On Theory of Information Functions Presence

Magda Tsintsadze

Dortmund University LS1
Iv.Javakhishvili Tbilisi State University

Dortmund 2006

On Theory of Information Functions Presence

The determination of the uncertainty nature is the main task of the initial information processing. In computer science the data is so-called set of information units [1]. Each information unit represents the four: (object, sign, value, plausibility). The origin of fuzziness might be in all components of the four. The fuzzy notion corresponding to the object is usually constructed either from value or plausibility. We offer Information Function Presentation of these complementary fuzzy subsets [2].

Suppose, $\tilde{A}$ fuzzy subset of Ω Universal set corresponds to A concept and suppose this concept is characterized by numerical parameter ξ. Consider ξ is quantitatively characterizing some property of $\tilde{A}$ - let's call it "color" $\wp$.

Main definition: The numerical characteristic of color $\xi(\wp_{\tilde{A}}[\omega])$ is a random quantity. Define appropriate distribution density of probabilities by $\rho_{\wp}(x;\omega)$.

Denote

$$x_{\omega}^{*} = M\xi\left(\wp_{\tilde{A}}[\omega]\right) = \int_{\Re} x\rho_{\wp}(x;\omega)dx \qquad (1^*)$$

as Calculated value of $\tilde{A}$ fuzzy subset membership function's modal value.

Except of $M\xi$, presence of color to ω is characterized by dispersion also:

$$\sigma_{\wp}^{2}(\omega) = \int_{\Re} \left(x - x_{\omega}^{*}\right)^{2} \rho_{\wp}(x;\omega)dx \qquad (2^*)$$

In our model exactly $\sigma_{\wp}^{2}(\omega)$ is connected with definition of presence $\wp$ color to ω. If $\sigma_{\wp}^{2}(\omega) \to 0$, we'll say $\wp$ has quite define value ω. The more $\sigma_{\wp}^{2}(\omega)$ is, the uncertain $\wp$ in ω. If $\sigma_{\wp}^{2}(\omega) \to \infty$ it means ω has no $\wp$ color.

suppose, $\chi_{\tilde{A}}(\omega), \omega \in \Omega$ denotes $\tilde{A}$ fuzzy subset's appropriate membership function.

note 1. Lets call expression

$$\sqrt{\rho_{\tilde{A}}(x;\omega)} \equiv \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle \qquad (1)$$

Information function.

Here we are using Dirac's nomenclature [3]. We need this function to represent the information in $\tilde{A}$ concept. Information function's magnitude square determines membership function (precisely the appropriate density):

$$\rho_{\tilde{A}}(x;\omega) = \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle \qquad (2)$$

Any $\tilde{A}$ fuzzy subset might be defined separate from some value's hidden parameters. Call such values (by Dirac nomenclature) ket-vector and denote as $\left| \tilde{A} \right\rangle$.

We may sum ket-vectors, also product ket-vectors as on scalar also on complex values – and receive ket-vectors again.

Suppose, $\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle \in L^{2}(\Re)$ (Hilbert space), consider Fourier transformation of this function:

$$F\left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle = \frac{1}{2\pi c} \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_{c}} dx \qquad (3)$$

where c is const. (3) expression is equal of information function at x_c presentation:

$$\hat{F}\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle = \left\langle x_{c}, x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle \qquad (4)$$

where $\tilde{A}^c$ is canonically conjugate fuzzy subset:

$$\chi_{\wp c}(\omega) = \int_{I_{\wp c}(\omega)} \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle^+ \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle dx_c =$$

$$= \int_{\Re} I_{I_{\wp c}(\omega)}(x_x) \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle^+ \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle dx_c \qquad (5)$$

At information function space $\left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle$, $\hat{\wp}$ operator is appropriate of $\wp$ color. If information about color is precise, than

$$\hat{\wp} \left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle = x \left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle \qquad (6)$$

analogically

$$\hat{\wp}_c \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle = x_c \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle \qquad (7)$$

Theorem1. Suppose $\left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle$ and $\frac{d}{dx} \left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle \in L^2(\Re)$, $\left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle = \hat{F} \left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle$, than the following expression is valid for $\hat{\wp}$ and $\hat{\wp}^c$ operators:

$$\hat{\wp}_c \left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle = -ic \frac{d}{dx} \left\langle x; x_\omega^* \middle| \tilde{A} \right\rangle \qquad (8)$$

And analogically

$$\hat{\wp} \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle = ic \frac{d}{dx_c} \left\langle x_c ; x_{c\omega}^* \middle| \tilde{A}^c \right\rangle \qquad (9)$$

Note that by (4),(5) and (2),(1*):

$$x_{\omega}^{*} = \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \hat{\wp}_{c} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle dx \qquad (10)$$

$$x_{c\omega}^{*} = \int_{\Re} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle^{+} \hat{\wp} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle dx_{c} \qquad (11)$$

Consider (11) , let us show that the equality is true:

$$x_{c\omega}^{*} = \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left(-ic \frac{d}{dx} \right) \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle dx_{c} =$$

$$= \int_{\Re} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle \hat{\wp}_{c} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle dx_{c} \qquad (12)$$

We have:

$$x_{c\omega}^{*} = \int_{\Re} dx_{c} \left[\frac{1}{\sqrt{2\pi c}} \int_{\Re} dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} x} \right] \hat{\wp}_{c} \left[\frac{1}{\sqrt{2\pi c}} \int_{\Re} dx' \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_{c}} \right] =$$

$$= \int_{\Re} dx_{c} \left\langle x_{c}; x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle x_{c} \frac{1}{\sqrt{2\pi c}} \int_{\Re} dx' \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{-\frac{i}{c} x x_{c}} =$$

$$= \frac{1}{2\pi c} \int_{\Re} \int_{\Re} \int_{\Re} dx_{c} dx dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+'} \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} x} \left(ic \frac{d}{dx'} e^{-\frac{i}{c} x_{c} x'} \right) =$$

$$= \frac{1}{2\pi c} \int_{\Re} \int_{\Re} \int_{\Re} dx_{c} dx dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle ic \frac{d}{dx'} e^{\frac{i}{c} x_{c} (x - x')} =$$

$$= \frac{i}{2\pi} \int_{\Re} dx_{c} \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left[\left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_{c} (x - x')} \Big|_{+\infty}^{-\infty} - \right.$$

$$-\int_{\Re} dx' \frac{d}{dx'}\left\langle x'; x_{\omega}^{*} \middle| \tilde{A} \right\rangle e^{\frac{i}{c} x_c (x - x')} \Bigg] =$$

$$= -\frac{i}{2\pi} \int_{\Re} dx \int_{\Re} dx' \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \int_{\Re} dx_c e^{\frac{i}{c} x_c (x - x')} =$$

$$= \int_{\Re} dx \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle^{+} \left(-ic \frac{d}{dx} \right) \left\langle x; x_{\omega}^{*} \middle| \tilde{A} \right\rangle$$

Note 2. The following are valid:

$$x_{\omega}^{*} = \left(\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle, \hat{\wp} \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle \right) =$$

$$= \left(\left\langle x_c, x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle, \hat{\wp} \left\langle x_c, x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle \right) \qquad (13)$$

$$x_{c\omega}^{*} = \left(\left\langle x_c, x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle, \hat{\wp}_c \left\langle x_c, x_{c\omega}^{*} \middle| \tilde{A}^{c} \right\rangle \right) =$$

$$= \left(\left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle, \hat{\wp}_c \left\langle x, x_{\omega}^{*} \middle| \tilde{A} \right\rangle \right) \qquad (14)$$

The proofs of these equalities can be done by the same way, so we aren't considering them here.

Theorem 2. Operators $\hat{\wp}$ and $\hat{\wp}_c$ are satisfying the following condition:

$$\hat{\wp}\,\hat{\wp}_c - \hat{\wp}_c \hat{\wp} = ic\hat{E} \qquad (15)$$

where $\hat{E}$ - identity operator.

Proof. : Suppose $\hat{\wp} f(x) = x f(x)$, $f(x)$, $x f(x)$ and $f'(x) \in L^2(\Re)$, then

$$\left(\hat{\wp}\,\hat{\wp}_C - \hat{\wp}_C\hat{\wp}\right)f(x) = \hat{\wp}\left(\hat{\wp}_C\, f(x)\right) - \hat{\wp}_C\left(\hat{\wp} f(x)\right) =$$

$$= -icx\frac{df(x)}{dx} + ic\frac{d}{dx}\left(xf(x)\right) = ic\hat{E}f(x)$$

Here we have used the following facts:

$$\hat{\wp}\left(\hat{\wp}_C\, f\right) = x\left(\hat{\wp}_C\, f\right) \quad \text{and} \quad \hat{\wp}_C\left(\hat{\wp} f\right) = -ic\frac{d}{dx}\left(\hat{\wp} f\right), \quad x \in \Re$$

according to: $\left(f, \hat{\wp}\,\hat{\wp}_C\, f\right) = \left(\hat{\wp} f, \hat{\wp}_C\, f\right) = \left(xf, \hat{\wp}_C\, f\right) = \left(f, x\hat{\wp}_C\, f\right)$

Connection between canonically conjugated colors

Connection between canonically conjugated colors is given with the following theorem:

Theorem 3. if $\wp$ and $\wp_C$ are canonically conjugated colors, then

$$\sigma^2_{\wp}\left(x^*_{\omega}\right)\sigma^2_{\wp_C}\left(x^*_{C\omega}\right) \geq \frac{c^2}{4} \qquad (16)$$

Proof. Let's enter designations

$$\hat{\alpha} \equiv \hat{\wp} - x^*_{\omega}\hat{E} \quad , \quad \hat{\beta} \equiv \hat{\wp}_C - x^* c\omega\hat{E} \qquad (17)$$

appropriately

$$\sigma^2_{\wp}\left(x^*_{\omega}\right) = \left\langle \hat{\alpha}^2 \right\rangle = \left(\left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle, \hat{\alpha}^2 \left\langle x, x^*_{\omega} \middle| \tilde{A} \right\rangle\right)$$

(18)

$$\sigma^2_{\wp_C}\left(x^*_{c\omega}\right)=\left\langle \hat\beta^2\right\rangle=\left(\left\langle x,x^*_\omega\middle|\tilde A\right\rangle,\hat\beta^2\left\langle x,x^*_\omega\middle|\tilde A\right\rangle\right)$$

We have

$$\sigma^2_{\wp}\left(x^*_\omega\right)\sigma^2_{\wp_C}\left(x^*_{c\omega}\right)=$$

$$=\int_{\Re}\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\hat\alpha^2\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\int_{\Re}\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\hat\beta^2\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx=$$

$$1=\int_{\Re}\hat\alpha^+\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\hat\alpha\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\int_{\Re}\hat\beta^+\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\hat\beta\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx \qquad (19)$$

Using Cauchy-Buniakovski inequality:

$$\int\left|f(x)\right|^2dx\int\left|g(x)\right|^2dx\geq\left|\int f(x)g(x)dx\right|^2 \qquad (20)$$

And suppose that:

$$\hat\alpha\left\langle x,x^*_\omega\middle|\tilde A\right\rangle\equiv f(x) \quad \text{and} \quad \hat\beta\left\langle x,x^*_\omega\middle|\tilde A\right\rangle\equiv g(x) \quad ,$$

We will have:

$$\sigma^2_{\wp}\left(x^*_\omega\right)\sigma^2_{\wp_C}\left(x^*_{c\omega}\right)\geq\left|\int_{\Re}\hat\alpha^+\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\hat\beta\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\right|^2=$$

$$=\left|\int_{\Re}\left\langle x,x^*_\omega\middle|\tilde A\right\rangle\hat\alpha\hat\beta\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\right|^2=$$

$$=\left|\int_{\Re}\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\left[\frac{1}{2}\left(\hat\alpha\hat\beta+\hat\beta\hat\alpha\right)+\frac{1}{2}\left(\hat\alpha\hat\beta-\hat\beta\hat\alpha\right)\right]\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\right|^2=$$

(21)

$$=\frac{1}{4}\left|\int_{\Re}\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\left(\hat\alpha\hat\beta+\hat\beta\hat\alpha\right)\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\right|^2+\frac{1}{4}\left|\int_{\Re}\left\langle x,x^*_\omega\middle|\tilde A\right\rangle^+\left(\hat\alpha\hat\beta-\hat\beta\hat\alpha\right)\left\langle x,x^*_\omega\middle|\tilde A\right\rangle dx\right|^2$$

Missed member is equal to 0, because $\hat\alpha^+=\hat\alpha$ and $\hat\alpha\hat\beta-\hat\beta\hat\alpha=ic\hat E$.

So using (17):

$$\left(\hat{\alpha}\hat{\beta}-\hat{\beta}\hat{\alpha}\right)\left\langle x, x_{\omega}^{*}\middle|\tilde{A}\right\rangle = -ic\left[x\frac{d}{dx}\left\langle x, x_{\omega}^{*}\middle|\tilde{A}\right\rangle - \frac{d}{dx}\left(\left\langle x, x_{\omega}^{*}\middle|\tilde{A}\right\rangle\right)\right] = ic\left\langle x, x_{\omega}^{*}\middle|\tilde{A}\right\rangle \qquad (22)$$

so, if at right side of equality (21) we'll ignore second summary (which ≥ 0) finally receive (16) .

As we see canonically conjugated colors are representing complementary fuzzy subsets, which will lead us to optimal fuzzy subset presentation [4], thus giving possibility to take into account information optimally.

This work was supported by the INTAS Grant Nr. 04-83-2971

Dortmund University LS1

References

1. **D.Dubois. H.A. Prade. Teoria Vozmojnostei. M.,1987 (Russian).**
2. **M.Tsintsadze. Proceedings of Iv.Javakhishvili Tbilisi State University, 365(24), 75-84. 2005, (Georgian).**
3. **A.Messia – Kvantovaia mexanika. M., (1982), (Russia)**
4. **M.Tsintsadze. Bulletin of the Georgian Academy of Sciences. Vol.171(3)(2005). Tbilisi**