Asymptotic Theory for Range-Based Estimation of Integrated Variance of a Continuous Semi-Martingale*

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Abstract

We provide a set of probabilistic laws for range-based estimation of integrated variance of a continuous semi-martingale. To accomplish this, we exploit the properties of the price range as a volatility proxy and suggest a new method for non-parametric measurement of return variation. Assuming the entire sample path realization of the log-price process is available - and given weak technical conditions - we prove that the high-low statistic converges in probability to the integrated variance. Moreover, with slightly stronger conditions, in particular a zero drift-term, we find an asymptotic distribution theory. To relax the mean-zero constraint, we modify the estimator using an adjusted range. A weak law of large numbers and central limit theorem is then derived under more general assumptions about drift. In practice, inference about integrated variance is drawn from discretely sampled data. Here, we split the sampling period into sub-intervals containing the same number of price recordings and estimate the true range. In this setting, we also prove consistency and asymptotic normality. Finally, we analyze our framework in the presence of microstructure noise.

JEL Classification: C10; C22; C80.

Keywords: Central Limit Theorem; Continuous Semi-Martingale; High-Frequency Data; Integrated Variance; Market Microstructure Noise; Quadratic Variation; Range-Based Variance; Realized Variance; Stochastic Volatility Diffusion; Volatility Measurement.

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1 Introduction

The latent security price volatility is an essential measure of unexpected return variation and a key ingredient in several pillars of financial economics. Some years ago, academia customarily adopted constant volatility (e.g., Black & Scholes (1973)), despite the data argued against this assumption (e.g., Mandelbrot (1963)). Today, the gathering of empirical evidence makes us recognize that the conditional variance is both time-varying and highly persistent. Such stylized facts have been uncovered by the development and application of strict parametric models, such as ARCH (see, e.g., Engle (1982), Bollerslev (1986), Nelson (1991), and Bollerslev, Engle & Nelson (1994)), through stochastic volatility models (e.g., Hull & White (1987), and Ghysels, Harvey & Renault (1996)), and more recently non-parametric methods based on high-frequency data, the most conspicuous idea being the notion of "realized variance" (see, e.g., Taylor & Xu (1997), Andersen & Bollerslev (1998), Andersen, Bollerslev & Diebold (2002), Barndorff-Nielsen & Shephard (2004*a*)).

Unlike the daily squared return, realized variance is computed by summing high-frequency squared returns over the sampling period. The motivation is the theory of quadratic variation, which states that given weak regularity conditions, realized variance converges uniformly in probability to the quadratic variation of all semi-martingales as the sampling frequency tends to infinity (e.g., Protter (2004)). Since realized variance is, in theory, a consistent estimator of the latent volatility it may, as such, be regarded largely free of error, which justifies treating volatility as observed. In fact, recent work on volatility has progressed mainly by the growing availability of high-frequency data, which are routinely used in research projects now (for an incomplete list, see the above and also, e.g., Andersen, Bollerslev, Diebold & Ebens (2001), Andersen, Bollerslev, Diebold & Labys (2003), and Barndorff-Nielsen & Shephard (2002*a*, 2002*b*)).

In practice, the assumptions behind realized variance breaks down. Data limitations prevent the sampling frequency from rising without bound and, more notably, market microstructure effects contaminate high-frequency asset prices and induce autocorrelation in returns. This (potentially) inflates realized variance with a large cumulative error, invalidates its asymptotic properties, and in the presence of noise realized variance is both biased and inconsistent (e.g., Bandi & Russell (2004*a*, 2004*b*), Aït-Sahalia, Mykland & Zhang (2004), and Hansen & Lunde (2006)). To control the magnitude of the error, an optimal sampling-scheme is often selected with MSE-criteria by inducing a bias/variance trade-off, resulting in moderate sampling and throwing data away. Though current research seeks to develop methods of making realized variance robust against microstructure noise, getting accurate estimates of price uncertainty remains unsettled. Set against this backdrop, we suggest a simple approach to reduce the impact of microstructure noise, using another volatility proxy: the price range.

Range-based estimation of volatility, as developed in, e.g., Feller (1951), Garman & Klass

(1980), Parkinson (1980), Ball & Torous (1984), Rogers & Satchell (1991), Kunitomo (1992), and Yang & Zhang (2000), is very efficient, since the extremes are formed from the entire curve of the process and reveal more information than points sampled at fixed intervals. For example, the daily range is about five times more efficient than the daily squared return. However, Andersen & Bollerslev (1998, footnote 20) remark that "...compared to the measurement errors reported in Table 3, this puts the accuracy of the high-low estimator around that afforded by the intra-day sample variance based on two- or three-hour returns." Despite the daily range is more precise than the daily squared return, a consequence of its measurement error against realized variance is that the class of range-based proxies remains neglected. This is unfortunate, as range-based variance encompasses both time-varying dynamics, multivariate interactions and, moreover, the range is somewhat robust against common forms of microstructure noise, including bid-ask bounce and asynchronous trading (e.g., Alizadeh, Brandt & Diebold (2002) and Brandt & Diebold (2004)).

Nonetheless, one subject, with the potential of markedly reducing the error in range-based volatility, remains uncharted territory: intra-day range-based estimation of stochastic volatility. That is, while the range is recognized as being highly efficient, no one has explored the properties of price ranges sampled within the trading day in the context of estimating integrated variance. With access to high-frequency data, however, low-frequency measures are also obsolete in the range-based context. Thus, we conjecture that the failure of range-based volatility to match realized variance is grounded on an unfair comparison. The current paper seeks to remedy this surprising fact: we propose sampling and summing intra-day price ranges to obtain more precise estimates of integrated variance. For example, with exchange-rate data available around the clock, what can we expect by using, properly transformed, high-frequency ranges? Direct extrapolation suggests that, if daily ranges are as accurate as realized variance based on two- or three-hour returns, then hourly ranges, say, achieve the accuracy of realized variance sampled at five- or ten-minute intervals.

The remainder of the paper is organized as follows. In the next section, we unfold the necessary diffusion theory, present various ways of measuring financial market volatility and advance our methodological contribution by suggesting the high-frequency range-based estimator. Under mild regularity conditions, we prove consistency for the estimation method, and also provide an asymptotic distribution theory under stronger conditions. Section 3 illustrates the approach in more detail with a simple Monte-Carlo analysis to discover its relative merits, and we consider the impact of market microstructure noise. Rounding up, section 4 offers conclusions and sketches several important directions for further research.

2 A Semi-Martingale Framework

In this section, we propose a new method based on the price range for consistently estimating the *integrated variance* (IV). The theory is developed for the log-price of a univariate asset evolving in continuous-time over some interval, say $p = \{p_t\}_{t\in[0,\infty)}$. p is defined on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P}\right)$ and adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$; i.e. a family of σ-fields with $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $s \leq t < \infty$.

The basic building block is that p constitutes a (special) semi-martingale. This may, for instance, be achieved by invoking standard assumptions of no-arbitrage and a finite expected return. A fundamental result states that semi-martingales admit a decomposition into a term with sample paths of bounded variation plus an infinite variation local martingale (e.g., Back (1991) or Protter (2004)). The first (last) component represents the expected (unexpected) return innovation and, as such, semi-martingales provide a rather general class of stochastic processes. Much work in both theoretical and empirical finance or time-series econometrics is cast within this setting (see, e.g., Andersen, Bollerslev & Diebold (2002) or Barndorff-Nielsen & Shephard (2005) for excellent reviews and several references).

We specialize by assuming that the sample path of p is continuous.[1] Hence, we write the time t price in the generic form:

$$p_t = p_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_{s-} \mathrm{d}W_s, \quad \text{for } 0 \leq t < \infty \tag{2.1}$$

where $\mu = \{\mu_t\}_{t\in[0,\infty)}$ (the instantaneous mean) is a locally bounded predictable process, $\sigma = \{\sigma_t\}_{t\in[0,\infty)}$ (the spot volatility) is a strictly positive càdlàg process, $W = \{W_t\}_{t\in[0,\infty)}$ is a standard Brownian motion, and $\sigma_{s-} = \lim_{t\to s, t<s} \sigma_t$.

Except for the continuity of the local martingale, here comprised by $\left\{\int_0^t \sigma_{s-} \mathrm{d}W_s\right\}_{t\in[0,\infty)}$, we impose little structure on the model.[2] In fact, for semi-martingales with continuous martingale component as above, the form $\left\{\int_0^t \mu_s \mathrm{d}s\right\}_{t\in[0,\infty)}$ is implicit, when the drift-term is predictable (in the absence of arbitrage). Note in passing that, without loss of generality, our conditions imply we can restrict the functions μ and σ to be bounded (e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2004)).

The objective is to estimate a suitable measure of the return variation over a sub-interval $[a, b] \subseteq [0, \infty)$, termed the sampling period or measurement horizon. We assume $[a, b] = [0, 1]$; this will be thought of as representing a trading day, but the choice is arbitrary and can serve as a normalization. At any two distinct sampling times t_{i-1} and t_i, with $0 \leq t_{i-1} < t_i \leq 1$, the intra-day (in general, intra-period) return over $[t_{i-1}, t_i]$ is denoted by $r_{t_i,\Delta_i} = p_{t_i} - p_{t_{i-1}}$,

[1]We adopt the continuity assumption as a starting point only. In concurrent work, we are analyzing the properties of our method, when p exhibits jumps.

[2]All continuous local martingales, whose quadratic variation (to be defined in a moment) is absolutely continuous has the stochastic volatility representation of the second term in equation (2.1), e.g., Doob (1953). We refer to Barndorff-Nielsen & Shephard (2004*a*) for a discussion of the details of this aspect.

where $\Delta_i = t_i - t_{i-1}$. As a convention, we call the return strecthing $[0, 1]$ for the inter-day (in general, inter-period) or daily return.

From the theory of stochastic calculus, it is well-known that *quadratic variation* (QV) - or conditional expectations thereof - are natural measures of variability for the class of semi-martingales. Specifically, for every semi-martingale, $X = \{X_t\}_{t \in [0,\infty)}$, there exists a unique increasing QV process, $[X, X] = \{[X, X]_t\}_{t \in [0,\infty)}$, given by:

$$[X, X]_t = X_t^2 - 2 \int_0^t X_{s-} \mathrm{d}X_s \tag{2.2}$$

with $X_{s-} = \lim_{t \to s, t<s} X_t$.

In the absence of jumps in p, QV is entirely induced by innovations to the continuous local martingale. Moreover, QV coincides with IV that is central to financial economics, whether in asset- and derivatives-pricing, portfolio selection or risk management (e.g., Hull & White (1987)). IV is the object of interest here, and we recall its definition:

$$IV = \int_0^1 \sigma_s^2 \mathrm{d}s \tag{2.3}$$

The econometrical problem is that IV is latent, which renders empirical estimation of this quantity a crucial issue in practice. We shall briefly review the literature on existing methods for measuring IV, before carrying on to suggest a new approach.

2.1 Return-Based Estimation of Integrated Variance

Not long ago, the daily squared return was employed as a non-parametric ex-post measure of IV. Although the estimator is (conditionally) unbiased under some auxiliary conditions, this method is not optimal, as the inter-period return is a noisy indicator of volatility. With the advent of high-frequency data, more recent work computes *realized variance* (RV), being the summation of squared intra-period returns sampled over non-overlapping intervals (see, e.g., Taylor & Xu (1997), Andersen & Bollerslev (1998), Andersen, Bollerslev & Diebold (2002), Andersen et al. (2003), and Barndorff-Nielsen & Shephard (2004*a*)).[3]

More formally, consider a (deterministic) partition $0 = t_0 < t_1 < \cdots < t_n = 1$. Then, we define RV at sampling times $\Xi = \{t_i \mid i = 0, 1, \ldots, n\}$ by setting:

$$RV^{\Xi} = \sum_{i=1}^{n} r_{t_i, \Delta_i}^2 \tag{2.4}$$

Intuitively, the daily squared return is the least efficient member of this class of estimators,

[3]The work of Poterba & Summers (1986), French, Schwert & Stambaugh (1987), Schwert(1989, 1990), and Hsieh (1991), among others, pioneered the construction of a volatility proxy by exploiting data sampled at a higher frequency than the measurement horizon.

and the justification for the RV procedure builds directly on the theory of QV.[4] An equivalent definition of QV is the probability limit of RV^{Ξ}, when the diameter of Ξ tends to zero (e.g., Protter (2004)). Hence, as $n \to \infty : \max_{1 \leq i \leq n}\{\Delta_i\} \to 0$, it follows that in our model:

$$RV^{\Xi} \xrightarrow{p} IV \tag{2.5}$$

The convergence is also locally uniform in time. Thus, given a complete record of p, IV is estimated with arbitrary accuracy, effectively making it observed.

Of course, we are forced to work with a set of discretely sampled data in applications, but the theory encourages using high-frequency proxies to reduce the measurement error. And though an irregular partition of the sampling period suffices for consistency, an equidistant time series of intra-period returns is often computed in practice by various approaches, such as linear interpolation (see, e.g., Andersen & Bollerslev (1997*a*, 1997*b*, 1998), and Andersen et al. (2001)), or the previous-tick method suggested in Wasserfallen & Zimmermann (1985).[5] The equidistant RV based on n high-frequency returns, sampled over non-overlapping intervals of length $\Delta = 1/n$, is defined as:

$$RV^{\Delta} = \sum_{i=1}^{n} r_{i\Delta,\Delta}^2 \tag{2.6}$$

Barndorff-Nielsen & Shephard (2002*b*) found a distribution theory for RV^{Δ} (in relation to IV). The limit law of the scaled difference between RV^{Δ} and IV is mixed Gaussian:

$$n^{1/2}\left(RV^{\Delta} - IV\right) \xrightarrow{d} MN\left(0, 2\int_0^1 \sigma_s^4 \mathrm{d}s\right) \tag{2.7}$$

where $\int_0^1 \sigma_s^4 \mathrm{d}s$ is the *integrated quarticity* (IQ). Thus, the size of the error bounds for RV^{Δ} is positively related to the level of σ. Barndorff-Nielsen & Shephard (2002*b*) also derived a feasible central limit theorem (CLT), where all quantities except IV can be computed directly from the data. This was done by simply replacing the latent IQ by a consistent estimator, like $RQ^{\Delta} = n/3 \sum_{i=1}^{n} r_{i\Delta,\Delta}^4$, making it possible to construct approximate confidence bands for RV to measure the estimation error involved with finite samples.

2.2 Range-Based Estimation of Integrated Variance

In practice, the choice of volatility proxy is less obvious, as financial markets are not frictionless and microstructure bias sneaks into RV. With noisy prices, for instance, RV is both biased and inconsistent, see, e.g., Bandi & Russell (2004*a*, 2004*b*), Aït-Sahalia et al. (2004), Hansen & Lunde (2006).[6] Thus, RV is error-free in theory, but in reality this condition is not satisfied.

[4] In general, the (non-normed) qth order realized power variation is defined as $\sum_{i=1}^{n} |r_{t_i,\Delta_i}|^q$, with $q > 0$. For Brownian semi-martingales, only $q = 2$ leads to a non-trivial limit (IV). Barndorff-Nielsen & Shephard (2003*b*, 2004*c*) use a normalizing sequence to compute $\sum_{i=1}^{n} \Delta_i^{1-q/2} |r_{t_i,\Delta_i}|^q$. They prove that, under suitable conditions, $\sum_{i=1}^{n} \Delta_i^{1-q/2} |r_{t_i,\Delta_i}|^q \xrightarrow{p} \mathbb{E}\left[|\phi|^q\right] \int_0^1 \sigma_s^q \mathrm{d}s$, where ϕ is a standard normal random variable.

[5] A side-effect of the linear interpolation method is that - with a fixed number of discretely sampled data - $RV^{\Xi} \xrightarrow{p} 0$ as $n \to \infty$. Intuitively, a straight line is the "minimum-variance" path between two points.

[6] Technically, with IID noise, RV diverges to infinity almost surely, i.e. $RV^{\Xi} \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$.

Academia has recognized this by developing bias-reducing techniques (e.g., pre-whitening of the return series with moving average or autoregressive filters as in Andersen et al. (2001) and Bollen & Inder (2002), or kernel-based estimation as in Zhou (1996) and Hansen & Lunde (2006)). In empirical work, the benefit of more frequent sampling is traded off against the damage caused by cumulating noise, and - using various criteria for picking the optimal sampling frequency - the result is often moderate sampling (e.g., at the 5-, 10-, or 30-minute frequency), whereby data are discarded.

The pitfalls of RV motivate our choice of another proxy with a long and colorful history in finance: the price range or high-low. Using a terminology similar to the above, we define the intra-period range at sampling times t_{i-1} and t_i, with $0 \leq t_{i-1} < t_i \leq 1$, as:

$$s_{p_{t_i,\Delta_i}} = \sup_{t_{i-1} \leq s,t \leq t_i} \{p_t - p_s\} \tag{2.8}$$

Compared to the return over $[t_{i-1}, t_i]$, r_{t_i,Δ_i}, the extra subscript p indicates that we are taking supremum of the price process. Below, we also need the range of a standard Brownian motion over $[t_{i-1}, t_i]$, which is denoted by $s_{W_{t_i,\Delta_i}} = \sup_{t_{i-1} \leq s,t \leq t_i} \{W_t - W_s\}$. We use the short-hand notation s_p and s_W for the inter-period ranges.

2.2.1 The Inter-Period Range

Newspapers usually report the high-low of the preceding trading day's security price next to the open-close. So without high-frequency data at hand, the daily range - printed freely in the business press - provides indirect access to the intra-day price information by screening all data over the sampling period. Thus, in a setup where tick-by-tick data are not accessible, this is a major advantage from using the range as a volatility proxy. In contrast, RV is restricted to inter-day estimation, rendering it inefficient.[7]

The nature of the daily range is appealing: suppose the asset price fluctuates wildly within the sampling period, but happens to end near the starting point; then an inter-period high-low, unlike the corresponding return, correctly reports the level of volatility as high. In technical analysis it also figures as a key indicator when constructing so-called "candlestick plots," e.g., Edwards & Magee (2001).

Its attractiveness is not based on just intuitive grounds, however. The theoretical underpinnings go a long way back.[8] Feller (1951) found the distribution of the range by using the

[7]Of course, high-frequency data are increasingly available, and we return to this setting shortly.

[8]There are basically two branches in the range-based volatility context: i) relies purely on the high-low, while ii) adds the open-close, e.g., Garman & Klass (1980), Beckers (1983), Ball & Torous (1984), Rogers & Satchell (1991) and Yang & Zhang (2000). Brown (1990) and Alizadeh et al. (2002) argue against inclusion of the latter on the grounds that they are highly contaminated by microstructure effects. Thus, throughout we only report on the high-low estimator.

theory of Brownian motion.[9] According to his work, the density of the range from a standard Brownian motion, over a general interval of length Δ_i ending at t_i, is given by:

$$\mathbb{P}\left[s_{W_{t_i,\Delta_i}} = r\right] = 8\sum_{x=1}^{\infty}(-1)^{x-1}\frac{x^2}{\sqrt{\Delta_i}}\phi\left(\frac{xr}{\sqrt{\Delta_i}}\right) \tag{2.9}$$

with $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. The infinite series is evaluated by a suitable truncation. In Figure 1, we plot the probability function of the daily range (taking $t_i = \Delta_i = 1$).

[INSERT FIGURE 1 ABOUT HERE]

In a historical context, another reason for selecting the daily high-low to estimate IV relates to its sampling stability. Viewed separatively, the density function does not reveal this feature. Therefore, the figure also displays the distribution of the daily absolute return. By comparing these proxies, the efficiency of the range, or in other words its lower variance vis-á-vis the return, is more evident.

Parkinson (1980) advanced Feller's insights by deriving the moment-generating function for the range of a Wiener diffusion with $\mu = 0$ and $\sigma_t = \sigma$.[10] For the rth moment:

$$\mathbb{E}\left[s^r_{p_{t_i,\Delta_i}}\right] = \lambda_r \Delta_i^{r/2}\sigma^r, \quad \text{for } r \geq 1 \tag{2.10}$$

where $\lambda_r = \mathbb{E}\left[s^r_W\right]$. In particular, $\lambda_2 = 4\ln(2)$ and $\lambda_4 = 9\zeta(3)$ are needed below.[11] For daily sampling (again with $t_i = \Delta_i = 1$), this gives an unbiased estimator of IV - equal to σ^2 in this simple model - by scaling s^2_p down with λ_2.

If $\mu \neq 0$, the inter-period range cannot distinguish drift from volatility and is upward biased. A number of methods have been suggested to accommodate non-zero, but constant, μ. Rogers & Satchell (1991) used the exponential distribution and Wiener-Hopf factorization of a Lévy process to produce a range-based estimator that is independent of μ. Yang & Zhang (2000) developed a multi-period estimator, while Kunitomo (1992) suggested the range of a Brownian bridge (from 0 to 0), which removes drift by construction.[12] The latter is also applied in the sequel (see, section 2.2.5).

Arguably, a process with constant μ and σ is irrelevant from an empirical point of view. The most critical aspect of range-based theory is perhaps the homoscedasticity constraint forced

[9]Assuming Gaussian increments makes the range more restricted vis-á-vis RV, as the latter is non-parametric and consistent for all return distributions. Nonetheless, the majority of empirical work is conducted within the log-normal diffusion framework.

[10]Note, σ does double-duty; representing either the process $\sigma = \{\sigma_t\}_{t\in[0,\infty)}$ or a constant diffusion parameter $\sigma_t = \sigma$. The meaning is clear from the context.

[11]The explicit formula for λ_r is: $\lambda_r = \frac{4}{\sqrt{\pi}}(1 - \frac{4}{2^r})2^{\frac{r}{2}}\Gamma(\frac{r+1}{2})\zeta(r-1)$, for real $r \geq 1$; where $\Gamma(x)$ and $\zeta(x)$ denote the Gamma and Riemann's zeta function, respectively.

[12]Making this transformation requires access to high-frequency data; a prerequisite, which is fundamental for our analysis below.

upon σ. An overwhelming amount of research indicates that the conditional variance is time-varying, see, e.g., Ghysels et al. (1996). Nonetheless, to our knowledge there exists no, or little, theory about range-based estimation of IV in the presence of a continually evolving diffusion parameter.[13] Previous work achieve (randomly) changing volatility by holding σ_t fixed within the trading day, while allowing for (stochastic) shifts between them (e.g., Alizadeh et al. (2002), Brunetti & Lildholdt (2002)).[14] Still, there are strong intra-day movements in σ_t (e.g., Andersen & Bollerslev (1997*b*)).

A main objective of this paper is therefore to extend the theoretical domain of the extreme value method to a more general class of stochastic processes. Contrary to the extant research, we develop a statistical framework for the Brownian semi-martingale in equation (2.1), featuring less restrictive dynamics for μ and σ.

The contribution is three-fold in this respect. First, we deal with range-based estimation of time-varying volatility, when μ and σ are (possibly) continuously evolving random functions. Such a model is capable of - but also necessary for - fitting the stylized facts of financial markets data; in particular the second moment structure of the conditional return distribution. Second, we develop a new method for non-parametric measurement of IV with a variant of the range-based estimator that, unlike the existing theory, uses high-frequency data more efficiently. This provides a framework for comparing our results to RV. Third, we formalize the approach by deriving a set of probabilistic laws for sampling intra-period high-lows.

2.2.2 An Intra-Period Range-Based Estimator

Despite the high-low is more efficient than RV on an inter-period basis, the sampling variation of a low-frequency proxy is too high. By analogy with the daily squared return, the daily squared range is noisy. Thus, while the inter-period range processes the entire price trajectory, it does so inefficiently. For example, using simulations Andersen & Bollerslev (1998) found that mean-squared error (MSE) of the daily range is about that afforded by RV computed from two- or three-hour returns.

In the presence of tick-by-tick data, we can, however, exploit the insights of RV to construct more precise range-based estimates of IV. The idea is simple enough: we split the measurement horizon in minor pieces and sample high-lows within the trading day. Accordingly, consider again the partition $0 = t_0 < t_1 < \cdots < t_n = 1$. We then propose a high-frequency *range-based*

[13] An exception is Gallant, Hsu & Tauchen (1999), who estimate two-factor stochastic volatility models in a general continuous-time framework. They derive the density function of the range in this setting, but do not otherwise explore its theoretical properties.

[14] Brunetti & Lildholdt (2002) consider a discrete-time model with GARCH dynamics and show that the scaled squared range is unbiased for the unconditional variance.

variance (RBV) estimator of IV at sampling times Ξ:

$$RBV^{\Xi} = \frac{1}{\lambda_2} \sum_{i=1}^{n} s_{p_{t_i},\Delta_i}^2 \tag{2.11}$$

or,

$$RBV^{\Delta} = \frac{1}{\lambda_2} \sum_{i=1}^{n} s_{p_{i\Delta},\Delta}^2 \tag{2.12}$$

where RBV^{Δ} is the equidistant version.

The intra-day range-based statistic has three advantages compared to the previous return- and range-based methods suggested in the extant research on volatility measurement. First, in the spirit of Aït-Sahalia et al. (2004), the approach uses all data points (regardless of the sampling frequency); whereby we avoid neglecting any information about IV. Second, the theoretical efficiency of our estimator is several times that obtained with RV, leading to narrower confidence bands for IV (see below). Third, to a certain extent the estimator is more efficient, compared to RV, in the presence of common forms of market microstructure noise. This is indicated by the simulation evidence in Alizadeh et al. (2002). We return to the impact of microstructure bias in the Monte Carlo section below.

2.2.3 Convergence in Probability to the Integrated Variance

At a minimum, the estimator should be consistent for IV. In the classic time-invariant driftless diffusion setting for the range, i.e. $\mu = 0$ and $\sigma_t = \sigma$, proving this property of RBV^{Δ} is trivial.[15] As the fill-in asymptotics start operating by letting $n \to \infty$, we achieve an increasing sequence of IID random variables. Suitably transformed to unbiased measures of σ^2, the consistency follows from a standard law of large numbers by averaging. To see this, note that $\mathbb{E}\left(RBV^{\Delta}\right) = \sigma^2$ and $\text{var}\left(RBV^{\Delta}\right) = \Lambda n^{-1}\sigma^4$, with $\Lambda = \left(\lambda_4 - \lambda_2^2\right)/\lambda_2^2$. Hence, MSE $\to 0$ as $n \to \infty$, which is sufficient.

If μ and σ are stochastic, establishing the large-sample properties of RBV^{Δ} is more involved, but nonetheless feasible. Overall, the basic idea extends to a general continuous semi-martingale, given the appropriate regularity conditions. To justify our approach, we therefore progress by deriving limit theorems for RBV^{Δ}. Its probability limit is stated first.[16]

Theorem 1 *Let μ and σ fulfil the conditions following equation* (2.1)*. As $n \to \infty$*

$$RBV^{\Delta} \xrightarrow{p} IV \tag{2.13}$$

[15] Henceforth, we use equidistant estimation, as this simplifies the notational burden in the proofs. All results can be generalized to irregular subdivisions of the sampling period with just a slight modification of the conditional variance in the CLT.

[16] Throughout the paper, proofs of the theorems are reserved for the appendix.

This result mirrors the consistency of RV that by definition converges in probability to the limit process IV. It speaks directly for more efficient use of high-frequency data in range-based estimation of IV by sampling intra-period ranges.

That Theorem 1 allows for general specifications of μ, for instance, is a consequence of the fact that for continuous-time arbitrage-free price processes, the expected move in p is an order of magnitude lower than variation induced by the local martingale; i.e. the stochastic volatility component $\left\{\int_0^t \sigma_{s-} \mathrm{d}W_s\right\}$ here. Thus, while the inter-period range is sensitive to drift, the mean component vanishes (sufficiently fast) as $n \to \infty$.

From the perspective of volatility measurement, the analysis extends the theory of RBV - where the time-invariant geometric Brownian motion is a leading model - much further. Except for weak technical conditions on σ, no knowledge about its dynamics is needed. Hence, we allow for very general continuous-time processes, including, but not limited to, volatility models that possess leverage, long-memory, jumps or diurnal effects. This is certainly not appreciated in the previous literature.

2.2.4 Asymptotic Mixed Normality of Range-Based Variance

In empirical work, we often compute confidence bands as a guide to the error made from estimation based on a finite sample. Theorem 1 does not reveal the precision IV is estimated with for moderate n. As an approximation to the finite-sample variation of RBV^{Δ}, we now develop an asymptotic distribution theory for it. Here, the weak assumptions on μ and σ are too general, and we need stronger conditions to prove a CLT. In particular, some smoothness of σ is required.

To avoid any confusion about our terminology, we first present the definition of a special mode of convergence that is probably not widely familiar to people in econometrics and finance; namely stable convergence in law.

Definition 1 *A sequence of random variables, $\{X_n\}_{n \in \mathbb{N}}$,* converges stably in law with limit *X, defined on an appropriate extension of $\left(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \in [0,\infty)}, \mathbb{P}\right)$, if and only if for every $\mathcal{F}$-measurable, bounded random variable Y and any bounded, continuous function g, the convergence* $\lim_{n \to \infty} \mathbb{E}\left[Y g\left(X_n\right)\right] = \mathbb{E}\left[Y g\left(X\right)\right]$ *holds.*

Throughout the paper, the symbol "$X_n \xrightarrow{d_s} X$" will be used to denote stable convergence in law (see, e.g., Rényi (1963) or Aldous & Eagleson (1978) for more details). Note that such asymptotic behavior is slightly stronger than - and implies - weak convergence, which may equivalently be defined by taking $Y = 1$. From this, we now state the next result, which is a non-standard CLT.

Theorem 2 *Suppose $\mu = 0$ and that σ is Hölder continuous of order $\gamma > 1/2$ in $\mathcal{L}^2\left(\mathbb{P}\right)$, i.e.*

$\mathbb{E}\Big[(\sigma_t - \sigma_s)^2\Big] = O\left(\mid t - s \mid^{2\gamma}\right)$. *Then,*

$$n^{1/2}\left(RBV^{\Delta} - IV\right) \xrightarrow{d_s} \Lambda^{1/2} \int_0^1 \sigma_s^2 \mathrm{d}B_s \tag{2.14}$$

where $B = \{B_t\}_{t \in [0,1]}$ *is a standard Brownian motion, independent of* σ *and* W.

Thus, RBV^{Δ} minus IV converges stably at rate $n^{1/2}$ to a process being the stochastic integral with respect to a Brownian motion, which is unrelated to the driving terms σ and W. A crucial feature of the theorem is the stochastic independence of B, as this implies $n^{1/2}\left(RBV^{\Delta} - IV\right)$ has a mixed Gaussian limit law, with σ governing the mixture, i.e.:

$$n^{1/2}\left(RBV^{\Delta} - IV\right) \xrightarrow{d} MN\left(0, \Lambda \int_0^1 \sigma_s^4 \mathrm{d}s\right) \tag{2.15}$$

Remark 1 *The* Λ *scalar in front of IQ in equation* (2.15) *is roughly 0.4. In contrast, the number appearing in the CLT for* RV^{Δ} *is* 2.

In reflection of this, we revise an incorrect conjecture heard in the literature; that RV theoretically outperforms the range. An assertion that originated from the fact that $RV^{\Delta} \xrightarrow{p} IV$ as $n \to \infty$, in contrast to the scaled squared inter-period range. RBV^{Δ} has the same property and, indeed, given a sufficiently fine partition of the sampling period, the error of RBV^{Δ} is about one-fifth of RV^{Δ}. This is not surprising: the range inspects all the data, whereas RV is based on high-frequency returns sampled at fixed points in time. As, for the moment, p is assumed fully observed, RV^{Δ} is neglecting a lot of information. But even with discrete-time data, RV^{Δ} is a lower bound for the efficiency of RBV^{Δ} (see, section 2.2.6).

Remark 2 *The Lipschitz-like condition on* σ *is a limitation; it excludes some models used in both theoretical and applied work. While this assumption is sufficient, it may not be necessary. In fact, it seems possible to relax it, but we haven't formally proved this yet. In the special case that* $\sigma \perp\!\!\!\perp W$, *it nevertheless suffices to assume that* σ *satisfies the condition:*

$$\sigma_t^2 = \sigma_0^2 + \int_0^t \mu_s^* \mathrm{d}s + \int_0^t \sigma_{s-}^* \mathrm{d}W_s^*, \qquad \text{for } 0 \leq t < \infty \tag{H}$$

where $\mu^* = \{\mu_t^*\}_{t \in [0,\infty)}$ *is locally bounded,* $\sigma^* = \{\sigma_t^*\}_{t \in [0,\infty)}$ *is càdlàg, and* $W^* = \{W_t^*\}_{t \in [0,\infty)}$ *is a Brownian motion (see "Proof of Remark" in the appendix). It comes at the cost of excluding real-world stylized facts such as leverage effects in stock markets.*

The integral appearing on the right-hand side in (2.15) is statistically infeasible and cannot be computed directly from the data. We resolve this by setting:

$$RBQ^{\Delta} = \frac{n}{\lambda_4} \sum_{i=1}^{n} s_{p_{i\Delta,\Delta}}^4 \tag{2.16}$$

With techniques similar to the proof of Theorem 1 we have that $RBQ^{\Delta} \xrightarrow{p} \int_0^1 \sigma_s^4 \mathrm{d}s$. Thus, by exploiting the properties of stable convergence, we get the next corollary.

Corollary 1 *Given the assumptions of Theorem 2, it holds that:*

$$\frac{n^{1/2}\left(RBV^{\Delta}-IV\right)}{\left(\Lambda RBQ^{\Delta}\right)^{1/2}} \xrightarrow{d} N(0,1) \tag{2.17}$$

2.2.5 A Central Limit Theorem with Drift

Although the probability limit of RBV^{Δ} is IV under weak regularity conditions on μ and σ, we imposed a mean-zero restriction to establish a CLT. But even though drift is small over shorter time-periods - like a day - assets trade with a positive expected return. Therefore, a more general distribution theory for the range is required.

We exploit a change-of-variables:

$$\tilde{p}_t = p_t - p_{t_{i-1}} - \frac{t-t_{i-1}}{\Delta_i} r_{t_i,\Delta_i}, \quad \text{for } t_{i-1} \le t \le t_i \text{ and } i = 1,\ldots,n \tag{2.18}$$

Note that the transformation maps p to 0 at all sampling times $t_i \in \Xi$. Its main purpose is to eliminate drift.

From $\tilde{p} = \{\tilde{p}_t\}_{t\in[0,1]}$ we sample new high-lows, termed "adjusted ranges." The theory for the adjusted range was developed by Kunitomo (1992), but a further analysis has been missing, as until recently high-frequency data were not available. Thus, with only daily data the approach is not feasible, because the intra-day trajectory of p is required to construct (a non-trivial) $\tilde{p}$.

We denote the intra-period adjusted range of $\tilde{p}$ over $[t_{i-1}, t_i]$ by:

$$s_{\tilde{p}_{t_i,\Delta_i}} = \sup_{t_{i-1}\le s,t\le t_i} \{\tilde{p}_t - \tilde{p}_s\} \tag{2.19}$$

and

$$s_{\tilde{W}_{t_i,\Delta_i}} = \sup_{t_{i-1}\le s,t\le t_i} \{\tilde{W}_t - \tilde{W}_s\} \tag{2.20}$$

with $\tilde{W}_t = W_t - W_{t_{i-1}} - (t-t_{i-1})/\Delta_i\left(W_{t_i} - W_{t_{i-1}}\right)$, is notation for the adjusted range of a standard Brownian motion over $[t_{i-1}, t_i]$, adopting the convention from above. $\{\tilde{W}_t\}_{t\in[t_{i-1},t_i]}$ connects $W_{t_{i-1}}$ at t_{i-1} with itself at t_i and normalizes by subtracting $W_{t_{i-1}}$. For daily sampling (taking $t_i = \Delta_i = 1$), $\tilde{W} = \{\tilde{W}_t\}_{t\in[0,1]}$ is a Brownian bridge on $[0,1]$, and $s_{\tilde{W}}$ is short-hand notation for the high-low of this process. Notice the remarkable sampling stability of $s_{\tilde{W}}$ by its density function, as plotted in Figure 1.

An equidistant estimator of IV is now defined by setting:

$$\widetilde{RBV}^{\Delta} = \frac{1}{\tilde{\lambda}_2}\sum_{i=1}^{n} s^2_{\tilde{p}_{i\Delta,\Delta}} \tag{2.21}$$

where $\tilde{\lambda}_r = \mathbb{E}\left[s^r_{\tilde{W}}\right]$. In particular, $\tilde{\lambda}_2 = \pi^2/6$ and $\tilde{\lambda}_4 = \pi^4/30$.[17] Below, we also construct the ratio $\tilde{\Lambda} = \left(\tilde{\lambda}_4 - \tilde{\lambda}_2^2\right)/\tilde{\lambda}_2^2 = 0.2$.

[17] A closed-form expression for $\tilde{\lambda}_r$ was found by Kunitomo (1992): $\tilde{\lambda}_r = 2^{1-r/2}\Gamma(\frac{r+2}{2})\zeta(r)$, for real $r \ge 2$, where $\Gamma(x)$ and $\zeta(x)$ are the Gamma and Riemann's zeta function. For $r = 1 : \tilde{\lambda}_1 = \sqrt{\frac{\pi}{2}}$.

The consistency of Theorem 1 holds for $\widetilde{RBV}^{\Delta}$, i.e. $\widetilde{RBV}^{\Delta} \xrightarrow{p} IV$ as $n \to \infty$ (proof omitted for brevity). Moreover, the next result gives a CLT under weaker conditions on μ.

Theorem 3 *If μ is continuous and σ is Hölder continuous of order $\gamma > 1/2$ in $\mathcal{L}^2(\mathbb{P})$:*

$$n^{1/2}\left(\widetilde{RBV}^{\Delta} - IV\right) \xrightarrow{d_s} \tilde{\Lambda}^{1/2}\int_0^1 \sigma_s^2 \mathrm{d}B_s \tag{2.22}$$

where $B \perp\!\!\!\perp \mu, \sigma, W$.

Remark 3 *The conditional variance of the limit process in* (2.22) *is $\tilde{\Lambda}\int_0^1 \sigma_s^4 ds$. Hence, $\widetilde{RBV}^{\Delta}$ is, asymptotically, ten times more efficient than RV^{Δ}.*

Theorem 3 leads to the mixed normal distribution:

$$n^{1/2}\left(\widetilde{RBV}^{\Delta} - IV\right) \xrightarrow{d} MN\left(0, \tilde{\Lambda}\int_0^1 \sigma_s^4 \mathrm{d}s\right) \tag{2.23}$$

with a feasible estimator of IQ defined by:

$$\widetilde{RBQ}^{\Delta} = \frac{n}{\tilde{\lambda}_4}\sum_{i=1}^{n} s_{\tilde{p}_{i\Delta},\Delta}^4 \tag{2.24}$$

If $\sigma \perp\!\!\!\perp W$, we also have asymptotic normality under the weaker assumption (H).

2.2.6 Discretely Sampled High-Frequency Data

Estimation of IV relies on discretely sampled high-frequency data in practice. As p is not continuously monitored, we cannot extract the supremum of the increments to the semi-martingale. Hence, if unaccounted for, the range is a downward biased measure of IV (e.g. Garman & Klass (1980)). With a fixed number of data points, the discretization error will be progressively more severe as n grows larger.

Rogers & Satchell (1991) improved upon the earlier work and simulation evidence from Garman & Klass (1980) to propose a technique for bias-correcting the range; a method that largely removed the error from a numerical perspective.

We develop the point by using two estimators that account for the number of transactions used in forming the high-low; thereby removing the source of the bias. To formalize this idea, a bit more notation is required. Assume, without loss of generality, that mn equidistant data are available. We split them into n intervals with m observations each and denote the maximum observed price difference by:

$$m_{p_{i\Delta},\Delta} = \max_{1 \le s,t \le m}\left\{p_{(i-1)/n+t/mn} - p_{(i-1)/n+s/mn}\right\} \tag{2.25}$$

$$m_{\tilde{p}_{i\Delta},\Delta} = \max_{1 \le s,t \le m}\left\{\tilde{p}_{(i-1)/n+t/mn} - \tilde{p}_{(i-1)/n+s/mn}\right\} \tag{2.26}$$

Also, let $m_W = \max_{1 \leq s,t \leq m} \{W_{t/m} - W_{s/m}\}$, and $m_{\tilde{W}} = \max_{1 \leq s,t \leq m} \{\tilde{W}_{t/m} - \tilde{W}_{s/m}\}$. Then, we define the new high-low statistics by setting:

$$RBV_m^{\Delta} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^{n} m_{p_{i\Delta},\Delta}^2 \tag{2.27}$$

and

$$\widetilde{RBV}_m^{\Delta} = \frac{1}{\tilde{\lambda}_{2,m}} \sum_{i=1}^{n} m_{\tilde{p}_{i\Delta},\Delta}^2 \tag{2.28}$$

where $\lambda_{r,m} = \mathbb{E}[m_W^r]$, $\tilde{\lambda}_{r,m} = \mathbb{E}[m_{\tilde{W}}^r]$. The constants appearing in these expressions are nothing more than (the reciprocal of) the rth moment of the range of a standard Brownian motion - or Brownian bridge for the latter - over a unit interval, when we only observe the continuous-time process at m points in time.

To the best of our knowledge, there are no explicit formulas for $\lambda_{r,m}$ and $\tilde{\lambda}_{r,m}$, but they are easily computed to any degree of accuracy from simple simulations. Figure 2 details this for the example $r = 2$ and selected values of m (see below).

[INSERT FIGURE 2 ABOUT HERE]

Of course, $\lambda_{2,m} \to \lambda_2$ and $\tilde{\lambda}_{2,m} \to \tilde{\lambda}_2$ as $m \to \infty$, but note also that $\lambda_{2,1} = 1$ (this defines RV^{Δ}). The downward bias reported from previous simulation studies on the range is a consequence of the fact that λ_2 was incorrectly applied in place of $\lambda_{2,m}$, as the bias is in one-to-one correspondence with the difference.

Having completed these preliminaries, we prove consistency and asymptotic normality for the estimators in equations (2.27) and (2.28) by letting $n \to \infty$. Note, m is not required to approach infinity; convergence to a (positive) integer is sufficient.[18]

Theorem 4 *Assume $n \to \infty$ and $m \to c \in \mathbb{N} \cup \{\infty\}$. Then,*

$$RBV_m^{\Delta} \xrightarrow{p} IV \tag{2.29}$$

Moreover, if $\mu = 0$ and σ is Hölder continuous of order $\gamma > 1/2$ in $\mathcal{L}^2(\mathbb{P})$:

$$n^{1/2}\left(RBV_m^{\Delta} - IV\right) \xrightarrow{d_s} \Lambda_c^{1/2} \int_0^1 \sigma_s^2 \mathrm{d}B_s$$

where $\Lambda_c = \left(\lambda_{4,c} - \lambda_{2,c}^2\right)/\lambda_{2,c}^2$ and $B \perp\!\!\!\perp \sigma, W$. Finally,

$$\frac{n^{1/2}\left(RBV_m^{\Delta} - IV\right)}{\left(\Lambda_m RBQ_m^{\Delta}\right)^{1/2}} \xrightarrow{d} N(0,1) \tag{2.30}$$

with $\Lambda_m = \left(\lambda_{4,m} - \lambda_{2,m}^2\right)/\lambda_{2,m}^2$ and

$$RBQ_m^{\Delta} = \frac{n}{\lambda_{4,m}} \sum_{i=1}^{n} m_{p_{i\Delta},\Delta}^4$$

[18]We only state the theorem for RBV_m^{Δ}. The version with $\widetilde{RBV}_m^{\Delta}$ is identical, except for the same modifications made in the previous theorem.

Remark 4 *A corollary worth pointing out is that our distribution theory nests RV, in the sense that for the special case $m = 1$, Theorem 4 provides a CLT for RV^{Δ}, as discussed in, e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2004) or Barndorff-Nielsen & Shephard (2004*b*).*

Again, if $\sigma \perp\!\!\!\perp W$, Theorem 4 delivers a mixed normal as the limit distribution, even when σ only satisfies assumption (H).

[INSERT FIGURE 3 ABOUT HERE]

Figure 3 provides an impression of the efficiency of RBV_m^{Δ} and $\widetilde{RBV}_m^{\Delta}$. Along the y-axis, it depicts Λ_m and $\tilde{\Lambda}_m$, respectively, as a function of the number of returns, m, on the x-axis. Several hundred recordings are needed to achieve a good fit to the asymptotic values of 0.4 and 0.2, but the steep initial decline renders the advantage of the high-low statistics huge compared to RV even for small m. For the case $m = 5$, say, the scalars appearing in front of IQ in the CLT for the range-based estimators equal roughly 0.8 and 0.5, making the confidence bands for IV about two and a half to four times more narrow than with RV^{Δ}.

3 Monte Carlo Exploration

In this section, we illustrate the workings of our theory by using repeated samples from a stochastic volatility model to further study its finite-sample performance and to document the asymptotic properties of the range-based statistics. The following bivariate system of stochastic differential equations is simulated:

$$\begin{aligned} \mathrm{d}p_t &= \sigma_t \mathrm{d}W_t \\ \mathrm{d}\ln\sigma_t^2 &= \theta(\omega - \ln\sigma_t^2)\mathrm{d}t + \eta \mathrm{d}B_t \end{aligned} \tag{3.1}$$

where W and B are independent Brownian motions, while (θ, ω, η) are parameters.[19] Thus, spot log-variance evolves as a mean-reverting Ornstein-Uhlenbeck process with mean ω, mean reversion parameter θ and volatility η (see, e.g., Gallant et al. (1999), Alizadeh et al. (2002), and Andersen, Benzoni & Lund (2002)). The vector (θ, ω, η) is from Andersen, Benzoni & Lund (2002), who apply Efficient Method of Moments (EMM) to calibrate numerous continuous-time diffusions; except that we standardize the (annualized) average of IV to unity, i.e. $(\theta, \omega, \eta) = (0.032, -0.103, 0.115)$.

The initial conditions are set to $p_0 = 0$ and $\ln\sigma_0^2 = \omega$, and our simulation design is completed by generating T daily replications from this model with mn price increments each, with T and mn depending on the setting (see below). Throughout, we continue to ignore the irregular spacing of high-frequency data and work with equidistant data.

[19] A discrete-time version of the continuous-time model in (3.1) is obtained with a standard Euler approximation scheme, i.e. $p_{t+\Delta} = p_t + \sigma_t\sqrt{\Delta}\phi_t$ and $\ln\sigma_{t+\Delta}^2 = \theta\omega\Delta + \ln\sigma_t^2(1-\theta\Delta) + \eta\sqrt{\Delta}\epsilon_t$, where ϕ_t and ϵ_t are orthogonal N(0,1) variates.

3.1 Simulation Results

To maintain a streamlined exposition, we start with the distributional implications derived for RBV_m^Δ.[20] After this, we elaborate on the consistency result of Theorem 4 by augmenting the simulated prices with microstructure noise.

In the current context we select $m = 5$, but our results are not that sensitive to specific values of m. In general, higher values of m lead to size properties of the asymptotic confidence bands. We use $n = 10, 50, 100$ for a total of $mn = 50, 250, 500$ increments each day. This allows us to visualize the gradual convergence in distribution to the standard normal for daily high-frequency sample sizes that resemble those of moderately liquid assets. As one simulation is quite fast with these specifications, a total of $T = 1,000,000$ replications are generated, providing very accurate estimation of the actual finite-sample density.

[INSERT FIGURE 4 ABOUT HERE]

Figure 4, upper panel, graphs smoothed densities for the standardized errors of RBV_m^Δ; cf. the ratio in equation (2.30). It details that for $n = 10$, the distribution is left-skewed with a poor behavior in both the center and tail areas compared to the superimposed N(0,1) reference density. The size properties improve by progressively increasing the sample and $n = 100$ tracks the tails quite closely.

Barndorff-Nielsen & Shephard (2003*a*) showed that log-based inference via standard linearization methods improved upon the raw distribution theory for RV. They found a better finite-sample behavior for the errors of the log-transform than those extracted with the feasible version of the CLT outlined in equation (2.7). The shape of the actual densities for the range-based statistic suggests this also applies in our setting. By the delta-rule, the log-version of the CLT for RBV_m^Δ takes the form:

$$n^{1/2}\left(\ln RBV_m^\Delta - \ln IV\right) \xrightarrow{d} MN\left(0, \Lambda_m \int_0^1 \sigma_s^4 \mathrm{d}s / IV^2\right) \tag{3.2}$$

In the lower panel of Figure 4, we plot the density function of the feasible log-based t-statistics. Apparently, the coverage of the limit theory in equation (3.2) is a much better guide for small values of n; with $n = 100$ providing a near-perfect fit to the N(0,1). Hence, the results for the range are broadly consistent with the findings for RV.

3.2 Market Microstructure Noise

To conclude our analysis of range-based estimation of IV, we consider the impact of market microstructure noise. This bias arises from a variety of sources, including price discreteness, bid-ask bounce and illiquidity. A lot of work can be conducted in this setting and we are

[20] The limit theory for the adjusted range is omitted for compactness, though all results are available upon request. In fact, for small n it affords a better description than the version with the range.

currently devoting a separate paper for further results and improvements of our theoretical framework to the presence of microstructure noise. Consequently, the exposition serves as a brief illustration.

Start with the decomposition:

$$p^*_{i/mn} = p_{i/mn} + \varepsilon_{i/mn} \tag{3.3}$$

where $\varepsilon = \left\{\varepsilon_{i/mn}\right\}_{0\leq i\leq mn}$ is an IID process with $\mathbb{E}\left[\varepsilon_{i/mn}\right] = 0$, $\mathbb{E}\left[\varepsilon^2_{i/mn}\right] = \sigma^2_\varepsilon$ and $\varepsilon \perp\!\!\!\perp p$. Here, the observed price $p^* = \left\{p^*_{i/mn}\right\}_{0\leq i\leq mn}$ equals the efficient price p plus the ε term due to microstructure bias. In this case, the p-part of p^* is asymptotically negligible.

For simplicity, assume $\varepsilon_{i/mn} \sim U\left([-\nu,\nu]\right)$, implying that $\sigma^2_\varepsilon = \nu^2/3$. We then compute some bounds for the estimators of IV. Starting with RV^Δ, we have:

$$\mathbb{E}\left[RV^\Delta\right] \approx \frac{2}{3}\nu^2 n \tag{3.4}$$

Likewise, the bound for RBV^Δ_m is:

$$\begin{aligned}
\mathbb{E}\left[RBV^\Delta_m\right] &\approx \mathbb{E}\left[\frac{1}{\lambda_{2,m}}\sum_{i=1}^{n}\max_{1\leq s,t\leq m}\left(\varepsilon_{(i-1)/n+t/mn} - \varepsilon_{(i-1)/n+s/mn}\right)^2\right] \\
&\leq \mathbb{E}\left[4\frac{1}{\lambda_{2,m}}\sum_{i=1}^{n}\max_{1\leq t\leq m}\varepsilon^2_{(i-1)/n+t/mn}\right] \\
&= 4\frac{1}{\lambda_{2,m}}\frac{m}{m+2}\nu^2 n
\end{aligned}$$

Hence, $\mathbb{E}\left[RBV^\Delta_m\right] \to \nu^2 n/\ln(2)$ for $m\to\infty$. For the expected value of $\widetilde{RBV}^\Delta_m$, we calculate an upper bound, which is probably not optimal,

$$\begin{aligned}
\mathbb{E}\left[\widetilde{RBV}^\Delta_m\right] &\approx \mathbb{E}\left[\frac{1}{\tilde{\lambda}_{2,m}}\sum_{i=1}^{n}\max_{1\leq s,t\leq m}\left(\varepsilon_{(i-1)/n+t/mn} - \varepsilon_{(i-1)/n+s/mn} - \frac{t-s}{m}\left(\varepsilon_{i/n} - \varepsilon_{(i-1)/n}\right)\right)^2\right] \\
&\leq \mathbb{E}\left[4\frac{1}{\tilde{\lambda}_{2,m}}\sum_{i=1}^{n}\max_{1\leq t\leq m}\varepsilon^2_{(i-1)/n+t/mn} + \varepsilon^2_{i/n}\right] \\
&= 4\frac{1}{\tilde{\lambda}_{2,m}}\left(\frac{m}{m+2}+\frac{1}{3}\right)\nu^2 n
\end{aligned}$$

and $\mathbb{E}\left[\widetilde{RBV}^\Delta_m\right] \to 32\nu^2 n/\pi^2$ for $m\to\infty$.

To interpret these results, compare $1/\ln(2) \approx 1.44$ and $32/\pi^2 \approx 3.24$ to the constant $2/3$ appearing in the expected value of RV^Δ. It means that given a sampling frequency n, the two range-based statistics are more biased than RV^Δ. The intuition is clear: the high-low is more sensitive to movements in the price and, hence, also microstructure fluctuations. However, $RBV^{3\Delta}_{3m}$ is less biased than RV^Δ in the model with noise and more efficient without noise (as $m\to\infty$). So instead of using RV sampled at, say, the 5-minute frequency, one could use the range at the 15-minute frequency to reduce bias, while retaining efficiency. For the adjusted range, $\widetilde{RBV}^{5\Delta}_{5m}$ has the same property.

We now provide a simple illustration of these insights by using a second set of repeated samples to compute the intra-day range-based statistics and RV^{Δ}. In this simulation, we use $T = 10,000$, $mn = 23,400$ and report the results for every sampling frequency n that divides mn evenly. Thus, we have estimates of IV for a total of 72 different frequencies. The same frequencies were used in the previous section. The simulated prices are contaminated by adding uniform noise and setting $\nu = 0.001$.

[INSERT FIGURE 5 ABOUT HERE]

The results of the Monte-Carlo experiments are plotted in Figure 5. To assess the performance of the volatility proxies, we plot the root mean squared error (RMSE) for the no-noise case, and a "signature plot" in the setting with noise.

With no microstructure noise, RMSE of both range-based statistics are always lower than RV, which is consistent with the theory outlined in this paper. By incorporating noise, all estimators of IV diverge to infinity as $n \to \infty$, as we are cumulating increasingly more microstructure noise. The asymptotic bounds hold approximately for frequent sampling (high values of n), but are less accurate in the right-most part of the figure, corresponding to low values of n.

In an independent and concurrent paper, Dijk & Martens (2005) use Monte Carlo experiments - closely related to ours - for the scaled Brownian motion (i.e. $\mu = 0$ and $\sigma_t = \sigma$). They specify more structural mechanisms for the microstructure noise and report that RMSE for the range-based estimator of equation (2.12) is lower than that of RV. Our evidence is somewhat mixed; in unreported studies the range achieves a lower RMSE in some settings, but RV also manages to outperform the range depending on specifications. There seems to be no systematic pattern, but it admits a further investigation.

All told, range-based estimation of IV offers several advantages compared to the standard method of computing RV; both from a theoretical and practical viewpoint. But, as a final remark, we acknowledge that if the data are subject to pricing errors, it is central with formal tools for getting consistent estimates of IV. Such techniques are already being developed in the context of RV, see, e.g., Aït-Sahalia et al. (2004) or Barndorff-Nielsen, Hansen, Lunde & Shephard (2004), and it presents a central topic for future research to verify whether this extends to our method.

4 Conclusions and Directions for Future Research

In this paper, we developed a framework for estimation of the latent integrated variance (IV) using the high-low as a volatility proxy. The setup is a continuous semi-martingale for the log-price of a security. Unlike existing range-based theory, where the (driftless) geometric Brownian motion is a leading model, we allow the instantaneous mean and variance to evolve stochastically. Both from a theoretical and empirical point of view, this extension is attractive. As a novelty we suggested a new approach for non-parametric estimation of IV, which extracts information in high-frequency data more efficiently than previous methods. The outline - inspired by realized variance (RV) - is the summation of price ranges sampled at a higher frequency than the measurement horizon.

As a third contribution, we provided a set of probabilistic laws for sampling intra-day high-lows. Under weak technical conditions, the estimator converges in probability to IV as the sampling frequency tends to infinity. With stronger conditions, in particular a mean-zero assumption, we also found a mixed Gaussian asymptotic distribution theory for the standardized statistic. Compared to RV, the conditional variance of the high-low is five times smaller. To relax the restriction on the drift-term, we transformed the price process and sampled an adjusted range. Consistency holds for the modified statistic and we proved a central limit theorem (CLT) under weaker conditions about the drift. Moreover, the greater sampling stability of the adjusted range makes it (asymptotically) ten times more efficient than RV. Finally, we analyzed the setting, where estimation of IV relies on discretely sampled data. Here, similar results were derived.

With Monte Carlo simulations, we investigated the finite-sample performance of our method and documented its asymptotic properties. The approximation to the normal law of the standardized errors is fairly accurate for moderate sample sizes, especially for the log-based version of the CLT. Moreover, if the high-low is constructed on the basis of just a few discrete-time increments - five or ten is sufficient - the efficiency of our estimators is quite high compared to RV. Thus, in a setting where true asset prices are subject to error, it may be preferable to sparsely sample an intra-day range-based statistic, as it holds the same efficiency as RV sampled at a higher frequency but reduces the bias arising from microstructure noise.

In future projects, we envision several extensions of our methodological contribution. First, there is plenty of evidence against the continuous sample path diffusion adopted in this paper; particularly at very high frequencies. We are convinced that the high-low statistics are capable of providing consistent estimates of quadratic variation, when the underlying process exhibits jumps. Second, if a jump component is added, it may be preferable to untangle the continuous and discontinuous innovation part in order to construct more robust volatility measures or analyze their separate properties; equivalently to the theory of realized power/bipower variation. Finally, a further theoretical examination of the range is needed, when the observed price is partly due to microstructure noise. As for RV, subsampling based methods for the range seem promising in, hopefully, providing consistent estimates of IV irrespective of the noise, while being more efficient.

A Appendix of Proofs

A.1 Proof of Theorem 1

First, we define:

$$\begin{aligned} \xi_i^n &= \frac{1}{\lambda_2}\sigma^2_{\frac{i-1}{n}} s^2_{W_{i\Delta,\Delta}} \\ U^n &= \sum_{i=1}^n \xi_i^n \end{aligned}$$

Note that:

$$\mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right] = \frac{1}{n}\sigma^2_{\frac{i-1}{n}}$$

so,

$$\sum_{i=1}^n \mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right] \xrightarrow{p} IV \tag{A.1}$$

Now, by setting

$$\eta_i^n = \xi_i^n - \mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]$$

we get:

$$\mathbb{E}\left[(\eta_i^n)^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] = \Lambda \frac{1}{n^2}\sigma^4_{\frac{i-1}{n}}$$

Therefore,

$$\sum_{i=1}^n \mathbb{E}\left[(\eta_i^n)^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] \xrightarrow{p} 0$$

Hence, the assertion $U^n \xrightarrow{p} IV$ follows directly from (A.1). As a sufficient condition in the next step, we deduce $RBV^\Delta - U^n \xrightarrow{p} 0$. Note, the equality:

$$\begin{aligned} RBV^\Delta - U^n &= \frac{1}{\lambda_2}\sum_{i=1}^n \left(s_{p_{i\Delta,\Delta}} - \sigma_{\frac{i-1}{n}} s_{W_{i\Delta,\Delta}}\right)\left(s_{p_{i\Delta,\Delta}} + \sigma_{\frac{i-1}{n}} s_{W_{i\Delta,\Delta}}\right) \\ &\equiv R_1^n + R_2^n \end{aligned}$$

with R_1^n and R_2^n defined by:

$$\begin{aligned} R_1^n &= \frac{2}{\lambda_2}\sum_{i=1}^n \sigma_{\frac{i-1}{n}} s_{W_{i\Delta,\Delta}}\left(s_{p_{i\Delta,\Delta}} - \sigma_{\frac{i-1}{n}} s_{W_{i\Delta,\Delta}}\right) \\ R_2^n &= \frac{1}{\lambda_2}\sum_{i=1}^n \left(s_{p_{i\Delta,\Delta}} - \sigma_{\frac{i-1}{n}} s_{W_{i\Delta,\Delta}}\right)^2 \end{aligned}$$

We decompose the second term further:

$$\begin{aligned} R_2^n &\leq \frac{1}{\lambda_2}\sum_{i=1}^n \left(\sup_{\frac{i-1}{n}\leq s,t\leq \frac{i}{n}} \mid \int_s^t \mu_u \mathrm{d}u + \int_s^t \left(\sigma_u - \sigma_{\frac{i-1}{n}}\right)\mathrm{d}W_u \mid\right)^2 \\ &\leq \frac{2}{\lambda_2}\sum_{i=1}^n \left(\sup_{\frac{i-1}{n}\leq s,t\leq \frac{i}{n}} \mid \int_s^t \mu_u \mathrm{d}u \mid\right)^2 + \frac{2}{\lambda_2}\sum_{i=1}^n \left(\sup_{\frac{i-1}{n}\leq s,t\leq \frac{i}{n}} \mid \int_s^t \left(\sigma_u - \sigma_{\frac{i-1}{n}}\right)\mathrm{d}W_u \mid\right)^2 \\ &\equiv R_{2.1}^n + R_{2.2}^n \end{aligned}$$

It is straightforward to verify the estimation $\mathbb{E}\left[R_{2.1}^n\right] = \mathrm{O}\left(n^{-1}\right)$. For the latter term, we exploit the Burkholder inequality (e.g., Revuz & Yor (1998)):

$$\begin{aligned} \mathbb{E}\left[R_{2.2}^n\right] &\leq \frac{2C}{\lambda_2}\sum_{i=1}^{n}\mathbb{E}\left[\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\sigma_u - \sigma_{\frac{i-1}{n}}\right)^2 \mathrm{d}u\right] \\ &= \frac{2C}{\lambda_2}\mathbb{E}\left[\int_0^1\left(\sigma_u - \sigma_{\frac{[nu]}{n}}\right)^2 \mathrm{d}u\right] \\ &= \mathrm{o}(1) \end{aligned}$$

for some constant $C > 0$. Thus, $R_2^n = \mathrm{o}_p(1)$. With a decomposition as above and the Cauchy-Schwarz inequality, we have $R_1^n = \mathrm{o}_p(1)$. By assembling the parts, $RBV^{\Delta} - U^n \xrightarrow{p} 0$. ■

A.2 Proof of Theorem 2

First, note:

$$\begin{aligned} \sum_{i=1}^{n}\mathbb{E}\left[\xi_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right] - IV &= \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\sigma_{\frac{i-1}{n}}^2 - \sigma_s^2\right)\mathrm{d}s \\ &= \mathrm{o}_p\left(n^{-1/2}\right) \end{aligned} \tag{A.2}$$

as σ has Hölder index of order $\gamma > 1/2$ in $\mathcal{L}^2(\mathbb{P})$. Using the Hölder continuity again and methods similar to the previous proof, yields $R_1^n = \mathrm{o}_p\left(n^{-1/2}\right)$ and $R_2^n = \mathrm{O}_p\left(n^{-1}\right)$. Combining this with (A.2), we obtain:

$$n^{1/2}\left(RBV^{\Delta} - IV\right) = n^{1/2}\sum_{i=1}^{n}\eta_i^n + \mathrm{o}_p(1)$$

We see that:

$$n\sum_{i=1}^{n}\mathbb{E}\left[(\eta_i^n)^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] \xrightarrow{p} \Lambda\int_0^1 \sigma_s^4 \mathrm{d}s$$

and by the scaling property of Brownian motion,

$$n^{1/2}\sum_{i=1}^{n}\mathbb{E}\left[\eta_i^n\left(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] \xrightarrow{p} \frac{\nu}{\lambda_2}IV$$

where $\nu = \mathbb{E}\left[W_1 s_W^2\right]$. Quite trivially, $\{W_t\}_{t\in[0,1]} \stackrel{d}{=} \{-W_t\}_{t\in[0,1]}$, with the consequence $\nu = -\nu$ and, hence, $\nu = 0$.

Next, let $N = \{N_t\}_{t\in[0,1]}$ be a bounded martingale on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, P\right)$, which is orthogonal to W (i.e., with quadratic covariation process $\{[W,N]_t\}_{t\in[0,1]} = 0$). Then,

$$n^{1/2}\sum_{i=1}^{n}\mathbb{E}\left[\eta_i^n\left(N_{\frac{i}{n}} - N_{\frac{i-1}{n}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0 \tag{A.3}$$

For this result, we use Clark's representation theorem (see, e.g., Karatzas & Shreve (1998), Appendix E):

$$s^2_{W_{i\Delta,\Delta}} - \frac{1}{n}\lambda_2 = \int_{\frac{i-1}{n}}^{\frac{i}{n}} H^n_s \mathrm{d}W_s \tag{A.4}$$

for some predictable function H^n_s. Notice $\mathbb{E}\left[\int_a^b f_s \mathrm{d}W_s \left(N_b - N_a\right) \mid \mathcal{F}_a\right] = 0$, for any $[a,b]$ and predictable f. To prove this assertion, take a partition $a = t^*_0 < t^*_1 < \ldots < t^*_n = b$ and compute:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n f_{t^*_{i-1}}\left(W_{t^*_i} - W_{t^*_{i-1}}\right)\left(N_b - N_a\right) \mid \mathcal{F}_a\right] = \mathbb{E}\left[\sum_{i=1}^n f_{t^*_{i-1}}\left(W_{t^*_i} - W_{t^*_{i-1}}\right) N_b \mid \mathcal{F}_a\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left[f_{t^*_{i-1}}\left(W_{t^*_i} - W_{t^*_{i-1}}\right) N_b \mid \mathcal{F}_{t^*_i}\right] \mid \mathcal{F}_{t^*_{i-1}}\right] \mid \mathcal{F}_a\right] \\
&= 0
\end{aligned}$$

From equation (A.4), (A.3) is attained. Finally, stable convergence in law follows by Theorem IX 7.28 in Jacod and Shiryaev (2003):

$$n^{1/2}\left(RBV^{\Delta} - IV\right) \xrightarrow{d_s} \Lambda^{1/2} \int_0^1 \sigma^2_s \mathrm{d}B_s$$

■

A.3 Proof of Theorem 3

For $\frac{i-1}{n} \leq s, t \leq \frac{i}{n}$, we have:

$$\begin{aligned}
\tilde{p}_t - \tilde{p}_s &= \int_s^t \mu_u \mathrm{d}u + \int_s^t \sigma_u \mathrm{d}W_u - n\left(t-s\right)\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \mu_u \mathrm{d}u + \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sigma_u \mathrm{d}W_u\right) \\
&= \int_s^t \left(\mu_u - \mu_{\frac{i-1}{n}}\right) \mathrm{d}u - n\left(t-s\right)\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\mu_u - \mu_{\frac{i-1}{n}}\right) \mathrm{d}u \\
&+ \int_s^t \sigma_u \mathrm{d}W_u - n\left(t-s\right)\int_{\frac{i-1}{n}}^{\frac{i}{n}} \sigma_u \mathrm{d}W_u
\end{aligned}$$

By the continuity of μ and Hölder continuity of σ (and methods similar to the above), we get the estimate:

$$n^{1/2}\left(\widetilde{RBV}^{\Delta} - IV\right) = n^{1/2}\sum_{i=1}^n \tilde{\eta}^n_i + \mathrm{o}_p\left(1\right)$$

with the sequence $\tilde{\eta}^n_i$ being defined by,

$$\tilde{\eta}^n_i = \frac{1}{\tilde{\lambda}_2}\sigma^2_{\frac{i-1}{n}}\left(s^2_{\tilde{W}_{i\Delta,\Delta}} - \frac{1}{n}\tilde{\lambda}_2\right)$$

Now, the CLT for $n^{1/2}\sum_{i=1}^n \tilde{\eta}^n_i$ is derived exactly as in the previous proof. ■

A.4 Proof of Theorem 4

The result is shown in the same manner as Theorem 1, 2, and 3. ■

A.5 Proof of Remark

Recall $\mu = 0$ and $\sigma \perp\!\!\!\perp W$. Notice, under assumption (H), that it suffices to verify:

$$
\begin{aligned}
RBV^{\Delta} - U^n &= \frac{1}{\lambda_2} \sum_{i=1}^{n} \left(s^2_{p_{i\Delta,\Delta}} - \sigma^2_{\frac{i-1}{n}} s^2_{W_{i\Delta,\Delta}} \right) \\
&= \mathrm{o}_p \left(n^{-1/2} \right)
\end{aligned}
$$

Set,

$$
\xi_i^n = \frac{1}{\lambda_2} \left(s^2_{p_{i\Delta,\Delta}} - \sigma^2_{\frac{i-1}{n}} s^2_{W_{i\Delta,\Delta}} \right)
$$

From the Markov and scaling property of Brownian motion, we get the identity:

$$
\mathbb{E} \left[\xi_i^n \mid \{\sigma_t\}_{t \in [0,1]} \right] = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\sigma_s^2 - \sigma^2_{\frac{i-1}{n}} \right) \mathrm{d}s
$$

and, by assumption (H), we deduce that $\mathbb{E}\left[\xi_i^n\right] = \mathrm{O}\left(n^{-2}\right)$, so:

$$
RBV^{\Delta} - U^n = \sum_{i=1}^{n} \left(\xi_i^n - \mathbb{E}\left[\xi_i^n\right] \right) + \mathrm{o}\left(n^{-1/2} \right)
$$

Thus,

$$
\begin{aligned}
\mathbb{E}\left[\left(RBV^{\Delta} - U^n \right)^2 \right] &\leq \mathbb{E}\left[\sum_{i=1}^{n} \mathbb{E}\left[(\xi_i^n)^2 \mid \{\sigma_t\}_{t \in [0,1]} \right] \right] + \mathrm{o}\left(n^{-1}\right) \\
&= \frac{\lambda_4}{\lambda_2^2} \mathbb{E}\left[\sum_{i=1}^{n} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\sigma_s^2 - \sigma^2_{\frac{i-1}{n}} \right) \mathrm{d}s \right)^2 \right] + \mathrm{o}\left(n^{-1}\right) \\
&= \mathrm{o}\left(n^{-1}\right)
\end{aligned}
$$

■

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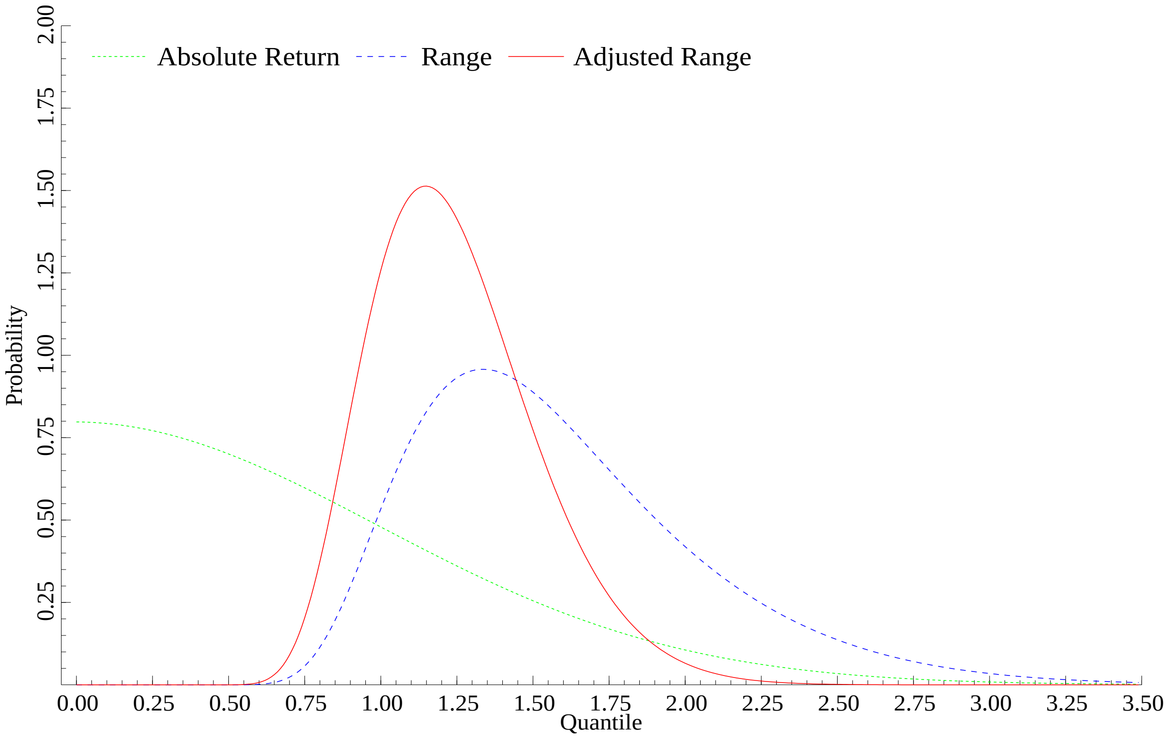
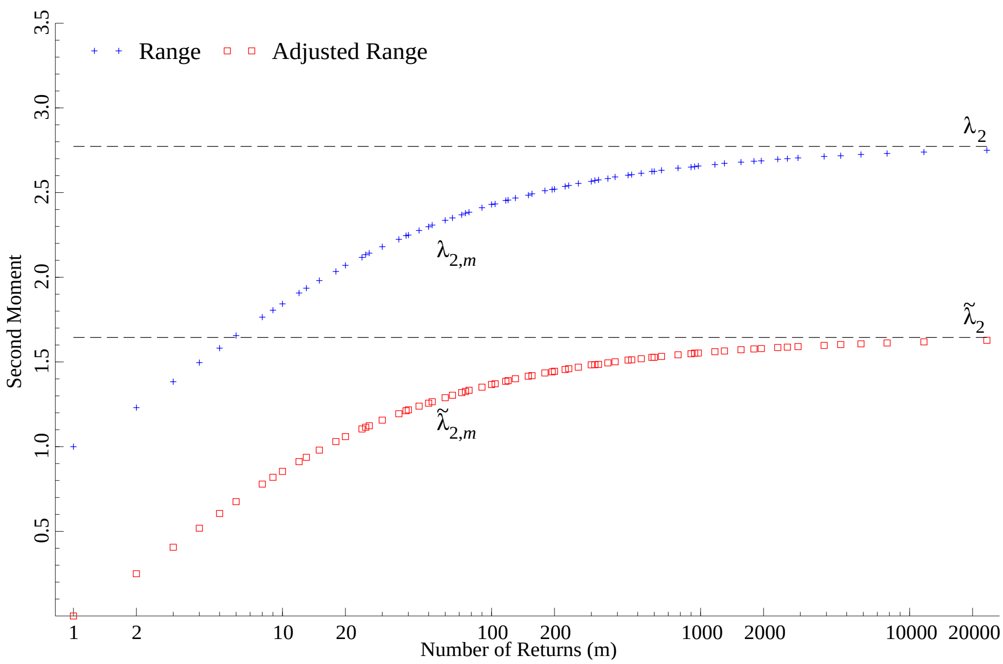
Figure 1: The Distribution of the Absolute Return, Range and Adjusted Range



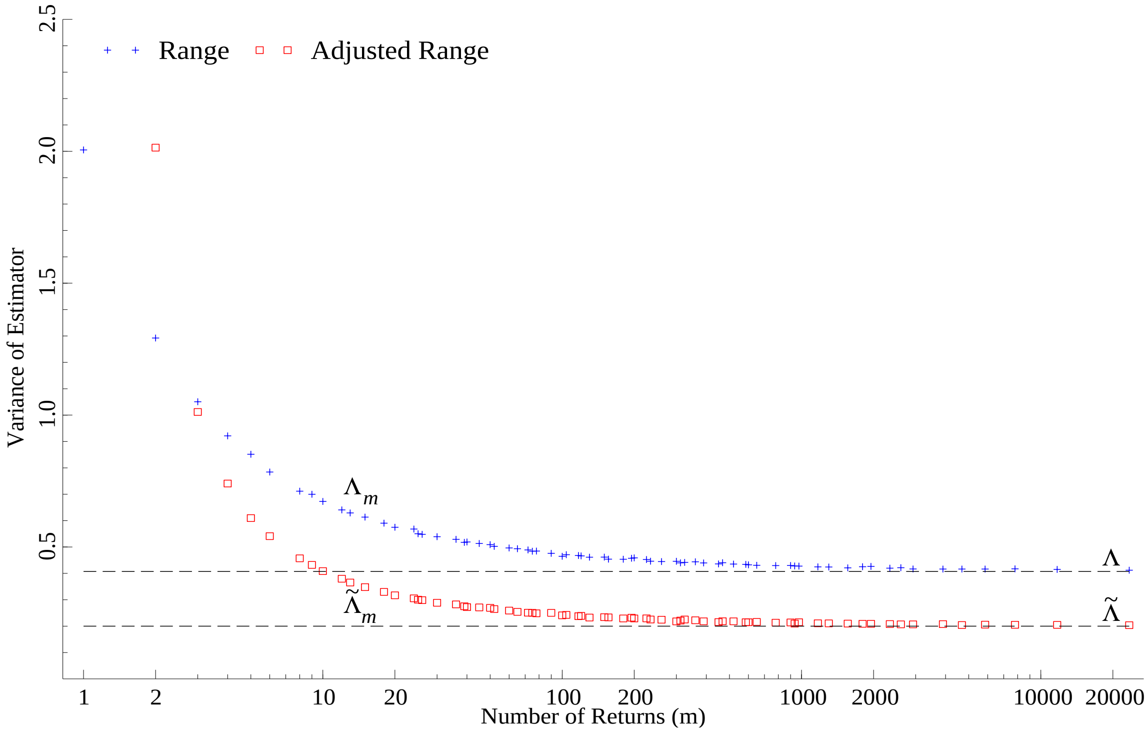
We present the distribution of the absolute return, range and adjusted range of a standard Brownian motion over an interval of unit length.

Figure 2: The Finite-Sample Expectation of the Squared Adjusted Range and Range of a Standard Brownian Motion



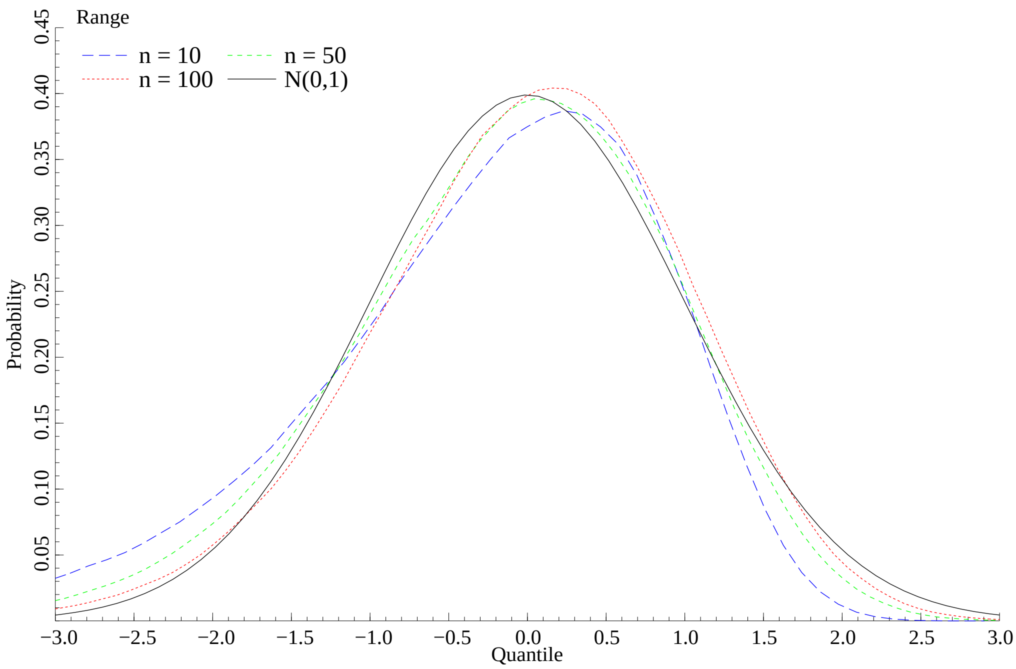
The figure details the second moment of the adjusted range and range of a standard Brownian motion over an interval of unit length, when the underlying continuous-time process is only observed at m (equidistant) points in time. The dashed lines represent the asymptotic values. The figure is based on a simulation with 1,000,000 repetitions.

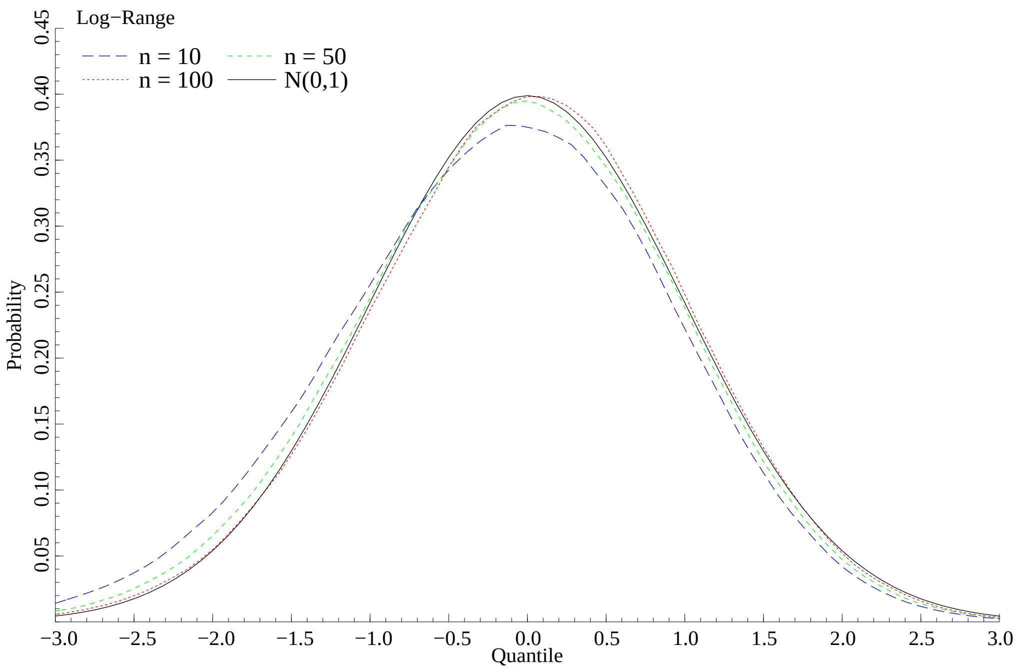
Figure 3: The Variance of the Adjusted Range- and Range-Based Estimator of Integrated Variance of a Standard Brownian Motion



We plot the variance of the adjusted range- and range- based estimator of IV of a standard Brownian motion over an interval of unit length, when the underlying continuous-time process is only observed at m (equidistant) points in time. The dashed lines represent the asymptotic values. The figure is based on a simulation with 1,000,000 repetitions.

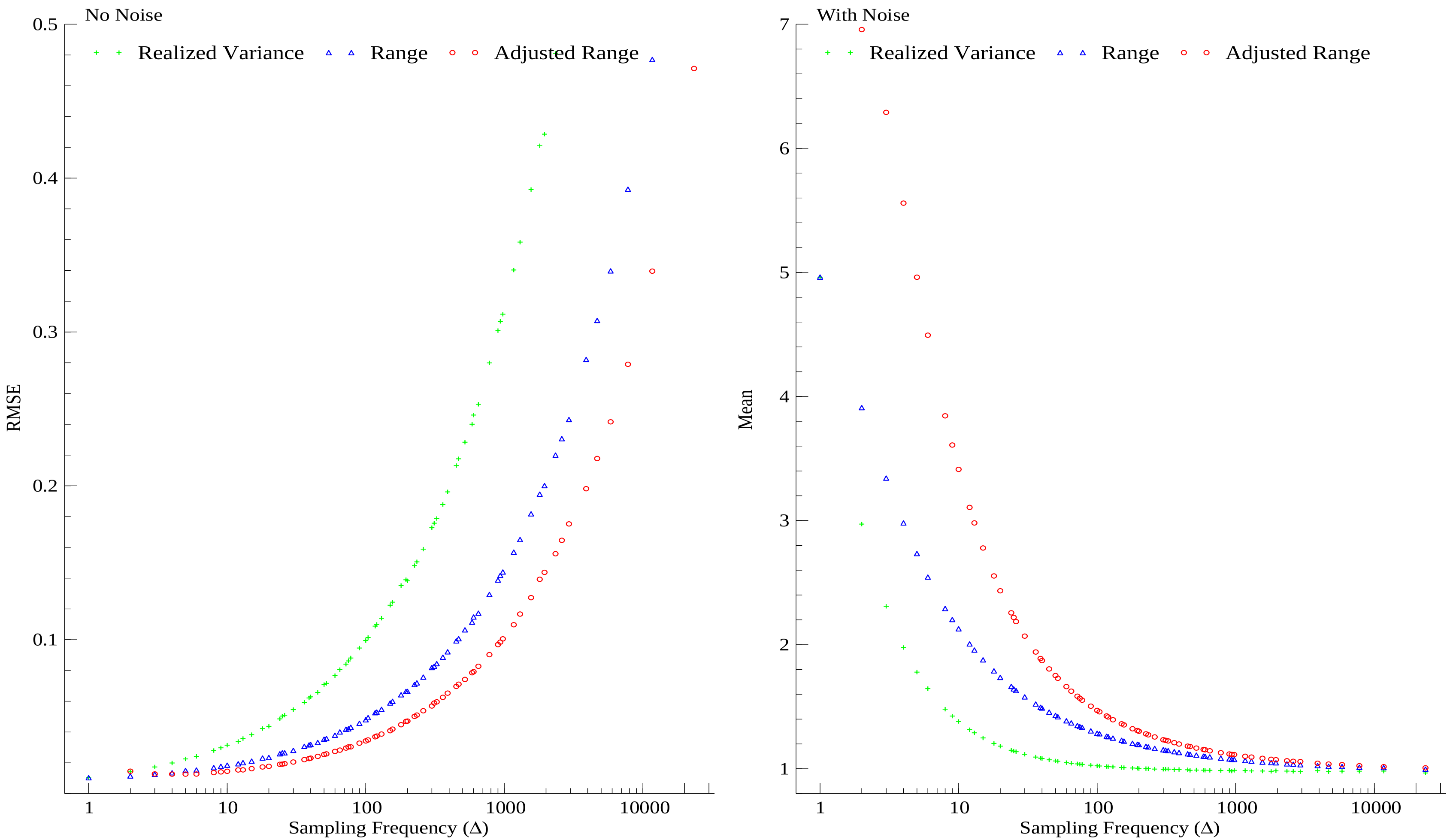
Figure 4: Asymptotic Normality for the Standardized Range-Based Statistics in Estimating Integrated Variance





We illustrate the asymptotic normality of the errors of the standardized range-based statistic RBV_m^{Δ} in estimating IV. The plots are smoothed densities - using a Gaussian kernel - for the finite-sample settings with $n = 10, 50, 100$ and $m = 5$, and the graphs are based on a simulation with 1,000,000 repetitions from a log-normal diffusion for σ, as detailed in the main text. In the upper panel, we depict the t-statistics of the feasible CLT for RBV_m^{Δ}, while the lower panel is the corresponding log-based version. The solid line is the density function of a standard normal random variable.

Figure 5: The Impact of Market Microstructure Noise



In the left part of Figure 5, we plot the RMSE of the three different estimators of IV without adding microstructure noise. In the right part of the figure, we have added a uniformly distributed noise term. Here, we create a ”signature plot” of the various estimators.