A Multi-Stream Process Capability Assessment Using a Nonconformity Ratio Based Desirability Function

Thesis for the Acquisition of the Degree of a Doctor in Natural Sciences of the University of Dortmund

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Chapter 1

Prerequisite

Quality of products in the current economy context of hard competition, is a brand image guarantor. The inspection of the quality of products or services provided by a company is an infallible proof of the presence of competences within the organization. Mastering the process of quality control is a guarantee for the company competitiveness.

Many definitions have been given to quality, but no one of them is an unanimous definition. In fact, everyone considers quality from his (her) own point of view. However, in the scope of this work, the definition given by Montgomery (1996) is adopted:"the fitness for use". Following this definition, a product has a good quality if it satisfies the customer expectations. Although this definition clarifies the quality concept, it seems to be vague for the manufacturers who want to integrate quality in their decision making process.

The quality of a product is expressed through some quality characteristics. The customer expectations concerning a measurable quality characteristic are often expressed in terms of the lower specification limit (LSL), the target value (T) and the upper specification limit (USL). When the quality characteristic measure falls between the specification limits then the customer expectations are fulfilled. However, an item presenting a quality characteristic measure which is outside the specification limits is non conform to the customer expectations, hence the customer is not satisfied. Moreover, the customer satisfaction is maximized when the quality characteristic measure equals the target value (T). T is the value that the designers of the product give to the quality characteristic in the aim to satisfy some needs of the customers. It is not possible that all the produced items have a measured quality characteristic which is equal to the target value (T). Indeed, it is admitted that the variability exists all around us, the same experiences are made in the same conditions, but, will not give necessarily the same results. Since nature offers to us this variability it is called a natural variability. Starting from this fact, it is understandable to admit that the process output presents a given variability and it is dangerous to start the production with a process presenting a large variability. Indeed, this kind of process gives an important proportion of non conforming products. The "enemy" of the perfect product is the uncontrolled variability. Manager efforts for improving the product quality should be oriented toward understanding variability causes, evaluating the variability impacts and trying to reduce this variability. As long as the expected value of a given quality characteristic is most likely to fall between the specification limits, reducing variability is equivalent to reducing the proportion of non conforming items, hence, increasing the customer satisfaction. In order to reach these goals, statistics becomes an important tool in quality improvement. Process Capability Indices (PCIs) are an important tool of statistical process control. PCI general form is

$$PCI = \frac{\text{Specification \quad limits \quad width}}{\text{Natural \quad process \quad variability}}.$$

This PCI form allows to summarize the ability of a process to meet the customer requirements. Figure 1.1 is a visualization of the specification band which represents the performance standard established by the customer and the tolerance band which represents the process performance.

The introduction of PCIs in the United States triggered off the extension of the use of PCIs. Indeed, each company wants to be sure that the products delivered by its suppliers meet its requirements. Hence, PCIs are considered as an important tool for the suppliers selection. Following this reasoning, the supplier who wants to win the customer confidence should have a process presenting an acceptable capability level.

In order to assess the process capability, the specification limits width called also specification width and the natural variability need to be computed. For that purpose the process capability index computation is based on some assumptions.

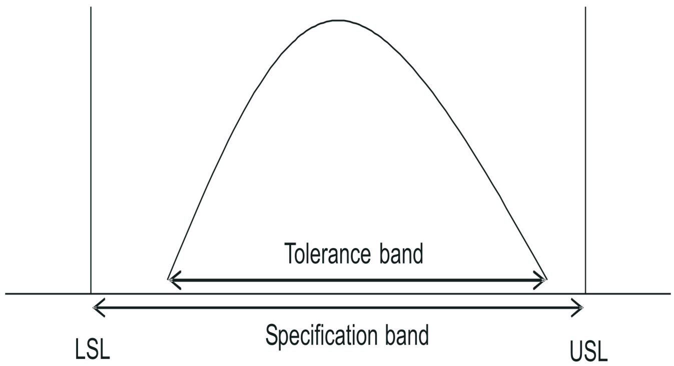

Figure 1.1: Visualization of the Specification Band and the Tolerance band.

1.1 PCI Assumptions

Measuring the process ability to meet the customer expectations is very important. The customers express their expectations by providing the specification limits for the quality characteristics of interest. The process performance is evaluated by the comparison of the process variability to the specification limits width. This comparison could be carried out using the histogram or the control chart. Hence, the process performance evaluation is done visually. However, managers need a value which summarizes this process performance, allows to follow the performance evolution and to compare it with the performance of other processes.

From the general form of the process capability indices it is clear that the rule of thumb for the PCIs is that the higher the process capability index value, the more able the process to satisfy the customer expectations. In order to compute PCIs the following assumptions are commonly admitted:

- The process is under statistical control.
- The underlying distribution is the normal distribution.

1.1.1 The Process is Under Statistical Control

Before computing the indices, data is collected by successive samples. Averages of the collected samples are represented on the control chart. If the represented points are within the control limits, the process is under statistical control. It should be noticed that the used sample size should be greater or equal 5 in order to be able to determine the control limits using the central limit theorem. The control limits for the $\bar{x}$ control chart are computed as follows:

$$UCL = \bar{\bar{x}} + 3\sigma_{\bar{x}},$$
$$\text{central} \quad \text{line} = \bar{\bar{x}},$$
$$LCL = \bar{\bar{x}} - 3\sigma_{\bar{x}}.$$

This means that the obtained data reflects a variability only due to the process and not to some external or special causes. Under this assumption the process capability index enables a comparison between the real process performance and the standard performance established by the customer or set by the engineers.

1.1.2 The Underlying Distribution is the Normal Distribution

The process variability is measured by the tolerance band. It is determined through two values between which there is an important fraction of the population. The width of this interval measures the natural variability of the process. Montgomery (1996) pointed out that if the normal distribution assumption holds, the interval $[\mu \pm 3\sigma]$ contains 99.73% of the population, where μ and σ are the expected value and the standard deviation of the distribution. Moreover, such interval can be constructed for other distributions, Lovelace and Kotz (1998) noticed that this is the reason which makes some authors extend this assumption to the existence of a probability distribution for the collected data. Hence, the natural variability is obtained through the estimation of the quantiles $X_{0.00135}$ and $X_{0.99865}$ of the identified probability distribution. It is important to recall that the quantile of order α, X_α, satisfies: $Pr[X \leq X_\alpha] = \alpha$. However, it is commonly admitted that the underlying distribution is the normal distribution in order to make the determination of the statistical properties of the indices tractable.

The widely used process capability indices are:

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1.1}$$

$$C_{pk} = \min\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\}, \tag{1.2}$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{1.3}$$

$$C_{pmk} = \frac{\min\{USL - \mu, \mu - LSL\}}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{1.4}$$

where, μ and σ are the parameters of the normal distribution.

1.2 The Index Structures

Since their first appearance in industry, the structures of the process capability indices have been revised several times. These changes aimed at taking into account the deviations from the PCI assumptions. Statisticians and quality engineers tried to improve the indices performance in reflecting the real process capability and to avoid misleading interpretations when using PCIs. The new generation of indices takes into account the particularity of some collected data, like autocorrelation and non normality. The index structures are still one of the basic and most important problem of the PCIs theory. This problem becomes more obvious for the multivariate case. Nevertheless, in their evolution the PCIs are still based mainly on two approaches: The nonconformity ratio approach and the loss function approach. In this work we focus on the deviation from the normal distribution assumption and only the nonconformity ratio approach is considered. In order to explain the reason for this option the nonconformity ratio approach and the loss function approach are explained and the relationship between both approaches is presented.

1.2.1 The Nonconformity Ratio Approach

The structure of the first and the second generation of indices is the most basic and the most simple. The classical indices belonging to these generations are the most known. However, these indices have different interpretations.

The Potential Capability Index

The index C_p given by (1.1) is considered as the first generation of PCIs. In (1.1) the specification width is fixed by the engineers or imposed by the customers. However, under the normality assumption 6σ is used as denominator. Indeed, with such assumption, the chosen denominator represents

99.73% of the population. Hence, if the C_p index is used, reducing the process variation guarantees a higher quality level, more capability to meet the specification limits and a higher value of C_p. But at which value of C_p the process is considered capable?
In this way Montgomery (1996) suggests (for an existing process) 1.33 as a minimum value for C_p. This value provides satisfactory capability of the process. When the normal distribution holds it corresponds to 0.0064% of non conforming products and to 4σ level in the six sigma theory as mentioned by Breyfogle III (1993).

In fact, care must be taken before concluding such interpretation, because, in the case of C_p index, the association of C_p value and the nonconformity ratio is not so direct and may not reflect the actual capability of the process even when the normal distribution holds.

Since the nonconformity ratio is the main interpretation of C_p index, it is interesting to integrate C_p in its computation in order to find a direct link between them. Considering a quality characteristic X, under the normality assumption the nonconformity ratio is expressed as:

$$P[X > USL] + P[X < LSL]$$
$$\Rightarrow P[\frac{X-\mu}{\sigma} > \frac{USL-\mu}{\sigma}] + P[\frac{X-\mu}{\sigma} < \frac{LSL-\mu}{\sigma}]$$
$$\Rightarrow \text{nonconformity ratio} = 1 - \Phi[\frac{USL-\mu}{\sigma}] + \Phi[\frac{LSL-\mu}{\sigma}]$$

If μ is substituted by $\frac{USL+LSL}{2}$ then:

$$\text{nonconformity ratio} = 2\Phi(-3C_p).$$

It is important to notice that the C_p index depends only on σ. If several processes having the same standard deviation but with different expected values are considered it is pointed out that these processes have the same index value and different nonconformity ratios.
Hence, the interpretation of the C_p value is reliable only for a fixed value of the location parameter μ. In order to avoid any confusion, when C_p index is used, it is assumed that μ is the midpoint of the specification limits. Indeed, given the symmetry of the normal distribution, it becomes very interesting to assume that the distribution mean is centered between the specification limits. In this way the nonconformity ratio is minimized. Under such assumption the nonconformity ratio corresponding to

the C_p value is the optimal ratio that the process can reach through the adjustment of the location parameter. Hence, it is assumed that C_p measures the "**potential process capability**". This means that C_p corresponds to the nonconformity ratio when the distribution mean is actually centered between the specification limits.

The Actual Capability Index

It was noticed that C_p structure does not take into account the effect of the location of the distribution on the process capability. That is the reason behind creating the second generation of indices: C_{pk} and C_{pm}.

C_{pk} is presented as follows :

$$C_{pk} = \min(CP_l, CP_u),$$

with

$$CP_u = \frac{USL - \mu}{3\sigma} \quad \text{and} \quad CP_l = \frac{\mu - LSL}{3\sigma}.$$

The following identity,

$$\min(x, y) = \frac{1}{2}(x + y) - \frac{1}{2}|x - y|$$

is used in order to give a more clear expression for C_{pk} as follows:

$$C_{pk} = \frac{USL - LSL}{6\sigma} - \frac{|\mu - M|}{3\sigma} \tag{1.5}$$

$$C_{pk} = C_p - \frac{|\mu - M|}{3\sigma},$$

where M is the midpoint of the specification limits with: $M = \frac{USL+LSL}{2}$.

From the structure of the C_{pk} index it is noted that the mean of the process divides the specification width into two areas and an index is computed for each area. The C_{pk} value is the minimum of these two indices. This means that the capability computation is based only on the closest side of the distribution to the specification limits.

The problem with this structure is that it evolves a simultaneous effect of the variance and the process mean. Indeed, several combinations of the distribution parameters give the same C_{pk} value. However, it is obvious that this value corresponds to different nonconformity ratios.

Since the nonconformity ratio is one of the most important interpretations of the process capability indices, it is very interesting to have a link between C_{pk} and the nonconformity ratio or the process yield with process yield $= 1 -$ nonconformity ratio. A glance at the C_{pk} structure reveals that this relationship is more complicated than for C_p index.

Boyles (1991) gives this link in form of bounds of the process yield for each value of C_{pk}. These bounds are:

$$100\{2\Phi(3C_{pk}) - 1\} \leq \%\text{yield} \leq 100\{\Phi(3C_{pk})\}.$$

These bounds are obtained as follows: Since

$$\%\text{yield} = 100[\Phi(\frac{USL - \mu}{\sigma}) - \Phi(\frac{LSL - \mu}{\sigma})],$$

$C_{pk} = \min(\frac{USL-\mu}{3\sigma}, \frac{LSL-\mu}{3\sigma})$ can be interpreted as follows:

$C_{pk} \leq \frac{USL-\mu}{3\sigma}$ and $C_{pk} \leq \frac{\mu-LSL}{3\sigma} \Leftrightarrow 2\Phi(3C_{pk}) - 1 \leq \Phi[\frac{USL-\mu}{\sigma}] - \Phi[\frac{LSL-\mu}{\sigma}]$

The upper bound is in fact an approximation:

$$C_{pk} = \frac{1}{3}\min(\Phi^{-1}[\%\text{yield} + \Phi(\frac{LSL - \mu}{\sigma})], \Phi^{-1}[\%\text{yield} + \Phi(\frac{\mu - USL}{\sigma})])$$

if $C_{pk} = \frac{1}{3}\min\{\Phi^{-1}[\%\text{yield} + \Phi(\frac{LSL-\mu}{\sigma})]\}$ then, $\%\text{yield} = \Phi(3C_{pk}) - \Phi(\frac{LSL-\mu}{\sigma})$.

Now if $C_{pk} = \frac{1}{3}\{\Phi^{-1}[\%\text{yield} + \Phi(\frac{\mu-USL}{\sigma})]\}$ then, $\%\text{yield} = \Phi(3C_{pk}) - \Phi(\frac{\mu-USL}{\sigma})$.

In both cases, $\%\text{yield} \leq \Phi(3C_{pk})$.

From (1.5) it is noted that $C_{pk} \leq C_p$ and $C_{pk} = C_p$ only when μ is centered between the specification limits. It becomes clear that centering the process mean between the specification limits improves the process capability and the process yield. Hence, C_{pk} is interpreted as the "**actual process capability**". If C_p and C_{pk} are used at the same time, weaknesses in their structures are covered. They give more information about the process behavior and directions for capability improvement. The simple structure of these indices make them easy to comprehend. Thus, they are the most frequently used indices in industry.

For the indices C_p and C_{pk} it is expected that their values increase when the nonconformity ratio decreases, it is said then that the indices respect the ”higher the better” rule. However, as it is shown in the literature mainly by Somerville and Montgomery (1996), Tang and Than (1999) and in this work, these indices are not indicative of the process capability when the normal distribution does not hold. Indeed, these classical indices do not respect the ”higher the better” rule when non normal distributions are considered. Furthermore, more weaknesses of these indices were proved when the loss function approach was incorporated in (1.3).

1.2.2 The Loss Function Approach

The loss function is considered in a point estimation context. First the UMVU estimator needs to be defined. An UMVU estimator is an unbiased estimator which presents the minimum variance among all other unbiased estimators for the same parameter. There are several methods which allow the determination of the UMVU estimator.

Let $x_1, ..., x_n$ be a random sample from a distribution with probability density function (p.d.f.) $f(x, \theta)$, $\theta \in \Omega$. Let $Y = u(x_1, ..., x_n)$ be a statistic giving the point estimate of θ, and let $\delta(y)$ be a function of observed value of Y. δ is a decision function and $\delta(y)$ is a decision. In order to measure the goodness of this point estimate we need to measure the difference between θ and $\delta(y)$ through the function $L[\delta(y), \theta]$: L is the loss function. The expected value of the loss function is a risk: $R[\theta, \delta]$.

If $g(y, \theta)$ is a p.d.f of Y we can write

$$R[\theta, \delta] = E[L[\theta, \delta(y)]] = \int_{-\infty}^{+\infty} L[\theta, \delta(y)]g(y, \delta)dy.$$

Then, we have to use point estimate of θ that minimizes $R[\theta, \delta] \forall \theta \in \Omega$. With the restriction $E[\delta(y)] = \theta$ and using $L[\theta, \delta(y)] = (\theta - \delta(y))^2$, $R[\theta, \delta]$ will be in fact the variance of $\delta(y)$. If $\delta(y)$ that minimizes $R[\theta, \delta]$ is found, we get an UMVU estimator of θ. Hence the loss function measures the deviation of the estimator from the parameter to be estimated.

The loss function approach was introduced in statistical process control by Hsiang and Taguchi (1985). They proposed to substitute the variance $\sigma^2 = E(X - \mu)^2$ by a new approach which considers a variation around the target value $\varsigma^2 = E(X - T)^2$. In this way, any deviation of the measured value x of the quality characteristic X from the target value T entails a monetary loss to the producer. This monetary loss can be expressed as $L(x) = k(x - T)^2$, where k is a positive constant related to the amount of the penalty supported by the company per product unity.

In (1.4) a pure monetary approach is not adopted, and k value is set to 1. The expected loss is given by: $E(L(X)) = E(X - T)^2 = \varsigma^2$.

ς^2 can also be written in the alternative form: $\varsigma^2 = \sigma^2 + (\mu - T)^2$. This form presents a process variability penalized by the deviation of the expected value of the process from the target value. C_{pm} is defined in (1.3).

From the C_{pm} structure it is deduced that a higher capability level is obtained through:

- The reduction of the dispersion around the mean.

- The adjustment of the mean to the target value.

This index incorporates a new component: the target value. It will be then more sensitive than other indices to departures from T. Hence, when the hypotheses explained in 1.1.1 and 1.1.2 hold this index gives more information about the process. Lovelace and Kotz (1998) noticed that there exist no reliable link between C_{pm} and the nonconformity ratio. Moreover, from the C_{pm} structure it is noticed that the index value reaches its maximum when μ is adjusted to T. In this case when T is the midpoint of the specification limits, the "higher the better" rule is respected in the loss function approach and the nonconformity ratio approach. However, when asymmetric specification limits are considered where T is not at the midpoint of the specification limits, the index C_{pm} reaches its maximum value when μ is adjusted to T. However, C_p and C_{pk} reach their maximum values when μ is adjusted at M. The nonconformity ratio approach and the loss function approach have in this case different purposes. The same problem arises when skewed distributions are considered with symmetric specification limits. In order to overcome this problem, the index C_{pmk} defined in (1.4) was introduced by Pearn et al. (1992) in the aim to integrate the nonconformity ratio approach and the loss function approach in one index.

The index C_{pmk} enables a compromise between both approaches. However, as for any compromise the goals of neither the nonconformity approach nor the loss function approach are reached. Table 1.1 gives the direction for quality improvement proposed by both approaches.

From Table 1.1 it is obvious that both approaches have different directions for quality improve-

Table 1.1: Direction for quality improvement when the normal distribution holds

	Nonconformity ratio approach	Loss function approach
Quality improvement directions	Adjusting the mean to M Reduce the variability around μ	Adjusting the mean to T Reduce the variability around μ

ment when $T \neq M$. Moreover, when $T = M$ the approaches are different in the presence of a non normal distribution. Indeed, in this case the quality of a product is improved when the mean is adjusted to M following the loss function approach. For the nonconformity approach adjusting the mean to M leads to misleading decision about the process capability. In this case the classical indices reach their maximum values when $\mu = M$. However, the nonconformity ratio is not minimized when $\mu = M$. Adopting a pure loss function approach in this case is dangerous as this approach still has no connection with the nonconformity ratio. A pure loss function approach should be adopted when its direction for quality improvement causes no serious degradation on the nonconformity ratio. Furthermore, the study of the relationship between the indices provides better understanding of the difference between both approaches even when $T = M$ and allows the presentation of the approaches properties and to improve the quality of a product.

1.3 Relationships Between Indices

It is important in process capability indices theory to show relationships between the different indices. This step gives clear ideas about the properties of the indices when they are faced to the same data, and more information about quality improvement.

An effective tool that allows theoretical comparison between capability indices is a (μ, σ) plot for $LSL \leq \mu \leq USL$ and $\sigma > 0$. This work is due to Boyles (1991) who illustrated five contours of C_p and C_{pk} in the (μ, σ) plan, and five contours of C_p and C_{pm} in the (μ, σ) plan.

From these illustrations it is noted that $C_p \geq C_{pk}$ and $C_p \geq C_{pm}$. C_{pk} and C_{pm} reach their maximum when μ is centered at the midpoint of the specification limits which is assumed to be the target value. At their maximum C_{pk} and C_{pm} are equal to C_p, and decrease when μ moves away from M in the case of symmetric specification limits.

It is noted that C_{pk} does not take into account the distance between μ and T and becomes arbitrarily large as σ approaches 0, independently of this distance.

However, C_p and C_{pk} could be used at the same time to overcome their weaknesses. Indeed, when there is a large difference between C_p and C_{pk} values, it would be better to center the process mean at the midpoint in the aim to have a higher capability of the process. Then, for more capability improvement, the variability around the mean should be reduced. This step would be applied also for C_{pm} only for the case when $T = M$.

Moreover, while C_{pk} increases without bounds when $\sigma \rightarrow 0$, C_{pm} is bounded and $C_{pm} < \frac{USL-LSL}{6|\mu-T|}$.

Since the absolute bound from which the process is judged as capable is $C_{pm} = 1$ (tolerance band=natural variability band) it is assumed that a necessary condition for the capability is: $|\mu - T| < \frac{USL-LSL}{6}$.

This condition is used as there exist no direct connection to the nonconformity ratio. It means that μ would be in the middle third of the specification range.

After this demonstration of the index properties, interrelationships between different indices are established. Some analytical relations can be shown like the one presented in the previous section:

$$C_{pk} = C_p - \frac{|\mu - M|}{3\sigma},$$

or $C_{pk} = (1-k)C_p$.

Under the assumption of $T = M$, the following interrelationships can be derived:

$$C_{pk} = C_p - \frac{1}{3}\sqrt{\frac{C_p^2}{C_{pm^2}} - 1} \tag{1.6}$$

$$C_{pm} = \frac{C_p}{\sqrt{1 + 9(C_p - C_{pk})^2}} \tag{1.7}$$

$$C_{pm} = \frac{C_p}{\sqrt{1 + \frac{(\mu-T)^2}{\sigma^2}}}$$

$$C_{pm} = \frac{C_{pk}}{(1 - \frac{|\mu-M|}{d})\sqrt{1 + \frac{(\mu-T)^2}{\sigma^2}}}$$

$$C_{pmk} = (1-k)C_{pm}$$

$$C_{pmk} = \frac{C_{pk}}{\sqrt{1 + \frac{(\mu-T)^2}{\sigma^2}}}.$$

Parlar and Wesolowsky (1999) illustrate C_{pk} as a function of C_p for a given C_{pm} using the relations (1.6) and (1.7). They noticed that $C_p = 2.8$ and $C_{pk} = 1.8$ give a C_{pm} of only 0.9. For the authors, the reason of this behavior is the fact that C_p and C_{pk} are essentially concerned with the nonconformity ratio rather than with adjusting the process mean on target.

From the second illustration, where they illustrate C_{pm} values as a function of C_p for a given value of C_{pk}, it is noted that for example for C_{pk} value of 0.9, $C_{pk} < C_{pm}$ if $C_p \in [0.9; 1.2]$; and for C_{pk} value of 1.2, $C_{pk} < C_{pm}$ if $C_p \in [1.2; 1.4]$. However, from the indices structures, it is seen that C_{pm} is built on the fact of penalizing the process variability by the amount of the process mean deviation from the target value. Hence, it is expected that C_{pm} presents a more restrictive measure of the process capability.

As it is seen, relations between indices are not so clear, and for some values of C_p, the index C_{pk} presents a more restrictive measure of the process capability than C_{pm}.

The fact that $C_{pk} < C_{pm}$, for given values of C_p, does not mean that C_{pk} becomes more sensitive to the departure of the process mean from the target value as the index does not depend on T.

In order to explain this, it is noticed that from the mathematical relations, C_{pm} values are obtained from the variation of C_p value and for a given value of C_{pk}. For a given value of C_{pk}, it is noted that $C_{pk} \leq C_{pm}$ as C_p decreases. This means that for sufficiently large process variability $C_{pk} \leq C_{pm}$. But, to keep the given value of C_{pk} constant, when σ increases, the process mean is moved away from the upper specification limit (or lower specification limit). This is equivalent to reducing the process mean deviation from the target value. So C_{pk} kept constant, but C_{pm} increases.

As C_{pk} index depends only upon the half of the specification width, $C_{pk} \leq C_{pm}$ is obtained for enough large σ and small process mean deviation from T. In fact, $C_{pk} < C_{pm}$ when

$$\sigma^2 \geq \frac{4(USL-\mu)^2(\mu-T)^2}{(USL-LSL)^2 - 4(USL-\mu)^2}.$$

Under this condition the effect of the penalty in the C_{pm} denominator is not important any more. In this case it is suggested to reduce the process variability since it has more important impact on the process capability than the adjustment of the process mean on the target value.

1.4 Summary

Under normality assumption reducing the deviation from the target (T) and minimizing the nonconformity ratio are equivalent, especially when $T = M$. However, when the underlying distribution is not normal, especially when it is skewed, the presented approaches have conflicting goals. For this reason, Pearn et al. (1992) proposed to use the index C_{pmk} as it provides a compromise between the two approaches. However, as for any compromise, none of the goals would be reached, neither the nonconformity ratio nor the deviation from the target T is minimized. From the structure of the indices it is deduced to start by reducing the variance and then tackling the problem of reducing the loss around T.

Adopting the loss function approach in case of departure from PCI hypotheses can lead to serious degradation in the product quality as it can increase considerably the nonconformity ratio. However, minimizing the deviations from the target T is still the supreme objective for any company which wants to produce high quality products. Nevertheless, it would be a mistake to adjust the process to the target T without taking into account the impact of such adjustment on the nonconformity ratio. Indeed, the loss function approach should be adopted when it has no ”significant” effect on the process capability, this means as long as the nonconformity ratio will not be less than 0.0064%. For this purpose any company which wants to adopt the loss function approach should master the nonconformity ratio first and should make sure that adjusting to the target T does not affect the process capability.

1.5 Objective

In this work focus is on the nonconformity ratio approach as it is an important step in the quality improvement process. Notice that when the PCI assumptions hold, classical indices C_p and C_{pk} respect the ”higher the better” rule: the higher the index value, the lower the nonconformity ratio, the better the process in meeting customers requirements. If the normality assumption does not hold this rule is not respected. Somerville and Montgomery (1996) studied the effect of non normal distributions on the classical capability indices and it is noticed that classical indices do not respect the ”higher the better” rule for such distributions. Tang and Than (1999) compare seven indices and methods in the presence of non normal distributions. The authors notice that the classical indices are indicative of the process capability for non normal process characteristics. Hence, increasing the PCI value can lead to misleading quality improvement directions.

A process capability index summarizes the ability of a process to meet the customer requirements in one value. Hence, the process capability assessment becomes easy to understand and to communicate inside each organization. However, the PCI computation is the output of a long procedure during which several resources are used. Hence, there is a strong need that the PCI computation is based on a reliable approach.
This work aims at highlighting some of the existing PCI shortcomings through a case study. In order to overcome these drawbacks a new capability index is proposed. Indeed, a nonconformity ratio based desirability function is considered as a univariate capability index. The extension of the proposed index to the multivariate case is discussed and some of its advantages when compared to other classical multivariate indices are proved. Finally, a design of experiments based approach is presented in order to allow the capability assessment when only a small sample size could be considered. First a case study is presented in Chapter 2. The presentation takes into account the process definition, the product definition and some steps in the process capability assessment. In Chapter 3, the shortcomings of the classical univariate indices are shown and a new index is proposed to overcome these shortcomings. A nonconformity ratio based desirability function is used as a univariate process capability index. In Chapter 4 an extension of the index to the multivariate case is presented and its properties are investigated. As the desirability function could be written as dependent on the influential factors the implementation of a capability analysis using an experimental design is studied in Chapter 5. This approach is justified as it allows to assess the process capability considering small sample sizes.

Chapter 2

Case Study

The willing of each company is to acquire new markets, to attract new customers with robust arguments like a compromise between the quality and the price of the product. For the product quality improvement, each organization should attend the state of its production process considering the fact that their resources are limited. Faced to variety and to complexity of tasks that they must carry out, some companies give the priority to the execution of some tasks at the expense of others.
Indeed, some companies when faced to the absence of technicians mastering statistical process control (SPC) tools, to the pressure of personnel charges, and to the requirements of a continuous investment in the aim to follow technological transfer rhythm, will relegate SPC practices to a second order priority.

It would be interesting to highlight the existing PCI shortcomings through a case study and to present a new index which respects the ”higher the better” rule under non normality. The case study takes place within a tunisian company. This company was the subject of the case study in Telmoudi and Limam (2000). In this work the same methodology is adopted. The intervention close to the tunisian company is done following these steps:

- Select a candidate for the study.
- Define the study object.
- Get the necessary resources for the study.

- Evaluate the measurement system.
- Prepare a control plan.
- Select a method for the study.
- Gather and analyze the data.
- Move out the assignable causes.
- Estimate process capability.
- Establish a plan for a continuous quality improvement.

This chapter involves all points except the last two points as they will be discussed in the next chapters.

2.1 Select a Candidate for the Study

Knowing the hypotheses the capability indices theory is based on, the chosen candidate should express at least a minimum level of interest to the SPC tools. Preferably, it would be familiar with the application of the control chart for some quality characteristics.

A custom controlled company, created by a german investment and settled in Tunisia, is our case study.

The german firm which chose to open a subsidiary in Tunisia considers the quality of its products as a strategic choice. The german firm imposes to its tunisian subsidiary the application of some SPC tools like the control chart and the realization of acceptable values of capability indices. In the aim to motivate the tunisian subsidiary to go ahead in this way, the german firm supplied a computer software (Qs-stat version 3.1). This software is able to make easier the representation of the control charts and the computation of the process capability indices. The tunisian company has understood its interest to consider the quality as a strategic choice, especially, that the majority of its customers are german and they grant a great care to the seriousness of their suppliers and to the quality of the products they receive. The tunisian company organizes many training seminars, training courses and creates a department of quality control which has a direct link with the top management in the aim to make the personnel more sensitive to quality and to adopt the quality as a part of the company culture.

Of course the company does not sell its products in the local market, but the personnel is tunisian, and it will be very interesting to note, while quantifying the process performances, how the behavior of this organization will be in applying SPC tools.
It is noted here, that one of the reasons of the success of a capability study is the fruitful communication with the engineers and with the technicians. In fact my knowledge in mechanics is very limited, and one of the reasons that pushes me to decide for an application within a company is the fact that the quality characteristics to be investigated are relatively simple to understand.

2.2 The Product Definition

The MARQUART company is specialized in the production of switches. All necessary materials to the production are supplied by a german firm and other foreign suppliers. In fact it is a question of several assembly chains that produce several switch versions. The investigated product is a power tool switch. This product will be exported to a german customer who will assemble it in an electrical drill.
It is known that the customer grants a special interest to the quality of the received product. It happened that the customer sent back a product that he ordered beforehand because of a high proportion of nonconforming products. The care accorded to the quality is then proved if it is known that among the other subsidiary of the german firm, the tunisian subsidiary offers the most satisfactory results. In this environment, the quality of the product becomes a key of success and the control of SPC tools becomes compulsory.
The investigated product is one for which the company establishes a control chart. It is formed by a superimposition of two plastic plinthes. Before closing them, on one of them, at the first part, some electrical conductors are assembled, and at the second part are assembled two screw supports. At the end of the assembly process the switches go through a machine which has to screw two screws, one on each support. Screwing will be made in such a way that a space will be kept between the inferior boundary of the screw and the inferior boundary of the screw support.
The customer will assemble this switch in an electrical drill by the penetration of two conductor cables in the cited spaces. The dimension of this space is in fact the area to be controlled. Figure 2.3 illustrates the quality characteristic to be controlled.

For lack of measuring this space directly when the switches are closed, it is evaluated through an other characteristic, which is the height of the screw. Indeed, the higher the screw, the more important the dimension of this space. Figure 2.5 shows the surrogate quality characteristic. In this case study it is proposed that the specification limits for the surrogate quality characteristic are given by $(LSL, T, USL) = (20.15\text{mm}, 20.85\text{mm}, 21.35\text{mm})$

2.3 The Process Definition

A production process is the set of activities that transform the input into an output by bringing an added value into it.

However, in practice the definition and the identification of the process is not an easy task. In order to understand how the process is working, at first we have to distinguish between the elements that belong to the process and the elements that do not bring any added value to the input. This is neither evident nor easy.

Indeed, sometimes when we point out the existence of some problems in the final product, we have several and different opinions about the possible origins of the problems. To convince other parts, everyone will try to make them understand his (her) own conception of the process. The process definition is in the heart of the problem and can lead to conflicting opinions.

The main part of the process is composed in fact of a control unit or a control machine, its task is controlling some performances of the switches and screwing. According to the performances of every switch unit, the control machine keeps the good items and reject the nonconforming items, however, the heights of both screws are not controlled by this control machine. Figure 2.1 shows the different stages of control at the control unit and the screw driver.

The screwing system is composed from a single automatic arm at the end of which exists the screwdriver. The same screwing system is used for both screw supports. The first part of the screwing system has the task to fix the screw on the top of the support. It is composed of a triangular piece. At the beginning of each screwing operation, this piece goes and stands at the top of the screw support. This piece is cut in the aim to get the screw through it. The screw is conducted to the triangular piece

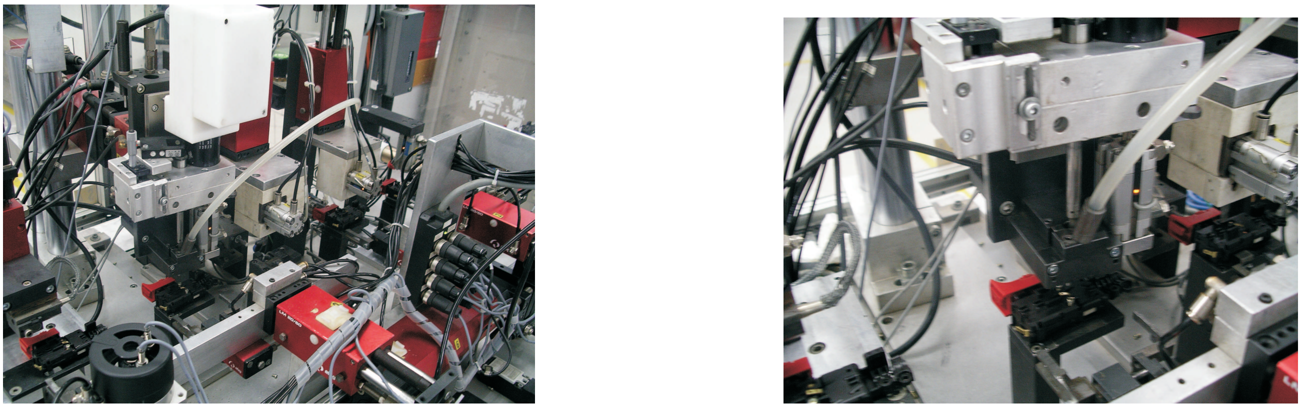

Figure 2.1: Control Unit Wide View and Zoom on the Screw Driver

by means of a plastic tube which is linked up to a tank placed at the right side of the control unit. This tank contains the screw stock.

The task of the second part of the screwing system is the screwing operation. This part is composed of two elements.

- A screwing releaser composed of a transmitter-receiver of a luminous ray. When an object is placed between the transmitter and the receiver, this object will prevent the luminous ray to reach the receiver, this will release the screwing operation.

- The screwdriver system is composed of a cylindrical box containing three elements: A metallic stem which will be in contact with the screw and will play the role of the screwdriver. This metallic stem is linked up to a plastic stem by a spring. This device is located underside the screwing release system.

The screwing system is working as follows:

The switch is fixed on the top of a metallic plinth and at the underside of the screwing system. After the fixation of a screw on the first screw support, the automatic arm at the end of which is located the screw system goes down until the metallic stem being at the level of the screw, and will be in the contact of the screw. After screwing, the automatic arm goes back up and slides until being just on the top of the second screw support. The same work will be done at the second screw support.

But if the screwing process is observed with more details, what happens exactly? In the case where there is no fixation of the screw on the top of the screw support, it is observed that the metallic stem goes down until reaching the level of the superior face of the switch plastic plinth without releasing the screwing operation.

In the case where there is a fixation of the screw on the top of the support, the automatic arm will go down the same distance as in the precedent case, but with the simple difference that the metallic stem will not continue its running until reaching the plastic plinth because it will meet the screw before. In this case, the metallic stem pushes the plastic stem by means of the spring. When the plastic stem reaches the level of the luminous ray, the screwing will start. As screwing goes along, the screw will penetrate into the support and the plastic stem will go down. When the receiver can receive the luminous ray, screwing will stop. It should be noticed that the same screwing procedure is used for both screw supports.

However, there are some other tasks which are achieved before screwing. It is more appropriate to consider the different steps for better understanding the process. Indeed, the process is formed by four stages:

- Setting stage: At each inferior plastic plinth of the switches there exist reserved places for the screw supports. At this stage, it is a question of pushing the screw supports into the plastic plinthes. Four machines are used at this stage.

- Assembly stage: It is a question of assembling manually some conductor pieces. 16 machines are used at this stage. Figure 2.2 shows the switch after the assembly stage and Figure 2.3 shows the space where the costumer will penetrate two cable conductors.

- Closing stage: It is a question of superimposing two plastic plinthes, to introduce them into the closing machine which makes a pressure on the top plinth to make it go down. When the top plinth reaches the level of the inferior plinth, the machine closes the switch. Five machines are used at this stage. Figure 2.4 shows the switch at the closing stage.

- Screwing stage: It is a question of introducing the switches into the control machine, then, the machine controls some characteristics of the switches and screws. We must note that the machine does not control the height of the screws. Figure 2.5 shows the switch at the screwing stage.

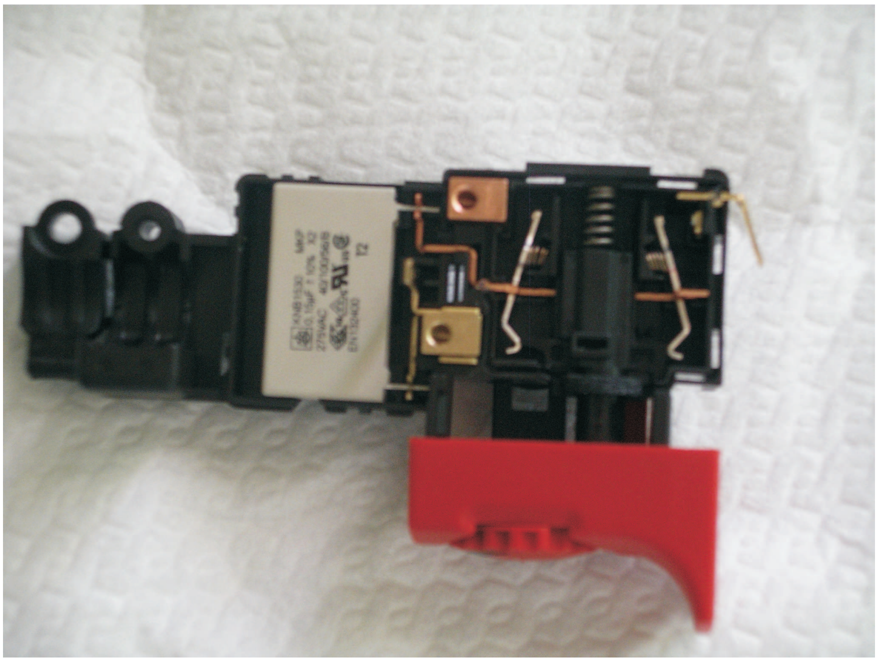

Figure 2.2: The Switch at the Assembly Stage

In order to get data the company does not have a specialized operator for collecting data. Here one question arises concerning the company SPC practices.

2.4 SPC Practices

The company does not have a specialized operator for collecting data. When there is a need of establishing a control chart for example, the operator working on the control unit was charged of this task. A look at the payment system reveals that the company fixed at each step of each assembly chain a given number of product items that the operator has to produce each day. When an operator reaches this product items number she can leave. The operator is not paid for the additional job of collecting data, then she (he) will be more concerned in passing further switches through the control machine than by wasting time in measuring the height of the screws. Moreover, in the previous capability reports, it was noticed that small sample sizes were considered and sampling is done with a very small frequency. The company measures a sample of five switches at the beginning of each customer

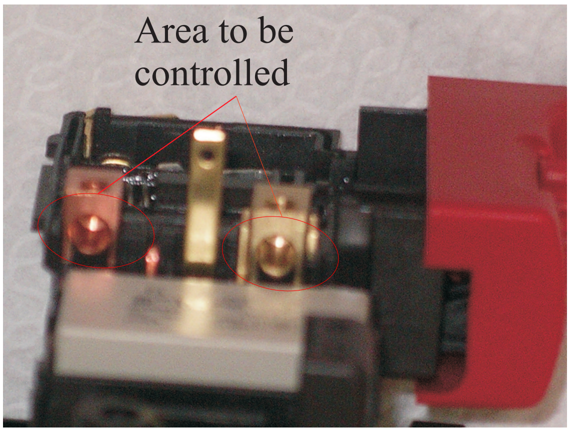

Figure 2.3: The Screw Supports

order. Some reports demonstrate that the company takes five switches every three days or every week.

It is noticed that according to Montgomery (1996), the most frequent sampling practice is to take a small sample but with a high frequency. The selection a sample size of $n = 5$ is due to the fact that using this sample size allows to us to detect a process mean shift on the first sample following the shift with a probability of 93%. This probability is the probability of detecting a 2σ process shift.
Now, Concerning the sampling frequency we must say that when we establish an $\bar{x}$ control chart, our goal is to maximize our chance to detect a process mean shift between samples. If we select a small sample size taken with a small frequency, it can happen that the process undergoes a process mean shift then comes back to the initial situation. If for example we select to use one sample a day we have great chances to not detect this shift.
In the same way, the number of samples necessary to detect a shift is measured by the ARL: Average Run Length. $ARL = \frac{1}{1-\beta}$ where β is the probability of not detecting a shift at the following sample. If $\beta = 0.75$ then $ARL = 4$, which means that we need four samples to detect a shift, if we use a sampling frequency of one sample a day we will need four days for detecting the shift. If we adopt a sampling frequency of a sample every 15 minutes we will detect the shift in one hour.
The ideal practice is to use a small sample size with a high frequency. The SPC practices of the company can lead to an important degradation of the quality of its products. We can understand the volume of this degradation if we know that the company produces a minimum of 4500 switches a day.

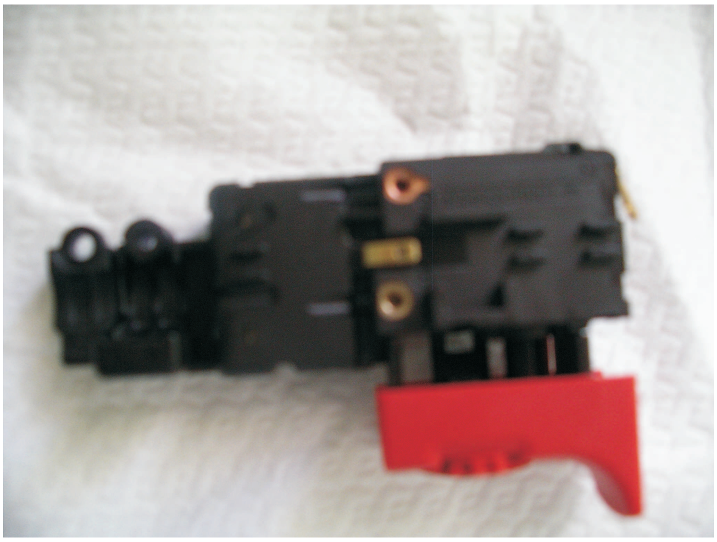

Figure 2.4: A Closed Switch

The second remark that we can make is about the process capability computation. The company uses a software to compute the indices. The same data, used in establishing the control chart, serves for computing the indices. If we look at the computation method, it is noticed that when the data follows a normal distribution the variance is estimated by $S^2 = \frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}(x_{ij}-\bar{\bar{x}})^2}{mn-1}$.

The goal of the variance computation is that it gives us an idea about the variability within samples, but the company in fact is using the following estimation of the variance S^2 where m is the total number of repetitions and n is the sample size.

When we use such estimator we integrate implicitly the variability between samples in the variance computation. In this way we will overestimate the variance and then underestimate the process capability indices.

If we want to use the same data for establishing the control chart and computing the process capability indices, Bissel (1990) uses the following estimator: $\sigma^{\star 2} = \frac{\bar{R}}{d_2}$, where R_i is the range of sample i, with $R_i = x_{max} - x_{min}$, $\bar{R} = \frac{\Sigma_{i=1}^{m} R_i}{m}$, and d_2 is a tabulated value depending upon the selected sample size.

There is a difference between sampling for computing the process capability indices and sampling for establishing the control chart. However, before starting collecting data the necessary resources for the study should be checked.

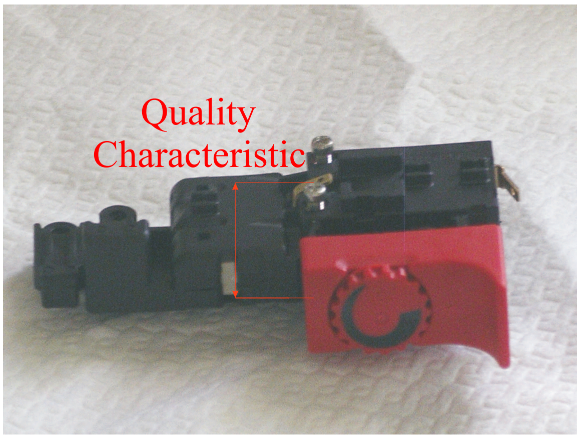

Figure 2.5: The Final Product

2.5 Get the Necessary Resources for the Study

A capability study requires significant expenses concerning material loss and human resources motivation. To make this study reach its goals, top management implication is required in the aim to make the planning of the different tasks easier, and to motivate different participants to the study.

The main constraint was about the planning of the tasks. Indeed, the company produces different versions of switches. Some versions have the same characteristics and some other have different characteristics. The production is done by order. For example if the customer makes an order where there are different versions of switches, the production can be made following three different cases:

- Finish the production of the first version then to start an other version production.
- Produce two different versions simultaneously.
- Start the first version production, interrupt the production and start the production of a new version.

This should be taken into consideration, mainly when there are more than one version in the order and these versions do not have the same characteristics. This factor can bother the sampling operation, especially when we decide to base the study upon a unique version of switch. The other problem which can disturb the data collection is the intervention of the technicians in order to adjust the control unit parameters. Hence, the data should be collected for the same switch and for the same control unit parameters.

2.6 The Effect of the Measurement System on the Capability Study

The reliability of the obtained results of a capability study depends upon the fact that the variability of the process is not contaminated by an additional variability due to the measurement system. The performance of the measurement system in reflecting the process variability must be checked before starting the study. McNeese et al.(1991) propose an approach in order to assess the effect of the measurement system upon the capability study.

2.6.1 The McNeese et al. (1991) Approach

Considering the measurement system as a process it is interesting to analyze its capability by evaluating its accuracy and precision. The accuracy refers to the exactitude of the the measurement system, and the precision is relative to the reproducibility of the measurements.

To isolate the variation caused by the measurement system, the same sample of size n is measured m times. Each time $\bar{x}$ of the sample is computed and plotted on the control chart. In the same way, the R chart is obtained by computing and plotting the range between consecutive results of controls (Moving range).

The accuracy of the measurement system is determined by comparing the center line of the $\bar{x}$ control chart to the true value of the standard. In some cases where there is no standard for the measurement system, it is assumed that the center line represents the true value of the standard.

The precision of the measurement system is also evaluated by measuring it from the R control chart. The measurement system standard deviation is: $\hat{\sigma}_{ms} = \frac{\bar{R}}{d_2}$, where d_2 is a tabulated value. Knowing that the procedure is based on the determination of the difference between consecutive controls, then, $\hat{\sigma}_{ms} = \frac{\bar{R}}{1.128}$.

Ford Motor Company, considers that a capable measurement system means " that the $\pm 3\sigma$ spread is equal to or less than 10% of the tolerance of the characteristic being evaluated."

McNeese et al. (1991) consider that this condition for the measurement system capability is very stringent. For the same purpose they define the percent of total variance due to the measurement system as $100(\frac{\sigma_{ms}}{\sigma_t})^2$. They also provide the following definition of a capable measurement system: " A capable measurement system is a system that is in statistical control with respect to the average and variation, where average value is equal to the true value, and that is responsible for less than 10% of the total process variance."

In the aim to explain the effect of the cited percent of the total variance on the process capability, it is assumed that $\hat{\sigma}_t^2 = \hat{\sigma}_{process}^2 + \hat{\sigma}_{ms}^2$, if $a = \frac{\hat{\sigma}_{ms}^2}{\hat{\sigma}_t^2}$ the total variance is expressed as $\hat{\sigma}_t^2 = \frac{\hat{\sigma}_p^2}{(1-a)}$. To demonstrate the effect of this percentage on the process capability C_p is expressed as: $C_p = \frac{(USL-LSL)\sqrt{1-a}}{6\hat{\sigma}_p}$. By representing C_p for various a values McNeese et al. (1991) find that a measurement system responsible for 10% of the total variance causes a 5% decrease in C_p. It is assumed, then, that the measurement system in this case presents an acceptable level of variation.

2.6.2 The Measurement System Assessment

The measurement system is composed of a metallic surface on which the operator puts the switch to be measured, and a metallic stem which slides through a graduated frame. Before starting the screw height measurement it should be checked that when the metallic stem is in contact of the metallic surface the graduated frame indicates zero in this initial position. It is noticed here that the quality characteristic is the height of the screw measured from the switch base.

The adopted methodology for the measurement system evaluation is given in McNeese et al. (1991). A sample of ten switches was considered for which the height of the screw was measured at 21 occasions. At each repetition $\bar{x}$ and the range R between two consecutive measures of the sample are computed. The results are presented in Table 2.1.

The control limits of the measurement system control chart are determined considering the sample mean of each sample measure as an individual measurement:

$$UCL = \bar{\bar{x}} + 3\sigma_{\bar{x}} = 20.8095$$
$$\text{center line} = \bar{\bar{x}} = 20.8038$$
$$LCL = \bar{\bar{x}} - 3\sigma_{\bar{x}} = 20.798.$$

Table 2.1: Measurement System Assessment

m	1	2	3	4	5	6	7	8	9	10
$\bar{x}$	20.804	20.802	20.804	20.803	20.802	20.806	20.807	20.807	20.804	20.804
R	-	0.002	0.002	0.001	0.001	0.004	0.001	0.000	0.003	0.000

11	12	13	14	15	16	17	18	19	20	21
20.803	20.805	20.802	20.805	20.804	20.805	20.804	20.799	20.805	20.804	20.801
0.001	0.002	0.003	0.003	0.001	0.001	0.001	0.005	0.006	0.001	0.003

For the range chart samples of size $n = 2$ are considered. The control limits are given by:

$$UCL = \bar{R}D_4 = 0.0067$$
$$\text{center}\quad\text{line} = \bar{R} = 0.00205$$
$$LCL = \bar{R} - D_3 = 0.$$

The measurement system is under statistical control and the variability due to the measurement system is given by: $\hat{\sigma_{ms}} = \frac{\bar{R}}{d_2} = 0.0018$. In order to give a judgement about the measurement system capability in considering the most severe rule adopted by Ford Motor Company, it is noticed that the range $\bar{\bar{x}} \pm 3\sigma_{\bar{x}}$ of the actual measurement system represents less than 1% of the tolerance range relative to the quality characteristic being measured.

2.7 Prepare a Control Plan

The setting of a control plan is relatively simple in this case, because the quality characteristics are already known. It is a question of measuring the height of the screws. It should be made sure during the study that the process operates normally. This allows to determine what can the process do if it operates the way it is designed to operates. For example the study should be implemented in an acceptable ambient environment with the removal of all potential sources of variability like the variability due to operators or materials.

According to the technicians, the material, especially the plastic plinthes are supplied by the same vendor. We were interested by the plastic plinthes because a great material variability can cause a great measurement system variability.
Concerning the operator, there is generally only one operator working on the control unit. At the other production stages, mainly at the assembly stage there are some manual operations, but it is noticed that the final level of the screw supports is determined at the closing stage and this level does not depend upon the support levels before the closing stage. This fact was confirmed by all the company technicians. Moreover a simple observation of how the process is operating can confirm this.
In order to avoid the treatment of a huge number of process streams and to limit the effect of the intervention of several operators, only two process stages which can have a direct effect on the height of the screw: The stage of the switch closing and the stage of screwing are considered. it is noticed that five machines are used at the closing stage. The output of each machine will go through the control unit. At the beginning of the study it is assumed that there are five different streams.

2.8 Select a Method for the Analysis

The adopted methodology during the study is as follows:

- Establish the control chart (the $\bar{x}$ and S control chart) for each closing machine and for the control unit, in the aim to verify if the process is under statistical control. If the control chart shows that there are some assignable causes, the reasons should be checked, tracked down and removed.

- If the hypothesis of data normality does not hold the probability distribution should be determined.

- Estimate the capability indices for each process stream and check whether the used indices respect the ”higher the better” rule.

First of all, it should be checked whether the data present some particularities especially the presence of autocorrelated data through the representation of the autocorrelation function. From these representations, it is clear that the observations of both quality characteristics do not reflect significant autocorrelation coefficients. It is noticed that the limits in Figure 2.6 and in Figure 2.7 are the two standard error limits.

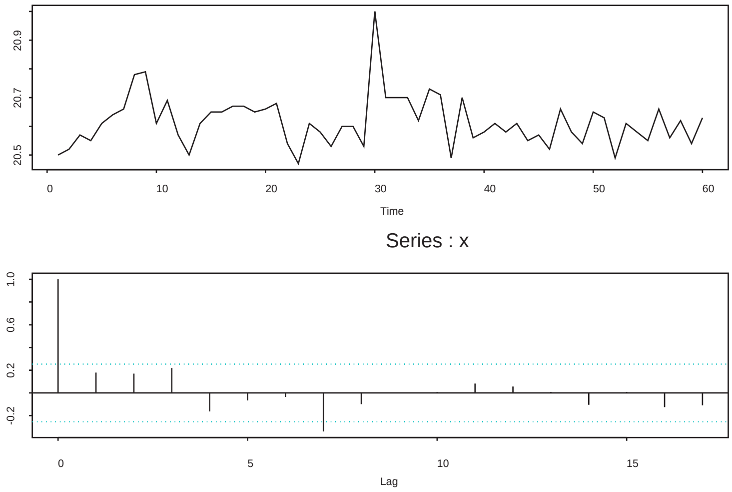

Figure 2.6: ACF for Screw 1

For that purpose a sample size of $n = 5$ was taken with a high frequency, almost a sample every 15 minutes. In order to establish the control chart 30 samples were taken from each machine. For estimating the process capability indices, a sample of size $n = 300$ was taken from each stream.
In the aim to reduce the variability due to the material, only a single switch version was considered. Thus, sampling should be done during the production of this version and before changing the produced version.

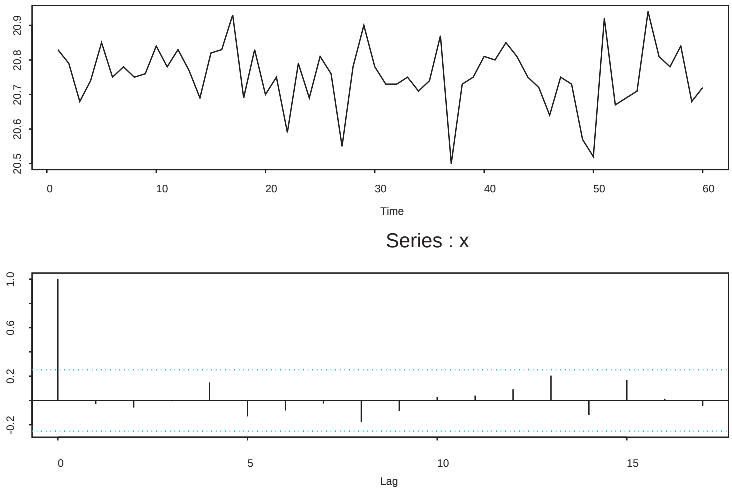

Figure 2.7: ACF for Screw 2

Chapter 3

The Univariate Process Capability Indices

All prerequisites for the indices computation were presented in the previous chapters. However, it is unavoidable to check whether the PCI hypotheses hold. This means that a control chart should be constructed for each stream and that the normal distribution assumption should be checked.

3.1 Control Chart

In this section the control charts for the control unit and for both screws are constructed. For each closing machine a control chart is established for both screw heights and for the screw heights. These control charts are in the appendix. The $\bar{x}$ control chart limits are computed as follow:

$$UCL = \bar{\bar{x}} + \frac{3S_{\bar{x}}}{c_4\sqrt{n}}$$
$$\text{center} \quad \text{line} = \bar{\bar{x}}$$
$$LCL = \bar{\bar{x}} - \frac{3S_{\bar{x}}}{c_4\sqrt{n}}.$$

For the S control chart the limits are given by:

$$UCL = \bar{S} + \frac{3\bar{S}_{\bar{s}}}{c_4\sqrt{n}}$$
$$\text{center} \quad \text{line} = \bar{S}$$
$$LCL = \bar{S} - \frac{3\bar{S}_{\bar{s}}}{c_4\sqrt{n}},$$

where c_4 is a tabulated value. For a sample size $n = 5$, $c_4 = 0.94$. $\bar{\bar{x}}$ is the sample average of the sample means of samples taken from the same stream. In the same way $\bar{S}$ is the average of a standard deviation of these samples.
Following Montgomery (1996) a process is out of control if one of the following cases holds:

- One point is out of the control limits.
- Two of three consecutive points outside the 2σ limits but still inside control limits.
- A run of at least eight points, where the type of run could be either a run up or down.
- Four of Five consecutive points beyond one σ limits.
- An unusual or non random pattern in the data.

From the control charts given in Appendix A it is noticed that the process is under statistical control. The PCIs could be computed then.

3.2 The Normal Distribution Assumption

In this section the data normality hypothesis is checked. Some graphical methods can be used. One of these methods is the quantile-quantile plot. It is a representation of the sample quantiles against the theoretical quantiles. If the j^{th} ordered sample quantile $x_{(j)}$ is considered, the proportion at or to the left of $x_{(j)}$ is often approximated by $\frac{(j-\frac{1}{2})}{n}$. The quantile-quantile plot is the representation of the pairs $(q_{(j)}, x_{(j)})$ with the same associated cumulative probability $\frac{(j-\frac{1}{2})}{n}$. If the data arises from a normal population, the pairs $(q_{(j)}, x_{(j)})$ will be approximately linearly related. Figure 3.1 and Figure 3.2 show the quantile-quantile plots for the normal distribution for screw 1 and screw 2 respectively. A glance to the plot can reveal that the normal distribution could not be rejected for both quality characteristics.

However, in order to decide in an objective way about the acceptance of the normal distribution hypothesis it would be better to use a goodness of fit test. In this section the normal distribution hypothesis is checked using the Shapiro-Wilk test for normality. The test is given by

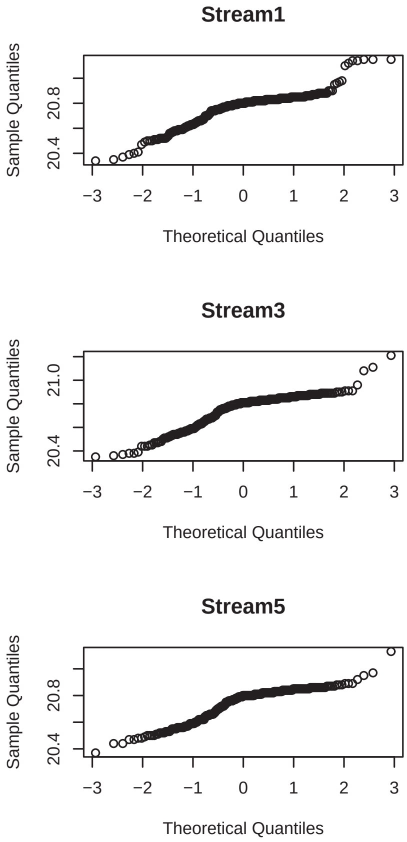

Figure 3.1: Normal QQ Plot for Screw 1

H_0:The data follows a normal distribution.

vs.

H_1: The data does not follow a normal distribution.

The p-values of the test for each stream are given in Table 3.1. It is noticed from Table 3.1 that the risk of rejecting H_0 is less than the significance level of the test which is 5% in this case.

It becomes obvious that the normal distribution assumption does not hold for any process stream. That is expected as no negative values are possible. Furthermore, in section 2.7 it is noticed that five streams are considered in the study. It is important to study the correlation between the different streams in order to check whether it is appropriate to study each stream separately.

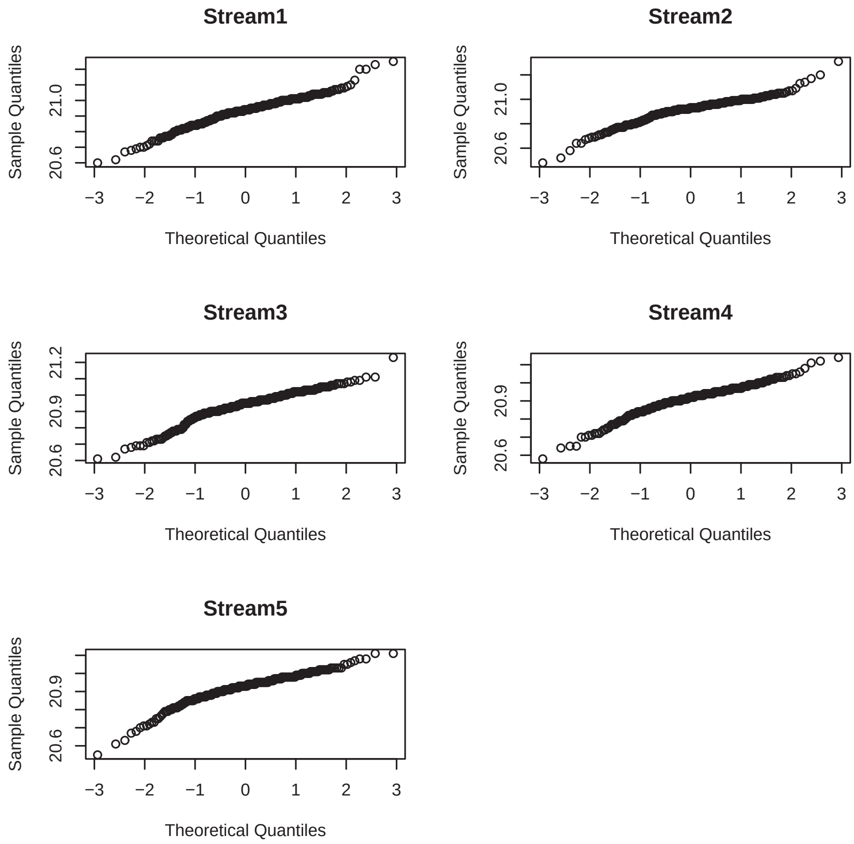

Figure 3.2: Normal QQ Plot for Screw 2

3.3 Correlation Between Streams

Before starting the study it is important to check whether the streams are correlated. This path is unavoidable in order to determine the methodology of the study. Correlations between streams could be checked visually using the scatter plot in order to detect trends. Figure 3.3 and Figure 3.4 show the scatter plots for screw 1 streams and screw 2 streams respectively. No linear correlations are observed in Figure 3.3 and Figure 3.4. However, it would be better to compute the correlation coefficients and to test whether the correlation is significant.

Knowing that the normal distribution hypothesis does not hold for all screw streams it would be better to compute a nonparametric correlation coefficient instead of the Pearson correlation coefficient. Indeed, it is known that in order to test the significance of the Pearson correlation coefficient the

Table 3.1: P-values for the Shapiro-Wilk Test

Stream	Screw 1	Screw 2
1	10^{-13}	8×10^{-6}
2	2×10^{-12}	10^{-9}
3	5×10^{-14}	2×10^{-9}
4	6×10^{-14}	6×10^{-8}
5	4×10^{-13}	10^{-10}

variables must be normally distributed. The most known nonparametric methodology for measuring the correlation are the contingency coefficient, the Spearman rank coefficient, the Kendall rank correlation, the Kendall partial rank correlation and Kendall coefficient of concordance as explained in Siegel (1956). However, because of the presence of tied observation the Kendall rank correlation which is also known as the Kendall tau-b (τ) is used in the study. Indeed, Siegel (1956) explains that this coefficient could take into account the effect of a large proportion of tied observations.

Kendall rank correlation is a nonparametric measure of association based on the number of concordance and discordance in paired observations. Concordance occurs when paired observations vary together, and discordance occurs when paired observations vary differently. The formula for the Kendall coefficient of concordance is given by:

$$\tau = \frac{G}{\sqrt{\frac{n(n-1)}{2} - T_X}\sqrt{\frac{n(n-1)}{2} - T_Y}},$$

where, $T_X = \frac{1}{2}\sum t_i(t_i - 1)$, t_i being the number of tied observations in the group of ties i on a X variable. $T_Y = \frac{1}{2}\sum u_i(u_i - 1)$, u_i being the number of tied observations in the group of ties i on a Y variable. G is the number of concordance minus the number of discordance. G is computed by arranging the ranks of X in their natural order and determining G for the corresponding order of ranks on variable Y. For that purpose (starting from left to right) for each rank belonging to Y the number of larger ranks to its right is counted, then, subtract from this number the number of smaller ranks to its right. The obtained value is P_i. Hence $G = \sum_{i=1}^{n}$ where n is the sample size. Kendall and Gibbons (1990) notice that ties in X contribute nothing to G. Table 3.2 and Table 3.3 show the correlation

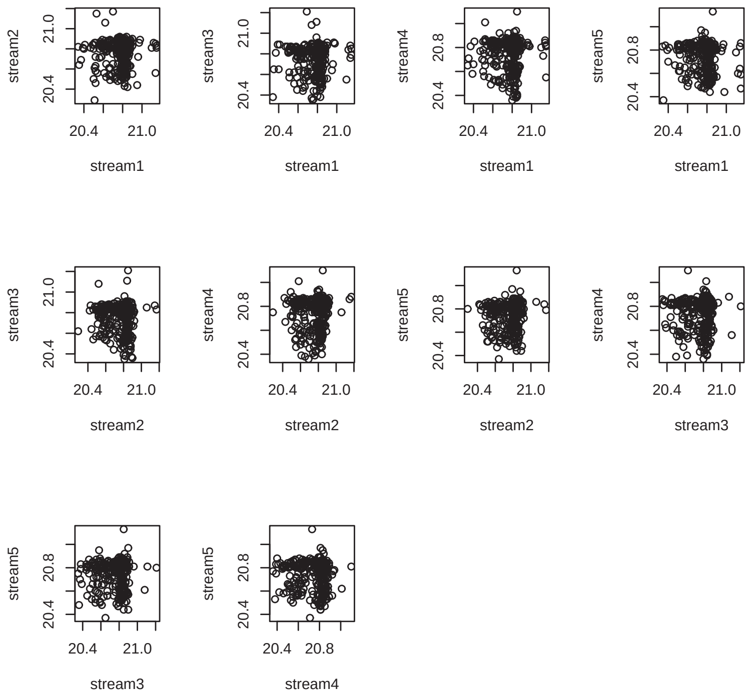

Figure 3.3: Scatter Plot for Screw 1

coefficients between streams τ_{ij}, with $i = 1, \ldots, 5$ and $j = 1, \ldots, 5$.

In order to test the hypothesis $H_0 : \tau_{ij} = 0$, Kendall and Gibbons (1990) explains that S follows a normal distribution for $n > 10$. The standard normal test statistic for $H_0 : \tau_{ij} = 0$ based on τ_{ij} is

$$z = \frac{3\tau_{ij}\sqrt{n(n-1)}}{\sqrt{2(2n+5)}}.$$

The p-values associated with a two sided test are given in Table 3.2 and Table 3.3 for screw 1 and screw

Table 3.2: τ_{ij} for Screw 1

	τ_{12}	τ_{13}	τ_{14}	τ_{15}	τ_{23}	τ_{24}	τ_{25}	τ_{34}	τ_{35}	τ_{45}
Correlation	-0.013	0.018	-0.005	-0.031	-0.021	0.054	0.068	0.003	0.010	-0.037
P-value	0.737	0.641	0.897	0.423	0.587	0.163	0.079	0.938	0.796	0.339

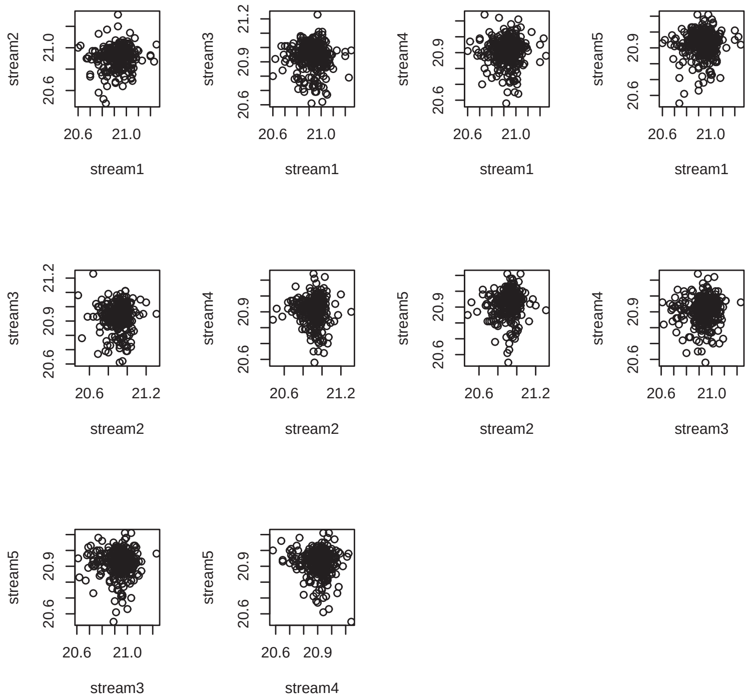

Figure 3.4: Scatter Plot for Screw 2

2 respectively. The risk associated with the rejection of H_0 when it is true is larger than the allowed risk of 5%. Hence, the null hypothesis is not rejected. The streams are considered uncorrelated for both screws and they are studied separately.

Table 3.3: τ_{ij} for Screw 2

	τ_{12}	τ_{13}	τ_{14}	τ_{15}	τ_{23}	τ_{24}	τ_{25}	τ_{34}	τ_{35}	τ_{45}
Correlation	0.061	0.046	0.024	0.006	0.011	0.001	0.045	0.061	-0.004	-0.027
P-value	0.115	0.234	0.535	0.876	0.776	0.979	0.245	0.115	0.917	0.485

Table 3.4: $\hat{\theta}$ Values for Screw 2

Stream	1	2	3	4	5
$\hat{\theta}$	21.34143	21.24993	21.21	21.26072	21.19468

3.4 Distribution Parameters Estimation

It should be noticed that only the distribution parameters for the height of screw 2 are explained in this section. The distribution parameters estimation for the height of screw 1 is given while explaining the goodness of fit test concerning the height of screw 1. This will be done in the next section.

3.4.1 Distribution Parameters Estimation for the Height of Screw 2

From previous reports prepared by the technicians using the Qs-stat 3.1 software, it is noted that the underlying distribution for screw 2 is most likely to be the Lognormal distribution. The considered distribution is a three parameter Lognormal distribution. The parameters are θ, ξ and σ, where $Z = log(\theta - X) \sim N(\xi, \sigma^2)$. Furthermore, the estimation of θ leads to the estimation of the other parameters of the distribution using maximum likelihood estimation. As explained in Johnson et al. (1994) θ is estimated using the quantile method. Following the quantile method the 100 α-th lower, 50th and 100 α-th upper percentiles of normal variable $Z = log(\theta - X)$ are considered. The corresponding percentiles of X are $x_{\{1\}} = exp(\xi - z\sigma) + \theta$, $x_{\{2\}} = exp(\xi) + \theta$, and $x_{\{3\}} = exp(\xi + z\sigma) + \theta$. Hence θ is obtained through

$$\theta = \frac{x_{\{1\}}x_{\{3\}} - x_{\{2\}}^2}{x_{\{1\}} - 2x_{\{2\}} + x_{\{3\}}}.$$

In Johnson et al. (1994) it was recommended that z should be chosen in the range 1.5 to 2. In this case it is considered that $z = 2$. It should be noticed that the common method to estimate a distribution parameters is the maximum likelihood estimation. However, the likelihood of a three parameter Lognormal distribution could be maximized only through numerical methods. Moreover, the main difficulty with this method is that θ becomes infinitely large and can lead to unrealistic solution. Table 3.4 gives the estimated θ values for the different streams. All the screw 2 streams should be fitted by Lognormal distributions such that $Z = log(\hat{\theta} - X)$ follows a normal distribution.

Table 3.5: Screw 2 Height Maximum Values

Stream	1	2	3	4	5
Maximum	21.25	21.31	21.23	21.14	21.11

Hence, it is expected that $\hat{\theta} > \max(x_1, \ldots, x_n)$ for each stream. Table 3.5 gives the maximum values of the samples. It is noticed that the estimated parameter θ does not represent a threshold parameter for the stream 2 and the stream 3. One possible reason for this behavior is the existence of outliers.

3.4.2 Outliers Detection for Screw 2

An outlier is an observation which deviates so much from the other observations as to arouse suspicions that it was generated by different mechanism. An inspection of a sample containing outliers would show up such characteristics as large gaps between ”outlying” and ”inlying” observations. More knowledge about the mechanism behind the outliers appearance is required before discarding them from the study. The causes of outliers are mainly:

- An extreme or relatively extreme value.
- A contaminant observation from other population.
- A legitimate but unexpected data value.
- A data value that was measured or recorded incorrectly.

Some graphical tools allow the detection of outliers. The box plot is one of these tools. The box plot is composed mainly by a box representing the interquartile (IQ) and two lines starting from the 25th percentile and the 75th percentile to given limits. These limits help in identifying outliers. They are computed as follows:

$$L_1 = \text{lower quartile} - 1.5 \times \text{IQ}$$

$$L_2 = \text{lower quartile} - 3 \times \text{IQ}$$

$$U_1 = \text{lower quartile} + 1.5 \times \text{IQ}$$

$$U_2 = \text{lower quartile} + 3 \times \text{IQ}.$$

Observations beyond L_1 and U_1 could be considered as outliers. Observations beyond L_2 and U_2 present stronger evidence to be outliers. However, as long as the mechanism behind the appearance of outliers is not identified the observation could not be removed. The final decision should be based on the interpretation of the user. Figure 3.5 illustrates the screw 2 height box plots for stream 2 and stream 3. L_1 and U_1 limits were used in the figures on the top. L_2 and U_2 limits were used in the figures on the bottom.

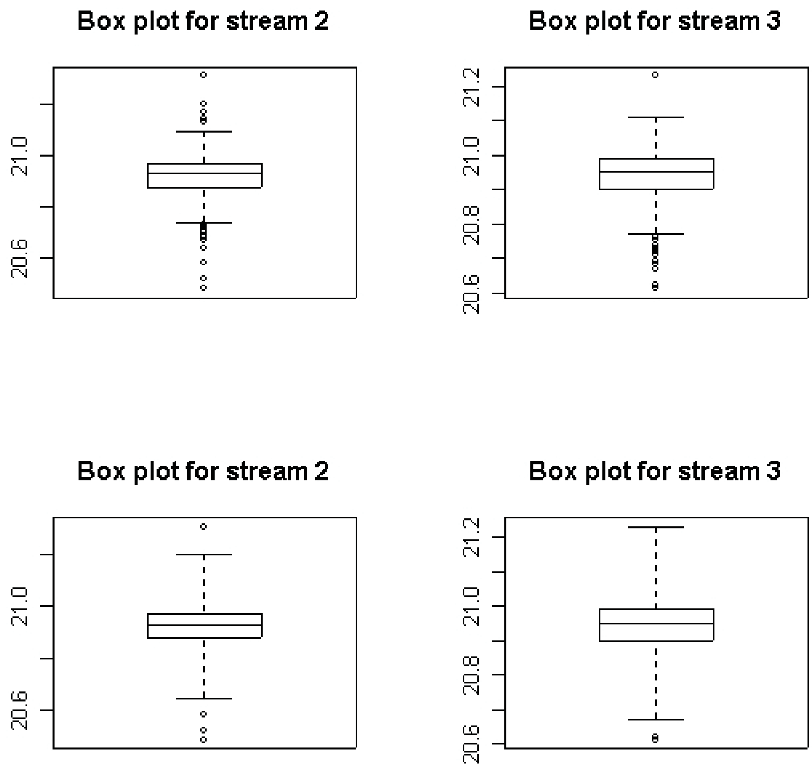

Figure 3.5: Screw 2 Height Box Plots for Stream 2 and Stream 3

From Figure 3.5 it is noticed that only few observations fall beyond the limit U_1 for stream 2 and stream 3. Exactly five observations for the stream 2 and only the maximum value for stream 3. For stream 2 there is a gap between the maximum value and the other four observations. However, it is noticed that for both streams there are more than ten observations which fall beyond the limit L_1 and there are no gaps between the observations. This is an indication that the observations reflect the natural variability of the process and do not belong to an other population.

When the limits L_2 and U_2 are used it is noticed that only the maximum is still beyond the U_2 limit for stream 2. However, no observation is beyond the U_2 limit for stream 3. In order to know if the maximum values could be considered as outliers the Camp-Meidell theorem is used. Following the Camp-Meidell theorem

$$Pr(X < \mu - k\sigma \text{ or } X > \mu + k\sigma) \leq \frac{1}{2.25k^2}.$$

Hence, the probability that an observation deviates from its expected value by 3σ is less than 5%. In order to implement the Camp-Meidell theorem, μ is estimated by $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ and σ is estimated by $S = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$. The upper limit for the Camp-Meidell theorem is given by $\bar{x} + 3S$ which is equal 21.21 and 21.20 for stream 2 and stream 3 respectively. Hence, only the maximum values are over these limits and they are considered as outliers and they are removed from the study.

3.5 Goodness of Fit Tests

The goodness of fit test is a test of hypotheses where the null hypothesis is that a given random variable X follows a stated law $F(X)$. The goodness of fit techniques are based on measuring in some way the conformity of the sample data to the hypothesized distribution, or equivalently its discrepancy from it.

3.5.1 Goodness of Fit Test for the Height of Screw 2

It is necessary to determine a probability distribution function for each process stream. From the history of the process concerning screw 2, it is noted that the underlying distribution is most likely to be the Lognormal distribution. Hence it becomes possible to test the Lognormal distribution at a significance level of 1%.

The chi squared goodness of fit test is used. In order to test the hypothesis that a random sample $x_1, \ldots, x_n$ has the distribution $F(X)$, the range of X is partitioned into ω bins, $b_1, \ldots, b_\omega$. If $N_1, \ldots, N_\omega$ are the number of observations in these bins, then the N_j has a binomial distribution with parameters n and $p_j = \Pr(x_i \in b_j)$, $i = 1, \ldots, n$ and $j = 1, \ldots, \omega$. The difference $N_j - np_j$ between the observed and expected frequencies express the lack of fit of the data to $F(X)$. This difference is reflected in the test statistic

$$X^2 = \sum_{j=1}^{\omega} \frac{(N_j - np_j)^2}{np_j}$$

which has approximately the $\chi^2_{\omega-1}$ distribution in large samples. This is in fact the Pearson Chi-squared statistic. Following Moore (1986) the number of bins is given by $2n^{(2/5)}$. When the tested distribution has K unknown parameters Moore (1986) assumes that the correct critical points for the test fall between those of $\chi^2_{(\omega-K-1)}$ and those of $\chi^2_{(\omega-1)}$. When the value of the test statistic exceeds the critical point value the tested distribution is rejected. In this case 19 bins are used for the goodness of fit test. It is common to use equiprobable bins in order to compute the test statistic. This means that all p_j are equal. Because of the problem of rounding arbitrarily some non equiprobable bins are considered. Indeed, rounding can affect the goodness of fit test, for example for the stream number four there was no observation in one of the bins although it is far from the distribution tails. This can increase considerably the test statistic.

In order to give an argument to the number of degrees of freedom for the test it is interesting to consider the estimation method of the Lognormal distribution parameters. In a previous section it was noticed that the estimation of θ leads to the estimation of the other parameters. Hence, only θ is considered as an unknown parameter and the degrees of freedom of the χ^2 distribution is set to 17. Table 3.6 gives the test statistic values and the number of non equiprobable bins (# non equiprobable bins. Table 3.7 gives the parameter $\hat{\theta}$, the expected value $\hat{\mu}$, the estimated variance of the Lognormal distribution $\hat{\mu}_2$, where $\hat{\mu}$ is the first moment around zero and $\hat{\mu}_2$ is the second central moment, the third central moment $\hat{\mu}_3$ and the probability $\hat{P}_x$ that an observation is less than its expected value. From Table 3.6 it is noticed that the Lognormal distributions are not rejected at the significance level of 5% as the test statistic is less than the critical value 27.58 for all streams.

Table 3.6: χ^2 Goodness of Fit Test for Screw 2

Stream	1	2	3	4	5
X^2	24.9	21.73	27.06	26.16	15.88
# non equiprobable bins	1	4	2	0	5

Table 3.7: $\hat{\theta}$, $\hat{P}_x$ and the Estimated Moments of the Lognormal Distribution for Screw 2

Stream	1	2	3	4	5
$\hat{\theta}$	21.34143	21.24754	21.17424	21.26072	21.19468
$\hat{\mu}$	0.412464	0.335991	0.241492	0.352547	0.273706
$\hat{\mu}_2$	0.066584	0.010860	0.008529	0.048190	0.029864
$\hat{\mu}_3$	0.055406	0.000999	0.000461	0.023006	0.006619
$\hat{P}_x$	0.549509	0.560224	0.573277	0.545623	0.556330

3.5.2 Goodness of Fit Test for the Height of Screw 1

For the first screw, the technicians confirm that during the examination of the switches they noted that because of the pressure of the screw driver the interior of some supports of the first screw were broken during the screwing operation. Only the screw supports used for screw 1 are broken as they are made from different material. Indeed they are made from an ally of copper and tin, however, the other supports are made from only copper. Hence, the material for the screw 1 supports is less resistent to scrape and to the screwing operation. From Figure 3.6 it is noticed that the encircled areas enclose unusual frequencies. It is interesting to check the presence of outliers in the data because of the use of a different screw support. These outliers are most likely to occur at the lower tail of the distribution.

Outliers Detection for Screw 1

Outliers are most likely to occur at the lower tail. Hence, the observations are removed from the upper tail if the box plot and the Camp-Meidell theorem allows simultaneously to treat the observations as outliers. At the lower tail the observations are removed if the box plot or the Camp-Meidell

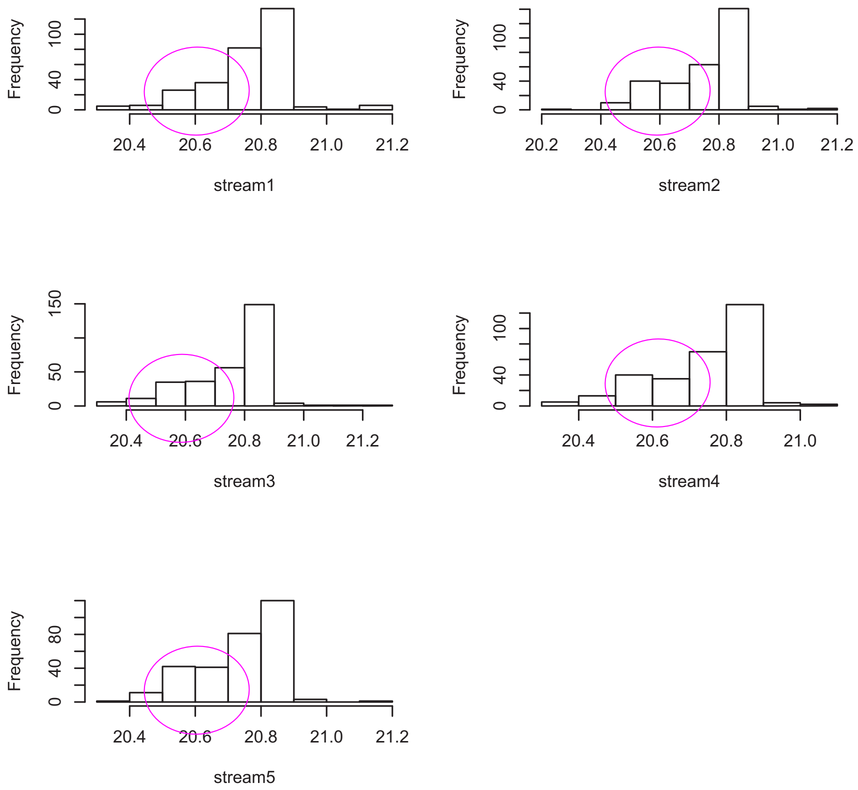

Figure 3.6: Histograms for Screw 1 Height Streams

theorem allow to treat them as outliers. Moreover, if an observation is identified as an outlier and this observation belongs to a group of observations which has a gap up to the other observations of the sample then all the group is removed. Figure 3.7-3.11 show the box plots for the different streams. Only outliers at the lower tails are removed. Only one outlier is removed for stream 2, 3 outliers are removed for stream 1, 6 outliers for stream 3 and no outliers are detected for stream 4 even if there are some unusual frequencies which are detected in the histogram. The minimum values are compared with the minimum values of the other streams and 4 values are considered as outliers and removed. 2 outliers are removed for stream 5. For this stream a decision was taken in order to consider the maximum value as an outlier as it has an important gap up to the other observations and the Camp-Meidell theorem allows the removal of this value.

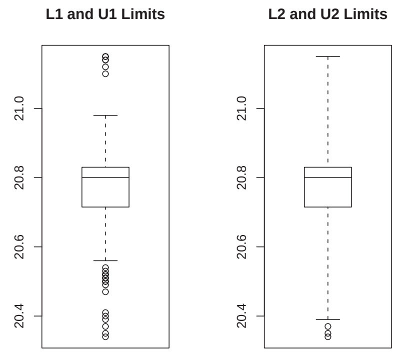

Figure 3.7: Box Plot for the Height of Screw 1 for Stream 1

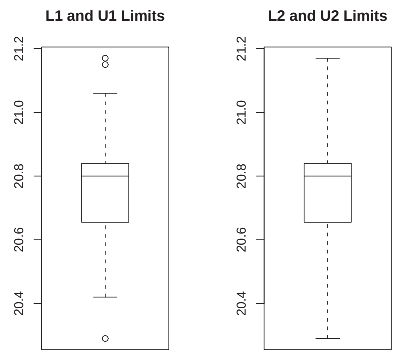

Figure 3.8: Box Plot for the Height of Screw 1 for Stream 2

Goodness of Fit Test and Parameters Estimation

In the case of screw 1 the classical goodness of fit tests fail in confirming an appropriate distribution for the data. However, the appropriate distribution will be checked using the tests based on regression. These tests as explained in Stephens (1986) are based on the representation of the order statistics $x_{(i)}$ on the vertical axis against I_i a suitable function of i on the horizontal axis. If $F(V)$ is the hypothesized continuous distribution, $v_1, \ldots, v_n$ a random sample is considered from $F(V)$ and I_i can be obtained by $I_i \equiv q_i = E(v_{(i)})$ where E denotes the expectation, or $I_i \equiv H_i = F^{-1}\{i/(n+1)\}$. V could be expressed as $V = \frac{(X-\xi)}{\gamma}$ where ξ is the location parameter and γ is the scale parameter. If

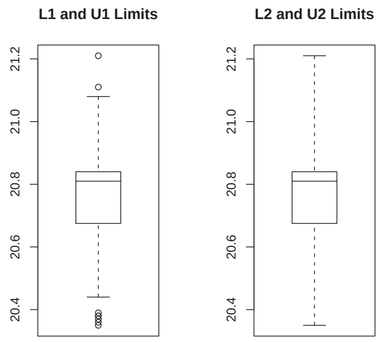

Figure 3.9: Box Plot for the Height of Screw 1 for Stream 3

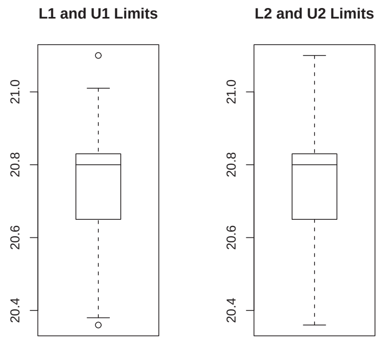

Figure 3.10: Box Plot for the Height of Screw 1 for Stream 4

v_i were taken from $F(V)$, a sample x_i is constructed by

$$x_i = \xi + \gamma v_i$$

If $q_i = E(v_{(i)}) \Rightarrow$

$$E(x_{(i)}) = \xi + \gamma q_i$$

and a plot of $x_{(i)}$ against q_i should be approximately a straight line. This formulation could be replaced by the model:

$$x_{(i)} = \xi + \gamma I_i + \epsilon_i$$

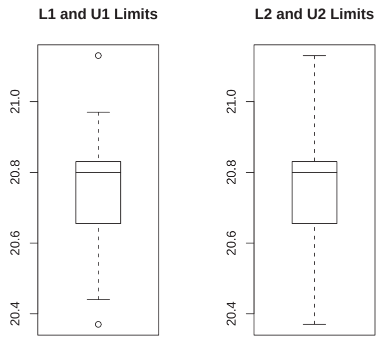

Figure 3.11: Box Plot for the Height of Screw 1 for Stream 5

where ϵ_i is an error parameter which for $I = q$ has mean zero. To be able to measure the fit the following sums are defined:

$$S(I,I) = \sum (I_i - \bar{I})^2$$
$$S(X,X) = \sum (X_{(i)} - \bar{X})^2$$
$$S(X,I) = \sum (X_{(i)} - \bar{X})(I_i - \bar{I})$$

$R^2 = \frac{S(X,I)^2}{S(X,X)S(I,I)}$ is computed when $I = q$, R^2 is an appealing statistic for measuring the fit of the model $x_{(i)} = \xi + \gamma I_i$. For a sample whose ordered values fall exactly at their expected values $R^2(X,q)$ will be equal to 1.

From the process history it is noticed that the distribution for screw 1 is most likely to be the Weibull distribution. A random variable X has a Weibull distribution if there are values of the parameters $c > 0$, $\alpha > 0$, and ξ_0 such that

$$Y = (\frac{X - \xi_0}{\alpha})^c$$

has the standard exponential distribution with probability density function

$$f_Y(y) = e^{-y}, \text{ for } y > 0.$$

The probability density function of The Weibull random variable X is then

$$f_X(x) = \frac{c}{\alpha}(\frac{x - \xi_0}{\alpha})^{c-1} e^{-(\frac{(x-\xi_0)}{\alpha})^c}, \text{ for } x > \xi_0,$$

where c, α and ξ_0 are the shape, the scale and the location parameters respectively. The parameters for the Weibull distribution are estimated in a way to maximize the adjustment coefficient for screw 1 R_1^2. Johnson et al. (1994) explained a modified moment estimation method for estimating the parameters of a three parameters Weibull distribution. This method is based on the following moments equations:

$$E[X] = \bar{X}$$

$$\text{Var}(X) = S^2$$

$$E[X'_{(1)}] = X'_{(1)}$$

for the simultaneous estimation of the parameters. $X'_{(1)}$ is the observed smallest order statistic. It is known that

$$E[X] = \xi_0 + \alpha\Gamma(1 + (1/c))$$

$$\text{Var}(X) = \alpha^2\{\Gamma(1 + (2/c)) - \Gamma^2(1 + (1/c))\}$$

$$E[X'_{(1)}] = \xi_0 + \frac{\alpha}{n^{1/c}}\Gamma(1 + (1/c)),$$

hence, the parameters are obtained from the following equations:

$$\frac{S^2}{(\bar{X} - X'_{(1)})^2} = \frac{\gamma(1 + (2/\hat{c})) - \gamma^2(1 + (1/\hat{c}))}{\{(1 - n^{-1/\hat{c}})\gamma(1 + (1/\hat{c}))\}^2} \tag{3.1}$$

$$\hat{\xi}_0 = \frac{n^{1/\hat{c}}X'_{(1)} - \bar{X}}{n^{1/\hat{c}} - 1}$$

$$\hat{\alpha} = \frac{n^{1/\hat{c}}(\bar{X} - X'_{(1)})}{(n^{1/\hat{c}} - 1)\gamma(1 + (1/\hat{c}))}.$$

Equation (3.1) need to be solved for $\hat{c}$ and subsequently $\hat{\xi}_0$ and $\hat{\alpha}$ can be determined. In order to determine the underlying distributions for screw 1 $\hat{c}$ is incremented and the remaining parameters are determined through the equations of the modified moment equations. The retained parameters are the parameters which maximize the correlation coefficient R_1^2. The results of the Weibull distribution parameter estimation and the corresponding adjustment coefficients are in Table 3.8 where R_1^2 are the adjustment coefficients between the observations of the screw 1 height and the estimated Weibull distributions for screw 1. R_2^2 are the adjustment coefficients between the observed screw 2 height and the estimated Lognormal distributions in Table 3.7. It is noticed that although the estimated Weibull distributions could not be confirmed by the chi squared goodness of fit test, they have higher

Table 3.8: The Estimated Weibull Distribution Parameters

Stream	$\hat{c}$	$\hat{\xi}_0$	$\hat{\alpha}$	R_1^2	R_2^2
1	16.80	19.4482	1.3647	0.916	0.916
2	13.43	19.7848	1.0094	0.940	0.835
3	23.65	19.2547	1.5435	0.933	0.792
4	33.42	18.5358	2.248015	0.944	0.889
5	20.87	19.4752	1.301143	0.942	0.826

Table 3.9: The Estimated Weibull Distribution Moments and $\hat{P}_x$

Stream	1	2	3	4	5
$\hat{\mu}$	1.322349	0.971201	1.508457	2.211126	1.267988
$\hat{\mu}_2$	0.009405	0.007796	0.006312	0.006904	0.005686
$\hat{\mu}_3$	2.348829	0.938267	3.460518	10.855610	2.059916
$\hat{P}_x$	0.445023	0.448799	0.440620	0.437433	0.442064

adjustment coefficients than the estimated Lognormal distributions. Table 3.9 gives the moments of the Weibull distributions and $\hat{P}_x$. The considered random variable in Table 3.9 is $X - \xi_0$.

3.6 The Process Capability Indices

With the parameters of the distributions already estimated it becomes possible to compute the process capability indices. For that purpose the specification limits are given by (LSL,T,USL)= (20.15mm, 20.85mm, 21.35mm).

3.6.1 Classical PCIs

The considered indices are the classical indices presented in the first chapter C_p and C_{pk}. The estimated PCI values are in Table 3.10 and Table 3.11 for screw 1 and screw 2 respectively where the

Table 3.10: PCI Values for Screw 1

Stream	$\hat{C}_p$	$\hat{C}_{pk}$	Nonconformity ratio
1	2.062	1.991	14.05×10^{-6}
2	2.265	2.242	1.175×10^{-6}
3	2.517	2.462	2.546×10^{-6}
4	2.407	2.394	15.58×10^{-6}
5	2.652	2.622	1.118×10^{-6}

Table 3.11: PCI Values for Screw 2

Stream	$\hat{C}_p$	$\hat{C}_{pk}$	Nonconformity ratio
1	0.775	0.543	5.747×10^{-6}
2	1.919	1.402	24.82×10^{-6}
3	2.165	1.506	21.06×10^{-6}
4	0.911	0.670	1.519×10^{-7}
5	1.157	0.827	5.622×10^{-7}

indices are estimated using the estimated distribution moments presented in previous sections.

From Table 3.10 and Table 3.11 we notice that the classical indices do not respect the higher the better rule, when the normality assumption does not hold. It is noticed that following the $\hat{C}_p$ and $\hat{C}_{pk}$ values all streams are more capable for the height of screw 1, however, following the nonconformity ratio stream 1, 4 and 5 are more capable for the height of screw 2. It is more appropriate to use process capability indices which are proposed in the literature in the aim to deal with non normal distributions.

3.6.2 PCIs for non Normal Distributions

It is well known that especially C_p and C_{pk} are not indicative of the process capability for non normal process characteristics and that the ”higher the better” rule is not respected. Some methods and indices were presented in the literature to deal with PCIs for non normal distributions. A new index C_s proposed by Wright (1995) incorporate a skewness correction factor to the index C_{pmk}, Choi and Bai (1996) proposed a weighted variance method which adjust the PCI value by considering the deviations above and below the process mean. Tang and Than (1999) compared seven indices and methods, including the C_s index and the weighted variance index, when the underlying distribution is non normal. The authors noticed that under the normality assumption C_p and C_{pk} jointly determine the proportion of nonconforming items. If the process distribution is non normal, this relation is no longer valid. Hence, any proposed PCI for non normal data should give an objective view of the real capability of the process in terms of the nonconformity ratio.

In what follows the process capability indices are computed using the index C_{pmk} introduced by Pearn et al. (1992), the index C_s introduced by Wright (1995) and the weighted variance indices C_{pw} and C_{pkw} introduced by Choi and Bai (1996). With

$$C_{pmk} = \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

where $d = (USL - LSL)/2$ and $M = (USL + LSL)/2$.

$$C_s = \frac{\min(USL - \mu, \mu - LSL)}{3\sqrt{\sigma^2 + (\mu - T)^2 + |\mu_3/\sigma|}},$$

where μ_3 is the third central moment. The weighted variance indices are given by:

$$C_{pw} = \frac{USL - LSL}{6\sigma W_x}$$

$$C_{pkw} = \min\{\frac{USL - \mu}{3\sqrt{2P_x}\sigma}, \frac{\mu - LSL}{3\sqrt{2(1 - P_x)}\sigma}\},$$

where $W_x = \sqrt{1 + |1 - 2P_x|}$ and P_x is the probability that the process variable X is less than or equal to its expected value μ. One of the most important features of the process capability indices is that their values increase when the proportion of nonconforming items decreases. It will be interesting to check whether the considered indices have this feature in the presence of non normal distributions. Tables (3.12)-(3.13) give the index values, the actual proportion of nonconforming items for each stream

r and the minimum proportion of nonconforming items r^{min} for the screw 1 and 2 respectively. The index estimates, the current nonconformity ratio estimates and the minimum of the nonconformity ratio estimates are obtained using the estimated distribution moments presented in previous sections. The minimum proportion of nonconforming items is obtained by shifting the location parameter between the specification limits. This is equivalent to shifting the specification limits while keeping the same specification limits width and the same distribution parameters. Hence, if F is the distribution function, then the actual nonconformity artio r is given by $r = F(LSL) + 1 - F(USL)$ and r^{min} is expressed as $r^{min} = F(LSL + h) + 1 - F(USL + h)$ with $h \in \mathbb{R}$ and $(-h)$ is the adjustment of the location parameter which gives r^{min}.

Example for the determination of r^{min}:

r^{min} is determined numerically. For that purpose consider a three parameters Weibull distribution with $c = 16.8$, $\xi_0 = 19.44$ and $\alpha = 1.3$. The LSL and USL are 20.15 and 21.35 respectively. First the sign of h should be determined. Two grid points are considered with $h = 0.001$ and $h = -0.001$. The computed nonconformity ratios for $h = 0.001$ and $h = -0.001$ are noted a_+ and a_-. In this example $r = 38.6 \times 10^{-6}$, $a_+ = 39.5 \times 10^{-6}$ and $a_- = 37.7 \times 10^{-6}$. Notice that only $a_- < r$, hence h has a negative sign. r^{min} is determined using the following steps.

Step0: Set $k = 1$.

Step1: Set $H = k \times h$.

Step2: Compute $a_k = F(LSL + H) + 1 - F(USL + H)$.

Step3: Set $H = (k + 1) \times h$.

Step4: Compute $a_{k+1} = F(LSL + H) + 1 - F(USL + H)$.

Step5: If $a_{k+1} < a_k$, set $k = k + 1$ and go to step 1. If $a_{k+1} > a_k$, set $r^{min} = a_k$, stop.

In this example $r^{min} = 6.881 \times 10^{-10}$ for $h = -0.344$.

From Tables (3.12)-(3.13) it becomes obvious that all considered indices fail in respecting ”the higher the better” rule when the current proportion of nonconforming items is used as a benchmark. All indices have different behaviors and give different results for the same data. For screw 1 the failures in respecting the ”higher the better” rule are observed in stream 2 and 4 for $\hat{C}_{pmk}$, in stream 2 and 3 for $\hat{C}_s$, in stream 3 and 4 for $\hat{C}_{pkw}$ and $\hat{C}_{pw}$. Moreover, the most capable stream is stream 2 for $\hat{C}_s$, stream 3 for $\hat{C}_{pkw}$, stream 5 for $\hat{C}_{pmk}$ and $\hat{C}_{pw}$. For screw 2 the failures in respecting the ”higher the

Table 3.12: Process Capability Indices Computation for Screw 1

Stream	1	2	3	4	5
$\hat{r}$	14.05×10^{-6}	1.175×10^{-6}	2.546×10^{-6}	15.58×10^{-6}	1.118×10^{-6}
$\hat{C}_{pmk}$	0.0097	0.01002	0.0098	0.01007	0.01001
$\hat{C}_s$	0.0094	0.0098	0.0093	0.0087	0.0096
$\hat{C}_{pkw}$	2.024	2.178	2.432	2.257	2.482
$\hat{C}_{pw}$	1.957	2.157	2.380	2.269	2.510
$\hat{r}^{min}$	5.454×10^{-9}	8.415×10^{-14}	6.704×10^{-11}	4.872×10^{-9}	4.908×10^{-13}

Table 3.13: Process Capability Indices Computation for Screw 2

Stream	1	2	3	4	5
$\hat{r}$	5.747×10^{-6}	24.82×10^{-6}	21.06×10^{-6}	1.519×10^{-7}	5.622×10^{-7}
$\hat{C}_{pmk}$	0.218	0.217	0.182	0.215	0.191
$\hat{C}_s$	0.177	0.215	0.181	0.194	0.185
$\hat{C}_{pkw}$	0.572	1.495	1.630	2.225	1.866
$\hat{C}_{pw}$	0.739	1.813	2.022	0.872	1.097
$\hat{r}^{min}$	1.030×10^{-6}	3.051×10^{-6}	2.221×10^{-6}	4.750×10^{-9}	2.034×10^{-8}

better" rule are observed in stream 1 and 2 for $\hat{C}_{pmk}$, in stream 2 and 3 for $\hat{C}_s$, in stream 1 for $\hat{C}_{pkw}$, in stream 2, 3 and 5 for $\hat{C}_{pw}$. Moreover, the most capable stream is stream 1 for $\hat{C}_{pmk}$, stream 2 for $\hat{C}_s$, stream 4 for $\hat{C}_{pkw}$, stream 3 for $\hat{C}_{pw}$. The index $\hat{C}_s$ has the best behavior as it gives the same results for screw 1 and 2. However, following this index stream 2 is more capable for the height of screw 2 when it has higher nonconformity ratio than screw 1. This behavior is also observed for the other indices. Furthermore, the threshold for capability judgment is not clear for the considered indices.

Several authors proposed new generations of PCIs. Johnson et al. (1994) and Boyles (1994) tried to provide indices presenting a compromise between the loss function approach and the nonconformity ratio approach. Vännman (1997) proposed different weights to the process mean deviation from the target value and from the midpoint of the specification limits in order to make process capability indices more sensitive to such deviations and to control such sensitivity using the new family of indices $C_p(u, v)$. However, Tang and Than (1999) noticed that $C_p(1, 1)$ which is in fact the index C_{pmk} is the most suited to evaluate process capability for non normal processes. Jessenberger (1999) proposed to use a new generation of indices based on the desirability function. She proposed to use the index EDU as a metric for capability assessment which is the expected value of the Derringer and Suich (1980) desirability function assuming normality.

None of the proposed indices succeeds to overcome the classical indices shortcomings. It will be interesting to provide a process capability index which succeeds in ordering the stream capabilities following the current nonconformity ratio, which take into account the minimum nonconformity ratio and which has a clear threshold for capability judgment. In the following section a nonconformity ratio based desirability function is used as a process capability index. This index is based on the Derringer and Suich (1980) desirability function.

3.7 Nonconformity Ratio Based Desirability Function

The Derringer and Suich (1980) desirability function evolves transformation of each response variable Y_i into a desirability value d_i between 0 and 1. The desirability of the response increases as it becomes closer to its target value T_i. It reaches the maximum value of 1 only if the response value is equal to the target T. The overall desirability is then given by the desirability index which is the geometric mean of the individual desirabilities. It is noticed that the definition of the desirability function does not depend on any distribution assumption. In what follows the nonconformity ratio is considered as a response variable and the property that the desirability value increases when the response becomes closer to its target is used to make the ”higher the better” rule hold for any type of distribution and for any type of specification limit.

The nonconformity based desirability function associated with the quality characteristic Y_1 is:

$$NCDU_1 = \begin{cases} 0 & \text{if} \quad r_1 \geq USL', \\ \frac{USL' - r_1}{USL' - r_1^{min}} & \text{if} \quad r_1^{min} < r_1 < USL', \\ 1 & \text{if} \quad r_1 \leq r_1^{min}. \end{cases} \qquad (3.2)$$

Here r_1 is the current nonconformity ratio associated with the quality characteristic Y_1. If $Y_1 \sim N(\mu_1, \sigma_1^2)$, then $r_1 = \Phi(\frac{LSL_1-\mu_1}{\sigma_1}) + 1 - \Phi(\frac{USL_1-\mu_1}{\sigma_1})$. r_1^{min} is the minimum of the nonconformity ratio, it is obtained when $\mu_1 = \frac{USL_1+LSL_1}{2}$. $USL^{'}$ is the upper limit for the nonconformity ratio beyond which the process is not capable. It is common to set $USL^{'} = 64p.p.m$ as it corresponds to 4σ level in six sigma theory. Hence, when the proportion of nonconforming items is less than 64 p.p.m the $NCDU$ is positive and the process is considered capable, whereas if the value of the index is 0 then the process is not capable.

It is important to note that using 0 instead of r_1^{min} can lead to misleading interpretations of the index, especially in the case when the index is used to compare between the capability of several processes. Indeed, assume that a comparison is carried out between the capability of two processes: process 1 and process 2. If 0 is used instead of r_1^{min} and r_2^{min} the comparison will be between r_1 and r_2, if $r_2 < r_1$ then process 2 is considered as more capable than process 1. However, including r_1^{min} and r_2^{min} with $r_1^{min} < r_2^{min}$ in the index computation gives the additional information that with some process adjustments process 1 is more capable than process 2. When $r_1^{min} < r_2^{min}$ we say that the potential capability of process 1 is higher than the potential capability of process 2. Hence, the index $NCDU$ is not only used for the comparison of the actual capability, it allows also to compare the potential capability. This is an analogy to the classical indices where the use of the index C_{pk} which assesses the actual capability is associated with the use of the index C_p which assesses the potential capability. Then, why not comparing individual nonconformity ratios and the individual minimum of the nonconformity ratios separately? This question is equivalent to the question why are we using capability indices. In fact the capability index is used to characterize in one value the ability of the process to meet the customer requirements. Hence, capability indices are easy to communicate inside each organization. The use of $NCDU$ avoids the use of two indices for the actual capability and for the potential capability separately as the computation of $NCDU$ is based already on the comparison between the actual capability and the potential capability. The $NCDU$ value gives an idea about how far away is the present nonconformity ratio from the maximum ability of the process to meet the customer requirements given by r^{min}. $NCDU$ is also easy to interpret as it is sufficient to notice that the index value is positive in order to judge that the process is capable.

Moreover, assume that two processes are considered. The quality of process 1 and process 2 is expressed in terms of the quality characteristics Y_1 and Y_2 respectively. r_1 and r_2 are the nonconformity ratios associated with Y_1 and Y_2 respectively. It is noticed that when r_1 and r_2 reach simultaneously their minimum, the assigned desirability is 1 for both processes, although both processes do not have the same capability. This happens because there is no comparison between r_1^{min} and r_2^{min}. To overcome this shortcoming it is proposed to consider $\min(r_1^{min}, r_2^{min})$ in the computation of the desirability function . Hence, if $r_1^{min} < r_2^{min}$ the nonconformity based desirability function associated with Y_2 is:

$$NCDU_2 = \begin{cases} 0 & \text{if} \quad r_2 \geq USL', \\ \frac{USL' - r_2}{USL' - r_1^{min}} & \text{if} \quad r_2^{min} < r_2 < USL', \\ \frac{USL' - r_2^{min}}{USL' - r_1^{min}} & \text{if} \quad r_2 \leq r_2^{min}. \end{cases} \tag{3.3}$$

Figure 3.12 shows the linear nonconformity based desirability functions. The solid line corresponds to the $NCDU_1$ and the dashed line corresponds to $NCDU_2$. Notice that in order to allow comparability between processes when $r_1^{min} < r_2^{min}$ only $NCDU_1$ can reach the maximum value of 1. This is due to the fact that the potential capability of process 1 is higher than the potential capability of process 2. However, $NCDU_2$ value will not exceed $\frac{USL' - r_2^{min}}{USL' - r_1^{min}}$.

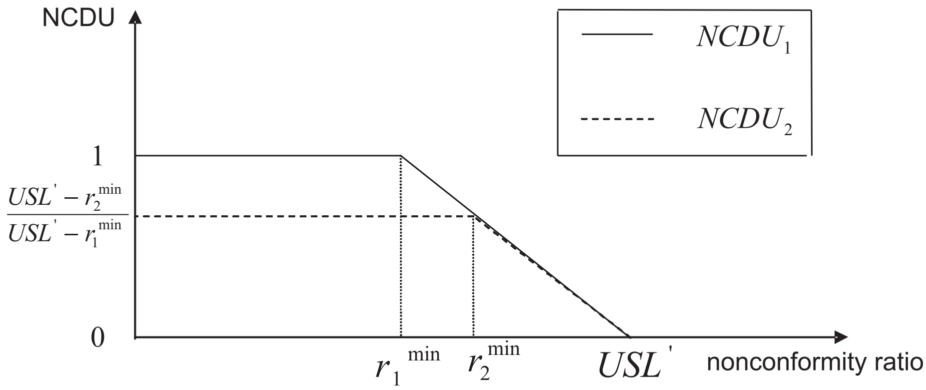

Figure 3.12: One Sided Linear Nonconformity Based Desirability functions

When more than 2 quality characteristics are considered, r^{min} is determined for each quality characteristic and the minimum among all r^{min} is used in the computation of each $NCDU$ as explained for (3.3). In the case where several quality characteristics express the quality of a single product, the natural extension of $NCDU$ is given by the desirability index $D(r_1, ..., r_p) = [\Pi_{j=1}^{p} NCDU_j]^{\frac{1}{p}}$

Table 3.14: $\widehat{NCDU}$ Values for Screw 1

Stream	1	2	3	4	5
$\widehat{NCDU}$	0.780	0.981	0.960	0.756	0.982

Table 3.15: $\widehat{NCDU}$ Values for Screw 2

Stream	1	2	3	4	5
$\widehat{NCDU}$	0.910	0.612	0.670	0.997	0.991

where $NCDU_j$ are defined in (3.2) and (3.3) and m is the number of quality characteristics. The geometric mean assigns an overall desirability of 0 if there exists at least one quality characteristic for which the nonconformity ratio exceeds the USL' value. $NCDU$ allows for capability judgment, it compares the actual capability of a process to its potential capability and it allows the comparison between the capability of several processes for any type of distribution and any type of specification limit. However, it will be interesting to compare the performance of $NCDU$ to the considered indices in the previous section. Table 3.14 and Table 3.15 show the index values for the different streams for screw 1 and screw 2 respectively. $\widehat{NCDU}$ is computed using the estimated distribution parameters presented in the previous sections. Notice that $\widehat{NCDU}$ succeeds in respecting "the higher the better" rule when the current nonconformity ratio is used as a benchmark. The $\widehat{NCDU}$ computation is not possible without a previous comparison between the minimums of the nonconformity ratios. Furthermore, $\widehat{NCDU}$ allows the comparison between the capability for the different quality characteristics for each stream. However, $NCDU$ is based on the Derringer and Suich (1980) desirability function which is interpreted as a loss function. $NCDU$ is a loss a function which measures how desirable is a nonconformity ratio. Although the threshold for capability judgment is clear for $NCDU$, it is common in the capability theory to construct confidence intervals for the indices and to base the capability judgment on the lower limit of the interval. In what follows a confidence interval is constructed for $NCDU$.

3.8 A Bootstrap Confidence Interval for $NCDU$

A confidence interval can be constructed by using bootstrapping technique. The principle of a bootstrap method is the following: if a sample of size n is considered with sample values $x_1, ..., x_n$ from this sample a random sample of size n' is chosen -with replacement- and a PCI is computed for the obtained sample, say, $\widehat{NCDU}_{[1]}$. This operation is repeated B times in the aim to have: $\widehat{NCDU}_{[1]}, \ldots, \widehat{NCDU}_{[b]}, \ldots, \widehat{NCDU}_{[B]}$ which compose the bootstrap distribution of $\widehat{NCDU}$. In this study $n = n' = 300$ and $B = 7500$.

There are many approaches to construct a bootstrap confidence interval. The approach adopted in this work is the quantile confidence interval. Efron and Tibshirani (1993) explain that a minimum value of $B = 1000$ is required for an acceptable estimation of the quantiles. The bootstrap confidence interval of intended coverage $(1 - (2\alpha))\%$ is given by

$$(\widehat{NCDU}_{lo}, \widehat{NCDU}_{up}) = (\widehat{NCDU}_{(\alpha)}, \widehat{NCDU}_{(1-\alpha)}),$$

where $\widehat{NCDU}_{(\alpha)}$ is the quantile of order α of the $\widehat{NCDU}$ bootstrap distribution. In this study $\alpha = 2.5\%$.

Bootstrap confidence intervals are constructed for $NCDU$ of screw 1 and screw 2. For that purpose the observations are gathered in a (300×10) matrix. New samples are obtained by choosing matrix lines with replacement. Then, the parameters of the distributions corresponding to each stream have to be estimated at each replication. In order to be able to estimate the parameters the outliers are detected and removed. For screw 1 the Camp-Meidell theorem is applied as explained in section 3.4.2 and section 3.5.2 on the outlier values already detected in section 3.5.2. For screw 2 observations greater than $\hat{\theta}$ are outliers and they are removed. After removing the outliers, the distribution parameters are estimated using quantile method and maximum likelihood estimators for screw 2 as explained in section 3.4.1 and using the maximization of the coefficient of determination for screw 1 as explained in section 3.5.2. The estimation of the distribution parameters allows the nonconformity ratios estimation. Moreover, the location parameter which allows the minimum of the nonconformity ratio determination is determined. A comparison between all minima of the nonconformity ratios is carried out and the minimum among all minima is used in the $\widehat{NCDU}$ computation for each stream. The comparison considers the minima corresponding both screws. In order to construct the bootstrap

Table 3.16: Bootstrap Confidence Intervals for $\widehat{NCDU}$ Corresponding to Screw 1

Stream	1	2	3	4	5
$[\widehat{NCDU}_{lo}, \widehat{NCDU}_{up}]$	[0.491, 1.000]	[0.000,1.000]	[0.546, 0.989]	[0.779, 0.999]	[0.904, 1.000]

Table 3.17: Bootstrap Confidence Intervals for $\widehat{NCDU}$ Corresponding to Screw 2

Stream	1	2	3	4	5
$[\widehat{NCDU}_{lo}, \widehat{NCDU}_{up}]$	[0.000, 0.999]	[0.000,0.999]	[0.000, 0.997]	[0.000, 0.999]	[0.000, 0.999]

confidence interval $\widehat{NCDU}$ is computed at each replication. In this case 7500 replications are considered. The results are given in Table 3.16 and Table 3.17 for screw 1 and screw 2 respectively. It is interesting to use a summary statistic of the $\widehat{NCDU}$ bootstrap distribution in order to compare the stream capabilities. Indeed, the median is a good indicator of the bootstrap distribution central tendency as it has the minimum of the average of the absolute deviations among other indicators of the distribution central tendency. The summary statistic is obtained through $MNCDU$ which is the median of $\widehat{NCDU}$ bootstrap distribution. It would be interesting to check whether $MNCDU$ respects the higher the better rule. $MNCDU$ and the median of the estimated nonconformity ratio $\hat{r}$ bootstrap distribution $\hat{r}_{0.5}$ are given in Table 3.18 and Table 3.19 for screw 1 and screw 2 respectively.

It should be noticed that the capability judgment is more reliable when it is based on the lower limit of the $NCDU$ confidence interval. A stream is said capable if the lower limit is higher than the threshold for capability judgment. Notice from Tables (3.16) and (3.17) that all streams are not

Table 3.18: Central Tendency of $\widehat{NCDU}$ Bootstrap Distribution for Screw 1

Stream	1	2	3	4	5
$MNCDU$	0.978	0.000	0.817	0.936	0.997
$\hat{r}_{0.5}$	1.392×10^{-6}	77.14×10^{-6}	11.66×10^{-6}	4.089×10^{-6}	1.721×10^{-7}

Table 3.19: Central Tendency of $\widehat{NCDU}$ Bootstrap Distribution for Screw 2

Stream	1	2	3	4	5
$MNCDU$	0.952	0.795	0.799	0.995	0.985
$\hat{r}_{0.5}$	3.020×10^{-6}	13.07×10^{-6}	12.85×10^{-6}	2.878×10^{-7}	9.413×10^{-7}

capable in screwing screw 2. Concerning screw 1 all streams are capable except stream 2 for which the lower confidence limit is 0. Figures 3.13 and 3.14 show the stream bootstrap distributions for screw 1 and screw 2 respectively.

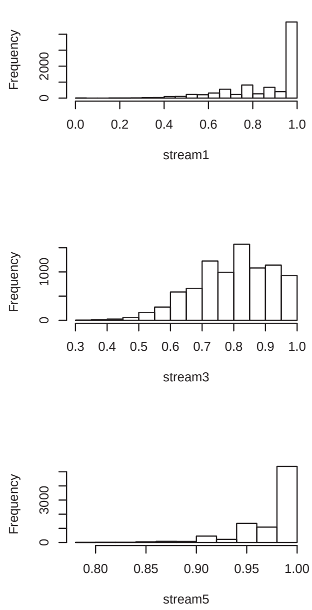

Figure 3.13: Different Stream $\widehat{NCDU}$ Bootstrap Distributions for Screw 1

Furthermore, $MNCDU$ is given in Table 3.18 and Table 3.19 for screw 1 and screw 2 respectively. In this case the order of the stream capabilities following $MNCDU$ is the same as the order given by the nonconformity ratio. Items produced by stream 2 have the worst quality. Moreover, following the order given by $MNCDU$ stream 5 and stream 4 give the most satisfactory capability for screw 1 and screw 2 respectively.

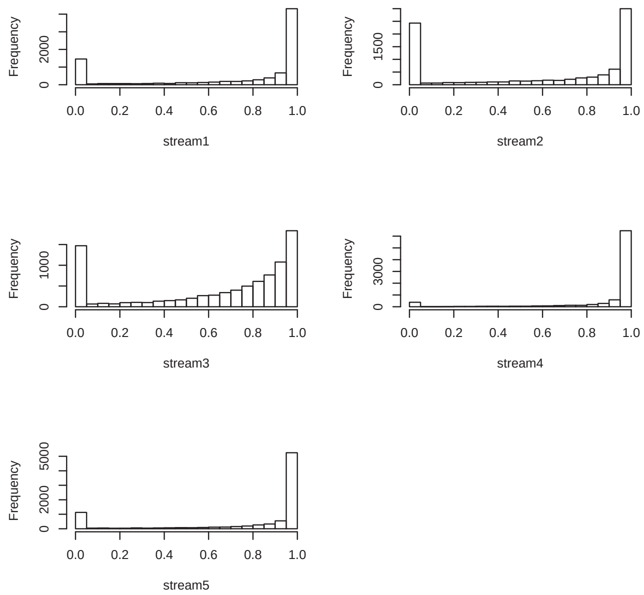

Figure 3.14: Different Stream $\widehat{NCDU}$ Bootstrap Distributions for Screw 2

3.9 Conclusion

A linear nonconformity based desirability function is presented as a process capability index. $NCDU$ avoids the use of two different indices for assessing the actual capability and the potential capability. The performance of this index is compared with the performance of other indices in the literature using non normal distributions and asymmetric specification limits. It was demonstrated that the index respects the ”higher the better” rule for any type of distribution and for any specification limits. The use of this approach in the multivariate case is possible using the desirability index. The proposed univariate index overcomes some shortcomings of the existing indices in the literature. However, in many cases the quality of a product is given through several quality characteristics. Hence the capability assessment is done using multivariate capability indices. It is interesting to present the multivariate extension of the proposed index.

Chapter 4

The Multivariate Process Capability Indices

Many approaches have been elaborated in the aim to implement multivariate capability indices. These approaches try to solve some theoretical and practical problems like multivariate specification limits and statistical properties of the indices. In what follows some of the existing approaches in the literature are explained before presenting a multivariate extension of the index $NCDU$, introduced in the previous chapter.

4.1 Review of the Literature

Lovelace and Kotz (1998) presented the multivariate process capability indices as a dangerous but unavoidable area. Dangerous because many of the existing multivariate process capability indices are in fact generalizations of the univariate classical indices. It is expected then that the proposed multivariate indices have the same shortcomings as the univariate indices. Wang et al. (2000) compared different multivariate indices and noticed that a current problem in multivariate quality control, there is no consensus about a methodology for assessing capability.

Multivariate PCIs are unavoidable especially when several quality characteristics determine the quality of a product. Several indices were proposed in the literature in order to deal with multivariate capability. Wierda (1993) proposed an extension of the index C_{pk}. The approximation of the multivariate index depends on the actual process yield. Chan et al. (1991) found on the ellipsoidal specification limits an extension of the index C_{pm}. In order to make the structure of the multivariate process capability index closer to the general structure of the process capability indices. Taam et al. (1993) proposed a multivariate analog to the index C_{pm}. The proposed index is given by the ratio of the volume of the specification region over a scaled 99.73% process region. Pearn et al.(1992) introduced an approach based on the extension of the index C_p. This approach was studied by Chen (1994) who proposed a multivariate index for the rectangular specification limits. Actually, the proposed multivariate index is the ratio of the tolerance region to that of the region needed to achieve the desired process yield. Wang et al. (1998) proposed to reduce the complexity of the problem when several quality characteristics are considered. For this purpose, the process capability indices were computed for some selected principle components.

In this work it is suggested the use of a nonconformity ratio based desirability function $NCDU$ as a capability index in the univariate case. In what follows the extension of $NCDU$ to the multivariate case is discussed.

4.2 The Multivariate Extension

When n quality characteristics are considered, the $NCDU$ index is defined for each quality characteristic as follows:

$$NCDU_i = \begin{cases} 0 & \text{if} \quad r_i \geq USL', \\ \frac{USL' - r_i}{USL' - \min_{j=1,\ldots,n}(r_j^{min})} & \text{if} \quad r_i^{min} < r_i < USL', \\ \frac{USL' - r_i^{min}}{USL' - \min_{j=1,\ldots,n}(r_j^{min})} & \text{if} \quad r_i \leq r_i^{min}. \end{cases} \tag{4.1}$$

where r_i is the actual nonconformity ratio for the quality characteristic i. r_i^{min} is the minimum of the nonconformity ratio for the quality characteristic i and $\min_{j=1,\ldots,n}(r_j^{min})$ is the minimum among all the minima of the nonconformity ratios. In the multivariate case the actual nonconformity ratio for a quality characteristic is computed on the basis of the marginal probability density function. Hence, if $X_1, \ldots, X_p$ are random variables with $f(x_1, \ldots, x_p)$ the joint probability density function, $LSL_1, \ldots, LSL_p$ the lower specification limits and $USL_1, \ldots, USL_p$ the upper specification limits then:

$$r_i = 1 - \int_{-\infty}^{+\infty} \cdots \int_{LSL_i}^{USL_i} f(x_1, \ldots, x_p) dx_1 \ldots dx_p$$

A natural extension of the $NCDU_i$ to the multivariate case is given by the desirability index. The desirability index is a function of the univariate $NCDU_i$. It will be considered as a multivariate capability index. However, several types of the desirability index were proposed in the literature. It will be interesting to check which type is more appropriate for capability assessment.

Harrington (1965) proposed the geometric mean of the individual desirabilities as a desirability index. It is defined as

$$D = [\Pi_{i=1}^{p} d_i]^{\frac{1}{p}}.$$

In this way if one quality characteristic has a desirability equal 0 than the overall desirability would be 0. Derringer (1994) proposed a weighted composite desirability which is given by

$$D = [\Pi_{i=1}^{p} d_i^{w_i}]^{\frac{1}{\Sigma_{i=1}^{p} w_i}},$$

where w_i corresponds to the importance of the quality characteristics i. The weights are determined by individual or group judgement. Kim (2000) proposed the minimum of the desirability values as an assessment for the overall desirability. It is given by

$$D = \min_{i=1,\ldots,p}(d_i).$$

One of the main features of the process capability indices is that it is possible to judge whether the process is capable or not from their values. However, this feature does not hold when the minimum of the desirabilities is considered as a multivariate process capability index. In order to prove that the following Lemma is formulated:

Lemma1:

If $r_i < USL' \; \forall i \nRightarrow R < USL'$, where R is the joint nonconformity ratio computed using a joint probability function f with infinite support. Hence, univariate capability $\nRightarrow$ multivariate capability.

Proof:

Suppose that $r_1, \ldots, r_p$ are the nonconformity ratios corresponding to the quality characteristics $X_1, \ldots, X_p$ respectively and that $f(x_1, \ldots, x_p)$ with infinite support. In the multivariate case the process is said capable when the joint nonconformity ratio $R \leq USL'$. The joint nonconformity ratio is given by:

$$R = 1 - \int_{LSL_1}^{USL_1} \ldots \int_{LSL_p}^{USL_p} f(x_1, \ldots, x_p) dx_1 \ldots dx_p$$

it is obvious that

$$\int_{-\infty}^{+\infty} \dots \int_{LSL_i}^{USL_i} f(x_1, \dots, x_p) dx_1 \dots dx_p > \int_{LSL_1}^{USL_1} \dots \int_{LSL_p}^{USL_p} f(x_1, \dots, x_p) dx_1 \dots dx_p$$

this means

$$r_i < R \ , \forall i.$$

when $r_i = USL^{'}, \forall i.$ this means that $R > USL^{'}$. Hence, the univariate capability does not imply the multivariate capability. □

When the minimum of the desirabilities is used as a multivariate process capability index, this is equivalent to reducing the multivariate case to the univariate case. Indeed, only the minimum of the $NCDU_i$ is considered. In this case the capability judgement rule is the same in the univariate and in the multivariate case. Hence, it becomes not appropriate to use the minimum of the $NCDU_i$ as a multivariate process capability index because it does not provide a reliable capability judgment rule.

The natural extension of $NCDU$ to the multivariate case becomes the geometric mean of the $NCDU_i$. The geometric mean ($NCDM$) is considered as a multivariate capability index for the correlated and the uncorrelated quality characteristics. The desirability index equals 0 when at least one quality characteristic has a nonconformity ratio higher than $USL^{'}$. Moreover, as it will be shown in the next section the desirability index could be written as a function of the joint nonconformity ratio for uncorrelated quality characteristics. When n uncorrelated quality characteristics are considered the joint nonconformity ratio is expressed as

$$R_p = 1 - [(1 - r_1)(1 - r_2) \dots (1 - r_{n-1})(1 - r_p)]. \tag{4.2}$$

Furthermore, when the geometric mean is used, it becomes possible to present a threshold for the desirability index over which the process is considered capable.

4.3 The Capability Threshold Setting

In this section the considered desirability index is the geometric mean of the univariate $NCDU_i$. It is important to notice that in the univariate case when $r < USL^{'}$ the univariate $NCDU$ is positive. A positive value of the capability index is sufficient in order to judge whether the process is capable or not. In the multivariate case when the desirability index equals 0 then the process is not capable. In order to derive a capability threshold for the desirability index it is interesting to express the desirability index as a function of the joint nonconformity ratio. The threshold for the capability judgment is given in Theorem 1. However, in order to be able to prove Theorem 1, the following lemmas should be formulated.

Lemma 2:

The general expression of $NCDM$ is given by

$$NCDM^p = \frac{\sum_{i=1}^{p-1}(USL')^{(p-i)}(-1)^i[\sum_{j=1}^{p-i+1}\sum_{k=2}^{p-i+2}\dots\sum_{m=l}^{p-i+l}\dots\sum_{\substack{q=p-1\\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{i,[1]}}r_k^{\mathbb{1}_{i,[2]}}\dots r_m^{\mathbb{1}_{i,[l]}}\dots r_q^{\mathbb{1}_{i,[p-1]}}]}{(USL'-C)^p}$$

$$+\frac{USL'^p + (-1)^p\Pi_{i=1}^p r_i}{(USL'-C)^p}.$$

p is the number of quality characteristics, r_i is the actual nonconformity ratio for the quality characteristic i, USL' is an upper limit for the actual nonconformity ratio, r_i^{min} is the minimum of the nonconformity ratio for the quality characteristic i and $C = min_{i=1,\dots,p}(r_i^{min})$. $\mathbb{1}_u$ is a $[(p-1)\times 1]$ vector and its elements are 0 and 1. Only the first u^{th} elements are 1. $\mathbb{1}_{u,[l]}$ is the l^{th} element of the vector and

$$\mathbb{1}_{u,[l]} = \begin{cases} 0 & \text{if} \quad l > u, \\ 1 & \text{if} \quad l \le u. \end{cases} \tag{4.3}$$

The proof of Lemma 2 is in Appendix B.

Lemma 3:

The general expression of the joint nonconformity ratio for uncorrelated quality characteristics is given by

$$R_p = -(\sum_{i=1}^{p-1}(-1)^i[\sum_{j=1}^{p-i+1}\sum_{k=2}^{p-i+2}\dots\sum_{m=l}^{p-i+l}\dots\sum_{\substack{q=p-1\\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{i,[1]}}r_k^{\mathbb{1}_{i,[2]}}\dots r_m^{\mathbb{1}_{i,[l]}}\dots r_q^{\mathbb{1}_{i,[p-1]}}]+(-1)^p\Pi_{i=1}^p r_i)$$

The proof of Lemma 3 is in Appendix B.

Lemma 4:

$NCDM$ is expressed as a function of the joint nonconformity ratio of uncorrelated quality characteristics as follows

$$(USL^{'} - C)^p NCDM^p = USL^{'p} + (-1)^p \Pi_{i=1}^{p} r_i$$
$$+ \sum_{i=1}^{p-1} USL^{'(p-i)} (-(\sum_{u=1_{u\neq i}}^{p-1} (-1)^u [\sum_{j=1}^{p-u+1} \sum_{k=2}^{p-u+2} \dots \sum_{m=l}^{p-u+l} \dots \sum_{\substack{q=p-1 \\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}} \dots r_m^{\mathbb{1}_{u,[l]}} \dots r_q^{\mathbb{1}_{u,[p-1]}}])$$
$$-(-1)^p \Pi_{i=1}^{p} r_i - R_p). \tag{4.4}$$

The proof of Lemma 5 is given in Appendix B.

Lemma 5:

The joint nonconformity ratio for uncorrelated quality characteristics is expressed as a function of $NCDM$ as follows:

$$R_p \sum_{i=1}^{p-1} USL^{'(p-i)} = -NCDM^p (USL^{'} - C)^p + USL^{'p} + \sum_{i=1}^{p-1} USL^{'(p-i)}$$
$$(-(\sum_{u=1_{u\neq i}}^{p-1} (-1)^u [\sum_{j=1}^{p-u+1} \sum_{k=2}^{p-u+2} \dots \sum_{m=l}^{p-u+l} \dots \sum_{\substack{q=p-1 \\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}} \dots r_m^{\mathbb{1}_{u,[l]}} \dots r_q^{\mathbb{1}_{u,[p-1]}}])$$
$$-(-1)^p \Pi_{i=1}^{p} r_i) + (-1)^p \Pi_{i=1}^{p} r_i.$$

The proof of Lemma 5 is given in Appendix B.

Theorem 1:

A process is capable if the desirability index satisfies the following condition:

$$(USL^{'} - C)^p NCDM^p \geq \sum_{i=1}^{p-1} USL^{'(p-i)} (-(\sum_{u=1_{u\neq i}}^{p-1} (-1)^u [\sum_{j=1}^{p-u+1} \sum_{k=2}^{p-u+2} \dots \sum_{m=l}^{p-u+l} \dots \sum_{\substack{q=p-1 \\ j<k<\dots<m<\dots<q}}^{p}$$

$$r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}} \dots r_m^{\mathbb{1}_{u,[l]}} \dots r_q^{\mathbb{1}_{u,[p-1]}}]) - (-1)^p \Pi_{i=1}^p r_i - USL^{'}) + USL^{'p} + (-1)^p \Pi_{i=1}^p r_i \quad (4.5)$$

where p is the number of uncorrelated quality characteristics, r_i is the actual nonconformity ratio for the quality characteristic i, $USL^{'}$ is an upper limit for the actual nonconformity ratio, r_i^{min} is the minimum of the nonconformity ratio for the quality characteristic i and $C = min_{i=1,\dots,p}(r_i^{min})$. $\mathbb{1}_u$ is an [(p-1)x1] vector and its elements are 0 and 1. Only the first u^{th} elements are 1. $\mathbb{1}_{u,[l]}$ is the l^{th} element of the vector and

$$\mathbb{1}_{u,[l]} = \begin{cases} 0 & \text{if} \quad l > u, \\ 1 & \text{if} \quad l \leq u. \end{cases} \quad (4.6)$$

Proof of Theorem 1 is given in Appendix B.

Although the desirability index is computed for correlated and uncorrelated quality characteristics, it is obvious from (4.4) that its use is more appropriate for uncorrelated quality characteristics. It will be interesting to check whether the threshold given in (4.5) could concern also correlated quality characteristics. For that purpose, it is interesting to highlight the effect of correlation on the joint nonconformity ratio. The relationship between the correlated and the uncorrelated case is studied assuming that the multivariate normal distribution holds.

Theorem 2:

Assume that $X_1, \dots, X_p$ are p correlated quality characteristics with variances $\sigma_1^2, \dots, \sigma_p^2$ respectively. The considered quality characteristics follow the $N(\mu, \Sigma)$ where μ is the mean vector and Σ the covariance matrix. If $|\Sigma| < \Pi_{i=1}^p \sigma_i^2$ then correlation will make joint nonconformity ratio smaller.

Proof:

Assume that $Z \sim N(\mu, \Sigma)$ where μ is $(p \times 1)$ vector mean and Σ is the $p \times p$ covariance matrix with variances σ_i^2 for $i = 1 \dots p$. Consider $\Sigma^{'} = diag(\sigma_1^2 \dots \sigma_p^2)$. The joint probability density function is given by

$$f(x) = \frac{1}{\sqrt{(2\pi)^p|\Sigma|}}\ exp(-\frac{1}{2}(x-\mu)^{'}\Sigma^{-1}(x-\mu)),$$

it depends on x only in the quadratic form $(x-\mu)^{'}\Sigma^{-1}(x-\mu)$ this means that

$$\text{if } (x-\mu)^{'}\Sigma^{-1}(x-\mu) = \text{constant} \Rightarrow f(x) = \text{constant}.$$

Notice that $(x-\mu)^{'}\Sigma^{-1}(x-\mu) \sim \chi^2_p$. Hence, the density is constant for $(x-\mu)^{'}\Sigma^{-1}(x-\mu) = \chi^2_{p,\alpha}$. This is in fact the equation of an ellipsoid. All realizations of the multivariate normal distribution on the border of the ellipsoid have the same probability. The volume of this ellipsoid is given by

$$V = \frac{\pi^{\frac{p}{2}}}{\Gamma(1+\frac{p}{2})}(\chi^2_{p,\alpha})^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}.$$

For fixed specification limits the smaller V, the tighter is the distribution and the smaller the joint nonconformity ratio.

When the quality characteristics are assumed to be uncorrelated, the volume of the ellipsoid is given by

$$V^{'} = \frac{\pi^{\frac{p}{2}}}{\Gamma(1+\frac{p}{2})}(\chi^2_{p,\alpha})^{\frac{p}{2}}|\Sigma^{'}|^{\frac{1}{2}},$$

$$\text{where } |\Sigma^{'}| = \Pi_{i=1}^{p}\sigma_i^2.$$

Hence, if $|\Sigma| < \Pi_{i=1}^{p}\sigma_i^2$ then $V < V^{'}$. In this case the correlation will make the distribution tighter. This means that more observations can fall between the specification limits. Hence, in this case the correlation makes the joint nonconformity ratio smaller. □

This means that if (4.5) holds for a process which has correlated quality characteristics with $|\Sigma| < \Pi_{i=1}^{p}\sigma_i^2$, then the correlation will not affect the capability judgement. Indeed, consider a process with correlated quality characteristics. The joint nonconformity ratio is computed assuming that the quality characteristics are uncorrelated with $R < USL^{'}$. The correlation do not affect the capability judgment as long as $|\Sigma| < \Pi_{i=1}^{p}\sigma_i^2$.

Example 1:

In the bivariate case $|\Sigma| < |\Sigma^{'}|$ is equivalent to $(1-\rho^2) < 1$. Hence, in the bivariate case any correlation coefficient value makes the joint nonconformity ratio smaller. Hence, when (4.5) holds this means that the process is capable independently of the correlation coefficient.

Example 2:

For $p = 3$, $|\Sigma| < |\Sigma^{'}|$ is equivalent to $0 < 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23} < 1$. This means that when (4.5) holds this condition should be checked first before judging the process capability. When more than three quality characteristics are considered a condition on the correlation coefficients should be derived.

In the case of multivariate normality when $|\Sigma| < |\Sigma^{'}|$ and the desirability index has a value which is higher than the threshold in (4.5) the process is said capable. Under these conditions the capability threshold concerns the correlated and uncorrelated quality characteristics. However, when the condition on the generalized variance holds and (4.5) does not hold, this does not mean that the process is not capable. Indeed, when the desirability index is under the threshold given in (4.5) the process capability is rejected only when the hypothesis of independence holds.

Furthermore, many multivariate indices are presented in the literature but it is still not clear whether these indices respect ”the higher the better” rule. In what follows ”the higher the better” rule is discussed when the $NCDM$ is used.

4.4 ”The Higher the Better” Rule Using the Desirability Index

When the capability of several processes are compared, $NCDM$ is written as follows:

$$NCDM_j = [\Pi_{i=1}^{p} NCDU_{ij}]^{\frac{1}{p}},$$

where p is the number of the quality characteristics and $NCDU_{ij}$ is the univariate index for the quality characteristic i in the process j. The geometric mean is used as a capability index and it was proved in Lemma 4 that in this way it is possible to write the capability index as a function of the joint nonconformity ratio for uncorrelated quality characteristics. Hence, it becomes possible to get a threshold for capability judgment in the multivariate case. The most important expected feature of a multivariate capability index is that its value should increase when the joint nonconformity ratio decreases. When such feature holds it is said that the capability index respects ”the higher the better” rule. It will be interesting to check under which condition the desirability index respects the higher the better rule. Indeed, under such condition it becomes possible to compare between the capability of different processes. This condition is given by the following Theorem.

Theorem 3:

Consider two processes, process 1 and process 2. The quality of these processes is expressed in terms of p uncorrelated quality characteristics. These processes have the joint nonconformity ratios R_1 and R_2 respectively with $R_1 < R_2$. The ”higher the better” rule is respected if

$$\begin{cases} NCDM_1^p - NCDM_2^p > \frac{A_1 - A_2}{(USL' - C)^p} > 0 & \text{if} \quad A_1 - A_2 > 0, \\ NCDM_1^p - NCDM_2^p > 0 & \text{if} \quad A_1 - A_2 < 0, \end{cases} \tag{4.7}$$

where

$$A_v = \sum_{i=1}^{p-1} USL^{'(p-i)} (-(\sum_{u=1_{u \neq i}}^{p-1} (-1)^u [\sum_{j=1}^{p-u+1} \sum_{k=2}^{p-u+2} \dots \sum_{m=l}^{p-u+l} \dots \sum_{\substack{q=p-1 \\ j<k<\dots<m<\dots<q}}^{p} r_{jv}^{\mathbb{1}_{u,[1]}} r_{kv}^{\mathbb{1}_{u,[2]}} \dots r_{mv}^{\mathbb{1}_{u,[l]}} \dots r_{qv}^{\mathbb{1}_{u,[p-1]}}]) - (-1)^p \Pi_{i=1}^p r_i) + (-1)^p \Pi_{i=1}^p r_i + USL^{'p}$$

where $v = 1, 2$, r_{kv} is the actual nonconformity ratio for the quality characteristic k in the process v, $USL^{'}$ is an upper limit for the actual nonconformity ratio, r_{k1}^{min} is the minimum of the nonconformity ratio for the quality characteristic k in the process 1, r_{k2}^{min} is the minimum of the nonconformity ratio for the quality characteristic k in the process 2 with $k = 1 \dots p$ and $C = \min(\min_{k=1,\dots,p}(r_{k1}^{min}), min_{k=1,\dots,p}(r_{k2}^{min}))$. $\mathbb{1}_{u,[l]}$ is an [(p-1)x1] vector, its elements are 0 or 1. Only the first u^{th} elements are 1. $\mathbb{1}_{u,[l]}$ is the l^{th} element of the vector and

$$\mathbb{1}_{u,[l]} = \begin{cases} 0 & \text{if} \quad l > u, \\ 1 & \text{if} \quad l \leq u. \end{cases} \tag{4.8}$$

Proof:

In Lemma 5 it is proved that the nonconformity ratio for p uncorrelated quality characteristics in a process v is given by

$$R_v = \frac{-NCDM_v^p (USL^{'} - C)^p + A_v}{\sum_{i=1}^{p-1} USL^{'(p-i)}},$$

then $R_1 < R_2$ becomes equivalent to

$$NCDM_1^p - NCDM_2^p > \frac{A_1 - A_2}{(USL' - C)^p}.$$

$NCDM_1$ and $NCDM_2$ should also respect the following condition

$$NCDM_1^p - NCDM_2^p > 0.$$

Hence, the ”the higher the better” rule is respected when

$$\begin{cases} NCDM_1^p - NCDM_2^p > \frac{A_1 - A_2}{(USL' - C)^p} > 0 & \text{if} \quad A_1 - A_2 > 0, \\ NCDM_1^p - NCDM_2^p > 0 & \text{if} \quad A_1 - A_2 < 0, \end{cases} \tag{4.9}$$

□

It is noticed that the ”higher the better” rule is based on the joint nonconformity ratio R. However, the condition under which the desirability index respects the ”higher the better” rule depends only on nonconformity ratios computed in the univariate case for each quality characteristic. Moreover, it is noticed that the fact that $NCDM_1 > NCDM_2$ does not mean that process 1 is more capable than process 2, that is true only when (4.8) holds.

4.5 Comparison of the Multivariate PCIs

It will be interesting to compare the performance of the proposed multivariate index with different indices from the literature. It is interesting to check wether the considered indices succeed in respecting the ”higher the better” rule. A simulated example is leading the comparison between the competing indices. In what follows a comparison is carried out between the indices $MVCp$, $MVCpm$ proposed by Taam et al. (1993) and the geometric mean of $NCDU_i$.
The indices $MVCp$ and $MVCpm$ are defined as follows:

$$MVCp = \frac{\text{vol(max.vol. ellipsoid in specification region)}}{\text{vol(process ellipsoid)}}.$$

The maximum volume ellipsoid embedded in the specification region is given by $\{x|(x-M)'H^{-1}(x-M) \leq 1\}$, where M is the vector formed by the midpoints of the specification limits, $H = diag(\varepsilon_1^2, \ldots, \varepsilon_p^2)$

Table 4.1: Example Processes

Process	A	B	C	D
σ_1^2	6	6	6	6
σ_2^2	12	12	12	12
σ_3^2	15	15	15	15

with $\varepsilon_i = \frac{USL_i - LSL_i}{2}$ and p the number of quality characteristics. When the multivariate normal distribution holds the process region is given by the following ellipsoid $\{x|(x-\mu)'\Sigma^{-1}(x-\mu) \leq \chi^2_{p,0.9973}\}$. Where Σ is the covariance matrix and $\chi^2_{p,0.9973}$ is a quantile of the chi square distribution with p degrees of freedom. Then,

$$MVCp = [\frac{|H|}{|\Sigma|(\chi^2_{p,0.9973})^p}]^{\frac{1}{2}}$$

The index MVCpm is given by

$$MVCpm = \frac{\text{vol(max.vol. ellipsoid in specification)}}{vol((x-T)'\Sigma_T^{-1}(x-T) \leq \chi^2_{p;0.9973})}$$

where $\Sigma_T = E[(X-T)(X-T)']$, hence,

$$MVCpm = MVCp/\sqrt{1+(\mu-T)'\Sigma^{-1}(\mu-T)}.$$

The comparison is implemented over four processes, each process has a trivariate normal distribution. The quality characteristics are assumed to be uncorrelated for all processes. Table 4.1 gives the variances σ_1^2, σ_2^2, σ_3^2 for the examined processes A, B, C and D.

For all quality characteristics in processes A and B symmetric specification limits are considered. Furthermore, The specification limits are the same for all quality characteristics in the processes A and C they are given by (LSL, USL)= (15, 50). For process A the specification limits are given by (LSL, T, USL)=(15, 32.5, 50) for all quality characteristics. The specification region is given by the Cartesian product of the univariate specification limits: (15,32.5,50)*(15,32.5,50)*(15,32.5,50). Notice that the Process A is centered and on-target with $\mu = T$ for all quality characteristics. However, for process C asymmetric specification limits are used, it is off-target and not centered with $T = 30$ and $\mu = 34$ for all quality characteristics.

For the process B the specification region is given by the following Cartesian product:

Table 4.2: Nonconformity Ratios and Index Values

Process	R	$MVCp$	$MVCpm$	$NCDM$	A_v
A	$6.666\ \ 10^{-6}$	3.061	3.061	0.9642	4.268×10^{-10}
B	$7.720\ \ 10^{-7}$	3.061	3.061	0.9959	4.966×10^{-11}
C	$2.046\ \ 10^{-5}$	3.061	1.243	0.8832	1.277×10^{-9}
D	$2.973\ \ 10^{-5}$	3.061	1.243	0.8320	1.885×10^{-9}

(15,27.5,40)*(15,32.5,50)*(15,39.5,64). Notice that the Process B is centered and on-target with $\mu = T$ for all quality characteristics. However, for the process D asymmetric specification limits are used, it is off-target and not centered. The specification region is given by the following Cartesian product: (15,29,40)*(15,34,50)*(15,42,64). The mean values are $\mu_1 = 25$, $\mu_2 = 30$ and $\mu_3 = 38$ for the quality characteristics 1, 2 and 3 respectively. Table 4.2 shows the joint nonconformity ratio and the values of the process capability indices.

Notice that the index $MVCp$ is constant for all processes and this is due to the fact that $|H|$ and $|\Sigma|$ are the same for all processes. However, the considered processes have different nonconformity ratios because they have different specification limits. The same shortcoming is observed for the index $MVCpm$ as the same deviation from the target values was considered for all quality characteristics. This comparison shows that the index $NCDM$ succeeds in respecting the higher the better rule when uncorrelated quality characteristics are considered. The higher the better rule is respected independently of the specification limits type as long as the condition in (4.8) is respected. Indeed, notice that the order given by A_v is the same order as $NCDM$. Hence, the second line of (4.8) is respected for all possible comparisons in the considered example.

4.6 $NCDM$ Implementation

In the previous chapter the capability of the streams was compared using $NCDU$. The objective was the determination of the most capable stream for each quality characteristic. However, $NCDU$ does not help for determining if the process is more capable in screwing screw 1 or screw 2. $NCDM$ appears to be more appropriate for this task. Table 4.3 shows the joint nonconformity ratio R_1 for

Table 4.3: $NCDM$ Computation

R_1	R_2	$NCDM_1$	$NCDM_2$
34.46×10^{-6}	52.34×10^{-6}	0.887	0.819

screw 1, the joint nonconformity ratio R_2 for screw 2, $NCDM_1$ for screw 1 and $NCDM_2$ for screw 2. As it is explained in section 3.3 the streams are uncorrelated. Hence, R_1 and R_2 are computed as in equation (4.2). Table 4.3 shows that $NCDM$ respects ”the higher the better” rule and that the process is more capable in screwing screw 1.

As explained in section 3.8, the capability judgment is based on the bootstrap confidence interval for $NCDM$. Following the same procedure as in 3.8 the bootstrap confidence intervals are [0.000, 0.973] and [0.000, 0.978] for screw 1 and screw 2 respectively. It is concluded that the process is not capable in screwing screw 1 and screw 2. However, it is shown in section 3.8 that stream 2 is the only stream which is not capable for screwing screw 1 and screw 2. When discarding stream 2 from the analysis the obtained confidence intervals are given by [0.751, 0.979] and [0.000, 0.989] for screw 1 and screw 2 respectively and the process is capable only for screwing screw 1. Hence, when stream 2 is discarded a loss function approach could be adopted for screw 1. In this case the supreme objective which is the adjustment to the target value could be reached. More quality improvement should be adopted for the screw 2 streams before tackling the adjustment to the target objective. For that purpose an experimental design should be implemented in order to determine the optimal operating conditions. When a confidence interval is constructed for $NCDM$ under optimal operating conditions it is definitely known whether the adoption of a loss function approach is possible. Figure 4.1 and Figure 4.2 show the bootstrap distribution of $NCDM_1$ and $NCDM_2$ when stream 2 is discarded form the analysis.

4.7 Conclusion

It was demonstrated that the index $NCDU$ respects the ”higher the better” when the other indices fail. The extension of the univariate process capability index $NCDU$ to the multivariate case is given by the desirability index. It was demonstrated that the geometric mean of the desirability functions

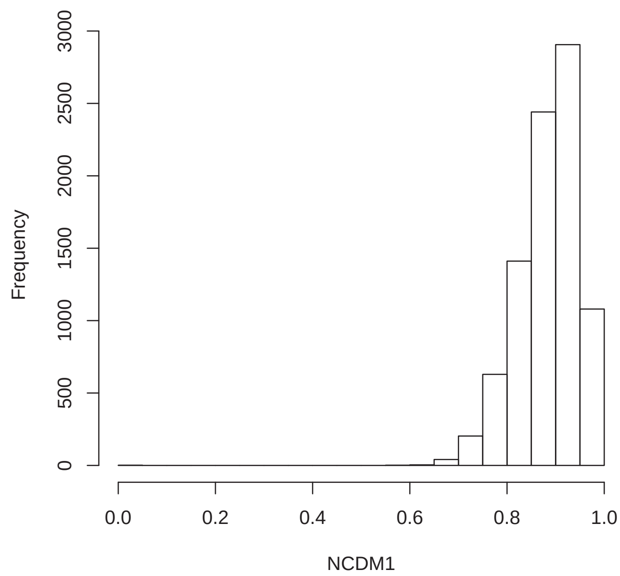

Figure 4.1: Histogram of $NCDM_1$

is suitable for process capability assessment. Indeed $NCDM$ is written as a function of the joint nonconformity ratio for p uncorrelated quality characteristics. A condition under which $NCDM$ respects the ”higher the better” rule is derived. Moreover, it is shown that it is possible to use $NCDM$ for some correlated quality characteristics. Finally, a threshold for capability judgment is proposed. Knowing that the desirability index is an important tool in the desirability optimization methodology, the presented approach becomes promising as its application together with experimental design is straightforward. Hence, the capability index will not be considered only as a tool for describing the process capability but also as a tool for minimizing the proportion of nonconforming items.

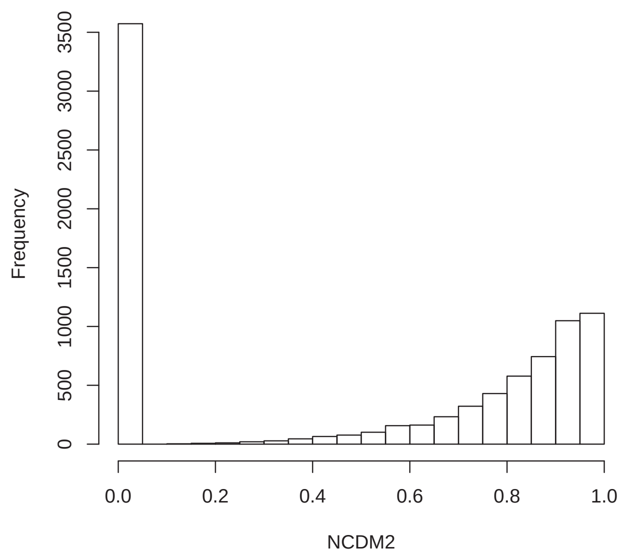

Figure 4.2: Histogram of $NCDM_2$

Chapter 5

Capability Assessment Under Optimal Operating Conditions

In the previous chapter it was noticed that the considered process is not capable in screwing screw 2. It was recommended to run an experimental design in order to improve the process capability. In what follows an algorithm is presented in order to maximize $NCDM$. A simulation study is carried out using the presented algorithm. The maximization of $NCDM$ means the process capability maximization. It allows the determination of the optimal operating condition. It would be interesting to test the process capability under this condition. The presented steps allow also to give an answer for whether the adoption of a loss function approach for quality continuing improvement purpose is possible for the simulated process.

5.1 The Algorithm for the Capability Assessment

As it was noticed, the desirability index is the geometric mean of the individual desirability functions which depend on some response variables, in this case the considered response variables are the individual nonconformity ratios. When the individual nonconformity ratios are expressed as functions of some factors $X_1, ..., X_s$ it becomes possible to express the desirability index on the factor space. The maximization of the desirability index on the factor space allows the determination of the optimal factor levels. These factor levels determine in fact the most desirable combinations of the individual nonconformity ratios. Hence, determining $NCDM$ under optimal operating conditions it becomes possible to compare different processes on the basis of the joint nonconformity ratio as long as (4.7) is respected. The challenge is to use the same technique for describing the process capability and determining the optimal operating conditions.

The following algorithm describes a simulation design. The objective is to estimate the factor levels which minimize the joint nonconformity ratio. This is done by the simultaneous optimization of individual nonconformity ratios. The desirability index corresponding to the optimal operating conditions is then considered as a metric for capability assessment. In this chapter two variables Y_1 and Y_2 are considered. It is supposed that these variables correspond to two streams of the same quality characteristic. It is supposed that the realizations of Y_1 and Y_2 depend on the levels of two factors X_1 and X_2. It is also assumed that the variables are independent.

Step1: experimental design

An excribed central composite experimental design is considered and the variables are considered as response variables. In the aim to be able to write the variables as functions of the factors: $Y_i = f_i(X_1, X_2) + \epsilon_i$ where ϵ_i, $i = 1, 2$, are the errors of the model, $E(\epsilon_i) = 0$ and $E(Y_i) = f_i(X_1, X_2)$. The model is supposed to be quadratic and the experimental design does not evolve replications.

Step2: data generation

One observation of each variable is assigned to each run of the experimental design. For the purpose of running a simulation, each observation could be considered as a realization of a random variable which follows a given distribution. A different distribution is considered at each run of the experimental design and for each response. In what follows non normal distributions are considered and it is assumed that the distributions considered for each response have the same variance.

Step3: transform data

In the aim to be able to use normality for the model coefficients, it is proposed to use the Box-Cox transformation as defined in Box and Cox (1964). The transformation is given by

$$y_i^{(\lambda_i)} = \begin{cases} \frac{y_i^{\lambda_i} - 1}{\lambda_i} & \text{if} \quad \lambda_i \neq 0, \\ log(y_i) & \text{if} \quad \lambda_i = 0. \end{cases} \tag{5.1}$$

It is important to notice that the mean squared error MSE is used to estimate σ^2. λ_i should be estimated by $\hat{\lambda}_i$ which is determined numerically. It is incremented in the range -3 to 3. The likelihood is computed for each value of $\hat{\lambda}_i$ and the retained $\hat{\lambda}_i$ is the one that maximizes the likelihood. In order to give an idea about the values of $\hat{\lambda}_i$ histograms are shown in section 5.2.

Step4: model the transformed data

The transformed data are supposed to follow the normal distribution. Hence, modelling the transformed data allows us to estimate the parameters of the normal distributions by $\hat{E}(y_i^{(\hat{\lambda}_i)})$ and MSE.

Step5: nonconformity ratios estimation

After estimating the parameters of the normal distributions it becomes necessary to transform the specification limits LSL_i and USL_i using the same transformation as for Y_i. The nonconformity ratio estimators at each run u are given by $\hat{r}_{iu} = \Phi(\frac{LSL_i^{(\hat{\lambda}_i)} - \hat{E}(y_{iu}^{(\hat{\lambda}_i)})}{\hat{\sigma_i}}) + 1 - \Phi(\frac{USL_i^{(\hat{\lambda}_i)} - \hat{E}(y_{iu}^{(\hat{\lambda}_i)})}{\hat{\sigma_i}})$, where u is the number of the experimental design run with $u = 1, ..., n_0$, $i = 1, 2$

Step6: model the nonconformity ratios

The nonconformity ratio is a value between 0 and 1. In the next step the nonconformity ratio is minimized. In order to avoid negative values of $\hat{r}_i^{min}$, $log(\hat{r}_i)$ are modelled instead of $\hat{r}_i$, where $\hat{r}_i$ gives the estimated nonconformity ratio for Y_i.

The model is supposed to be quadratic and the coefficients of the model are estimated using ordinary least squares. The adequacy of the model is checked using the F statistic. The F statistic measures the goodness of fit of the model with

$$F_i = \frac{\sum_{u=1}^{n_0} (\widehat{log(\hat{r}_{iu})} - \overline{log(\hat{r}_i)})^2 (n - p - 1)}{\sum_{u=1}^{n_0} (log(\hat{r}_{iu}) - \widehat{log(\hat{r}_{iu})}) p},$$

where $\widehat{log(\hat{r}_i)} = g_i(X_1, X_2)$, with the number of variable $p = 2$ and the number of the experimental design runs $n_0 = n = 9$ as there is one observation at each run. The runs are showed in Table 5.1.The model is judged appropriate if the F statistic is 10 times greater than the F percentage point as noticed Box and Draper (1987, p.280). In the simulation study in section 5.2 if the condition on the F statistic is fulfilled, then step 7 is started otherwise the algorithm is restarted from step 1.

Step7: nonconformity ratios minimization

Each nonconformity ratio $\hat{r}_i$ is minimized using a grid search. It is important to notice that the log transformation avoids to have a minimum of the nonconformity ratio which is negative. Knowing the minimum of each nonconformity ratio allows the determination of the desirability functions as

Table 5.1: Distributions and Nonconformity Ratios for Y_1 and Y_2

Run number	Distributions of Y_1	Distributions of Y_2	r_{1u}	r_{2u}
1	Lognormal (1.90,0.5)	Lognormal (1.92,0.45)	1.310×10^{-3}	7.994×10^{-3}
2	Lognormal (1.98,0.5)	Lognormal (1.51,0.45)	2.260×10^{-3}	4.795×10^{-4}
3	Lognormal (0.90,0.5)	Lognormal (1.50,0.45)	2.735×10^{-7}	4.265×10^{-4}
4	Lognormal (1.15,0.5)	Lognormal (1.25,0.45)	3.255×10^{-6}	5.002×10^{-5}
5	Lognormal (1.00,0.5)	Lognormal (1.12,0.45)	7.582×10^{-7}	1.529×10^{-5}
6	Lognormal (1.35,0.5)	Lognormal (1.30,0.45)	1.986×10^{-5}	7.861×10^{-5}
7	Lognormal (1.50,0.5)	Lognormal (1.38,0.45)	6.975×10^{-5}	1.581×10^{-4}
8	Lognormal (1.30,0.5)	Lognormal (2.05,0.45)	1.282×10^{-5}	1.770×10^{-2}
9	Lognormal(1.60,0.5)	Lognormal(1.50,0.45)	1.536×10^{-4}	4.265×10^{-4}

defined in (3.3).

Step8: desirability index maximization

Using the models in step 6 and the minimum in step 7 it becomes possible to express the desirability index as a function of the factors $NCDM(X_1, X_2) = [\Pi_{i=1}^{p} NCDU_i(X_1, X_2)]^{\frac{1}{p}}$. A grid search is performed to find the optimum operating conditions which minimize the joint nonconformity ratio and the corresponding desirability index is considered as reflecting the maximum process capability.

5.2 Simulation Study

Focus is on non normal distributions, hence, different Lognormal distributions are chosen for Y_1 and Y_2 for each run of the experimental design as shown in Table 5.1. It is assumed at this stage that the parameters of the distributions are known. Furthermore the specification limits are set for each response they are given by $(LSL_1, USL_1) = (0.1, 30.1)$ and $(LSL_2, USL_2) = (0.1, 20.1)$ for Y_1 and Y_2 respectively. The parameters of the distributions are set in a way that the models for $Log(r_i)$ have high F statistics and in this case $F_1 = 1396$ and $F_2 = 914.7$. It should be noticed that as the parameters of the distributions are known there is no need for computing the nonconformity ratio for the transformed data. This means that the Box-Cox transformation is not used for the construction of

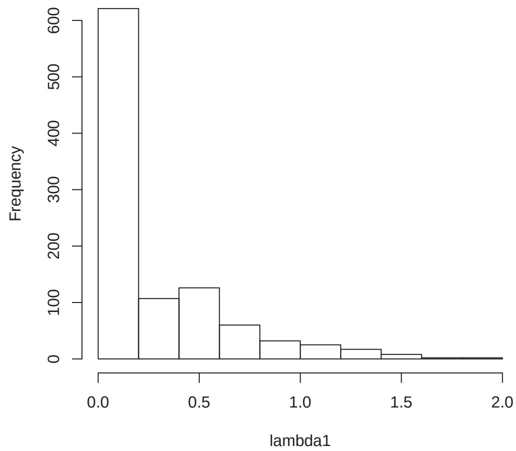

Figure 5.1: Histogram of $\hat{\lambda}_1$.

what are supposed ”theoretical” models. The models are determined on the basis of the nonconformity ratios shown in Table 5.1. After the minimization of r_i and the determination of $NCDU_i$, the maximization of the desirability index provides the optimal factor levels and the desirability index is considered as an indicator for the maximum process capability. In this case $NCDM = 0.888$, the optimal factor levels are $X_1 = -0.246$ and $X_2 = -0.066$. These settings are considered as a theoretical optimum. The location parameters corresponding to these settings are 1.0303 and 1.1082 for Y_1 and Y_2 respectively. In this simulation study it is assumed that the target values $T_1 = 1.0303$ and $T_2 = 1.1082$ for Y_1 and Y_2 respectively. Indeed, with this assumption the loss function approach and the nonconformity ratio approach have no conflicting goals.
In order to assess the validity of this approach under the mentioned conditions a confidence interval is constructed for the desirability index. For this purpose the same distributions are considered and an observation for each response Y_i from each distribution at each run u is generated. The generated data are transformed using the Box-Cox transformation. Figure 5.1 and Figure 5.2 show the histograms for $\hat{\lambda}_1$ and $\hat{\lambda}_2$ for Y_1 and Y_2 respectively.

It is noticed that large proportions of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are equal zero and that only positive values of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ occur. The algorithm as described in the last section is repeated 1000 times. This number of iterations allows to have an idea about the distribution of the desirability index and the construction

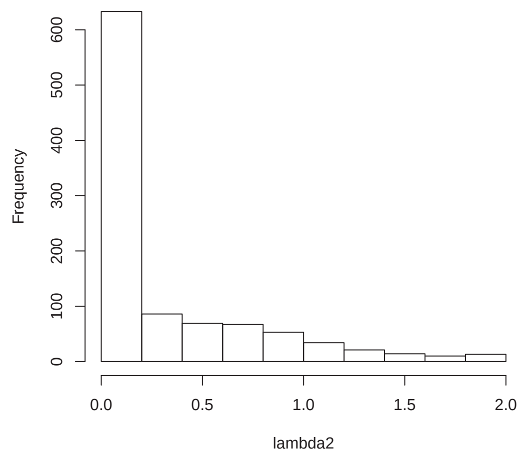

Figure 5.2: Histogram of $\hat{\lambda}_2$.

of the confidence interval based on quantiles as explained in section 3.8. The histogram in Figure 5.3 gives an idea about the desirability index distribution. Moreover, at each iteration a repeat routine is used, this routine is broken only when the condition on the F statistic given in step 6 is fulfilled.

Following this procedure the constructed confidence interval is [0.60,1.00]. Notice that when the number of iterations is greater or equal 1000 the lower confidence limit could be approximated by the order statistic $NCDM_{[25]}$. Hence, it becomes possible to know the factor levels corresponding to the lower confidence limit and they are $X_1 = -0.2828$ and $X_2 = 1.414$. Substituting the factor levels into what was considered the ”theoretical” models, it becomes possible to compute the nonconformity ratio for each variable Y_1 and Y_2 and to compute the joint nonconformity ratio which is $R = 24.55 \times 10^{-5}$. Hence, the considered process in this simulation study is not capable. Neither the loss function approach nor the nonconformity ratio approach can improve the process capability. In this case other influential factors should be taken into account. A loss function approach could be adopted for the process capability improvement when the process is still capable when the deviations from the target values are minimized.

Notice that the confidence interval succeeds in capturing the theoretical optimum which is in favor of the statistical validity of this approach. It would be interesting to check the impact of the goodness of fit of the model on the confidence interval. For that purpose the condition presented in step 6 that the

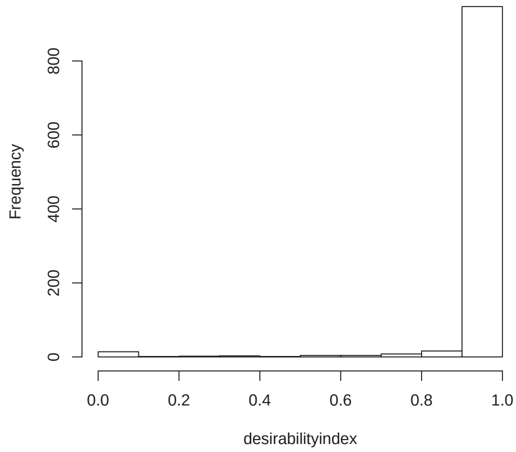

Figure 5.3: Histogram of the Desirability Index.

F statistic should be 10 times greater than the F percentage point is released. Figure 5.4 shows the new distribution of the desirability index. The constructed confidence interval is given by [0.84, 1.00]. The obtained confidence interval has shorter length than the one constructed previously. When releasing the condition on the goodness of fit the confidence interval still contains the theoretical optimum and is more accurate.

In order to know whether the condition on the F statistic has a significant effect on the desirability index distribution the Levene's test and the Mann Whitney test are used.

5.2.1 The Levene's Test for Equality of Variances

The Levene's test is used to test if k samples have equal variances. Notice that in this study it is tested whether $NCDM$ when the condition on the F statistic is used has the same variance with $NCDM$ when the condition on the F statistic is released. First, the Levene's test is presented in the general case. The Levene's test is known to be less sensitive to normality. The Levene's test is defined as:

$$H_0 : \sigma_1 = \sigma_2 = \ldots = \sigma_k$$

vs.

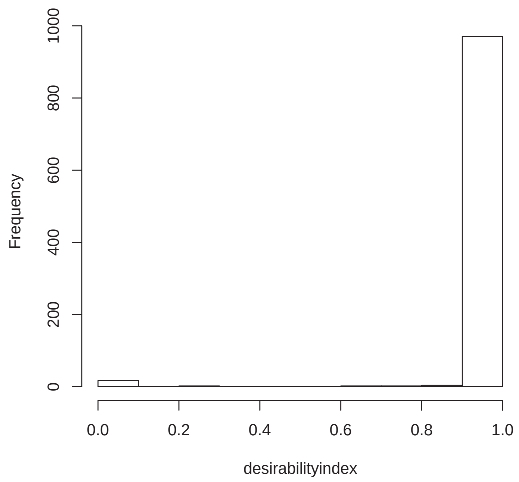

Figure 5.4: Histogram of the Desirability Index When the Goodness of Fit Condition is Released.

$$H_1 : \sigma_i \neq \sigma_j \text{ for at least one pair } (i, j).$$

Given the variables $Y_1, \ldots, Y_k$ with sample sizes $n_1, \ldots, n_k$ and $n = \sum_{i=1}^{k} n_i$, the Levene's test statistic is defined as

$$W = \frac{(n-k) \sum_{i=1}^{k} n_i (\bar{Z}_i - \bar{Z})^2}{(k-1) \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Z_{ij} - \bar{Z}_i)^2},$$

where $Z_{ij} = |Y_{ij} - \tilde{\mathrm{Y}}_i|$ with $\tilde{\mathrm{Y}}_i$ the median of Y_i, $i = 1, \ldots, k$. Moreover, $\bar{Z}_i$ is the mean of Z_{ij} and $\bar{Z}$ is the overall mean of the k samples. The Levene's test statistic follows the Fisher distribution with $k-1$ and $n-k$ degrees of freedom. The Levene's test statistic rejects H_0 at the significance level α if $W > F(\alpha, k-1, n-k)$, where $F(\alpha, k-1, n-k)$ is the upper critical value of the F distribution. In this study $W = 0.902$ and $F(0.95, 1, 1998) = 3.846$. Hence, the goodness of fit of the model has no significant effect on the $NCDM$ distribution spread.

5.2.2 The Mann Whitney Test

The Mann Whitney test is a nonparametric equivalent for the t-test. The Mann Whitney is used to test whether the considered variables have the same median. In this case the variables are $NCDM$ when the condition on the F statistic is used and $NCDM$ when the condition on the F statistic is released. First the test is presented in the general case. If two variables are considered with sample sizes n_1 and n_2 respectively. The following hypotheses are tested

$$H_0 : \tilde{Y}_1 = \tilde{Y}_2$$

vs.

$$H_1 : \tilde{Y}_1 \neq \tilde{Y}_2.$$

The Mann Whitney U test statistic is obtained by ranking all (n_1+n_2) observations in ascending order. Then, the sums of the ranks corresponding to each variable are computed say T_a and T_b. Hence, the U statistic is given by $U = min(U_a, U_b)$ where

$$U_a = n_1 n_2 + 0.5 n_1 (n_1 + 1) - T_a,$$

$$U_b = n_1 n_2 + 0.5 n_2 (n_2 + 1) - T_b.$$

For sample sizes larger than 20, $U \sim N(E(U), \sigma^2)$, with $E(U) = 0.5 n_1 n_2$ and $\sigma^2 = \frac{[n_1 n_2 (n_1+n_2+1)]}{12}$. H_0 is rejected at the significance level α if $\frac{U-E(U)}{\sigma} < z_{\frac{\alpha}{2}}$ or $\frac{U-E(U)}{\sigma} > z_{1-\frac{\alpha}{2}}$, where $z_{\frac{\alpha}{2}}$ is the quantile of order $\frac{\alpha}{2}$ of the standard normal distribution. In this simulation study the z score associated with the U statistic is -17.52. Hence, the goodness of fit of the model has no effect on the desirability index distribution spread, but it has a significant effect on the $NCDM$ distribution location at a significance level of 5%. Moreover, the significant effect on the distribution location can provide an explanation to the shorter length of the confidence interval in this case. Indeed, notice that in both confidence intervals the upper limit is 1.00, this value could not be exceeded by $NCDM$. Hence, the change in the confidence interval is observed only at the lower limit and in this case the confidence interval has a shorter length.

5.3 Conclusion

Knowing that the desirability index can be used for optimization, then, the presented approach becomes promising as its use with experimental design is straightforward. Hence, the capability index will not be considered only as a tool for describing the process capability but also as a tool for minimizing the proportion of nonconforming items. For this purpose an algorithm is defined which associates the use of the capability index with experimental design implementation. The assessment of the approach based on the algorithm was done in the bivariate case and the statistical validity of the approach was shown.

Chapter 6

General Conclusion

In this work a linear nonconformity ratio based desirability function ($NCDU$) is presented as a process capability index. $NCDU$ avoids the use of two different indices for assessing the actual capability and the potential capability. Based on a real case study the performance of this index is compared to other indices in the literature. It was demonstrated that $NCDU$ respects the ”higher the better” rule for any type of distribution and for any specification limits. Moreover, a bootstrap confidence interval is constructed for $NCDU$. The lower bootstrap confidence limit was used for capability judgment. The presented univariate index overcomes some shortcomings of the existing indices. However, in many cases the quality of a product is given through several quality characteristics. Hence an extension to the multivariate case of $NCDU$ is given by the desirability index. Moreover, it was demonstrated that the geometric mean of the univariate indices is suitable for process capability assessment as it is proved that it could be written as a function of the joint nonconformity ratio for uncorrelated quality characteristics. A threshold for the capability judgment for the multivariate index ($NCDM$) and a condition under which the multivariate index respects the ”higher the better” rule were proposed. Furthermore, a condition under which the threshold for capability judgment could be used for correlated quality characteristics is presented. The performance of $NCDM$ is compared to other multivariate indices from the literature through a simulated example. The implementation of $NCDM$ revealed that it respects the ”higher the better rule” in the case study. Moreover, a bootstrap confidence interval was constructed for $NCDM$ and the lower limit was used for capability judgment. The case study revealed that the capability of the streams needs improvement. A Monte Carlo simulation is performed in order to assess the ability of using $NCDM$ with experimental design. A confidence interval is constructed for max($NCDM$) and it appears that the constructed interval succeeds in capturing the "theoretical" max($NCDM$). Moreover, it is shown in this case that the model adjustment has a significant effect on the max($NCDM$) distribution but not on the spread of the distribution which provided an explanation for the width of the confidence interval in the considered simulation.

A nonconformity ratio approach is used in order to assess a multi-stream screwing process. The considered univariate index $NCDU$ reflects better the state of the process than other considered univariate indices. Moreover, it was noticed that although the material of the screw 1 support is less resistant to the screwing operation, the screw 1 streams present better capability than screw 2 streams. This judgment is based on the lower limit of the bootstrap confidence interval of $NCDU$. It would be interesting to check the technical possibility of using the same material for both screws. However, many outliers are observed for screw1 and this due mainly to the fact that the interior of some supports are broken during the screwing operation. The elimination of outliers would improve considerably the quality of the final product. Furthermore, even for screw 1, stream 2 does not present an acceptable capability and the closing machine 2 should be checked and its settings compared to the other machine settings.
As the streams are not correlated, the properties of $NCDM$ proved in Chapter 4, make of it a suitable index in order to assess the capability of multi-stream processes. The interest of using $NCDU$ and $NCDM$ is based on the fact that they are used for a wide range of non normal distributions and specification limits as they do not depend directly on the distributions parameters.
An alternative to the use of the same screw support material is to run an experimental design for screw 2 streams. In Chapter 5, the statistical validity of this approach is showed through a simulation study. The use of the experimental design provides an answer to whether a loss function approach could be adopted or not. However, more investigations are required in order to give such answer. The investigations should take into account specially the case when the loss function approach and the nonconformity ratio approach have conflicting goals. Moreover, the investigations should cover the effect of the adjustment of the model, and transformations on final results. That is a challenging topic

which should be associated to the adaptation of $NCDM$ to correlated quality characteristics.

Appendix A

Capabilité Procédé

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Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument	
Evaluation de	01/01/70 / 00:00:00	à 28/10/00 / 10:27:07	
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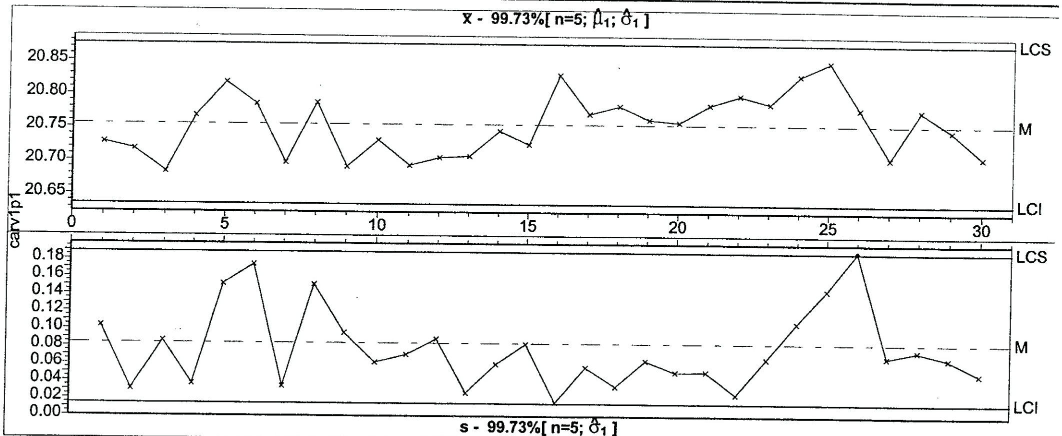

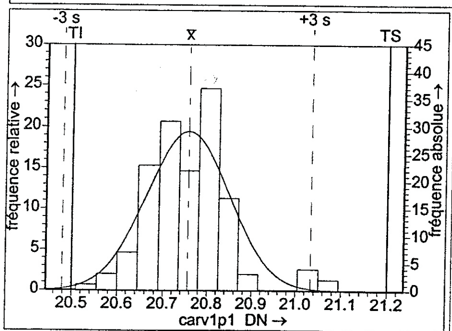

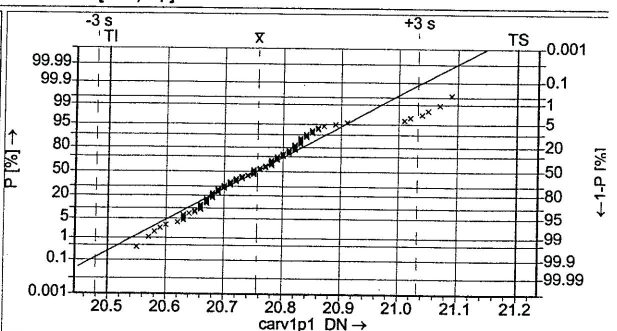

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Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.7559
TI	20.50	x_{min}	20.55	$\bar{x}$ - 3s	20.4790
TS	21.20	x_{max}	21.09	$\bar{x}$ + 3s	21.0328
Tolérance	0.70	R	0.54	6 · s	0.5538
		$n_{<T>}$	150	$n_{<T>}$	99.72158 %
		$n_{>TS}$	0	$p_{>TS}$	0.00007 %
		$n_{<TI}$	0	$p_{<TI}$	0.27834 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DN**	**Distr. Norm.**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x̄-99.865%) (1)**
Indice de capabilité	**P_p**	**1.26**
Indice de capabilité	**P_{pk}**	**0.92**

Capabilité Process non confirmée !

Q-DAS 1

Capabilité Procédé

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Usine	Dept./Achat./Prod. τ	Nom NN	Date 28/10/00
Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Caracarv1p2v	Val. Nom20.85 0.35	TS 21.20
Mach.No.	Carc. N° 2500	Unité -0.35	TI 20.50
Type Carac. variable	Classe de Caracsignificatif	Taille d'échan2.	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrum0ent	
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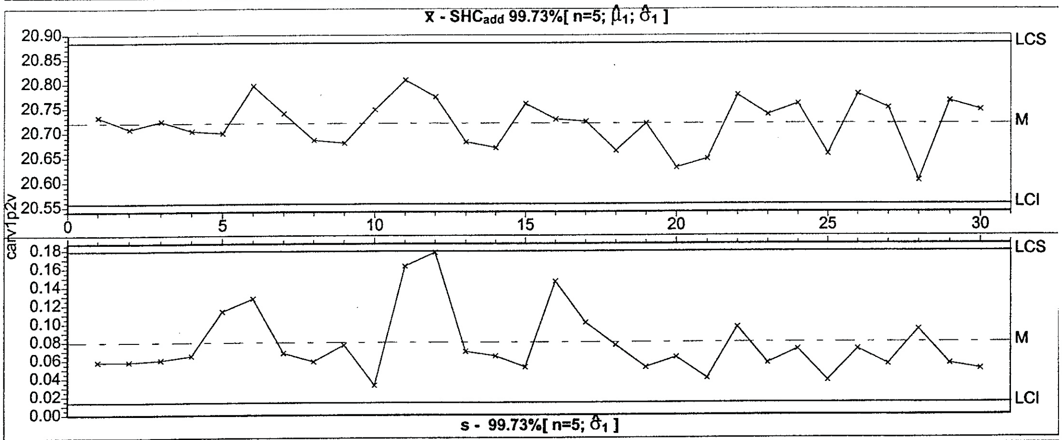

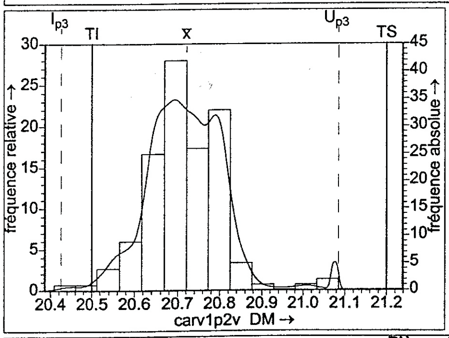

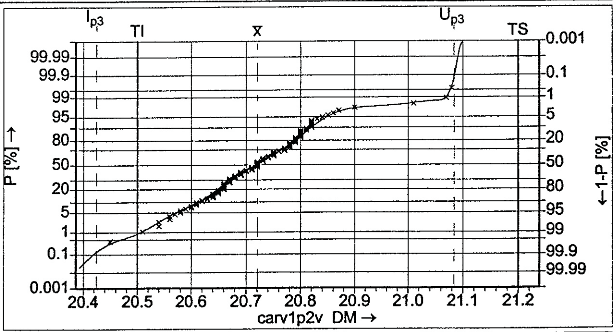

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.7203
TI	20.50	x_{min}	20.45	l_{p3}	20.4247
TS	21.20	x_{max}	21.08	U_{p3}	21.0843
Tolérance	0.70	R	0.63	$U_{p3} - l_{p3}$	0.6596
		$n_{<T>}$	149	$n_{<T>}$	99.12561 %
		$n_{>TS}$	0	$p_{>TS}$	0.00000 %
		$n_{<TI}$	1	$p_{<TI}$	0.87439 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DM**	**Distr. Mixe**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-X-99.865%) (1)**
Indice de capabilité	**T_p**	**1.06**
Indice de capabilité	**T_{pk}**	**0.75**

Capabilité Process non confirmée !

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Capabilité Procédé

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Page 1 / 1

Usine	Dept./Achat./Prod. τ	Nom NN	Date 28/10/00

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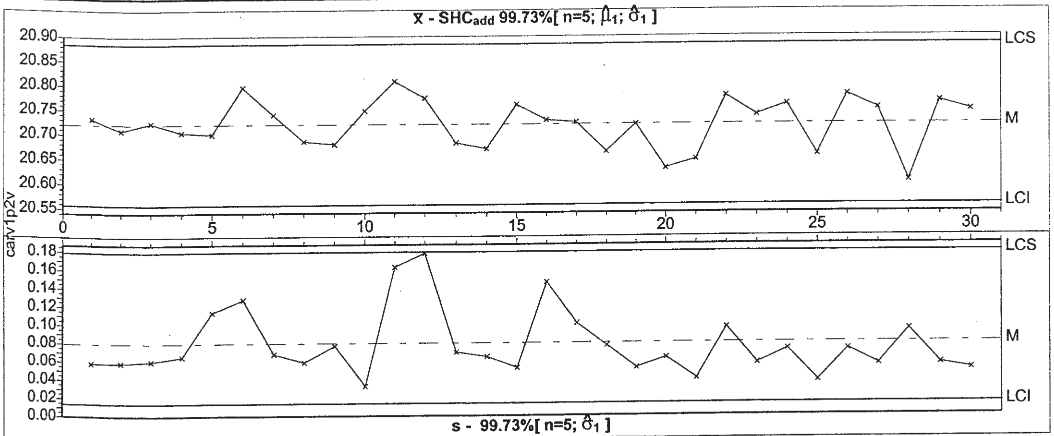

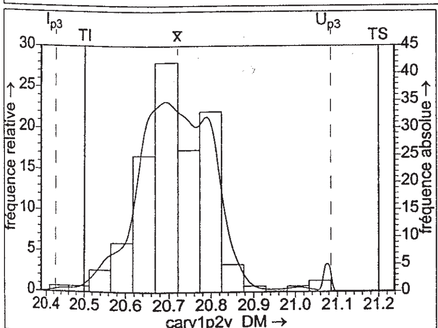

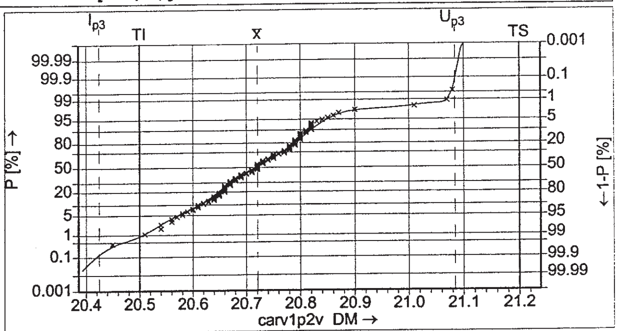

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

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TI	20.50	x_{min}	20.45	I_{p3}	20.4247
TS	21.20	x_{max}	21.08	U_{p3}	21.0843
Tolérance	0.70	R	0.63	$U_{p3} - I_{p3}$	0.6596
		$n_{<T>}$	149	$n_{<T>}$	99.12561 %
		$n_{>TS}$	0	$p_{>TS}$	0.00000 %
		$n_{<TI}$	1	$p_{<TI}$	0.87439 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DM**	**Distr. Mixe**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x̄-99.865%) (1)**
Indice de capabilité	T_p	**1.06**
Indice de capabilité	T_{pk}	**0.75**

Capabilité Process non confirmée !

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Capabilité Procédé

Q-DAS	Capabilité Procédé		Page 1 / 1
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Type de production	Instrument N°	Résol. Instrument 0

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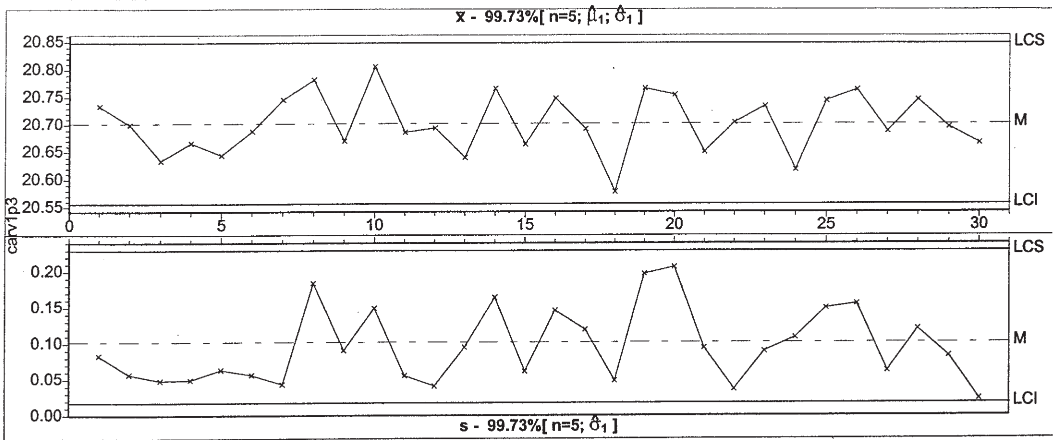

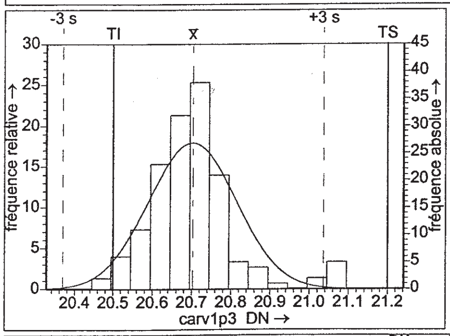

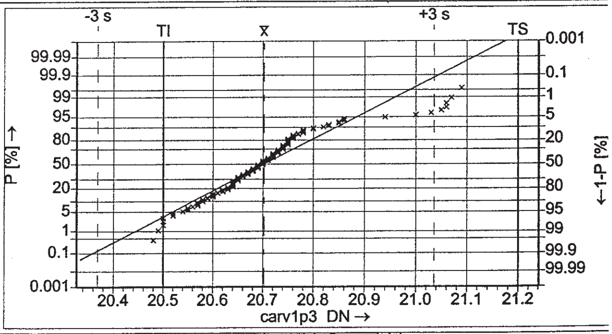

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.7023
TI	20.50	x_{min}	20.48	$\bar{x}$ - 3s	20.3686
TS	21.20	x_{max}	21.09	$\bar{x}$ + 3s	21.0359
Tolérance	0.70	R	0.61	6 · s	0.6673
		$n_{<T>}$	148	$n_{<T>}$	96.55070 %
		$n_{>TS}$	0	$p_{>TS}$	0.00038 %
		$n_{<TI}$	2	$p_{<TI}$	3.44892 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DN**	**Distr. Norm.**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x̄-99.865%) (1)**
Indice de capabilité	T_p	**1.05**
Indice de capabilité	T_{pk}	**0.61**

Capabilité Process non confirmée !

Q-DAS 1

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Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Carac. carv1p4	Val. Nom. 20.85	0.35 TS 21.20
Mach.No.	Carc. N° 2500	Unité	-0.35 TI 20.50
Type Carac. variable	Classe de Carac significatif	Taille d'échant. 5	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument 0	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 10:42:06

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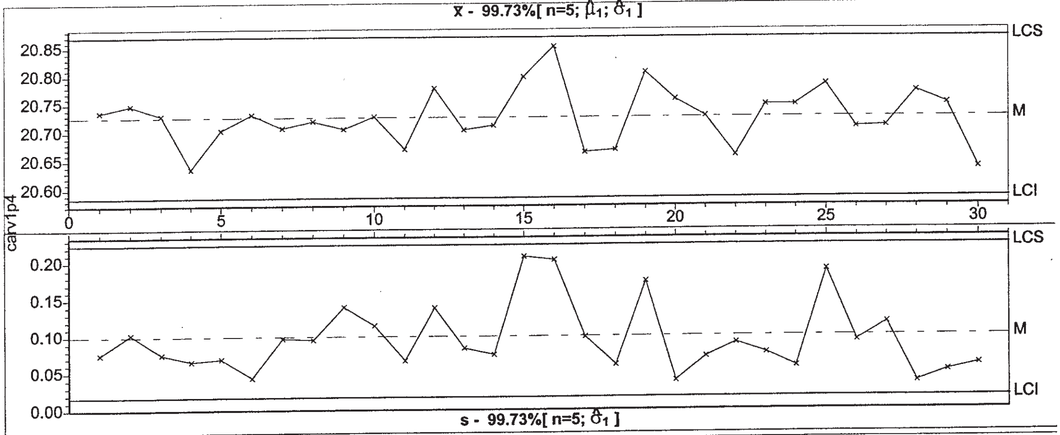

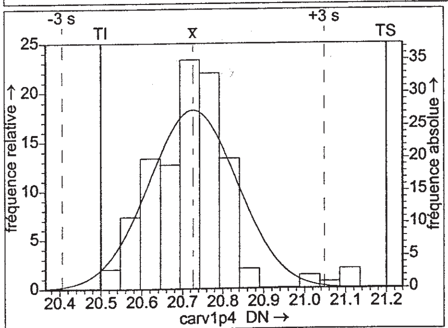

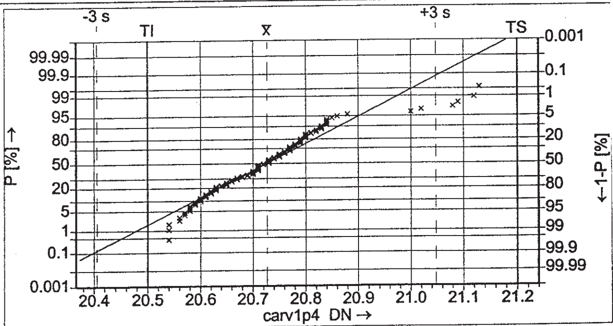

Valeurs — Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.7267
TI	20.50	x_{min}	20.54	$\bar{x}$ - 3s	20.4055
TS	21.20	x_{max}	21.13	$\bar{x}$ + 3s	21.0478
Tolérance	0.70	R	0.59	6 · s	0.6423
		$n_{<T>}$	150	$n_{<T>}$	98.28755 %
		$n_{>TS}$	0	$p_{>TS}$	0.00049 %
		$n_{<TI}$	0	$p_{<TI}$	1.71195 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DN**	**Distr. Norm.**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x̄-99.865%) (1)**
Indice de capabilité	P_p	**1.09**
Indice de capabilité	P_{pk}	**0.71**

Capabilité Process non confirmée !

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Capabilité Procédé

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Usine	Dept./Achat./Prod. x	Nom NN	Date 28/10/00
Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Caract. carv1p6	Val. Nom. 20.85 0.35	TS 21.20
Mach.No.	Carc. N° 2500	Unité -0.35	TI 20.50
Type Carac. variable	Classe de Carac. significatif	Taille d'échant. 5	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument 0	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 10:44:32

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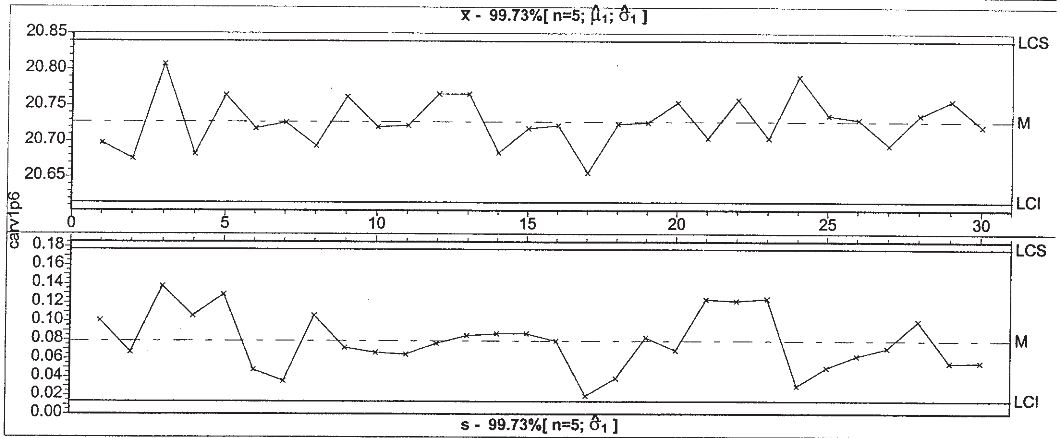

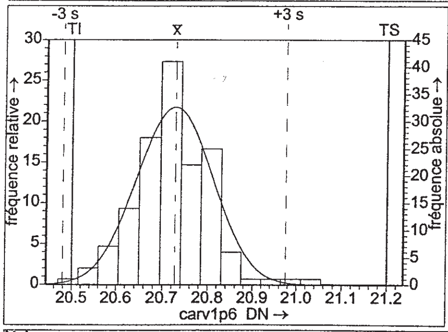

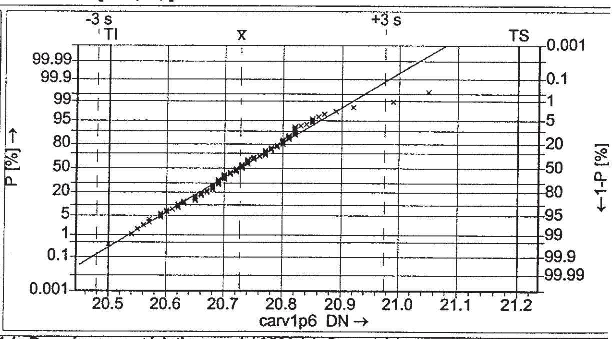

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.7271
TI	20.50	x_{min}	20.50	$\bar{x}$ - 3s	20.4789
TS	21.20	x_{max}	21.05	$\bar{x}$ + 3s	20.9753
Tolérance	0.70	R	0.55	6 · s	0.4964
		$n_{<T>}$	150	$n_{<T>}$	99.69779 %
		$n_{>TS}$	0	$p_{>TS}$	0.00000 %
		$n_{<TI}$	0	$p_{<TI}$	0.30221 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DN**	**Distr. Norm.**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x-99.865%) (1)**
Indice de capabilité	P_p	**1.41**
Indice de capabilité	P_{pk}	**0.92**

Capabilité Process non confirmée !

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Capabilité Procédé

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Usine	Dept./Achat./Prod.	Nom NN	Date 28/10/00
Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Carac. carv2p1	Val. Nom. 20.85	0.35 TS 21.20
Mach.No.	Carc. N° 2500	Unité	-0.35 TI 20.50
Type Carac. variable	Classe de Carac. significatif	Taille d'échant. 5	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument 0	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 12:04:00

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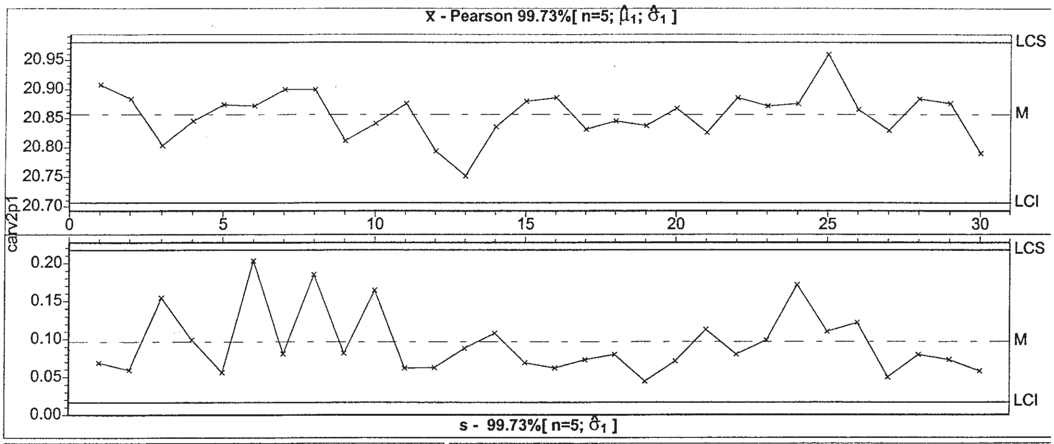

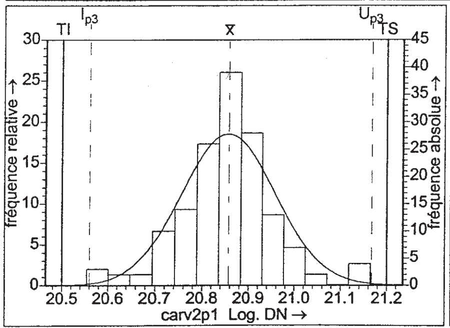

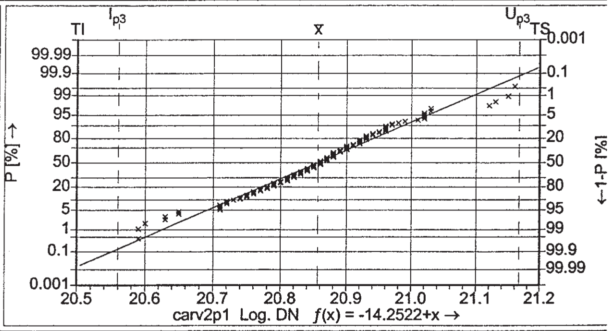

Valeurs		Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1			
Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85 [nt]			$\bar{x}$	20.8572 [nt]
TI	20.50 [nt]	x_{min}	20.59 [nt]	I_{p3}	20.5597 [rt]
TS	21.20 [nt]	x_{max}	21.16 [nt]	U_{p3}	21.1671 [rt]
Tolérance	0.70 [nt]	R	0.57 [nt]	$U_{p3} - I_{p3}$	0.6074 [rt]
		$n_{<T>}$	150	$n_{<T>}$	99.93857 %
		$n_{>TS}$	0	$p_{>TS}$	0.04673 %
		$n_{<TI}$	0	$p_{<TI}$	0.01470 %
		n_{tot}	150	n_{eff}	150
Modèle de distribution		**Log. DN**		**Distr. Log-Norm.**	
Transformation		**g(x)**		**f(x) = ln(a+x)**	
Méthode de calcul				**Percentile (0.135%-x̄-99.865%) (1)**	
Indice de capabilité		**P_p**		**1.15**	
Indice de capabilité		**P_{pk}**		**1.11**	

Capabilité Process non confirmée !

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Usine	Dept./Achat./Prod.	Nom NN	Date 28/10/00

Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Caract. carv2p2	Val. Nom. 20.85 / 0.35	TS 21.20
Mach.No.	Carc. N° 2500	Unité / -0.35	TI 20.50
Type Carac. variable	Classe de Carac. significatif	Taille d'échant. 5	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument 0	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 12:06:41

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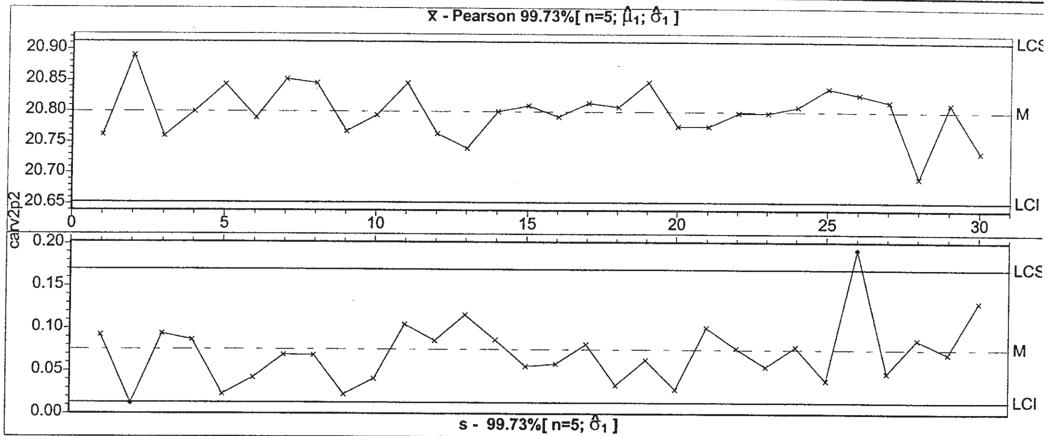

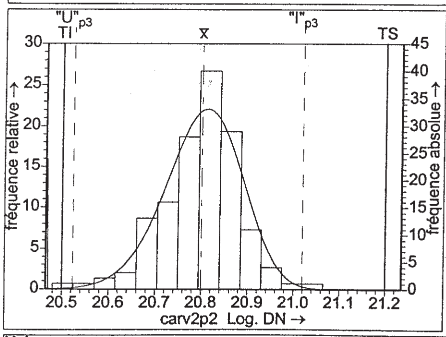

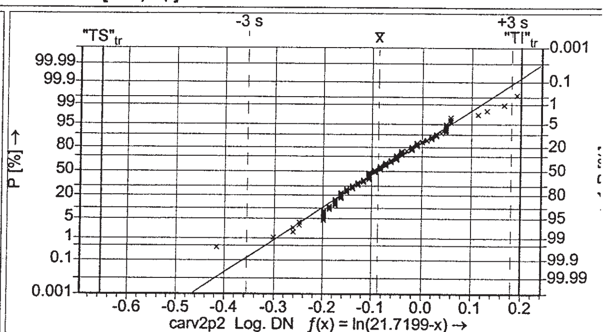

Valeurs — Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85 [nt]			$\bar{x}$	20.8001 [nt]
TI	20.50 [nt]	x_{min}	20.51 [nt]	l_{p3}	20.5228 [rt]
TS	21.20 [nt]	x_{max}	21.06 [nt]	U_{p3}	21.0188 [rt]
Tolérance	0.70 [nt]	R	0.55 [nt]	$U_{p3} - l_{p3}$	0.4959 [rt]
		$n_{<T>}$	150	$n_{<T>}$	99.93411 %
		$n_{>TS}$	0	$p_{>TS}$	0.00000 %
		$n_{<TI}$	0	$p_{<TI}$	0.06589 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**Log. DN**	**Distr. Log-Norm.**
Transformation	**g(x)**	**$f(x) = \ln(a-x)$**
Méthode de calcul		**Percentile (0.135%-X-99.865%) (1)**
Indice de capabilité	**P_p**	**1.41**
Indice de capabilité	**P_{pk}**	**1.08**

Capabilité Process non confirmée !

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Capabilité Procédé

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Usine	Dept./Achat./Prod. τ	Nom NN	Date 28/10/00

Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Caracarv2p3	Val. Nom20.85 0.35	TS 21.20
Mach.No.	Carc. N° 2500	Unité -0.35	TI 20.50
Type Carac. variable	Classe de Caracsignificatif	Taille d'échant.	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 12:19:09

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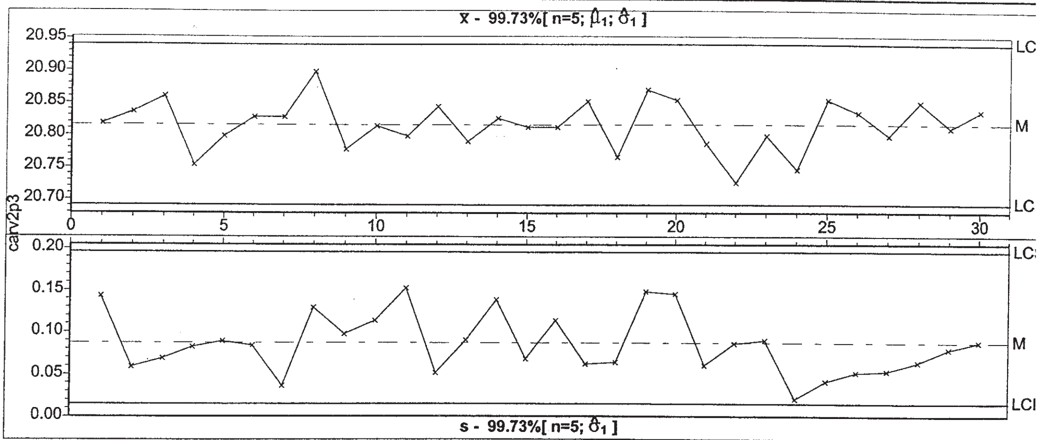

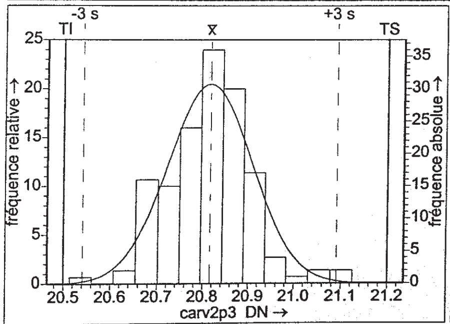

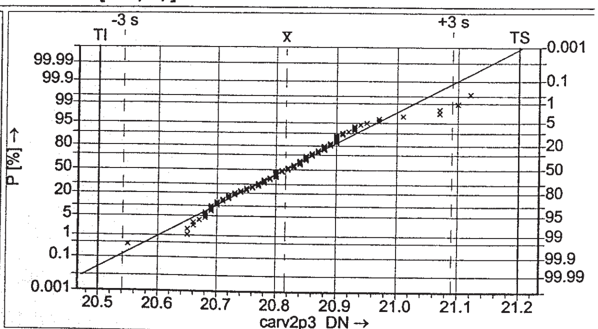

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.8158
TI	20.50	x_{min}	20.55	$\bar{x}$ - 3s	20.5411
TS	21.20	x_{max}	21.12	$\bar{x}$ + 3s	21.0905
Tolérance	0.70	R	0.57	6 · s	0.5494
		$n_{<T>}$	150	$n_{<T>}$	99.97049 %
		$n_{>TS}$	0	$p_{>TS}$	0.00136 %
		$n_{<TI}$	0	$p_{<TI}$	0.02815 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DN**	**Distr. Norm.**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x̄-99.865%) (1)**
Indice de capabilité	P_p	**1.27**
Indice de capabilité	P_{pk}	**1.15**

Capabilité Process non confirmée !

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Capabilité Procédé

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Usine	Dept./Achat./Prod.	Nom NN — Date 28/10/00

Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Caract. carv2p4	Val. Nom. 20.85	0.35 TS 21.20
Mach.No.	Carc. N° 2500	Unité	-0.35 TI 20.50
Type Carac. variable	Classe de Carac. significatif	Taille d'échant. 5	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument 0	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 12:21:51

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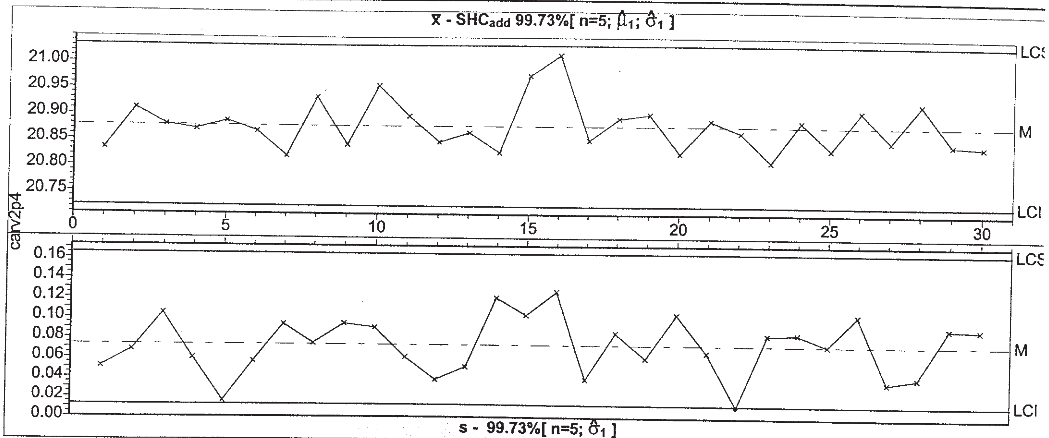

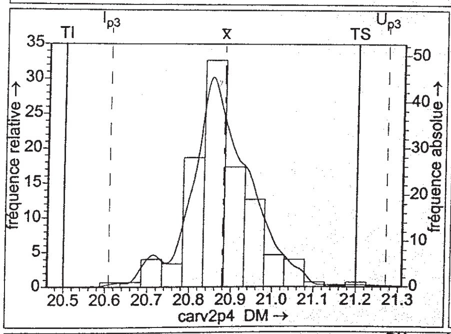

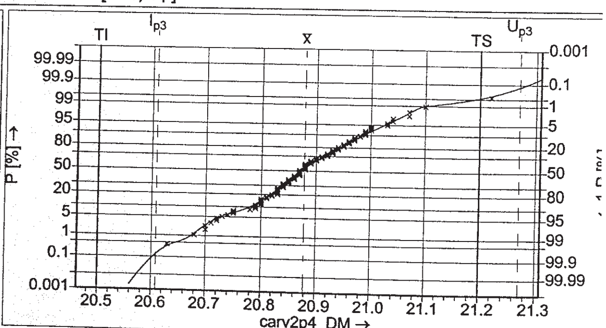

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85				
TI	20.50	x_{min}	20.63	$\bar{x}$	20.8788
TS	21.20	x_{max}	21.22	l_{p3}	20.6086
Tolérance	0.70	R	0.59	U_{p3}	21.2705
		$n_{<T>}$	149	$U_{p3} - l_{p3}$	0.6619
		$n_{>TS}$	1	$n_{<T>}$	99.54742 %
		$n_{<TI}$	0	$p_{>TS}$	0.45258 %
		n_{tot}	150	$p_{<TI}$	0.00000 %
				n_{eff}	150

Modèle de distribution	**DM**	**Distr. Mixe**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-$\bar{x}$-99.865%) (1)**
Indice de capabilité	T_p	**1.06**
Indice de capabilité	T_{pk}	**0.82**

Capabilité Process non confirmée !

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Capabilité Procédé

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Usine	Dept./Achat./Prod. τ	Nom NN	Date 28/10/00
Pièce N°	Descr. Pièce		Pt. Ctrl. n
Descr.Mach.	Descr. Caracarv2p6	Val. Nom20.85 0.35	TS 21.20
Mach.No.	Carc. N° 2500	Unité -0.35	TI 20.50
Type Carac. variable	Classe de Caracsignificatif	Taille d'échant.	Résolution 2
Descr. Type Prod.	Grp. d'Instruments	Descrip. Instrument	
Type de production	Instrument N°	Résol. Instrument	

Evaluation de 01/01/70 / 00:00:00 à 28/10/00 / 12:28:48

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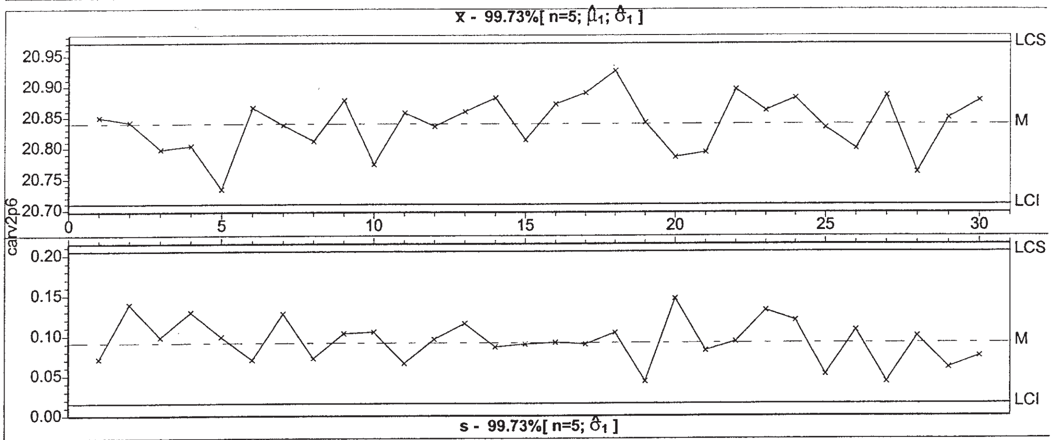

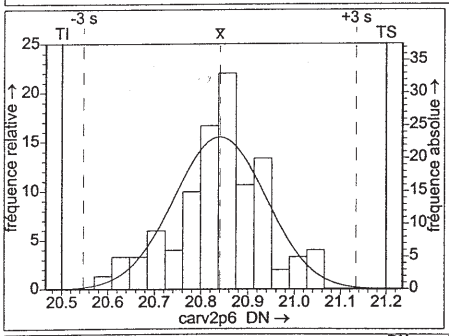

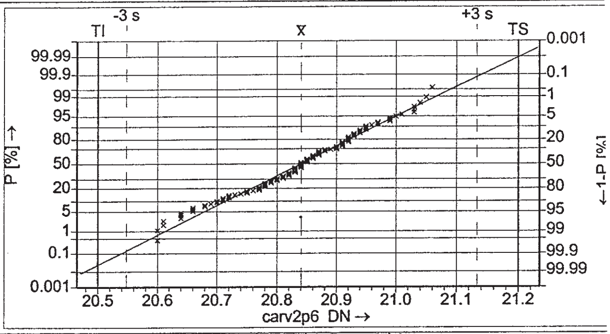

Valeurs — **Pièce : 1 / ; Données caractéristiques : 1 / 2500 / 1; Page 1 / 1**

Valeurs dessinées		Valeurs saisies		Statistiques	
Moyenne tolérance	20.85			$\bar{x}$	20.8402
TI	20.50	x_{min}	20.60	$\bar{x}$ - 3s	20.5472
TS	21.20	x_{max}	21.06	$\bar{x}$ + 3s	21.1332
Tolérance	0.70	R	0.46	6 · s	0.5860
		$n_{<T>}$	150	$n_{<T>}$	99.96371 %
		$n_{>TS}$	0	$p_{>TS}$	0.01150 %
		$n_{<TI}$	0	$p_{<TI}$	0.02479 %
		n_{tot}	150	n_{eff}	150

Modèle de distribution	**DN**	**Distr. Norm.**
Transformation	**g(x)**	
Méthode de calcul		**Percentile (0.135%-x̄-99.865%) (1)**
Indice de capabilité	P_p	**1.19**
Indice de capabilité	P_{pk}	**1.16**

Capabilité Process non confirmée !

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Appendix B

Proof of Lemma 2:

Notice that

$$NCDM^{p} = NCDM^{(p-1)}\frac{USL^{'} - r_p}{USL^{'} - C},$$

the general expression of $NCDM$ is proved by induction.

For $p = 2$ the desirability index is given by

$$NCDM^{2} = \frac{USL^{'2} + USL^{'}(-r_1 - r_2) + r_1 r_2}{(USL^{'} - C)^2}$$

it is supposed that for p quality characteristics the desirability index is given by

$$NCDM^{p} = \frac{\sum_{i=1}^{p-1}(USL^{'})^{(p-i)}(-1)^{i}[\sum_{j=1}^{p-i+1}\sum_{k=2}^{p-i+2}\ldots\sum_{m=l}^{p-i+l}\ldots\sum_{\substack{q=p-1\\ j<k<\ldots<m<\ldots<q}}^{p} r_j^{\mathbb{1}_{i,[1]}} r_k^{\mathbb{1}_{i,[2]}}\ldots r_m^{\mathbb{1}_{i,[l]}}\ldots r_q^{\mathbb{1}_{i,[p-1]}}]}{(USL^{'} - C)^{p}}$$

$$+\frac{USL^{'p} + (-1)^{p}\Pi_{i=1}^{p} r_i}{(USL^{'} - C)^{p}}.$$

In what follows it will be checked whether this expression is still true for $p+1$ quality characteristics.

$$NCDM^{p+1} = NCDM^{(p)}\frac{USL^{'} - r_{p+1}}{USL^{'} - C}$$

$$(USL^{'} - C)^{(p+1)}NCDM^{(p+1)} = (USL^{'} - r_{p+1})[USL^{'p} - USL^{'(p-1)}(r_1 + r_2 + \ldots + r_p) + USL^{'(p-2)}$$

$$(r_1 r_2 + r_1 r_3 + \ldots + r_1 r_p + r_2 r_3 + \ldots + r_2 r_p + \ldots + r_{p-1} r_p) + \ldots + USL^{'}(-1)^{(p-1)}$$

$$(r_1 r_2 r_3 \ldots r_{p-1} + r_1 r_2 r_4 \ldots r_p + \ldots + r_2 r_3 \ldots r_p) + (-1)^{p}\Pi_{i=1}^{p} r_i]$$

$$(USL^{'} - C)^{(p+1)}NCDM^{(p+1)} = USL^{'(p+1)} - r_{p+1}USL^{'p} - USL^{'p}(r_1 + r_2 + \ldots + r_p)$$

$$+r_{p+1}USL^{'(p-1)}(r_1 + r_2 + \ldots + r_p) + USL^{'(p-1)}(r_1 r_2 + r_1 r_3 + \ldots + r_1 r_p + r_2 r_3 + \ldots + r_2 r_p + \ldots + r_{p-1} r_p) + \ldots$$

$$+(-1)r_{p+1}USL^{'}(-1)^{(p-1)}(r_1 r_2 r_3 \ldots r_{p-1} + r_1 r_2 r_4 \ldots r_p + \ldots + r_2 r_3 \ldots r_p)$$

$$+USL^{'}(-1)^{p}\Pi_{i=1}^{p} r_i + (-1)r_{p+1}(-1)^{p}\Pi_{i=1}^{p} r_i$$

$$(USL^{'}-C)^{(p+1)}NCDM^{(p+1)} = USL^{'(p+1)} + (-1)^{(p+1)}\Pi_{i=1}^{p+1}r_i - USL^{'(p)}(r_1+r_2+\ldots+r_p+r_{p+1})$$

$$+USL^{'(p-1)}(r_1r_2+r_1r_3+\ldots+r_1r_p+r_1r_{p+1}+r_2r_3+\ldots+r_2r_p+r_2r_{p+1}+\ldots+r_{p-1}r_{p+1}+r_pr_{p+1})$$

$$+\ldots+USL^{'}(-1)^p(r_1r_2r_3\ldots r_{p-1}r_p+r_1r_2r_3\ldots r_{p-1}r_{p+1}+r_1r_2r_4\ldots r_pr_{p+1}+\ldots+r_2r_3\ldots r_pr_{p+1})$$

$$(USL^{'}-C)^{(p+1)}NCDM^{(p+1)} = USL^{'p+1} + (-1)^{(p+1)}\Pi_{i=1}^{(p+1)}r_i$$
$$+\sum_{i=1}^{p}(USL^{'})^{(p+1-i)}(-1)^i[\sum_{j=1}^{p-i+2}\sum_{k=2}^{p-i+3}\ldots\sum_{m=l}^{p+1-i+l}\ldots\sum_{\substack{q=p\\ j<k<\ldots<m<\ldots<q}}^{p+1} r_j^{\mathbb{1}_{i,[1]}}r_k^{\mathbb{1}_{i,[2]}}\ldots r_m^{\mathbb{1}_{i,[l]}}\ldots r_q^{\mathbb{1}_{i,[p]}}].$$

The expression is confirmed for $p+1$ quality characteristics □

Proof of Lemma 3:

Notice that the joint nonconformity ratio for p uncorrelated quality characteristics is given by

$$R_p = 1 - [(1-r_1)(1-r_2)\ldots(1-r_{p-1})(1-r_p)]$$

$$R_p = 1 - [(1-R_{p-1})(1-r_p)]$$

The general expression of $(-R_p)$ is obtained by induction.

p=4:

$$-R_4 = -r_1-r_2-r_3-r_4+r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4-r_1r_2r_3-r_1r_2r_4-r_2r_3r_4+r_1r_2r_3r_4$$

It is supposed that for p quality characteristics $(-R_p)$ is written as follows

$$-R_p = \sum_{i=1}^{p-1}(-1)^i[\sum_{j=1}^{p-i+1}\sum_{k=2}^{p-i+2}\ldots\sum_{m=l}^{p-i+l}\ldots\sum_{\substack{q=p-1\\ j<k<\ldots<m<\ldots<q}}^{p} r_j^{\mathbb{1}_{i,[1]}}r_k^{\mathbb{1}_{i,[2]}}\ldots r_m^{\mathbb{1}_{i,[l]}}\ldots r_q^{\mathbb{1}_{i,[p-1]}}]+(-1)^p\Pi_{i=1}^{p}r_i$$

In what follows it will be checked whether this expression is still true for $p+1$ quality characteristics.

$$R_{p+1} = 1 - [(1-R_p)(1-r_{p+1})]$$

$$R_{(p+1)} = 1 - [(1-r_1-r_2-\ldots-r_p+r_1r_2+r_1r_3+\ldots+r_1r_p+r_2r_3+\ldots+r_2r_p+\ldots+r_{p-1}r_p$$
$$+\ldots+(-1)^{(p-1)}(r_1r_2r_3\ldots r_{p-1}+r_1r_2r_4\ldots r_p+\ldots+r_2r_3\ldots r_p)+(-1)^p\Pi_{i=1}^{p}r_i)(1-r_{p+1})]$$

$$-R_{(p+1)} = -r_1 - r_2 - \ldots - r_p - r_{p+1} + r_1r_2 + r_1r_3 + \ldots + r_1r_p + r_1r_{p+1} + r_2r_3 + \ldots + r_2r_p + r_2r_{p+1}$$

$$+ \ldots + r_{p-1}r_p + r_pr_{p+1} + \ldots + (-1)^p(r_1r_2r_3 \ldots r_{p-1}r_p + r_1r_2r_4 \ldots r_pr_{p+1} + \ldots + r_2r_3 \ldots r_pr_{p+1})$$

$$+(-1)^{p+1}\Pi_{i=1}^{p+1}r_i$$

$$-R_{(p+1)} = \sum_{i=1}^{p}(-1)^i[\sum_{j=1}^{p-i+2}\sum_{k=2}^{p-i+3}\ldots\sum_{m=l}^{p+1-i+l}\ldots\sum_{\substack{q=p\\ j<k<\ldots<m<\ldots<q}}^{p+1} r_j^{\mathbb{1}_{i,[1]}}r_k^{\mathbb{1}_{i,[2]}}\ldots r_m^{\mathbb{1}_{i,[l]}}\ldots r_q^{\mathbb{1}_{i,[p]}}]+$$

$$(-1)^{(p+1)}\Pi_{i=1}^{(p+1)}r_i$$

The expression is confirmed for $p+1$ quality characteristics □

Proof of Lemma 4:

From Lemma 2 the general expression of $NCDM$ is given by

$$NCDM^p = \frac{\sum_{i=1}^{p-1}(USL')^{(p-i)}(-1)^i[\sum_{j=1}^{p-i+1}\sum_{k=2}^{p-i+2}\ldots\sum_{m=l}^{p-i+l}\ldots\sum_{\substack{q=p-1\\ j<k<\ldots<m<\ldots<q}}^{p} r_j^{\mathbb{1}_{i,[1]}}r_k^{\mathbb{1}_{i,[2]}}\ldots r_m^{\mathbb{1}_{i,[l]}}\ldots r_q^{\mathbb{1}_{i,[p-1]}}]}{(USL'-C)^p}$$

$$+\frac{USL'^p + (-1)^p\Pi_{i=1}^{p}r_i}{(USL'-C)^p}.$$

This means that

$$(USL^{'}-C)^pNCDM^p = USL^{'p} + (-1)^p\Pi_{i=1}^{p}r_i - USL^{'(p-1)}(r_1 + r_2 + \ldots + r_{p-1} + r_p)$$

$$+USL^{'(p-2)}(r_1r_2 + r_1r_3 + \ldots + r_1r_{p-1} + r_1r_p + r_2r_3 + \ldots + r_2r_{p-1} + r_2r_p + \ldots + r_{p-2}r_{p-1} + r_{p-1}r_p)$$

$$+ \ldots + USL^{'}(-1)^{p-1}(r_1r_2r_3 \ldots r_{p-2}r_{p-1} + r_1r_2r_3 \ldots r_{p-2}r_p + r_1r_2r_4 \ldots r_{p-1}r_p + \ldots + r_2r_3 \ldots r_{p-1}r_p)$$

$NCDM$ could be written as follows

$$(USL^{'}-C)^pNCDM^p = USL^{'p} + (-1)^p\Pi_{i=1}^{p}r_i + USL^{'(p-1)}(-r_1 - r_2 - \ldots - r_{p-1} - r_p + r_1r_2 + r_1r_3 +$$

$$\ldots + r_1r_{p-1} + r_1r_p + r_2r_3 + \ldots + r_2r_{p-1} + r_2r_p + \ldots + r_{p-2}r_{p-1} + r_{p-1}r_p + \ldots$$

$$+(-1)^{p-1}(r_1r_2r_3 \ldots r_{p-2}r_{p-1} + r_1r_2r_4 \ldots r_{p-1}r_p + \ldots + r_2r_3 \ldots r_{p-1}r_p) + (-1)^p\Pi_{i=1}^{p}r_i$$

$$-(r_1r_2+r_1r_3+\ldots+r_1r_{p-1}+r_1r_p+r_2r_3+\ldots+r_2r_{p-1}+r_2r_p+\ldots+r_{p-2}r_{p-1}+r_{p-1}r_p+\ldots$$
$$+(-1)^{p-1}(r_1r_2r_3\ldots r_{p-2}r_{p-1}+r_1r_2r_4\ldots r_{p-1}r_p+\ldots+r_2r_3\ldots r_{p-1}r_p)+(-1)^p\Pi_{i=1}^p r_i))$$

$$+\ldots+USL^{'}(-1)^{p-1}(-r_1-r_2-\ldots-r_{p-1}-r_p+r_1r_2+r_1r_3+\ldots+r_1r_{p-1}+r_1r_p+r_2r_3+\ldots$$
$$+r_2r_{p-1}+r_2r_p+\ldots+r_{p-2}r_{p-1}+r_{p-1}r_p+\ldots+(-1)^{p-1}(r_1r_2r_3\ldots r_{p-2}r_{p-1}+r_1r_2r_4\ldots r_{p-1}r_p+\ldots$$
$$+r_2r_3\ldots r_{p-1}r_p)+(-1)^p\Pi_{i=1}^p r_i-(-r_1-r_2-\ldots-r_{p-1}-r_p+r_1r_2+r_1r_3+\ldots+r_1r_{p-1}+r_1r_p$$
$$+r_2r_3+\ldots+r_2r_{p-1}+r_2r_p+\ldots+r_{p-2}r_{p-1}+r_{p-1}r_p+\ldots$$
$$+(-1)^{p-2}(r_1r_2r_3\ldots r_{p-3}r_{p-2}+r_1r_2r_4\ldots r_{p-2}r_{p-1}+\ldots+r_3r_4\ldots r_{p-1}r_p)+(-1)^p\Pi_{i=1}^p r_i))$$

Using the expression of the joint nonconformity ratio in Lemma 3, the expression of $NCDM$ is then given by

$$(USL^{'}-C)^pNCDM^p=USL^{'p}+(-1)^p\Pi_{i=1}^p r_i+USL^{'(p-1)}(-R_p-(r_1r_2+r_1r_3+\ldots+r_1r_{p-1}+r_1r_p$$
$$+r_2r_3+\ldots+r_2r_{p-1}+r_2r_p+\ldots+r_{p-2}r_{p-1}+r_{p-1}r_p+\ldots+(-1)^{p-1}(r_1r_2r_3\ldots r_{p-2}r_{p-1}$$
$$+(r_1r_2r_4\ldots r_{p-1}r_p)+\ldots+r_2r_3\ldots r_{p-1}r_p)+(-1)^p\Pi_{i=1}^p r_i)$$

$$+\ldots+USL^{'}(-1)^{p-1}(-R_p-(-r_1-r_2-\ldots-r_{p-1}-r_p+r_1r_2+r_1r_3+\ldots+r_1r_{p-1}+r_1r_p+r_2r_3$$
$$+\ldots+r_2r_{p-1}+r_2r_p+\ldots+r_{p-2}r_{p-1}+r_{p-1}r_p+\ldots$$
$$+(-1)^{p-2}(r_1r_2r_3\ldots r_{p-3}r_{p-2}+r_1r_2r_4\ldots r_{p-2}r_{p-1}+\ldots+r_3r_4\ldots r_{p-1}r_p)+(-1)^p\Pi_{i=1}^p r_i))$$

$$(USL^{'}-C)^pNCDM^p=USL^{'p}+(-1)^p\Pi_{i=1}^p r_i$$
$$\sum_{i=1}^{p-1}USL^{'(p-i)}(-(\sum_{u=1_{u\neq i}}^{p-1}(-1)^u[\sum_{j=1}^{p-u+1}\sum_{k=2}^{p-u+2}\ldots\sum_{m=l}^{p-u+l}\ldots\sum_{\substack{q=p-1\\ j<k<\ldots<m<\ldots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}}r_k^{\mathbb{1}_{u,[2]}}\ldots r_m^{\mathbb{1}_{u,[l]}}\ldots r_q^{\mathbb{1}_{u,[p-1]}}])$$
$$-(-1)^p\Pi_{i=1}^p r_i-R_p).$$

$\square$

Proof of Lemma 5

In Lemma 4 it is proved that

$$(USL^{'}-C)^pNCDM^p = USL^{'p} + (-1)^p\Pi_{i=1}^{p} r_i$$
$$+\sum_{i=1}^{p-1} USL^{'(p-i)}(-(\sum_{u=1_{u\neq i}}^{p-1}(-1)^u[\sum_{j=1}^{p-u+1}\sum_{k=2}^{p-u+2}\dots\sum_{m=l}^{p-u+l}\dots\sum_{\substack{q=p-1\\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}}\dots r_m^{\mathbb{1}_{u,[l]}}\dots r_q^{\mathbb{1}_{u,[p-1]}}])$$
$$-(-1)^p\Pi_{i=1}^{p} r_i - R_p).$$

This means that

$$(USL^{'}-C)^pNCDM^p = USL^{'p} + (-1)^p\Pi_{i=1}^{p} r_i$$
$$+\sum_{i=1}^{p-1} USL^{'(p-i)}(-(\sum_{u=1_{u\neq i}}^{p-1}(-1)^u[\sum_{j=1}^{p-u+1}\sum_{k=2}^{p-u+2}\dots\sum_{m=l}^{p-u+l}\dots\sum_{\substack{q=p-1\\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}}\dots r_m^{\mathbb{1}_{u,[l]}}\dots r_q^{\mathbb{1}_{u,[p-1]}}])$$
$$-(-1)^p\Pi_{i=1}^{p} r_i) - R_p\sum_{i=1}^{p-1} USL^{'(p-i)}.$$

Hence,

$$R_p\sum_{i=1}^{p-1} USL^{'(p-i)} = -NCDM^p(USL^{'}-C)^p + USL^{'p} + \sum_{i=1}^{p-1} USL^{'(p-i)}$$
$$(-(\sum_{u=1_{u\neq i}}^{p-1}(-1)^u[\sum_{j=1}^{p-u+1}\sum_{k=2}^{p-u+2}\dots\sum_{m=l}^{p-u+l}\dots\sum_{\substack{q=p-1\\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}}\dots r_m^{\mathbb{1}_{u,[l]}}\dots r_q^{\mathbb{1}_{u,[p-1]}}])$$
$$-(-1)^p\Pi_{i=1}^{p} r_i) + (-1)^p\Pi_{i=1}^{p} r_i.$$

$\square$

Proof of Theorem 1

Theorem 1 concerns two components, the desirability index and the joint nonconformity ratio for uncorrelated quality characteristics. The general expression of both components is proved by induction in Lemmas 2-4.

Following Lemma 5, the joint nonconformity ratio for uncorrelated quality characteristics is expressed as follows:

$$R_p \sum_{i=1}^{p-1} USL^{'(p-i)} = -NCDM^p (USL^{'} - C)^p + USL^{'p} + \sum_{i=1}^{p-1} USL^{'(p-i)}$$

$$(-(\sum_{u=1_{u \neq i}}^{p-1} (-1)^u [\sum_{j=1}^{p-u+1} \sum_{k=2}^{p-u+2} \dots \sum_{m=l}^{p-u+l} \dots \sum_{\substack{q=p-1 \\ j<k<\dots<m<\dots<q}}^{p} r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}} \dots r_m^{\mathbb{1}_{u,[l]}} \dots r_q^{\mathbb{1}_{u,[p-1]}}])$$

$$-(-1)^p \Pi_{i=1}^p r_i) + (-1)^p \Pi_{i=1}^p r_i.$$

However, the capability is confirmed only if $R_p \leq USL^{'}$ and this means that the capability is confirmed if

$$(USL^{'} - C)^p NCDM^p \geq \sum_{i=1}^{p-1} USL^{'(p-i)} (-(\sum_{u=1_{u \neq i}}^{p-1} (-1)^u [\sum_{j=1}^{p-u+1} \sum_{k=2}^{p-u+2} \dots \sum_{m=l}^{p-u+l} \dots \sum_{\substack{q=p-1 \\ j<k<\dots<m<\dots<q}}^{p}$$

$$r_j^{\mathbb{1}_{u,[1]}} r_k^{\mathbb{1}_{u,[2]}} \dots r_m^{\mathbb{1}_{u,[l]}} \dots r_q^{\mathbb{1}_{u,[p-1]}}]) - (-1)^p \Pi_{i=1}^p r_i - USL^{'}) + USL^{'p} + (-1)^p \Pi_{i=1}^p r_i.$$

□

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Declaration

Hereby I declare that the available thesis represents an independent research achievement, and that I used no different than the quoted sources and aids.