A note on some extremal problems for trigonometric polynomials

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Abstract

We consider the problem of finding the trigonometric polynomial

$$\vartheta_0 + \sum_{j=1}^{m} \vartheta_{2j-1} \sin(jx) + \vartheta_{2j} \cos(jx)$$

with minimal sup-norm on the interval $[-\pi, \pi]$, where one coefficient in the polynomial, say ϑ_k $(0 \leq k \leq 2m)$, has been fixed. A complete solution of this problem is given, which depends sensitively on the ratio k/m, and the problem of uniqueness is discussed. Several examples are presented to illustrate the main properties of the extremal polynomials.

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1 Introduction

Consider the class

$$(1.1)\ \mathcal{T} = \left\{ \varphi_m(x) = \vartheta_0 + \sum_{j=1}^{m} \vartheta_{2j-1} \sin(jx) + \vartheta_{2j} \cos(jx) \middle| x \in [-\pi, \pi]; \vartheta_i \in \mathbb{R}, \quad i = 0, \ldots, 2m \right\}$$

of all trigonometric polynomials of degree m, and let

$$\|f\|_\infty = \max_{|x| \leq \pi} |f(x)|$$

denote the sup-norm of a continuous function on the interval $[-\pi, \pi]$. In the present paper we derive an analytic solution of the extremal problem

$$\min\{\|\varphi_m\|_\infty \mid \varphi_m \in \mathcal{T},\ |\vartheta_k| = 1\}; \tag{1.2}$$

where $k \in \{0, \ldots, 2m\}$ is arbitrary but fixed. Extremal problems of this type have been originally studied for ordinary polynomials by Chebyshev (1859), who determined the minimal sup-norm of all polynomials with a fixed degree and leading coefficient 1 [see e.g. Achieser (1956) or Natanson (1955) among others]. In the context of algebraic and trigonometric polynomials numerous extremal problems have been studied in the literature [see Rivlin (1974) or Milavonović, Mitrinović and Rassias (1994) among many others] but to the knowledge of the authors a solution of the above problem is not available. Note that the extremal problem (1.2) can be reformulated to the problem of finding the best constant, say $c_{m,\infty}$, such that the inequality

$$1 = |\vartheta_k| \leq c_{k,m,\infty}\|\varphi_m\|_\infty \quad \forall\ \varphi_m \in \mathcal{T} \tag{1.3}$$

holds. This problem is related to the problem of determining the best constant $c_{m,p}$ such that

$$\max_{j=0}^{2m} |\vartheta_j| \leq c_{m,p}\|\vartheta_m\|_p \quad \forall\ \varphi_m \in \mathcal{T}, \tag{1.4}$$

(here $\|f\|_p = (\int_{-\pi}^{\pi} |f(x)|^p dx)^{1/p}$ denotes the L^p-norm), which was recently solved by Marshall and Ganzburg (1999). Similar problems for the L_1-norm of nonnegative trigonometric polynomials were studied by Peherstorfer (1982). We finally note that the extremal problem (1.2) is also equivalent to the problem

$$\max\{|\vartheta_k| \mid \varphi_m \in \mathcal{T}, \|\varphi_m\|_\infty \leq 1\} \tag{1.5}$$

in the following sense. If φ_m is a solution of the extremal problem (1.2), then $\varphi_m/||\varphi_m||_\infty$ is a solution of the extremal problem (1.5) and vice versa. It is also easy to see that $-\varphi_m$ is a solution of the extremal problem (1.2) [or equivalently (1.3) or (1.5)] if and only if φ_m is a solution, and throughout this paper we do not distinguish these two solutions.

It is the purpose of the present note to derive an explicit solution of the extremal problem (1.2), [or equivalently (1.3), (1.5)]. In Section 2 we provide a necessary and sufficient condition for a trigonometric polynomial to be a solution of the extremal problem. It turns out that this solution depends sensitively on the value of $k \in \{0, \ldots, 2m\}$, and two different scenarios are identified in Section 3 and Section 4 corresponding to the cases $\lfloor \frac{k+1}{2} \rfloor \leq m/3$ and $\lfloor \frac{k+1}{2} \rfloor > m/3$, respectively. Finally, in Section 5 the problem of uniqueness is discussed and it is demonstrated that the extremal problem (1.2) can have uncountable many solutions. Moreover, several examples are presented, which illustrate some peculiar properties of the extremal polynomials in the case $\lfloor \frac{k+1}{2} \rfloor < m/3$.

2 A necessary and sufficient condition for the solution of the extremal problem

For the sake of a transparent notation we introduce the functions

$$f_j(x) = \begin{cases} 1 & \text{if } j = 0 \\ \sin(\ell x) & \text{if } j = 2\ell - 1; \quad \ell = 1, \ldots, m \\ \cos(\ell x) & \text{if } j = 2\ell; \quad \ell = 1, \ldots, m \end{cases} \tag{2.1}$$

and rewrite a trigonometric polynomial $\varphi_m \in \mathcal{T}$ as

$$\varphi_m(x) = \sum_{j=0}^{2m} \vartheta_j f_j(x). \tag{2.2}$$

The following result characterizes the solution of the extremal problem (1.5).

Theorem 2.1. *A trigonometric polynomial of the form (2.2) with $\|\varphi_m\|_\infty \leq 1$ is a solution of the extremal problem (1.5) if and only if there exist points $x_1, \ldots, x_N \in [-\pi, \pi]$ and real constants $A_1, \ldots, A_N$ ($N \in \mathbb{N}$) such that $N \leq 2m$ and*

(i) *sign*$(A_j)\varphi_m(x_j) = 1$ *whenever* $A_j \neq 0$ $(j = 1, \ldots, N)$;

(ii) $\sum_{j=1}^{N} A_j f_i(x_j) = 0$; *for all* $i \in \{0, 1, \ldots, 2m\} \backslash \{k\}$;

(iii) $\sum_{j=1}^{N} A_j f_k(x_j) = 1$.

Moreover, if $\hat{\varphi}_m = \sum_{j=0}^{2m} \hat{\vartheta}_j f_j$ is a solution of the extremal problem (1.5) the coefficient $\hat{\vartheta}_k$ corresponding to the function f_k in the polynomial $\hat{\varphi}_m$ is given by

$$\hat{\vartheta}_k = \sum_{j=1}^{N} |A_j|. \tag{2.3}$$

Proof. Using Theorem 1.3 in Singer (1970) it follows that a trigonometric polynomial $\tilde{\varphi}_m$ is a solution of the extremal problem (1.2) if and only if there exist $N \leq 2m$ points $x_1, \ldots, x_N$ and N scalars $\mu_1, \ldots, \mu_N$ with $\sum_{i=1}^{N} |\mu_i| = 1$ such that

$$\sum_{j=1}^{N} \mu_j f_i(x_j) = 0 \quad i \in \{0, \ldots, 2m\} \backslash \{k\} \tag{2.4}$$

$$\sum_{j=1}^{N} \mu_j \tilde{\varphi}_m(x_j) = \|\tilde{\varphi}_m\|_\infty \tag{2.5}$$

$$\tilde{\varphi}_m(x_j) \operatorname{sign}(\mu_j) = \|\tilde{\varphi}_m\|_\infty \quad j \in \{0, \ldots, N\} \tag{2.6}$$

Define $A_j = \mu_j/\|\tilde{\varphi}_m\|_\infty$, note that $\tilde{\varphi}_m(x) = f_k(x) + \sum_{i\neq k} \vartheta_i f_i(x)$ and define $\hat{\varphi}_m(x) = \tilde{\varphi}_m(x)/\|\tilde{\varphi}_m\|_\infty$. Because the extremal problems (1.2) and (1.5) are equivalent, it is now easy to see that the trigonometric polynomial $\hat{\varphi}_m$ is a solution of the extremal problem (1.5), if and only if there exist points $x_1, \ldots, x_N$ and constants $A_1, \ldots, A_N$ such that the conditions (i) – (iii) of Theorem 2.1 are satsified. Finally, we obtain from the conditions (i) – (iii) the representation

$$1 = \sum_{j=1}^{N} A_j f_k(x_j) = \frac{1}{\hat{\vartheta}_k} \sum_{j=1}^{N} A_j \hat{\varphi}_m(x_j) = \frac{1}{\hat{\vartheta}_k} \sum_{j=1}^{N} |A_j|,$$

which proves the remaining assertion (2.3) of Theorem 2.1. □

3 Extremal polynomials in the case $k = 0, m/3 < \lfloor \frac{k+1}{2} \rfloor \leq m$

In this section we provide a solution of the extremal problem (1.2) for the case $k = 0$ and "large" values of k. It turns out that in this case the solution is very transparent.

Theorem 3.1. *Assume that $k = 0$ or $m/3 < \lfloor \frac{k+1}{2} \rfloor \leq m$, then the trigonometric polynomial $\hat{\varphi}_m(x) = f_k(x)$ is a solution of the extremal problem (1.2). In particular the constant $c_{m,\infty}$ in (1.3) is given by $c_{m,\infty} = 1$ if $k = 0$ or $m/3 < \lfloor \frac{k+1}{2} \rfloor \leq m$.*

Proof. Consider the case $k = 2\ell$ with $\ell > m/3$, for which we have to show that the solution of the extremal problem is given by $\hat{\varphi}_m(x) = f_k(x) = \cos(\ell x)$. Obviously, we have $\|\hat{\varphi}_m\|_\infty = 1$. Define

$$\begin{aligned} x_j &= \frac{j-1}{\ell}\pi - \pi; \quad j = 1, \ldots, 2\ell, \\ A_j &= \frac{1}{2\ell}\cos(\ell x_j) = \frac{(-1)^{\ell-j+1}}{2\ell}; \quad j = 1, \ldots, 2\ell, \end{aligned} \tag{3.1}$$

and observe the identities

$$\sum_{j=1}^{q} \cos\Big(s\Big(\frac{j-1}{q}\pi - \pi\Big)\Big) \cos\Big(i\frac{j-1}{q}\pi - \pi\Big)\Big) = 0 \tag{3.2}$$

for all $i \in \{0, \ldots, s-1, s+1, \ldots, m\}$ such that $s + i$ and $|s - i|$ are not multipliers of $2q$ [see e.g. Rivlin (1974), Exercise 1.5.28]. Using these equalities it is easy to see that

$$\sum_{j=1}^{2\ell} A_j f_{2i}(x_j) = \frac{1}{2\ell} \sum_{j=1}^{2\ell} \cos(\ell x_j) \cos(i x_j) = 0, \tag{3.3}$$

whenever $i \in \{1, \ldots, m\} \backslash \{\ell\}$ [note that we require the condition $\ell > m/3$ for the application of formula (3.2)]. On the other hand it follows from the definition of the points x_j in (3.1) that

$$x_{2\ell-j} + x_{j+2} = 0, \quad j = 0, \ldots, \ell - 1; \quad x_1 = -\pi; x_{\ell+1} = 0,$$

and a direct calculation shows

$$\sum_{j=1}^{2\ell} A_j f_{2i-1}(x_j) = \frac{1}{2\ell} \sum_{j=1}^{2\ell} \cos(\ell x_j) \sin(i x_j) = 0,$$

which proves the conditions (i) and (ii) of Theorem 2.1. Finally, the remaining condition (iii) of this Lemma is obvious from

$$\sum_{j=1}^{2\ell} A_j f_{2\ell}(x_j) = \frac{1}{2\ell} \sum_{j=1}^{2\ell} \cos^2((j-1-\ell)\pi) = 1,$$

and by Theorem 2.1 the function $\varphi_m(x) = f_{2\ell}(x) = \cos(\ell x)$ is a solution of the extremal problem (1.5) which is equivalent to the extremal problem (1.2).
The case $k = 0$ is obtained exactly in the same way. Finally, the remaining case $k = 2\ell - 1, \ell > m/3$ is treated by similar arguments using the the points

$$x_j = \frac{2j-1}{2\ell}\pi - \pi, \quad j = 1, \dots, 2\ell,$$

the weights

$$A_j = \frac{1}{2\ell} \sin(\ell x_j), \quad j = 1, \dots, 2\ell$$

in (3.1) and observing the identity

$$\sum_{j=1}^{q} \sin\Big(s\Big(\frac{2j-1}{2q}\pi - \pi\Big)\Big) \sin\Big(i\Big(\frac{2j-1}{2q}\pi - \pi\Big)\Big) = 0$$

for all $i \in \{0, \dots, s-1, s+1, \dots, m\}$ such that $s+i$ and $|s-i|$ are not multipliers of $2q$ [see Rivlin (1974)]. The details are omitted for the sake of brevity.

□

4 Extremal polynomials in the case $1 \leq \lfloor \frac{k+1}{2} \rfloor \leq m/3$

It will be demonstrated in the present section that the solution of the extremal problem (1.2) in the case $1 \leq \lfloor \frac{k+1}{2} \rfloor \leq m/3$ is substantially more complicated. Throughout this section we define the index

(4.1) $$p = \lfloor \frac{m+\ell}{2\ell} \rfloor + 1,$$

which will be an important quantity for the classification of the solution of the extremal problem. In particular we have to distinguish the following three cases

(4.2) $$k = 2\ell, \quad p \text{ is odd}$$
(4.3) $$k = 2\ell - 1$$
(4.4) $$k = 2\ell, \quad p \text{ is even}$$

Trigonometric polynomials of the form

$$(4.5)\quad \varphi(x,\vartheta) = \begin{cases} \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)} f_{2\ell(2j-1)}(x) = \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)} \cos(\ell(2j-1)x) & \text{if } k = 2\ell \\ \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)-1} f_{2\ell(2j-1)-1}(x) = \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)-1} \sin(\ell(2j-1)x) & \text{if } k = 2\ell - 1 \end{cases}$$

will serve as appropriate candidates for the extremal polynomial [recall the definition of the function $f_k(x)$ in (2.1)]. The following result is the basic tool for the solution of the extremal problem (1.2).

Lemma 4.1. *Assume that* $1 \le \lfloor \frac{k+1}{2} \rfloor \le \frac{m}{3}$.

(a) If condition (4.2) is satisfied, define

$$(4.6)\qquad x_i = \Big(2i - 1 + 2\lfloor \frac{i-1}{p-1} + \frac{1}{2} \rfloor\Big) \frac{\pi}{2p\ell}; \quad i = 1, \ldots, p-1,$$

then there exists a unique trigonometric polynomial of the form (4.5), which is determined by of the system of equations

$$(4.7)\qquad \varphi(x_i,\vartheta) = \frac{f_k(x_i)}{|f_k(x_i)|}; \quad i = 1, \ldots, p-1,$$

$$(4.8)\qquad \varphi'(x_i,\vartheta) = 0; \quad i = 1, \ldots, p-1.$$

(b) If condition (4.3) is satisfied, define

$$(4.9)\qquad x_i = \Big(i + \lfloor \frac{i-1}{p-1} \rfloor\Big) \frac{\pi}{p\ell}; \quad i = 1, \ldots, p-1,$$

then there exists a unique trigonometric polynomial of the form (4.5) which is determined by the equations (4.7) and (4.8) .

(c) If condition (4.4) is satisfied, define

$$(4.10)\qquad x_i = \Big(i - 1 + \lfloor \frac{i-1}{p-1} + \frac{1}{2} \rfloor\Big) \frac{\pi}{p\ell}; \quad i = 1, \ldots, p,$$

then there exists a unique trigonometric polynomial of the form (4.5) which is determined by (4.8) and

$$(4.11)\qquad \varphi(x_i,\vartheta) = \frac{f_k(x_i)}{|f_k(x_i)|}; \quad i = 1, \ldots, p.$$

Proof. We only prove case (a), the other cases are shown exactly in the same way. Recall that in this case $k = 2\ell, \ell = \lfloor \frac{k+1}{2} \rfloor$ and p is odd which obviously implies $p \geq 3$. We define the $(p-1) \times (p-1)$-matrix

$$
(4.12) \qquad B = \begin{pmatrix} \cos(\ell x_1) & \cos(3\ell x_1) & \dots & \cos((2p-3)\ell x_1) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(\ell x_{\frac{p-1}{2}}) & \cos(3\ell x_{\frac{p-1}{2}}) & \cdots & \cos((2p-3)\ell x_{\frac{p-1}{2}}) \\ -\ell \sin(\ell x_1) & -3\ell \sin(3\ell x_1) & \cdots & -(2p-3)\ell \sin((2p-3)\ell x_1) \\ \vdots & \vdots & \vdots & \vdots \\ -\ell \sin(\ell x_{\frac{p-1}{2}}) & -3\ell \sin(3\ell x_{\frac{p-1}{2}}) & \cdots & -(2p-3)\ell \sin((2p-3)\ell x_{\frac{p-1}{2}}) \end{pmatrix}
$$

and the vector

$$
(4.13) \qquad e = \Big(\frac{\cos(\ell x_1)}{|\cos(\ell x_1)|}, \dots, \frac{\cos(\ell x_{\frac{p-1}{2}})}{|\cos(\ell x_{\frac{p-1}{2}})|}, 0, \dots, 0 \Big)^T \in \mathbb{R}^{p-1}.
$$

Observing the fact that

$$
x_i + x_{p-i} = \frac{\pi}{\ell} \quad i = 1, \dots, \frac{p-1}{2}
$$

[by definition (4.6)] and the identities

$$
\begin{aligned}
\cos((2j-1)\ell(x - \frac{\pi}{\ell})) &= -\cos((2j-1)\ell x), \\
\sin((2j-1)\ell(x - \frac{\pi}{\ell})) &= -\sin((2j-1)\ell x),
\end{aligned}
$$

it follows that the system of equations in (4.7) and (4.8) for an arbitrary trigonometric polynomial of the form

$$
\varphi(x, \vartheta) = \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)} f_{2\ell(2j-1)}(x) = \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)} \cos((2j-1)\ell x)
$$

is equivalent to the system

$$
(4.14) \qquad B\vartheta = e.
$$

Therefore the assertion in part (a) of Lemma 4.1 follows, if it can be shown that the matrix B is non-singular. For this purpose consider for $s \in \mathbb{N}$ the matrix

$$
A = \begin{pmatrix} \cos x_1 & \cos(3x_1) & \dots & \cos((4s-1)x_1) \\ \cos x_2 & \cos(3x_2) & \dots & \cos((4s-1)x_2) \\ \vdots & \vdots & \vdots & \vdots \\ \cos x_{2s} & \cos(3x_{2s}) & \dots & \cos((4s-1)x_{2s}) \end{pmatrix} = (u_1, \dots, u_{2s}) \in \mathbb{R}^{2s \times 2s},
$$

where $x_1 < x_2 < \dots < x_{2s}$ are arbitrary numbers and the vector u_ℓ is defined by

$$
u_\ell = (\cos((2\ell-1)x_1), \dots, \cos((2\ell-1)x_{2s}))^T; \quad \ell = 1, \dots, 2s.
$$

Let $U_{2\ell-1}(x) = \sin(2\ell \arccos x)/\sin(\arccos x)$ denote the $(2\ell-1)$th Chebyshev polynomial of the second kind [see Szegö (1975)]. Observing the identity

$$\sum_{j=1}^{\ell} \cos((2j-1)x) = \frac{1}{2}\frac{\sin(2\ell x)}{\sin x} = \frac{1}{2}U_{2\ell-1}(z)$$

with $z = \cos x$ [see Jolley (1961), formula 420] it follows that

$$\begin{aligned} \det A &= \det(u_1, u_1+u_2, u_1+u_2+u_3, \ldots, u_1+\ldots+u_{2s}) \\ &= \Big(\frac{1}{2}\Big)^{2s} \det\Big(U_{2j-1}(z_i)\Big)_{i,j=1}^{2s} \end{aligned}$$

where $z_i = \cos x_i$ $(i = 1, \ldots, 2s)$. It is well known that the leading coefficient of $U_j(x)$ is 2^j [see Szegö (1975)], and we obtain (using the Vandermonde determinant formula)

$$\begin{aligned} \det A &= (\frac{1}{2})^{2s} 2^{\sum_{j=1}^{2s}(2j-1)} \det(z_i^{2j-1})_{i,j=1}^{2s} \\ &= 2^{2s(2s-1)} \prod_{i=1}^{2s} \cos x_i \prod_{1\le i<j\le 2s} (\cos^2 x_j - \cos^2 x_i). \end{aligned}$$

Subtracting the ith row of the matrix A from the $(i+1)$th row $(i = 1, \ldots, s)$ it follows that the determinant of the matrix B defined by (4.12) (with $s = \frac{p-1}{2}$) can be obtained as

$$\begin{aligned} \det B &= \lim_{x_{i+s}\to x_i; i=1,\ldots,s} \frac{\det A}{\prod_{i=1}^{s}(x_{i+s}-x_i)} \\ &= -2^{2s(2s-1)} \prod_{i=1}^{s} (\cos^2 x_i \sin 2x_i) \prod_{1\le i<j\le s} (\cos^2 x_i - \cos^2 x_j)^4. \end{aligned}$$

Finally, if $s = \frac{p-1}{2}$ and the points $x_1, \ldots, x_{\frac{p-1}{2}}$ are given by (4.6), then it follows that $x_j \in (0, \frac{\pi}{2})$ $(j = 1, \ldots, \frac{p-1}{2})$, which implies $\det B \neq 0$. This proves the assertion of Lemma 4.1. □

Theorem 4.2. *Assume that $m > 2$. For any k such that $1 \le \lfloor \frac{k+1}{2} \rfloor \le \frac{m}{3}$ the trigonometric polynomial defined by Lemma 4.1 is a solution of the extremal problem (1.2). Moreover, the constant $c_{m,\infty}$ in the equivalent problem (1.3) depends only on p and is given by*

$$c_{m,\infty} = \Big\{\frac{2}{p}\cot\Big(\frac{\pi}{2p}\Big)\Big\}^{-1},$$

where p is defined by (4.1).

Proof. Again we restrict ourselves to the case (4.2) where $k = 2\ell$ and p is odd. We consider the trigonometric polynomial $\hat{\varphi}(x) = \varphi(x, \hat{\vartheta})$ defined by the first part of Lemma 4.1 and show that

$$|\varphi(x, \hat{\vartheta})| \le 1 \quad \forall\, x \in [-1, 1]. \tag{4.15}$$

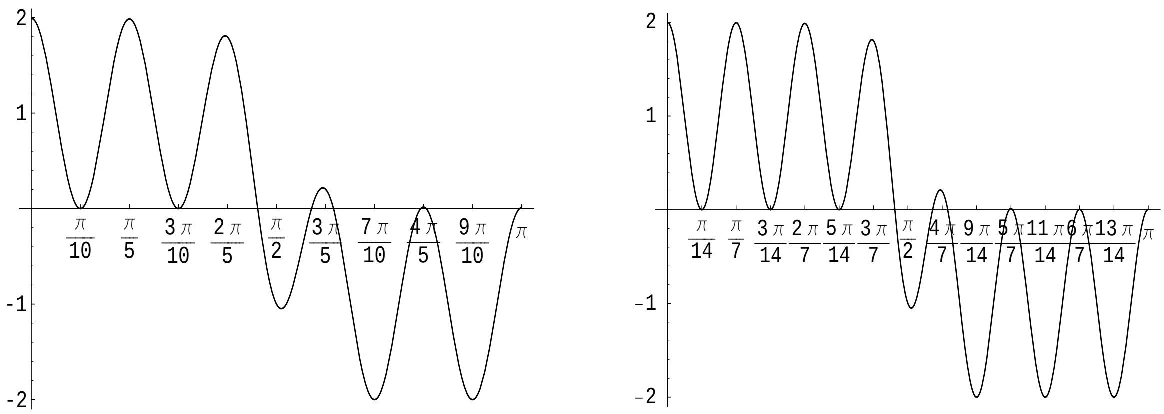

Figure 4.1: *The function $\bar{Q}$ defined in (4.17) for $p = 5$ (left panel) and $p = 7$ (right panel), which is considered in the proof of Theorem 4.2.*

For this we introduce the function

$$Q(x) = \varphi(x, \hat{\vartheta}) + \cos(2p\ell x), \tag{4.16}$$

and note that it follows from the representation (4.5) that

$$Q(x) = Q(\ell u) = \bar{\varphi}(u) + \cos(2pu) =: \bar{Q}(u) \tag{4.17}$$

where $u = \ell x$ and the function $\bar{\varphi}$ is defined by

$$\bar{\varphi}(u) = \sum_{j=1}^{p-1} \vartheta_{2\ell(2j-1)} \cos((2j-1)u).$$

Similarly, define $u_i = \ell x_i$ ($i = 1, \ldots, p-1$), where x_i is given by (4.6), note that

$$u_i = \frac{2i-1}{2p}\pi;\ i = 1, \ldots, \frac{p-1}{2},$$

and that $u_1, \ldots, u_{\frac{p-1}{2}} \in (0, \frac{\pi}{2})$ are roots of trigonometric polynomial $\bar{Q}(u)$ with at least multiplicity two, by Lemma 4.1. Moreover, $\bar{\varphi}(u)$ changes in the interval $[u_{\frac{p-1}{2}}, \frac{\pi}{2}]$ from 1 to 0, while $-\cos(2pu)$ changes in the interval $[u_{p-1}, \frac{p-1}{p}\pi] = [\frac{2p-3}{2p}\pi, \frac{p-1}{p}\pi]$ from 1 to -1 and in the interval $[\frac{p-1}{2p}\pi, \frac{\pi}{2}]$ from -1 to 1 (note that p is odd). Therefore there exists at least one more root of the trigonometric polynomial $\bar{Q}(u)$ in the interval $(u_{\frac{p-1}{2}}, \frac{\pi}{2})$. A similar argument shows that $\bar{Q}(u)$ has at least p roots in the interval $(\frac{\pi}{2}, \pi)$ (counted with multiplicities). Therefore the polynomial $\bar{Q}(u)$ has at least $2p$ roots in the interval $(0, \pi)$. A typical behaviour of $\bar{Q}(u)$ is depicted in Figure 4.1 for the case $p = 5$ and $p = 7$. As a consequence we obtain that $|\bar{\varphi}(u)| \leq 1 \quad \forall\ u \in [0, \pi]$ because otherwise the polynomial $\bar{Q}(u)$ would have at least $2p + 1$ roots which is impossible. By symmetry it follows that $\|\bar{\varphi}\|_\infty \leq 1$ and we also have $\|\hat{\varphi}\|_\infty \leq 1$.

We now use Theorem 2.1 to prove that the trigonometric polynomial $\hat{\varphi}$ is a solution of the extremal problem (1.5). For this we consider the points

$$x_i \;=\; \Big(2i-1+2\lfloor\frac{i-1}{p-1}+\frac{1}{2}\rfloor\Big)\frac{\pi}{2p\ell} \quad i=1,\ldots,n=\ell(p-1)$$

where $n=\ell(p-1)$, and

$$x_{n+j}=-x_j \quad j=1,\ldots,n.$$

Define $N=2n$ and the weights $A_1,\ldots,A_N$ by

$$A_j=\frac{1}{p\ell}\cos(\ell x_j) \quad j=1,\ldots,N.$$

With these notations it follows for $i=2s$, $s\in\{0,\ldots,m\}\backslash\{\ell\}$ that

$$\begin{aligned}\sum_{j=1}^{N}A_jf_{2s}(x_j) \;&=\; \frac{2}{p\ell}\sum_{j=1}^{(p-1)\ell}\cos(\ell x_j)\cos(sx_j)\\ &=\; \frac{2}{p\ell}\sum_{j=1}^{p\ell}\cos\Big(\ell\frac{2j-1}{2p\ell}\pi\Big)\cos\Big(s\frac{2j-1}{2p\ell}\pi\Big)=0,\end{aligned}$$

where we have used the identity

$$\sum_{j=1}^{q}\cos\Big(\ell\frac{2j-1}{2q}\pi\Big)\cos\Big(s\frac{2j-1}{2q}\pi\Big)=0 \quad \text{if} \quad s\neq\ell$$

[see Rivlin (1974), Exercise 1.5.26]. On the other hand we obtain

$$\begin{aligned}\sum_{j=1}^{N}A_jf_{2\ell}(t_j) \;&=\; \frac{2}{p\ell}\sum_{j=1}^{p\ell}\cos^2\Big(\ell\frac{2j-1}{2pl}\pi\Big)=1+\frac{1}{p\ell}\sum_{j=1}^{p\ell}\cos\Big(\ell\frac{2j-1}{p\ell}\pi\Big)\\ &=\; 1+\frac{1}{2}\frac{\sin(2\ell\pi)}{\sin(\pi/p)}=1,\end{aligned}$$

[see Jolley (1961), formula 420] and by symmetry (using the fact that $A_{j+n}=A_j$)

$$\sum_{j=1}^{N}A_jf_{2s-1}(x_j)=\sum_{j=1}^{N}A_j\sin(sx_j)=0, \quad s\in\{1,\ldots,m\}.$$

This shows that conditions (i) - (iii) of Theorem 2.1 are satisfied and consequently the trigonometric polynomial $\hat{\varphi}(x)=\varphi(x,\hat{\vartheta})$ is a solution of the extremal problem (1.5). For the calculation of the value $\hat{\vartheta}_{2\ell}$ corresponding to function $\cos(\ell x)$ in the trigonometric polynomial $\hat{\varphi}(x)$ we use the representation (2.3) in Theorem 2.1 and obtain

$$\begin{aligned}\hat{\vartheta}_{2\ell}=\sum_{j=1}^{N}|A_j| \;&=\; \frac{1}{p\ell}\sum_{j=1}^{N}|\cos\ell x_j|=\frac{2}{p\ell}\sum_{j=1}^{\ell(p-1)}|\cos\ell x_j|\\ &=\; \frac{2}{p\ell}\sum_{j=1}^{p\ell}\Big|\cos\Big(\frac{2j-1}{2p}\pi\Big)\Big|=\frac{4}{p}\sum_{j=1}^{\frac{p-1}{2}}\cos\Big(\frac{2j-1}{2p}\pi\Big)\\ &=\; \frac{2}{p}\cot\Big(\frac{\pi}{2p}\Big),\end{aligned}$$

where the last line follows again from formula 420 in Jolley (1961). This completes the proof of Theorem 4.2.

□

Remark 4.3. Note that it follows from the proof of Theorem 4.2 that the extremal polynomial defined by (4.5) has $2\ell(p-1)$ extremal points, if condition (4.2) or (4.3) is satisfied, while there are $2\ell(p-1)+3$ extremal points in the case (4.4). Moreover, it follows from the representation (4.10) that the points $-\pi$ and π are extremal points if condition (4.4) is satisfied. In the two other cases the boundary points of the interval $[-\pi, \pi]$ are not extremal points.

Remark 4.4. It is interesting to note that Theorem 3.1 is contained in Theorem 4.2 (note that our proof does not require the condition $1 \leq \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{3}$. To see this note in the case $\ell > \frac{m}{3}$ we have $p = \lfloor \frac{m+3\ell}{2\ell} \rfloor = 2$, by an elementary calculation, and therefore the polynomial in (4.5) reduces to $\cos(\ell t)$ or $\sin(\ell t)$ corresponding to the case $k = 2\ell$ and $k = 2\ell - 1$, respectively. In order to emphasize the different structure of the solution of the extremal problem (1.2) in the cases $1 \leq \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{3}$ and $\frac{m}{3} < \lfloor \frac{k+1}{2} \rfloor \leq m$ both scenarios are presented in different sections.

Remark 4.5. The extremal polynomial in Theorem 4.2 can be represented explicitly as

$$\hat{\varphi}(x) = \sum_{j=1}^{n} \operatorname{sign}(f_k(x_j))\ell_{jk}(x), \tag{4.18}$$

where the functions $\ell_{jk}(t)$ are defined as

$$\ell_{jk}(x) = \prod_{i \neq j} \Big(\frac{f_k(x) - f_k(x_i)}{f_k(x_j) - f_k(x_i)} \Big)^{s_i} \{a_j(f_k(x) - f_k(x_j)) + 1\}$$

with

$$a_j = - \sum_{i \neq j} \frac{s_i}{f_k(x_i) - f_k(x_j)}$$

and $s_1 = \ldots = s_n = 2, n = p-1$, if (4.2) or (4.3) holds and $s_1 = s_n = 1, s_2 = \ldots = s_{n-1} = 2, n = p$, if (4.4) holds. This follows by a tedious calculation showing that the function on the right hand side of (4.18) satisfies the conditions (4.7) and (4.8) which determine the extremal polynomial of the form (4.5) uniquely.

Remark 4.6. Note that by Theorem 4.2 the optimal constant in (1.3) is given by

$$c_{k,m,\infty} = \begin{cases} \{\frac{2}{p}\cot(\frac{\pi}{2p})\}^{-1} & \text{if} \quad 1 \leq \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{3} \\ 1 & \text{else} \end{cases} .$$

Therefore if $m \to \infty$ it follows for any fixed $k \in \mathbb{N}$ that

$$\lim_{m\to\infty} c_{k,m,\infty} = \lim_{p\to\infty} \Big\{ \frac{2}{p} \cot\Big(\frac{\pi}{2p}\Big) \Big\}^{-1} = \frac{\pi}{4}.$$

In other words the optimal constant $c_{k,m,\infty}$ in the extremal problem (1.5) exhibits the same asymptotic behaviour as the optimal constants $c_{m,p}$ in the extremal problem (1.4) [see Marshall and Ganzburg (1999)].

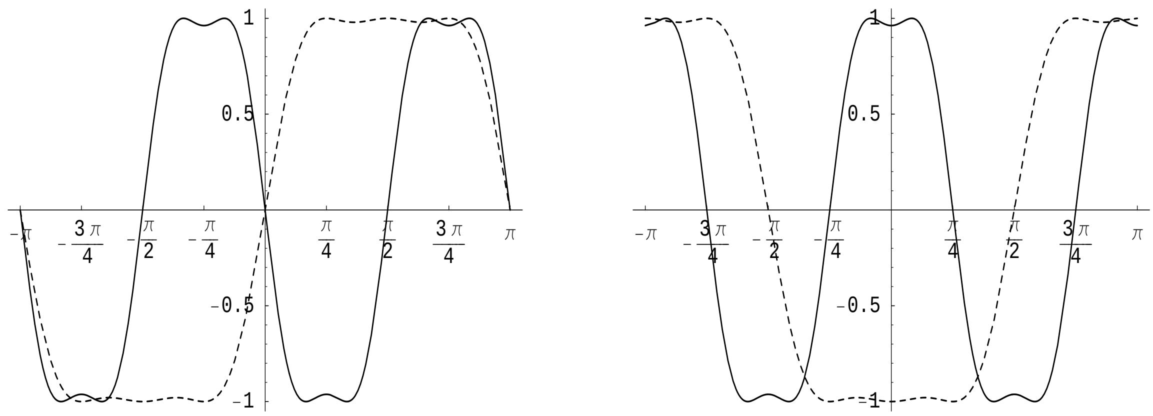

Figure 4.2: *The solution of the extreml problem (1.2) in the case $m = 6$. The left part of the figure shows the extremal polynomials for $k = 1$ (dotted line) and $k = 3$ (solid line), while the right part displays the solution for $k = 2$ (dotted line) and $k = 4$ (solid line).*

Example 4.7. Consider the case $m = 6$, where Theorem 4.2 can be applied for the cases $k = 1, 2, 3, 4$. If $\ell = 1$ we have $p = 4$, and the trigonometric polynomial in (4.5) is given by

$$\varphi(t) \;=\; \frac{\sqrt{3+\sqrt{2}}}{2}\Big(\sin t + \frac{8-5\sqrt{2}}{4}\sin(3t) + \frac{3\sqrt{2}-4}{4}\sin(5t)\Big)\ , \tag{4.19}$$

$$\varphi(t) \;=\; \frac{\sqrt{3+\sqrt{2}}}{2}\Big(-\cos t + \frac{8-5\sqrt{2}}{4}\cos(3t) - \frac{3\sqrt{2}-4}{4}\cos(5t)\Big) \tag{4.20}$$

corresponding to the cases $k = 1$ and $k = 2$, respectively. If $\ell = 2$ the solution of the extremal problem (1.2) is simpler and given by

$$\varphi(t) \;=\; \frac{\sqrt{3}}{2}\Big(\sin(2t) + \frac{1}{6}\sin(6t)\Big) \tag{4.21}$$

$$\varphi(t) \;=\; \frac{\sqrt{3}}{2}\Big(\cos(2t) - \frac{1}{6}\cos(6t)\Big) \tag{4.22}$$

corresponding to the cases $k = 3$ and $k = 4$, respectively. These polynomials are depicted in Figure 4.2. In all other cases the solution of the extremal problem (1.2) is given in Theorem 3.1.

5 Uniqueness and examples

In the present section we will demonstrate that the solution of the extremal problem (1.2) [or equivalently (1.3) or (1.5)] is not necessarily unique. Our first result characterizes the set of all extremal polynomials.

Theorem 5.1. *Let $\hat{\varphi}(x)$ denote the extremal polynomial described in Theorem 3.1 or 4.2 and define $\{x_1, \ldots, x_N\}$ as the set of all extremal points of $\hat{\varphi}(x)$. An arbitrary trigonometric polynomial $\varphi(x)$ with $\|\varphi\|_\infty \le 1$ is a solution of the extremal problem (1.2) if and only if it can be represented as*

$$\varphi(x) = \hat{\varphi}(x) + \gamma Q(x). \tag{5.1}$$

Here γ is a real constant and $Q(x)$ is a trigonometric polynomial with roots $x_1, \dots, x_N$, where the multiplicity of a root x_i is two whenever $|x_i| \neq \pi$.

Proof. Assume that φ is a solution of the extremal problem (1.2), such that it has the same coefficient ϑ_k as $\hat{\varphi}$. Then the trigonometric polynomial

$$\tilde{\varphi}(x) = \frac{1}{2}(\varphi(x) + \hat{\varphi}(x))$$

is also a solution of the extremal problem (1.2) because it has the same coefficient of ϑ_k and satisfies $\|\tilde{\varphi}\|_\infty \leq 1$. If $\tilde{x}_1, \dots, \tilde{x}_{\tilde{N}}$ are the extremal points of $\tilde{\varphi}$, then it follows from the inequality

$$|\tilde{\varphi}(x)| \leq \frac{1}{2}\{|\varphi(x)| + |\hat{\varphi}(x)|\} \leq 1$$

that

$$\{\tilde{x}_1, \dots, \tilde{x}_{\tilde{N}}\} \subset \{x_1, \dots, x_N\}.$$

Moreover, $\tilde{x}_1, \dots, \tilde{x}_N$ are also extremal points of the trigonometric polynomial $\varphi(x)$, by the same argument. Note that $\{f_0, \dots, f_{2m}\}$ is a Chebyshev system on the interval $[-\pi, \pi)$ and that $N \leq 2m < 2m + 1$. As a consequence there exists a unique solution $A_1, \dots, A_N$ of the system of equations (ii) and (iii) in Theorem 2.1, which implies

$$\{\tilde{x}_1, \dots, \tilde{x}_N\} = \{x_1, \dots, x_N\}.$$

Consequently, we obtain $\tilde{\varphi}(x_j) = \hat{\varphi}(x_j) = \text{sign}(A_j)$, $j = 1, \dots, N$, and the function

$$\tilde{\varphi}(x) - \hat{\varphi}(x) = \frac{1}{2}(\varphi(x) - \hat{\varphi}(x))$$

has roots at the points $x_1, \dots, x_N$, where the multiplicity of a root x_i is (at least) two, whenever $|x_i| \neq \pi$. Therefore the polynomial φ can be written in the form (5.1), which proves the first part of the assertion.

The converse part of the assertion can be proved in a similar way and is omitted for the sake of brevity.

□

Example 5.2. Assume that $\frac{m}{3} < \ell = \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{2}$, then, if $k = 2\ell$ is even, by Theorem 3.1 the polynomial $\hat{\varphi}(x) = \cos(\ell x)$ is a solution of the extremal problem (1.2) and the extremal points are given by $x_j^* = (j - \ell - 1)\pi/\ell$ $(j = 1, \dots, 2\ell)$. These points are roots of multiplicity 2 of the polynomial $Q(x) = 1 - \cos(2\ell x)$ and a further extremal polynomial is of the form

$$\varphi_\gamma(x) = \cos(\ell x) + \gamma(1 - \cos(2\ell x)), \tag{5.2}$$

where the constant γ is chosen such that $\|\varphi_\gamma\|_\infty = 1$. By a substitution of $u = \cos(\ell x)$ it is easy to see that the possible range for γ is the interval $[-\frac{1}{4}, \frac{1}{4}]$. Extremal polynomials of the form (5.2) for various values of ℓ and γ are depicted in Figure 5.1 in the case $\ell = 2$ and $\ell = 5$ (note that the solution of the extremal problem (1.2) depends only on m by the inequality $\frac{m}{3} < \ell \leq \frac{m}{2}$).

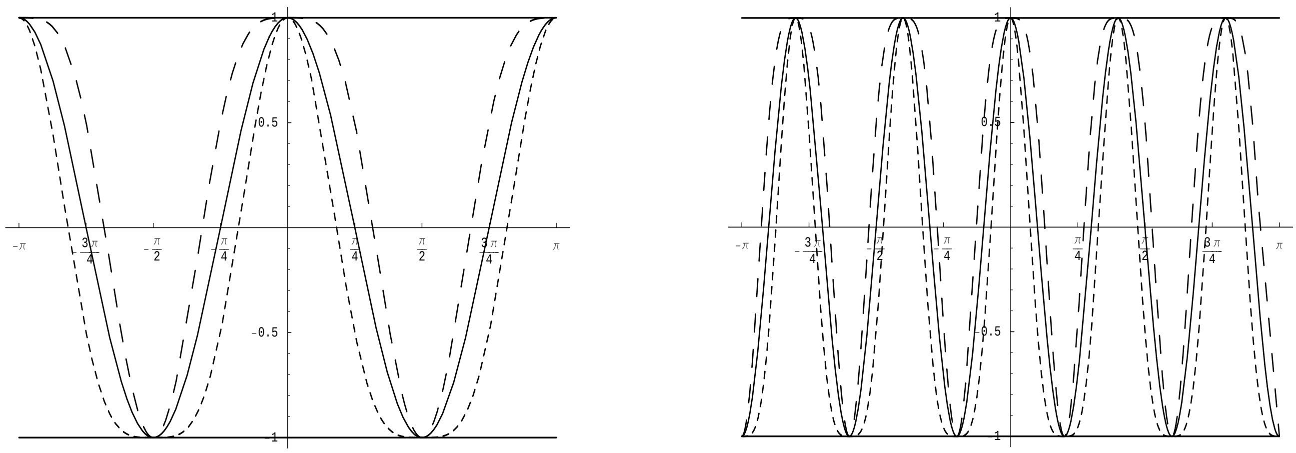

Figure 5.1: *Various solutions of the form (5.2) for the extremal problem (1.2) with $k = 2\ell$. Solid lines $\gamma = 0$; dashed lines: $\gamma = 1/4$; dotted lines: $\gamma = -1/4$. The left part of the figure corresponds to the case $\ell = 2$, while the right part represents the case $\ell = 5$.*

It can be shown by similar arguments that in the case $k = 2\ell - 1; m/3 \leq \ell \leq m/2$ trigonometric polynomials of the form

$$\varphi_\gamma(x) = \sin(\ell x) + \gamma(1 + \cos(2\ell x)); \quad \gamma \in \left[-\frac{1}{4}, \frac{1}{4}\right] \tag{5.3}$$

are solutions of the extremal problem (1.2), and some representative examples are depicted in Figure 5.2 (in the case $\ell = 2$ and $\ell = 5$).

Example 5.3. Assume that $1 \leq \ell = \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{3}$ and that either $k = 2\ell - 1$ and $\frac{m+3\ell}{2\ell}$ is an integer or that $k = 2\ell$ and $\frac{m+3\ell}{2\ell}$ is odd. If $\frac{m+3\ell}{2\ell}$ is an integer we obtain that $2p - 3 = \frac{m}{\ell}$ and consequently the trigonometric polynomial in (4.5) is of degree m. In this case it follows from Remark 4.3 that the polynomial Q in Theorem 5.1 must have $2\ell(p-1) = m + \ell$ roots with multiplicity no less than 2 and its degree must be larger than m. As a consequence, there exists a unique solution of the extremal problem (1.2) in this case.
On the other hand, if $\frac{m+3\ell}{2\ell}$ is not an integer, we obtain $\frac{m+3\ell}{2\ell} - 1 < p < \frac{m+3\ell}{2\ell}$ or equivalently $m - 2\ell < \ell(2p-3) < m$. Because the degree of the polynomial in (4.5) is $\ell(2p-3) < m$ there exists more than one solution of the extremal problem (1.2). To be more specific consider the case m even $k = 2\ell$ and assume that p is odd, then the extremal polynomial $\hat{\varphi}$ from Lemma 4.1 is of the form

$$\hat{\varphi}(x) = \sum_{j=1}^{p-1} \vartheta_j \cos(\ell(2j-1)x) \tag{5.4}$$

Let Q denote the positive trigonometric polynomial, which has precisely roots of multiplicity two at the extreme points of $\hat{\varphi}$. Because $k = 2\ell$ and p is odd it follows from Remark 4.3 that the extreme points are not located at the boundary of the interval $[-\pi, \pi]$ and by the above discussion the degree of Q is not larger than m. By Theorem 5.1 the trigonometric polynomial

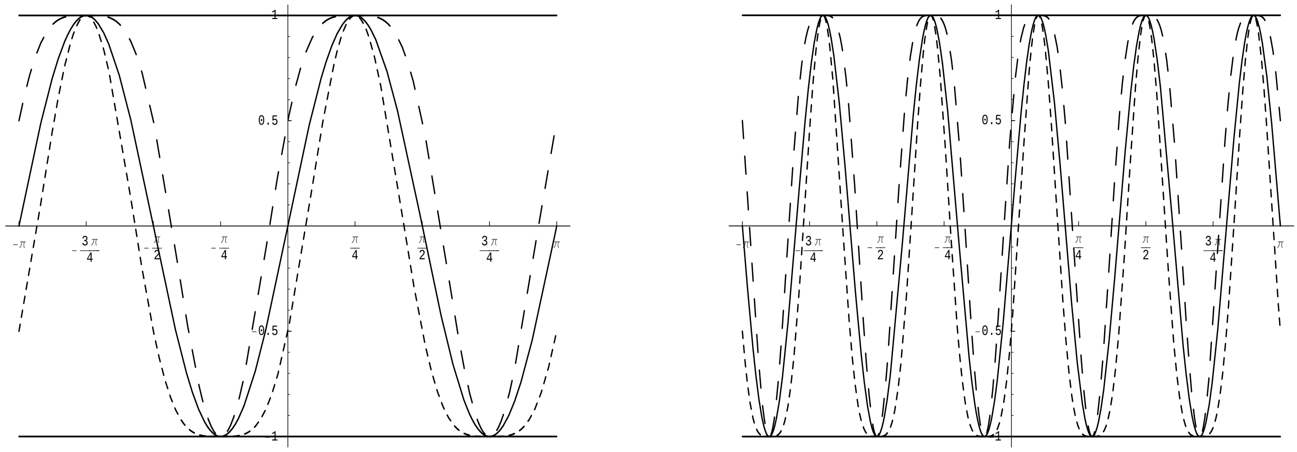

Figure 5.2: *Various solutions of the form (5.3) for the extremal problem (1.2) with $k = 2\ell - 1$. Solid lines $\gamma = 0$; dashed lines: $\gamma = 1/4$; dotted lines: $\gamma = -1/4$. The left part of the figure corresponds to the case $\ell = 2$, while the right part represents the case $\ell = 5$.*

$\varphi_\gamma(x) = \hat{\varphi}(x) + \gamma Q(x)$ is a solution of the extremal problem (1.2) if and only if $\|\varphi_\gamma\|_\infty \leq 1$. In order to derive conditions on the parameter γ such that this inequality is satisfied let $\bar{x}_1, \ldots, \bar{x}_{N_1}$ denote the extreme points of $\hat{\varphi}(x)$ located in the interval $[0, \pi)$ with $\hat{\varphi}(x_j) = 1$, define the trigonometric polynomial

$$U(x) = \prod_{j=1}^{N_1} (\cos x - \cos \bar{x}_j)^2, \tag{5.5}$$

and consider the functions

$$H_1(x) = \frac{\hat{\varphi}(x) - 1}{U(x)}; \quad H_2(x) = \frac{Q(x)}{U(x)}. \tag{5.6}$$

Note that H_1 and H_2 are trigonometric polynomials and that the function H_2 is nonnegative. The inequality $\|\varphi_\gamma\|_\infty \leq 1$ is obviously equivalent to the inequalities

$$\hat{\varphi}(x) - 1 + \gamma Q(x) = (H_1(x) + \gamma H_2(x))U(x) \leq 0 \tag{5.7}$$

$$\hat{\varphi}(x) + 1 + \gamma Q(x) \geq 0 \tag{5.8}$$

for all $x \in [-\pi, \pi]$. We consider the case (5.7) in more detail, the second case is treated similarly by an appropriate modification of the definition of the functions H_1 and H_2. Because $U(x)$ is a positive function it follows that $H_1(x)$ is nonpositive and (5.7) is equivalent to the inequality

$$H_1(x) + \gamma H_2(x) \leq 0 \tag{5.9}$$

for all $x \in [-\pi, \pi]$. Define

$$\bar{\gamma} = \min\Big\{ -\frac{H_1(x)}{H_2(x)} \mid x \in [-\pi, \pi] \Big\},$$

and note that $\bar{\gamma} > 0$, then it is easy to see that (5.9) is satisfied if and only if $\gamma \leq \bar{\gamma}$. Similarly, it can be shown that (5.8) holds if and only if $\gamma \geq -\bar{\gamma}$. Consequently, the polynomial $\varphi_\gamma(x) = \hat{\varphi}(x) + \gamma Q(x)$ constructed in this way is a solution of the extremal problem (1.2) if and only if $|\gamma| \leq \bar{\gamma}$.
As a concrete example consider the case $m = 4$, $k = 2\ell = 2$, where $p = 3$ and the extremal polynomial defined in (4.5) is given by

$$\hat{\varphi}(x) = \frac{2}{\sqrt{3}}\Big\{\cos x - \frac{\cos(3x)}{6}\Big\}.$$

The extremal points of $\hat{\varphi}$ are $-\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$ and polynomial Q is calculated as

$$Q(x) = 16(\cos^2 x - \frac{3}{4})^2 = 2\cos(4x) - 4\cos(2x) + 3$$

(note that $\cos(\frac{\pi}{6}) = -\cos(\frac{5}{6}\pi) = \frac{\sqrt{3}}{2}$). The polynomial U is given by $U(x) = (\cos x - \frac{\sqrt{3}}{2})^2$, which yields

$$\begin{aligned} H_2(x) &= 16\Big(\cos x + \frac{\sqrt{3}}{2}\Big)^2, \\ H_1(x) &= -\frac{4}{3}\Big(1 + \frac{\cos x}{\sqrt{3}}\Big), \end{aligned}$$

and for $\bar{\gamma}$ the expression

$$\bar{\gamma} = \min\Big\{\frac{1 + (\cos x)/\sqrt{3}}{12(\cos x + \sqrt{3}/2)^2}\Big| x \in [-\pi, \pi]\Big\} = 1 - \frac{5}{9}\sqrt{3}.$$

Several solutions of the extremal problem (1.2) are depicted in Figure 5.3.

Example 5.4. If $\ell > \frac{m}{2}$, the solution of the extremal problem (1.2) is given by $\hat{\varphi}(x) = \cos(\ell x)$ if $k = 2\ell$ and by $\hat{\varphi}(x) = \sin(\ell x)$ if $k = 2\ell - 1$. In the second case the corresponding polynomial Q must have (at least) $2 \cdot 2\ell > 2m$ roots (counted with multiplicities), while in the first case there are $2\ell - 1$ roots with multiplicity two (corresponding to the extreme points in the interval $(-\pi, \pi)$ and one root at the point π, which gives (counted with multiplicities) $2(2\ell - 1) + 1 > 2m$ roots for the polynomial Q. In both cases the polynomial Q would not be a trigonometric polynomial of degree m and therefore the solution of the extremal problem is unique if $\ell > \frac{m}{2}$.

We conclude this paper summarizing the discussion presented in the previous examples.

Theorem 5.5. *The solution of the extremal problem (1.2) is unique if one of the following conditions is satisfied*

(a) $\lfloor \frac{k+1}{2} \rfloor \geq \frac{m}{2}$

(b) $1 \leq \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{3}$, *k is even,* $\frac{m+3\ell}{2\ell} \in \mathbb{N}$ *and is odd*

(c) $1 \leq \lfloor \frac{k+1}{2} \rfloor \leq \frac{m}{3}$ *k is odd,* $\frac{m+3\ell}{2\ell} \in \mathbb{N}$

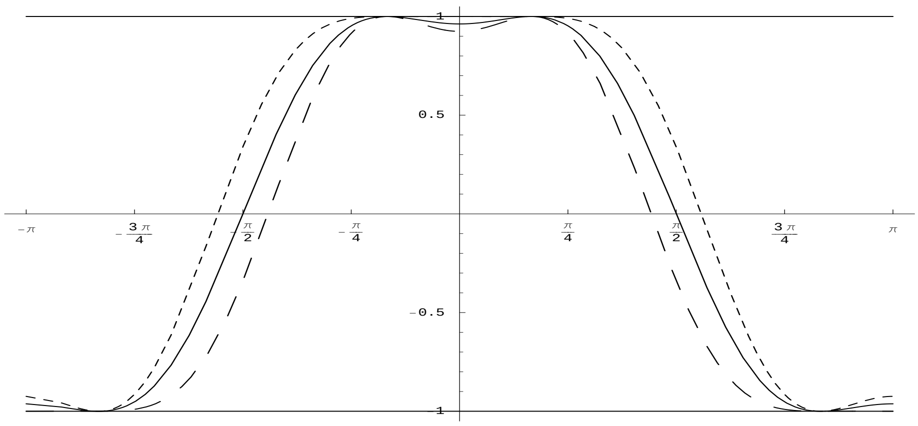

Figure 5.3: *Various solutions of the form (5.1) of the extremal problem (1.2) for* $m = 4$ *and* $k = 2$ ($\ell = 1$)*. Solid line* $\gamma = 0$*; dashed line:* $\gamma = -1 + \frac{5}{9}\sqrt{3}$*; dotted lines:* $\gamma = 1 - \frac{5}{9}\sqrt{3}$.

In all other cases, the solution of the extremal problem (1.2) is not unique.

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