a label (of course, this would also be possible if $\mathfrak{T} \Subset \mathfrak{D}$). Some labels of this type are sketched in Figure 5.4. Their meaning is discussed in the following.

True. The label denoted by ℓ^{T} stands for an exact correspondence between a formula's truth value and validity degree. It is defined to be the *identity mapping* from T into D, i. e.

$$\ell^{\mathrm{T}}(t) =_{\mathrm{def}} t. \qquad (t \in T)$$

By definition, for every $\mathrm{Val} \in \mathfrak{S}$ and $x \in \mathrm{Frm}$,

$$\mathrm{Val} \models_{\overline{\mathrm{Val}(x)}} \left[x, \ell^{\mathrm{T}}\right].$$

This means that the label True does not really represent an *assessment* of the validity of a formula, depending on its truth value. It simply translates the truth value directly into the validity degree.

This type of label appears in the works of L. A. Zadeh (see [105], for instance) under the name "formulae of Type IV" (*truth qualifications*). See also J. F. Baldwin's *truth value restrictions* [2].

[3] Remember the intersection of labels corresponds to the union of fuzzy filters.

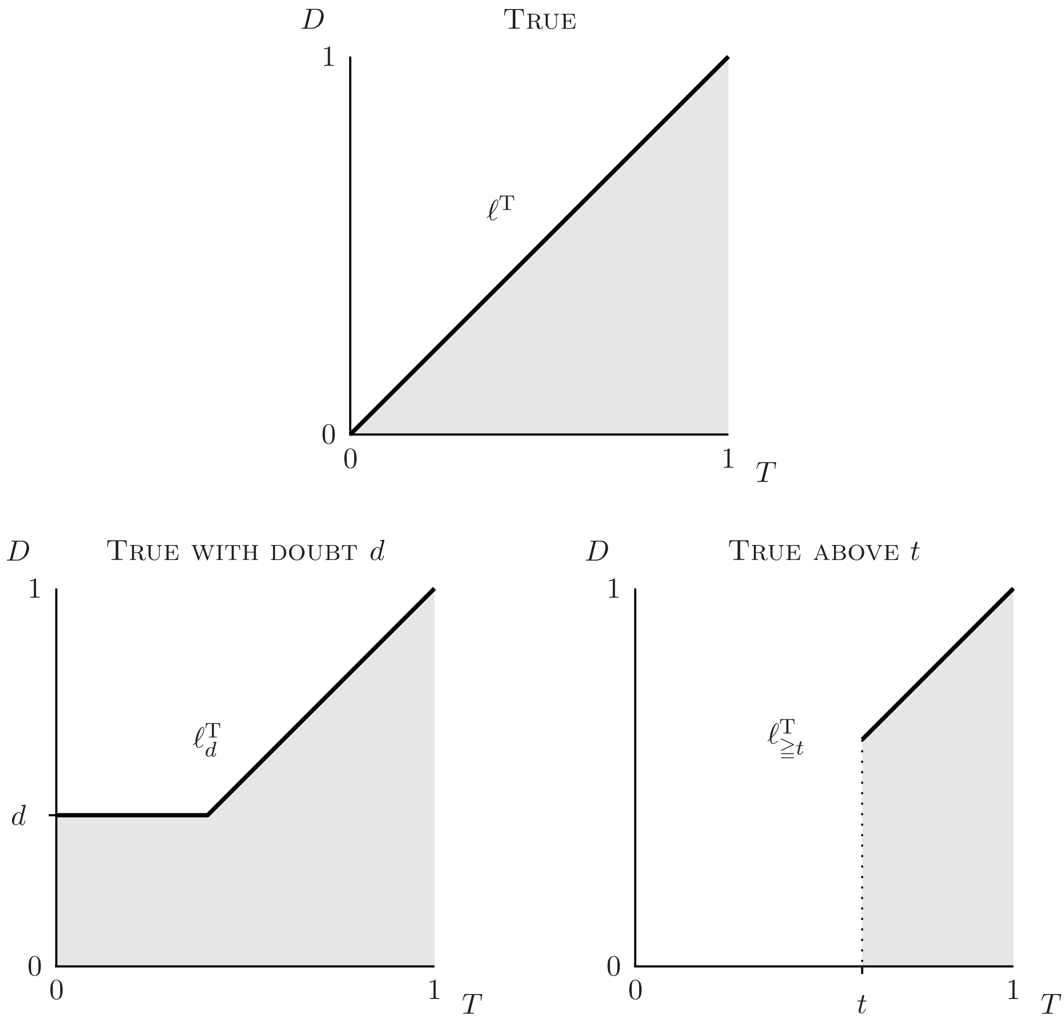


Figure 5.4: Labels requiring $\mathfrak{T} = \mathfrak{D}$.

Remark
Assume that $\operatorname{rg}\mathcal{X} \subseteqq \{\ell^{\mathrm{U}}, \ell^{\mathrm{T}}\}$. Then

$$\mathcal{X} \Vdash \left[x, \ell^{\mathrm{T}}\right] \text{ iff for all Val} \in \mathfrak{S} \text{ and all } y \in \mathrm{Frm} \text{ with } \mathcal{X}(y) = \ell^{\mathrm{T}},\ \mathrm{Val}(y) \underline{\sqsubseteq}_{\mathfrak{T}} \mathrm{Val}(x).$$

This kind of entailment corresponds to level $1'$ in the development of *graded trust assessment* mentioned in section 3.4.2. It is interesting because it abstracts completely from the idea from classical many-valued logic to have a set of *designated truth values* inducing the model relation. It has been studied for instance by R. C. T. Lee and C. L. Chang [64]. •

True with doubt d. Another class of labels representing **degrees of validity** is defined as follows. For each $d \in D$, the label TRUE WITH DOUBT d is denoted by ℓ^{T}_d and derived by taking the intersection of ℓ^{T} and ℓ_d.

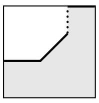
A labelled formula $\left[x, \ell^{\mathrm{T}}_d\right]$ means that in general, the validity degree should be derived immediately from the truth value of x, but the information represented by x is trusted only to a certain degree d, so the validity degree of the labelled formula should not be allowed to drop below d (thus giving x the benefit of the doubt in case x is indeed a misrepresentation of the actual facts).

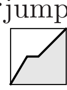
For embedding *logics of graded trust assessment* into *logics of graded truth and graded trust assessment* where the underlying logic is many-valued instead of two-valued (compare section 5.3), labels of type ℓ_d^{T} would probably be the best choice (instead of ℓ_d) for representing "necessity labels" (see [18]). See also section 5.7.1 where it is made plausible that labelled formulae of *possibilistic logic with vague predicates* can be embedded into a logic of graded truth and graded trust assessment by restricting labels exactly to the class of all labels ℓ_d^{T}.

True above *t*. Another class of labels representing **truth values** is defined as follows. For each $t \in T$, the label TRUE ABOVE t is denoted by $\ell_{\geqq t}^{\mathrm{T}}$ and derived by taking the union of ℓ^{T} and $\ell_{\geqq t}$.

A labelled formula $\left[x, \ell_{\geqq t}^{\mathrm{T}}\right]$ means that in general, the validity degree should be derived immediately from the truth value of x, but the truth value of x is not allowed to drop below a certain level for the labelled formula to be assigned any validity at all.

Piecewise linear labels. As a matter of fact, almost every combination of ℓ^{T} with different instances of ℓ_d^t has some useful interpretation. When designing a knowledge representation system using the type of logics presented in this dissertation (augmented by automated deduction in the form of a logic programming language), it would possibly be best to attach to every formula a pictorial representation of its label, as it is very easy to intuitively grasp the logical meaning of such a picture.

 for instance could denote a label derived from TRUE, allowing a certain degree of doubt, and for a situation where it is not important to reach the highest possible level of truth, so full validity is assigned even before full truth is reached.

From a computational complexity point of view, it would probably be best to allow all piecewise linear labels whose points of non-differentiability (i. e. where the function 'jumps' or changes gradient) are taken from a fixed, finite set. This would also allow labels like .

5.5.1.3 Label Languages

In [105], L. A. ZADEH describes a *language* with which labels could be expressed. This means starting from some simple labels (like ℓ_d^t and ℓ^{T}), combining these with operators like *and* and *or* (interpreted, for instance, by meet and join in $\mathfrak{L}$), and providing *modifiers* to adjust the meaning of a label to a certain context (see also [83]). For giving some examples of such modified labels, assume that $\mathfrak{T}$ and $\mathfrak{D}$ are both interpreted by $\left[\langle 0,1\rangle, \min, \max\right]$, hence a label is a monotone function on the unit interval which reaches 1. This means functions like *square* or *square root* can be used for defining modifiers.

Some *composite labels* which are derived by taking the union or intersection of other labels have already been presented above. In Figure 5.5 some modifications of the label ℓ^{T} using the modifiers VERY (interpreted by taking the *square*) and MORE OR LESS (interpreted by taking the *square root*) are sketched. These modifiers can be iterated, yielding for instance VERY VERY TRUE.

It is obvious that VERY *strengthens* a label (by making it smaller) while MORE OR LESS *weakens* a label (by making it larger).

These modified labels can then again be combined with labels from the family ℓ_d^t, for instance.

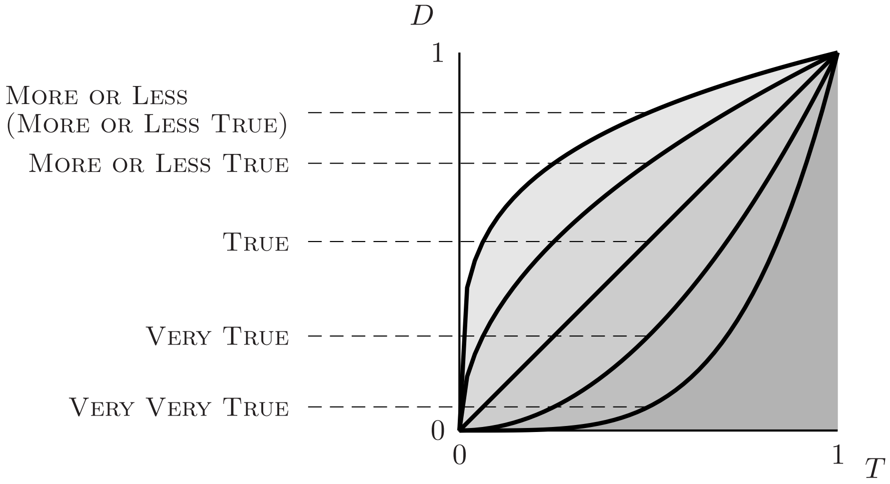


Figure 5.5: Modifications of True.

5.5.2 Examples of Inferences

In this section, a small example is given for knowledge modelling with fuzzy filter-based labelled logics, demonstrating in particular the notion of *semantic consequence* from a particular $\mathfrak{L}$-fuzzy set of formulae.

The example given here necessarily has to be a toy example, because without first establishing a correct and complete *syntactic derivation system* which characterises the semantic consequence operator, calculating semantic consequences is a tedious task and not feasible for realistic-sized knowledge bases.

The development of fuzzy filter-based logics has been carried out with *resolution-based automated reasoning* in mind, and the labelled formulae employed here are especially well suited for defining a *layered normal form* on which a resolution-based automated reasoning system can be based (compare [72] and section 6.2.1.2). But the full development of an automated reasoning system is not carried out in this dissertation, so the discussion of 'realistic' examples has to be left for future publications.

Small Natural Numbers

The idea to model the concept **small natural number** using axioms is based on an ancient logical paradox called *sorites paradox* (bald man). Classically, it is formulated like this:

> *A person with only a very few hairs on their head can be called "bald". If a single hair is added to the head of a bald person, they are still bald.*
>
> *Consequently (by mathematical induction), every person is bald.*

A simple way of 'dissolving' this paradox is to use a many-valued logic and define a many-valued concept `baldness` as above, making sure that the implication is not 'completely true'. This reflects the fact that adding a single hair to the head of a bald person indeed makes them a little 'less bald'. That the 'degree of baldness' decreases while the number of hairs increases then follows naturally from the number of times the (not completely true) implication has to be applied.

Using a *logic of graded truth* for this modelling task has the significant advantage that the relaxation of the necessary degree of truth for the implication can be expressed immediately by the label, while in a non-labelled logic, truth constants and a *residuated implication* (or even more complex constructs) have to be employed.

An arithmetic counterpart to this paradox is the axiomatic definition of a **small natural number**. Obviously, the number 1 is *small* to the highest degree, and if 1 is added to a small number, then it will stay at least somewhat small. But every natural number can be reached by successively adding up 1, starting at 1, so the 'degree of smallness' should decrease when adding 1 to a small number.

In the following, an axiomatic characterisation for the concept **small natural number** is given, using *fuzzy filter-based first order predicate logic*. Different variants for modelling this concept are given and their influence on the results is discussed.

For this characterisation, let $\mathfrak{T} = \mathfrak{D} = \mathfrak{F}$ (compare Example 1.3.1.2), so truth values as well as degrees of validity are taken from the *real unit interval*, the most commonly used structure for graded truth and graded trust. For convenience, assume that $\mathfrak{L} = F\mathscr{F}(\mathfrak{F})$, the full lattice of all $\mathfrak{F}$-fuzzy filters of $\mathfrak{F}$ as discussed in Example 2.3.1.3 (compare also Observation 2.3.5). Observe that $F\mathscr{F}(\mathfrak{F})$ consists of all monotone unary functions on $\mathfrak{F}$ mapping 1 to 1, ordered by the fuzzy superset relation.

Assume the *standard syntax of first order predicate logic*, i. e.

$$\mathrm{Frm} = \mathrm{FOFrms_S} = \mathrm{FOFrm}\,(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega_{\mathrm{S}}, \mathrm{Ar}_{\mathrm{S}})$$

as defined in Example 3.1.3, for a given (non-empty) set IV of individual variables, $\mathrm{Func} =_{\mathrm{def}} \{\mathtt{1}, \mathtt{inc}\}$ with $\mathrm{Ar}_{\mathrm{Func}}(\mathtt{1}) = 0, \mathrm{Ar}_{\mathrm{Func}}(\mathtt{inc}) = 1$ and $\mathrm{Pred} =_{\mathrm{def}} \{\mathtt{small}\}$ with $\mathrm{Ar}_{\mathrm{Pred}}(\mathtt{small}) = 1$.

The interpretation of the logical operator symbols $\neg, \wedge, \vee$ from Ω_{S} is defined as in Example 3.2.4.2, i. e. for $s, t \in \langle 0, 1\rangle$:

$$\begin{aligned}\varphi_{\neg}(t) &= 1 - t,\\ \varphi_{\wedge}(s,t) &= \min(s,t),\\ \varphi_{\vee}(s,t) &= \max(s,t).\end{aligned}$$

An interpretation of $\rightarrow$ is not fixed at this point; several variants are discussed in the following.

The semantics employed is (of course) the one of *first order predicate logic* as defined in Example 3.2.2. For the extent of this example, let the *domain* of each interpretation be fixed to be the set $\mathbb{N}$ of all natural numbers (compare remark 2 on page 46 concerning restricted interpretations), let the interpretation of $\mathtt{1}$ be fixed to be the natural number 1 and let the interpretation of $\mathtt{inc}$ be fixed to be the *successor function* of the natural numbers, i. e. for any admissible interpretation $[\mathbb{N}, \Pi, \Phi]$ and $n \in \mathbb{N}$, assume that

$$\Phi_{\mathtt{inc}}(n) = n + 1.$$

This leaves only the interpretation of $\mathtt{small}$ open, which will be characterised axiomatically.

Summarising, for a given (non-empty) set IV and a given function $\varphi_{\rightarrow} : \langle 0,1\rangle^2 \rightarrow \langle 0,1\rangle$, the **semantics** $\mathfrak{S} \subsetneqq T^{\mathrm{FOFrms}}$ considered in this subsection is defined (wrt. Ω_{S}, Ar_{S}, Func, $\mathrm{Ar}_{\mathrm{Func}}$, Pred, $\mathrm{Ar}_{\mathrm{Pred}}$ as mentioned above, see Example 3.1.2) by (3.6), taking into consideration the restrictions on the set of admissible interpretations described above.

Furthermore, assume for convenience that the isomorphism ι employed in the definition of the model relation $\models\!\!=$ (see (4.1)) is *identity*, i. e. for $\mathrm{Val} \in \mathfrak{S}$, $[x, \ell] \in \mathrm{LFrm}$ and $d \in \langle 0, 1\rangle$,

$$\mathrm{Val} \mathrel{\models\!\!=_{d}} [x, \ell] \;\text{ iff }\; d = \ell\left(\mathrm{Val}(x)\right). \tag{5.125}$$

In the following, iterations of $\mathtt{inc}$ will be used frequently. For a more convenient notation, the following abbreviations for terms are introduced:

$$\mathtt{2} =_{\mathrm{def}} \mathtt{inc}(\mathtt{1})$$

$$\begin{aligned}
\mathtt{3} &=_{\text{def}} \mathtt{inc}\left(\mathtt{inc}(\mathtt{1})\right)\\
\mathtt{4} &=_{\text{def}} \mathtt{inc}\Big(\mathtt{inc}\left(\mathtt{inc}(\mathtt{1})\right)\Big)\\
\mathtt{5} &=_{\text{def}} \mathtt{inc}\bigg(\mathtt{inc}\Big(\mathtt{inc}\left(\mathtt{inc}(\mathtt{1})\right)\Big)\bigg)
\end{aligned}$$

Next, several variants of the axiomatic characterisation of the predicate `small` in fuzzy filter-based logic are discussed.

Variant 1. Let $\varphi_{\rightarrow} =_{\text{def}} \mathrm{imp}_{\mathrm{G}}$, i.e. the implication connective is interpreted by the *r-implication* of min (compare (3.12)).

For the characterisation of `small`, the $\mathfrak{L}$-fuzzy set $\mathcal{X}_1$ of formulae is used, where for some $v \in \mathrm{IV}$,

$$\begin{aligned}
\mathcal{X}_1\left(\mathtt{small}(\mathtt{1})\right) &=_{\text{def}} \ell^{\mathrm{AT}},\\
\mathcal{X}_1\bigg(\forall v\Big(\mathtt{small}(v) \rightarrow \mathtt{small}\left(\mathtt{inc}(v)\right)\Big)\bigg) &=_{\text{def}} \ell_{\geqq 0.9}
\end{aligned}$$

and $\mathcal{X}_1(x) =_{\text{def}} 0$ for all $x \in \mathrm{Frm} \setminus \Big\{\mathtt{small}(\mathtt{1}), \forall v\Big(\mathtt{small}(v) \rightarrow \mathtt{small}\left(\mathtt{inc}(v)\right)\Big)\Big\}$.

This definition means that $\mathtt{small}(\mathtt{1})$ is forced by $\mathcal{X}_1$ to be *absolutely true* while it is sufficient for the implication $\forall v\Big(\mathtt{small}(v) \rightarrow \mathtt{small}\left(\mathtt{inc}(v)\right)\Big)$ to be true at least to degree 0.9. It is to be expected that if $\mathtt{small}(v)$ is *true* to some degree, then this will be carried over to $\mathtt{small}\left(\mathtt{inc}(v)\right)$ in some sense, though the truth value that $\mathtt{small}\left(\mathtt{inc}(v)\right)$ necessarily has might be smaller than that of $\mathtt{small}(v)$ if the implication is not completely true.

Note further that both labels employed are *two-valued*, assuming only the validity degrees 0 and 1 (compare section 5.5.1.1). Hence, any valuation from $\mathfrak{S}$ can either be a model for $\mathcal{X}_1$ to the highest degree or to the degree 0. So this variant of modelling the concept small natural number is done entirely by *graded truth assessment*.

Now, what does $\mathrm{Cons}(\mathcal{X}_1)$ look like? As explained above, it would be extremely tedious to even sketch the full extent of $\mathrm{Cons}(\mathcal{X}_1)$ without a syntactic proof system at hand. Hence, only few examples are discussed in the following.

Recall the definition of Cons in Definition 4.3.1, for $x \in \mathrm{Frm}$:

$$\begin{aligned}
&\mathrm{Cons}(\mathcal{X}_1)(x)\\
&\quad= \bigsqcup_{\mathrm{L}}\left\{\ell \,\middle|\, \ell \in L \text{ and } \mathcal{X}_1 \Vdash [x,\ell]\right\} && \text{(by (4.98))}\\
&\quad= \bigsqcup_{\mathrm{L}}\Big\{\ell \,\Big|\, \ell \in L \text{ and } \mathrm{Mod}(\mathcal{X}_1) \subseteqq \mathrm{Mod}\left([x,\ell]\right)\Big\} && \text{(by (4.38), (4.41), and (5.125))} \qquad (5.126)\\
&\quad= \bigsqcup_{\mathrm{L}}\left\{\ell \,\middle|\, \begin{array}{l}\ell \in L \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S}:\\ \min\left(\begin{array}{l}\ell^{\mathrm{AT}}\Big(\mathrm{Val}\left(\mathtt{small}(\mathtt{1})\right)\Big),\\ \ell_{\geqq 0.9}\bigg(\mathrm{Val}\bigg(\forall v\Big(\mathtt{small}(v) \rightarrow \mathtt{small}\left(\mathtt{inc}(v)\right)\Big)\bigg)\bigg)\end{array}\right)\\ \qquad\qquad\qquad\qquad \leqq \ell\left(\mathrm{Val}(x)\right)\end{array}\right\}
\end{aligned}$$

and, because ℓ^{AT} and $\ell_{\geqq 0.9}$ only take the values 0 and 1,

$$= \boxed{\mathrm{L}} \left\{ \ell \;\middle|\; \begin{array}{l} \ell \in L \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S} : \\ \text{if } \mathrm{Val}\left(\mathtt{small(1)}\right) = 1 \text{ and } \mathrm{Val}\left(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\right) \geqq 0.9, \\ \hfill \text{then } \ell\left(\mathrm{Val}\left(x\right)\right) = 1 \end{array} \right\}$$

(5.127)

$$= \boxed{\mathrm{L}} \left\{ \ell \;\middle|\; \begin{array}{l} \ell \in L \text{ and for all interpretations } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi] : \\ \text{if } \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \varphi_{\to}\left(\Pi_{\mathtt{small}}(n), \Pi_{\mathtt{small}}\left(n+1\right)\right) \geqq 0.9, \\ \hfill \text{then } \ell\left(\mathrm{Val}_{\mathfrak{I}}\left(x\right)\right) = 1 \end{array} \right\}.$$

It is a simple observation that in all cases,

$$\mathrm{Cons}(\mathcal{X}_1)(x) \in \left\{\ell_{\geqq t} \middle| t \in \langle 0, 1\rangle\right\} \cup \left\{\ell_{>t} \middle| t \in \langle 0, 1\rangle \setminus \{1\}\right\},$$

because in the above construction, ℓ is only restricted by the requirement of being equal to 1 in some places. Hence, by analysing (5.127), it holds that

(5.128)

$$\mathrm{Cons}(\mathcal{X}_1)(x) = \begin{cases} \ell_{\geqq t}, & \text{if } t \in \langle 0, 1\rangle \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}\left(x\right) \geqq t \text{ if and only if} \\ & \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \varphi_{\to}\left(\Pi_{\mathtt{small}}(n), \Pi_{\mathtt{small}}\left(n+1\right)\right) \geqq 0.9 \\ \ell_{>t}, & \text{if } t \in \langle 0, 1\rangle \setminus \{1\} \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}\left(x\right) > t \text{ if and only if} \\ & \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \varphi_{\to}\left(\Pi_{\mathtt{small}}(n), \Pi_{\mathtt{small}}\left(n+1\right)\right) \geqq 0.9 \end{cases}$$

This is the most expanded form which can be reached without specifying $\varphi_{\to}$ or x. In this variant, $\varphi_{\to} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{G}}$ has been chosen, given for $s, t \in \langle 0, 1\rangle$ by

(5.129)
$$\mathrm{imp}_{\mathrm{G}}(s, t) = \begin{cases} 1, & \text{if } s \leqq t \\ t, & \text{if } s > t \end{cases}$$

Obviously, $\mathrm{imp}_{\mathrm{G}}(s, t) \geqq 0.9$ holds trivially if the first case occurs. Hence, from (5.128) it is obtained that

$$\mathrm{Cons}(\mathcal{X}_1)(x) = \begin{cases} \ell_{\geqq t}, & \text{if } t \in \langle 0, 1\rangle \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}\left(x\right) \geqq t \text{ if and only if } \Pi_{\mathtt{small}}(1) = 1 \text{ and} \\ & \forall n \in \mathbb{N} : \Pi_{\mathtt{small}}(n) \leqq \Pi_{\mathtt{small}}\left(n+1\right) \text{ or } \Pi_{\mathtt{small}}\left(n+1\right) \geqq 0.9 \\ \ell_{>t}, & \text{if } t \in \langle 0, 1\rangle \setminus \{1\} \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}\left(x\right) > t \text{ if and only if } \Pi_{\mathtt{small}}(1) = 1 \text{ and} \\ & \forall n \in \mathbb{N} : \Pi_{\mathtt{small}}(n) \leqq \Pi_{\mathtt{small}}\left(n+1\right) \text{ or } \Pi_{\mathtt{small}}\left(n+1\right) \geqq 0.9 \end{cases}$$

As $\Pi_{\mathtt{small}}(1) = 1$ in all interpretations under consideration, it is clear that for all $n \in \mathbb{N}$, $\Pi_{\mathtt{small}}\left(n+1\right) \geqq 0.9$ is the weaker constraint than $\Pi_{\mathtt{small}}(n) \leqq \Pi_{\mathtt{small}}\left(n+1\right)$, hence the equation reduces to

$$\mathrm{Cons}(\mathcal{X}_1)(x) = \begin{cases} \ell_{\geqq t}, & \text{if } t \in \langle 0,1\rangle \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}(x) \geqq t \text{ if and only if} \\ & \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \Pi_{\mathtt{small}}(n+1) \geqq 0.9 \\ \ell_{>t}, & \text{if } t \in \langle 0,1\rangle \setminus \{1\} \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}(x) > t \text{ if and only if} \\ & \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \Pi_{\mathtt{small}}(n+1) \geqq 0.9 \end{cases} \tag{5.130}$$

From this, it follows immediately that

$$\begin{aligned} \mathrm{Cons}(\mathcal{X}_1)\big(\mathtt{small}\,(1)\big) &= \ell^{\mathrm{AT}} \\ \mathrm{Cons}(\mathcal{X}_1)\big(\mathtt{small}\,(2)\big) &= \ell_{\geqq 0.9} \\ \mathrm{Cons}(\mathcal{X}_1)\big(\mathtt{small}\,(3)\big) &= \ell_{\geqq 0.9} \\ \mathrm{Cons}(\mathcal{X}_1)\big(\mathtt{small}\,(4)\big) &= \ell_{\geqq 0.9} \\ &\vdots \end{aligned}$$

This means that when $\varphi_{\rightarrow}$ is interpreted by $\mathrm{imp}_{\mathrm{G}}$, then $\mathcal{X}_1$ does not give a good characterisation of the concept of **small number** because every natural number, however high, will be considered *small* with at least the truth value 0.9.

Variant 2. Let $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{KD}}$, i.e. the implication connective is interpreted by the *s-implication* of max and $1-x$ (compare (3.11)). It is given for $s, t \in \langle 0,1\rangle$ by

$$\mathrm{imp}_{\mathrm{KD}}(s,t) = \max(1-s,t). \tag{5.131}$$

For the characterisation of `small`, the same $\mathfrak{L}$-fuzzy set $\mathcal{X}_1$ as in the previous variant is used. This means that the derivation of $\mathrm{Cons}(\mathcal{X}_1)$ is identical to that of the previous variant until equation (5.128). Applying the definition of $\varphi_{\rightarrow}$ in this case, it follows that

$$\mathrm{Cons}(\mathcal{X}_1)(x) = \begin{cases} \ell_{\geqq t}, & \text{if } t \in \langle 0,1\rangle \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}(x) \geqq t \text{ if and only if} \\ & \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \max\big(1-\Pi_{\mathtt{small}}(n), \Pi_{\mathtt{small}}(n+1)\big) \geqq 0.9 \\ \ell_{>t}, & \text{if } t \in \langle 0,1\rangle \setminus \{1\} \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \mathrm{Val}_{\mathfrak{I}}(x) > t \text{ if and only if} \\ & \Pi_{\mathtt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \max\big(1-\Pi_{\mathtt{small}}(n), \Pi_{\mathtt{small}}(n+1)\big) \geqq 0.9 \end{cases}$$

By a simple inductive argument starting with $\Pi_{\mathtt{small}}(1) = 1$, it is observed that for all $n \in \mathbb{N}$, $1 - \Pi_{\mathtt{small}}(n) < 0.9$, hence (5.130) holds in this case also, leading to the same result as in the previous variant.

This leads to the conclusion that $\mathrm{imp}_{\mathrm{KD}}$ also does not yield a good characterisation of the concept of **small number** in combination with $\mathcal{X}_1$.

Variant 3. For this variant, a new implication function is employed, which can not be defined by means of the lattice connectives in the way $\mathrm{imp}_{\mathrm{G}}$ and $\mathrm{imp}_{\mathrm{KD}}$ can.

Łukasiewicz's implication $\mathrm{imp}_{\mathrm{Ł}}$ is given for $s, t \in \langle 0,1\rangle$ by

$$\mathrm{imp}_{\mathrm{Ł}}(s,t) = \min(1, 1-s+t). \tag{5.132}$$

It is a connective of very high expressive power, equipping the truth value lattice with an **MV algebra** structure. Let $\varphi_{\rightarrow} =_{\text{def}} \text{imp}_{\text{L}}$. Again, the $\mathfrak{L}$-fuzzy set $\mathcal{X}_1$ is used.

This means that the derivation of $\text{Cons}(\mathcal{X}_1)$ is again identical to that of variant 1 until equation (5.128). Applying the definition of $\varphi_{\rightarrow}$ in this case, it follows that

$$\text{Cons}(\mathcal{X}_1)(x) = \begin{cases} \ell_{\geqq t}, & \text{if } t \in \langle 0,1\rangle \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \text{Val}_{\mathfrak{I}}(x) \geqq t \text{ if and only if} \\ & \Pi_{\texttt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \min\big(1, 1 - \Pi_{\texttt{small}}(n) + \Pi_{\texttt{small}}(n+1)\big) \geqq 0.9 \\ \ell_{>t}, & \text{if } t \in \langle 0,1\rangle \setminus \{1\} \text{ and for every interpretation } \mathfrak{I} = [\mathbb{N}, \Pi, \Phi], \\ & \text{Val}_{\mathfrak{I}}(x) > t \text{ if and only if} \\ & \Pi_{\texttt{small}}(1) = 1 \text{ and } \forall n \in \mathbb{N} : \min\big(1, 1 - \Pi_{\texttt{small}}(n) + \Pi_{\texttt{small}}(n+1)\big) \geqq 0.9 \end{cases}$$

A simple induction on the value of $\Pi_{\texttt{small}}(n)$ for $n \in \mathbb{N}$, starting with $\Pi_{\texttt{small}}(1) = 1$, yields

$$\begin{aligned} \text{Cons}(\mathcal{X}_1)\big(\texttt{small(1)}\big) &= \ell^{\text{AT}} \\ \text{Cons}(\mathcal{X}_1)\big(\texttt{small(2)}\big) &= \ell_{\geqq 0.9} \\ \text{Cons}(\mathcal{X}_1)\big(\texttt{small(3)}\big) &= \ell_{\geqq 0.8} \\ \text{Cons}(\mathcal{X}_1)\big(\texttt{small(4)}\big) &= \ell_{\geqq 0.7} \\ \text{Cons}(\mathcal{X}_1)\big(\texttt{small(5)}\big) &= \ell_{\geqq 0.6} \\ &\vdots \end{aligned}$$

It is clear that $\mathcal{X}_1$ together with the interpretation of $\varphi_{\rightarrow}$ by imp_{L} leads to a proper modelling of the concept of **small number**. As the numbers increase, the truth value required of `small` for them decreases, until it vanishes to 0 for the number 11. There are other implications which also lead to adequate results, for instance **Goguen's implication**, but studying them and their differences would lead to a study of truth value structures, which is not intended in this dissertation.

Variant 4. Let $\varphi_{\rightarrow} =_{\text{def}} \text{imp}_{\text{G}}$ as in variant 1.

For the characterisation of `small`, this time a 'specially designed' label is employed. Let the label $\ell_{\gtrapprox 0.9}$ be given as sketched in Figure 5.6. Looking at Figure 5.2, this label is comparable with $\ell_{\geqq 0.9}$. It postulates that the truth value of a labelled formula must be above 0.9 for achieving full validity. But if this constraint is not met, then validity does not drop to 0 immediately, but goes down gradually, expressing some uncertainty about the place where the correct boundary should be.

The fuzzy set from which inferences are drawn is the $\mathfrak{L}$-fuzzy set $\mathcal{X}_2$ of formulae, where for some $v \in \text{IV}$,

$$\begin{aligned} \mathcal{X}_2\big(\texttt{small(1)}\big) &=_{\text{def}} \ell^{\text{AT}}, \\ \mathcal{X}_2\Big(\forall v\Big(\texttt{small}(v) \rightarrow \texttt{small}\big(\texttt{inc}(v)\big)\Big)\Big) &=_{\text{def}} \ell_{\gtrapprox 0.9} \end{aligned} \tag{5.133}$$

and $\mathcal{X}_2(x) =_{\text{def}} 0$ for all $x \in \text{Frm} \setminus \Big\{\texttt{small(1)}, \forall v\Big(\texttt{small}(v) \rightarrow \texttt{small}\big(\texttt{inc}(v)\big)\Big)\Big\}$.

This definition means that `small(1)` is forced by $\mathcal{X}_2$ to be *absolutely true* while there is some uncertainty about the truth of the implication $\forall v\Big(\texttt{small}(v) \rightarrow \texttt{small}\big(\texttt{inc}(v)\big)\Big)$. As in the preceding three variants, it is completely sufficient for the implication to be true at least to degree 0.9. But even if this constraint is not met, some benefit of the doubt is given because

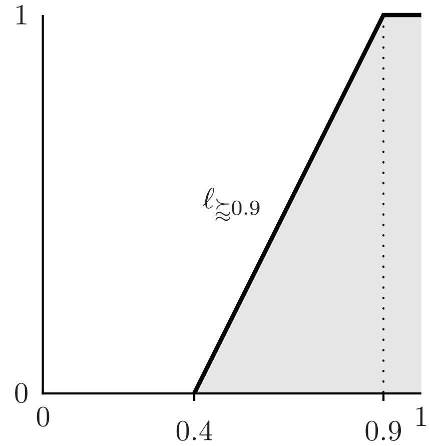


Figure 5.6: Special label Approximately truer than 0.9.

it is not known exactly whether a sharp restriction to be above 0.9 adequately represents the knowledge that with a certain doubt, if a number is small, then its successor is also small.

Again, the result of $\mathrm{Cons}(\mathcal{X}_2)$ is evaluated. The expansion given in variant 1 can be carried out analogously until equation (5.126), so the considerations on this variant can start at

$$\mathrm{Cons}(\mathcal{X}_2)(x) = \boxed{\mathrm{L}}\left\{\ell \;\middle|\; \begin{array}{l} \ell \in L \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S}: \\ \min\left(\begin{array}{l}\ell^{\mathrm{AT}}\Big(\mathrm{Val}\left(\mathtt{small(1)}\right)\Big), \\ \ell_{\gtrapprox 0.9}\bigg(\mathrm{Val}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big)\bigg)\end{array}\right) \leqq \ell\left(\mathrm{Val}\,(x)\right)\end{array}\right\}$$

$$= \boxed{\mathrm{L}}\left\{\ell \;\middle|\; \begin{array}{l}\ell \in L \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S}: \text{if } \mathrm{Val}\left(\mathtt{small(1)}\right) = 1, \\ \text{then } \ell_{\gtrapprox 0.9}\bigg(\mathrm{Val}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big)\bigg) \leqq \ell\left(\mathrm{Val}\,(x)\right)\end{array}\right\} \tag{5.134}$$

Now, assume that for every $t \in \langle 0, 1\rangle$, there exists $\mathrm{Val}_{\mathcal{X}_2,x,t} \in \mathfrak{S}$ such that

(i) $\mathrm{Val}_{\mathcal{X}_2,x,t}(x) = t$ and

(ii) $\mathrm{Val}_{\mathcal{X}_2,x,t}\left(\mathtt{small(1)}\right) = 1$ and

(iii) for all $\mathrm{Val} \in \mathfrak{S}$ with $\mathrm{Val}(x) = t$ and $\mathrm{Val}\left(\mathtt{small(1)}\right) = 1$, it holds that

$$\mathrm{Val}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big) \leqq \mathrm{Val}_{\mathcal{X}_2,x,t}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big)$$

and

(iv) if $t, t' \in \langle 0, 1\rangle$ such that $t \leqq t'$, then

$$\mathrm{Val}_{\mathcal{X}_2,x,t}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big) \leqq \mathrm{Val}_{\mathcal{X}_2,x,t'}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big).$$

It is easily observed that in this case, (5.134) reduces to the following equation, for every $t \in \langle 0, 1\rangle$:

$$\mathrm{Cons}(\mathcal{X}_2)(x)(t) = \ell_{\gtrapprox 0.9}\bigg(\mathrm{Val}_{\mathcal{X}_2,x,t}\Big(\forall v\left(\mathtt{small}(v) \to \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\Big)\bigg). \tag{5.135}$$

Next, some instances of x are considered.

1. $x = \texttt{small}(1)$.

   For this formula, items (i) and (ii) are contradictory, so for $t \neq 1$, no valuation $\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(1),t} \in \mathfrak{S}$ fulfilling (i)–(iv) exists. But from the fact that always, $\mathrm{Val}(x) = 1$ in (5.134), it is obvious that the set in (5.134) is L, the least upper bound of which is ℓ^{AT}, yielding

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}(1)\right) = \ell^{\mathrm{AT}}.$$

2. $x = \texttt{small}\,(2)$.

   It is easily established that for every $t \in \langle 0, 1\rangle$ there exists $\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t} \in \mathfrak{S}$ such that

$$\begin{aligned}
\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}\left(\texttt{small}(1)\right) &= 1\\
\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}\left(\texttt{small}\,(2)\right) &= t\\
\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}\left(\texttt{small}\,(3)\right) &= t\\
&\vdots
\end{aligned}$$

   Proving that $\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}$ as defined here indeed fulfils all conditions (i)–(iv) is simple and not carried out in detail here.

   It remains to evaluate $\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(2)\right)$ using (5.135). Let the interpretation $\mathfrak{I} = [\mathbb{N}, \Pi, \Phi]$ be such that $\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t} = \mathrm{Val}_{\mathfrak{I}}$. From the above assumptions about $\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}$, it is clear that

$$\begin{aligned}
\Pi_{\texttt{small}}(1) &= 1\\
\Pi_{\texttt{small}}(2) &= t\\
\Pi_{\texttt{small}}(3) &= t\\
&\vdots
\end{aligned}$$

   hence by (5.129) (as $\rightarrow$ is interpreted by $\mathrm{imp}_{\mathrm{G}}$),

$$\begin{aligned}
\varphi_{\rightarrow}\left(\Pi_{\texttt{small}}(1), \Pi_{\texttt{small}}(2)\right) &= t\\
\varphi_{\rightarrow}\left(\Pi_{\texttt{small}}(2), \Pi_{\texttt{small}}(3)\right) &= 1\\
\varphi_{\rightarrow}\left(\Pi_{\texttt{small}}(3), \Pi_{\texttt{small}}(4)\right) &= 1\\
&\vdots
\end{aligned}$$

   It follows by (3.4) that

$$\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}\left(\forall v\left(\texttt{small}(v) \rightarrow \texttt{small}\left(\texttt{inc}(v)\right)\right)\right) = t,$$

   thus for every $t \in \langle 0, 1\rangle$,

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(2)\right)(t) = \ell_{\gtrapprox 0.9}\,(t)\,,$$

   meaning

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(2)\right) = \ell_{\gtrapprox 0.9}.$$

3. $x = \texttt{small}\,(3)$.

   In this case, the same reasoning as in the previous item shows that for every $t \in \langle 0, 1\rangle$, setting $\mathrm{Val}_{\mathcal{X}_2,\texttt{small}(3),t} =_{\mathrm{def}} \mathrm{Val}_{\mathcal{X}_2,\texttt{small}(2),t}$ yields exactly the same results, establishing

$$\mathrm{Cons}(\mathcal{X}_2)\,\big(\texttt{small}\,(3)\big) = \ell_{\gtrapprox 0.9}.$$

In the same manner, it is demonstrated that

$$\begin{aligned}\mathrm{Cons}(\mathcal{X}_2)\,\big(\texttt{small}\,(4)\big) &= \mathrm{Cons}(\mathcal{X}_2)\,\big(\texttt{small}\,(5)\big)\\ &= \ldots\\ &= \ell_{\gtrapprox 0.9}.\end{aligned}$$

As had to be expected, when $\varphi_{\rightarrow}$ is interpreted by $\mathrm{imp}_{\mathrm{G}}$, then a good characterisation of the concept of **small number** is impossible even when labels expressing *graded uncertainty* are employed. Apparently, the uncertainty of the conclusion of any reasoning, however remote from the original facts, is the infimum of the uncertainties of all information employed. In this case, only ℓ^{AT} and $\ell_{\gtrapprox 0.9}$ are employed and $\ell_{\gtrapprox 0.9} \sqsubseteq \ell^{\mathrm{AT}}$, so the result is always $\ell_{\gtrapprox 0.9}$.

Variant 5. Let $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{KD}}$ as in variant 2.

For the characterisation of $\texttt{small}$, again the $\mathfrak{L}$-fuzzy set $\mathcal{X}_2$ of formulae is employed as defined in (5.133). This means all reasoning from the previous variant can be recycled up until equation (5.135).

Next, some instances of x are considered.

1. $x = \texttt{small(1)}$.

   Exactly the same reasoning as in variant 4 yields

$$\mathrm{Cons}(\mathcal{X}_2)\,\big(\texttt{small(1)}\big) = \ell^{\mathrm{AT}}.$$

2. $x = \texttt{small}\,(2)$.

   In this case, let $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t} \in \mathfrak{S}$ be such that

$$\begin{aligned}\mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t}\,\big(\texttt{small(1)}\big) &= 1\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t}\,\big(\texttt{small}\,(2)\big) &= t\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t}\,\big(\texttt{small}\,(3)\big) &= 1\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t}\,\big(\texttt{small}\,(4)\big) &= 1\\ &\vdots\end{aligned}$$

   Again, the proof that $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t}$ as defined here indeed fulfils all conditions (i)–(iv) is simple and is omitted here.

   Let the interpretation $\mathfrak{I} = [\mathbb{N}, \Pi, \Phi]$ be such that $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t} = \mathrm{Val}_{\mathfrak{I}}$. From the above assumptions about $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(2)},t}$, it is clear that

$$\begin{aligned}\Pi_{\texttt{small}}(1) &= 1\\ \Pi_{\texttt{small}}(2) &= t\\ \Pi_{\texttt{small}}(3) &= 1\\ &\vdots\end{aligned}$$

hence by (5.131) (as $\rightarrow$ is interpreted by $\mathrm{imp}_{\mathrm{KD}}$),

$$\begin{aligned}
\varphi_{\rightarrow}\big(\Pi_{\mathtt{small}}(1), \Pi_{\mathtt{small}}(2)\big) &= t \\
\varphi_{\rightarrow}\big(\Pi_{\mathtt{small}}(2), \Pi_{\mathtt{small}}(3)\big) &= 1 \\
\varphi_{\rightarrow}\big(\Pi_{\mathtt{small}}(3), \Pi_{\mathtt{small}}(4)\big) &= 1 \\
&\vdots
\end{aligned}$$

As in variant 4, it follows that

$$\mathrm{Cons}(\mathcal{X}_2)\big(\mathtt{small}\,(2)\big) = \ell_{\gtrapprox 0.9}.$$

3. $x = \mathtt{small}\,(3)$.

   In this case, let $\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t} \in \mathfrak{S}$ be such that

$$\begin{aligned}
\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}\big(\mathtt{small(1)}\big) &= 1 \\
\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}\big(\mathtt{small}\,(2)\big) &= \max\left(t, \frac{1}{2}\right) \\
\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}\big(\mathtt{small}\,(3)\big) &= t \\
\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}\big(\mathtt{small}\,(4)\big) &= 1 \\
\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}\big(\mathtt{small}\,(5)\big) &= 1 \\
&\vdots
\end{aligned}$$

   The proof that $\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}$ as defined here indeed fulfils all conditions (i)–(iv) is omitted.

   Let the interpretation $\mathfrak{I} = [\mathbb{N}, \Pi, \Phi]$ be such that $\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t} = \mathrm{Val}_{\mathfrak{I}}$. From the above assumptions about $\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}$, it is clear that

$$\begin{aligned}
\Pi_{\mathtt{small}}(1) &= 1 \\
\Pi_{\mathtt{small}}(2) &= \max\left(t, \frac{1}{2}\right) \\
\Pi_{\mathtt{small}}(3) &= t \\
\Pi_{\mathtt{small}}(4) &= 1 \\
&\vdots
\end{aligned}$$

   hence by (5.131) (as $\rightarrow$ is interpreted by $\mathrm{imp}_{\mathrm{KD}}$),

$$\begin{aligned}
\varphi_{\rightarrow}\big(\Pi_{\mathtt{small}}(1), \Pi_{\mathtt{small}}(2)\big) &= \max\left(t, \frac{1}{2}\right) \\
\varphi_{\rightarrow}\big(\Pi_{\mathtt{small}}(2), \Pi_{\mathtt{small}}(3)\big) &= \max\left(t, \frac{1}{2}\right) \\
\varphi_{\rightarrow}\big(\Pi_{\mathtt{small}}(3), \Pi_{\mathtt{small}}(4)\big) &= 1 \\
&\vdots
\end{aligned}$$

   It follows by (3.4) that

$$\mathrm{Val}_{\mathcal{X}_2,\mathtt{small(3)},t}\left(\forall v\left(\mathtt{small}(v) \rightarrow \mathtt{small}\left(\mathtt{inc}(v)\right)\right)\right) = \max\left(t, \frac{1}{2}\right),$$

thus for every $t \in \langle 0, 1 \rangle$,

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(3)\right)(t) = \ell_{\gtrapprox 0.9}\left(\max\left(t, \frac{1}{2}\right)\right),$$

meaning

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(3)\right) = \ell'_{\gtrapprox 0.9},$$

where the label $\ell'_{\gtrapprox 0.9}$ is defined as in Figure 5.7.

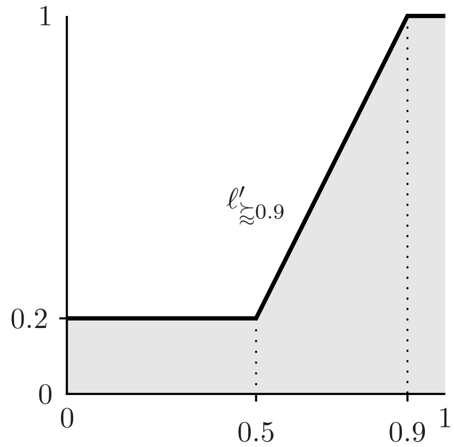


Figure 5.7: Label resulting from inference with Kleene-Dienes implication.

In the same manner, it is demonstrated that no further change occurs in further inferences, i. e.

$$\begin{aligned}\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(4)\right) &= \mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(5)\right)\\ &= \dots\\ &= \ell'_{\gtrapprox 0.9}.\end{aligned}$$

It can be observed that interpreting $\rightarrow$ by $\mathrm{imp}_{\mathrm{KD}}$ does not allow for an adequate characterisation of the concept of **small number**, because as numbers get very large, no further reduction of the label with which they can be derived takes place.

Still, a significant difference to variant 2 can be observed. While variant 2 is essentially identical to variant 1, the results for variant 5 are quite different from those for variant 4. In the third inference step, the uncertainty of the derived formula suddenly increases, an effect which didn't happen in variant 2.

It seems that below $\frac{1}{2}$, truth values cannot be distinguished from each other, so all of them get the same degree of trust. The fact that the value $\frac{1}{2}$ plays an important role in Lees fuzzy logic (which is the underlying many-valued logic in this case) has been pointed out already in section 5.4. Here, it plays the role of the threshold below which all truth values are equally trusted.

In this special case, the label $\ell_{\gtrapprox 0.9}$ completely rules out truth values between 0 and 0.4, i. e. in no model of $\mathcal{X}_2$ (to a degree above zero) can for instance $\texttt{small}\,(2)$ have a truth value strictly below 0.4. $\ell'_{\gtrapprox 0.9}$ is a weaker label allowing even a truth value of 0 for $\texttt{small}\,(3)$ to lead to a validity degree of 0.2, the same as for the truth value $\frac{1}{2}$, because an inference of three steps with $\mathrm{imp}_{\mathrm{KD}}$ introduces too much uncertainty to make this distinction.

This example also shows how *fuzzy labels* can help overcome deficiencies of the underlying many-valued logic. If only two-valued labels of the type $\ell_{\geqq t}$ are used as in variant 2 (which corresponds to Pavelka-style logics; compare sections 5.2.1.1 and 5.5.1.1), then values of t below $\frac{1}{2}$ lead to meaningless inferences, because $\texttt{small}\,(\texttt{3})$ would already be assigned the *zero element* ℓ^{U} of the label lattice by Cons. When *fuzzy labels* like $\ell_{\gtrapprox 0.9}$ are used, then truth values below $\frac{1}{2}$ are dealt with gracefully by assigning them the same degree of trust as $\frac{1}{2}$, but inference does not become meaningless as long as the degree of trust assigned to the truth value $\frac{1}{2}$ is below 1.

Variant 6. Let $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{L}}$ as in variant 3.

For the characterisation of `small`, again the $\mathfrak{L}$-fuzzy set $\mathcal{X}_2$ of formulae is employed as defined in (5.133). This means all reasoning from variant 4 can be recycled up until equation (5.135).

Next, some instances of x are considered. For $x = \texttt{small(1)}$ and $x = \texttt{small}\,(\texttt{2})$, exactly the same reasoning as in variant 4 yields

$$\begin{aligned}\mathrm{Cons}(\mathcal{X}_2)\,\big(\texttt{small(1)}\big) &= \ell^{\mathrm{AT}},\\ \mathrm{Cons}(\mathcal{X}_2)\,\big(\texttt{small}\,(\texttt{2})\big) &= \ell_{\gtrapprox 0.9}.\end{aligned}$$

The results diverge when reasoning 'farther away' from the assumptions.

1. $x = \texttt{small}\,(\texttt{3})$.

   In this case, let $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t} \in \mathfrak{S}$ be such that

$$\begin{aligned}\mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}\,\big(\texttt{small(1)}\big) &= 1\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}\,\big(\texttt{small}\,(\texttt{2})\big) &= \frac{t+1}{2}\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}\,\big(\texttt{small}\,(\texttt{3})\big) &= t\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}\,\big(\texttt{small}\,(\texttt{4})\big) &= 1\\ \mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}\,\big(\texttt{small}\,(\texttt{5})\big) &= 1\\ &\vdots\end{aligned}$$

   The proof that $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}$ as defined here indeed fulfils all conditions (i)–(iv) is omitted.

   Let the interpretation $\mathfrak{I} = [\mathbb{N}, \Pi, \Phi]$ be such that $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t} = \mathrm{Val}_{\mathfrak{I}}$. From the above assumptions about $\mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}$, it is clear that

$$\begin{aligned}\Pi_{\texttt{small}}(1) &= 1\\ \Pi_{\texttt{small}}(2) &= \frac{t+1}{2}\\ \Pi_{\texttt{small}}(3) &= t\\ \Pi_{\texttt{small}}(4) &= 1\\ &\vdots\end{aligned}$$

   hence by (5.132) (as $\rightarrow$ is interpreted by $\mathrm{imp}_{\mathrm{L}}$),

$$\varphi_{\rightarrow}\,\big(\Pi_{\texttt{small}}(1), \Pi_{\texttt{small}}(2)\big) = \frac{t+1}{2}$$

$$\varphi_{\rightarrow}\big(\Pi_{\texttt{small}}(2), \Pi_{\texttt{small}}(3)\big) = \frac{t+1}{2}$$
$$\varphi_{\rightarrow}\big(\Pi_{\texttt{small}}(3), \Pi_{\texttt{small}}(4)\big) = 1$$
$$\vdots$$

It follows by (3.4) that

$$\mathrm{Val}_{\mathcal{X}_2,\texttt{small(3)},t}\left(\forall v\left(\texttt{small}(v) \rightarrow \texttt{small}\left(\texttt{inc}(v)\right)\right)\right) = \frac{t+1}{2},$$

thus for every $t \in \langle 0, 1\rangle$,

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(3)\right)(t) = \ell_{\gtrapprox 0.9}\left(\frac{t+1}{2}\right),$$

meaning

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(3)\right) = \ell_{\gtrapprox 0.8},$$

where the label $\ell_{\gtrapprox 0.8}$ is defined as in Figure 5.8.

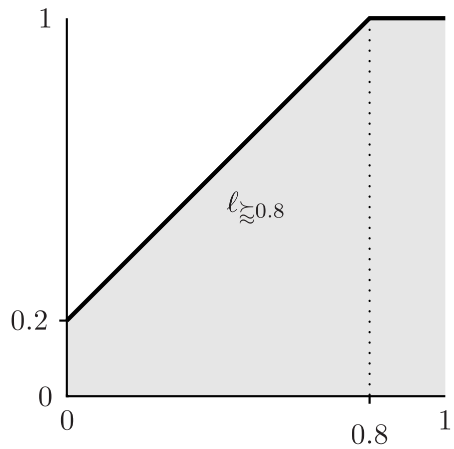


Figure 5.8: Label resulting from inference with Łukasiewicz's implication.

In the same manner, it is demonstrated that the resulting label gets more and more uncertain in further inferences, i. e.

$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(4)\right) = \ell_{\gtrapprox 0.7},$$
$$\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(5)\right) = \ell_{\gtrapprox 0.6},$$

where the labels $\ell_{\gtrapprox 0.7}, \ell_{\gtrapprox 0.6}$ are defined as in Figure 5.9.

Note furthermore that the first number `n` (respectively the term representing the corresponding iteration of `inc`) for which $\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(\texttt{n})\right)$ yields the zero element ℓ^{U} of $\mathfrak{L}$ is `11`, exactly as in variant 3. But unlike variant 3, the labels yielded by $\mathrm{Cons}(\mathcal{X}_2)\left(\texttt{small}\,(\texttt{n})\right)$ for numbers below `11` get increasingly uncertain.

For comparison, the inference results of variants 4–6 are summarised graphically in Figure 5.10.

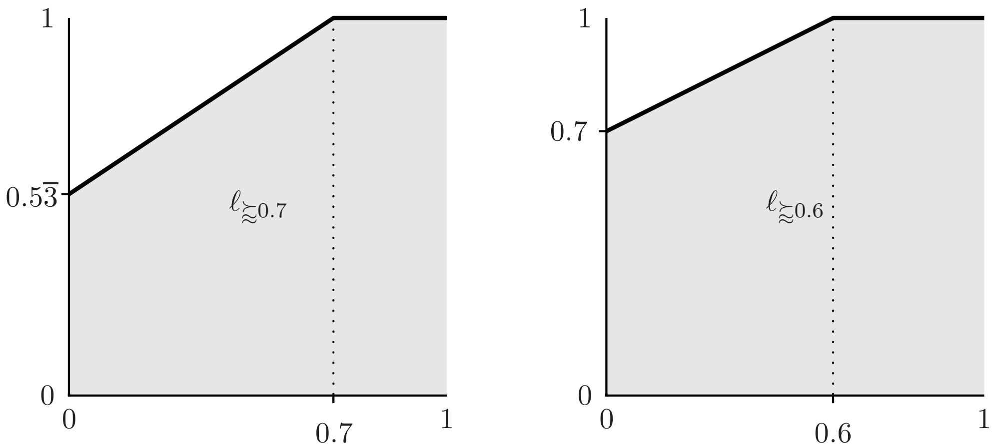


Figure 5.9: Further labels resulting from inference with ŁUKASIEWICZ's implication.

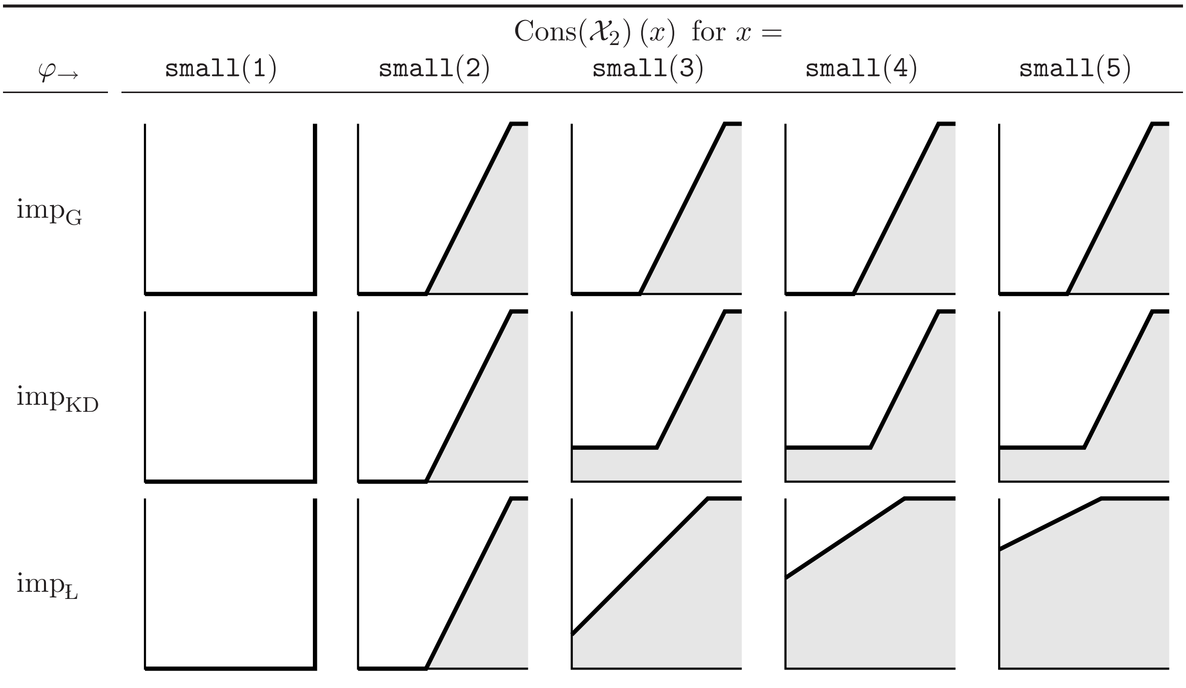


Figure 5.10: Graphical summary of inference results with 'fuzzy' labels.

Remarks

1. As a conclusion, it can be stated that when implication is interpreted by imp_G, then the length of a chain of inferences is irrelevant; the conclusion of such a chain will always have an uncertainty corresponding to the infimum of the uncertainties of all evidence used in the chain.

   When implication is interpreted by imp_KD, then the situation is similar, but at a certain 'distance' from the original evidence, low truth values become indistinguishable, so uncertainty raises until all truth values below $\frac{1}{2}$ are assigned the same degree of trust. Considering section 5.5.1.2, this can be interpreted as a certain degree of *doubt* which is introduced when reasoning beyond the given evidence.

   When implication is interpreted by $\mathrm{imp}_\mathrm{Ł}$, then uncertainty is increased along a chain of inferences, so it can be expected that if an implication is labelled with anything but ℓ^{AT}, then inferences drawn from it will receive labels converging towards ℓ^{U} as the number of applications of said implication which are necessary to achieve the conclusion increases.

2. Two points should have become clear from this exercise in deriving semantic consequences directly from the definition. First, there is no big difference to doing the same in classical mathematical logic (only the quantifiers $\forall, \exists$ are replaced by $\inf, \sup$, respectively, and different interpretations of $\varphi_{\rightarrow}$ have to be considered). Secondly, it is infeasible to do this for real applications.

3. Using the sound inference rules from section 6.2.1.1, at least part of the results obtained above can be reconstructed. In Example 6.2.2, it is demonstrated how the same inferences can be drawn by syntactic derivation. •

5.6 On the Issue of Compositionality

D. Dubois and H. Prade have, on several occasions [23, 26–28], gone to great lengths explaining why *possibilistic logic* is not **compositional** wrt. the *necessity measure* induced by a *possibility distribution* (see equation (5.91) in section 5.3.1.1).

Compositionality here means that given a *necessity measure* $N : \mathrm{Frm} \to \langle 0, 1 \rangle$ and a composite formula $x = \omega\, x_1 \dots x_n \in \mathrm{Frm}$, the value $N(x)$ can not (always) be calculated from the values $N(x_1), \dots, N(x_n)$.

While the comprehensive considerations in [28] are very useful in themselves, clarifying side issues like common misunderstandings about probability logics or belief measures in expert systems and contemplating 'almost compositional' belief measures, it is the firm belief of this dissertation's author that the whole issue vaporises in the light of the characterisation $N = \mathrm{Cons}(\mathcal{S})$, where $\mathcal{S}$ is the *possibility distribution* inducing N.

By exposing the *necessity measure* N to be, by nature, a fuzzy set of semantic consequences, it is evident that N cannot be 'compositional'.

To see this, consider the situation in classical two-valued logic. Let X be a set of logical formulae and $\mathrm{Cons}(X)$ the set of *semantic consequences* in the sense of mathematical logic. It is well-known that for a conjunction $x \wedge y$, it holds that $x \wedge y \in \mathrm{Cons}(X)$ if and only if $x \in \mathrm{Cons}(X)$ and $y \in \mathrm{Cons}(X)$, but for other formulae, Cons is not 'compositional'. For instance, for $x \vee y$ it cannot be determined whether $x \in \mathrm{Cons}(X)$ and/or $y \in \mathrm{Cons}(X)$ knowing only whether or not $x \vee y \in \mathrm{Cons}(X)$. Also, for $\neg x$ it cannot be determined whether $\neg x \in \mathrm{Cons}(X)$ knowing only whether or not $x \in \mathrm{Cons}(X)$.

The same situation, which is basic knowledge for classical logic, has been established (not surprisingly) in Proposition 4.3.20 for the semantic consequence operator Cons of fuzzy filter-based logic, i. e. when $\wedge, \vee$ are interpreted by the lattice connectives (which is the case for possibilistic logic), then $\mathrm{Cons}(\mathcal{X})(x \wedge y)$ can be determined from $\mathrm{Cons}(\mathcal{X})(x)$ and $\mathrm{Cons}(\mathcal{X})(y)$ while for $\mathrm{Cons}(x \vee y)$, only a lower estimate exists (corresponding to the fact that classically, $x \in \mathrm{Cons}(X)$ or $y \in \mathrm{Cons}(X)$ implies $x \vee y \in \mathrm{Cons}(X)$). For $\neg$, obviously the situation depends on the many-valued interpretation of $\neg$, and in particular the susceptibility of the logic to **inconsistency** (compare section 5.4.4).

After accepting that in *possibilistic logic*, which is a special case of a *logic of graded trust assessment*, the *necessity measure* induced by a *possibility distribution* is really a 'measure of semantic consequence', the fact that it is not 'compositional' goes without saying.

In fact, exactly this observation lies behind the illustration in [28] for non-compositionality of two-valued belief measures using an example from the area of *rational agents*. An agent is given a set of formulae K called *knowledge*. The agent is assumed to *believe* in all formulae from

K and additionally all formulae which follow semantically from K. In [28], a two-valued 'belief measure' N on Frm is introduced which is 1 for a formula x iff x follows from K. Concerning the belief of an agent in a formula x and/or its negation $\neg x$, four cases can be distinguished:

1. $N(x) = 1$ and $N(\neg x) = 0$, i. e. the agent believes in x and rejects $\neg x$,
2. $N(x) = 0$ and $N(\neg x) = 1$, i. e. the agent rejects x and believes in $\neg x$,
3. $N(x) = 0$ and $N(\neg x) = 0$, interpreted as *ignorance*,
4. $N(x) = 1$ and $N(\neg x) = 1$, i. e. the agent inconsistently believes in x **and** $\neg x$.

Obviously, all four cases can occur (though the last one only if K is inconsistent, which is normally prevented). This means N is not compositional wrt. $\neg$.

This illustration is in fact complementary to the reasoning above. In both cases, 'belief' or 'necessity' measures are revealed to be 'measures of semantic consequence' to justify why they are not compositional.

To summarise, it is the authors belief that the method in which fuzzy filter-based logics are defined adhering strictly to the conventions of mathematical logic helps clarifying the issue of compositionality:

- On the lowest level, there is the underlying many-valued logic where truth values are given by valuations Val mapping formulae to T, which are (usually) compositional wrt. logical connectives, according to the *principle of extensionality*.
- On the next level, there are fuzzy sets of formulae mapping formulae to L which are completely arbitrary because the labels are chosen by the user.
- On the next level, there are model fuzzy sets Mod mapping labelled formulae to D which are by nature compositional wrt. the lattice meet (compare Proposition 4.1.1.4 and (4.42)) but not wrt. the other connectives because the labels (being fuzzy filters) guarantee 'well-behaviour' only wrt. the lattice meet.
- On the highest level, there is the fuzzy set of semantic consequences of a fuzzy set of formulae mapping formulae to L which is compositional only wrt. $\wedge$ if it is interpreted by the lattice meet. This is the case in classical two-valued logic as well.

It should be clear that the same holds for logics using a different paradigm of *uncertainty*, for instance, probability or belief measures, as long as the general form of construction is the same: an underlying compositional logic, labels expressing an a priori assessment of the uncertainty of certain formulae, and an 'uncertainty' measure which is given by semantic consequence for labelled formulae.

5.7 Other Paradigms for Defining Logics of Graded Truth and Graded Trust Assessment

This section is intended to give a brief survey of other logics for combining aspects of **vagueness** and **uncertainty** within a single framework.

In fact, only two variants of **possibilistic logic** based on many-valued logics are presented in a little more detail. *Possibilistic logic with vague predicates* is based on Lees fuzzy logic as presented in Example 3.2.4.2. *Possibilistic logic with fuzzy constants* is based on an interesting variant of many-valued logic where *terms* are fuzzy and predicates are crisp. The handling of labels is quite similar in both: Only *necessity degrees* are given as labels, and the definition of the model relation involves a quantification where the truth value of the formula part of a labelled formula and the necessity degree of the label part of a labelled formula are combined. Only the operator used for the combination is different.

The only other alternative presented here is *qualitative fuzzy possibilistic logic*, which is mostly skimmed over because the approach is quite different from fuzzy filter-based logic semantically, hence a detailed comparison would take much preparation.

A deeper study and systematic comparison of alternative approaches is left for future research.

5.7.1 Possibilistic Logic with Vague Predicates

Possibilistic logic with vague predicates is a labelled logic where the underlying logic is many-valued first order logic and the labels are necessity degrees, defined in [19, section 4.5].

Concerning the semantics of labels, possibilistic logic with vague predicates is based on a slightly different approach than fuzzy filter-based logic, as will be explained in the following.

For this subsection, let $\mathrm{Frm} = \mathrm{FOFrm}_{\mathrm{S}}$, i.e. employ the language of **first order logic**[4]. Let $\mathfrak{T} = \mathfrak{F} = \left[\langle 0,1\rangle, \min, \max\right]$ and $\mathfrak{S} =_{\mathrm{def}} \mathfrak{S}_{\mathrm{F}}^{\mathrm{F}}$ as defined in Example 3.2.4.2.

Furthermore, let $\mathfrak{D} = \mathfrak{L} = \mathfrak{F}$. It follows immediately that possibilistic logic with vague predicates does not fit the definition of a *fuzzy filter-based logic*. In Observation 4.1.4.2, it is established that $\mathfrak{T}$ and $\mathfrak{D}$ can't both be many-valued while $\mathfrak{L}$ is a chain. Consequently, $\mathfrak{L}$ in possibilistic logic with vague predicates is 'too weak' for it to be a fuzzy filter-based logic.

In the following, the definition of semantic consequence given in [19] is translated to the conceptual framework of this dissertation, as was done in section 5.3.1.1 for standard possibilistic logic.

Let an $\mathfrak{F}$-fuzzy set $\mathcal{S} : \mathfrak{S} \to \langle 0,1\rangle$ (*possibility distribution* on $\mathfrak{S}$) be given. Then the $\mathfrak{F}$-fuzzy set $\mathrm{Cons}_{\mathrm{PLVP}}(\mathcal{S})$ of **semantic consequences** of $\mathcal{S}$ is given, for $x \in \mathrm{Frm}$, by

$$\mathrm{Cons}_{\mathrm{PLVP}}(\mathcal{S})(x) =_{\mathrm{def}} \inf\left\{\mathrm{imp}\left(\mathcal{S}(\mathrm{Val}), \mathrm{Val}(x)\right) \,\middle|\, \mathrm{Val} \in \mathfrak{S}\right\}, \tag{5.136}$$

where imp is a given **implication function**.

In [19], the following definition of imp is suggested, for $s, t \in \langle 0,1\rangle$:

$$\mathrm{imp}(s,t) =_{\mathrm{def}} \begin{cases} 1, & \text{if } s \leqq t \\ 1-s, & \text{if } s > t \end{cases}$$

(This connective is called *reciprocal of* Gödel *implication.*)

Expanding imp in (5.136) yields

$$\mathrm{Cons}_{\mathrm{PLVP}}(\mathcal{S})(x) =_{\mathrm{def}} \inf\left\{1 - \mathcal{S}(\mathrm{Val}) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \text{ and } \mathcal{S}(\mathrm{Val}) > \mathrm{Val}(x)\right\}.$$

By Proposition 4.3.1.9, this leads to the following definition for **entailment** in possibilistic logic with vague predicates, for $[x,d] \in \mathrm{LFrm}$:

$$\mathcal{S} \Vdash_{\mathrm{PLVP}} [x,d] \text{ iff for every } \mathrm{Val} \in \mathfrak{S} : d \leqq \mathrm{imp}\left(\mathcal{S}(\mathrm{Val}), \mathrm{Val}(x)\right),$$

which is, by the definition of imp, equivalent with

$$\text{iff for every } \mathrm{Val} \in \mathfrak{S} : \mathcal{S}(\mathrm{Val}) \leqq \max\left(1-d, \mathrm{Val}(x)\right).$$

[4] Individual variables, function and predicate symbols can be arbitrary.

This leads to the interesting observation that $\mathcal{S} \Vdash_{\overline{\text{PLVP}}} [x, d]$ holds if and only if $\mathcal{S} \Vdash \left[x, \ell^{\mathrm{T}}_{1-d}\right]$ in the corresponding logic of graded truth and graded trust assessment (see Figure 5.3 for the definition of the label ℓ^{T}_{1-d}).

Hence, it can be safely claimed that possibilistic logic with vague predicates can be embedded into logic of graded truth and graded trust assessment by a simple transformation of the labels, and consequently, possibilistic logic with vague predicates forms a special case of the logics of graded truth and graded trust assessment studied in this dissertation.

5.7.2 Possibilistic Logic with Fuzzy Constants (PLFC)

Possibilistic Logic with Fuzzy Constants (PLFC) is a labelled logic where the underlying logic is a special variant of many-valued first order logic and the labels are necessity degrees.

The semantics of labels in PLFC is the same as for possibilistic logic with vague predicates (see previous subsection), only for a different choice of imp.

PLFC is studied in several publications by Sandra Sandri and others [1, 29].

The syntax and semantics of the underlying many-valued logic of PLFC are defined next. The logic varies significantly from the first order many-valued logic presented in Example 3.2.2, so it is interesting to compare them. Afterwards, the semantics of labels in PLFC is defined and briefly compared with fuzzy filter-based logic. This presentation is based on [1] which covers less issues than [29] but gives slightly more 'extractable' definitions. The notation used in [1] is adapted to ease comparison with the notation introduced in this dissertation.

5.7.2.1 Syntax

In [1], a many-sorted first order clausal form without function symbols is used. Here, the sorts are left out because they are not important for the comparison. Hence, the *logical language* of PLFC is determined by

1. Non-empty sets IV, IC, FC of **individual variables**, **individual constants**, and **fuzzy constants**, respectively.
2. A non-empty set Pred of **predicate symbols**.
3. A mapping $\mathrm{Ar}_{\mathrm{Pred}} : \mathrm{Pred} \to \mathbb{N}$ giving the **arity** of each predicate.
4. A unary **operator symbol** (or **connective**) $\neg$ and a binary **operator symbol** $\wedge$.

Definition 5.7.1 (Formulae of PLFC)
The set $\mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$ *of all* ***well-formed formulae of PLFC*** *with respect to the sets* IV, IC, FC, Pred *and the mapping* $\mathrm{Ar}_{\mathrm{Pred}}$ *as defined above is the smallest set such that*

1. *For each* $p \in \mathrm{Pred}$ *and symbols* $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \in \mathrm{IV} \cup \mathrm{IC} \cup \mathrm{FC}$, *the symbol sequences*

$$\begin{aligned} & p\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \\ \textit{and} \quad & \neg p\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \end{aligned}$$

   are contained in $\mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$.

   Formulae of either form are called ***literals****. The set of all literals (for fixed* IV, IC, FC, Pred, $\mathrm{Ar}_{\mathrm{Pred}}$*) is denoted by* Lit.

   Formulae of the first type are called ***atomic formulae****.*

2. *For formulae* $x_1, x_2 \in \mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$, *the symbol sequence*

$$\vee\, x_1 x_2$$

is contained in $\mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$.

Note that every formula which is not a literal is a superposition of disjunctions of literals.

5.7.2.2 Semantics

For defining the semantics of PLFC, first of all, the truth value lattice is fixed to be $\mathfrak{T} =_{\mathrm{def}} \mathfrak{F} = \left[\langle 0,1\rangle, \min, \max\right]$.

As in Example 3.2.2, the semantics for the language $\mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$ is a set of valuation functions induced by *interpretations* which specify a *domain* containing all *individuals* under consideration and assign *relations* (on the domain) to *predicate symbols*, *individuals* (from the domain) to *individual constants*, and *fuzzy sets* (on the domain) to *fuzzy constants*.

Definition 5.7.2 (Interpretations in PLFC)
Given a ***logical language*** $\mathrm{Frm} = \mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$ *(see* Definition 5.7.1 *for a definition of* IV, IC, FC, Pred, *and* $\mathrm{Ar}_{\mathrm{Pred}}$*), an* ***interpretation*** *for* Frm *is given by a tuple*

$$\mathfrak{I} = [U, \Pi, \Gamma_{\mathrm{I}}, \Gamma_{\mathrm{F}}]$$

where

1. U *is an arbitrary non-empty set called* ***domain*** *or* ***universe***.
2. $\Pi : \mathrm{Pred} \to \bigcup\{\mathfrak{P}U^n \,|\, n \in \mathbb{N}\}$ *such that for every* $p \in \mathrm{Pred}$, $\Pi(p) \in \mathfrak{P}U^n$.
3. $\Gamma_{\mathrm{I}} : \mathrm{IC} \to U$.
4. $\Gamma_{\mathrm{F}} : \mathrm{FC} \to \langle 0,1\rangle^U$.

With every *interpretation* $\mathfrak{I} = [U, \Pi, \Gamma_{\mathrm{I}}, \Gamma_{\mathrm{F}}]$ as specified above, a valuation function $\mathrm{Val}_{\mathfrak{I}}$ is associated inductively as follows.

Definition 5.7.3 (Valuation of formulae in PLFC)
Let a ***logical language*** $\mathrm{Frm} = \mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$ *and an* ***interpretation*** $\mathfrak{I} = [U, \Pi, \Gamma_{\mathrm{I}}, \Gamma_{\mathrm{F}}]$ *for* Frm *be given.*

For this definition, ***assignments*** $\sigma : \mathrm{IV} \to U$ *are used in exactly the same manner as in* Definition 3.2.3. *See there for details.*

The interpretation of PLFC formulae is defined as follows.

1. *Given an assignment* $\sigma : \mathrm{IV} \to U$ *and a Formula* $x \in \mathrm{Frm}$, *the* ***truth value*** *associated with* x *by* $\mathfrak{I}$ *and* σ *is denoted by* $\mathrm{Val}(x, \mathfrak{I}, \sigma) \in T$ *and defined inductively as follows.*
   1.1. *Let* $p \in \mathrm{Pred}$ *and* $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \in \mathrm{IV} \cup \mathrm{IC} \cup \mathrm{FC}$ *such that* $x = p\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}$. *Wlg assume that there exist* $n, m \in \mathbb{N}$ *such that* $n \leqq m \leqq \mathrm{Ar}_{\mathrm{Pred}}(p)$ *and* $t_1, \ldots, t_n \in \mathrm{IV}$, $t_{n+1}, \ldots, t_m \in \mathrm{IC}$, *and* $t_{m+1}, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \in \mathrm{FC}$. *Then*

$$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}}$$
$$\sup\left\{ \min\left(\begin{array}{l} \Gamma_{\mathrm{F}}\left(t_{m+1}\right)\left(u_{m+1}\right), \\ \vdots, \\ \Gamma_{\mathrm{F}}\left(t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}\right)\left(u_{\mathrm{Ar}_{\mathrm{Pred}}(p)}\right) \end{array} \right) \middle| \left[\begin{array}{l} \sigma(t_1), \ldots, \sigma(t_n), \\ \Gamma_{\mathrm{I}}(t_{n+1}), \ldots, \Gamma_{\mathrm{I}}(t_m), \\ u_{m+1}, \ldots, u_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \end{array} \right] \in \Pi(p) \right\}.$$

1.2. Let $p \in \mathrm{Pred}$ *and* $t_1, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \in \mathrm{IV} \cup \mathrm{IC} \cup \mathrm{FC}$ *such that* $x = \neg p\, t_1 \ldots t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}$. *As above, assume that there exist* $n, m \in \mathbb{N}$ *such that* $n \leqq m \leqq \mathrm{Ar}_{\mathrm{Pred}}(p)$ *and* $t_1, \ldots, t_n \in \mathrm{IV}$, $t_{n+1}, \ldots, t_m \in \mathrm{IC}$, *and* $t_{m+1}, \ldots, t_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \in \mathrm{FC}$. *Then*

$$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}}$$

$$\sup \left\{ \min \left(\begin{array}{l} \Gamma_{\mathrm{F}}\,(t_{m+1})\,(u_{m+1})\,, \\ \vdots, \\ \Gamma_{\mathrm{F}}\left(t_{\mathrm{Ar}_{\mathrm{Pred}}(p)}\right)\left(u_{\mathrm{Ar}_{\mathrm{Pred}}(p)}\right) \end{array} \right) \middle| \left[\begin{array}{l} \sigma(t_1), \ldots, \sigma(t_n), \\ \Gamma_{\mathrm{I}}(t_{n+1}), \ldots, \Gamma_{\mathrm{I}}(t_m), \\ u_{m+1}, \ldots, u_{\mathrm{Ar}_{\mathrm{Pred}}(p)} \end{array} \right] \notin \Pi(p) \right\}.$$

1.3. For $x_1, x_2 \in \mathrm{Frm}$ *such that* $x = \vee\, x_1 x_2$,

$$\mathrm{Val}(x, \mathfrak{I}, \sigma) =_{\mathrm{def}} \max\left(\mathrm{Val}(x_1, \mathfrak{I}, \sigma), \mathrm{Val}\,(x_2, \mathfrak{I}, \sigma)\right).$$

2. The valuation function $\mathrm{Val}_{\mathfrak{I}} : \mathrm{Frm} \to T$ *induced by* $\mathfrak{I}$ *is now defined as follows. Let* $x \in \mathrm{Frm}$ *be given. Then*

(5.137) $$\mathrm{Val}_{\mathfrak{I}}(x) =_{\mathrm{def}} \inf \left\{ \mathrm{Val}\,(x, \mathfrak{I}, \sigma) \middle|\, \sigma : \mathrm{IV} \to U \right\}$$

Assuming IV, IC, FC, Pred, and $\mathrm{Ar}_{\mathrm{Pred}}$ to be given, the **semantics** $\mathfrak{S}$ for $\mathrm{PLFCFrm}(\mathrm{IV}, \mathrm{IC}, \mathrm{FC}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}})$ is defined to be

(5.138) $$\mathfrak{S} =_{\mathrm{def}} \left\{ \mathrm{Val}_{\mathfrak{I}} \middle| \mathfrak{I} = [U, \Pi, \Gamma_{\mathrm{I}}, \Gamma_{\mathrm{F}}] \text{ as defined in Definition 5.7.2} \right\}.$$

For the example of PLFC, fulfilling assumption (3.2) is not as trivial as in first order logic, but considering that Pred and FC are both required to be non-empty, it is easy to observe that every combination of Frm and $\mathfrak{S}$ in PLFC fulfils assumption (3.2) (by choosing an interpretation with a suitable combination of interpretation of some predicate symbol and fuzzy constant symbol, respectively).

5.7.2.3 Semantics of Labels

The definition of labelled formulae as well as the semantics of labels in PLFC is identical with possibilistic logic with vague predicates (section 5.7.1) up to equation (5.136).

In PLFC, $\mathrm{imp} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{KD}}$ (compare (3.11)) is used, yielding, for $x \in \mathrm{Frm}$,

(5.139) $$\mathrm{Cons}_{\mathrm{PLFC}}(\mathcal{S})(x) =_{\mathrm{def}} \inf \left\{ \max\left(1 - \mathcal{S}(\mathrm{Val}), \mathrm{Val}(x)\right) \middle| \mathrm{Val} \in \mathfrak{S} \right\},$$

leading to the following definition for **entailment** in PLFC, for $[x, d] \in \mathrm{LFrm}$:

(5.140) $$\mathcal{S} \mathrel{\Vdash_{\mathrm{PLFC}}} [x, d] =_{\mathrm{def}} \text{for every } \mathrm{Val} \in \mathfrak{S} : d \leqq \max\left(1 - \mathcal{S}(\mathrm{Val}), \mathrm{Val}(x)\right).$$

Let's try to compare these definitions with fuzzy filter-based logics.

From (4.115), it is clear that entailment by a fuzzy set $\mathcal{S} : \mathfrak{S} \to D$ has to be of the form

(5.141) $$\mathcal{S} \Vdash [x, \ell] =_{\mathrm{def}} \text{for every } \mathrm{Val} \in \mathfrak{S} : \mathcal{S}(\mathrm{Val}) \leqq \iota(\ell)\left(\mathrm{Val}(x)\right),$$

which cannot be equivalent with (5.140), for any value of $\iota(\ell)$.

Rather, the semantics of PLFC is based on a different approach, which could be named **degree of entailment** approach. Inspecting (5.139), the formula defining $\mathrm{Cons}_{\mathrm{PLFC}}$ is a *fuzzification* of the classical formula $\forall\, \mathrm{Val} \in \mathfrak{S}\left(\mathrm{Val} \in \mathcal{S} \to \mathrm{Val} \models x\right)$, where $\forall$ is interpreted by inf and $\to$ by KLEENE-DIENES implication $\mathrm{imp}_{\mathrm{KD}}$. The value $\mathrm{Cons}_{\mathrm{PLFC}}(\mathcal{S})(x)$ can thus be interpreted as the degree to which

$$\mathcal{S} \Vdash x$$

holds. $\mathcal{S} \mathrel{\Vdash_{\mathrm{PLFC}}} [x, d]$ is then true by (5.140) iff d is below the degree of $\mathcal{S} \Vdash x$.

Remarks

1. In [1], the definitions of $\mathrm{Cons}_{\mathrm{PLFC}}$ and $\models\!\!\!-_{\mathrm{PLFC}}$ are relative to a given **context** fixing some parts of the interpretation. The context, formally defined in [1], is an explicit form of the method of *fixing* described in remark 2 on page 46.

2. In addition to labels which are *necessity degrees*, in [1] a more complex form of label called *variable weight* is discussed, which is essentially a mapping from assignments of individuals to necessity degrees. The same concept exists for *possibilistic logic* [19, section 4.2]; it is neglected here because it is not comparable with the notion of label employed in this dissertation. •

5.7.2.4 Conclusions

Obviously, the previous two subsections leave more questions open that answered. The brief glimpse given here, however, makes it clear that possibilistic logic with vague predicates as well as PLFC are very interesting types of logics for the representation of vagueness and possibilistic uncertainty, which are very much in need of a detailed comparison with each other and with fuzzy filter-based logics. The most pressing open questions are:

1. What is the significance of imp? Possibilistic logic with vague predicates and PLFC merely represent two special cases. It is intriguing to ask in which way the properties of semantic consequence in 'fuzzy possibilistic logic' defined by (5.136) depend on properties of the implication function (or reciprocal thereof) inserted for imp. This leads to the subquestions:

   - When is the resulting logic a special case of fuzzy filter-based logic (as for possibilistic logic with vague predicates)?
   - When is the resulting semantic consequence operator a *fuzzy closure operator*?

2. What about semantic consequences of fuzzy sets of formulae? In accordance with the literature, only semantic consequences of fuzzy sets of valuations were studied in the previous two subsections. In [1], a general definition for $\mathcal{X} \models\!\!- [x, d]$ is given by

$$\mathcal{X} \models\!\!- [x, d] \text{ iff for every } \mathcal{S} \in \langle 0, 1\rangle^{\mathfrak{S}} : (\mathcal{S} \models\!\!- \mathcal{X}) \to (\mathcal{S} \models\!\!- [x, d]),$$

   which is obviously applicable to both possibilistic logic with vague predicates and PLFC, but no 'closed form' is given.

3. How to express *graded truth assessment*? The label lattice of possibilistic logic with vague predicates and PLFC is 'weaker' than that of fuzzy filter-based logic because a label consists only of a necessity degree. Truth values are handled by combining them with validity degrees in $\mathrm{imp}\big(\mathcal{S}(\mathrm{Val}), \mathrm{Val}(x)\big)$. It has to be asked what exactly the significance of truth values is in such logics.

Investigation of these aspects, together with comparison of the logics wrt. concrete application examples, is left for future research.

5.7.3 Qualitative Fuzzy Possibilistic Logic

In [55, 56], Petr Hájek and several other authors introduce an approach to the combination of many-valued truth and graded possibility based on modal logic.

That is, Łukasiewicz's continuously many-valued logic is equipped with modal operators $\Diamond$ (possibly) and $\Box$ (necessarily) in a straightforward way (compare also H. Thiele [93]).

The *truth value* yielded by these operators is now interpreted as the *degree of possibility* and *degree of necessity*, respectively, of the formula the operator operates on. Additionally, a *binary modality* $\lhd$ is introduced meaning *less possible than.* Properties of the resulting modal logic are studied in [55, 56].

Comparisons between several approaches to 'approximate reasoning' (most of them involving modal logic) are given in [50, 51, 55]. See also [52, chapter eight]

Here, this approach is not studied in any more detail, as a detailed comparison between the modal concepts from [56] and the label-based concepts used in this dissertation would require a lot of preparation. Just some remarks:

- By making possibility degrees coincide with truth values of certain formulae, the distinction between graded truth and graded trust, enforcing which was one of the main goals of this dissertation, is weakened. This approach has advantages and disadvantages.

  On the one hand, the possibility of combining formulae containing modal operators freely with each other via many-valued logical operators creates expressive power not present in approaches based on labelled formulae.

  On the other hand, all advantages of a strict distinction between graded truth and graded trust are lost, for instance, the possibility to choose completely different algebraic structures for both. Furthermore, modal logic is much more difficult to handle in *automated deduction*. Another open question is how the expressive power stemming from labels which are arbitrary *fuzzy filters* is emulated using logical formulae from modal logic.

- In principle, the fuzzy modal logic referred to above fits nicely in the general outlook on syntax and semantics of the underlying many-valued logic taken in this dissertation, so one could 'plug in' qualitative fuzzy possibilistic logic into fuzzy filter-based logic and see what happens. This would, however, make the confusion concerning semantics of values complete and should be left open until the relationship between both approaches is understood better.

A detailed comparison between the modal fuzzy logic approach and the labelled fuzzy logic approach to the representation of graded truth assessment and graded trust assessment is left for future research.

6 Summary, Conclusions, and Future Work

In this chapter, the results obtained in this dissertation are summarised and possible extensions and starting points for future work are sketched.

In the next section, the most significant results are grouped by subject and summarised. Their significance for the corresponding area of research is assessed.

In section 6.2, several possible extensions are described.

The most significant development which has not been achieved in this dissertation is to establish a correct and complete syntactical **derivation system** for *fuzzy filter-based logics*, an indispensable part of every logic to be used for knowledge representation. Of particular interest are derivation systems which can be used for **automated deduction** on a computer. In section 6.2.1, first steps towards automated deduction systems for fuzzy filter-based logics are described.

In section 6.2.2, it is described how the lattice $\mathfrak{D}$ (corresponding to a modelling of **uncertainty** by **possibility measures**) can be replaced by algebraic structures supporting the use of other measure-theoretic concepts (for instance, **probability measures** or **Dempster-Shafer uncertainty measures**) for uncertainty modelling.

Section 6.2.3 gives some hints towards applications of *fuzzy filter-based logics* in knowledge representation.

6.1 Summary and Conclusions

In this section, the results achieved in this dissertation are bundled by the area of research they belong to. Their impact on the respective area is estimated.

6.1.1 Contributions to the Theory of Fuzzy Filters in Lattices

The idea of studying *fuzzy filters* of a lattice is not new. References to publications ranging back to the year 1988 are given in the introduction to chapter 2. Some of the results given in chapter 2 appear in this (or in slightly different) form in the literature. Other results, especially in section 2.1, are purely technical, obtained by expanding definitions, and do not represent a significant contribution to the theory of fuzzy filters in lattices.

As a whole, however, chapter 2 represents a significant contribution to the theory of fuzzy filters in lattices. To the author's knowledge, it represents the most comprehensive study of fuzzy filters from a completely general, purely *lattice-theoretic* point of view. Some particular aspects are summarised in the following. Selected results from chapter 2 have been reported by the author in [71].

6.1.1.1 Using Arbitrary Complete Lattices as Domain and Range of Fuzzy Filters

In all publications about fuzzy filters in lattices known to the author, either the lattice $\mathfrak{L}$ representing the **domain** of *fuzzy filters* (see Definition 2.1.1) or the lattice $\mathfrak{L}'$ representing the **range** of fuzzy filters are restricted more or less severely.

In [103], for instance, $\mathfrak{L}'$ is assumed to be the **real unit interval** $\mathfrak{F}$. Furthermore, B. Yuan and W. Wu do not assume a condition like Definition 2.1.1.3 assuring the *non-emptiness* of fuzzy filters. Consequently, the *empty fuzzy set* is a fuzzy filter in the sense of [103], destroying the compatibility with the two valued case.

In [32, 39], $\mathfrak{L}$ is assumed to be the lattice $\left[{L'}^U, \cap, \cup\right]$, where L' is the domain of $\mathfrak{L}'$, U is an arbitrary non-empty set and $\cap, \cup$ are defined on the basis of meet and join as in (1.17), (1.18).

A lot of the definitions and results from the literature can be reproduced for the general case that $\mathfrak{L}, \mathfrak{L}'$ are arbitrary lattices, but in some cases, special conditions are required to reproduce the results from the literature. For instance, the definition of fuzzy filter given in [32, 39] contains the condition $\mathcal{F}\left(\bar{d}\right) \sqsupseteq d$, where $\bar{d}$ denotes the *constant mapping* from U to L' such that $\bar{d}(u) = d$ for all $u \in U$. Obviously, this condition can not reproduced if $\mathfrak{L}$ is not a lattice of $\mathfrak{L}'$-fuzzy sets, and furthermore, this condition is genuinely stronger than the condition $\mathcal{F}(1) = 1$ used in Definition 2.1.1.

In chapter 2, several results from the literature are reproduced for the most general definition of fuzzy filter, and in other cases, it is pointed out which special properties of $\mathfrak{L}, \mathfrak{L}'$ are needed to achieve the results (see for instance Theorem 2.2.2 and Observation 2.2.4).

6.1.1.2 'Extensional' Definition of Supremum in the Lattice of Fuzzy Filters

The representation (2.9) of the least upper bound in the lattice of fuzzy filters has to the author's knowledge not been mentioned yet in the literature (though it might follow from the considerations in [39, section 3.2]). It is a straightforward fuzzification of the classical equation (1.16), though the proof in the fuzzy case is not completely straightforward. The proof presented in Theorem 2.2.2 requires that $\mathfrak{L}'$ is *completely distributive* wrt. $\boxed{\mathrm{D}}$. It is not clear whether this is a necessary condition in the case that $\mathfrak{L}$ is an arbitrary lattice (in the case that $\mathfrak{L}$ is a chain, it is not necessary; see Observation 2.2.4).

The essential property of the representation (2.9) is its *extensionality*: The value of $\mathcal{F} \cup \mathcal{G}$ in a certain point c can be calculated *only* from the values of $\mathcal{F}$ and $\mathcal{G}$ in certain other points. This property is vitally important for the axiomatic characterisation of *fuzzy filter-based logics* by *logics of graded truth and graded trust assessment* in Observation 4.1.2. Without the representation (2.9), the validity of axiom 11 from Definition 4.1.3 could not be established, which in turn is essential for proving the reverse characterisation in Theorem 4.1.3. As the logical axioms in Definition 4.1.3 can only 'access' the 'values' of labels at certain truth values, a representation like (2.8) of the least upper bound in the lattice of fuzzy filters would not allow to formulate an equivalent logical axiom.

6.1.1.3 Lattices of Principal Fuzzy Filters and their Embedding into the Lattice of Fuzzy Filters

Principal fuzzy filters in the sense of (2.1) are defined and studied in [39, section 4]. Some of the results presented in chapter 2 can be found there (for instance, Lemma 2.1.3 and Lemma 2.1.8 from chapter 2 correspond to [39, proposition 4.1])

For the investigations in section 2.3 on the *expansion* of one lattice by another one, a 'tool-box' of results about lattices of principal fuzzy filters, ways for *embedding* lattices of principal fuzzy filters into the corresponding lattice of *all* fuzzy filters, and *isomorphisms* between the lattices $\mathfrak{L}, \mathfrak{L}'$ and lattices of principal fuzzy filters based on these lattices is needed. This tool-box is provided in chapter 2 (in particular by Theorem 2.2.10, Theorem 2.2.11, Observation 2.2.12, and Observation 2.2.13), but the corresponding results are not found yet in the literature.

6.1.1.4 Expanding a Lattice by another Lattice

The concept of **expansion** (see Definition 2.3.1) is new and specially tailored to provide a convenient structure for *labels* in *labelled fuzzy logic* (compare section 3.4 and Definition 3.5.1).

The concept is based on fuzzy filters for obvious reasons. First, a fuzzy filter provides a combination of two lattices (in this case, a lattice of *truth values* and a lattice of *degrees of validity*) and has the property of being *monotone* which is essential for uncertainty modelling (when a formula gets more true, it gets more valid). Secondly, fuzzy filters possess a *complete lattice structure* which is essential for defining certain operations in labelled logics (compare Definition 4.3.1).

That all *principal fuzzy filters* are required to be contained in every expansion by Definition 2.3.1.1 assures embedding properties (see Proposition 2.3.1) and a minimal level of expressive power required of the label lattice for logical reasons (see for instance Theorem 4.3.3).

The fact that the identity with a sublattice of the dual lattice of all fuzzy filters is required only *up to isomorphism* in Definition 2.3.1 is for a more convenient representation of the respective labelled logics. For instance, in the special case $\mathfrak{D} = \mathfrak{B}$, the result of Corollary 2.3.3 allows to label formulae with *truth values* if validity is two-valued (see section 5.2.1).

The results about expansions given in section 2.3 mainly concern some special cases, for instance when one of the lattices under consideration is *two-valued* (Proposition 2.3.2, Proposition 2.3.4) or a chain (Proposition 2.3.6). These results immediately lead to corresponding special cases of labelled logics (see Observation 4.1.4 and sections 5.2 and 5.3).

6.1.2 Separating Degrees of Truth and Degrees of Validity

The idea of treating *many-valued truth* and *many-valued validity* as completely separate and independent concepts with correspondingly independent algebraic structures is to the author's knowledge unheard of in literature on many-valued logics (apart from special cases).

It has hopefully been demonstrated in this dissertation that the independence of these concepts can yield interesting theoretical results and offers rich expressive power with respect to applications in knowledge representation. In particular, the roles played by these concepts in logical systems are quite distinct.

In sections 1.1, 3.4, and 5.1, the relationship of and differences between degrees of truth and degrees of validity and their uses in knowledge modelling under uncertainty are discussed. Part of this discussion and parts of chapter 5 illustrating the concepts have been published by the author in [73].

Section 5.6 also demonstrates that degrees of truth and degrees of validity act on different *levels* in logical systems: While degrees of truth are located on a *lower* level and are subject to truth-theoretic, compositional logical operators, degrees of validity are located on a *higher* level and are subject to quantifying operators like semantic consequence where compositionality is never present (not even in the traditional case of two-valued validity).

6.1.2.1 Identification and Comparison of Special Cases

One of the most encouraging results of the separation of *many-valued truth* and *many-valued validity* is the fact that the most popular systems for modelling *vagueness* and *uncertainty* in logic, namely PAVELKA-*style logics* (also known as *fuzzy logic in narrow sense*, compare [84]) and *possibilistic logic*, are obtained as (the simplest possible) special cases of *logics of graded truth and graded trust assessment*.

In the special case of **two-valued validity**, a class of *logics of graded truth assessment* is obtained where formulae are labelled by filters of the truth value lattice. This class of logics is studied in section 5.2. From this class, PAVELKA-style logic is the simplest one where only *principal filters* are admitted as labels (see section 5.2.1). The most expressive logic from this class is the one where all filters are admitted as labels (see section 5.2.2). It corresponds to HÄHNLE's *regular logics* [47]. Other choices of label lattices yield logics 'between' PAVELKA-style logic and HÄHNLE's regular logics.

In the special case of **two-valued truth**, a class of *logics of graded trust assessment* is obtained where formulae are labelled by degrees of validity (but note that the order of the label lattice is the reverse of the order of the validity degree lattice). This class of logics, which corresponds to possibilistic logic, is studied in section 5.3. Note that in section 5.3, most results from [19] could be reproduced even for the general case that $\mathfrak{D}$ is an arbitrary complete lattice (in [19], $\mathfrak{D}$ is assumed to be equal to $\mathfrak{F}$).

The relationship between *many-valued logic* and *possibilistic logic* has always been of interest to the logic community, and several attempts to comparing them exist in the literature [14]. The way both are presented here as special cases of a more general concept makes a systematic comparison particularly easy. This comparison is carried out in section 5.4.

The second big advantage of defining PAVELKA-style logic and possibilistic logic as special cases of the more general concept *logic of graded truth and graded trust assessment* is that the two types of knowledge representation which are characteristic for both types of logics can be combined in one knowledge base, and even mixtures of both types of information in one single label are possible. In section 5.5.1, it is demonstrated how *vagueness* and *uncertainty* can be represented in different types of labels. Section 5.5.2 contains a small example of knowledge representation with different types of labels.

6.1.2.2 Using Arbitrary Complete Lattices for Truth Values and Validity Degrees

In most examples of logics for the representation of vagueness or uncertainty, very strong restrictions are placed on the algebraic structures which are admitted for *truth values* and *degrees of validity*. One of them is usually even *two-valued*, as explained above, but still the other one is not an arbitrary complete lattice.

For PAVELKA*'s logic*, in the beginning of [85] indeed $\mathfrak{T}$ is assumed to be an arbitrary complete lattice, but soon it is argued that $\mathfrak{T}$ has to be a chain, and further on $\mathfrak{T}$ is even restricted to be the *real unit interval* $\mathfrak{F}$. In [84], $\mathfrak{T}$ is assumed to be equal to $\mathfrak{F}$ from the outset.

In *possibilistic logic*, for the (most intensively studied) necessity-valued case (discussed in section 5.3.1.1), $\mathfrak{D}$ is assumed to be equal to $\mathfrak{F}$.

The approach of this dissertation to allow arbitrary complete lattices[1] for $\mathfrak{T}$ and $\mathfrak{D}$ has several advantages:

1. It can be investigated which are the minimal additional requirements to be placed on the respective algebraic structures for certain logical properties to hold. Characterisation results like Observation 4.1.2, Theorem 4.3.7, Proposition 4.3.8, Proposition 4.3.12, Proposition 4.3.15, and Corollary 5.3.6 would not be possible if $\mathfrak{T}$ and/or $\mathfrak{D}$ were fixed to be equal to $\mathfrak{F}$.

2. That in chapter 4, most of the basic results of mathematical logic about the model and semantic entailment relation could be reproduced even in the most general case that both $\mathfrak{T}$ and $\mathfrak{D}$ are arbitrary complete lattices is valuable as an insight into the nature of mathematical logic itself.

   In the classical case of two-valuedness, much more powerful tools are available for carrying out proofs. For instance, a classical proof by *case distinction* wrt. the cases **true** / **not**

[1]For several reasons, being a complete lattice is the absolutely minimal requirement for both structures. In both cases, the existence of a partial order and of a least upper and greatest lower bound for an arbitrary subset of the respective structure is necessary for being able to define even the most basic logical concepts.

true or **valid** / **not valid** cannot be adapted to the case that values are taken from an arbitrary lattice. The same holds for proofs which might allow many values, but assume that all values are *comparable*, i. e. the case of a **chain**.

It has turned out that for most basic properties of mathematical logic, the strong assumptions of two-valuedness or comparability are unnecessary, for *truth values* as well as for *validity degrees*. It suffices to assume the notion of *ordering* provided by a complete lattice. This insight can be considered to be a (small) contribution to the foundations of mathematical logic.

3. Obviously, admitting a larger class of algebraic structures for degrees of truth and degrees of validity offers a wider choice for applications.

   It can be argued that for the modelling of *vagueness*, the lattice of truth values should be a chain, as it is hard to conceive what it would mean for two truth values not to be *comparable*. But on the one hand, the choice of an arbitrary chain leaves the choice between *finitely many-valued logics* and *infinitely many-valued logics*. On the other hand, logics where the truth values are themselves **fuzzy sets** [94] provide simple examples for truth value structures which are **not** chains.

   For the lattice of validity degrees, a lot of scenarios are conceivable where a lattice which is not a chain is profitable for applications. The simplest example is the *Cartesian product* of two chains, for instance to store *evidence values* [3].

   Other occasions for employing lattices which are not chains can arise from **knowledge acquisition**.

   Assume that a knowledge base stems from two phases of knowledge acquisition, both with different questionnaires. On the first questionnaire, experts were asked to rate their trust in the information given on a *continuous scale* (given, for instance, by a graphical representation). On the second questionnaire, only five degrees of trust were allowed:

   NON-TRUSTWORTHY, RATHER NON-TRUSTWORTHY,
   MEDIUM TRUSTWORTHY, RATHER TRUSTWORTHY, TRUSTWORTHY,

   which are assumed to be linearly ordered. It is decided to equate NON-TRUSTWORTHY to 0 on the continuous scale, TRUSTWORTHY to 1, and MEDIUM TRUSTWORTHY to $\frac{1}{2}$ because an accumulation of choices around $\frac{1}{2}$ on the continuous scale bears evidence that the value $\frac{1}{2}$ is recognised as a "distinguished degree of trust" by the experts. Between these three points, no significant accumulation of choices is observed, so to avoid an arbitrary identification, it is decided to leave RATHER NON-TRUSTWORTHY and RATHER TRUSTWORTHY incomparable with the degrees from the continuous scale. This leads to the lattice of validity degrees sketched in Figure 6.1. There, the dotted line denotes the continuous scale. This lattice is obviously not *distributive* (not even *modular*), so it provides a good example that there are realistic cases where $\mathfrak{D}$ is not a chain and not even distributive.

   Note that in the case that $\mathfrak{T}$ is two-valued, Corollary 5.3.6.1 plays an important role wrt. this issue. In the above example of $\mathfrak{D}$ obviously the **infinite distributive law** does *not* hold, hence (by Corollary 5.3.8.1), *semantic consequence* in the corresponding *logic of graded trust* is not reducible to semantic consequence in two-valued logic (equation (5.62)).

   Still, Corollary 5.3.6 lists a lot of cases where $\mathfrak{D}$ is not a chain and still semantic consequence is reducible to semantic consequence in two-valued logic.

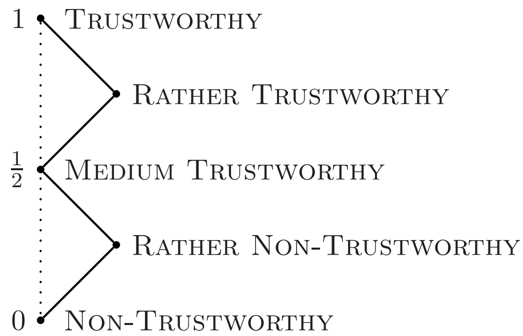


Figure 6.1: A validity degree lattice which is not a chain

Some more examples of 'non-standard' validity structures and their possible applications are listed in [19, section 4.3].

6.1.3 Development of Fuzzy Filter-Based Logics

As the concept of **fuzzy filter-based logic** is introduced in this dissertation, naturally no prior mention of it exists in the literature. In fact, definitions 4.1.1 and 4.3.1 can be seen to establish the semantics of a completely new, as yet unknown class of logical systems.

The idea of using *fuzzy sets of truth values* is mentioned at several places under different names ('truth qualifications' in L. A. Zadeh's paper [105]; 'truth value restrictions' in J. F. Baldwin's [2]). Apart from a few systems where fuzzy sets play the role of *truth values* (see for instance H. Thiele [94]; note that by the fact that those fuzzy sets form a *lattice structure*, this represents an allowed interpretation of $\mathfrak{T}$), there doesn't seem to exist a study in the context of *mathematical logic* yet.

The concept of *possibilistic logic with vague predicates* is mentioned by Dubois and Prade [19], but only very few results exist, as for *possibilistic logic with fuzzy constants* [29]. Other mentions for instance in [56] are in a completely different setting (fuzzy modal logic).

See section 5.7 for a brief survey of existing approaches to the simultaneous representation of *vagueness* and *uncertainty* in logical systems.

Some early definitions and results on fuzzy filter-based logics, mainly from chapter 4, have been published by the author in [69, 70].

6.1.3.1 Properties of the Model and Semantic Entailment Relations

Proposition 4.1.1, Proposition 4.2.1, Proposition 4.2.2, Observation 4.2.3, Observation 4.2.4, Observation 4.2.5, Theorem 4.2.6, Proposition 4.3.1, Theorem 4.3.2, Theorem 4.3.3, Proposition 4.3.4, Proposition 4.3.5, and Proposition 4.3.6 all establish semantic properties of the basic logical concepts like **model relation**, **semantic equivalence**, **semantic entailment**, **validity**, **inconsistency**. Sometimes, more general definitions for the concepts had to be chosen because of the presence of *validity degrees*, but it is easily checked that in all cases, the canonical definition was chosen.

These results represent the foundation on which the more advanced concepts like *normal forms* or *refutation* are based. Also, further developments, mainly in the area of **syntactic derivation systems** and **automated deduction**, need to make use of these basic results.

It is very interesting to note that all of these results hold unconditionally in the most general case where $\mathfrak{T}$ and $\mathfrak{D}$ are arbitrary complete lattices — it seems that the complete lattice structure as the basis of truth values and validity degrees is sufficient for most basic semantic properties of mathematical logic, though classically, much more restricted structures are used (one or both of $\mathfrak{T}$ and $\mathfrak{D}$ are two-valued).

6.1.3.2 Axiomatic Characterisation of Logics of Graded Truth and Graded Trust Assessment

Definition 4.1.2 gives a definition of a *labelled logic* called *fuzzy filter-based logic* where the label lattice is fixed to be an **expansion** of the truth value lattice by the validity degree lattice. The **graded model relation** of this logic is immediately derived from the isomorphism by means of which $\mathfrak{T}$ is expanded to $\mathfrak{L}$ by $\mathfrak{D}$.

In contrast with this definition, Definition 4.1.3 defines a *labelled logic* called *logic of graded truth and graded trust assessment* where the label lattice is characterised (apart from the fact that it is a complete lattice) solely by **axioms** on the **graded model relation**.

The latter characterisation is more intuitive because the meaning of the axioms for representing *vagueness* and *uncertainty* can be evaluated.

Observation 4.1.2 and Theorem 4.1.3 state a striking relationship between both definitions: every *logic of graded truth and graded trust assessment* is a *fuzzy filter-based logic* and if the lattices $\mathfrak{T}$ and $\mathfrak{D}$ possess certain properties (which do not represent a very severe restriction), then every *fuzzy filter-based logic* is a *logic of graded truth and graded trust assessment.*

This equivalence means that the *algebraic property* of the *label lattice* to be an expansion of $\mathfrak{T}$ by $\mathfrak{D}$ is characterised by certain *logical properties* of the resulting *graded model relation.* Note that the result of Theorem 2.2.2 is vital for this characterisation, hence the requirements placed in Observation 4.1.2.

In the case $\mathfrak{D} = \mathfrak{B}$, Observation 5.2.1 and Corollary 5.2.2 provide an even stronger characterisation. Definition 5.2.1 gives a specialised set of axioms which provides *necessary and sufficient* conditions for fuzzy filter-based logics in this special case. No restriction has to be placed on $\mathfrak{T}$ in this case because $\mathfrak{B}$ is *completely distributive* wrt. its least upper bound. Furthermore, the class of possible label lattices is characterised precisely.

In the case $\mathfrak{T} = \mathfrak{B}$, Observation 5.3.1 and Corollary 5.3.2 provide an equivalent result wrt. Definition 5.3.1. Furthermore, there is even only **one** possible label lattice (up to isomorphism) in this case.

6.1.3.3 Investigation of Normal Forms

Fortunately, the existence of a **normal form** on the underlying many-valued logic could be transferred to fuzzy filter-based logics. In Theorem 4.2.8 and Corollary 4.3.22, it is proved that any normal form on the underlying many-valued logic leads to a corresponding normal form for the labelled logics studied here.

This means that the well-developed theory of normal forms for many-valued logics can be applied without changes to fuzzy filter-based logics.

In Example 4.2.1 and Example 4.3.1, this is made concrete by establishing the well-known **clausal form** for fuzzy filter-based logics with a *lattice-based underlying many-valued logic.*

See also section 6.2.1.2 where the more advanced **layered normal form** which is applicable to a much larger class of underlying many-valued logics is mentioned.

6.1.3.4 Investigation of Refutation

Refutation, which is discussed in section 4.3.3, is a good example for the effect that a simple concept of classical logic can become quite complicated when studied in a more general system.

In classical two-valued logic, refutation means that to establish that a formula is entailed by a set of formulae is equivalent with establishing that adding the *negation* of the formula to said set of formulae makes it *inconsistent*.

For classical (non-labelled) many-valued logic, usually a refutation system does not exist, because the requirement that a formula has to assume the truth value 1 to be considered valid leads to the *dual* requirement for the negated formula to assume a truth value which is *strictly above* 0, a property which can be formalised only in a minority of all many-valued logics.

The situation is only marginally better for PAVELKA-style logics because there is no canonical method of calculating the *label* of the negated formula to be added. For some particular underlying many-valued logics, an appropriate label can be calculated under certain preconditions (compare Theorem 5.4.8.1 and [78]), but the method of calculating the label depends on the algebraic properties of the logical operators of the underlying many-valued logics.

In *fuzzy filter-based logics*, the situation becomes even more complicated because not only *truth values*, but also *degrees of validity* have to be considered. In particular, it is not sufficient to ask whether an $\mathfrak{L}$-fuzzy set of formulae is *inconsistent*, because consistency is a matter of *degree* (compare Definition 4.2.2.2, Definition 4.3.3, and Proposition 4.3.5).

Still, the expressive power of labels is high enough to provide a canonical **refutation system** (Definition 4.3.4). This definition raises two problems:

1. The definition (4.133) not always yields (the ι-image of) a label. This observation leads to studying which labels **admit refutation**.

   Theorem 4.3.7 and Proposition 4.3.8 give some results in this direction. It is analysed which properties of $\mathfrak{T}$ and $\mathfrak{D}$ assure that labels admit refutation.

   Note that from these results, it follows that 'standard' PAVELKA's logic (where truth values are taken from the real unit interval $\mathfrak{F}$) does not allow to apply the general refutation system given in Definition 4.3.4 (Observation 5.2.5 and Observation 5.2.8) while the system from Definition 4.3.4 is fully applicable in a slight generalisation of PAVELKA's logic where arbitrary filters are allowed as labels (see section 5.2.2; compare also [72]), as long as the truth value lattice is an arbitrary *chain* (Observation 5.2.9 and Corollary 5.2.10).

2. Even if a label *admits refutation*, then it is not a matter of course that *entailment* can be characterised by *refutation*.

   Theorem 4.3.10, Observation 4.3.11, Proposition 4.3.12, Theorem 4.3.13, Proposition 4.3.14, Proposition 4.3.15, Corollary 4.3.16, Corollary 4.3.17, Corollary 4.3.18, and Corollary 4.3.19 give some results in this direction. It is analysed which properties of $\mathfrak{T}$ and $\mathfrak{D}$ assure that *entailment* can be characterised by *refutation*.

   Note that the well-known refutation system for *possibilistic logic with necessity-valued formulae* (see Observation 5.3.14) is a special case of these results. A slightly less special version for arbitrary *logics of graded trust assessment* is given in Observation 5.3.10.

The two items above illustrate one benefit of choosing different and arbitrary complete lattices for $\mathfrak{T}$ and $\mathfrak{D}$: The results of section 4.3.3 provide a direct connection between properties of the corresponding lattices and properties of the refutation system established in Definition 4.3.4. As the refutation system from Definition 4.3.4 is a direct generalisation of all known refutation systems, these results can be regarded as new insights into the nature of refutation itself.

6.2 Extensions and Future Work

After carefully studying this dissertation, the reader will without doubt find that some essential subjects which have to be part of the thorough study of a new type of logical system have been neglected. As summarised in section 6.1, the following have been provided:

1. Algebraic foundations for *truth degrees*, *validity degrees*, and their fusion into **labels**, from a lattice-theoretic point of view.

2. Foundations for the study of **semantics**, in particular with respect to the central concepts of *model* and *semantic consequence*, and additional concepts like *semantic equivalence* and *refutation*.

3. Study of special cases and examples to illustrate the new concepts. Comparison of special cases.

It has hopefully become clear that the idea of differentiating between many-valued truth and many-valued validity and their possible combination has merit of providing new means for expressing uncertain and vague knowledge while at the same time possessing precisely defined semantics and preserving the basic laws of mathematical logic.

The most important subjects which have not been covered in this dissertation but are indispensable for a full account of a new type of logical system and which are needed for a reader to fully appreciate the merits of the new system are the following:

4. Further study of the **semantics of logics of graded truth and graded trust assessment**.

   So far, only the basic semantic properties of logics of graded truth and graded trust assessment have been made precise. In particular, most results given here are capable of illustrating the relationship between the lattices $\mathfrak{T}$, $\mathfrak{D}$, and $\mathfrak{L}$ and conditions to be placed on these lattices for certain properties to hold, but are largely independent of the syntax and semantics of the underlying many-valued logic. The examples in section 5.5.2 show that the underlying many-valued logic has (of course) a significant influence on the results of labelled inferences.

   The relationship between the truth value structure employed by the underlying many-valued logic (for instance, a BL-algebra or MV-algebra, compare P. Hájek [53]) and the labelled inference process has to be studied intensively.

   Another important interaction between labels and the underlying logic is the use of *variable labels* as in PLFC (see section 5.7.2). Of course, this is only possible if the underlying logic is some variant of first order logic.

   Furthermore, in this dissertation, only propositional and classical many-valued first order logic have been studied. Other interesting approaches of incorporating fuzziness or possibilistic quantifiers into the underlying many-valued logic have not been investigated yet. The literature is rich in examples of such special many-valued logics, for instance logics with *fuzzy constants* (see section 5.7.2) or fuzzy modal logics (see section 5.7.3). These systems have been mentioned only briefly here; the benefits of combining such a special logic (as an underlying many-valued logic) with the fuzzy filter-based labels employed in this dissertation should be studied.

   Finally, it has been pointed out in several places that validity degrees are basically a *measure-theoretic* concept, where in this dissertation, only the case of a *possibility measure* has been considered. Other types of measure, for instance, *probability*, *uncertainty* or *belief measures*, should be considered as an algebraic structure of degrees of validity.

5. **Syntactic derivation** and **automated deduction**.

   This is by far the most important missing subject in the study of fuzzy filter-based logics. So far, only *semantic* properties of the logics under consideration have been investigated. To obtain a tool for *knowledge representation*, however, there has to be a means of making inferences by *syntactic derivation*, and building *automated deduction* systems for processing the represented knowledge on a computer, maybe even a specialised *logic programming* language.

There have been some general preparations here like the investigation of *normal forms* (section 4.2) and *refutation* (section 4.3.3), but for establishing a *sound* and *complete syntactic derivation system* or an algorithm for *automated deduction*, of course first of all the underlying many-valued logic has to be fixed, something which has been avoided as much as possible in this dissertation.

Results for a special case exist (see [68, 72]), which can hopefully be generalised.

6. **Applications**.

   It has to be investigated what actual *applications* of the concepts developed here in knowledge representation and approximate reasoning can be. In particular, the possibility of *combining* vague and ill-known evidence stemming from different logical paradigms (Pavelka-style logics and possibilistic logic, say) is intriguing.

Some of the above-mentioned subjects are discussed further in the remainder of this section, presenting possible approaches for solving the problems at hand, but a deeper study and eventual complete solution of said problems is left for future investigations.

Note that proofs for propositions in this section will be sketched briefly or left out. Some propositions are to be considered as a 'proof of concept'; their preconditions were strengthened to yield a simpler proof. A deeper study, making the results more generally applicable, is left for future research.

6.2.1 Syntactic Derivation and Automated Deduction

Immediately after defining and justifying the semantics of a new system of logic, the most important task is to establish a **syntactic derivation system**, that is, a means of calculating semantic consequences purely by syntactic manipulations on the language of formulae.

It has hopefully become clear in section 5.5.2 that the calculation of semantic consequences by expanding the semantic definition of the concept (Definition 4.3.1) does not yield an effective method of finding all consequences of a given fuzzy set of formulae.

Hence, classically an **axiomatisation** of a given logic is a recursively enumerable procedure based on syntactically manipulating formulae, which characterises exactly the language of all semantic consequences of a given set of formulae.

For the labelled logics discussed in this dissertation, of course the sought procedure has to manipulate labelled formulae[2] to yield the fuzzy set of consequences of a fuzzy set of formulae.

Roughly, a **syntactic derivation system** consists of two parts:

1. An **axiom system**, i. e. a designated fuzzy set of formulae (with a recursively enumerable representation);
2. a (recursively enumerable) set of **inference rules**, each of which takes a finite number of *premises* (in the form of labelled formulae) and allows to derive a *conclusion* (another labelled formula).

[2] In he brief presentation of syntactic derivation in this section, problems arising from the *representation of labels* are neglected completely. Obviously, if $\mathfrak{T}$ and $\mathfrak{D}$ are sufficiently large, there is a large number of labels which do not allow for an effective finite representation. On the other hand, all labels used in examples in this dissertation obviously allow for a finite representation, and it is easy to establish subclasses of labels which allow for an effective representation and which are not left by a finite number of applications of the operations on labels discussed here. A thorough investigation of this issue is left for future research.

The process of syntactically deriving a consequence from a fuzzy set $\mathcal{X}$ of formulae then consists of an iteration of applications of inference rules, such that the premises are

- taken from $\mathcal{X}$ or
- taken from the axiom system or
- derived as conclusions in earlier steps of the derivation.

The language of derived labelled formulae is then the set of all labelled formulae which can be derived in finitely many steps in the manner sketched above.

This informal description will be made more precise in the following subsection.

What happens further with derived formulae depends on the nature of the derivation system.

In **Hilbert style derivation** or **Gentzen style derivation**, derived formulae form a counterpart to *semantic consequences*, i. e. the goal is to derive exactly those labelled formulae which follow semantically from the given fuzzy set.

Another class of derivation systems is based on **refutation**. That is, first the question whether some labelled formula is a semantic consequence of a fuzzy set of formulae is reduced to the question of what the degree of consistency of a fuzzy set of formulae is (see section 4.3.3). Then this degree is determined by syntactic derivation, i. e. the goal is to derive insatisfiable formulae with labels as large as possible. Derivation systems of this class are, for instance, based on **semantic tableaux** and the **resolution rule**.

After a derivation system has been defined, it remains to establish that it can really characterise semantic consequence. This is done in two steps.

Soundness. It has to be established that the derivation system is not too strong, i. e. nothing can be derived which is not a semantic consequence. Practically this means to establish that

1. Every labelled formula in the axiom system is *valid*, i. e. it is a semantic consequence of *every* fuzzy set of formulae (see Proposition 4.3.4.1) and
2. Every *inference rule* is sound, i. e. it will go from semantic consequences of any $\mathcal{X}$ only to semantic consequences of $\mathcal{X}$.

Completeness. It has to be established that the derivation system is not too weak, i. e. every semantic consequence can indeed be derived.

This is the hard part, and completeness results are usually very deep theorems.

In the following subsection, some sound inference rules suitable for Hilbert style derivation are presented. Resolution needs a bit more preparation. First steps towards a resolution-based system are presented in subsections 6.2.1.2–6.2.1.4.

Completeness results are not given in this section. After some hints at the necessary proceedings, the matter is left for future research.

6.2.1.1 Labelled Rules of Inference

Let a **logic of graded truth and graded trust assessment** $\Lambda = \left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=\right]$ be fixed as defined in Definition 4.1.3.

Given $n \in \mathbb{N}$, an n-ary **labelled inference rule** for Λ is a relation[3] $\mathscr{R}$ on LFrm^{n+1}.

Given labelled formulae $[x_1, \ell_1], \ldots, [x_n, \ell_n]$, if there exists

$$\left[[x_1, \ell_1], \ldots, [x_n, \ell_n], [x_{n+1}, \ell_{n+1}]\right] \in \mathscr{R},$$

[3] It is implicitly assumed that membership in this relation is recursive wrt. a suitable representation of formulae.

then $[x_{n+1}, \ell_{n+1}]$ is said to be a **conclusion** (or inference result) of $\mathscr{R}$ for $[x_1, \ell_1], \ldots, [x_n, \ell_n]$.

Usually, it is possible to denote a labelled inference rule by a scheme

$$\mathscr{R}: \boxed{\begin{array}{ll} \xi_1 & \ell_1 \\ & \vdots \\ \xi_n & \ell_n \\ \hline \zeta & \mathscr{R}_{\mathrm{L}}(\ell_1, \ldots, \ell_n) \end{array}}$$

where $\xi_1, \ldots, \xi_n, \zeta$ are 'patterns' for formulae, specifying for which types of formulae a conclusion of $\mathscr{R}$ exists, and how the formula part of the conclusion is composed from the premises. $\ell_1, \ldots, \ell_n$ are place-holders for labels, and $\mathscr{R}_{\mathrm{L}}$ is a mapping from L^n into L.

A typical example of an inference rule is **modus ponens**, which can be defined if Frm contains a binary operator symbol $\rightarrow$:

$$\mathrm{MP}: \boxed{\begin{array}{ll} A \rightarrow B & \ell_1 \\ A & \ell_2 \\ \hline B & \ell_1 * \ell_2 \end{array}}$$

The notation above means that for all formulae $x, y \in \mathrm{Frm}$, the triple $\left[[x \rightarrow y, \ell_1], [x, \ell_2], [y, \ell_1 * \ell_2]\right]$ is an element of the relation MP. What mapping $*$ is to be employed, depending on the interpretation of $\rightarrow$, is clarified later.

Definition 6.2.1 (Syntactic derivation system)
Let a fuzzy set $\mathrm{AX} \in L^{\mathrm{Frm}}$ *and a set* IR *of inference rules be given*[4]*. Then a* ***syntactic derivation operator*** $\vdash_{\mathrm{AX,IR}}$ *based on* AX *and* IR *is defined recursively as follows.*

Let $\mathcal{X} \in L^{\mathrm{Frm}}$, $n \in \mathbb{N}$ *and* $[x, \ell] \in \mathrm{LFrm}$. $[x, \ell]$ *is said to be* ***derivable*** *from* $\mathcal{X}$ *in* n *steps (denoted* $\mathcal{X} \vdash^{n}_{\mathrm{AX,IR}} [x, \ell]$*) if*

(i) $\ell = \mathcal{X}(x)$ *or*

(ii) $\ell = \mathrm{AX}(x)$ *or*

(iii) $n > 0$ *and* $\mathcal{X} \vdash^{n-1}_{\mathrm{AX,IR}} [x, \ell]$ *or*

(iv) $n > 0$ *and there exist* $m \in \mathbb{N}$ *and* $\mathfrak{x}_1, \ldots, \mathfrak{x}_m \in \mathrm{LFrm}$ *and* $\mathscr{R} \in \mathrm{IR}$ *such that*

$$\begin{array}{c} \mathcal{X} \vdash^{n-1}_{\mathrm{AX,IR}} \mathfrak{x}_1 \\ \vdots \\ \mathcal{X} \vdash^{n-1}_{\mathrm{AX,IR}} \mathfrak{x}_m \end{array}$$

and $\left[\mathfrak{x}_1, \ldots, \mathfrak{x}_m, [x, \ell]\right] \in \mathscr{R}$.

Finally, $[x, \ell]$ *is said to be* ***derivable*** *from* $\mathcal{X}$ *(denoted* $\mathcal{X} \vdash_{\mathrm{AX,IR}} [x, \ell]$*) if there exists* $n \in \mathbb{N}$ *such that* $\mathcal{X} \vdash^{n}_{\mathrm{AX,IR}} [x, \ell]$.

[4] Again, it is assumed implicitly that both sets allow for a recursively enumerable representation.

Definition 6.2.2 (Soundness and completeness)
Let the semantic entailment operator $\Vdash$ *be defined for the given logic* Λ *as in* (4.97).

1. *An* ***inference rule*** $\mathscr{R}$ *is said to be* ***sound***
   $=_{\text{def}}$ *for every* $[\mathfrak{x}_1, \ldots, \mathfrak{x}_m, \mathfrak{y}] \in \mathscr{R}$, $\mathfrak{x}_1 \cup \ldots \cup \mathfrak{x}_m \Vdash \mathfrak{y}$.

2. *A* ***syntactic derivation operator*** $\vdash_{\overline{\text{AX,IR}}}$ *(as defined above) is said to be* ***sound***
   $=_{\text{def}}$ *for every* $\mathcal{X} \in L^{\text{Frm}}$ *and* $\mathfrak{x} \in \text{LFrm}$, *if* $\mathcal{X} \vdash_{\overline{\text{AX,IR}}} \mathfrak{x}$, *then* $\mathcal{X} \Vdash \mathfrak{x}$.

3. *A* ***syntactic derivation operator*** $\vdash_{\overline{\text{AX,IR}}}$ *(as defined above) is said to be* ***complete***
   $=_{\text{def}}$ *for every* $\mathcal{X} \in L^{\text{Frm}}$ *and* $\mathfrak{x} \in \text{LFrm}$, *if* $\mathcal{X} \Vdash \mathfrak{x}$, *then* $\mathcal{X} \vdash_{\overline{\text{AX,IR}}} \mathfrak{x}$.

Observation 6.2.1 (Soundness)
A syntactic derivation operator $\vdash_{\overline{\text{AX,IR}}}$ based on an axiom system $\text{AX} \in L^{\text{Frm}}$ and a set IR of inference rules is **sound** if and only if

1. for every $x \in \text{Frm}$, $\big[x, \text{AX}(x)\big] \in \text{Valid}$ and
2. every $\mathscr{R} \in \text{IR}$ is sound. •

Proof

"if". Let $\mathcal{X} \in L^{\text{Frm}}$ and $[x, \ell] \in \text{LFrm}$ be given such that $\mathcal{X} \vdash_{\overline{\text{AX,IR}}} [x, \ell]$. Let $n \in \mathbb{N}$ be given such that $\mathcal{X} \vdash^{n}_{\overline{\text{AX,IR}}} [x, \ell]$.

It is proved by induction on n that $\mathcal{X} \Vdash [x, \ell]$.

1. If $\ell = \mathcal{X}(x)$, then $\mathcal{X} \Vdash [x, \ell]$ follows from Theorem 4.3.2.1.
2. If $\ell = \text{AX}(x)$, then $\mathcal{X} \Vdash [x, \ell]$ follows from Proposition 4.3.4.1 by the fact that $\big[x, \text{AX}(x)\big] \in \text{Valid}$.
3. The case that $n > 0$ and $\mathcal{X} \vdash^{n-1}_{\overline{\text{AX,IR}}} [x, \ell]$ is trivial by the induction hypothesis.
4. In the case that $n > 0$ and there exist $m \in \mathbb{N}$ and $\mathfrak{x}_1, \ldots, \mathfrak{x}_m \in \text{LFrm}$ and $\mathscr{R} \in \text{IR}$ such that

$$\begin{gathered} \mathcal{X} \vdash^{n-1}_{\overline{\text{AX,IR}}} \mathfrak{x}_1 \\ \vdots \\ \mathcal{X} \vdash^{n-1}_{\overline{\text{AX,IR}}} \mathfrak{x}_m \end{gathered}$$

   and $\big[\mathfrak{x}_1, \ldots, \mathfrak{x}_m, [x, \ell]\big] \in \mathscr{R}$, $\mathcal{X} \Vdash [x, \ell]$ follows from the soundness of $\mathscr{R}$ by the induction hypothesis and items 2 and 3 of Theorem 4.3.2.

"only if" Assume $\vdash_{\overline{\text{AX,IR}}}$ is sound.

Obviously, for every $x \in \text{Frm}$, $\emptyset \vdash_{\overline{\text{AX,IR}}} \big[x, \text{AX}(x)\big]$, so $\emptyset \Vdash \big[x, \text{AX}(x)\big]$, from which it follows by Proposition 4.3.4.1 that $\big[x, \text{AX}(x)\big] \in \text{Valid}$.

Furthermore, for every $[\mathfrak{x}_1, \ldots, \mathfrak{x}_m, \mathfrak{y}] \in \mathscr{R}$, $\mathfrak{x}_1 \cup \ldots \cup \mathfrak{x}_m \vdash_{\overline{\text{AX,IR}}} \mathfrak{y}$, so $\mathfrak{x}_1 \cup \ldots \cup \mathfrak{x}_m \Vdash \mathfrak{y}$ follows by the soundness of $\vdash_{\overline{\text{AX,IR}}}$. □

By the observation above, checking whether a derivation operator is sound is as simple as checking whether $\big[x, \text{AX}(x)\big] \in \text{Valid}$ for every $x \in \text{Frm}$ and checking the soundness of every individual inference rule.

Establishing completeness involves a deep theorem for all but the simplest logics. The issue of completeness will not be investigated any further in this section; it is left for future research.

Examples of valid axioms and sound inference rules are given in the remainder of this subsection.

In the introductive chapter of this dissertation, it has been promised that insights and methods of classical many-valued logic could be applied to fuzzy filter-based logics by the fact that a many-valued logic in the usual sense forms the basis of fuzzy filter-based logic.

This promise conjures the important problem of how rules of inference which are *sound* wrt. the underlying many-valued logic can be 'lifted' to the corresponding fuzzy filter-based logic.

To make the 'lifting' process as easy as possible, the inference rules for the underlying many-valued logic under consideration will be so-called **many-valued inference rules** as introduced by J. Pavelka [85] and studied, for instance, in [53, 84].

For presenting these *many-valued inference rules*, let Λ_{T} be the *logic of graded truth assessment* $[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{T}, \models]$ where $\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}$ are the same as for the given logic Λ and $\models$ is given by (5.24). In section 5.2.1, it has been demonstrated that this type of logic is equivalent with Pavelka-type logics and their successors, so the inference rules which are sound for these logics and described in [53, 84, 85] are sound for Λ_{T} also (provided the logic consisting of $\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}$ is compatible with the systems on which the inference rules are defined).

For defining the lifting procedure, one more tool is needed for extending the operations on labels taken from $\mathfrak{T}$ to labels taken from $\mathfrak{L}$. The **extension principle** is well-known from fuzzy logic for transferring an operation on the domain of fuzzy sets to the fuzzy sets themselves.

For simplicity, assume that $\mathfrak{L}$ is identical with the dual lattice $\left[\mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T}), \cup, \cap\right]$ mentioned in Definition 2.3.1 and the model relation $\models$ of Λ is defined, for $\mathrm{Val} \in \mathfrak{S}$, $x \in \mathrm{Frm}$, $\ell \in \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$, and $d \in D$, by

$$\mathrm{Val} \models_{d} [x, \ell] =_{\mathrm{def}} d = \ell\left(\mathrm{Val}(x)\right).$$

The extension principle is defined for arbitrary $\mathfrak{D}$-fuzzy sets on $\mathfrak{T}$ at first. When it preserves fuzzy filters is clarified by the next proposition.

Definition 6.2.3 (Extension)

Let $n \in \mathbb{N}$ and $\varphi : T^n \to T$. Then the **extension** *of φ to D^T is denoted $\widehat{\varphi} : \left(D^T\right)^n \to D^T$ and defined for $\mathcal{F}_1, \ldots, \mathcal{F}_n \in D^T$ and $t \in T$ by*

$$\widehat{\varphi}\left(\mathcal{F}_1, \ldots, \mathcal{F}_n\right)(t) =_{\mathrm{def}} \bigsqcup_{\mathrm{D}} \left\{\mathcal{F}_1(t_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1, \ldots, t_n \in T \text{ and } \varphi(t_1, \ldots, t_n) \sqsubseteq_{\mathrm{T}} t\right\}. \tag{6.1}$$

Proposition 6.2.2 (When does extension preserve filters?)

Let $n \in \mathbb{N}$ and $\varphi : T^n \to T$ be given. If $\mathfrak{D}$ is **completely distributive** *wrt. $\bigsqcup_{\mathrm{D}}$ and φ fulfils the following conditions:*

(i) for all $s_1, \ldots, s_n, t_1, \ldots, t_n \in T$, $\varphi\left(s_1 \sqcap_{\mathrm{T}} t_1, \ldots, s_n \sqcap_{\mathrm{T}} t_n\right) = \varphi(s_1, \ldots, s_n) \sqcap_{\mathrm{T}} \varphi(t_1, \ldots, t_n)$;

(ii) $\varphi(1, \ldots, 1) = 1$;

then for all $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$, it holds that $\widehat{\varphi}\left(\mathcal{F}_1, \ldots, \mathcal{F}_n\right) \in \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$.

Proof

Let $n \in \mathbb{N}$ and $\varphi : T^n \to T$ be given as specified above. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathfrak{D}\text{-}\mathrm{Fl}(\mathfrak{T})$.

Note that from condition (i), the **monotonicity** of φ follows.

For establishing $\widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n) \in \mathfrak{D}\text{-Fl}(\mathfrak{T})$, the conditions 1 and 3 from Definition 2.1.1 and condition 2a from Proposition 2.1.6 are checked:

$$\widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(1) = 1 \tag{6.2}$$

$$\widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(s) \sqcap_{\mathfrak{D}} \widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(t) \sqsubseteq_{\mathfrak{D}} \widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(s \sqcap_{\mathfrak{T}} t) \qquad (s,t \in T) \tag{6.3}$$

$$\text{if } s \sqsubseteq_{\mathfrak{T}} t\text{, then } \widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(s) \sqsubseteq_{\mathfrak{D}} \widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(t) \qquad (s,t \in T) \tag{6.4}$$

ad (6.2). Expanding definition (6.1) yields

$$\widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(1) = \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1,\ldots,t_n \in T \text{ and } \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} 1 \right\}.$$

By condition (ii),

$$\mathcal{F}_1(1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(1) \in \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1,\ldots,t_n \in T \text{ and } \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} 1 \right\}.$$

From $\mathcal{F}_1,\ldots,\mathcal{F}_n \in \mathfrak{D}\text{-Fl}(\mathfrak{T})$, it follows that $\mathcal{F}_1(1) = \cdots = \mathcal{F}_n(1) = 1$, hence

$$\mathcal{F}_1(1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(1) = 1 \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} 1 = 1,$$

hence

$$\bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1,\ldots,t_n \in T \text{ and } \varphi(t_1,\ldots,t_n) = 1 \right\} \sqsupseteq_{\mathfrak{T}} 1.$$

ad (6.3). Let $s,t \in T$. Expanding definition (6.1) yields

$$\begin{aligned}
&\widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(s) \sqcap_{\mathfrak{D}} \widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(t)\\
&\quad = \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(s_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(s_n) \,\middle|\, s_1,\ldots,s_n \in T \text{ and } \varphi(s_1,\ldots,s_n) \sqsubseteq_{\mathfrak{T}} s \right\}\\
&\qquad \sqcap_{\mathfrak{D}} \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1,\ldots,t_n \in T \text{ and } \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} t \right\},
\end{aligned}$$

from which it follows by the complete distributivity of $\mathfrak{D}$ wrt. $\bigsqcup_{\mathfrak{D}}$ that

$$= \bigsqcup_{\mathfrak{D}} \left\{ \begin{array}{l} \mathcal{F}_1(s_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(s_n) \\ \sqcap_{\mathfrak{D}} \mathcal{F}_1(t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(t_n) \end{array} \,\middle|\, \begin{array}{l} s_1,\ldots,s_n,t_1,\ldots,t_n \in T \\ \text{and } \varphi(s_1,\ldots,s_n) \sqsubseteq_{\mathfrak{T}} s \text{ and } \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} t \end{array} \right\},$$

from which it follows by the fact that $\mathcal{F}_1,\ldots,\mathcal{F}_n \in \mathfrak{D}\text{-Fl}(\mathfrak{T})$ that

$$= \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(s_1 \sqcap_{\mathfrak{T}} t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(s_n \sqcap_{\mathfrak{T}} t_n) \,\middle|\, \begin{array}{l} s_1,\ldots,s_n,t_1,\ldots,t_n \in T \\ \text{and } \varphi(s_1,\ldots,s_n) \sqsubseteq_{\mathfrak{T}} s \text{ and } \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} t \end{array} \right\}.$$

As $\varphi(s_1,\ldots,s_n) \sqcap_{\mathfrak{T}} \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} s \sqcap_{\mathfrak{T}} t$ follows from $\varphi(s_1,\ldots,s_n) \sqsubseteq_{\mathfrak{T}} s$ and $\varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} t$, it holds that

$$\sqsubseteq_{\mathfrak{T}} \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(s_1 \sqcap_{\mathfrak{T}} t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(s_n \sqcap_{\mathfrak{T}} t_n) \,\middle|\, \begin{array}{l} s_1,\ldots,s_n,t_1,\ldots,t_n \in T \\ \text{and } \varphi(s_1,\ldots,s_n) \sqcap_{\mathfrak{T}} \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} s \sqcap_{\mathfrak{T}} t \end{array} \right\},$$

from which it follows by assumption (i) that

$$= \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(s_1 \sqcap_{\mathfrak{T}} t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(s_n \sqcap_{\mathfrak{T}} t_n) \,\middle|\, \begin{array}{l} s_1,\ldots,s_n,t_1,\ldots,t_n \in T \\ \text{and } \varphi(s_1 \sqcap_{\mathfrak{T}} t_1,\ldots,s_n \sqcap_{\mathfrak{T}} t_n) \sqsubseteq_{\mathfrak{T}} s \sqcap_{\mathfrak{T}} t \end{array} \right\}.$$

Now, because s_i, t_i are arbitrary for all $i \in \{1,\ldots,n\}$, obviously each $s_i \sqcap_{\mathfrak{T}} t_i$ covers all T in the above quantification. Hence, it is justified to write

$$\begin{aligned}
&= \bigsqcup_{\mathfrak{D}} \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathfrak{D}} \ldots \sqcap_{\mathfrak{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1,\ldots,t_n \in T \text{ and } \varphi(t_1,\ldots,t_n) \sqsubseteq_{\mathfrak{T}} s \sqcap_{\mathfrak{T}} t \right\}\\
&= \widehat{\varphi}(\mathcal{F}_1,\ldots,\mathcal{F}_n)(s \sqcap_{\mathfrak{T}} t).
\end{aligned}$$

ad (6.4)**.** Let $s, t \in T$ such that $s \sqsubseteq_{\mathrm{T}} t$. Expanding definition (6.1), it is to be proved that

$$\bigsqcup_{\mathrm{D}} \left\{ \mathcal{F}_1(s_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \mathcal{F}_n(s_n) \,\middle|\, s_1, \ldots, s_n \in T \text{ and } \varphi(s_1, \ldots, s_n) \sqsubseteq_{\mathrm{T}} s \right\}$$
$$\sqsubseteq_{\mathrm{D}} \bigsqcup_{\mathrm{D}} \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1, \ldots, t_n \in T \text{ and } \varphi(t_1, \ldots, t_n) \sqsubseteq_{\mathrm{T}} t \right\}.$$

From $s \sqsubseteq_{\mathrm{T}} t$, it follows immediately that for all $s_1, \ldots, s_n \in T$ such that $\varphi(s_1, \ldots, s_n) \sqsubseteq_{\mathrm{T}} s$, it holds that

$$\mathcal{F}_1(s_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \mathcal{F}_n(s_n) \in \left\{ \mathcal{F}_1(t_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \mathcal{F}_n(t_n) \,\middle|\, t_1, \ldots, t_n \in T \text{ and } \varphi(t_1, \ldots, t_n) \sqsubseteq_{\mathrm{T}} t \right\},$$

establishing the result. □

Proposition 6.2.3 (Lifting many-valued inference rules to labelled inference rules)
Assume that $\mathfrak{D}$ is ***completely distributive****. Let $n \in \mathbb{N}$ and let*

$$\mathscr{R}_{\mathrm{T}} : \boxed{\begin{array}{ll} \xi_1 & t_1 \\ & \vdots \\ \xi_n & t_n \\ \hline \zeta & \varphi(t_1, \ldots, t_n) \end{array}}$$

be a ***sound many-valued inference rule*** *wrt. Λ_{T}, such that $\varphi : T^n \to T$ has the properties (i)–(ii) from* Proposition 6.2.2.

Then

$$\mathscr{R} : \boxed{\begin{array}{ll} \xi_1 & \ell_1 \\ & \vdots \\ \xi_n & \ell_n \\ \hline \zeta & \widehat{\varphi}(\ell_1, \ldots, \ell_n) \end{array}}$$

is a ***sound labelled inference rule*** *wrt. Λ.*

Proof
From Proposition 6.2.2, it follows that $\widehat{\varphi}(\ell_1, \ldots, \ell_n)$ is a label, so that $\mathscr{R}$ defines a labelled inference rule at all.

For establishing soundness, it is to be proved that for all $x_1, \ldots, x_n, y$ matching the patterns $\xi_1, \ldots, \xi_n, \zeta$ and all $\ell_1, \ldots, \ell_n \in L$, it holds that

$$[x_1, \ell_1] \cup \ldots \cup [x_n, \ell_n] \Vdash \left[y, \widehat{\varphi}(\ell_1, \ldots, \ell_n)\right],$$

which means by Definition 4.3.1.1 that

$$\operatorname{Mod}\left([x_1, \ell_1] \cup \ldots \cup [x_n, \ell_n]\right) \subseteqq \operatorname{Mod}\left(\left[y, \widehat{\varphi}(\ell_1, \ldots, \ell_n)\right]\right),$$

which means by (4.42) that

$$\operatorname{Mod}\left([x_1, \ell_1]\right) \cap \ldots \cap \operatorname{Mod}\left([x_n, \ell_n]\right) \subseteqq \operatorname{Mod}\left(\left[y, \widehat{\varphi}(\ell_1, \ldots, \ell_n)\right]\right),$$

which means by (4.41) and (4.1) and (6.1) that for every $\mathrm{Val} \in \mathfrak{S}$,

(6.5)

$$\ell_1\left(\mathrm{Val}(x_1)\right) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \ell_n\left(\mathrm{Val}(x_n)\right) \sqsubseteq_{\mathrm{D}} \bigsqcup_{\mathrm{D}} \left\{ \ell_1(t_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \ell_n(t_n) \,\middle|\, \begin{array}{l} t_1, \ldots, t_n \in T \\ \text{and } \varphi(t_1, \ldots, t_n) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(y) \end{array} \right\}.$$

For some arbitrary fixed $\mathrm{Val} \in \mathfrak{S}$, it is now proved that (6.5) holds. The assumption that $\mathscr{R}_{\mathrm{T}}$ is sound means that for all $t_1, \ldots, t_n \in T$,

$$[x_1, t_1] \cup \ldots \cup [x_n, t_n] \Vdash \left[y, \varphi(t_1, \ldots, t_n)\right],$$

which means by the same reasoning as above (taking into account the semantics of labels for Λ_{T} as defined in section 5.2.1) that for every $\mathrm{Val}' \in \mathfrak{S}$,

$$\text{if } t_1 \sqsubseteq_{\mathrm{T}} \mathrm{Val}'(x_1) \text{ and } \ldots \text{ and } t_n \sqsubseteq_{\mathrm{T}} \mathrm{Val}'(x_n)\,, \text{ then } \varphi(t_1, \ldots, t_n) \sqsubseteq_{\mathrm{T}} \mathrm{Val}'(y).$$

As $t_1, \ldots, t_n \in T$ as well as $\mathrm{Val}' \in \mathfrak{S}$ in the above equation are completely arbitrary, inserting Val for Val′ and $\mathrm{Val}(x_i)$ for t_i $(i \in \{1, \ldots, n\})$ yields

$$\varphi(\mathrm{Val}(x_1), \ldots, \mathrm{Val}(x_n)) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(y).$$

From this it immediately follows that

$$\ell_1\left(\mathrm{Val}(x_1)\right) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \ell_n\left(\mathrm{Val}(x_n)\right) \in \left\{ \ell_1(t_1) \sqcap_{\mathrm{D}} \ldots \sqcap_{\mathrm{D}} \ell_n(t_n) \,\middle|\, \begin{array}{l} t_1, \ldots, t_n \in T \\ \text{and } \varphi(t_1, \ldots, t_n) \sqsubseteq_{\mathrm{T}} \mathrm{Val}(y) \end{array} \right\},$$

establishing (6.5). □

Example 6.2.1 (Some sound many-valued inference rules and their lifted counterparts) For this example, let $\mathrm{Frm} = \mathrm{FOFrm}_{\mathrm{S}}$, i. e. employ the language of **first order logic**[5]. Let $\mathfrak{T} =_{\mathrm{def}} \mathfrak{F} = \left[\langle 0, 1\rangle, \min, \max\right]$. Let $\tau : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$ be a continuous **t-norm**, i. e. a continuous, commutative, associative, monotone function with neutral element 1 (compare [61]).

Let $\varphi_{\rightarrow}$ be the **r-implication** of τ, i. e. the mapping $\varphi_{\rightarrow} : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$ is defined for $s, t \in \langle 0, 1\rangle$ by

$$\varphi_{\rightarrow}(s, t) =_{\mathrm{def}} \inf \left\{ r \,\middle|\, r \in \langle 0, 1\rangle \text{ and } \tau(s, r) \leqq t \right\}.$$

Then the **modus ponens**

$$\mathrm{MP}_{\mathrm{T}} : \boxed{\begin{array}{ll} A \to B & t_1 \\ A & t_2 \\ \hline \quad\;\; B & \tau(t_1, t_2) \end{array}}$$

is *sound* for Λ_{T}. See [84] for details.

Other inference rules sound for Λ_{T} are **generalisation**:

$$\mathrm{GEN}_{\mathrm{T}} : \boxed{\begin{array}{rl} A & t \\ \hline \forall v A & t \end{array}}$$

[5] Individual variables, function and predicate symbols can be arbitrary.

(where v can be replaced by any individual variable) and **specialisation**:

$$\mathrm{SPC_T}: \boxed{\begin{array}{cc} \forall v A & t \\ \hline A_{v:=T} & t \end{array}}$$

(where v can be replaced by any individual variable and T by any term wrt. Frm and $A_{v:=T}$ denotes the formula obtained by replacing every **free**[6] occurrence of v in A by T).

Both τ and the identical mapping involved in $\mathrm{GEN_T}$ and $\mathrm{SPC_T}$ fulfil the properties (i)–(ii) from Proposition 6.2.2, hence if $\mathfrak{D}$ is *completely distributive*, then

$$\mathrm{MP}: \boxed{\begin{array}{cc} A \to B & \ell_1 \\ A & \ell_2 \\ \hline B & \widehat{\tau}(\ell_1, \ell_2) \end{array}}$$

and

$$\mathrm{GEN}: \boxed{\begin{array}{cc} A & \ell \\ \hline \forall v A & \ell \end{array}}$$

and

$$\mathrm{SPC}: \boxed{\begin{array}{cc} \forall v A & \ell \\ \hline A_{v:=T} & \ell \end{array}}$$

are *sound* for Λ. •

Remark
The restriction $\mathfrak{T} = \mathfrak{F}$ is not necessary for the soundness of the two rules in the example above, it just simplifies the presentation. Especially the modus ponens would require much preparation otherwise (discussion of residuated lattice-ordered monoids, see [61]). Also, τ does not necessarily have to be a continuous t-norm for $\mathrm{MP_T}$ to be sound, but establishing the criteria for the soundness of $\mathrm{MP_T}$ as well as the properties (i)–(ii) from Proposition 6.2.2 would be quite tedious. •

Example 6.2.2 (Syntactic derivation) Let Λ be given as specified in section 5.5.2, i.e. let $\mathfrak{T} = \mathfrak{D} = \mathfrak{F}$, $\mathfrak{L} = F\mathscr{F}(\mathfrak{F})$,

$$\mathrm{Frm} = \mathrm{FOFrm_S} = \mathrm{FOFrm}\,(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega_{\mathrm{S}}, \mathrm{Ar}_{\mathrm{S}})\,,$$

for a given (non-empty) set IV of individual variables, Func $=_{\mathrm{def}}$ $\{\mathtt{1}, \mathtt{inc}\}$ with $\mathrm{Ar}_{\mathrm{Func}}(\mathtt{1}) = 0, \mathrm{Ar}_{\mathrm{Func}}(\mathtt{inc}) = 1$ and Pred $=_{\mathrm{def}}$ $\{\mathtt{small}\}$ with $\mathrm{Ar}_{\mathrm{Pred}}(\mathtt{small}) = 1$. Recall the abbreviations $\mathtt{2}, \mathtt{3}, \mathtt{4}, \mathtt{5}$ for terms defined on page 180. Let $\mathfrak{S} =_{\mathrm{def}} \mathfrak{S}^{\mathrm{F}}_{\mathrm{F}}$ as defined in Example 3.2.4.2, with the exception that the interpretation of $\to$ is fixed later.

As in section 5.5.2, let the *domain* of each interpretation be fixed to be the set $\mathbb{N}$ of all natural numbers, let the interpretation of $\mathtt{1}$ be fixed to be the natural number 1 and let the interpretation of $\mathtt{inc}$ be fixed to be the *successor function* of the natural numbers.

[6] The syntactic concept of *free variables* is not defined formally here. Intuitively, the meaning should be clear. See for instance [45] for details.

For the characterisation of `small`, the fuzzy set from which inferences are drawn is the $\mathfrak{L}$-fuzzy set $\mathcal{X}_2$ of formulae employed in variants 4–6 of section 5.5.2, where for some $v \in \mathrm{IV}$,

$$\mathcal{X}_2\left(\texttt{small}(\texttt{1})\right) =_{\mathrm{def}} \ell^{\mathrm{AT}},$$
$$\mathcal{X}_2\left(\forall v\left(\texttt{small}(v) \rightarrow \texttt{small}\left(\texttt{inc}(v)\right)\right)\right) =_{\mathrm{def}} \ell_{\gtrapprox 0.9}$$

where the label $\ell_{\gtrapprox 0.9}$ is given by Figure 5.6.

Next, it is investigated to which extent the results about $\mathrm{Cons}(\mathcal{X}_2)$ derived in section 5.5.2 by purely semantic means can be reproduced by syntactic derivation.

Let $\mathrm{AX} =_{\mathrm{def}} \emptyset$ and $\mathrm{IR} =_{\mathrm{def}} \{\mathrm{SPC}, \mathrm{MP}\}$ (as defined in Example 6.2.1), i. e. for this derivation, no axioms and only the rules of *modus ponens* and *specialisation* are needed.

Note that more often than not, *specialisation* is not introduced as an inference rule, but as a logical axiom (compare [53]), like this:

$$\mathrm{AX}\left(\forall v A \rightarrow A_{v:=T}\right) =_{\mathrm{def}} \ell^{\mathrm{AT}}.$$

But this only works reliably if $\varphi_{\rightarrow}$ is the r-implication of a left-continuous t-norm, which is not the case for $\mathrm{imp}_{\mathrm{KD}}$, for instance.

For the first derivation, consider $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{G}}$. As $\mathrm{imp}_{\mathrm{G}}$ is the r-implication of min, let $\tau =_{\mathrm{def}} \min$ in the definition of MP.

Sorted by the number of steps necessary for the derivation, the following is obtained:

$$\mathcal{X}_2 \mathrel{\big|\!\frac{0}{\mathrm{AX,IR}}} \left[\texttt{small}(\texttt{1}), \ell^{\mathrm{AT}}\right] \tag{6.6}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{0}{\mathrm{AX,IR}}} \left[\forall v\left(\texttt{small}(v) \rightarrow \texttt{small}\left(\texttt{inc}(v)\right)\right), \ell_{\gtrapprox 0.9}\right] \tag{6.7}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{1}{\mathrm{AX,IR}}} \left[\left(\texttt{small}(\texttt{1}) \rightarrow \texttt{small}\left(\texttt{2}\right)\right), \ell_{\gtrapprox 0.9}\right] \qquad \text{SPC on (6.7)} \tag{6.8}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{1}{\mathrm{AX,IR}}} \left[\left(\texttt{small}(\texttt{2}) \rightarrow \texttt{small}\left(\texttt{3}\right)\right), \ell_{\gtrapprox 0.9}\right] \qquad \text{SPC on (6.7)} \tag{6.9}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{1}{\mathrm{AX,IR}}} \left[\left(\texttt{small}(\texttt{3}) \rightarrow \texttt{small}\left(\texttt{4}\right)\right), \ell_{\gtrapprox 0.9}\right] \qquad \text{SPC on (6.7)} \tag{6.10}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{2}{\mathrm{AX,IR}}} \left[\texttt{small}\left(\texttt{2}\right), \ell_{\gtrapprox 0.9}\right] \qquad \text{MP on (6.8) and (6.6)} \tag{6.11}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{3}{\mathrm{AX,IR}}} \left[\texttt{small}\left(\texttt{3}\right), \ell_{\gtrapprox 0.9}\right] \qquad \text{MP on (6.9) and (6.11)} \tag{6.12}$$

$$\mathcal{X}_2 \mathrel{\big|\!\frac{4}{\mathrm{AX,IR}}} \left[\texttt{small}\left(\texttt{4}\right), \ell_{\gtrapprox 0.9}\right] \qquad \text{MP on (6.10) and (6.12)} \tag{6.13}$$

It has to be explained how the labels in applications of MP are obtained. Consider (6.11). When applying MP, by (6.1) the new label is

$$\widehat{\min}\left(\ell_{\gtrapprox 0.9}, \ell^{\mathrm{AT}}\right)(t) = \sup\left\{\min\left(\ell_{\gtrapprox 0.9}(t_1), \ell^{\mathrm{AT}}(t_2)\right) \,\middle|\, t_1, t_2 \in \langle 0, 1\rangle \text{ and } \min(t_1, t_2) \sqsubseteq t\right\}.$$

It is easy to see that the supremum is reached in the point $\min\left(\ell_{\gtrapprox 0.9}(t), \ell^{\mathrm{AT}}(1)\right)$, hence $\widehat{\min}\left(\ell_{\gtrapprox 0.9}, \ell^{\mathrm{AT}}\right) = \ell_{\gtrapprox 0.9}$.

For (6.12),

$$\widehat{\min}\left(\ell_{\gtrapprox 0.9}, \ell_{\gtrapprox 0.9}\right)(t) = \sup\left\{\min\left(\ell_{\gtrapprox 0.9}(t_1), \ell_{\gtrapprox 0.9}(t_2)\right) \,\middle|\, t_1, t_2 \in \langle 0, 1\rangle \text{ and } \min(t_1, t_2) \sqsubseteq t\right\}$$

and the supremum is reached in the point $\min\left(\ell_{\gtrapprox 0.9}(t), \ell_{\gtrapprox 0.9}(t)\right)$, hence $\widehat{\min}\left(\ell_{\gtrapprox 0.9}, \ell_{\gtrapprox 0.9}\right) = \ell_{\gtrapprox 0.9}$. The same goes for (6.13).

For the second derivation, consider $\varphi_{\rightarrow} =_{\text{def}} \text{imp}_{\text{L}}$. imp_{L} is the r-implication of the **bold conjunction** et_{b}, which is given for $s, t \in \langle 0, 1\rangle$ by

$$\text{et}_{\text{b}}(s,t) = \max(0, s+t-1).$$

Hence, let $\tau =_{\text{def}} \text{et}_{\text{b}}$ in the definition of MP.

Sorted by the number of steps necessary for the derivation, the following is obtained:

$$\begin{array}{lll}
(6.14) & \mathcal{X}_2 \mathrel{\vdash^{0}_{\text{AX,IR}}} \left[\texttt{small}(1), \ell^{\text{AT}}\right] & \\
(6.15) & \mathcal{X}_2 \mathrel{\vdash^{0}_{\text{AX,IR}}} \left[\forall v \left(\texttt{small}(v) \rightarrow \texttt{small}\left(\texttt{inc}(v)\right)\right), \ell_{\gtrapprox 0.9}\right] & \\
(6.16) & \mathcal{X}_2 \mathrel{\vdash^{1}_{\text{AX,IR}}} \left[\left(\texttt{small}(1) \rightarrow \texttt{small}\,(2)\right), \ell_{\gtrapprox 0.9}\right] & \text{SPC on (6.15)} \\
(6.17) & \mathcal{X}_2 \mathrel{\vdash^{1}_{\text{AX,IR}}} \left[\left(\texttt{small}(2) \rightarrow \texttt{small}\,(3)\right), \ell_{\gtrapprox 0.9}\right] & \text{SPC on (6.15)} \\
(6.18) & \mathcal{X}_2 \mathrel{\vdash^{1}_{\text{AX,IR}}} \left[\left(\texttt{small}(3) \rightarrow \texttt{small}\,(4)\right), \ell_{\gtrapprox 0.9}\right] & \text{SPC on (6.15)} \\
(6.19) & \mathcal{X}_2 \mathrel{\vdash^{2}_{\text{AX,IR}}} \left[\texttt{small}\,(2), \ell_{\gtrapprox 0.9}\right] & \text{MP on (6.16) and (6.14)} \\
(6.20) & \mathcal{X}_2 \mathrel{\vdash^{3}_{\text{AX,IR}}} \left[\texttt{small}\,(3), \ell_{\gtrapprox 0.8}\right] & \text{MP on (6.17) and (6.19)} \\
(6.21) & \mathcal{X}_2 \mathrel{\vdash^{4}_{\text{AX,IR}}} \left[\texttt{small}\,(4), \ell_{\gtrapprox 0.7}\right] & \text{MP on (6.18) and (6.20)}
\end{array}$$

(where $\ell_{\gtrapprox 0.8}$ and $\ell_{\gtrapprox 0.7}$ are given in figures 5.8 and 5.9)

It has to be explained how the labels in applications of MP are obtained. Consider (6.19). When applying MP, by (6.1) the new label is

$$\widehat{\text{et}_{\text{b}}}\left(\ell_{\gtrapprox 0.9}, \ell^{\text{AT}}\right)(t) = \sup\left\{\min\left(\ell_{\gtrapprox 0.9}(t_1), \ell^{\text{AT}}(t_2)\right) \,\middle|\, t_1, t_2 \in \langle 0,1\rangle \text{ and } \text{et}_{\text{b}}(t_1,t_2) \mathrel{\underline{\mathbb{T}}} t\right\}.$$

It is easy to see that the supremum is reached in the point $\min\left(\ell_{\gtrapprox 0.9}(t), \ell^{\text{AT}}(1)\right)$, hence $\widehat{\text{et}_{\text{b}}}\left(\ell_{\gtrapprox 0.9}, \ell^{\text{AT}}\right) = \ell_{\gtrapprox 0.9}$.

For (6.20),

$$\widehat{\text{et}_{\text{b}}}\left(\ell_{\gtrapprox 0.9}, \ell_{\gtrapprox 0.9}\right)(t) = \sup\left\{\min\left(\ell_{\gtrapprox 0.9}(t_1), \ell_{\gtrapprox 0.9}(t_2)\right) \,\middle|\, t_1, t_2 \in \langle 0,1\rangle \text{ and } \text{et}_{\text{b}}(t_1,t_2) \mathrel{\underline{\mathbb{T}}} t\right\}.$$

As $\ell_{\gtrapprox 0.9}$ is monotone, $\min\left(\ell_{\gtrapprox 0.9}(t_1), \ell_{\gtrapprox 0.9}(t_2)\right)$ is highest when $t_1 = t_2$. It thus remains to find, for every $t \in \langle 0, 1\rangle$, the highest t_1 such that $t = \text{et}_{\text{b}}(t_1, t_1) = \min(0, 2t_1 - 1)$. 'Reversing' this expression yields

$$t_1 = \frac{t+1}{2},$$

hence

$$\widehat{\text{et}_{\text{b}}}\left(\ell_{\gtrapprox 0.9}, \ell_{\gtrapprox 0.9}\right)(t) = \ell_{\gtrapprox 0.9}\left(\frac{t+1}{2}\right) \mathrel{\underline{\mathbb{T}}} \ell_{\gtrapprox 0.8}.$$

Similar reasoning yields $\widehat{\mathrm{et}_{\mathrm{b}}}\left(\ell_{\gtrapprox 0.9}, \ell_{\gtrapprox 0.8}\right) = \ell_{\gtrapprox 0.7}$ for (6.21).

Next, consider $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{KD}}$. As $\mathrm{imp}_{\mathrm{KD}}$ is **not** the r-implication of a continuous t-norm, the statement made above about the soundness of MP does not hold. This case is not considered any further here. •

Remark
Note that for $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{G}}$ and $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{L}}$, exactly the same results have been derived syntactically in the above example as those produced by semantic analysis in section 5.5.2.

By the soundness of the employed derivation operator, for every derived labelled formula $\mathfrak{x}$ it has been established that $\mathcal{X}_2 \Vdash \mathfrak{x}$, for instance $\mathcal{X}_2 \Vdash \left[\mathtt{small}(4), \ell_{\gtrapprox 0.7}\right]$ in the case that $\varphi_{\rightarrow} =_{\mathrm{def}} \mathrm{imp}_{\mathrm{L}}$.

But by Observation 4.3.1.9, this only means $\ell_{\gtrapprox 0.7} \sqsubseteq \mathrm{Cons}(\mathcal{X}_2)\left(\mathtt{small}(4)\right)$. For establishing $\mathrm{Cons}(\mathcal{X}_2)\left(\mathtt{small}(4)\right) \sqsubseteq \ell_{\gtrapprox 0.7}$, the **completeness** of the derivation operator would have to be established, which is not the case yet. •

Example 6.2.3 (A complete derivation operator for logics of graded trust assessment)
For this example, let $\mathrm{Frm} = \mathrm{FOFrm}_{\mathrm{S}}$, i. e. employ the language of **first order logic**[7]. Let $\mathfrak{T} =_{\mathrm{def}} \mathfrak{B} = \left[\{0,1\}, \mathrm{and}, \mathrm{or}\right]$ and $\mathfrak{S} =_{\mathrm{def}} \mathfrak{S}_{\mathrm{B}}^{\mathrm{F}}$ as defined in Example 3.2.4.1. Let $\mathfrak{D}$ be such that (5.62) holds (compare Corollary 5.3.8). Let Λ be a logic of graded trust assessment as given by Corollary 5.3.2. Consequently, assume $\mathfrak{L} = \mathscr{D}(\mathfrak{D})$.

In [19, section 3.6], a sound and complete derivation operator for necessity-valued possibilistic logic is given which can be easily adapted to arbitrary logics of graded trust assessment under the restrictions mentioned above.

Let $\mathrm{AX} \in L^{\mathrm{Frm}}$ be given such that for all $x, y, z \in \mathrm{Frm}$, $v \in \mathrm{IV}$, and $t \in \mathrm{Term}$

$$\begin{aligned}
\mathrm{AX}\left(x \rightarrow (y \rightarrow x)\right) &= \ell_0 \\
\mathrm{AX}\left(\left(x \rightarrow (y \rightarrow z)\right) \rightarrow \left((x \rightarrow y) \rightarrow (x \rightarrow z)\right)\right) &= \ell_0 \\
\mathrm{AX}\left((\neg x \rightarrow \neg y) \rightarrow \left((\neg x \rightarrow y) \rightarrow x\right)\right) &= \ell_0 \\
\mathrm{AX}\left(\forall v(x \rightarrow y) \rightarrow (x \rightarrow \forall v\, y)\right) &= \ell_0 \qquad \text{if } v \text{ is not free in } x \\
\mathrm{AX}\left(\forall v\, x \rightarrow x_{v:=t}\right) &= \ell_0 \qquad {}^{8}
\end{aligned}$$

and $\mathrm{AX}(x) = \ell_1$ for every formula $x \in \mathrm{Frm}$ not matching any of the patterns given above.

Remark
Note that according to the notation used in section 5.3, the label ℓ_0 is the **unit element** of $\mathfrak{L}$ and ℓ_1 is the **zero element** of $\mathfrak{L}$. •

Let $\mathrm{IR} =_{\mathrm{def}} \{\mathrm{MP}, \mathrm{GEN}, \mathrm{WK}\}$ where $\mathrm{MP}, \mathrm{GEN}$ are defined in Example 6.2.1, setting $\tau =_{\mathrm{def}}$ and in MP. Note that in the special case $\mathfrak{T} = \mathfrak{B}$, $\widehat{\mathrm{and}}$ reduces to $\sqcap$.

Remark
Note that when disregarding the labels, the given axiom system together with the inference rules $\mathrm{MP}, \mathrm{GEN}$ constitutes a complete derivation operator for classical two-valued first order logic. In fact, any other axiom system and set of inference rules which are sound and complete for two-valued first order logic can be used, provided that in inference rules, labels are combined with $\sqcap$ to form the label of the rule's conclusion. •

[7] Individual variables, function and predicate symbols can be arbitrary.

[8] Concerning the notion of a free variable and the notation $x_{v:=t}$, see the remark following the definition of $\mathrm{SPC}_{\mathrm{T}}$ on page 218.

WK is the **weakening rule**

$$\mathrm{WK}: \boxed{\begin{array}{cc} A & \ell \\ \hline B & \ell' \end{array}} \quad \text{for any } \ell' \in L \text{ with } \ell' \sqsubseteq \ell$$

Conjecture 6.2.4 (Completeness)
$\vdash_{\overline{\mathrm{AX,IR}}}$ is **sound** and **complete** for the logic Λ as defined above if the set of fuzzy sets $\mathcal{X} \in L^{\mathrm{Frm}}$ considered is restricted to those for which $\mathrm{rg}\,\mathcal{X}$ is *finite*. •

Proof
No formal proof is given here, but the claim should be evident considering the following facts:

1. When disregarding the labels, the given axiom system together with the inference rules MP, GEN constitutes a complete derivation operator for classical two-valued first order logic.

2. By Corollary 5.3.4, semantic consequence in this logic of graded trust is a matter of *threshold*.

For the special case of *possibilistic logic with necessity-labelled formulae*, completeness of this derivation operator has been proved in [62].

For understanding the completeness of $\vdash_{\overline{\mathrm{AX,IR}}}$, it is helpful to visualise the fuzzy set $\mathcal{X}$ as a 'stack' made up from the family of its cuts. By Corollary 5.3.4.2, it is sufficient to do derivations exactly as in two-valued logic, only making sure that every derived formula is put on a 'level' where all formulae needed for its derivation are present. This is exactly what is achieved by the MP rule: By giving the formula derived from $[x \to y, \ell_1]$ and $[x, \ell_2]$ the label $\ell_1 \sqcap \ell_2$, it is placed on the highest level below both levels of the premises. The weakening rule WK is there to simulate the property of CUT_ℓ of containing all formulae on levels above ℓ.

As an aside note, indeed there can be no sound and complete derivation operator for Λ without restriction on the admissible fuzzy sets $\mathcal{X}$, by the fact that even in the special case of *possibilistic logic with necessity-labelled formulae*, $\Vdash$ is not compact (Theorem 5.4.1.1).

If for all $\mathcal{X} \in L^{\mathrm{Frm}}, \mathfrak{x} \in \mathrm{LFrm}$, the relation $\mathcal{X} \Vdash \mathfrak{x}$ were equivalent with $\mathcal{X} \vdash_{\overline{\mathrm{AX,IR}}} \mathfrak{x}$, compactness of $\Vdash$ would follow immediately, because $\vdash_{\overline{\mathrm{AX,IR}}}$ is trivially compact. This is established as follows: $\mathcal{X} \vdash_{\overline{\mathrm{AX,IR}}} \mathfrak{x}$ means that there exists $n \in \mathbb{N}$ such that $\mathcal{X} \vdash^{n}_{\overline{\mathrm{AX,IR}}} \mathfrak{x}$. Hence, case (i) of Definition 6.2.1 is applied less than n times in the derivation of $\mathfrak{x}$. By just joining all labelled formulae stemming from applications of case (i) of Definition 6.2.1, easily a fuzzy set $\mathcal{X}_{\mathrm{fin}}$ is obtained for which it holds that $\mathcal{X}_{\mathrm{fin}} \vdash_{\overline{\mathrm{AX,IR}}} \mathfrak{x}$ and hence $\mathcal{X}_{\mathrm{fin}} \Vdash \mathfrak{x}$. □

Of course, it is intriguing to ask whether every sound and complete Hilbert-style derivation system for a many-valued logic (see [53] for examples) can be turned into a sound and complete derivation system for the corresponding fuzzy filter-based logic by using the 'extended' version of inference rules and adding the rule WK. For establishing this result, first a many-valued equivalent to Corollary 5.3.4 would have to be proved. Carrying out this idea is left for future investigations.

6.2.1.2 Normal Forms

In Theorem 4.2.8, it has been proved that any normal form existing on formulae of the underlying many-valued logic can be transferred to the labelled formulae of fuzzy filter-based logics.

When working towards **automated deduction** with resolution, then the most interesting normal forms are **conjunctive normal form** and **clausal form**.

The result of Theorem 4.2.8 means that for all logics of graded truth and graded trust assessment the semantics of which is based purely on the lattice of truth values, all fuzzy sets of formulae can be equivalently transferred to fuzzy sets of **clauses** (see Example 4.2.1).

The subclass of fuzzy filter-based logics with lattice-based many-valued semantics covers some which are interesting and usable for applications, like all variants of LEE*'s labelled fuzzy logic* or the whole class of *logics of graded trust* (compare sections 5.3 and 5.4).

Still, for real applications, logics with more expressive power of the underlying many-valued logic are desirable. In particular, a fuzzy filter-based ŁUKASIEWICZ logic is not contained in said subclass.

In [72], the author has presented a **layered normal form** which accommodates a large subclass of underlying many-valued logics and is suitable for resolution-based inference.

It is applicable to all many-valued logics where all operators can be reduced to a t-norm, a t-conorm (see [61]) and an involutive negation such that DE MORGAN's laws hold. Roughly, it works by first converting all formulae into **negation normal form** and then *disentangling* nested formulae (which is necessary because a t-norm and t-conorm are distributive iff they are equal to the lattice connectives) by introducing new propositional constants (which are also used as labels, avoiding the need for a residuated implication).

The layered normal form is **satisfiability-equivalent** with the original formula (because of the new constants), which is preserved by 'lifting' to labelled formulae, so layered normal form should be applicable to all fuzzy filter-based logics provided the underlying many-valued logic fulfils the necessary criteria.

6.2.1.3 Refutation

The subject of refutation has been studied in detail in section 4.3.3 and discussed for special cases throughout chapter 5.

For truly employing refutation in resolution-based deduction, it is necessary to identify exactly the class of labels which admit refutation and allow semantic consequence to be characterised by refutation. It needs to be investigated what expressive power is retained when restricting labels to this class.

Furthermore, it needs to be clarified whether 'mock' refutation systems like the one for LEE's fuzzy logic (Theorem 5.4.8.1) can be devised for other logics where refutation is not attainable by the system described in Definition 4.3.4.

With respect to resolution-based derivation, it has to be pointed out that for refutation, syntactic derivation reduces to determining the **degree of satisfiability** of a fuzzy set of formulae, so a normal form which is (only) satisfiability-equivalent with the original fuzzy set, and hence *layered normal form* from the previous subsection, can be applied.

6.2.1.4 Resolution-based Derivation

Resolution in *logics of graded trust* poses no principal problem, similar to HILBERT-style reasoning (compare Example 6.2.3). This class of logics has the additional advantage of being unproblematic wrt. *clausal form* and *refutation* (see the previous two subsections). A complete resolution-based derivation operator for necessity-valued possibilistic logic was presented in [19, 22, 25].

When the underlying logic is many-valued, things get a little more complicated. A many-valued logic based on the lattice connectives (like LEEs fuzzy logic) has the advantage of admitting the usual clausal form, but is slightly problematic wrt. refutation. Resolution-based derivation for lattice-based many-valued logic has been investigated by several authors [34,63,91]. Because of the semantic simplicity of these systems, it can be expected that the existing resolution-based derivation systems can be 'lifted' to corresponding fuzzy filter-based logics.

Other many-valued logics pose severe problems for resolution-based derivation, partly because the operators are neither idempotent nor distributive. See [48] for a survey or the state of the art.

In [72], the author has presented a **resolution-based syntactic derivation operator** for a logic of graded truth based on ŁUKASIEWICZ's logic. There, *layered normal form* is used to amend the non-distributivity of the connectives. ŁUKASIEWICZ's logic has the advantage for resolution-based derivation that the law of excluded middle holds, the residual implication can be eliminated and both conjunction and disjunction can be represented in a single clause construct. Hence, it is the only many-valued logic apart from LEE's logic for which earnest attempts at resolution-based derivation exist. Other approaches can be found in [60, 78, 101], but none of them is so universal and so well suited for labelled logic as the one presented by the author in [72].

After having clarified the issues mentioned in sections 6.2.1.2 and 6.2.1.3 above, the system developed in [72] needs to be adapted to fuzzy filter-based logic based on ŁUKASIEWICZ's logic.

Next, it should be investigated how the resolution procedure can be adapted to fuzzy filter-based logics with other underlying many-valued logics.

As soon as a working **resolution-based theorem prover**, or even **logic programming language** for fuzzy filter-based logics exist, applications can truly be attempted.

A knowledge representation system without an automatic **inference engine** or **logic programming language** is useless.

6.2.1.5 Handling First Order Logic

The biggest problem with using first order many-valued logics for knowledge representation is that most of the interesting ones are not axiomatiseable (for instance ŁUKASIEWICZ's infinitely many-valued first order logic; see [89]).

Of course, some axiomatiseable first order many-valued logics exist, for instance LEE's fuzzy logic.

Still, in the interest of expressive power, the goal of finding a suitably restricted version of ŁUKASIEWICZ's first order logic which allows for (resolution-based) axiomatisation is paramount.

When working towards this goal, specific problems of first order resolution-based derivation like Skolemisation and unification have to be solved.

6.2.2 Measure-Theoretic Interpretation of Validity Degrees

In this dissertation, the algebraic structure chosen for validity degrees is that of a complete lattice, which is the most general one imaginable which allows to talk about 'higher' or 'lower' validity and allows all the necessary operations to be defined.

In fact, however, when analysing the usage of validity degrees and the most important concepts dealing with validity degrees, it seems that validity degrees are really *measure-theoretic* in nature, unlike truth values which are clearly truth-theoretic in nature.

Considering a complete lattice as a measure-theoretic object leads to the concept of **possibility measure**. Once this point of view is attained, it is intriguing to see what happens when the concept of possibility measure is replaced by another one like **probability measure** or **belief measure**.

On the one hand, this replacement needs a lot of additional preparation, because a possibility measure is so simple that in this presentation, some important measure-theoretic issues like dependence or additivity have been neglected. Taking these issues into account would complicate some definitions considerably.

On the other hand, an abstract scheme for defining logics which combine truth-theoretic with measure-theoretic concepts could open the unique possibility to obtain a holistic paradigm for defining measure-based fuzzy logics, with the special cases *possibilistic fuzzy logic*, *probabilistic fuzzy logic*, *fuzzy uncertainty logic* and others.

6.2.3 Applications

Before applications can be discussed in earnest, the tools for defining a full-fledged **knowledge representation system** on the basis of fuzzy filter-based logics have to be created. Besides a tool for defining *knowledge bases*, in particular an *inference engine* and a *logic programming language* are needed. The development of a complete resolution-based derivation operator for fuzzy filter-based logics is thus an important prerequisite for applications.

As soon as the tools are available, applications of knowledge representation using labelled formulae can be developed. These can be found wherever complex knowledge in the presence of vagueness and uncertainty has to be handled.

Some areas where this is the case are:

1. Natural language understanding.
2. Expert systems, in particular in complex domains like medicine.
3. Data mining for complex data.
4. Planning in 'natural' environments (robotics).
5. 'Thinking' agents with very irregular/unreliable communication (distributed via Internet).

6.2.4 Tasks at Hand

In the previous part of this section, several starting points for further research have been mentioned. Here, the most immediate tasks are listed, roughly in the order they could or should be approached.

Comparison between logics for uncertainty representation: Throughout chapter 5, well-known logics able to represent one or the other form of uncertainty have been described relative to the paradigm of *logics of graded truth and graded trust assessment* and compared. Section 5.4 gives a detailed comparison between one specific logic of graded truth assessment and one specific logic of graded trust assessment. In section 5.7, several other logical systems for uncertainty representation are presented and briefly compared with logics of graded truth and graded trust assessment.

Still, a lot of work is left to be done in this sector.

Some logical systems have not been described here in any depth at all, for instance *possibilistic logic with possibility and necessity valuations* (see [19]) or ZADEH*'s fuzzy logic* (see [102]), not to mention logics based on completely different types of uncertainty, like *probabilistic logics* or *belief logics*.

All in all, a large variety of **logics for dealing with uncertainty** has been developed recently and described in the literature, but without a lot of interaction or competition between them so far. It seems to be high time for taking stock and doing a deep comparison of the relative merits of different systems.

Identification of suitable label classes: Subclasses of labels which

- are effectively representable on a computer and
- admit refutation and
- have a precisely defined meaning

have to be identified as a preparation for the development of automated derivation.

Development of complete derivation systems: In section 6.2.1.1, some first approaches at developing sound and complete **syntactic derivation operators** for fuzzy filter-based logics have been presented.

These need to be developed further and led to fruitition with highest priority.

Layered normal form: The **layered normal form** developed in [72] for ŁUKASIEWICZ's labelled logic needs to be generalised and adapted to fuzzy filter-based logics.

Resolution-based derivation. The **resolution-based derivation operator** developed in [72] for ŁUKASIEWICZ's filter-labelled logic needs to be generalised and adapted to fuzzy filter-based logics. Furthermore, it is necessary to adapt the resolution procedure to other than ŁUKASIEWICZ's logic.

Logic programming and knowledge representation: The ultimate goal of developing fuzzy filter-based logic is to create a knowledge representation system capable of representing vague and uncertain knowledge in the form of labelled formulae.

But a knowledge representation system is incomplete without an *inference engine* and a *query language* or *logic programming language* for programming dynamic tasks.

As soon as resolution-based derivation works, this should be the basis for developing a logic programming language on the basis of fuzzy filter-based logic, in which the necessary tools can be implemented.

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Symbols, Notation, and Glossary of Concepts

Words in **bold face** in the lists of symbols and notations refer to concepts explained in the glossary.

List of Symbols and Variable Names

Miscellaneous symbols

$\mathbf{0}$	*Zero element* of a complete **lattice** $[L, \sqcap, \sqcup]$ — $0 =_{\text{def}} \bigsqcap L$	page 8
$\mathbf{1}$	*Unit element* of a complete **lattice** $[L, \sqcap, \sqcup]$ — $1 =_{\text{def}} \bigsqcup L$	page 8
$\bot$	Special **formula** such that for all $\mathrm{Val} \in \mathfrak{S}$, $\mathrm{Val}(\bot) = 0$, wrt a given **semantics** $\mathfrak{S}$	page 96
$\varnothing$	*Empty Set*	page 7
$\not{\phi}$	Empty $\mathfrak{L}$-fuzzy set on a universe U — for every $u \in U$, $\not{\phi}(u) =_{\text{def}} 0$	page 12
$\mathbb{0}$	**Zero element** in the complete lattice $\left[\mathfrak{L}'\text{-Fl}\left(\mathfrak{L}\right), \cap, \uplus\right]$ of all **$\mathfrak{L}'$-fuzzy filters** of $\mathfrak{L}$ — $\mathbb{0}(a) = 0, \text{if } a \neq 1; \mathbb{0}(a) = 1, \text{if } a = 1$ (wrt $a \in L$)	page 24
$\mathbb{1}$	**Unit element** in the complete lattice $\left[\mathfrak{L}'\text{-Fl}\left(\mathfrak{L}\right), \cap, \uplus\right]$ of all **$\mathfrak{L}'$-fuzzy filters** of $\mathfrak{L}$ — $\mathbb{1}(a) = 1$ for all $a \in L$	page 24

Unary operators

$\neg$	Symbol for logical *negation* operator	page 42

Binary operators

$\boxed{\sqcap}$	*Meet* in the complete lattice $\mathfrak{D} = [D, \boxed{\sqcap}, \boxed{\sqcup}]$ of *degrees of trust* or *validity*	page 53, 59
$\boxed{\sqcup}$	*Join* in the complete lattice $\mathfrak{D} = [D, \boxed{\sqcap}, \boxed{\sqcup}]$ of *degrees of trust* or *validity*	page 53, 59
$\cap$	Set intersection. When applied to **fuzzy sets**, meet in the complete lattice $\left[L^U, \cap, \cup\right]$ of all **$\mathfrak{L}$-fuzzy sets** on U — $(\mathcal{F} \cap \mathcal{G})(u) =_{\text{def}} \mathcal{F}(u) \sqcap \mathcal{G}(u)$ wrt the complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$	page 12
$\cup$	Set union. When applied to **fuzzy sets**, join in the complete lattice $\left[L^U, \cap, \cup\right]$ of all **$\mathfrak{L}$-fuzzy sets** on U — $(\mathcal{F} \cup \mathcal{G})(u) =_{\text{def}} \mathcal{F}(u) \sqcup \mathcal{G}(u)$ wrt the complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$	page 12
$\circledast$	Join in the lattice of principal **fuzzy filters** — For $\mathcal{P}, \mathcal{P}' \in \mathfrak{L}'\text{-PFl}\left(\mathfrak{L}\right)$ wrt complete lattices $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$, $\mathcal{P} \circledast \mathcal{P}' =_{\text{def}} {}^{\left(\delta(\mathcal{P}) \curlyvee \delta(\mathcal{P}')\right)}\overline{\alpha(\mathcal{P}) \sqcap \alpha(\mathcal{P}')}$	page 26

$\dot\cup$ Join in the complete lattice $\left[\mathrm{Fl}(\mathfrak{L}), \cap, \dot\cup\right]$ of all **filters** of $\mathfrak{L}$ —
$F \dot\cup G =_{\mathrm{def}} \{c | c \in L$ and there are $a \in F$, $b \in G$ such that $a \sqcap b \sqsubseteq c\}$.
Join in the complete lattice $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \dot\cup\right]$ of all **$\mathfrak{L}'$-fuzzy filters** of $\mathfrak{L}$ —
$\mathcal{F} \dot\cup \mathcal{G} =_{\mathrm{def}} \bigcap \left\{\mathcal{H} \middle| \mathcal{H} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L})\right.$ and $\left.\mathcal{F} \cup \mathcal{G} \subseteqq \mathcal{H}\right\}$ .... page 10, 21

$\sqcap$ **Lattice** *meet* .... page 8

$\sqcup$ **Lattice** *join* .... page 8

$\longrightarrow$ Symbol for logical *implication* operator .... page 42

$\wedge$ Symbol for logical *conjunction* operator .... page 42

$\vee$ Symbol for logical *disjunction* operator .... page 42

$\sqcap_T$ *Meet* in the complete lattice $\mathfrak{T} = [T, \sqcap_T, \sqcup_T]$ of *truth values* .... page 43

$\sqcup_T$ *Join* in the complete lattice $\mathfrak{T} = [T, \sqcap_T, \sqcup_T]$ of *truth values* .... page 43

$\sqcap_L$ *Meet* in the complete lattice $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ of *labels* for **logical formulae** .... page 57

$\sqcup_L$ *Join* in the complete lattice $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ of *labels* for **logical formulae** ..... page 57

Equivalence relations

$\cong$ Relation of *consistency-equivalence* of $\mathfrak{L}$-fuzzy sets of **formulae**, wrt a label lattice $\mathfrak{L}$
— for $\mathcal{X}, \mathcal{Y} \in L^{\mathrm{Frm}}$, $\mathcal{X} \cong \mathcal{Y} =_{\mathrm{def}} \mathrm{cst}(\mathcal{X}) = \mathrm{cst}(\mathcal{Y})$ .... page 79

$\equiv$ Relation of *semantic equivalence* for logical **formulae** and *sets of formulae* — for
$x, y \in \mathrm{Frm}$, $x \equiv y$ iff for every $\mathrm{Val} \in \mathfrak{S}$, $\mathrm{Val}(x) = \mathrm{Val}(y)$; for $X, Y \subseteqq \mathrm{Frm}$,
$X \equiv Y$ iff for every $\mathrm{Val} \in \mathfrak{S}$, $\underset{x \in X}{\sqcap_T} \mathrm{Val}(x) = \underset{y \in Y}{\sqcap_T} \mathrm{Val}(y)$.
Also used to denote the *semantic equivalence* of $\mathfrak{L}$-fuzzy sets of **formulae**, wrt a label lattice $\mathfrak{L}$ — for $\mathcal{X}, \mathcal{Y} \in L^{\mathrm{Frm}}$, $\mathcal{X} \equiv \mathcal{Y} =_{\mathrm{def}} \mathrm{Mod}(\mathcal{X}) = \mathrm{Mod}(\mathcal{Y})$ .... page 49, 79

Partial order relations

$\sqsubseteq_D$ *Induced partial order* of the complete lattice $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$ of *degrees of trust* or *validity* .... page 59

$\subseteqq$ Subset relation. When applied to **fuzzy sets**, induced subset relation of the complete lattice $\left[L^U, \cap, \cup\right]$ of all **$\mathfrak{L}$-fuzzy sets** on U —
$\mathcal{F} \subseteqq \mathcal{G} =_{\mathrm{def}} (\mathcal{F}(u) \sqsubseteq \mathcal{G}(u)$ for every $u \in U)$ wrt the complete lattice
$\mathfrak{L} = [L, \sqcap, \sqcup]$ .... page 12

$\leqq$ Standard order "less than or equal" of the real numbers .... page 20

$\sqsubseteq$ *Partial order relation* induced by a **lattice** .... page 8

$\Subset$ Denotes the relation of being a *sublattice* .... page 9

$\sqsubseteq_T$ *Induced partial order* of the complete lattice $\mathfrak{T} = [T, \sqcap_T, \sqcup_T]$ of *truth values* .... page 59

$\sqsubseteq_L$ Induced partial order (strength) of the complete lattice $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ of *labels* for **logical formulae** .... page 57

Roman letters

c, d, e Denote *validity degrees* from a complete lattice $\mathfrak{D}$ .... page 61

c Denotes a *clause* in **propositional logic** .... page 86

D_{Val} Special subset of the set D of *degrees of validity* in a *logic of graded trust* — Given $\mathrm{Val} \in \mathfrak{S}$,
$D_{\mathrm{Val}} =_{\mathrm{def}} \left\{d \middle| \text{There exists } y \in \mathrm{Frm} \text{ such that } \mathrm{Val} \not\models y \text{ and } \ell_d \sqsubseteq \mathcal{X}(y)\right\}$ .... page 143

D Denotes the set of all *degrees of trust* or *validity*.
Also used to denote the set $D \subseteqq T$ of all *designated truth values* in many-valued logic .... page 52, 59

d Denotes the usual *metric* of the real line — $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $d(r, s) = |r - s|$ .... page 158

F, G, H Denote **filters** of a **lattice** .... page 9

f Denotes a *mapping*.
Also used to denote a *function symbol* in **first order predicate logic** .... page 7, 40

h Denotes a *lattice homomorphism* .... page 34

L Denotes the *domain* of a **lattice**
Also used to denote the set of all **labels** .... page 8, 57

l Denotes a *literal* in **propositional logic** — given a set PV of *propositional variables* and a *negation symbol* $\neg$, $l \in \mathrm{PV}$ or there exists $p \in \mathrm{PV}$ such that $l = \neg p$ .... page 86

N Denotes a *necessity measure* on Frm .... page 152

n Denotes a *natural number* from $\mathbb{N}$ .... page 156

p, q Denote *propositional variables* .... page 42

p Denotes a *predicate symbol* in **first order predicate logic** .... page 41

r, s Denote *real numbers* from the *real unit interval* $\langle 0, 1\rangle$ .... page 156

s, t Denote *truth values* from a complete lattice $\mathfrak{T}$ .... page 43

T Denotes the set of all *truth values* .... page 43

t Denotes a *term* in **first order predicate logic** .... page 40

U Denotes the universe of a **fuzzy set** (U is an arbitrary non-empty set).
Also used to denote the *domain* of an *interpretation* in **first order predicate logic** and **PLFC** .... page 11, 44, 196

u Denotes an element of a universe U .... page 11

v, w Denote *individual variables* of **first order predicate logic** .... page 41

X, Y Denote sets of logical **formulae** .... page 49

$X_{\equiv}$ *Quotient set* of the set X of **formulae** wrt. the relation $\equiv$ of *semantic equivalence* — $X_{\equiv} =_{\mathrm{def}} \left\{[x] \cap X \middle| x \in X\right\}$. .... page 50

$\boldsymbol{x, y, z, x_1, x_2, \ldots}$ Denote logical **formulae** .... page 40

$\boldsymbol{x_t}$ A special **formula**; for every *truth value* t, it is assumed that there exists a *truth valuation* Val_t and a formula x_t such that $\mathrm{Val}_t(x_t) = t$ (see (3.2)). .... page 43

$\boldsymbol{x_{\mathrm{Cnf}}}$ Given $x \in \mathrm{PFrm_S}$, $x_{\mathrm{Cnf}} \in \mathrm{Cnf}$ denotes a formula in *conjunctive normal form* such that $x \equiv x_{\mathrm{Cnf}}$ (which exists by Proposition 4.2.10.1) .... page 87

Script letters

$\boldsymbol{\ell}$ Denotes a **label** from a *label lattice* $\mathfrak{L}$ .... page 56

$\boldsymbol{\ell^t}$ Special **label** corresponding to the principal **fuzzy filter** ${}^1\overline{t}$, i.e. $\ell^t = \ell^t_1$. See Definition 5.2.1.8 .... page 128

$\boldsymbol{\ell^t_d}$ Special **label** corresponding to the principal **fuzzy filter** ${}^d\overline{t}$. See Definition 4.1.3.8 .... page 66, 174

$\boldsymbol{\ell_d}$ Special **label** corresponding to the principal **fuzzy filter** ${}^d\overline{0}$, i.e. $\ell_d = \ell^0_d$. See Definition 5.3.1.8.
Called Doubted to degree d in section 5.5.1 because it represents the degree of trust d .... page 138, 173

$\boldsymbol{\ell_{>t}}$ Special **label** called True to more than t. $\ell_{>t}$ (for $t \in T \setminus \{1\}$) corresponds to the fuzzy filter $\mathcal{F}$ defined for $t' \in T$ by $\mathcal{F}(t') =_{\mathrm{def}} \begin{cases} 1, & \text{if } t \sqsubseteq_{\mathfrak{T}} t' \text{ and } t \neq t' \\ 0, & \text{if } t = t' \text{ or not } t \sqsubseteq_{\mathfrak{T}} t' \end{cases}$ .... page 172

$\boldsymbol{\ell_{\geqq t}}$ Special **label** called True at least to t. $\ell_{\geqq t}$ represents the truth value t and is defined by $\ell_{\geqq t} =_{\mathrm{def}} \ell^t_1$ .... page 171

$\boldsymbol{\ell^{\mathrm{AT}}}$ The strongest **label**, called Absolutely True. $\ell^{\mathrm{AT}} =_{\mathrm{def}} \ell^1_1$ .... page 169

$\boldsymbol{\ell^{\mathrm{T}}_d}$ Special **label** called True with doubt d. $\ell^{\mathrm{T}}_d =_{\mathrm{def}} \ell^{\mathrm{T}} \sqcap_{\mathfrak{L}} \ell_d$ .... page 176

$\boldsymbol{\ell^{\mathrm{T}}_{\geqq t}}$ Special **label** called True above t. $\ell^{\mathrm{T}}_{\geqq t} =_{\mathrm{def}} \ell^{\mathrm{T}} \sqcap_{\mathfrak{L}} \ell_{\geqq t}$ .... page 177

$\boldsymbol{\ell^{\mathrm{T}}}$ Special **label** called True. ℓ^{T} stands for an exact correspondence between a formula's truth value and validity degree and is defined by $\ell^{\mathrm{T}}(t) =_{\mathrm{def}} t$ (only applicable if $\mathfrak{T} = \mathfrak{D}$) .... page 175

$\boldsymbol{\ell^{\mathrm{U}}}$ The weakest **label**, called Unknown. $\ell^{\mathrm{U}} =_{\mathrm{def}} \ell^1_0$ .... page 169

$\mathscr{T}$ Denotes an *operator* $\mathscr{T} : X \to \mathfrak{P}\mathrm{Frm}$ wrt $X \subseteqq \mathrm{Frm}$ .... page 50

$\mathscr{T}_{\mathbf{Cls}}$ Denotes a *syntax transformation operator* $\mathscr{T}_{\mathrm{Cls}} : \mathrm{PFrm_S} \to \mathfrak{P}\mathrm{PFrm_S}$ for transforming into *clausal form* .... page 87

Calligraphic letters (used for fuzzy sets)

$\boldsymbol{\mathcal{A}, \mathcal{B}}$ Assignments $\mathcal{A}, \mathcal{B} : \mathrm{PV} \to T$ of *truth values* to *propositional variables* .... page 43

$\boldsymbol{\mathcal{D}}$ Denotes the fuzzy set $\mathcal{D} : T \to D$ associating with every truth value its *degree of designation*, for a given complete lattice $\mathfrak{T} =_{\mathrm{def}} [T, \sqcap_{\mathfrak{T}}, \sqcup_{\mathfrak{T}}]$ of *truth values* and a given complete lattice $\mathfrak{D} = [D, \sqcap_{\mathfrak{D}}, \sqcup_{\mathfrak{D}}]$ of *validity degrees* .... page 54

$\boldsymbol{\mathcal{F}, \mathcal{G}, \mathcal{H}}$ Denote **fuzzy sets** or **fuzzy filters** .... page 11, 15

$\mathcal{F}_\ell$ Fuzzy filter calculated from a label ℓ for *refutation.* If $\mathcal{F}_\ell$ is in $\operatorname{rg}\iota$ for the given *fuzzy filter-based logic*, ℓ is said to *admit refutation.* See Definition 4.3.4 .... page 100

$\mathcal{P}$ Denotes a *principal fuzzy filter* .... page 25

$\mathcal{S}$ Denotes a $\mathfrak{D}$-fuzzy set on $\mathfrak{S}$, for a given complete lattice $\mathfrak{D}$ of *degrees of validity* and a given **semantics** $\mathfrak{S}$ .... page 93

$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ Denote $\mathfrak{L}$-fuzzy sets of formulae from L^{Frm} .... page 74

$\mathcal{X}_{\mathrm{Cls}}$ Given $\mathcal{X} \in L^{\mathrm{Frm}}$, $\mathcal{X}_{\mathrm{Cls}} \in L^{\mathrm{Cls}}$ denotes an $\mathfrak{L}$-fuzzy set of formulae in *clausal form* such that $\mathcal{X} \equiv \mathcal{X}_{\mathrm{Cls}}$ (which exists by Proposition 4.2.10.5) .... page 87

$\mathcal{X}_{\mathrm{Cnf}}$ Given $\mathcal{X} \in L^{\mathrm{Frm}}$, $\mathcal{X}_{\mathrm{Cnf}} \in L^{\mathrm{Cnf}}$ denotes an $\mathfrak{L}$-fuzzy set of formulae in *conjunctive normal form* such that $\mathcal{X} \equiv \mathcal{X}_{\mathrm{Cnf}}$ (which exists by Proposition 4.2.10.4) .... page 87

$\mathcal{X}_{\mathrm{fin}}$ Denotes a *finite* $\mathfrak{L}$-fuzzy set of formulae, i. e. $\mathcal{X}_{\mathrm{fin}} : \mathrm{Frm} \to L$ such that $\operatorname{supp}\mathcal{X}_{\mathrm{fin}}$ is *finite* .... page 156

Fraktur letters (used for higher order objects, algebrae and tuples)

$\mathfrak{B}$ Two-valued Boolean **lattice** — $\mathfrak{B} =_{\mathrm{def}} \left[\{0,1\}, \mathrm{and}, \mathrm{or}\right]$ .... page 11

$\mathfrak{D}$ Lattice $\mathfrak{D} = [D, ⩀, ⩁]$ (with induced partial order ⊑) of *degrees of trust* or *validity* .... page 33, 39, 53, 56

$\mathfrak{F}$ Complete **lattice** formed by the real unit interval — $\mathfrak{F} =_{\mathrm{def}} \left[\langle 0,1\rangle, \min, \max\right]$ page 11

$\mathfrak{I}$ Interpretation $[U, \Pi, \Phi]$ for a **first order language** $\mathrm{Frm} = \mathrm{FOFrm}(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\mathrm{Pred}}, \Omega, \mathrm{Ar})$, where U is an arbitrary non-empty set, $\Pi : \mathrm{Pred} \to \bigcup \{\mathfrak{P}U^n \,|\, n \in \mathbb{N}\}$ such that for every $p \in \mathrm{Pred}$, $\Pi(p) \in \mathfrak{P}U^{\mathrm{Ar}_{\mathrm{Pred}}(p)}$, and $\Phi : \mathrm{Func} \to \bigcup \left\{ U^{U^n} \,\middle|\, n \in \mathbb{N} \right\}$ such that for every $f \in \mathrm{Func}$, $\Phi(f) \in U^{U^{\mathrm{Ar}_{\mathrm{Func}}(f)}}$.
Also used for interpretations in PLFC (see Definition 5.7.2). .... page 44, 196

$\mathfrak{L}$ Denotes a **lattice** $[L, \sqcap, \sqcup]$.
Also used for the lattice $[L, ⩀, ⩁]$ (with induced partial order ⊑) of *labels* for logical **formulae** .... page 8, 33, 39, 56

$\mathfrak{S}$ Many-valued **semantics** for some given set Frm of **formulae** wrt a given lattice $\mathfrak{T} =_{\mathrm{def}} [T, ⩀, ⩁]$ of *truth values*, defined to be an arbitrary set $\mathfrak{S} \subseteq T^{\mathrm{Frm}}$ of *valuation functions* $\mathrm{Val} : \mathrm{Frm} \to T$ .... page 43

$\mathfrak{S}^{\mathbf{F}}_{\mathbf{B}}$ The semantics of classical Boolean **first order predicate logic**, where the operator symbols are interpreted as in Example 3.2.4.1. .... page 48

$\mathfrak{S}^{\mathbf{F}}_{\mathbf{F}}$ The semantics of Lee's fuzzy first order logic, where the operator symbols are interpreted as in Example 3.2.4.2. .... page 48

$\mathfrak{S}^{\mathbf{P}}_{\mathbf{B}}$ The semantics of classical Boolean **propositional logic** — $\mathfrak{S}^{\mathrm{P}}_{\mathrm{B}} =_{\mathrm{def}} \left\{ \mathrm{Val}_{\mathcal{A}} \,\middle|\, \mathcal{A} : \mathrm{PV} \to \{0,1\} \right\}$, where the operator symbols are interpreted as in Example 3.2.4.1. .... page 48

$\mathfrak{S}^{\mathbf{P}}_{\mathbf{F}}$ The semantics of Lee's fuzzy propositional logic — $\mathfrak{S}^{\mathrm{P}}_{\mathrm{F}} =_{\mathrm{def}} \left\{ \mathrm{Val}_{\mathcal{A}} \,\middle|\, \mathcal{A} : \mathrm{PV} \to \langle 0,1\rangle \right\}$, where the operator symbols are interpreted as in Example 3.2.4.2. .... page 48

$\mathfrak{T}$ Lattice $\mathfrak{T} = [T, \boxed{\sqcap}, \boxed{\sqcup}]$ (with induced partial order $\boxed{\sqsubseteq}$) of *truth values* .. page 33, 39, 43

$\mathfrak{X}, \mathfrak{Y}$ Denote *sets* $\mathfrak{X} \subsetneqq L^{\mathrm{Frm}}$ of $\mathfrak{L}$-fuzzy sets of formulae .... page 76

$\mathfrak{x}, \mathfrak{y}$ Denote **labelled formulae** .... page 57

Doublestroke letters

$\mathbb{N}$ Set of all *natural numbers* — $\mathbb{N} =_{\mathrm{def}} \{0, 1, \ldots\}$ .... page 7

$\mathbb{R}$ Set of all *real numbers* .... page 7

Greek letters

$\boldsymbol{\alpha}$ A mapping from ${L'}^L$ to L, wrt complete lattices $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$ — for $\mathcal{F} \in {L'}^L$, $\alpha(\mathcal{F}) =_{\mathrm{def}} \bigsqcap \left\{ b \middle| b \in L \text{ and } \mathcal{F}(b) \neq 0 \right\}$. α is supposed to yield the parameter $a \in L$ of the principal fuzzy d-filter ${}^{d}\overline{a}$, for $d \in L'$ .... page 25

$\boldsymbol{\Delta}$ Denotes a set of sets of *degrees of validity* .... page 143

$\boldsymbol{\delta}$ A mapping from ${L'}^L$ to L', wrt complete lattices $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$ — for $\mathcal{F} \in {L'}^L$, $\delta(\mathcal{F}) =_{\mathrm{def}} \bigcurlyvee \left\{ \mathcal{F}(b) \middle| b \in L \setminus \{1\} \right\}$. δ is supposed to yield the parameter $d \in L'$ of the principal fuzzy d-filter ${}^{d}\overline{a}$ of $a \in L$ .... page 25

$\boldsymbol{\Gamma_{\mathrm{F}}}$ Denote a mapping which assigns fuzzy sets to fuzzy constant symbols in **PLFC** .... page 196

$\boldsymbol{\Gamma_{\mathrm{I}}}$ Denote a mapping which assigns individuals to individual constant symbols in **PLFC** .... page 196

$\boldsymbol{\iota}$ Denotes a *lattice isomorphism* .... page 33

$\boldsymbol{\Lambda}$ Denotes a **labelled logic**. In this dissertation, a labelled logic is defined by a *tuple* containing everything needed for a complete characterisation of the logic. In every case, this includes a *logical language* Frm, a **semantics** $\mathfrak{S}$, and a **label lattice** $\mathfrak{L}$. Most logics also include a *truth value lattice* $\mathfrak{T}$ (not in *logics of graded trust assessment*) and a *validity degree lattice* $\mathfrak{D}$ (not in *logics of graded truth assessment*). The following types of logics are defined in this dissertation:
Fuzzy filter-based logics $\Lambda = [\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \iota]$. ι is a lattice isomorphism such that $\mathfrak{T}$ is **expanded** to $\mathfrak{L}$ by $\mathfrak{D}$, by means of ι. See Definition 4.1.2.
Logics of graded truth and graded trust assessment $\Lambda = \left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=\right]$. $\models\!\!=$ is the (graded) model relation (characterised by axioms). See Definition 4.1.3.
Logics of graded truth assessment $\Lambda = \left[\mathrm{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models\right]$. $\models$ is the (non-graded) model relation (characterised by axioms). See Definition 5.2.1.
Logics of graded trust assessment $\Lambda = \left[\mathrm{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=\right]$. $\models\!\!=$ is the (graded) model relation (characterised by axioms). See Definition 5.3.1. .... page 65, 66, 128, 138

$\boldsymbol{\Lambda_{\mathrm{L}}}$ Denotes the **labelled logic** characterising LEE's fuzzy logic with truth value-labelled formulae — $\Lambda_{\mathrm{L}} =_{\mathrm{def}} \left[\mathrm{PFrm}_{\mathrm{S}}, \mathfrak{F}, \mathfrak{S}^{\mathrm{P}}_{\mathrm{F}}, \mathfrak{F}, \overset{\mathrm{L}}{\models}\right]$. .... page 135

$\boldsymbol{\Lambda_{\mathrm{P}}}$ Denotes the **labelled logic** characterizing possibilistic logic with necessity-valued formulae — $\Lambda_{\mathrm{P}} =_{\mathrm{def}} \left[\mathrm{PFrm}_{\mathrm{S}}, \mathfrak{S}^{\mathrm{P}}_{\mathrm{B}}, \mathfrak{F}, \mathfrak{F}, \overset{\mathrm{P}}{\models\!\!=}\right]$. .... page 151

$\boldsymbol{\nu_{\mathfrak{D}}}$ Denotes an order-reversing, involutive mapping on the lattice $\mathfrak{D}$ of *validity degrees*. See Definition 4.3.4 .... page 100

$\nu_{\mathfrak{T}}$ Denotes an order-reversing, involutive mapping on the lattice $\mathfrak{T}$ of *truth values*. See Definition 4.3.4 .... page 100

ν Denotes a complementation on a lattice. .... page 9

ω Denotes a *logical operator symbol* .... page 40, 41

Ω The set of *operator symbols* or *connectives* for **logical formulae** .... page 39, 40

Ω_{S} The *standard* set of *logical operator symbols* .... page 42

Φ_{Δ} For a given set Δ of *sets*, Φ_{Δ} denotes the set of all *choice functions* for Δ, i. e. the set of all mappings $\varphi : \Delta \to \bigcup \Delta$ such that for every $D \in \Delta$, $\varphi(D) \in D$. .... page 145

Φ Denotes a set of **fuzzy sets**.
Also used to denote a mapping which assigns functions to function symbols in **first order predicate logic** .... page 12, 44

Π Denotes a mapping which assigns *fuzzy relations* to predicate symbols in **first order predicate logic** and **PLFC** .... page 44, 196

ρ Denotes the uniform metric on $\langle 0,1\rangle^{\mathrm{PV}}$ — $\rho : \langle 0,1\rangle^{\mathrm{PV}} \times \langle 0,1\rangle^{\mathrm{PV}} \to \mathbb{R}$ with $\rho(\mathcal{A},\mathcal{B}) =_{\mathrm{def}} \mathrm{Sup}\left\{ d\left(\mathcal{A}(p),\mathcal{B}(p)\right) \middle| p \in \mathrm{PV}\right\}$ .... page 158

σ Denotes an *assignment* $\sigma : \mathrm{IV} \to U$ of elements of the domain to individual variables in **first order predicate logic** and **PLFC** .... page 44, 196

φ_{ω} *Truth value function* $\varphi_{\omega} : T^{\mathrm{Ar}(\omega)} \to T$ associated with operator symbol ω in propositional logic. .... page 43

Multiletter names

$\mathrm{Ar}_{\mathrm{Func}}$ A mapping from Func to $\mathbb{N}$ giving the *arity* of each *function* in **first order predicate logic** .... page 40

$\mathrm{Ar}_{\mathrm{Pred}}$ A mapping from Pred to $\mathbb{N}$ giving the *arity* of each *predicate* in **first order predicate logic** .... page 40

Ar A mapping from Ω to $\mathbb{N}$ giving the *arity* of each *operator symbol* in **logical formulae** .... page 39, 40

Ar_{S} Defines the arities of the symbols from Ω_{S} .... page 42

Cls Denotes the set $\mathrm{Cls} \subseteqq \mathrm{PFrm}_{\mathrm{S}}$ of all *clauses* — $\mathrm{Cls} = \left\{\bigvee_{i=1}^{n} l_i \middle| n \in \mathbb{N}, n \geqq 1, l_1, \ldots, l_n \in \mathrm{Lit}\right\}$. For convenience, provided the chosen semantics permits this (compare Example 4.2.1), clauses from Cls are identified with sets $\{l_1, \ldots, l_n\}$ of *literals* .... page 86

$\mathrm{Cls}_{\mathrm{S}}$ Language of all *clauses* wrt. classical Boolean **propositional logic** — $\mathrm{Cls}_{\mathrm{S}} = \left\{c \middle| c \subseteqq \mathrm{Lit}_{\mathrm{S}} \text{ and } c \neq \varnothing \text{ and } c \text{ is finite}\right\}$ .... page 156

Cnf Denotes the set $\mathrm{Cnf} \subseteqq \mathrm{PFrm}_{\mathrm{S}}$ of all formulae in *conjunctive normal form* — $\mathrm{Cnf} = \left\{\bigwedge_{i=1}^{n} c_i \middle| n \in \mathbb{N}, n \geqq 1, c_1, \ldots, c_n \in \mathrm{Cls}\right\}$ .... page 86

FC The set of *fuzzy constants* for **PLFC** .... page 195

$\mathbf{FOFrm_S}$ The standard language of well-formed **formulae** of **first order predicate logic** — $\mathrm{FOFrm_S} =_{\mathrm{def}} \mathrm{FOFrm}\,(\mathrm{IV}, \mathrm{Func}, \mathrm{Ar}_{\,\mathrm{Func}}, \mathrm{Pred}, \mathrm{Ar}_{\,\mathrm{Pred}}, \Omega_{\mathrm{S}}, \mathrm{Ar}_{\,\mathrm{S}})$ .... page 42

Frm Denotes the set of **formulae** of the underlying **logic** of the logic of **labelled formulae** .... page 39

$\boldsymbol{\mathcal{P}}$Frm Denotes a special set of *fuzzy sets of formulae*, i. e. $\mathcal{P}\mathrm{Frm} \subseteqq L^{\mathrm{Frm}}$ .... page 171

Func The set of *function symbols* for **first order predicate logic** .... page 40

IC The set of *individual constants* for **PLFC** .... page 195

$\mathbf{imp_G}$ Denotes the *binary truth value function* on the real unit interval $\langle 0,1\rangle$ called Gödel implication — $\mathrm{imp_G}(s,t) = \begin{cases} 1, & \text{if } s \leqq t \\ t, & \text{if } s > t \end{cases}$ for $s,t \in \langle 0,1\rangle$ .... page 48

$\mathbf{imp_{KD}}$ Denotes the *binary truth value function* on the real unit interval $\langle 0,1\rangle$ called Kleene-Dienes implication — $\mathrm{imp_{KD}}(s,t) = \max(1-s,t)$ for $s,t \in \langle 0,1\rangle$ .. page 48

$\mathbf{imp_Ł}$ Denotes the *binary truth value function* on the real unit interval $\langle 0,1\rangle$ called Łukasiewicz implication — $\mathrm{imp_Ł}(s,t) = \min(1, 1-s+t)$ for $s,t \in \langle 0,1\rangle$ .... page 49

Incons The set of all *inconsistent* $\mathfrak{L}$-fuzzy sets of formulae — $\mathrm{Incons} =_{\mathrm{def}} \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \mathrm{cst}(\mathcal{X}) = 0\right\}$ .... page 77

$\mathbf{Incons_L}$ The set of all *inconsistent* $\mathfrak{F}$-fuzzy sets of clauses in Lee's fuzzy logic with truth value-labelled formulae — $\mathrm{Incons_L} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \forall\,\mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}} : \mathrm{Val} \not\models \mathcal{X}\right\}$ .... page 135

$\mathbf{Incons_P^{\boldsymbol{d}}}$ The set of all $\mathfrak{F}$-fuzzy sets of clauses in possibilistic logic with necessity-valued formulae which are at least d-inconsistent — $\mathrm{Incons}_{\mathrm{P}}^{d} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls_S}} \text{ and } \forall d' > d : \mathrm{CUT}_{d'}(\mathcal{X}) \in \mathrm{Sat_B}\right\}$ .... page 162

$\mathbf{Incons_P}$ The set of all *inconsistent* $\mathfrak{F}$-fuzzy sets of clauses in possibilistic logic with necessity-valued formulae — $\mathrm{Incons_P} = \left\{\mathcal{X} \,\middle|\, \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \sup\left\{d \,\middle|\, \mathrm{CUT}_d(\mathcal{X}) \notin \mathrm{Sat_B}\right\} = 1\right\}$ .... page 153

IV The set of *individual variables* for **first order predicate logic** .... page 40

$\mathbf{LCls_S}$ The set of all *labelled clauses* wrt. $\mathrm{Cls_S}$ and the label set $\langle 0,1\rangle$ .... page 156

LFrm The set of all **labelled formulae**, for a given set Frm of **formulae** and a given *label lattice* $\mathfrak{L}$ .... page 57

Lit Set of all *literals* in propositional logic, wrt a set PV of *propositional variables* and a *negation symbol* $\neg$ — $\mathrm{Lit} =_{\mathrm{def}} \mathrm{PV} \cup \{\neg p | p \in \mathrm{PV}\}$ .... page 86

$\mathbf{Lit_S}$ Set of all *literals* wrt. classical Boolean **propositional logic** — $\mathrm{Lit_S} = \mathrm{PV} \cup \{\neg p | p \in \mathrm{PV}\}$ .... page 156

$\mathbf{PFrm_S}$ The standard language of well-formed **formulae** of **propositional logic** — $\mathrm{PFrm_S} =_{\mathrm{def}} \mathrm{PFrm}(\mathrm{PV}, \Omega_{\mathrm{S}}, \mathrm{Ar}_{\,\mathrm{S}})$ .... page 42

Pred The set of *predicate symbols* for **first order predicate logic** .... page 40

PV The set of *propositional variables* for **propositional logic** .... page 39

Sat The set of all *satisfiable sets of formulae* in two-valued logic —
$\mathrm{Sat} =_{\mathrm{def}} \left\{X \middle| X \subseteqq \mathrm{Frm} \text{ and } \exists \mathrm{Val} \in \mathfrak{S} \text{ such that } \mathrm{Val} \models X\right\}$ .... page 140

Sat$_{\mathbf{B}}$ The set of all *satisfiable sets of clauses* in two-valued logic —
$\mathrm{Sat}_{\mathrm{B}} =_{\mathrm{def}} \left\{C \middle| C \subseteqq \mathrm{Cls} \text{ and } \exists \mathrm{Val} \in \mathfrak{S}_{\mathrm{B}}^{\mathrm{P}} \text{ such that } \mathrm{Val} \models C\right\}$ .... page 153

Taut The set of all *tautologies* — $\mathrm{Taut} =_{\mathrm{def}} \left\{x \middle| x \in \mathrm{Frm} \text{ and } \mathrm{taut}(x) = 1\right\}$ .... page 51

Taut$_{\mathbf{B}}$ The set of all *tautologies* wrt. Cls and $\mathfrak{S}_{\mathrm{B}}^{\mathrm{P}}$ —
$\mathrm{Taut}_{\mathrm{B}} = \left\{c \middle| c \in \mathrm{Cls} \exists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c\right\}$ .... page 153

Taut$_{\mathbf{F}}$ The set of all *tautologies* wrt. Cls and $\mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$ — $\mathrm{Taut}_{\mathrm{F}} = \emptyset$ .... page 135

Val Truth valuation function $\mathrm{Val} : \mathrm{Frm} \to T$, wrt a set Frm of **formulae** and a given lattice $\mathfrak{T} =_{\mathrm{def}} [T, ⊓, ⊔]$ of *truth values* .... page 43

Val$_{\mathcal{A}}$ *Valuation function* induced by assignment $\mathcal{A}$ in *propositional logic*. .... page 43

Val$_{t}$ A special *truth valuation function*; for every *truth value* t, it is assumed that there exists a truth valuation Val_t and a **formula** x_t such that $\mathrm{Val}_t(x_t) = t$ (see (3.2)). .... page 43

Val$_{\mathfrak{I}}$ *Valuation function* induced by *interpretation* $\mathfrak{I}$ in **first order predicate logic** and **PLFC**. .... page 44, 196

Valid The set of all *valid* $\mathfrak{L}$-fuzzy sets of formulae —
$\mathrm{Valid} =_{\mathrm{def}} \left\{\mathcal{X} \middle| \mathcal{X} \in L^{\mathrm{Frm}} \text{ and } \mathrm{valid}(\mathcal{X}) = 1\right\}$ .... page 77

Valid$_{\mathbf{L}}$ The set of all *valid* $\mathfrak{F}$-fuzzy sets of clauses in Lee's fuzzy logic with truth value-labelled formulae —
$$\mathrm{Valid}_{\mathrm{L}} = \left\{\mathcal{X} \middle| \begin{array}{l} \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \forall c \in \mathrm{Cls} : \\ \text{if } \mathcal{X}(c) > 0 \text{, then } \mathcal{X}(c) \leqq \frac{1}{2} \text{ and } \exists p \in \mathrm{PV} : \{p, \neg p\} \subseteqq c \end{array}\right\}$$. page 135

Valid$_{\mathbf{P}}^{d}$ The set of all $\mathfrak{F}$-fuzzy sets of clauses in possibilistic logic with necessity-valued formulae which are at least d-valid —
$\mathrm{Valid}_{\mathrm{P}}^{d} = \left\{\mathcal{X} \middle| \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Clss}} \text{ and } \forall d' > 1 - d : \mathrm{CUT}_{d'}(\mathcal{X}) \subseteqq \mathrm{Taut}_{\mathrm{B}}\right\}$ .... page 160

Valid$_{\mathbf{P}}$ The set of all *valid* $\mathfrak{F}$-fuzzy sets of clauses in possibilistic logic with necessity-valued formulae — $\mathrm{Valid}_{\mathrm{P}} = \left\{\mathcal{X} \middle| \mathcal{X} \in \langle 0,1\rangle^{\mathrm{Cls}} \text{ and } \mathrm{supp}\,\mathcal{X} \subseteqq \mathrm{Taut}_{\mathrm{B}}\right\}$ .... page 153

Notation

Miscellaneous

⊓ M *Greatest lower bound* in the complete lattice $\mathfrak{D} = [D, ⊓, ⊔]$ of *validity degrees*. page 61

⊔ M *Least upper bound* in the complete lattice $\mathfrak{D} = [D, ⊓, ⊔]$ of *validity degrees*. .. page 61

$\bigcap \Phi$ Greatest lower bound in the complete lattice $\left[L^U, \cap, \cup\right]$ of all **$\mathfrak{L}$-fuzzy sets** on U, wrt the complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$, where $\Phi \subseteqq L^U$ .... page 12

$\bigcup \boldsymbol{\Phi}$ Least upper bound in the complete lattice $\left[L^U, \cap, \cup\right]$ of all **$\mathfrak{L}$-fuzzy sets** on U, wrt the complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$, where $\Phi \subseteqq L^U$. .... page 12

$\dot{\bigcup} \boldsymbol{\Phi}$ Least upper bound in the complete lattice $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \dot{\cup}\right]$ of all **$\mathfrak{L}'$-fuzzy filters** of $\mathfrak{L}$ — $\dot{\bigcup}\Phi = \bigcap\left\{\mathcal{H} \middle| \mathcal{H} \in \mathfrak{L}'\text{-Fl}(\mathfrak{L}) \text{ and } \bigcup\Phi \subseteqq \mathcal{H}\right\}$, where $\Phi \subseteqq \mathfrak{L}'\text{-Fl}(\mathfrak{L})$. .... page 22

$\bigsqcap \boldsymbol{M}$ *Greatest lower bound* (in some partially ordered set $[L, \sqsubseteq]$) of $M \subseteqq L$. .... page 8

$\bigsqcup \boldsymbol{M}$ *Least upper bound* (in some partially ordered set $[L, \sqsubseteq]$) of $M \subseteqq L$. .... page 8

$\curlywedge \boldsymbol{M}$ Greatest lower bound in the complete lattice $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$, where $M \subseteqq L'$. . . page 21

$\curlyvee \boldsymbol{M}$ Least upper bound in the complete lattice $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$, where $M \subseteqq L'$. ... page 22

$\bigwedge_{i=1}^{n} \boldsymbol{x_i}$ Denotes an *iterated conjunction* wrt classical *propositional syntax* —
$\bigwedge_{i=1}^{n} x_i = \Big(\big(\ldots(x_1 \wedge x_2) \wedge \ldots\big) \wedge x_n\Big)$ .... page 86

$\bigvee_{i=1}^{n} \boldsymbol{x_i}$ Denotes an *iterated disjunction* wrt classical *propositional syntax* —
$\bigvee_{i=1}^{n} x_i = \Big(\big(\ldots(x_1 \vee x_2) \vee \ldots\big) \vee x_n\Big)$ .... page 86

$\boxed{\mathrm{T}} \boldsymbol{M}$ *Greatest lower bound* in the complete lattice $\mathfrak{T} = [T, \boxed{\sqcap}, \boxed{\sqcup}]$ of *truth values*. .... page 49

$\boxed{\mathrm{L}} \boldsymbol{\Phi}$ *Least upper bound* in the complete lattice $\mathfrak{L} = [L, \boxed{\sqcap}, \boxed{\sqcup}]$ of *labels*. .... page 61

$\overline{\boldsymbol{l}}$ Denotes the *complement* of a *literal* $l \in \text{Lit}$ —
$\overline{l} =_{\text{def}} \begin{cases} \neg l & \text{if } l \in \text{PV} \\ p & \text{if } l = \neg p \text{ and } p \in \text{PV} \end{cases}$. .... page 86

${}^{d}\overline{\boldsymbol{a}}$ *Principal fuzzy d-filter* on a **lattice** $\mathfrak{L} = [L, \sqcap, \sqcup]$ of $a \in L$ wrt a lattice $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$ and $d \in L'$ — ${}^{d}\overline{a}(b) = 1$, if $b = 1$; ${}^{d}\overline{a}(b) = d$, if $b \neq 1$ and $a \sqsubseteq b$; ${}^{d}\overline{a}(b) = 0$, if **not** $a \sqsubseteq b$ (wrt $b \in L$). .... page 16

$\overline{\boldsymbol{a}}$ *Principal filter* in a **lattice** $[L, \sqcap, \sqcup]$ of $a \in L$ — $\overline{a} =_{\text{def}} \{b \mid b \in L \text{ and } a \sqsubseteq b\}$. . page 9

$\widetilde{\boldsymbol{\ell}}$ Label calculated from a given label ℓ for *refutation* (if ℓ *admits refutation*). See Definition 4.3.4. .... page 100

Brackets (used for equivalence classes and tuples)

$[\boldsymbol{x}]$ *Equivalence class* of the **formula** x wrt. the relation $\equiv$ of *semantic equivalence* — $[x] =_{\text{def}} \{y \mid y \in \text{Frm} \text{ and } x \equiv y\}$. .... page 50

$[\boldsymbol{a}, \boldsymbol{b}]$ Ordered pair of a and b. .... page 7

$[\boldsymbol{L}, \sqsubseteq]$ Partially ordered set with *domain* L and *partial order relation* $\sqsubseteq$. .... page 8

$[\boldsymbol{L}, \sqcap, \sqcup]$ **Lattice** with *domain* L, *meet* $\sqcap$, and *join* $\sqcup$. .... page 8

$\left[\boldsymbol{\mathfrak{L}'\text{-Fl}(\mathfrak{L})}, \cap, \dot{\cup}\right]$ Lattice of all **$\mathfrak{L}'$-fuzzy filters** of $\mathfrak{L}$. .... page 21

$\left[\mathbf{Fl}(\boldsymbol{\mathfrak{L}}), \cap, \dot{\cup}\right]$ Lattice of all **filters** of $\mathfrak{L}$. .... page 10

$\left[\mathscr{P}(\boldsymbol{\mathfrak{L}'}, \boldsymbol{\mathfrak{L}}), \cap, \dot{\cup}\right]$ Denotes the sublattice of $\left[\mathfrak{L}'\text{-Fl}(\mathfrak{L}), \cap, \dot{\cup}\right]$ generated by $\mathfrak{L}'\text{-PFl}(\mathfrak{L})$. .... page 32

$\left[\mathbf{PFl}(\mathfrak{L}), \cap, \Cup\right]$ Lattice of all principal **filters** of $\mathfrak{L}$. .... page 10

$\left[\mathfrak{L}'\text{-}\mathbf{PFl}(\mathfrak{L}), \cap, \uplus\right]$ Lattice of all principal $\mathfrak{L}'$-**fuzzy filters** of $\mathfrak{L}$. .... page 26

$\left[\boldsymbol{L^U}, \cap, \cup\right]$ Complete lattice of all $\mathfrak{L}$-**fuzzy sets** on U, wrt the complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$. .... page 12

$\left[\mathbf{Frm}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=\right]$ A *logic of graded trust assessment* with *logical language* Frm, *semantics* $\mathfrak{S}$, *validity degree lattice* $\mathfrak{D}$, and **label lattice** $\mathfrak{L}$. $\models\!\!=$ is the (graded) model relation (characterized by axioms). See Definition 5.2.1. .... page 138

$[\mathbf{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \boldsymbol{\iota}]$ A *fuzzy filter-based logic* with *logical language* Frm, *truth value lattice* $\mathfrak{T}$, *semantics* $\mathfrak{S}$, *validity degree lattice* $\mathfrak{D}$, and **label lattice** $\mathfrak{L}$. ι is a lattice isomorphism such that $\mathfrak{T}$ is **expanded** to $\mathfrak{L}$ by $\mathfrak{D}$, by means of ι. See Definition 4.1.2. .... page 65

$\left[\mathbf{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{D}, \mathfrak{L}, \models\!\!=\right]$ A *logic of graded truth and graded trust assessment* with *logical language* Frm, *truth value lattice* $\mathfrak{T}$, *semantics* $\mathfrak{S}$, *validity degree lattice* $\mathfrak{D}$, and **label lattice** $\mathfrak{L}$. $\models\!\!=$ is the (graded) model relation (characterised by axioms). See Definition 4.1.3. .... page 66

$\left[\mathbf{Frm}, \mathfrak{T}, \mathfrak{S}, \mathfrak{L}, \models\right]$ A *logic of graded truth assessment* with *logical language* Frm, *truth value lattice* $\mathfrak{T}$, *semantics* $\mathfrak{S}$, and **label lattice** $\mathfrak{L}$. $\models$ is the (binary) model relation (characterized by axioms). See Definition 5.2.1. .... page 128

$\left[\langle \mathbf{0}, \mathbf{1}\rangle, \boldsymbol{d}\right]$ Metric space of truth values, with $d(r, s) = |r - s|$. .... page 158

$\left[\langle \mathbf{0}, \mathbf{1}\rangle^{\mathbf{PV}}, \boldsymbol{\rho}\right]$ Metric space of *assignments*, with $\rho(\mathcal{A}, \mathcal{B}) =_{\text{def}} \operatorname{Sup}\left\{ d\left(\mathcal{A}(p), \mathcal{B}(p)\right) \middle| p \in \text{PV}\right\}$. .... page 158

$[\boldsymbol{x}, \boldsymbol{\ell}]$ **Labelled formula** consisting of the **formula** x and the *label* ℓ. .... page 52, 57

Parentheses (used for intervals of real numbers)

$(\boldsymbol{r}, \boldsymbol{s})$ Open Interval of all real numbers $t \in \mathbb{R}$ with $r < t < s$. .... page 7

$(\boldsymbol{r}, \boldsymbol{s}\rangle$ Half-Open Interval of all real numbers $t \in \mathbb{R}$ with $r < t \leqq s$. .... page 7

$\langle \boldsymbol{r}, \boldsymbol{s})$ Half-Open Interval of all real numbers $t \in \mathbb{R}$ with $r \leqq t < s$. .... page 7

$\langle \boldsymbol{r}, \boldsymbol{s}\rangle$ Closed Interval of all real numbers $t \in \mathbb{R}$ with $r \leqq t \leqq s$. .... page 7

$\langle \mathbf{0}, \mathbf{1}\rangle$ Real unit interval. .... page 11

Expressions involving variables

$\mathfrak{P}\boldsymbol{S}$ Power set of the set S — $\mathfrak{P}S =_{\text{def}} \left\{S' \middle| S' \subseteqq S\right\}$. .... page 7

$\boldsymbol{\mathcal{F}(u)}$ The membership degree of u in the **fuzzy set** $\mathcal{F}$. .... page 11

$\boldsymbol{\mathcal{S}} \Vdash \boldsymbol{\mathfrak{x}}$ Denotes that the **labelled formula** $\mathfrak{x} \in \text{LFrm}$ is a **semantic consequence** of the $\mathfrak{D}$-fuzzy set $\mathcal{S}$ of valuation functions, for a given set $\mathfrak{S}$ of *valuation functions* and complete **lattice** $\mathfrak{D} = [D, \sqcap, \sqcup]$ of **degrees of validity** such that $\mathcal{S} \in D^{\mathfrak{S}}$ — $\mathcal{S} \Vdash \mathfrak{x}$ iff $\mathcal{S} \subseteqq \text{Mod}(\mathfrak{x})$. .... page 94

$\boldsymbol{\mathcal{S}} \Vdash_{\overline{\mathrm{P}}} [\boldsymbol{x}, \boldsymbol{d}]$ Denotes that the *necessity-labelled formula* $[x, d]$ is a **semantic consequence** of the $\mathfrak{F}$-fuzzy set $\mathcal{S}$ of valuations in possibilistic logic with necessity-valued formulae — $\mathcal{S} \Vdash_{\overline{\mathrm{P}}} [x, d]$ iff for every $\mathrm{Val} \in \mathfrak{S}_{\mathrm{B}}^{\mathrm{P}}$, if $\mathrm{Val} \not\models x$, then $\mathcal{S}(\mathrm{Val}) \leqq 1 - d$. .... page 152

$\boldsymbol{\mathcal{X}} \Vdash \boldsymbol{\mathfrak{x}}$ Denotes that the **labelled formula** $\mathfrak{x}$ is a **semantic consequence** of the $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae, for a given set of **formulae** Frm and complete **lattice** $\mathfrak{L} = [L, \sqcap, \sqcup]$ of **labels** such that $\mathfrak{x} \in \mathrm{LFrm}$ and $\mathcal{X} \in L^{\mathrm{Frm}}$ — $\mathcal{X} \Vdash \mathfrak{x}$ iff $\mathrm{Mod}(\mathcal{X}) \subseteqq \mathrm{Mod}(\mathfrak{x})$. page 88

$\boldsymbol{\mathcal{X}} \Vdash_{\overline{\mathrm{L}}} [\boldsymbol{x}, \boldsymbol{t}]$ Denotes that the *truth-value labelled formula* $[x, t]$ is a **semantic consequence** of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of formulae in Lee's fuzzy logic with truth value-labelled formulae — $\mathcal{X} \Vdash_{\overline{\mathrm{L}}} [x, t]$ iff $\forall\, \mathrm{Val} \in \mathfrak{S}_{\mathrm{F}}^{\mathrm{P}}$, if $\mathrm{Val} \overset{\mathrm{L}}{\models} \mathcal{X}$, then $\mathrm{Val} \overset{\mathrm{L}}{\models} [x, t]$. .... page 135

$\boldsymbol{\mathcal{X}} \Vdash_{\overline{\mathrm{P}}} [\boldsymbol{x}, \boldsymbol{d}]$ Denotes that the *necessity-labelled formula* $[x, d]$ is a **semantic consequence** of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of formulae in possibilistic logic with necessity-valued formulae — $\mathcal{X} \Vdash_{\overline{\mathrm{P}}} [x, d]$ iff $d \leqq \sup \left\{ d \middle| d \in \langle 0, 1\rangle \text{ and } \mathrm{CUT}_d(\mathcal{X}) \Vdash x \right\}$. .... page 152

$\boldsymbol{\mathcal{X}} \cap \boldsymbol{X}$ Given $\mathcal{X} \in L^{\mathrm{Frm}}$ and $X \subseteqq \mathrm{Frm}$, $\mathcal{X} \cap X$ denotes the $\mathfrak{L}$-fuzzy set of formulae derived from $\mathcal{X}$ by 'intersecting' it with X — $\mathcal{X} \cap X = \mathcal{X} \setminus (\mathrm{Frm} \setminus X)$. .... page 81

$\boldsymbol{\mathcal{X}} \setminus \boldsymbol{X}$ Given $\mathcal{X} \in L^{\mathrm{Frm}}$ and $X \subseteqq \mathrm{Frm}$, $\mathcal{X} \setminus X$ denotes the $\mathfrak{L}$-fuzzy set of formulae derived from $\mathcal{X}$ by 'removing' all elements of X — for $x \in \mathrm{Frm}$, $(\mathcal{X} \setminus X)(x) = 0$, if $x \in X$; $(\mathcal{X} \setminus X)(x) = \mathcal{X}(x)$, if $x \notin X$. .... page 81

$\mathscr{D}(\boldsymbol{\mathfrak{L}})$ *Dual* of the **lattice** $\mathfrak{L}$ — $\mathscr{D}([L, \sqcap, \sqcup]) =_{\mathrm{def}} [L, \sqcup, \sqcap]$. .... page 8

$\mathscr{F}(\boldsymbol{\mathfrak{L}})$ Denotes the *dual lattice* $\mathscr{D}\Big(\big[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\big]\Big) = \big[\mathrm{Fl}(\mathfrak{L}), \uplus, \cap\big]$ of the complete lattice structure $\big[\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\big]$ for filters of $\mathfrak{L}$ described in Observation 1.3.1. The induced partial order of this lattice is the *superset relation* $\supseteqq$. .... page 34

$\boldsymbol{F}\mathscr{F}(\boldsymbol{\mathfrak{L}})$ Denotes the *dual lattice* $\mathscr{D}\Big(\big[\mathfrak{L}\text{-}\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\big]\Big) = \big[\mathfrak{L}\text{-}\mathrm{Fl}(\mathfrak{L}), \uplus, \cap\big]$ of the complete lattice structure $\big[\mathfrak{L}\text{-}\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\big]$ for **$\mathfrak{L}$-fuzzy filters** of $\mathfrak{L}$ established in Theorem 2.2.1. The induced partial order of this lattice is the *superset relation* $\supseteqq$ for *$\mathfrak{L}$-fuzzy sets*. .... page 34

$\mathscr{P}(\boldsymbol{\mathfrak{L}'}, \boldsymbol{\mathfrak{L}})$ Denotes the smallest subset of $\mathfrak{L}'\text{-}\mathrm{Fl}(\mathfrak{L})$ which contains $\mathfrak{L}'\text{-}\mathrm{PFl}(\mathfrak{L})$ and is closed wrt. the lattice operations of $\big[\mathfrak{L}'\text{-}\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\big]$ — $\mathscr{P}(\mathfrak{L}', \mathfrak{L}) =_{\mathrm{def}} \bigcap \left\{ \Phi \middle| \mathfrak{L}'\text{-}\mathrm{PFl}(\mathfrak{L}) \subseteqq \Phi \text{ and } [\Phi, \cap, \uplus] \Subset \big[\mathfrak{L}'\text{-}\mathrm{Fl}(\mathfrak{L}), \cap, \uplus\big] \right\}$. ...... page 32

$\boldsymbol{\sigma_{v:=u}}$ For a given assignment $\sigma : \mathrm{IV} \to U$, an individual variable $v \in \mathrm{IV}$ and an element $u \in U$ of the domain, $\sigma_{v:=u}$ denotes the assignment given for $w \in \mathrm{IV}$ by

$$\sigma_{v:=u}(w) =_{\mathrm{def}} \begin{cases} u, & \text{if } w = v \\ \sigma(w), & \text{if } w \neq v. \end{cases}$$ .... page 44

$\boldsymbol{f : S \longrightarrow T}$ Denotes that f is a mapping from the set S into the set T. .... page 7

$\boldsymbol{M_{[c,r]}}$ The set of all *satisfying assignments* for the labelled clause $[c, r]$ in Lee's fuzzy logic with truth values as labels — $M_{[c,r]} = \left\{ \mathcal{A} \middle| \mathcal{A} \in \langle 0, 1\rangle^{\mathrm{PV}} \text{ and } \mathrm{Val}_{\mathcal{A}} \overset{\mathrm{L}}{\models} [c, r] \right\}$. page 158

$\boldsymbol{T^S}$ Set of all mappings from the set S into the set T — $T^S =_{\mathrm{def}} \{ f \,|\, f : S \to T \}$. .. page 7

$\boldsymbol{X} \Vdash \boldsymbol{x}$ Denotes that the **formula** x is a **semantic consequence** of the set X of formulae, for a given set of **formulae** Frm such that $x \in \mathrm{Frm}$ and $X \subseteqq \mathrm{Frm}$ — $X \Vdash x$ iff $\forall\, \mathrm{Val} \in \mathfrak{S}$, if $\mathrm{Val} \models X$, then $\mathrm{Val} \models x$. .... page 140

$\boldsymbol{Y \leqq X}$ Denotes that X is a *semantic covering* of Y — for every $y \in Y$, there exists $x \in X$ such that $y \equiv x$. .... page 50

Expressions involving multiletter names

Cons($\boldsymbol{\mathcal{S}}$) Denotes the $\mathfrak{L}$-fuzzy set of **consequences** of the $\mathfrak{D}$-fuzzy set $\mathcal{S}$ of valuation functions, for a given set $\mathfrak{S}$ of *valuation functions* and complete **lattice** $\mathfrak{D} = [D, \sqcap, \sqcup]$ of **degrees of validity** such that $\mathcal{S} \in D^{\mathfrak{S}}$ — $\mathrm{Cons}(\mathcal{S}) = \bigcup \left\{\mathfrak{x} \middle| \mathfrak{x} \in \mathrm{LFrm} \text{ and } \mathcal{S} \mathrel{|\!\!\vdash} \mathfrak{x}\right\}$. .... page 94

Cons($\boldsymbol{\mathcal{X}}$) Denotes the $\mathfrak{L}$-fuzzy set of **consequences** of the $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae, for a given set of **formulae** Frm and complete **lattice** $\mathfrak{L} = [L, \sqcap, \sqcup]$ of **labels** such that $\mathcal{X} \in L^{\mathrm{Frm}}$ — $\mathrm{Cons}(\mathcal{X}) = \bigcup \left\{\mathfrak{x} \middle| \mathfrak{x} \in \mathrm{LFrm} \text{ and } \mathcal{X} \mathrel{|\!\!\vdash} \mathfrak{x}\right\}$. .... page 88

Cons$_{\mathbf{L}}$($\boldsymbol{\mathcal{X}}$) Denotes the $\mathfrak{F}$-fuzzy set of **consequences** of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of formulae in Lee's fuzzy logic with truth value-labelled formulae — $\mathrm{Cons}_{\mathrm{L}}(\mathcal{X})(x) = \sup \left\{t \middle| t \in \langle 0, 1\rangle \text{ and } \mathcal{X} \mathrel{|\!\!\vdash_{\mathrm{L}}} [x, t]\right\}$. .... page 135

Cons$_{\mathbf{P}}$($\boldsymbol{\mathcal{S}}$) Denotes the $\mathfrak{F}$-fuzzy set of **consequences** of the $\mathfrak{F}$-fuzzy set $\mathcal{S}$ of valuations in possibilistic logic with necessity-valued formulae — $\mathrm{Cons}_{\mathrm{P}}(\mathcal{S})(x) = \inf \left\{1 - \mathcal{S}(\mathrm{Val}) \middle| \mathrm{Val} \in \mathfrak{S}^{\mathrm{P}}_{\mathrm{B}} \text{ and } \mathrm{Val} \not\models x\right\}$. .... page 152

Cons$_{\mathbf{P}}$($\boldsymbol{\mathcal{X}}$) Denotes the $\mathfrak{F}$-fuzzy set of **consequences** of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of formulae in possibilistic logic with necessity-valued formulae — $\mathrm{Cons}_{\mathrm{P}}(\mathcal{X})(x) = \sup \left\{d \middle| d \in \langle 0, 1\rangle \text{ and } \mathrm{CUT}_d(\mathcal{X}) \mathrel{|\!\!\vdash} x\right\}$. .... page 152

cst($\boldsymbol{\mathcal{X}}$) *Consistency index* of the $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae — $\mathrm{cst}(\mathcal{X}) =_{\mathrm{def}} \bigsqcup \left\{\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \middle| \mathrm{Val} \in \mathfrak{S}\right\}$. .... page 77

cst$_{\mathbf{P}}$($\boldsymbol{\mathcal{X}}$) *Consistency index* of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of clauses in possibilistic logic with necessity-valued formulae — $\mathrm{cst}_{\mathrm{P}}(\mathcal{X}) = \inf \left\{d \middle| \mathrm{CUT}_{1-d}(\mathcal{X}) \notin \mathrm{Sat}_{\mathrm{B}}\right\}$. ...... page 153

CUT$_{\boldsymbol{a}}$($\boldsymbol{\mathcal{F}}$) a-cut of the **fuzzy set** $\mathcal{F} \in L^U$, for $a \in L$ — $\mathrm{CUT}_a(\mathcal{F}) =_{\mathrm{def}} \left\{u \middle| u \in U \text{ and } a \sqsubseteq \mathcal{F}(u)\right\}$. .... page 12

$\boldsymbol{\mathfrak{L}'}$-Fl ($\boldsymbol{\mathfrak{L}}$) Set of all **$\mathfrak{L}'$-fuzzy filters** of the **lattice** $\mathfrak{L}$. .... page 15

Fl($\boldsymbol{\mathfrak{L}}$) Set of all **filters** of the **lattice** $\mathfrak{L}$. .... page 9

FOFrm(IV, Func, Ar$_{\mathbf{Func}}$, Pred, Ar$_{\mathbf{Pred}}$, $\boldsymbol{\Omega}$, Ar) The language of all well-formed **formulae** of **first order predicate logic** wrt the sets IV, Func, Pred, Ω and the mappings Ar, Ar$_{\mathrm{Func}}$, Ar$_{\mathrm{Pred}}$ .... page 40

inc($\boldsymbol{\mathcal{X}}$) *Inconsistency distribution* of the $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae — $\mathrm{inc}(\mathcal{X}) =_{\mathrm{def}} \mathrm{Cons}(\mathcal{X})(\bot)$, where for all $\mathrm{Val} \in \mathfrak{S}$, $\mathrm{Val}(\bot) = 0$. .... page 96

inc$_{\mathbf{P}}$($\boldsymbol{\mathcal{X}}$) *Inconsistency index* of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of clauses in possibilistic logic with necessity-valued formulae — $\mathrm{inc}_{\mathrm{P}}(\mathcal{X}) = 1 - \mathrm{cst}_{\mathrm{P}}(\mathcal{X})$. .... page 153

Ind($\boldsymbol{t, \mathfrak{I}, \sigma}$) The *individual* associated with the term t by the interpretation $\mathfrak{I}$ of first order logic and the assignment σ of individuals to individual variables. .... page 45

Mod($\boldsymbol{\mathcal{X}}$) The $\mathfrak{D}$-fuzzy set $\mathrm{Mod}(\mathcal{X}) \in D^{\mathfrak{S}}$ of models of $\mathcal{X}$, for $\mathcal{X} \in L^{\mathrm{Frm}}$ — for $\mathrm{Val} \in \mathfrak{S}$ and $d \in D$, $\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = d =_{\mathrm{def}} \mathrm{Val} \models_{d} \mathcal{X}$. .... page 75

$\mathbf{Mod_P(\mathcal{X})}$ The $\mathfrak{F}$-fuzzy set $\mathrm{Mod_P}(\mathcal{X}) \in \langle 0,1\rangle^{\mathfrak{S}_B^P}$ of models of $\mathcal{X}$, for $\mathcal{X} \in \langle 0,1\rangle^{\mathrm{PFrms}}$ — for $\mathrm{Val} \in \mathfrak{S}_B^P$ and $d \in \langle 0,1\rangle$,
$\mathrm{Mod}(\mathcal{X})(\mathrm{Val}) = \inf\left\{1 - \mathcal{X}(x) \middle| x \in \mathrm{Frm} \text{ and } \mathrm{Val} \not\models x\right\}$. .... page 151

$\mathbf{PFl(\mathfrak{L})}$ Set of all *principal filters* of the **lattice** $\mathfrak{L}$. .... page 9

$\mathbf{\mathfrak{L}'\text{-}PFl\,(\mathfrak{L})}$ Set of all *principal fuzzy filters* of the **lattice** $\mathfrak{L}$. .... page 16

$\mathbf{\mathfrak{L}'\text{-}PFl\,(\boldsymbol{a})}$ Set of all principal $\mathfrak{L}'$-**fuzzy filters** of $a \in L$, wrt lattices $\mathfrak{L} = [L, \sqcap, \sqcup]$,
$\mathfrak{L}' = [L', \curlywedge, \curlyvee]$ — $\mathfrak{L}'\text{-PFl}\,(a) =_{\mathrm{def}} \left\{ {}^{d}\overline{a} \middle| d \in L' \right\}$. .... page 31

$\boldsymbol{d}$-PFl $\mathbf{(\mathfrak{L})}$ Set of all *principal fuzzy d-filters* of the **lattice** $\mathfrak{L}$. .... page 16

$\mathbf{PFrm(PV, \Omega, Ar)}$ The language of all well-formed propositional **formulae** wrt the sets PV, Ω and the mapping Ar. .... page 40

$\mathbf{PLFCFrm(IV, IC, FC, Pred, Ar_{Pred})}$ The language of all well-formed **formulae** of **PLFC** wrt the sets IV, IC, FC, Pred and the mapping $\mathrm{Ar}_{\mathrm{Pred}}$ .... page 195

$\mathbf{rg}\,\boldsymbol{f}$ Denotes the *range* of the mapping $f : S \to T$, for sets S, T —
$\mathrm{rg}\, f = \left\{t \middle| t \in T \text{ and } \exists s \in S : t = f(s)\right\}$. .... page 7

$\mathbf{sat}(\boldsymbol{x})$ *Satisfiability index* of the formula x — $\mathrm{sat}(x) =_{\mathrm{def}} \boxed{\mathrm{T}} \left\{\mathrm{Val}(x) \middle| \mathrm{Val} \in \mathfrak{S}\right\}$. .... page 51

$\mathbf{supp}\,\boldsymbol{\mathcal{F}}$ Support of an $\mathfrak{L}$-**fuzzy set** $\mathcal{F} \in L^U$ — $\mathrm{supp}\,\mathcal{F} =_{\mathrm{def}} \left\{u \middle| u \in U \text{ and } \mathcal{F}(u) \neq 0\right\}$. $\mathcal{F}$ is said to be *finite* iff $\mathrm{supp}\,\mathcal{F}$ is finite. .... page 13

$\mathbf{taut}(\boldsymbol{x})$ *Tautology index* (or *inherent truth*) of the formula x —
$\mathrm{taut}(x) =_{\mathrm{def}} \boxed{\sqcap} \left\{\mathrm{Val}(x) \middle| \mathrm{Val} \in \mathfrak{S}\right\}$. .... page 51

$\mathbf{Term(IV, Func, Ar_{Func})}$ The language of all well-formed *terms* in **first order predicate logic** wrt the sets IV, Func and the mapping $\mathrm{Ar}_{\mathrm{Func}}$ .... page 40

$\mathbf{Val}(\boldsymbol{x}, \boldsymbol{\mathfrak{I}}, \boldsymbol{\sigma})$ The *truth value* associated with the **first order formula** (or PLFC formula) x by the interpretation $\mathfrak{I}$ of first order logic (or PLFC) and the assignment σ of individuals to individual variables. .... page 45, 196

$\mathbf{Val} \models \boldsymbol{\mathcal{X}}$ Denotes that Val is a **model** for the $\mathfrak{L}$-fuzzy set of formulae $\mathcal{X}$ in a *logic of graded truth assessment* — $\mathrm{Val} \models \mathcal{X}$ iff $\forall x \in \mathrm{Frm} : \mathrm{Val} \models [x, \mathcal{X}(x)]$. See (5.11). .. page 131

$\mathbf{Val} \models [\boldsymbol{x}, \boldsymbol{\ell}]$ Denotes that Val is a **model** for the labelled formula $[x, \ell]$ in a *logic of graded truth assessment*, for a given **semantics** $\mathfrak{S}$ such that $\mathrm{Val} \in \mathfrak{S}$ and a set of **$\mathfrak{L}$-labelled formulae** LFrm such that $[x, \ell] \in \mathrm{LFrm}$. In logics of graded truth assessment, validity degrees are neglected, so the model relation is a binary one. See section 3.4.1 for a motivation and section 5.2 (esp. equation (5.1)) for a formal definition and systematic study. .... page 52, 128

$\mathbf{Val} \models \boldsymbol{X}$ Denotes that Val is a **model** for the set X of formulae —
$\mathrm{Val} \models X$ iff $\forall x \in X, \mathrm{Val} \models x$. See Definition 5.3.2. .... page 140, 171

$\mathbf{Val} \models \boldsymbol{x}$ Denotes that the valuation Val is a **model** for the formula x, for a given **semantics** $\mathfrak{S}$ such that $\mathrm{Val} \in \mathfrak{S}$ and a set of **formulae** Frm such that $x \in \mathrm{Frm}$. Several definitions for $\models$ are discussed in section 3.4.1. See also Definition 5.3.2. page 51, 140

Val $\models^{\mathbf{L}} \boldsymbol{\mathcal{X}}$ Denotes that Val is a **model** for the $\mathfrak{F}$-fuzzy set of formulae $\mathcal{X}$ in LEE's fuzzy logic with truth value-labelled formulae — Val $\models^{\mathrm{L}} \mathcal{X}$ iff $\forall x \in \mathrm{PFrm}_{\mathrm{S}} : \mathcal{X}(x) \leqq \mathrm{Val}(x)$. See (5.28). .... page 135

Val $\models^{\mathbf{L}} [\boldsymbol{x}, \boldsymbol{t}]$ Denotes that Val is a **model** for the labelled formula $[x, t]$ in LEE's fuzzy logic with truth value-labelled formulae — Val $\models^{\mathrm{L}} [x, t]$ iff $t \leqq \mathrm{Val}(x)$. See (5.27). page 135

Val $\models^{\mathbf{P}}_{\boldsymbol{d'}} [\boldsymbol{x}, \boldsymbol{d}]$ Denotes that Val is a **model** for the **labelled formula** $[x, d]$ to the degree d', in possibilistic logic with necessity-valued formulae —
Val $\models^{\mathrm{P}}_{d'} [x, d]$ iff $d' = \begin{cases} 1, & \text{if Val} \models x \\ 1 - d, & \text{if Val} \not\models x \end{cases}$. See (5.27). .... page 151

Val $\models^{\mathbf{P}}_{\boldsymbol{d}} \boldsymbol{\mathcal{X}}$ Denotes that Val is a **model** for the $\mathfrak{F}$-fuzzy set of formulae $\mathcal{X}$ to the degree d, in possibilistic logic with necessity-valued formulae —
Val $\models^{\mathrm{P}}_{d} \mathcal{X}$ iff $d = \inf\left\{1 - \mathcal{X}(x) \,\middle|\, x \in \mathrm{Frm} \text{ and } \mathrm{Val} \not\models x\right\}$. See (5.85). .... page 151

Val $\models_{\boldsymbol{d}} \boldsymbol{x}$ Denotes that the valuation Val is a **model** for the formula x to the degree d, for a given **semantics** $\mathfrak{S}$ such that Val $\in \mathfrak{S}$, a set of **formulae** Frm such that $x \in$ Frm and a given complete lattice $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$ of *validity degrees* such that $d \in D$.
Note that in this dissertation, a *graded model relation* is usually defined on *labelled formulae*. Possibilities of defining a graded model relation for non-labelled formulae are briefly discussed in section 3.4.2 (levels $1'$ and $2'$). .... page 54

Val $\models_{\boldsymbol{d}} \boldsymbol{\mathcal{X}}$ Denotes that Val is a **model** for the $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae to the degree d, for a given **semantics** $\mathfrak{S}$ such that Val $\in \mathfrak{S}$, a set of **formulae** Frm, a **label lattice** $\mathfrak{L} = [L, \sqcap_L, \sqcup_L]$ such that $\mathcal{X} \in \mathrm{Frm}^L$ and a given complete lattice $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$ of *validity degrees* such that $d \in D$ — Val $\models_{d} \mathcal{X}$ holds iff
$d = \sqcap_D \left\{ d' \,\middle|\, x \in \mathrm{Frm} \text{ and } \mathrm{Val} \models_{d'} [x, \mathcal{X}(x)] \right\}$. .... page 74

Val $\models_{\boldsymbol{d}} [\boldsymbol{x}, \boldsymbol{\ell}]$ Denotes that the valuation Val is a **model** for the **labelled formula** $[x, \ell]$ to the degree d, for a given **semantics** $\mathfrak{S}$ such that Val $\in \mathfrak{S}$, a set of **$\mathfrak{L}$-labelled formulae** LFrm such that $[x, \ell] \in$ LFrm and a given complete lattice $\mathfrak{D} = [D, \sqcap_D, \sqcup_D]$ of *validity degrees* such that $d \in D$. When Val $\models_{d} [x, \ell]$ holds is defined on the basis of **fuzzy filters** in Definition 4.1.1 and by logical axioms in Definition 4.1.3. See also section 3.4.2 for motivating remarks. .... page 55, 59, 66

Val $\not\models [\boldsymbol{x}, \boldsymbol{\ell}]$ Denotes that Val is **not** a **model** for the labelled formula $[x, \ell]$ in a *logic of graded truth assessment*. Compare (5.1). .... page 128

Val $\not\models \boldsymbol{x}$ Denotes that Val is **not** a **model** for the formula x in a *two-valued logic*. Compare Definition 5.3.2. .... page 140

valid($\boldsymbol{\mathcal{X}}$) *Validity index* (or *inherent validity*) of the $\mathfrak{L}$-fuzzy set $\mathcal{X}$ of formulae —
$\mathrm{valid}(\mathcal{X}) =_{\mathrm{def}} \sqcap_D \left\{ \mathrm{Mod}(\mathcal{X})(\mathrm{Val}) \,\middle|\, \mathrm{Val} \in \mathfrak{S} \right\}$. .... page 77

valid$_{\mathbf{P}}$($\boldsymbol{\mathcal{X}}$) *Validity index* (or *inherent validity*) of the $\mathfrak{F}$-fuzzy set $\mathcal{X}$ of clauses in possibilistic logic with necessity-valued formulae —
$\mathrm{valid}_{\mathrm{P}}(\mathcal{X}) = \inf \left\{ d \,\middle|\, \mathrm{CUT}_{1-d}(\mathcal{X}) \subsetneqq \mathrm{Taut}_{\mathrm{B}} \right\}$. .... page 153

Glossary of Concepts

Expansion Given two lattices $\mathfrak{L}_1 = [L_1, \sqcap, \sqcup]$, $\mathfrak{L}_2 = [L_2, \curlywedge, \curlyvee]$, an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$ is a lattice isomorphic to a lattice *between* the $\mathfrak{L}_2$-fuzzy *principal filters* of $\mathfrak{L}_1$ and the $\mathfrak{L}_2$-fuzzy *filters* of $\mathfrak{L}_1$. That means $\mathfrak{L}_3$ is an expansion of $\mathfrak{L}_1$ by $\mathfrak{L}_2$ iff there exists a lattice $\mathfrak{L}_3'$ isomorphic with $\mathfrak{L}_3$ such that $\left[\mathscr{P}(\mathfrak{L}_2, \mathfrak{L}_1), \Cup, \Cap\right] \Subset \mathfrak{L}_3' \Subset \left[\mathfrak{L}_2\text{-Fl}\left(\mathfrak{L}_1\right), \Cup, \Cap\right]$. See Definition 2.3.1 .... page 33

Filters of a lattice For a lattice $[L, \sqcap, \sqcup]$, filters are defined to be nonempty subsets F of L such that for all $a, b \in L$, $a, b \in F$ iff $a \sqcap b \in F$. (See Definition 1.3.2) .... page 9

First Order Predicate Logic The most common *logical language*, where **formulae** may contain *individual variables*, *function symbols*, *predicate symbols*, and furthermore *logical operator symbols* and *quantifiers*, and where **semantics** consist of *valuations* induced by *interpretations* which fix a *domain* for individuals and assign *functions* on the domain to function symbols and (many-valued) *predicates* on the domain to predicate symbols. See Example 3.1.2 and Example 3.2.2 .... page 40

Formulae The *formal language* of a **logic**. The structural description of the set of all formulae is called **syntax**. In this dissertation, the set Frm of all formulae is assumed to be given as an arbitrary nonempty set. .... page 39

Fuzzy filters of a lattice For two lattices $\mathfrak{L} = [L, \sqcap, \sqcup]$, $\mathfrak{L}' = \left[L', \curlywedge, \curlyvee\right]$, $\mathfrak{L}'$-fuzzy filters of $\mathfrak{L}$ are defined to be $\mathfrak{L}'$**-fuzzy sets** $\mathcal{F}$ on L such that $\mathcal{F}(1) = 1$ and for all $a, b \in L$, $\mathcal{F}(a) \curlywedge \mathcal{F}(b) = \mathcal{F}(a \sqcap b)$. (See Definition 2.1.1) .... page 15

Fuzzy set A mapping $\mathcal{F} : U \to L$ from a universe U to the set L of degrees of membership, with respect to a given complete lattice $\mathfrak{L} = [L, \sqcap, \sqcup]$. $\mathcal{F}$ is said to be an $\mathfrak{L}$-fuzzy set; for some $u \in U$, the value $\mathcal{F}(u)$ is said to be the *degree of membership* of u in $\mathcal{F}$. (See section 1.4) .... page 11

Labelled Formulae In this dissertation, usually *labelled formulae* are considered, i. e. **formulae** from a given many-valued **logic** augmented by a *label* which *assesses* the *validity* of the formula. Consequently, all the significant logical concepts like **model** and **semantic consequence** are formulated wrt $\mathfrak{L}$-fuzzy sets of formulae, for a given complete **lattice** $\mathfrak{L}$ of *labels*. A *labelled formula* $\mathfrak{x}$ is a special $\mathfrak{L}$-fuzzy set of formulae the *support* of which is a one-element set. It is identified with the *ordered pair* $\left[x, \mathfrak{x}(x)\right]$, where x is the (single) formula in the support of $\mathfrak{x}$. .... page 57

Lattice A triple $[L, \sqcap, \sqcup]$ with a non-empty set L (*domain*) and two binary operations $\sqcap, \sqcup$ (*meet* and *join*) such that $\sqcap, \sqcup$ are *commutative* and *associative* and fulfil the *absorption laws* (see Definition 1.3.1). .... page 8

Logic An ambiguous term used with many meanings in this dissertation. In the most formal sense, a logic is an arbitrary nonempty set Frm (called **formulae**) together with an arbitrary **closure operator** Cons (called **semantic consequence**) on this set. In the most colloquial sense, a logic is any formal system in which somehow *true* statements can be derived from other true statements. A logic is called *fuzzy* if the truth of statements or the validity of their derivation are subject to *vagueness* or *uncertainty*. In this dissertation, two notions of logic are distinguished. The *underlying logic* of all logical systems considered here is a usual *many-valued logic* (two-valued in special cases), but semantic consequence in the underlying logic is not studied. Instead, upon the underlying logic a *labelled logic* is defined in which *formulae* of the underlying logic are paired with *labels*. Semantic consequence always refers to labelled formulae. See also **Syntax**, **Semantics**, **Formulae**, **Labelled Formulae**, **Model**, **Semantic Consequence**. See chapter 3 ... page 39

Model In this dissertation, the *model relation* is the most basic high level logical concept, on which the **semantic consequence** operator is based. Intuitively, a *valuation* is said to be a *model* of a formula (or *satisfy* a formula) iff the formula is *true* under this valuation. In this dissertation, this definition is fuzzified in two ways. First, a formula can be *many-valued*, which means there might be a *range* of truth values for which the formula is considered to be *satisfied* by the valuation, and secondly, *satisfaction* can be given by degree, so that different truth values lead to different degrees of satisfaction (or modelness) of the given valuation for the formula. A correspondence between truth values being attained by the formula and degrees of satisfaction is given by the *label* of a labelled formula. As a conclusion, in this thesis, the *model relation* is a *ternary* one between the set $\mathfrak{S}$ of valuations, the set LFrm of labelled formulae and the set D of *degrees of validity*. ... page 59

PLFC *Possibilistic logic with fuzzy constants* (PLFC) is a labelled logic where the underlying logic is a special variant of many-valued first order logic and the labels are necessity degrees. PLFC is studied in several publications by Sandra Sandri and others [1, 29]. ... page 195

Propositional Logic One of the simplest **logics**, where **formulae** consist only of *propositional variables* and *operator symbols*, and where **semantics** consist of *valuations* induced by *assignments* of truth values to propositional variables. See Example 3.1.1 and Example 3.2.1 ... page 39

Semantic Consequence The central semantic concept of any **logic**. Formally, a *closure operator* on the set of all logical **formulae**. Intuitively, allows to derive *true* statements from other true statements. In this dissertation, the *semantic consequence relation* is applied to **labelled formulae**, where the label with which a labelled formula follows from an $\mathfrak{L}$-fuzzy set of formulae is an indicator for the *strength* (of a constraint on *truth* and *validity* of the formula) with which the formula is a consequence of the *knowledge* expressed by the $\mathfrak{L}$-fuzzy set of formulae. ... page 88

Semantics Intuitively, the *semantics* of a **logic** have to give *meaning* to the **formulae**. Ultimately, the definition of semantics has to provide a basis for the definition of the central semantic concept of **semantic consequence**. In this dissertation, the semantics are given by explicitly defining a set $\mathfrak{S}$ of *valuation functions* $\mathrm{Val} : \mathrm{Frm} \to T$, for a given lattice $\mathfrak{T} =_{\mathrm{def}} [T, \sqcap, \sqcup]$ of *truth values*. See section 3.2 ... page 43

Syntax The *syntax* of a **logic** describes the structure of logical **formulae**. In this dissertation, the syntax is given by explicitly defining the set Frm of **formulae**. See section 3.1 ... page 39

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