Well-posedness results for dispersive equations with derivative nonlinearities

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Contents

Introduction

The present work is devoted to the study of Cauchy problems for nonlinear evolution equations with initial data in Sobolev spaces of low regularity which describe the propagation of nonlinear dispersive waves.

We are interested in a well-posedness theory for these problems, i.e. for given initial data we try to find

(i) unique

(ii) solutions

(iii) whose initial regularity persists

(iv) and which depend continuously on the initial data.

We are challenged to prove results with regularity assumptions on the initial data which are as weak as possible[1]. It is part of the problem to find an adequate way to express all these four aims precisely and consistently in a low regularity context.

The examples discussed here arise as one-dimensional model equations for nonlinear wave propagation in water wave theory (Benjamin-Ono type equations) or plasma physics (derivative nonlinear Schrödinger equation).

In order to introduce the principle of *dispersion*[2] let us consider the linear equation

$$\partial_t u + \partial_x^3 u = 0 \quad \text{(Airy)}$$

We may calculate explicit solutions $u : [-T, T] \times \mathbb{R} \to \mathbb{R}$ with the help of Fourier analysis: Let the periodic initial datum be given by $u_0(x) = \sum c_k e^{ikx}$, then the periodic solution is

$$u(t,x) = \sum c_k e^{i(kx+tk^3)}$$

[1]On the Sobolev scale H^s this means that we try to choose $s \in \mathbb{R}$ as small as possible.

[2]In the present work we focus on the analytical effects of special dispersion relations and do not give a formal definition of dispersive waves in general cp. [Whi74], pp.363–369.

which shows that the k-th Fourier mode of the initial datum propagates with group velocity $3k^2$. The different speed of Fourier modes has certain regularizing effects such as higher integrability and in the nonperiodic case the gain of fractional derivatives in L^2_{loc}. On the other hand, for each $t > 0$ we observe $\|u(t)\|_{H^s} = \|u_0\|_{H^s}$ and the solution operator is unitary in the Sobolev spaces H^s such that the solution has exactly the same regularity as the initial datum in the H^s sense.

Let us consider a nonlinear version of this equation, the Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u = \partial_x u^2 \quad \text{(KdV)}$$

One idea to establish well-posedness for nonlinear equations is to apply the Picard iteration scheme to a related integral equation as in the case of ordinary differential equations. The nonlinearity has to be small in a suitable sense such that the Duhamel term is a strict contraction and its influence on the linear solution is not too strong. To control all possible nonlinear interactions in the quadratic term and to gain the derivative one may exploit the above mentioned dispersive properties of the linear equation. This strategy was used by C.E. Kenig - G. Ponce - L. Vega [KPV93c] to prove well-posedness results for (generalized) KdV equations in the non-periodic case. In [Bou93] J. Bourgain developed a general approach to the well-posedness of nonlinear dispersive equations which reduces the problem to multi-linear estimates in spaces which are defined according to the symbol of the linear equation and inherit the dispersive properties of the solutions.

These harmonic analysis techniques in combination with the Picard iteration found applications in many different situations and lead to strong well-posedness results in spaces of low regularity. All of them share the property that the flow map (data upon solution) is necessarily analytic[3].

Later, it was observed that there are many interesting equations where a smooth dependence on the data or at least multi-linear estimates fail to hold [MST01, KT05b] even for regular data, although there are well-posedness results which include the continuous dependence on the data and which are based on energy type arguments.

Another approach to many nonlinear dispersive equations, such as the Korteweg-de Vries, the Benjamin-Ono or the derivative nonlinear Schrödinger equation is provided by the inverse scattering theory, cp. [AC91]. However, the results in the present work do not rely on inverse scattering techniques.

Here, we are particularly interested in situations where standard multi-linear estimates for the nonlinear term cannot be true and a direct approach

[3]This holds true if the nonlinear terms are analytic.

via the Picard iteration is not applicable. Our aim is to overcome these difficulties by identifying the strongest interactions and modifying the method accordingly.

In Chapter 1 the basic notation and a notion of well-posedness is introduced. In Chapter 2 the dispersive properties of solutions to the linear equations

$$(\partial_t - |D|^\alpha \partial_x) u = 0$$

in the non-periodic and

$$(\partial_t - i\partial_x^2) u = 0$$

in the periodic setting as well as related function spaces are discussed.

In Chapter 3 derivative nonlinear Schrödinger equations in the periodic setting are considered, in particular

$$\begin{aligned} \partial_t u(t) - i\partial_x^2 u(t) &= \partial_x(|u|^2 u)(t) \quad \text{for } t \in (-T,T) \\ u(0) &= u_0 \end{aligned}$$

A local well-posedness result for initial data in $H^s(\mathbb{T})$ for all $s \geq \frac{1}{2}$ is proved which extends to global well-posedness for $s \geq 1$ and data which satisfies a L^2 smallness condition, cp. [Her05a]. A detailed uniqueness statement is given and it is shown that the flow map is not uniformly continuous on balls in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$, but locally Lipschitz on subsets of data with fixed L^2 norm. Similarly, for a version of this equation with a regularized nonlinear term well-posedness follows with real analytic dependence on the intial data. The results are shown to be sharp in certain directions.

In Chapter 4 equations of Benjamin-Ono type[4]

$$\begin{aligned} \partial_t u(t) - |D|^\alpha \partial_x u(t) + \frac{1}{2}\partial_x u^2(t) &= 0 \quad \text{for } t \in (-T,T) \\ u(0) &= u_0 \end{aligned}$$

are studied. Section 4.2 deals with the cases $1 < \alpha < 2$ in the non-periodic setting. Local well-posedness for initial data in spaces $H^{(s,\omega)}(\mathbb{R})$ for $s > -\frac{3}{4}(\alpha-1)$ and $\omega = \frac{1}{\alpha} - \frac{1}{2}$ and global well-posedness for real valued data in the range $s \geq 0$ and $\omega = \frac{1}{\alpha} - \frac{1}{2}$ is shown, cp. [Her05b]. These spaces correspond to the usual Sobolev spaces $H^s(\mathbb{R})$ with an additional low frequency condition $\dot{H}^{-\omega}(\mathbb{R})$. The result includes the analyticity of the flow map, which fails under a weaker low frequency assumption. A smoothing property is used to prove that the nonlinear equation is satisfied in the sense of distributions.

[4]For $\alpha = 1$ this is the Benjamin-Ono equation and for $\alpha = 2$ this is the Korteweg-de Vries equation. In the literature, these equations are known as dispersion generalized Benjamin-Ono equations.

In Section 4.3 counterexamples are constructed which prove the failure of bilinear estimates related to the Benjamin-Ono equation ($\alpha = 1$) in the periodic case for real valued functions with zero mean value. This complements a recent well-posedness result of L. Molinet [Mol06].

In Section 4.4 it is remarked that in the case $0 < \alpha < 1$ a slight modification of the arguments of H. Koch - N. Tzvetkov [KT03b] for $\alpha = 1$ also leads to well-posedness in this range.

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Chapter 1

Cauchy problems and well-posedness

1.1 Basic function spaces

In this section we introduce some well-known function spaces and the Fourier transformation in order to fix notation.

We will both study problems involving functions (or distributions) on the real line being either spatially periodic or non-periodic. Let $C^\infty(\mathbb{R}^n)$ be the linear space of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ and for $u \in C^\infty(\mathbb{R}^n)$ we define the semi-norms

$$[u]_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)|$$

for all (multi-)indices $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$.

We start with defining Schwartz functions.

Definition 1.1.1. We define the Fréchet space of smooth, rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) \mid [u]_{k,\mathbb{R}^n} := \max_{|\alpha|,|\beta| \leq k} [u]_{\alpha,\beta} < \infty,\ k \in \mathbb{N}_0\}$$

Definition 1.1.2. The linear space $\mathcal{S}'(\mathbb{R}^n)$ is defined as the topological dual of $\mathcal{S}(\mathbb{R}^n)$. We write $u(\phi) = \langle u, \phi \rangle$ for $u \in \mathcal{S}'(\mathbb{R}^n), \phi \in \mathcal{S}(\mathbb{R}^n)$.

We identify $f \in L^2(\mathbb{R}^n)$ and $\tilde{f} \in \mathcal{S}'(\mathbb{R}^n)$, $\langle \tilde{f}, \phi \rangle = \int f \phi \, dx$.

1.1.1 The Fourier transformation and Sobolev spaces

Before we turn to the the definition of the L^2 based Sobolev spaces, let us quickly review the definition and basic properties of the Fourier transformation.

Proposition 1.1.3. *For $u \in \mathcal{S}(\mathbb{R}^n)$ we define*

$$\widehat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\xi} u(x)dx, \quad \xi \in \mathbb{R}^n \tag{1.1}$$

Then, $\widehat{\cdot} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism with inverse

$$\check{u}(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} u(\xi)d\xi, \quad x \in \mathbb{R}^n \tag{1.2}$$

and

$$\int \widehat{u}\phi dx = \int u\widehat{\phi} dx$$

holds true. We define the Fourier transformation

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \ \langle \mathcal{F}u, \phi \rangle = \langle u, \widehat{\phi} \rangle$$

which is an isomorphism such that $\mathcal{F} \mid_{L^1(\mathbb{R}^n)}$ and $\mathcal{F}^{-1} \mid_{L^1(\mathbb{R}^n)}$ are given by the formulas (1.1) *and* (1.2)*, respectively.*

Proposition 1.1.4. *$\mathcal{F}(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^n)$. Moreover, $\mathcal{F} \mid_{L^2(\mathbb{R}^n)} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator and in particular $\|\mathcal{F}u\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$.*

We write $\langle x \rangle := (1 + |x|^2)^{1/2}$ and define the Bessel potential operator

$$J^s : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad \langle \mathcal{F}J^s u, \phi \rangle = \langle \mathcal{F}u, \langle \xi \rangle^s \phi \rangle$$

Definition 1.1.5. Let $s \in \mathbb{R}$. We define the Sobolev spaces $H^s(\mathbb{R}^n)$ as the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$, such that $J^s u \in L^2(\mathbb{R}^n)$, endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|J^s u\|_{L^2(\mathbb{R}^n)} = \left(\int \langle \xi \rangle^{2s} |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

We remark that $H^s(\mathbb{R}^n)$ is a Hilbert space with scalar product

$$(u, v)_{H^s(\mathbb{R}^n)} := \int \langle \xi \rangle^{2s} \mathcal{F}u(\xi) \overline{\mathcal{F}v(\xi)} d\xi$$

Moreover, $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ is dense and $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for all $s \geq 0$. We may identify $H^{-s}(\mathbb{R}^n)$ with the dual of $H^s(\mathbb{R}^n)$ by the Riesz representation theorem.

1.1.2 The periodic case

We say that $u \in \mathcal{S}'(\mathbb{R}^n)$ is periodic, if

$$\langle u, \phi \rangle = \langle u, \phi(\cdot + 2\pi k) \rangle , \quad k \in \mathbb{Z}^n$$

Now, the Fourier transformation of a periodic $u \in \mathcal{S}'(\mathbb{R}^n)$ has the following form

$$\langle \mathcal{F}u, \psi \rangle = \sum_{\xi \in \mathbb{Z}^n} a_\xi \psi(\xi)$$

for a unique family $(a_\xi)_{\xi \in \mathbb{Z}^n}$ which grows at most polynomially. Hence, it is a sum of point measures and we identify a_ξ and $\mathcal{F}u(\xi)$, see [Hör83], p. 178 or [ST87], Section 3.2.

Let $L^2(\mathbb{T}^n)$ denote the Hilbert space of all $f : \mathbb{R}^n \to \mathbb{C}$ such that $f = f(\cdot + 2\pi k)$, $k \in \mathbb{Z}^n$ and $f|_{[0,2\pi]^n} \in L^2([0, 2\pi]^n)$ with scalar product

$$(f, g)_{L^2(\mathbb{T}^n)} = \int_{[0,2\pi]^n} f(x)\overline{g(x)}dx$$

We identify $f \in L^2(\mathbb{T}^n)$ and the periodic $\tilde{f} \in \mathcal{S}'(\mathbb{R}^n)$, $\langle \tilde{f}, \phi \rangle = \int f\phi \, dx$.

Proposition 1.1.6. *$\mathcal{F}(L^2(\mathbb{T}^n)) = l^2(\mathbb{Z}^n)$. Moreover, $\mathcal{F} |_{L^2(\mathbb{T}^n)}: L^2(\mathbb{T}^n) \to l^2(\mathbb{Z}^n)$ is unitary and in particular $\|\mathcal{F}u\|_{l^2(\mathbb{Z}^n)} = \|u\|_{L^2(\mathbb{T}^n)}$. For $u \in L^2(\mathbb{T}^n)$ we have*

$$\mathcal{F}u(\xi) = (2\pi)^{-\frac{n}{2}} \int_{[0,2\pi]^n} u(x)e^{-ix\xi}dx , \quad \xi \in \mathbb{Z}^n$$

and

$$u(x) = (2\pi)^{-\frac{n}{2}} \sum_{\xi \in \mathbb{Z}^n} \mathcal{F}u(\xi)e^{ix\xi} \quad \text{in } L^2(\mathbb{T}^n)$$

Definition 1.1.7. Let $s \in \mathbb{R}$. We define the Sobolev spaces $H^s(\mathbb{T}^n)$ as the space of all periodic $u \in \mathcal{S}'(\mathbb{R}^n)$, such that $J^s u \in L^2(\mathbb{T}^n)$, endowed with the norm

$$\|u\|_{H^s(\mathbb{T}^n)} := \|J^s u\|_{L^2(\mathbb{T}^n)} = \left(\sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\mathcal{F}u(\xi)|^2 \right)^{\frac{1}{2}}$$

Notice that $H^s(\mathbb{T}^n)$ is a Hilbert space with scalar product

$$(u, v)_{H^s(\mathbb{T}^n)} := \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} \mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}$$

Definition 1.1.8. We define the Fréchet space of smooth periodic functions

$$\mathcal{S}(\mathbb{T}^n) := \{u \in C^\infty(\mathbb{R}^n) \mid u = u(\cdot + 2\pi z), z \in \mathbb{Z}^n,\ [u]_{k,\mathbb{T}^n} := [u]_{0,k} < \infty\}$$

and of smooth functions which are spatially periodic

$$\begin{aligned}\mathcal{S}(\mathbb{R} \times \mathbb{T}^n) := \{u \in C^\infty(\mathbb{R}^{n+1}) \, |u(t,x) = u(t, x + 2\pi z), z \in \mathbb{Z}^n,\\ [u]_{k,\mathbb{R}\times\mathbb{T}^n} = \max_{k_1,|\beta|\leq k} [u]_{(k_1,0\ldots,0),\beta} < \infty,\ k \in \mathbb{N}_0\}\end{aligned}$$

We observe that $\mathcal{S}(\mathbb{T}^n) \subset H^s(\mathbb{T}^n)$ is dense. Moreover, $H^s(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$ for all $s \geq 0$. Furthermore, we may identify $H^{-s}(\mathbb{T}^n)$ with the dual of $H^s(\mathbb{T}^n)$ by the Riesz representation theorem.

Definition 1.1.9. The linear spaces $\mathcal{S}'(\mathbb{T}^n)$ and $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^n)$ are defined as the topological duals of $\mathcal{S}(\mathbb{T}^n)$ and $\mathcal{S}(\mathbb{R} \times \mathbb{T}^n)$, respectively. We write $u(\phi) = \langle u, \phi\rangle$.

The periodic elements of $\mathcal{S}'(\mathbb{R}^n)$ may be identified with $\mathcal{S}'(\mathbb{T}^n)$ via their Fourier representation, see [ST87], Section 3.2.3.

Definition 1.1.10. For $T > 0$ we define the linear spaces

$$\mathcal{S}^T(\mathbb{R}^n) = \{u|_{[-T,T]\times\mathbb{R}^n} \mid u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^n)\}$$

and

$$\mathcal{S}^T(\mathbb{T}^n) = \{u|_{[-T,T]\times\mathbb{R}^n} \mid u \in \mathcal{S}(\mathbb{R} \times \mathbb{T}^n)\}$$

1.1.3 Conventions

In the sequel we will deal with both the spatially periodic and the non-periodic setting. At the beginning of each logical subunit we will explicitly declare the setting to avoid confusion.

There are parts where we want to treat both cases simultaniously. In this case, in order to keep the exposition short we omit the letters $\mathbb{R}$ and $\mathbb{T}$ from the above definition of the spaces. E.g. we write $\mathcal{S}$ for $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{T}^n)$, $\mathcal{S}^T$ for $\mathcal{S}^T(\mathbb{R}^n)$ and $\mathcal{S}^T(\mathbb{T}^n)$, H^s for $H^s(\mathbb{R}^n)$ and $H^s(\mathbb{T}^n)$, L^p for $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{T}^n)$, respectively. Moreover, dx denotes integration with respect to the Lebesgue measure on $\mathbb{R}^n$ and its restriction to $[0, 2\pi]^n$, respectively. Similarly, $d\xi$ denotes integration with respect to the Lebesgue measure on $\mathbb{R}^n$ and the counting measure on $\mathbb{Z}^n$, respectively.

As an example, the well-known Sobolev embedding and multiplication theorems in both cases read as follows:

Proposition 1.1.11. *We have*

(i) $H^s \subset C^k$ *for* $s > k + \frac{n}{2}$, $k \in \mathbb{N}_0$

(ii) $H^s \subset L^p$ *for* $s \geq \frac{n}{2} - \frac{n}{p}$, $2 \leq p < \infty$

(iii) $L^p \subset H^s$ *for* $s \leq \frac{n}{2} - \frac{n}{p}$, $1 < p \leq 2$

(iv) $L^1 \subset H^s$ *for* $s < -\frac{n}{2}$

with continuous embeddings.

We omit the proof (for the non-periodic case see [Tri83], Section 2.7.1, for the periodic case see [ST87], Section 3.5.5.).

Corollary 1.1.12. *Let* $s \geq 0$ *and assume that*

$$s \overset{[\leq]}{<} s_1, s_2, \quad s \overset{[<]}{\leq} s_1 + s_2 - \frac{n}{2}$$

Then, there exists $c > 0$ *such that*

$$\|u_1 u_2\|_{H^s} \leq c\|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}\,, \quad u_1 \in H^{s_1}, u_2 \in H^{s_2} \tag{1.3}$$

In particular, H^s *is a Banach algebra for* $s > \frac{n}{2}$.

Proof. Define $v_i = \mathcal{F}^{-1}|\mathcal{F}u_i|$. The point-wise estimate on the Fourier side $\langle\xi\rangle^s \leq c\langle\xi_1\rangle^s + c\langle\xi - \xi_1\rangle^s$ shows

$$\begin{aligned}\|u_1 u_2\|_{H^s} \leq & \Big\|\langle\xi\rangle^s \int \mathcal{F}u_1(\xi - \xi_1)\mathcal{F}u_2(\xi_1)d\xi_1\Big\|_{L^2_\xi} \\ \leq & c\|J^s v_1 v_2\|_{L^2} + c\|v_1 J^s v_2\|_{L^2} \\ \leq & c\|J^s v_1\|_{L^{p_1}}\|v_2\|_{L^{p_2}} + c\|v_1\|_{L^{q_1}}\|J^s v_2\|_{L^{q_2}}\end{aligned}$$

for $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$. We start with the case $s = s_1 + s_2 - \frac{n}{2}$. Then, necessarily $0 < s_i < \frac{n}{2}$ and we choose $p_1 = \frac{n}{s_2}$ and $q_2 = \frac{n}{s_1}$ and the claim follows from Proposition 1.1.11 and the fact that $\|v_i\|_{H^{s_i}} = \|u_i\|_{H^{s_i}}$.

In the case $s < s_1 + s_2 - \frac{n}{2}$ we use the estimate

$$\langle\xi\rangle^s \leq c\langle\xi_1\rangle^{s-s_2}\langle\xi - \xi_1\rangle^{s_2} + c\langle\xi_1\rangle^{s_1}\langle\xi - \xi_1\rangle^{s-s_1}$$

and arrive at

$$\begin{aligned}\|u_1 u_2\|_{H^s} \leq & c\|J^{s-s_2} v_1 J^{s_2} v_2\|_{L^2} + c\|J^{s_1} v_1 J^{s-s_1} v_2\|_{L^2} \\ \leq & c\|J^{s-s_2} v_1\|_{L^\infty}\|J^{s_2} v_2\|_{L^2} + c\|J^{s_1} v_1\|_{L^2}\|J^{s-s_1} v_2\|_{L^\infty}\end{aligned}$$

Because $H^{s_1+s_2-s} \subset L^\infty$ by Proposition 1.1.11 the claim follows. □

1.2 Linear equations

Let $s \in \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function of polynomial growth. We define $\phi(D)$ to be the Fourier multiplier operator

$$\phi(D) : H^s \supset \mathcal{D}_{\phi,s} \to H^s$$
$$\phi(D)f = \mathcal{F}_x^{-1}\phi\mathcal{F}_x f$$

which has dense domain $\mathcal{D}_{\phi,s} := \{f \in H^s \mid \langle\xi\rangle^s\phi\mathcal{F}_x f \in L^2\}$.

1.2.1 Linear homogeneous equations

Let $s \in \mathbb{R}$, $u_0 \in H^s$ and define $W(t)u_0$ by

$$\mathcal{F}_x W(t)u_0(\xi) := e^{it\phi(\xi)}\mathcal{F}_x u_0(\xi)$$

Then, $\|W(t)u_0\|_{H^s} = \|u_0\|_{H^s}$ and

$$W(t) : H^s \to H^s$$

is a well-defined, linear and isometric operator. For $u_0 \in \mathcal{D}_{\phi,s}$ the function $u(t) := W(t)u_0$ satisfies $u \in C^1(\mathbb{R}, H^s)$ and solves the Cauchy problem

$$\begin{aligned} \partial_t u(t) - i\phi(D)u(t) &= 0 \qquad \text{for } \ t \in (-T,T) \\ u(0) &= u_0 \end{aligned} \tag{1.4}$$

The following proposition summarizes important properties of $W(t)$ (cp. [CH98] Theorem 3.2.3).

Proposition 1.2.1. *Let $s \in \mathbb{R}$. The one-parameter family $(W(t))_{t\in\mathbb{R}} \subset L(H^s)$ is a group of unitary operators. Moreover,*

(i) $\|W(t)u_0\|_{H^s} = \|u_0\|_{H^s}$, $u_0 \in H^s$

(ii) $t \mapsto W(t)u_0 \in C(\mathbb{R}, H^s)$, $u_0 \in H^s$

(iii) $t \mapsto W(t)u_0 \in C(\mathbb{R}, \mathcal{D}_{\phi,s}) \cap C^1(\mathbb{R}, H^s)$, $u_0 \in \mathcal{D}_{\phi,s}$

(iv) $W(0) = \mathrm{Id}$, $W(t+t') = W(t)W(t')$, $W(t)^* = W(-t)$

and for $u_0 \in \mathcal{D}_{\phi,s}$ the function $u(t) = W(t)u_0$ satisfies (1.4).

Because of its importance let us rewrite this solution for $u_0 \in \mathcal{S}$ in an explicit form

$$u(t,x) = W(t)u_0(x) = (2\pi)^{-n}\int\int e^{i(x-y)\xi+it\phi(\xi)}u_0(y)\,dyd\xi \tag{1.5}$$

Examples 1.2.2. (i) $\phi(\xi) = -|\xi|^2$: the equation

$$\partial_t u(t) - i\Delta u(t) = 0$$

is called the (linear) Schrödinger equation.

(ii) $n = 1$ and $\phi(\xi) = \xi^3$: The equation

$$\partial_t u(t) + \partial_x^3 u(t) = 0$$

is called the Airy equation.

1.2.2 Linear inhomogeneous equations

Next, we recall Duhamel's principle for the linear inhomogeneous problem (see [CH98], Section 4.1). Compare also Proposition 1.3.3.

Proposition 1.2.3. *Let $T > 0$ and $s \in \mathbb{R}$. Assume that $u_0 \in \mathcal{D}_{\phi,s}$ and $f \in C([-T,T], H^s) \cap L^1([-T,T], \mathcal{D}_{\phi,s})$ as well as $u \in C([-T,T], \mathcal{D}_{\phi,s}) \cap C^1((-T,T), H^s)$. Then, the following statements are equivalent:*

(i) u solves

$$\begin{aligned} \partial_t u(t) - i\phi(D)u(t) &= f(t)\,,\ t \in (-T,T) \\ u(0) &= u_0 \end{aligned} \tag{1.6}$$

in H^s.

(ii) u satisfies

$$u(t) = W(t)u_0 + \int_0^t W(t-t')f(t')\,dt'\,,\ t \in (-T,T) \tag{1.7}$$

in H^s.

Remark 1.2.4. More generally, for initial data in H^s and $f \in L^1([-T,T], H^s)$ the integral equation above defines a function $u \in C([-T,T], H^s)$.

1.3 The nonlinear Cauchy problem and well-posedness

We are interested in nonlinear Cauchy problems of the type

$$\begin{aligned} \partial_t u(t) - i\phi(D)u(t) &= F(u(t)) \quad \text{for } t \in (-T,T) \\ u(0) &= u_0 \end{aligned} \tag{1.8}$$

Let us recall the four aims formulated in the introduction: For given initial data in L^2 based Sobolev spaces we try to find

(i) unique

(ii) solutions

(iii) whose initial regularity persists

(iv) and which depend continuously on the initial data.

In this section we will give a precise mathematical meaning to the nonlinear Cauchy problem (1.8) and well-posedness. Since (1.8) is nonlinear and we want to study these equations in a low regularity framework, there is no unified notion of solutions like the theory of distributions provides for linear equations. To give an example, consider the Korteweg-de Vries equation

$$\partial_t u(t) + \partial_x^3 u(t) = \partial_x u^2(t)$$

for periodic initial data $u_0 \in H^s(\mathbb{T})$ for $s < 0$, where it is not clear how to define the product $u \cdot u$. Nevertheless, C.E. Kenig - G. Ponce - L. Vega [KPV96] and T. Kappeler - P. Topalov [KT03a] and others derived well-posedness results in some range where the generalized solutions are defined by an extension procedure.[1]

In this section we will define a rather weak general notion of well-posedness, related to [KT03a]. In each particular application considered here, we will be able to strengthen this in certain directions by using the specific structure of the problem. One main point here is the regularity of the flow map.

Examples 1.3.1. (i) In Chapter 3 we consider a Schrödinger equation in one space dimension with the derivative nonlinearity

$$\partial_t u(t) - i\partial_x^2 u(t) = \partial_x(|u|^2 u)(t)$$

(ii) The nonlinear equations in one space dimension

$$\partial_t u(t) + \partial_x^3 u(t) = \frac{\pm 1}{k+1}\partial_x u^{k+1}(t)$$

are called Korteweg - de Vries equation (KdV) if $k = 1$, modified Korteweg - de Vries equation (mKdV) if $k = 2$ or generalized Korteweg - de Vries equation of order k for $k \geq 3$.

[1]Concerning uniqueness, cp. Remarks in [Chr05].

(iii) $n = 1$ and $\phi(\xi) = \xi|\xi|$: The equations

$$\partial_t u(t) - |D|\partial_x u(t) = \frac{\pm 1}{k+1}\partial_x u^{k+1}(t)$$

are called Benjamin - Ono equation (BO) if $k = 1$, modified Benjamin - Ono equation (mBO) if $k = 2$ or generalized Benjamin - Ono equation of order k for $k \geq 3$.

(iv) $n = 1$ and $\phi(\xi) = \xi|\xi|^\alpha$, $1 < \alpha < 2$: We call

$$\partial_t u(t) - |D|^\alpha \partial_x u(t) = \partial_x u^2(t)$$

equation of Benjamin - Ono type, see Chapter 4.

Assumption. Let us assume throughout this work that there exists a number $k \geq 0$ such that $H^k \subset \mathcal{D}_{\phi,0}$ and $F : H^s \to H^{s-k}$ is locally Lipschitz continuous for all $s \geq k$.

Obviously, for functions $u \in C([-T,T], H^k) \cap C^1((-T,T), L^2)$ the expressions

$$\partial_t u(t)\ ,\ \phi(D)u(t)\ ,\ F(u(t)) \in L^2$$

are well-defined for all $t \in (-T,T)$.

Definition 1.3.2. A function $u \in C([-T,T], H^k)$ is called a *regular solution* of (1.8), iff $u \in C^1((-T,T), L^2)$ and

$$\partial_t u(t) - i\phi(D)u(t) = F(u(t))$$

is fulfilled in L^2 for every $t \in (-T,T)$.

Let us review Duhamel's principle.

Proposition 1.3.3. *The following statements are equivalent:*

(i) $u \in C([-T,T], H^k) \cap C^1((-T,T), L^2)$ *is a regular solution*

(ii) $u \in C([-T,T], H^k)$ *solves*

$$u(t) = W(t)u(0) + \int_0^t W(t-t')F(u(t'))dt', \quad t \in (-T,T)$$

Proof. Assume (i). Then, for $v(t) = W(-t)u(t) \in C^1((-T,T), L^2)$ we have

$$\begin{aligned}\partial_t v(t) &= -i\phi(D)W(-t)u(t) + W(-t)\partial_t u(t)\\ &= W(-t)F(u(t)), \quad t \in (-T,T)\end{aligned}$$

which implies

$$v(t) = v(0) + \int_0^t W(-t')F(u(t'))dt'$$

and the claim (ii) follows. Now, assume (ii). Then, by assumption

$$\int_0^t W(-t')F(u(t'))dt' \in C^1((-T,T),L^2)$$

and $v(t) = W(-t)u(t) \in C^1((-T,T),L^2)$ solves

$$\partial_t v(t) = W(-t)F(u(t)), \quad t \in (-T,T)$$

which gives (i). □

In most of the cases which are interesting to us, the existence and uniqueness of smooth solutions will be well-known (or at least straightforward to show) by classical means like a regularization procedure and energy estimates. However, we will not have to rely on any results about regular solutions[2].

Definition 1.3.4. Let $s \in \mathbb{R}$ and $H \hookrightarrow H^s$ such that $\mathcal{S} \subset H$ is dense and let $B_R = \{u_0 \in H \mid \|u_0\|_H < R\}$. We say that the Cauchy problem (1.8) is *locally [globally] well-posed in H (in the minimal sense)* iff there exists a non-increasing function $T^* : (0,\infty) \to (0,\infty)$ such that for all $R > 0$ and $0 < T \leq T^*(R)$ [for all $R > 0$ and all $T > 0$] the following conditions are satisfied:

(i) There exists a continuous map

$$S_{R,T} : B_R \to C([-T,T],H)$$

(ii) If $s' \geq s$, then $S_{R,T}(B_R \cap H^{s'}) \subset C([-T,T],H^{s'} \cap H)$ and

$$S_{R,T} \mid_{B_R \cap H^{s'}} : B_R \cap H^{s'} \to C([-T,T],H^{s'} \cap H)$$

is continuous.

(iii) There exists $s_1 \geq k$ such that for $u_0 \in B_R \cap H^{s_1}$ the function $S_{R,T}(u_0)$ is the unique regular solution in $C([-T,T],H \cap H^{s_1})$ of (1.8).

The map $S_{R,T}$ is called the *flow map* or *solution map.*

Let $M \subset H$ be an open subset. We say that the Cauchy problem is *locally [globally] well-posed in H for data in M (in the minimal sense)*, iff (i)-(iii) hold true with B_R replaced by $B_R \cap M$.

[2]Except for Section 4.4.

Roughly speaking, we may summarize this definition of well-posedness as follows: The map data $\mapsto$ solution is well-defined for smooth data and extends to a continuous map from $H \cap H^{s'}$ to $C([-T,T], H \cap H^{s'})$ for all $s' \geq s$.

We are mainly interested in the cases where $H = H^s$ for some $s \in \mathbb{R}$. But sometimes, it is also interesting to consider subsets of H^s endowed with slightly stronger norms, see Chapter 4.

Proposition 1.3.5. *Assume that the Cauchy problem* (1.8) *is locally well-posed in a space $H \hookrightarrow H^s$ for some $s \in \mathbb{R}$.*

(i) Let $R > 0, 0 < T \leq T^*(R)$ and $u_0 \in B_R$. If $v_n \to v$ in $C([-T,T], H)$ for a sequence $v_n \in C([-T,T], H \cap H^{s_1})$ of unique regular solutions of (1.8) *with $v_n(0) \to u_0$ in H, it follows $v = S_{R,T}(u_0)$.*

(ii) Assume $u_0 \in B_{R_1}$ and $R_2 \geq R_1$ and let $T_1 = T^*(R_1)$. Then, for all $0 < T_2 \leq T^*(R_2)$ we have

$$S_{R_1,T_1}(u_0)|_{[-T_2,T_2]} = S_{R_2,T_2}(u_0)$$

Proof. The first claim directly follows from parts (ii) and (iii) of Definition 1.3.4

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} S_{R,T}(v_n(0)) = S_{R,T}(\lim_{n\to\infty} v_n(0)) = S_{R,T}(u_0)$$

where the limits are taken in $C([-T,T], H)$ and H, respectively.

For the proof of the second claim it suffices to consider smooth data $u_0 \in H \cap H^{s_1}$ by continuous dependence and density. But then the claim immediately follows from part (iii) of Definition 1.3.4. $\square$

Remark 1.3.6. We conclude that well-posedness in the minimal sense implies that there exist unique limits of smooth, regular solutions whose initial regularity persists and which depend continuously on the initial data. Hence, the two aims existence and uniqueness are fulfilled in a very weak limiting sense which is inherited from existence and uniqueness for smooth, regular solutions. This is natural under the assumption we made on F which is only defined for quite smooth functions.

Remark 1.3.7. In the applications we will always try to strengthen the well-posedness results in several directions, in particular we want to verify that $S_{R,T}(u_0)$ fulfills the equation at least in a distributional sense and to specify a uniqueness class. Moreover, we are interested in the regularity properties of the flow map. All this depends mainly on suitable estimates for the nonlinear expression.

Remark 1.3.8. We defined the time of existence only with respect to balls around the origin, such that the flow map is then defined on these balls with a common time of existence. Alternatively, one could define $T^* : H \to (0, \infty]$ to be a lower semi-continuous function and require that the flow maps exist locally with a common time of existence, which will not be considered here.

1.4 Analytic maps between Banach spaces

In this section we will introduce differentiability and analyticity in the infinite dimensional setting. This will be relevant when studying the regularity of the flow maps in the later applications. The aim here is to provide some well-known facts for future reference.

Let X, Y be Banach spaces over $\mathbb{R}$ [over $\mathbb{C}$], $U \subset X$ open. We define $C^1(U, Y)$ as the set of all [complex] differentiable maps $F : U \to Y$, such that $F' : U \to L(X, Y)$ is continuous and inductively we define $C^k(U, Y)$ for $k = 2, \dots, \infty$. Let us recall

Proposition 1.4.1. *Let $F : U \to Y$ be differentiable. Assume that for $u \in U$ there exists $\varepsilon > 0$ such that $F' : B_\varepsilon(u) \to L(X, Y)$ is bounded. Then, $F |_{B_\varepsilon(u)}$ is Lipschitz continuous. In particular, if $F \in C^1(U, Y)$, then F is locally Lipschitz continuous.*

The following definition is the straightforward generalization of analyticity to Banach spaces.

Definition 1.4.2. Let X, Y be Banach spaces over $\mathbb{C}$ [over $\mathbb{R}$], $U \subset X$ open and $F : U \to Y$. Then, we say that F is analytic [real analytic], iff for every $u \in U$ there exists $r > 0$ with $B_r(u) \subset U$ such that for every $k \in \mathbb{N}_0$ there exists a continuous k-linear map $L_k : X \times \ldots \times X \to Y$ and with $L^{(k)}(x) = L_k(x, \dots, x)$

$$F(x) = \sum_{k=0}^{\infty} L^{(k)}(x - u), \quad x \in B_r(u)$$

holds true with uniform convergence in $B_r(u)$.

Let us discuss two trivial examples, which will be used in the sequel.

Examples 1.4.3. Let X, Y be Banach spaces over $\mathbb{C}$ [over $\mathbb{R}$].

(i) Let $T : X \times \ldots \times X \to Y$ be k-linear and continuous. Then, the map $X \to X, x \mapsto T(x, \dots, x)$ is analytic [real analytic].

(ii) Let X be a Banach algebra. The exponential map $X \to X, x \mapsto e^x$ is analytic [real analytic] by definition.

In the next theorem we summarize some properties of analytic maps.

Theorem 1.4.4. *Let X, Y be Banach spaces over $\mathbb{C}$, $U \subset X$ open and $F : U \to Y$. The following three statements are equivalent:*

(i) F is analytic.

(ii) F is complex differentiable in U.

(iii) F is locally bounded and for all $y' \in Y'$ and all $u \in U, v \in X$ the maps

$$\{z \in \mathbb{C} \mid u + zv \in U\} \to \mathbb{C}, \quad z \mapsto \langle y', F(u + zv)\rangle$$

are analytic.

Proposition 1.4.5. *Let X, Y, Z be Banach spaces over $\mathbb{C}$ [over $\mathbb{R}$], $U \subset X, V \subset Y$ open and $F : U \to V \subset Y$, $G : V \to Z$ analytic [real analytic]. Then, $G \circ F : U \to Z$ is analytic [real analytic].*

Finally, we state an implicit function theorem.

Theorem 1.4.6. *Let $k \in \mathbb{N}$, X, Y, Z be Banach spaces [over $\mathbb{C}$; over $\mathbb{R}$], $U \subset X$ and $V \subset Y$ neighborhoods of $x_0 \in X$ and $y_0 \in Y$, respectively, and let $F \in C^k(U \times V, Z)$ [let F be analytic; let F be real analytic], such that $F(x_0, y_0) = 0$ and $D_yF(x_0, y_0)$ is invertible. Then, there exist balls $B_r(x_0) \subset U, B_R(y_0) \subset V$ and exactly one map $G \in C^k(B_r(x_0), Y)$ [one analytic map $G : B_r(x_0) \to Y$; one real analytic map $G : B_r(x_0) \to Y$] with $G(B_r(x_0)) \subset B_R(y_0)$ such that $G(x_0) = y_0$ and $F(x, G(x)) = 0$ for $x \in B_r(x_0)$.*

1.5 Notes and References

The contents of this chapter is well-known and can be found in several textbooks, with the only exception of Section 1.3.

For rigorous definitions, identifications and further properties of the objects defined in Section 1.1 we refer to the textbooks of L. Hörmander [Hör83] (Chapter VII), W. Kaballo [Kab99] (Kapitel VII) and J. Bergh - J. Löfström [BL76] (Chapter 6), R.J. Iório - V. Iório [II01] as well as the detailed treatments of H.-J. Schmeisser - H. Triebel [ST87] (Chapter 3) and H. Triebel [Tri83] (Chapter 2). The books of T. Cazenave - A. Haraux [CH98] and A.

Pazy [Paz83] serve as a general reference for Section 1.2 as they provide an exhaustive introduction to linear evolution equations and semi group theory.

The aim of Section 1.3 is to formulate the minimal requirements which have to be fulfilled when we want to discuss well-posed Cauchy problems in the later applications. For different notions of well-posedness and ill-posedness for dispersive equations we refer to the influential works of J.L. Bona - R. Smith [BS75], T. Kato [Kat75, Kat83], J. Bourgain [Bou93], C.E. Kenig - G. Ponce - L. Vega [KPV93c, KPV93a, KPV96, KPV01], L. Molinet - J.-C. Saut - N. Tzvetkov [MST01], M. Christ - J. Colliander - T. Tao [CCT03], H. Koch - N. Tzvetkov [KT05b], D. Tataru [Tat05], M. Christ [Chr05], T. Kappeler - P. Topalov [KT05b] as well as A.D. Ionescu - C.E. Kenig [IK05]. Moreover, an overview of results on local and global well-posedness is provided by web pages maintained by J. Colliander - M. Keel - G. Staffilani - H. Takaoka - T. Tao [CKS+].

For the contents of Section 1.4 we refer to the textbooks of J. Mujica [Muj86], Chapters I-II, IV and K. Deimling [Dei85], §7.7 and §15. For the definition of analyticity, see e.g. [Dei85], Def. 15.1 and [Muj86], Def. 5.1. For the proof of the first part of Theorem 1.4.4 see [Muj86], Thm. 13.16 and Def. 13.1. The second part of Theorem 1.4.4 follows from combining [Muj86], Thm. 8.12, Prop. 8.6 and Thm. 8.7. For Banach spaces over $\mathbb{C}$ Proposition 1.4.5 follows from the chain rule, see [Muj86], Thm. 13.6 and the equivalence of Theorem 1.4.4 (i) and (ii). For Banach spaces over $\mathbb{R}$, we first complexify the spaces, extend the analytic maps to open sets of the complex spaces and apply the result for $\mathbb{C}$, similar to the process in the proof of [Dei85], Thm. 15.3 on pp.151-152. The proof of Theorem 1.4.6 is given in [Dei85], Thm. 15.1, Cor. 15.1 and Thm. 15.3 (a).

Chapter 2

Dispersive estimates and Bourgain spaces

2.1 Dispersive estimates

In this section we will discuss certain space-time estimates which display the dispersive character of the equations in consideration. Although the linear solution operator $W(t)$ is an isometry in H^s, it has certain smoothing effects.

2.1.1 The periodic case: The Schrödinger equation

In this subsection we recall the beautiful $L^4(\mathbb{T}^2)$ estimate for the Schrödinger equation in the periodic case in one spatial dimension, i.e. the phase function now is $\phi : \mathbb{Z} \to \mathbb{R}, \phi(\xi) = -\xi^2$. The following result is due to A. Zygmund [Zyg74] (formulated as a restriction theorem for the Fourier transform), see also J. Bourgain [Bou93].

Theorem 2.1.1. *Let $u_0 \in L^2(\mathbb{T})$. Then,*

$$\left\| \frac{1}{\sqrt{2\pi}} \sum_{\xi \in \mathbb{Z}} e^{i(\xi x - t\xi^2)} \mathcal{F}u_0(\xi) \right\|_{L^4(\mathbb{T}^2)} \leq \sqrt[4]{2} \|u_0\|_{L^2(\mathbb{T})} \tag{2.1}$$

Proof. Let $f(t,x) = \frac{1}{\sqrt{2\pi}} \sum_{\xi\in\mathbb{Z}} e^{i(\xi x - t\xi^2)} \mathcal{F}u_0(\xi)$. Then,

$$\|f\overline{f}\|_{L^2(\mathbb{T}^2)} = \left\| \frac{1}{2\pi} \sum_{(\xi_1,\xi_2)\in\mathbb{Z}^2} \mathcal{F}u_0(\xi_1)\overline{\mathcal{F}u_0(\xi_2)} e^{it(\xi_2^2-\xi_1^2)+i(\xi_1-\xi_2)x} \right\|_{L^2(\mathbb{T}^2)}$$

We rewrite this sum as a Fourier series in (t,x) variables

$$\frac{1}{2\pi} \sum_{(\tau,\xi)\in\mathbb{Z}^2} a(\tau,\xi) e^{ix\xi+it\tau}$$

where

$$a(\tau,\xi) = \sum_{(\xi_1,\xi_2)\in P(\tau,\xi)} \mathcal{F}u_0(\xi_1)\overline{\mathcal{F}u_0(\xi_2)}$$

and $P(\tau,\xi) = \{(\xi_1,\xi_2) \mid \xi_2^2 - \xi_1^2 = \tau, \xi_1 - \xi_2 = \xi\}$. Now, for given pair of frequency variables $(\tau,\xi) \neq (0,0)$ of the form $(\tau,\xi) = (\xi_2^2 - \xi_1^2, \xi_1 - \xi_2)$ there is at most one solution (ξ_1,ξ_2). Moreover $a(0,0) = \sum_{\xi\in\mathbb{Z}} |\mathcal{F}u_0(\xi)|^2$, and by Plancherel

$$\|f\overline{f}\|_{L^2(\mathbb{T}^2)} = \left(\sum_{(\tau,\xi)\in\mathbb{Z}^2} |a(\tau,\xi)|^2 \right)^{1/2} \leq \sqrt{2} \sum_{\xi\in\mathbb{Z}} |\mathcal{F}u_0(\xi)|^2$$

and the claim follows. □

2.1.2 The non-periodic case: Generalized dispersion

In this subsection we consider the non-periodic case and the phase function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(\xi) = \xi|\xi|^\alpha$ for $\alpha > 0$. Let $W_\alpha u_0(t,x) = W_\alpha(t)u_0(x)$ be the linear group defined by

$$\mathcal{F}W_\alpha(t)u_0(\xi) = e^{it\xi|\xi|^\alpha} \mathcal{F}u_0(\xi)$$

Definition 2.1.2. Let $p \in [4,\infty], q \in [2,\infty]$ and

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$$

Then (p,q) is called an *admissible pair.*

The following theorem is a special case of [KPV91a], Thm. 2.1.

Theorem 2.1.3. *Let $\alpha > 0$ and (p, q) be an admissible pair. Then we have*

$$\||D|^{\frac{\alpha-1}{p}} W_\alpha f\|_{L^p_t L^q_x} \le c\|f\|_{L^2_x} \tag{2.2}$$

$$\left\|\int_0^t |D|^{\frac{\alpha-1}{p}} W_\alpha(t-s) f(s)\, ds\right\|_{L^p_t L^q_x} \le c\|f\|_{L^1_t L^2_x} \tag{2.3}$$

Proof. The estimate

$$\||D|^{\frac{\alpha-1}{p}} W_\alpha f\|_{L^p_t L^q_x} \le c\|f\|_{L^2_x}$$

directly follows from [KPV91a], Thm. 2.1, formula (2.3). Then, using Minkowski's inequality,

$$\begin{aligned}
&\left\|\int_0^t |D|^{\frac{\alpha-1}{p}} W_\alpha(t-s) f(s)\, ds\right\|_{L^p_t L^q_x} \le \left\||D|^{\frac{\alpha-1}{p}} W_\alpha(t-s) f(s)\right\|_{L^p_t L^q_x L^1_s} \\
&\le c \left\||D|^{\frac{\alpha-1}{p}} W_\alpha(t) W_\alpha(-s) f(s)\right\|_{L^1_s L^p_t L^q_x} \\
&\le c\,\|W_\alpha(-s) f(s)\|_{L^1_s L^2_x} = c\|f\|_{L^1_t L^2_x}
\end{aligned}$$

where we exploited (2.2) and the fact that $W_\alpha(-s)$ is an isometry in L^2. □

The next theorem, which is due to C.E. Kenig - G. Ponce - L. Vega (cp. Lemma 2.1 in [KPV91b] or [KPV91a], Theorem 4.1) describes the sharp local smoothing effect. The proof is nothing else but a change of variables and Plancherel in t.

Theorem 2.1.4. *Let $\alpha > 0$. Then, for $u \in \mathcal{S}(\mathbb{R})$*

$$\int_{\mathbb{R}} ||D|^{\frac{\alpha}{2}} W_\alpha(t) u_0(x)|^2 dt = \frac{1}{1+\alpha}\|u_0\|^2_{L^2}\ , \quad x \in \mathbb{R} \tag{2.4}$$

which shows $\||D|^{\frac{\alpha}{2}} W_\alpha u_0\|_{L^\infty_x L^2_t} = \sqrt{\frac{1}{1+\alpha}}\|u_0\|_{L^2}$.

We will now provide a useful and well-known formula for approximate identities.

Lemma 2.1.5. *Let $g(x) = \pi^{-\frac{1}{2}} e^{-x^2}$, $g_\varepsilon(x) = \varepsilon^{-1} g(\varepsilon^{-1}x)$ and let $f \in L^1(\mathbb{R})$ be continuous. Moreover, let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Assume that $\varphi(x) = 0$, $x \in \operatorname{supp}(f)$ iff $x \in \{x_1, \ldots, x_n\}$ and $\varphi'(x_i) \neq 0$ and* $\operatorname{supp}(f)$ *is compact or* $\liminf_{x\to\pm\infty} |\varphi(x)| > 0$. *Then,*

$$\lim_{\varepsilon\to 0} \int g_\varepsilon(\varphi(x)) f(x)\, dx = \sum_{i=1}^n \frac{f(x_i)}{|\varphi'(x_i)|} \tag{2.5}$$

Proof. Within the support of f there are finitely many zeros of φ, these are simple, and φ stays away from zero at infinity if f is not compactly supported. Hence, there exists a $\delta > 0$ such that with $I_i := (x_i - \delta, x_i + \delta)$

$$\varphi(x) = 0, x \in \operatorname{supp}(f) \iff x = x_i,\ |\varphi'(x)| \geq c > 0\ , x \in I_i$$
$$\text{and} \quad |\varphi(x)| \geq c > 0 \ \text{ for } \ x \in V := \operatorname{supp}(f) \setminus \bigcup_{i=1}^{n} I_i$$

Then,

$$\int_{I_i} g_\varepsilon(\varphi(x)) f(x)\, dx = \int_{\varphi(I_i)} g_\varepsilon(y) \frac{f(\varphi|_{I_i}^{-1}(y))}{|\varphi'(\varphi|_{I_i}^{-1}(y))|}\, dy \to \frac{f(x_i)}{|\varphi'(x_i)|} \quad (\varepsilon \to 0)$$

because $\varphi|_{I_i}^{-1}(0) = x_i$ and $(g_\varepsilon)_{\varepsilon>0}$ is an approximate identity. In V we have

$$\left| \int_V g_\varepsilon(\varphi(x)) f(x)\, dx \right| \leq \sup_{x \in V} g_\varepsilon(\varphi(x)) \int_V |f(x)|\, dx$$
$$\leq \varepsilon^{-1} e^{-\varepsilon^{-2} c^2} \|f\|_{L^1} \to 0 \quad (\varepsilon \to 0)$$

because $|\varphi| \geq c$ in V. □

We will now apply this in the proof of a sharp bilinear smoothing estimate. Roughly speaking, the bilinear operator defined below controls $\alpha/2$ derivatives on the product of two solutions at different frequency. This is particularly useful for the study of quadratic nonlinearities involving derivatives and it is a generalization of previous estimates for Schrödinger and KdV equations by A. Grünrock [Grü01, Grü05a], which in turn were related to work by J. Bourgain [Bou98]. For $\delta > 0$ let $|x|_\delta := \zeta(x/\delta)|x|$ for an even function $\zeta \in C^\infty$ with $\zeta|_{[-1,1]} \equiv 0$ and $\zeta|_{\mathbb{R}\setminus[-2,2]} \equiv 1$ and $0 \leq \zeta \leq 1$.

Theorem 2.1.6. *We define the bilinear operator I_δ^s via*

$$\mathcal{F}_x\, I_\delta^s(u_1, u_2)(\xi) = \int_{\xi = \xi_1 + \xi_2} \left| |\xi_1|^{2s} - |\xi_2|^{2s} \right|_\delta^{\frac{1}{2}} \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2)\, d\xi_1.$$

for all $u_1, u_2 \in \mathcal{S}(\mathbb{R})$. Then, for all $\delta > 0$

$$\left\| I_\delta^{\frac{\alpha}{2}}(W_\alpha u_1, W_\alpha u_2) \right\|_{L^2_{xt}} \leq \sqrt{\frac{2}{1+\alpha}} \|u_1\|_{L^2_x} \|u_2\|_{L^2_x}, \tag{2.6}$$

Proof. For fixed $t \in \mathbb{R}$ we use Plancherel in x and calculate

$$\begin{aligned}
&\left\| I_\delta^{\frac{\alpha}{2}}(W_\alpha(t)u_1, W_\alpha(t)u_2)\right\|_{L_x^2}^2 \\
=&\frac{1}{2\pi}\int\left|\int_{\xi=\xi_1+\xi_2} ||\xi_1|^\alpha - |\xi_2|^\alpha|_\delta^{\frac{1}{2}}\, e^{it(\xi_1|\xi_1|^\alpha+\xi_2|\xi_2|^\alpha)}\widehat{u}_1(\xi_1)\widehat{u}_2(\xi_2)\,d\xi_1\right|^2 d\xi \\
=&\frac{1}{2\pi}\int\int\int e^{itP(\xi,\xi_1,\eta_1)} f(\xi,\xi_1,\eta_1)\,d\eta_1 d\xi_1 d\xi \qquad (2.7)
\end{aligned}$$

with the phase function

$$P(\xi,\xi_1,\eta_1) = \xi_1|\xi_1|^\alpha + (\xi-\xi_1)|\xi-\xi_1|^\alpha - \eta_1|\eta_1|^\alpha - (\xi-\eta_1)|\xi-\eta_1|^\alpha$$

and

$$\begin{aligned}
&f(\xi,\xi_1,\eta_1) \\
&= ||\xi_1|^\alpha - |\xi-\xi_1|^\alpha|_\delta^{\frac{1}{2}}\, ||\eta_1|^\alpha - |\xi-\eta_1|^\alpha|_\delta^{\frac{1}{2}}\, \widehat{u}_1(\xi_1)\widehat{u}_2(\xi-\xi_1)\overline{\widehat{u}_1(\eta_1)\widehat{u}_2(\xi-\eta_1)}
\end{aligned}$$

For fixed ξ, ξ_1 the function $P_1(\eta_1) = P(\xi,\xi_1,\eta_1)$ has only two simple roots ξ_1, $\xi-\xi_1$ in the support of f. Moreover,

$$|P_1'(\eta_1)| = (1+\alpha)||\xi-\eta_1|^\alpha - |\eta_1|^\alpha| \geq (1+\alpha)\delta \ \text{ in } \operatorname{supp}(f) \qquad (2.8)$$

and

$$|P_1'(\xi_1)| = |P_1'(\xi-\xi_1)| = (1+\alpha)||\xi-\xi_1|^\alpha - |\xi_1|^\alpha|.$$

For the approximate identity (g_ε) from Lemma 2.1.5 we observe $\mathcal{F}g_\varepsilon \uparrow (2\pi)^{-\frac{1}{2}}$. By Fubini's theorem and the Fourier inversion formula

$$\begin{aligned}
I(\varepsilon) &:= (2\pi)^{-\frac{1}{2}}\int \mathcal{F}g_\varepsilon(t)\int\int\int e^{itP(\xi,\xi_1,\eta_1)} f(\xi,\xi_1,\eta_1)\,d\eta_1 d\xi_1 d\xi dt \\
&= \int\int\int g_\varepsilon(P(\xi,\xi_1,\eta_1)) f(\xi,\xi_1,\eta_1)\,d\eta_1 d\xi_1 d\xi
\end{aligned}$$

Now, because of (2.8) we may use the dominated convergence theorem to show

$$\lim_{\varepsilon\to 0} I(\varepsilon) = \int\int \lim_{\varepsilon\to 0}\int g_\varepsilon(P(\xi,\xi_1,\eta_1)) f(\xi,\xi_1,\eta_1)\,d\eta_1 d\xi_1 d\xi \qquad (2.9)$$

By Lemma 2.1.5 we conclude that this is equal to

$$\begin{aligned}
&\int\int \frac{f(\xi,\xi_1,\xi_1)}{|P_1'(\xi_1)|} + \frac{f(\xi,\xi_1,\xi-\xi_1)}{|P_1'(\xi-\xi_1)|}\,d\xi_1 d\xi \\
\leq&\tfrac{1}{1+\alpha}\int\int |\widehat{u_1}(\xi_1)|^2|\widehat{u_2}(\xi-\xi_1)|^2 + |\widehat{u_1}(\xi_1)\widehat{u_2}(\xi_1)||\widehat{u_1}(\xi-\xi_1)\widehat{u_2}(\xi-\xi_1)|\,d\xi_1 d\xi \\
\leq&\tfrac{2}{1+\alpha}\|u_1\|_{L_x^2}^2\|u_2\|_{L_x^2}^2
\end{aligned}$$

On the other hand, by the monotone convergence theorem and (2.7) we see

$$\lim_{\varepsilon\to 0} I(\varepsilon) = \left\| I_\delta^{\frac{\alpha}{2}}(W_\alpha u_1, W_\alpha u_2)\right\|_{L^2_{xt}}^2$$

which implies (2.6). □

Remark 2.1.7. The proof shows that this estimate is sharp in the sense that for $u \in \mathcal{S}(\mathbb{R})$

$$\lim_{\delta\to 0} \left\| I_\delta^{\frac{\alpha}{2}}(W_\alpha u, W_\alpha u)\right\|_{L^2_{xt}} = \sqrt{\frac{2}{1+\alpha}}\|u\|_{L^2}^2$$

2.2 Fourier restriction norm spaces

Now, we define function spaces which are built according to the symbol of the linear equation and therefore comprise much information about the dispersive properties of their solutions, cp. [Bou93].

We will introduce these spaces with the intention to apply the results in Chapters 3 and 4 and we will treat both cases separately, mainly because of the non-standard low frequency condition used in the case of equations of Benjamin-Ono type. We start with the case of the one dimensional Schrödinger equation on $\mathbb{T}$.

2.2.1 The periodic case: The Schrödinger equation

In this subsection we consider the periodic case and the phase function $\phi : \mathbb{Z} \to \mathbb{R}$, $\phi(\xi) = -\xi^2$, associated to the Schrödinger equation

$$\partial_t u(t) - i\partial_x^2 u(t) = 0$$

and $W(t)$ denotes the corresponding group of unitary solution operators

$$\mathcal{F}_x W(t) u_0(\xi) = e^{-it\xi^2}\mathcal{F}_x u_0(\xi)$$

Assume that $u(t,x) = \chi(t)W(t)u_0(x)$ where $\chi \in C_0^\infty(-2,2)$, $\chi \equiv 1$ in $[-1,1]$, so u is a solution on $[-1,1]$. Then,

$$\mathcal{F}u(\tau,\xi) = \widehat{\chi}(\tau+\xi^2)\mathcal{F}_x u_0(\xi)$$

Now, because $\widehat{\chi}$ is a Schwartz function, we notice that $\mathcal{F}u$ is highly localized near the discrete parabola $\tau = -\xi^2$ and decays in τ-direction faster than any polynomial.

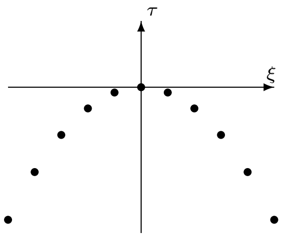

Figure 2.1: The parabola $\{(\xi, -\xi^2) \mid \xi \in \mathbb{Z}\}$ essentially supports the Fourier transform of short time solutions.

Definitions and basic properties of the spaces

The above observation motivates the following definition, which is essentially due to Bourgain [Bou93], see also [Gin96, GTV97, CKS⁺04, Grü00].

Definition 2.2.1. Let $s, b \in \mathbb{R}$. The Bourgain space $X_{s,b}$ associated to the Schrödinger operator $\partial_t - i\partial_x^2$ is defined as the completion of the space $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ with respect to the norm

$$\|f\|_{X_{s,b}} := \left(\sum_{\xi \in \mathbb{Z}} \int_{\mathbb{R}} \langle \xi \rangle^{2s} \langle \tau + \xi^2 \rangle^{2b} |\mathcal{F}f(\tau, \xi)|^2 \, d\tau \right)^{1/2} \tag{2.10}$$

$X^-_{s,b}$ is defined similarly by replacing $\langle \tau + \xi^2 \rangle$ with $\langle \tau - \xi^2 \rangle$.

Moreover, $Y_{s,b}$ is defined as the completion of the space $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ with respect to

$$\|f\|_{Y_{s,b}} := \left(\sum_{\xi \in \mathbb{Z}} \left(\int_{\mathbb{R}} \langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s |\mathcal{F}f(\tau, \xi)| \, d\tau \right)^2 \right)^{1/2} \tag{2.11}$$

and the space $Z_s := X_{s,\frac{1}{2}} \cap Y_{s,0}$ with norm

$$\|u\|_{Z_s} := \|u\|_{X_{s,\frac{1}{2}}} + \|u\|_{Y_{s,0}} \tag{2.12}$$

For $T > 0$ we define the restriction norm space

$$Z_s^T := \{u|_{[-T,T]} \mid u \in Z_s\}$$

with norm

$$\|u\|_{Z_s^T} = \inf\{\|\widetilde{u}\|_{Z_s} \mid u = \widetilde{u}|_{[-T,T]}, \, \widetilde{u} \in Z_s\}$$

We observe that the spaces $X_{s,b}$ and $X^-_{s,b}$ are isometrically isomorphic via complex conjugation $\|\overline{u}\|_{X_{s,b}} = \|u\|_{X^-_{s,b}}$.

Moreover, for a function $u \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ and $v(t) = W(-t)u(t)$ we have

$$\|J_t^b J_x^s v\|_{L^2}^2 = \sum_{\xi\in\mathbb{Z}} \int \langle\tau\rangle^{2b}\langle\xi\rangle^{2s}|\mathcal{F}v(\tau,\xi)|^2 d\tau$$

and

$$\mathcal{F}v(\tau,\xi) = \mathcal{F}u(\tau-\xi^2,\xi)$$

which shows

$$\|J_t^b J_x^s v\|_{L^2} = \|u\|_{X_{s,b}}$$

Therefore, $X_{s,b}$ is isometrically isomorphic to L^2 and $(X_{s,b})'$ is isometrically isomorphic to $X_{-s,-b}$. Every element of $X_{s,b}$ and $Y_{s,b}$ can be identified with a distribution in $\mathcal{S}'(\mathbb{R}\times\mathbb{T})$.

Lemma 2.2.2. *Let $s\in\mathbb{R}$, $T>0$. $\mathcal{S}^T(\mathbb{T})$ is a dense subset of Z_s^T.*

Proof. Let $u \in Z_s^T$. There exists $\widetilde{u} \in Z_s$ such that $u = \widetilde{u}|_{[-T,T]}$. Because $\mathcal{S}(\mathbb{R}\times\mathbb{T}) \subset Z_s$ is dense we find a sequence $\widetilde{u}_n \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ such that $\widetilde{u}_n \to \widetilde{u}$. With $u_n = \widetilde{u}_n|_{[-T,T]}$ it follows

$$\|u-u_n\|_{Z_s^T} \le \|\tilde{u}-\tilde{u}_n\|_{Z_s} \to 0$$

because $(\tilde{u}-\tilde{u}_n)|_{[-T,T]} = u-u_n$. □

Linear estimates

The following proposition contains the well-known and frequently used embedding estimates of Sobolev type.

Proposition 2.2.3.

$$\text{If } 2\le p<\infty, b\ge \frac{1}{2}-\frac{1}{p} : \|u\|_{L_t^pH^s} \le c\|u\|_{X_{s,b}} \tag{2.13}$$

$$\text{If } 2\le p,q<\infty, b\ge \frac{1}{2}-\frac{1}{p}\ ,\ s\ge \frac{1}{2}-\frac{1}{q} : \|u\|_{L_t^pL_x^q} \le c\|u\|_{X_{s,b}} \tag{2.14}$$

$$\text{If } 1<p\le 2\ ,\ b\le \frac{1}{2}-\frac{1}{p} : \|u\|_{X_{s,b}} \le c\|u\|_{L_t^pH^s} \tag{2.15}$$

We may replace $X_{s,b}$ by $X_{s,b}^-$. Moreover,

$$\|u\|_{C(\mathbb{R},H^s(\mathbb{T}))} \le c\|u\|_{Z_s} \quad , s\in\mathbb{R} \tag{2.16}$$

$$\|u\|_{Y_{s,b_1}} \le c\|u\|_{X_{s,b_2}} \quad , b_2 > b_1 + \frac{1}{2} \tag{2.17}$$

Proof. We consider $v = W(-\cdot)J_x^s u$ for $u \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$. Then, by Minkowski's and Sobolev's inequality

$$\|u\|_{L_t^p H_x^s} = \|v\|_{L_t^p L_x^2} \le \|v\|_{L_x^2 L_t^p} \le c\|J_t^b v\|_{L_x^2 L_t^2} = c\|u\|_{X_{s,b}}$$

and the claim (2.13) follows. Combining this with another application of Sobolev's inequality in the space variable $\|v(t)\|_{L_x^q} \le c\|J_x^s v(t)\|_{L_x^2}$ gives (2.14). Estimate (2.15) follows by duality from (2.13). The estimates for $X_{s,b}^-$ follow from the invariance of $L_t^p H_x^s$ and $L_t^p L_x^q$ under complex conjugation. To prove (2.16) it suffices to prove an estimate for the sup norm for $u \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ by density. We write for $t \in \mathbb{R}$

$$\mathcal{F}_x u(t,\xi) = c\int_{\mathbb{R}} e^{it\tau}\mathcal{F}u(\tau,\xi)\,d\tau$$

by the Fourier inversion formula. This yields

$$\|u(t)\|_{H^s} = c\left\|\int_{\mathbb{R}} e^{it\tau}\langle\xi\rangle^s\mathcal{F}u(\tau,\xi)\,d\tau\right\|_{L_\xi^2} \le c\|\langle\xi\rangle^s\mathcal{F}u(\tau,\xi)\|_{L_\xi^2 L_\tau^1}$$

Now we take the supremum with respect to t. The last estimate follows from the Cauchy-Schwarz inequality in τ:

$$\begin{aligned}\|u\|_{Y_{s,b_1}}^2 &= \sum_{\xi\in\mathbb{Z}}\left(\int_{\mathbb{R}}\langle\tau+\xi^2\rangle^{b_1}\langle\xi\rangle^s|\mathcal{F}f(\tau,\xi)|\,d\tau\right)^2\\ &\le \sum_{\xi\in\mathbb{Z}}\int_{\mathbb{R}}\langle\tau+\xi^2\rangle^{2b_1-2b_2}\,d\tau\int_{\mathbb{R}}\langle\tau+\xi^2\rangle^{2b_2}\langle\xi\rangle^{2s}|\mathcal{F}f(\tau,\xi)|^2\,d\tau\end{aligned}$$

Since by assumption $2b_1 - 2b_2 < -1$, there exists $c > 0$, such that for all ξ

$$\int_{\mathbb{R}}\langle\tau+\xi^2\rangle^{2b_1-2b_2}\,d\tau \le c$$

which finishes the proof. □

We review the L^4 estimate from Theorem 2.1.1 as well as an L^6 estimate in the framework of $X_{s,b}$ due to J. Bourgain [Bou93], but in versions of A. Grünrock [Grü00] which are global in time.

Proposition 2.2.4. *For $-b', b > \frac{3}{8}$ there exists $c > 0$, such that*

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{T})} \le c\|u\|_{X_{0,b}} \tag{2.18}$$

and its dual version

$$\|u\|_{X_{0,b'}} \leq c\|u\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})} \tag{2.19}$$

hold true. Moreover, for $b > \frac{1}{2}$ and any $\varepsilon > 0$ there exists $c > 0$, such that

$$\|u\|_{L^6(\mathbb{R}\times\mathbb{T})} \leq c\|u\|_{X_{\varepsilon,b}} \tag{2.20}$$

and its dual version hold true. Finally, for all $\varepsilon > 0$ and $2 \leq p < 6$ there exists $c > 0$ such that

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{T})} \leq c\|u\|_{X_{\varepsilon,\frac{1}{2}}} \tag{2.21}$$

In all estimates we may replace $X_{s,b}$ by $X^-_{s,b}$.

Proof. The estimates (2.18), (2.19) and (2.20) can be found as Lemma 2.1 and Lemma 2.2 in [Grü00]. To prove (2.21) we fix $\varepsilon > 0$ and $2 \leq p < 6$ and interpolate (see e.g. [Grü02] Lemma 1.4) between (2.20)

$$\|u\|_{L^6(\mathbb{R}\times\mathbb{T})} \leq c\|u\|_{X_{\delta_1,\frac{1}{2}+\delta_2}}$$

for small enough $\delta_1, \delta_2 > 0$ and the trivial statement

$$\|u\|_{L^2(\mathbb{R}\times\mathbb{T})} \leq \|u\|_{X_{0,0}}$$

and obtain (2.21). That the estimates hold both for $X_{s,b}$ and $X^-_{s,b}$ results from the invariance of L^p spaces under complex conjugation. □

We summarize the behavior of the $X_{s,b}, Y_{s,0}$ norms under multiplication with cutoffs in time. Let $\chi \in C_0^\infty((-2,2))$ denote a symmetric function with $\chi \equiv 1$ in $[-1,1]$ and $\chi_T(t) = \chi(t/T)$. For the following lemma see e.g. J. Ginibre - Y. Tsutsumi - G. Velo [GTV97], Lemma 2.5.

Lemma 2.2.5. *Let $s \in \mathbb{R}$ and $0 < T \leq 1$. There exists $c > 0$, such that*

$$\|\chi_T u\|_{Y_{s,0}} \leq c\|u\|_{Y_{s,0}}$$

Moreover, for $0 \leq b_1 < b_2 < \frac{1}{2}$ or $-\frac{1}{2} < b_1 < b_2 \leq 0$ there exists $c > 0$, such that

$$\|\chi_T u\|_{X_{s,b_1}} \leq cT^{b_2-b_1}\|u\|_{X_{s,b_2}}$$

and for any $\delta > 0$ there exists $c > 0$, such that

$$\|\chi_T u\|_{X_{s,\frac{1}{2}}} \leq cT^{-\delta}\|u\|_{X_{s,\frac{1}{2}}}$$

Proof. It suffices to consider smooth u. The first estimate follows from Young's inequality in τ: For fixed ξ we have

$$\begin{aligned}\|\mathcal{F}(\chi_T u)(\cdot,\xi)\|_{L^1} =& c\| \int_{\mathbb{R}} \widehat{\chi_T}(\cdot-\tau_1)\mathcal{F}u(\tau_1,\xi)\,d\tau_1\|_{L^1}\\ \leq& c\|\widehat{\chi_T}\|_{L^1}\|\mathcal{F}u(\cdot,\xi)\|_{L^1}\end{aligned}$$

Because $\|\widehat{\chi_T}\|_{L^1} = \|\widehat{\chi}\|_{L^1}$ the estimate follows by taking the weighted L^2 norms with respect to ξ on both sides. For the proof of the second estimate it suffices to consider $0 \leq b_1 < b_2 < \frac{1}{2}$ by duality. We define $v = W(-\cdot)u$ and fix ξ. Then,

$$\|\chi_T \mathcal{F}_x v(\cdot,\xi)\|_{H^{b_1}} \leq c\|\chi_T\|_{H^b}\|\mathcal{F}_x v(\cdot,\xi)\|_{H^{b_2}}$$

for admissible b according to Corollary 1.1.12. In the case $0 \leq b_1 < b_2 < \frac{1}{2}$ we may choose $b = \frac{1}{2} - (b_2 - b_1)$ and in the case of $b_1 = b_2 = \frac{1}{2}$ we may choose $b = \frac{1}{2} + \delta$ for $\delta > 0$. Now, taking the weighted L^2 norms with respect to the ξ variable we arrive at

$$\|\chi_T v\|_{H^s_x H^{b_1}_t} \leq c\|\chi_T\|_{H^b}\|v\|_{H^s_x H^{b_2}_t}$$

Now, a short calculation shows that $\|\chi_T\|_{H^b} \leq \sqrt{2}T^{\frac{1}{2}}(1+T^{-b})\|\chi\|_{H^b}$ and the second and third estimate follow. $\square$

The next proposition contains estimates for the linear homogeneous and inhomogeneous problem with a proof based on the strategy from J. Colliander - M. Keel - G. Staffilani - H. Takaoka - T. Tao [CKS+04], Lemma 3.1.

Proposition 2.2.6. *Let $s \in \mathbb{R}$. There exists $c > 0$, such that for $u_0 \in H^s(\mathbb{T})$*

$$\|\chi W(t)u_0\|_{Z_s} \leq c\|u_0\|_{H^s} \tag{2.22}$$

and for all $f \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$

$$\left\|\chi\int_0^t W(t-t')f(t')\,dt'\right\|_{Z_s} \leq c\|f\|_{Y_{s,-1}} + c\|f\|_{X_{s,-\frac{1}{2}}} \tag{2.23}$$

Proof. It suffices to consider smooth u_0. Let us write

$$\mathcal{F}(\chi W(\cdot)u_0)(\tau,\xi) = \widehat{\chi}(\tau+\xi^2)\mathcal{F}_x u_0(\xi)$$

Then, because $\widehat{\chi}$ is a Schwartz function the estimate (2.22) follows. Now we turn to the estimate (2.23) for the linear inhomogeneous equation. We may assume that $\operatorname{supp}(f) \subset \{(t,x) \mid |t| \leq 2\}$, since the complementary part does not contribute. Then, we have

$$\chi(t) \int_0^t W(t-t')f(t')\,dt' = F_1(t) + F_2(t)$$

with

$$F_1(t) = \frac{1}{2}\chi(t)W(t) \int_{\mathbb{R}} \varphi(t')W(-t')f(t')\,dt'$$
$$F_2(t) = \frac{1}{2}\chi(t) \int_{\mathbb{R}} \varphi(t-t')W(t-t')f(t')\,dt'$$

and $\varphi(t') = \chi(t'/10)\operatorname{sign}(t')$. Moreover,

$$|\mathcal{F}\varphi(\tau)| \leq c\langle\tau\rangle^{-1} \tag{2.24}$$

Now, by estimate (2.22)

$$\|F_1\|_{Z_s} \leq c \left\| \int_{\mathbb{R}} \varphi(t')W(-t')f(t')\,dt' \right\|_{H^s(\mathbb{T})}$$

and by Parseval's equality

$$\mathcal{F}_x \left(\int_{\mathbb{R}} \varphi(t')W(-t')f(t')\,dt' \right)(\xi) = \int_{\mathbb{R}} \overline{\mathcal{F}\varphi}(\tau+\xi^2)\mathcal{F}f(\tau,\xi)\,d\tau$$

which implies

$$\left\| \int_{\mathbb{R}} \varphi(t')W(-t')f(t')\,dt' \right\|_{H^s(\mathbb{T})} \leq c\|f\|_{Y_{s,-1}}$$

by (2.24). In order to show the estimate for F_2 we first apply Lemma 2.2.5 with $T=1$

$$\|F_2\|_{Z_s} \leq c \left\| \int_{\mathbb{R}} \varphi(t-t')W(t-t')f(t')\,dt' \right\|_{Z_s}$$

and observe

$$\mathcal{F} \left(\int_{\mathbb{R}} \varphi(t-t')W(t-t')f(t')\,dt' \right)(\tau,\xi) = \mathcal{F}\varphi(\tau+\xi^2)\mathcal{F}f(\tau,\xi)$$

Because of (2.24) the estimate

$$|\mathcal{F}\varphi(\tau+\xi^2)\mathcal{F}f(\tau,\xi)| \leq c\langle\tau+\xi^2\rangle^{-1}|\mathcal{F}f(\tau,\xi)|$$

holds true and the claim follows. □

2.2.2 The non-periodic case: Benjamin-Ono type equations

In this subsection we consider the non-periodic case and the phase function $\phi : \mathbb{R} \to \mathbb{R}, \phi(\xi) = \xi|\xi|^\alpha$ for $1 \le \alpha \le 2$. The spaces defined here are related to the linear equations

$$\partial_t u(t) - |D|^\alpha \partial_x u(t) = 0$$

which in the endpoints coincide with the linear Benjamin-Ono equation for $\alpha = 1$ and with the Airy equation for $\alpha = 2$, respectively. This will be applied in Chapter 4 to prove well-posedness of these equations with the quadratic nonlinearities $\partial_x u^2$.

Definitions and basic properties

We start with defining a space of initial data slightly smaller than $H^s(\mathbb{R})$.

Definition 2.2.7. For $s \in \mathbb{R}$ and $0 \le \omega < \frac{1}{2}$ we define the Sobolev space $H^{(s,\omega)}$ as the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|u\|^2_{H^{(s,\omega)}} := \int_{\mathbb{R}} \langle\xi\rangle^{2s+2\omega} |\xi|^{-2\omega} |\mathcal{F}u(\xi)|^2 \, d\xi \tag{2.25}$$

Remark 2.2.8. Since this norm is stronger than the corresponding $H^s(\mathbb{R})$ norm[1] from Definition 1.1.5, it is continuously embedded in $H^s(\mathbb{R})$ and in particular all elements define distributions in $\mathcal{S}'(\mathbb{R})$. Moreover, $H^{(s,\omega)}$ is a Hilbert space.

Remark 2.2.9. Notice that since $0 \le \omega < \frac{1}{2}$ we have the continuous embedding

$$L^p(\mathbb{R}) \cap H^s(\mathbb{R}) \subset H^{(s,\omega)}$$

for $1 \le p \le 2$ and $\omega < \frac{1}{p} - \frac{1}{2}$ by the Hausdorff-Young inequality, see [Hör83], Theorem 7.1.13.

Next, we introduce resolution spaces which are adaptions of the Bourgain spaces [Bou93] to our setting. In particular they inherit the low frequency condition from the space of initial data.

Definition 2.2.10. For $0 \le \omega < \frac{1}{2}$ and $s, b \in \mathbb{R}$ we define the space $X_{s,\omega,b}$ as the completion of $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm $\|u\|_{X_{s,\omega,b}}$ defined via

$$\begin{aligned} &\|u\|^2_{X_{s,\omega,b}} \\ &= \int_{\mathbb{R}^2} |\xi|^{-2\omega} \langle\xi\rangle^{2s-2\alpha\omega} \langle|\tau| + |\xi|^{1+\alpha}\rangle^{2\omega} \langle\tau - \xi|\xi|^\alpha\rangle^{2b} |\mathcal{F}u(\tau,\xi)|^2 \, d\tau d\xi \end{aligned} \tag{2.26}$$

[1]With the notion of homogeneous Sobolev spaces this space would be $H^s(\mathbb{R}) \cap \dot{H}^{-\omega}(\mathbb{R})$

Moreover, we define for $T > 0$ the restriction norm space

$$X^T_{s,\omega,b} := \{u|_{[-T,T]} \mid u \in X_{s,\omega,b}\}$$

with norm

$$\|u\|_{X^T_{s,\omega,b}} = \inf\{\|\widetilde{u}\|_{X_{s,\omega,b}} \mid u = \widetilde{u}|_{[-T,T]}, \widetilde{u} \in X_{s,\omega,b}\}.$$

Remark 2.2.11. $X_{s,\omega,b}$ is closed under complex conjugation due to the symmetry of the weights.

Similar to Lemma 2.2.2 we observe

Lemma 2.2.12. *Let $s, b \in \mathbb{R}$, $0 \leq \omega < \frac{1}{2}$ and $T > 0$. $\mathcal{S}^T(\mathbb{R})$ is a dense subset of $X^T_{s,\omega,b}$.*

Linear and bilinear estimates

Proposition 2.2.13. *Let $b > \frac{1}{2}$, $0 \leq \omega < \frac{1}{2}$ and $s \in \mathbb{R}$. Then,*

$$X_{s,\omega,b} \subset C\big(\mathbb{R}, H^{(s,\omega)}\big)$$

and

$$\|u\|_{C(\mathbb{R},H^{(s,\omega)})} \leq c\|u\|_{X_{s,\omega,b}} \tag{2.27}$$

Proof. Let $u \in \mathcal{S}(\mathbb{R}^2)$. Then, by the Fourier inversion formula we have

$$\|u(t)\|_{H^{(s,\omega)}} \leq c\||\xi|^{-\omega}\langle\xi\rangle^{s+\omega}\mathcal{F}u\|_{L^2_\xi L^1_\tau}$$

for $t \in \mathbb{R}$, and by Cauchy-Schwarz

$$\begin{aligned}\||\xi|^{-\omega}\langle\xi\rangle^{s+\omega}\mathcal{F}u\|_{L^2_\xi L^1_\tau} \leq& c\||\xi|^{-\omega}\langle\xi\rangle^{s+\omega}\langle\tau - \xi|\xi|^\alpha\rangle^b\mathcal{F}u\|_{L^2}\\ \leq& c\||\xi|^{-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle|\tau| + |\xi|^{\alpha+1}\rangle^\omega\langle\tau - \xi|\xi|^\alpha\rangle^b\mathcal{F}u\|_{L^2}\end{aligned}$$

and the estimate (2.27) follows. The claim follows by density. □

Let $\chi \in C^\infty_0((-2,2))$ be symmetric, $\chi \equiv 1$ in $[-1,1]$ and $\chi_T(t) = \chi(t/T)$.

Proposition 2.2.14. *Let $0 \leq \omega < \frac{1}{2}$, $s, b \in \mathbb{R}$. Then,*

$$\|\chi W_\alpha u_0\|_{X_{s,\omega,b}} \leq c\|u_0\|_{H^{(s,\omega)}} \tag{2.28}$$

for all $u_0 \in H^{(s,\omega)}$.

Proof. We may assume $u_0 \in \mathcal{S}(\mathbb{R}) \subset H^{(s,\omega)}$ by density and calculate

$$\mathcal{F}(\chi W_\alpha u_0)(\tau,\xi) = \mathcal{F}_t\chi(\tau - \xi|\xi|^\alpha)\widehat{u_0}(\xi).$$

Let $N \in \mathbb{N}$ be a positive integer with $b < N-1$. Since $\mathcal{F}_t\chi$ is a Schwartz function, we conclude for $s = \omega = 0$

$$\begin{aligned} \|\chi W_\alpha u_0\|^2_{X_{0,0,b}} &= c\int_{\mathbb{R}^2} \langle \tau - \xi|\xi|^\alpha\rangle^{2b} |\mathcal{F}_t\chi(\tau - \xi|\xi|^\alpha)\widehat{u_0}(\xi)|^2 \, d\tau d\xi \\ &\le c\int_{\mathbb{R}}\int_{\mathbb{R}} \langle \tau - \xi|\xi|^\alpha\rangle^{2b} \langle \tau - \xi|\xi|^\alpha\rangle^{-2N} \, d\tau |\widehat{u_0}(\xi)|^2 \, d\xi \\ &\le c\|u_0\|^2_{L^2} \end{aligned}$$

Now let $\omega \ge 0, s, b \in \mathbb{R}$. By using the inequality

$$\langle |\tau| + |\xi|^{1+\alpha}\rangle^\omega \le c(\langle \tau - \xi|\xi|^\alpha\rangle^\omega + \langle\xi\rangle^{(1+\alpha)\omega}) \tag{2.29}$$

we estimate

$$|\xi|^{-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle |\tau| + |\xi|^{1+\alpha}\rangle^\omega |\widehat{u_0}(\xi)| \le c\langle \tau - \xi|\xi|^\alpha\rangle^\omega |\widehat{v_0}(\xi)|,$$

where $\widehat{v_0}(\xi) = |\xi|^{-\omega}\langle\xi\rangle^{s+\omega}\widehat{u_0}(\xi)$. With $b' = b + \omega$ this gives

$$\|\chi W_\alpha u_0\|_{X_{s,\omega,b}} \le c\|\chi W_\alpha v_0\|_{X_{0,0,b'}} \le c\|v_0\|_{L^2} = c\|u_0\|_{H^{(s,\omega)}}.$$

which proves the proposition. □

Now, we prove an estimate for the linear, inhomogeneous problem.

Proposition 2.2.15. *Let $0 \le \omega < \frac{1}{2}$, $s \in \mathbb{R}$ and $-\frac{1}{2} < b' \le 0 \le b < b' + 1$ as well as $b' \le -\omega$. There exists $\varepsilon > 0$, such that for all $0 < T \le 1$*

$$\left\| \chi_T(t)\int_0^t W_\alpha(t-t')f(t')\,dt' \right\|_{X_{s,\omega,b}} \le cT^\varepsilon \|f\|_{X_{s,\omega,b'}} \tag{2.30}$$

for all $f \in \mathcal{S}(\mathbb{R}^2)$.

Proof. In the case $\omega = 0$ this is a well known estimate, see e.g. [Gin96], Lemme 3.2. We will reduce (2.30) to this case. Define

$$\mathcal{F}_x g(t)(\xi) := |\xi|^{-\omega}\mathcal{F}_x f(t)(\xi).$$

and

$$I(f)(t) := \chi_T(t)\int_0^t W_\alpha(t-t')f(t')\,dt$$

Using (2.29) we have

$$\|I(f)\|_{X_{s,\omega,b}} \leq c\|I(g)\|_{X_{s-\alpha\omega,0,b+\omega}} + c\|I(g)\|_{X_{s+\omega,0,b}}$$

where on the right hand side the usual Bourgain type norms appear. Thus, by our restrictions on b, b' and ω and using the standard estimate

$$\begin{aligned}
\|I(g)\|_{X_{s-\alpha\omega,0,b+\omega}} &\leq cT^{\varepsilon}\, \|g\|_{X_{s-\alpha\omega,0,b'+\omega}} \\
\|I(g)\|_{X_{s+\omega,0,b}} &\leq cT^{\varepsilon}\, \|g\|_{X_{s+\omega,0,b'}}
\end{aligned}$$

Because of the inequalities

$$\begin{aligned}
\langle \tau - \xi|\xi|^{\alpha}\rangle^{b'+\omega} &\leq \langle \tau - \xi|\xi|^{\alpha}\rangle^{b'} \langle |\tau| + |\xi|^{1+\alpha}\rangle^{\omega} \\
\langle \xi\rangle^{s+\omega} &\leq c\langle \xi\rangle^{s-\alpha\omega} \langle |\tau| + |\xi|^{1+\alpha}\rangle^{\omega}
\end{aligned}$$

we find the upper bound

$$\|I(f)\|_{X_{s,\omega,b}} \leq cT^{\varepsilon}\, \|f\|_{X_{s,\omega,b'}}$$

as desired. □

This proof shows that the space $X_{s,\omega,b}$ is an intersection of the usual Bourgain spaces (in addition to the low frequency condition), see also Remark 4.2.7. Next, we insert the endpoint Strichartz and the local smoothing estimate into the $X_{0,0,b}$ setting.

Lemma 2.2.16. *For $b > \frac{1}{2}$ we have*

$$\|J^{\frac{\alpha-1}{4}} u\|_{L^4_t L^\infty_x} \leq c\|u\|_{X_{0,0,b}} \tag{2.31}$$

$$\|J^{\frac{\alpha}{2}} u\|_{L^\infty_x L^2_t} \leq c\|u\|_{X_{0,0,b}} \tag{2.32}$$

Proof. From Theorem 2.1.3 we know that

$$\||D|^{\frac{\alpha-1}{4}} W_\alpha(t)u_0\|_{L^4_t L^\infty_x} \leq c\|u_0\|_{L^2} \tag{2.33}$$

Now we use a general property of Bourgain spaces $X_{0,0,b}$ with $b > \frac{1}{2}$, see e.g. [Gin96] Lemme 3.3. By the Fourier inversion formula we may write $u \in X_{0,0,b}$ as

$$u(t) = c\int e^{it\tau} W_\alpha(t)\mathcal{F}_t(W_\alpha(-\cdot)u)(\tau)\, d\tau \tag{2.34}$$

and this implies

$$\||D|^{\frac{\alpha-1}{4}}u\|_{L_t^4L_x^\infty} \leq c\int \||D|^{\frac{\alpha-1}{4}}W_\alpha(t)\mathcal{F}_t(W_\alpha(-\cdot)u)(\tau)\|_{L_t^4L_x^\infty}\,d\tau$$

Using (2.33) and Cauchy-Schwarz we arrive at

$$\begin{aligned}\||D|^{\frac{\alpha-1}{4}}u\|_{L_t^4L_x^\infty} \leq & c\left(\int\langle\tau\rangle^{-2b}d\tau\right)^{\frac{1}{2}}\left(\int\langle\tau\rangle^{2b}\|\mathcal{F}_t(W_\alpha(-\cdot)u)(\tau)\|_{L_x^2}^2\,d\tau\right)^{\frac{1}{2}}\\ \leq & c\left(\int\int\langle\tau-\xi|\xi|^\alpha\rangle^{2b}|\mathcal{F}u(\tau,\xi)|^2\,d\tau d\xi\right)^{\frac{1}{2}}\end{aligned}$$

since $b>\frac{1}{2}$ and therefore

$$\||D|^{\frac{\alpha-1}{4}}u\|_{L_t^4L_x^\infty} \leq c\|u\|_{X_{0,0,b}} \tag{2.35}$$

By smooth cutoffs in frequency, we split u into a low frequency part u^{low}

$$\mathcal{F}u^{low}(\tau,\xi) = \chi(\xi)\mathcal{F}u(\tau,\xi)$$

and a high frequency part $u^{high} := u - u^{low}$. Then,

$$\|J^{\frac{\alpha-1}{4}}u\|_{L_t^4L_x^\infty} \leq \|J^{\frac{\alpha-1}{4}}u^{low}\|_{L_t^4L_x^\infty} + \|J^{\frac{\alpha-1}{4}}u^{high}\|_{L_t^4L_x^\infty}$$

By an application of the Sobolev inequality, the first part is bounded by

$$c\|J^{\frac{\alpha+1}{4}+\varepsilon}u^{low}\|_{L_t^4L_x^2} \leq c\|u\|_{L_t^4L_x^2} \leq c\|u\|_{X_{0,0,b}}$$

whereas the second term is bounded by

$$c\||D|^{-\frac{\alpha-1}{4}}J^{\frac{\alpha-1}{4}}u^{high}\|_{X_{0,0,b}} \leq c\|u\|_{X_{0,0,b}}$$

due to (2.35), which gives the desired estimate (2.31). The second claim follows in a similar way: As above we use (2.4) and arrive at

$$\||D|^{\frac{\alpha}{2}}u\|_{L_x^\infty L_t^2} \leq c\|u\|_{X_{0,0,b}}$$

which proves the desired estimate for the high frequency part. For the low frequency part we use

$$\|J^{\frac{\alpha}{2}}u^{low}\|_{L_x^\infty L_t^2} \leq c\|J^{\frac{\alpha}{2}}u^{low}\|_{L_t^2L_x^\infty} \leq c\|J^{\frac{\alpha+1+\varepsilon}{2}}u^{low}\|_{L_t^2L_x^2}$$

which is obviously bounded by $\|u\|_{X_{0,0,b}}$. $\square$

The next statement directly follows from our bilinear smoothing estimate from Theorem 2.1.6.

Corollary 2.2.17. *For $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$ we define the bilinear operator I_*^s via*

$$\mathcal{F}\, I_*^s(u_1,u_2)(\tau,\xi) = \int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \left||\xi_1|^{2s} - |\xi_2|^{2s}\right|^{\frac{1}{2}} \mathcal{F}u_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\, d\tau_1 d\xi_1$$

For $b > \frac{1}{2}$ there exists a unique, bilinear extension $I_*^{\frac{\alpha}{2}}$ with

$$\left\| I_*^{\frac{\alpha}{2}}(u_1,u_2)\right\|_{L^2_{xt}} \le c\|u_1\|_{X_{0,0,b}}\|u_2\|_{X_{0,0,b}} \quad , u_1, u_2 \in X_{0,0,b} \tag{2.36}$$

For $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$ we define the operator $K_*^{\frac{\alpha}{2}}$

$$\mathcal{F}\, K_*^{\frac{\alpha}{2}}(u_1,u_2)(\tau,\xi) = \int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \left||\xi|^{\alpha} - |\xi_1|^{\alpha}\right|^{\frac{1}{2}} \mathcal{F}\overline{u}_1(\tau_1,\xi_1)\mathcal{F}u_2(\tau_2,\xi_2)\, d\tau_1 d\xi_1$$

$K_*^{\frac{\alpha}{2}}$ is the formal adjoint of $u_2 \mapsto I_*^{\frac{\alpha}{2}}(u_1,u_2)$ with respect to L^2_{xt} and for $b > \frac{1}{2}$ there exists a unique, bilinear extension $K_*^{\frac{\alpha}{2}}$ with

$$\left\| K_*^{\frac{\alpha}{2}}(u_1,u_2)\right\|_{X_{0,0,-b}} \le c\|u_1\|_{X_{0,0,b}}\|u_2\|_{L^2_{xt}} \quad , u_1 \in X_{0,0,b}\, , u_2 \in L^2_{xt} \tag{2.37}$$

Proof. We may assume that $u_1, u_2, v \in \mathcal{S}(\mathbb{R}^2)$, then

$$\|I_*^{\frac{\alpha}{2}}(u_1,u_2)\|_{L^2_{xt}} = \lim_{\delta\to 0} \|I_\delta^{\frac{\alpha}{2}}(u_1,u_2)\|_{L^2_{xt}}$$

We write u_1, u_2 as in (2.34) and estimate

$$\begin{aligned}
&\|I_\delta^{\frac{\alpha}{2}}(u_1,u_2)\|_{L^2_{xt}}\\
\le& \int\!\!\int \left\| I_\delta^{\frac{\alpha}{2}}(W_\alpha \mathcal{F}_t(W_\alpha(-\cdot)u_1)(\tau_1), W_\alpha \mathcal{F}_t(W_\alpha(-\cdot)u_2)(\tau_2))\right\|_{L^2_{xt}} d\tau_1 d\tau_2\\
\le& c\int\!\!\int \|\mathcal{F}_t(W_\alpha(-\cdot)u_1)(\tau_1)\|_{L^2_x} \|\mathcal{F}_t(W_\alpha(-\cdot)u_2)(\tau_2)\|_{L^2_x}\, d\tau_1 d\tau_2
\end{aligned}$$

where we used the estimate (2.6) for the last inequality. Next, we insert $\langle\tau_i\rangle^{-2b}\langle\tau_i\rangle^{2b}$ in each integral and use Cauchy-Schwarz to obtain

$$\left\| I_\delta^{\frac{\alpha}{2}}(u_1,u_2)\right\|_{L^2_{xt}} \le c\|u_1\|_{X_{0,0,b}}\|u_2\|_{X_{0,0,b}}$$

with a constant independent of $\delta > 0$, which proves the first claim. Now we calculate the adjoint of $I_*^{\frac{\alpha}{2}}(u_1, \cdot)$ with respect to L^2_{xt} for Schwartz functions. By Plancherel

$$
\begin{aligned}
&\left(I_*^{\frac{\alpha}{2}}(u_1,u_2),v\right)_{L^2_{xt}}\\
&=\int\int ||\xi_1|^\alpha-|\xi-\xi_1|^\alpha|^{\frac12}\widehat{u}_1(\tau_1,\xi_1)\widehat{u}_2(\tau-\tau_1,\xi-\xi_1)\,d\tau_1 d\xi_1\overline{\widehat{v}(\tau,\xi)}\,d\tau d\xi\\
&=\int \widehat{u}_2(\tau-\tau_1,\xi-\xi_1)\overline{||\xi_1|^\alpha-|\xi-\xi_1|^\alpha|^{\frac12}\widehat{\overline{u}}_1(-\tau_1,-\xi_1)\widehat{v}(\tau,\xi)}\,d\tau_1 d\xi_1 d\tau d\xi
\end{aligned}
$$

The change of variables $(\tau_1, \xi_1, \tau, \xi) \mapsto (-\tau_1, -\xi_1, \tau - \tau_1, \xi - \xi_1)$ yields

$$
\begin{aligned}
&\left(I_*^{\frac{\alpha}{2}}(u_1,u_2),v\right)_{L^2_{xt}}\\
&=\int \widehat{u}_2(\tau,\xi)\overline{\int ||\xi_1|^\alpha-|\xi|^\alpha|^{\frac12}\widehat{\overline{u}}_1(\tau_1,\xi_1)\widehat{v}(\tau-\tau_1,\xi-\xi_1)\,d\tau_1 d\xi_1}\,d\tau d\xi\\
&=\left(u_2,K_*^{\frac{\alpha}{2}}(u_1,v)\right)_{L^2_{xt}}
\end{aligned}
$$

due to the Plancherel identity. Therefore, (2.37) is dual to (2.36). □

2.3 Notes and References

The linear L^pL^q estimates are well-known and often referred to as Strichartz estimates due to related work of R.S. Strichartz [Str77] on the wave equation. There has been a lot of progress in generalizing these estimates and in this direction we would like to refer in particular to the works of C.E. Kenig - G. Ponce - L. Vega [KPV91a], M. Keel - T. Tao [KT98] and H. Koch - D. Tataru [KT05a] and the references therein. The smoothing properties of dispersive equations were discovered and developed by T. Kato [Kat83], S. N. Kruzhkov - A.V. Faminskii [KF83], P. Constantin - J.-C. Saut [CS88], P. Sjölin [Sjö87], L. Vega [Veg88], C.E. Kenig - G. Ponce - L. Vega [KPV91a] and others. The main reference for this chapter is [KPV91a]. The bilinear smoothing estimate from Theorem 2.1.6 generalizes the results from [Grü05a], Lemma 1, see also [Grü01], Def. 2.1 and Lemma 2.4.

The Fourier restriction norm spaces in connection with the KdV and Schrödinger equations were introduced by J. Bourgain in [Bou93] (and in connection with well-posedness for wave equations by S. Klainerman - M. Machedon [KM93, KM95]) and found application in several situations. The papers of J. Ginibre [Gin96], J. Ginibre - Y. Tsutsumi - G. Velo [GTV97],

J. Colliander - M. Keel - G. Staffilani - H. Takaoka - T. Tao [CKS+04] and the first chapter of A. Grünrock's thesis [Grü02] are the main references for this chapter concerning facts about the Fourier restriction norm method.

Chapter 3

Derivative nonlinear Schrödinger equations

Throughout this chapter we consider the one dimensional periodic case and the phase function $\phi : \mathbb{Z} \to \mathbb{R}$, $\phi(\xi) = -\xi^2$.

3.1 Motivation and main results

In this chapter we discuss the Cauchy problem for the derivative nonlinear Schrödinger (DNLS) equation with the periodic boundary condition

$$\begin{aligned} \partial_t u(t) - i\partial_x^2 u(t) &= \partial_x(|u|^2 u)(t)\,, \quad t \in (-T, T) \\ u(0) &= u_0 \in H^s(\mathbb{T}) \end{aligned} \tag{3.1}$$

The L^2 norm is a conserved quantity and we also consider the modified equation

$$\begin{aligned} \partial_t u(t) - i\partial_x^2 u(t) &= 2\left(|u|^2 - \frac{1}{2\pi}\int_0^{2\pi} |u|^2 dx\right)\partial_x u(t) + u^2 \partial_x \overline{u}(t) \\ u(0) &= u_0 \in H^s(\mathbb{T}) \end{aligned} \tag{3.2}$$

which we will refer to as (DNLS_0).

Our aim is to prove local and global well-posedness in low regularity Sobolev spaces and to analyze the regularity of the flow maps. We will show that both problems are well-posed for $s \geq \frac{1}{2}$. But there is one remarkable difference: Heuristically, (DNLS_0) should be viewed as a regularized equation since the term in front of $\partial_x u$ has zero mean value, hence the zero frequency term, which leads to a strong transport effect, is not present. This will be expressed more precisely by the regularity properties of the flows. Both equations are linked via a translation, see Section 3.2. Moreover, for both problems it is impossible to prove useful tri-linear estimates directly, see Subsection 3.4.3. We resolve this with the help of a nonlinear transformation, see Section 3.2.

Theorem 3.1.1. *There exists a non-increasing function $T^* : (0,\infty) \to (0,\infty)$, such that for all $R > 0$ and $0 < T \leq T^*(R)$ there exists a continuous map*

$$S_{R,T} : B_R = \{u_0 \in H^{\frac{1}{2}}(\mathbb{T}) \mid \|u_0\|_{H^{\frac{1}{2}}} < R\} \to C\big([-T,T], H^{\frac{1}{2}}(\mathbb{T})\big)$$

with the properties:

(i) For all $u_0 \in B_R$ the function $u = S_{R,T}(u_0)$ is a solution of the integral equation

$$u(t) = W(t)u_0 + \int_0^t W(t-t')\partial_x(|u|^2 u)(t')dt', \quad t \in (-T,T) \tag{3.3}$$

associated to the Cauchy problem (3.1).

(ii) For every $s \geq \frac{1}{2}$ we have $S_{R,T}(B_R \cap H^s(\mathbb{T})) \subset C([-T,T], H^s(\mathbb{T}))$ and

$$S_{R,T}\,|_{B_R \cap H^s(\mathbb{T})} : B_R \cap H^s(\mathbb{T}) \to C\big([-T,T], H^s(\mathbb{T})\big)$$

is continuous. For $C_r(0) = \{u_0 \in H^s \mid \|u_0\|_{H^s} \leq r\}$, its restriction to $C_r(0)$ is not uniformly continuous.

(iii) For $s > \frac{7}{6}$ the function $u = S_{R,T}(u_0)$ is the unique solution of (3.3) in

$$\{v \in C([-T,T], H^1(\mathbb{T})) \mid v_x \in L^1([-T,T], L^\infty(\mathbb{T}))\}$$

with initial datum $u_0 \in H^s(\mathbb{T})$. Moreover, for $u_0 \in H^{\frac{1}{2}}(\mathbb{T})$ the function $S_{R,T}(u_0)$ is the unique limit of $C([-T,T], H^{\frac{3}{2}+\varepsilon}(\mathbb{T}))$ solutions with initial data converging to u_0 in $H^{\frac{1}{2}}(\mathbb{T})$.

(iv) Let $B_{R,\mu} = \{u_0 \in H^{\frac{1}{2}}(\mathbb{T}) \mid \|u_0\|_{H^{\frac{1}{2}}} \leq R,\ \|u_0\|_{L^2} = \mu\}$ for fixed $\mu > 0$. Then, for all $s \geq \frac{1}{2}$

$$S_{R,T}\,|_{B_{R,\mu} \cap H^s(\mathbb{T})} : B_{R,\mu} \cap H^s(\mathbb{T}) \to C\big([-T,T], H^s(\mathbb{T})\big)$$

is locally Lipschitz continuous.

Remark 3.1.2. The proof will show another (more technical) uniqueness statement, see Subsection 3.4.2.

Corollary 3.1.3. *(i) Cauchy problem* (3.1) *is locally well-posed in* $H^{\frac{1}{2}}(\mathbb{T})$.

(ii) The Cauchy problem (3.1) *is globally well-posed in* $H^1(\mathbb{T})$ *for data in* $\{u_0 \in H^1(\mathbb{T}) \mid \|u_0\|^2_{L^2(\mathbb{T})} < 2\pi\}$.

Remark 3.1.4. In the exposition we focus on the DNLS equation but we remark that the same approach is also applicable to slightly more general nonlinearities e.g.

$$\lambda_1 |u|^2 \partial_x u + \lambda_2 u^2 \partial_x \overline{u} + \textit{polynomial}, \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

with not necessarily gauge invariant polynomials, see also Proposition 3.3.5. We also remark that the strategy of proof of the tri-linear estimate presented here is also applicable in the non-periodic case, cp. [Tak99].

The general outline of the proof will be as in [Tak99], i.e. we will link the DNLS with another Cauchy problem via the nonlinear transformation defined in Section 3.2, cp. N. Hayashi - T. Ozawa [HO92, Hay93, HO94]. This problem enjoys better properties than the original problem in the sense that it is possible to prove multi-linear estimates in Section 3.3. All this is used to solve the corresponding integral equation via a fixed point argument and transfer the results back to the DNLS.

Moreover, for the problem (3.2) it follows

Theorem 3.1.5. *The Cauchy problem* (3.2) *is locally well-posed in* $H^{\frac{1}{2}}(\mathbb{T})$. *The flow map is locally Lipschitz, moreover it is real analytic.*

Under the assumption that the Cauchy problem (3.2) *is locally well-posed in* $H^s(\mathbb{T})$ *for some* $s < \frac{1}{2}$ *the corresponding flow map is* not C^3.

Hence, our result is sharp with respect to the smoothness of the flow map.

3.2 The gauge transformation

We start with the following observation.

Proposition 3.2.1. *Let* $T > 0$, $s \geq 0$. *The translations*

$$\tau : \mathbb{R} \times C([-T,T], H^s(\mathbb{T})) \to C([-T,T], H^s(\mathbb{T})), \; \tau(h,u)(t,x) = u(t, x + ht)$$

are continuous.

Proof. 1. We first show that

$$\mathbb{R} \times H^s(\mathbb{T}) \to H^s(\mathbb{T}),\ (h, f) \mapsto f(\cdot + h)$$

is a continuous operator. For $f_n \to f \in H^s$ and $h_n \to h$ we write

$$\|f_n(\cdot+h_n)-f(\cdot+h)\|_{H^s} \le \|f_n(\cdot+h_n)-f(\cdot+h_n)\|_{H^s}+\|f(\cdot+h_n)-f(\cdot+h)\|_{H^s}$$

and the first term is equal to $\|f_n - f\|_{H^s} \to 0$ because a translation by a fixed amount is an isometry in H^s and for the second term we observe

$$\|f(\cdot + h_n) - f(\cdot + h)\|^2_{H^s(\mathbb{T})} = \sum_{\xi\in\mathbb{Z}} \langle\xi\rangle^{2s} \left|e^{i\xi h_n} - e^{i\xi h}\right|^2 |\mathcal{F}_x f(\xi)|^2$$

For every $\xi \in \mathbb{Z}$ we have $e^{i\xi h_n} \to e^{i\xi h}$, whence the first claim follows from the dominated convergence theorem.

2. Fix $u \in C([-T, T], H^s(\mathbb{T}))$ and $u_n \to u$, $h_n \to h$. Let $\varepsilon > 0$. Because u is uniformly continuous, we find $N \in \mathbb{N}$ and points $-T = t_0 < \ldots < t_N = T$ such that

$$\sup_{t\in[t_i,t_{i+1}]} \|u(t) - u(t_i)\|_{H^s(\mathbb{T})} < \varepsilon/2, \quad i = 0, \ldots, N-1$$

and on every subinterval $I_i = [t_i, t_{i+1}]$ we have

$$\begin{aligned}&\|u_n(t, \cdot + h_n t) - u(t, \cdot + ht)\|_{H^s}\\ \le&\|u_n(t, \cdot + h_n t) - u(t, \cdot + h_n t)\|_{H^s} + \|u(t, \cdot + h_n t) - u(t_i, \cdot + h_n t)\|_{H^s}\\ +&\|u(t_i, \cdot + h_n t) - u(t_i, \cdot + ht)\|_{H^s} + \|u(t_i, \cdot + ht) - u(t, \cdot + ht)\|_{H^s}\end{aligned}$$

Now, the supremum in $t \in I_i$ of the first term tends to zero as $n \to \infty$ and the supremum in $t \in I_i$ of the second and fourth term is smaller than $\varepsilon/2$. By part 1 of the proof the third term

$$\|u(t_i, \cdot + h_n t) - u(t_i, \cdot + ht)\|_{H^s(\mathbb{T})}$$

is continuous in t. Therefore, we find $t_n^* \in I_i$ such that

$$\begin{aligned}&\sup_{t\in I_i} \|u(t_i, \cdot + h_n t) - u(t_i, \cdot + ht)\|_{H^s}\\ =&\|u(t_i, \cdot + h_n t_n^*) - u(t_i, \cdot + ht_n^*)\|_{H^s}\\ =&\|u(t_i, \cdot + (h_n - h)t_n^*) - u(t_i)\|_{H^s}\end{aligned}$$

and this tends to zero as $(h_n - h)t_n^* \to 0$, again by part 1 of the proof. Hence,

$$\lim_{n\to\infty} \sup_{t\in I_i} \|u_n(t, \cdot + h_n t) - u(t, \cdot + ht)\|_{H^s(\mathbb{T})} < \varepsilon$$

This is true for every subinterval I_i, $i = 0, \ldots, N-1$. Because $\varepsilon > 0$ may be chosen arbitrarily small the continuity of τ follows. □

Proposition 3.2.2. *For $u \in C([-T,T], L^2(\mathbb{T}))$ define $\mu(u) = \frac{1}{2\pi}\|u(0)\|^2_{L^2(\mathbb{T})}$. Then, for $s \geq 0$*

$$\tau^{\mp} : C([-T,T], H^s(\mathbb{T})) \to C([-T,T], H^s(\mathbb{T})), \quad u \mapsto \tau(\mp 2\mu(u), u)$$

are homeomorphisms.

Moreover, $u \in C([-T,T], H^2(\mathbb{T})) \cap C^1((-T,T), L^2(\mathbb{T}))$ is a solution of (3.1) *if and only if $v = \tau^-(u) \in C([-T,T], H^2(\mathbb{T})) \cap C^1((-T,T), L^2(\mathbb{T}))$ is a solution of* (3.2). *For $t > 0$ and every $r > 0$ the maps*

$$\{u \in H^s(\mathbb{T}) \mid \|u\|_{H^s} \leq r\} \to H^s(\mathbb{T}), \quad u \mapsto u(\cdot \mp 2t\|u\|^2_{L^2})$$

are not uniformly continuous.

Proof. 1. Obviously, $\tau_+ = (\tau^-)^{-1}$ and the continuity statements follow from Proposition 3.2.1.

2. Taking into account that

$$\partial_t \tau^-(u) = \tau^-(\partial_t u) - 2\mu(u)\tau^-(\partial_x u)$$

the second claim follows from the L^2 conservation law from Lemma A.5.1.

3. Fix $r > 0$. Consider for $j = 1, 2$, $n \in \mathbb{N}$ the functions

$$u_{n,j}(x) = \frac{1}{\sqrt{2\pi}}(rn^{-s}e^{inx} + c_{n,j})$$

where $c_{n,j} = \frac{1}{\sqrt{2n}}$ for $j = 1$ and $c_{n,j} = 0$ for $j = 2$. Then, for large n we have $\|u_{n,j}\|_{H^s(\mathbb{T})} \leq 2r$ and $\|u_{n,j}\|^2_{L^2(\mathbb{T})} = r^2n^{-2s} + (c_{n,j})^2$ and

$$\begin{aligned}
&\|u_{n,1}(\cdot - 2t(r^2n^{-2s} + (2n)^{-1})) - u_{n,2}(\cdot - 2tr^2n^{-2s})\|_{H^s(\mathbb{T})} \\
\geq &\frac{1}{\sqrt{2\pi}}\left\| rn^{-s}e^{inx}e^{-2intr^2n^{-2s}}\left(e^{-it} - 1\right) + \frac{1}{\sqrt{2n}}\right\|_{H^s(\mathbb{T})} \\
\geq &\frac{r}{2}|\sin(t)| - \frac{1}{\sqrt{2n}} > 0, \text{ for } t > 0, n \text{ large}
\end{aligned}$$

in spite of $\|u_{n,1} - u_{n,2}\|_{H^s(\mathbb{T})} = (2n)^{-\frac{1}{2}} \to 0$ which contradicts the uniform continuity for every neighborhood of the origin. The same argument works with the + sign. □

We reduced the problem (3.1) to (3.2) up to a homeomorphism. In the following, we will slightly adjust the gauge transformation, developed by N. Hayashi - T. Ozawa [HO92, Hay93, HO94] in the non-periodic case, to the periodic setting.

Definition 3.2.3. For $f \in L^2(\mathbb{T})$ we define

$$\mathcal{G}_0(f)(x) = e^{-i\mathcal{I}(f)} f\,(x)$$

where

$$\mathcal{I}(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \int_\theta^x |f(y)|^2 - \frac{1}{2\pi} \|f\|^2_{L^2(\mathbb{T})}\, dy d\theta$$

For $u \in C([-T,T], L^2(\mathbb{T}))$ we define

$$\mathcal{G}_0(u)(t,x) := \mathcal{G}_0(u(t))(x) \tag{3.4}$$

Remark 3.2.4. The function $\mathcal{G}_0(f)$ is 2π-periodic because

$$|f(y)|^2 - \frac{1}{2\pi} \|f\|^2_{L^2(\mathbb{T})}$$

has zero mean value and therefore

$$\int_0^{2\pi} \int_\theta^x |f(y)|^2 - \frac{1}{2\pi} \|f\|^2_{L^2(\mathbb{T})}\, dy d\theta$$

is 2π-periodic.

Now, we summarize important properties of the nonlinear operator $\mathcal{G}_0$.

Proposition 3.2.5. *For $s \geq 0$ the map*

$$\mathcal{G}_0 : C([-T,T], H^s(\mathbb{T})) \to C([-T,T], H^s(\mathbb{T}))$$

is locally Lipschitz. More precisely, for $r > 0$ there exists $c > 0$ such that for

$$u, v \in B_r = \left\{ u \in C([-T,T], H^s(\mathbb{T})) \mid \sup_{|t| \leq T} \|u(t)\|_{H^s(\mathbb{T})} \leq r \right\}$$

the map $\mathcal{G}_0$ satisfies

$$\|\mathcal{G}_0(u)(t) - \mathcal{G}_0(v)(t)\|_{H^s(\mathbb{T})} \leq c\|u(t) - v(t)\|_{H^s(\mathbb{T})} \quad , t \in [-T,T] \tag{3.5}$$

The inverse map is given by

$$\mathcal{G}_0^{-1}(v) = e^{i\mathcal{I}(v)} v$$

and $\mathcal{G}_0^{-1}$ also satisfies (3.5) *on B_r, hence $\mathcal{G}_0$ is bi-Lipschitz on bounded subsets. The maps $\mathcal{G}_0, \mathcal{G}_0^{-1}$ are real analytic.*

Proof. We fix $s \geq 0$. There exists $c > 0$ such that for $f, g, h \in H^s(\mathbb{T})$

$$\left\| (e^{\pm i\mathcal{I}(f)} - e^{\pm i\mathcal{I}(g)})h \right\|_{H^s} \leq c e^{c\|f\|_{H^s}^2 + c\|g\|_{H^s}^2} (\|f\|_{H^s} + \|g\|_{H^s}) \|f - g\|_{H^s} \|h\|_{H^s}$$

This is proved in Appendix A.2. We apply this estimate to $u, v \in B_r$ and obtain

$$\begin{aligned}
&\|\mathcal{G}_0(u)(t) - \mathcal{G}_0(v)(t)\|_{H^s(\mathbb{T})} \\
\leq &\left\| (e^{-i\mathcal{I}(u(t))} - e^{-i\mathcal{I}(v(t))})u(t) \right\|_{H^s} + \left\| (e^{-i\mathcal{I}(v(t))} - 1)(u - v)(t) \right\|_{H^s} \\
&+ \|(u - v)(t)\|_{H^s} \\
\leq &(2cre^{2cr^2} + cre^{cr^2} + 1) \|(u - v)(t)\|_{H^s}
\end{aligned}$$

which shows the Lipschitz estimate (3.5) on B_r.

If $v = \mathcal{G}_0(u)$, then $|v(t, x)| = |u(t, x)|$ and the inversion formula follows. For $\mathcal{G}_0^{-1}$ the Lipschitz estimate on subsets B_r follows as above by replacing $-$ by $+$ in the exponential. Concerning the analyticity statement we remark that

$$\int_0^{2\pi} \int_\theta^x u_1(t, y)\overline{u}_2(t, y) - \frac{1}{2\pi} \int_0^{2\pi} u_1(t, z)\overline{u}_2(t, z) dz dy d\theta$$

is bilinear in u_1, u_2 (over $\mathbb{R}$). If $s > 0$ we can show that this is continuous as a map

$$C([-T, T], H^s(\mathbb{T})) \times C([-T, T], H^s(\mathbb{T})) \to C([-T, T], H^{s'}(\mathbb{T}))$$

for some $s' = \max\{s, \frac{1}{2} + \varepsilon\}$, similar to (A.4). In the case $s = 0$ we easily show the continuity with range in $C([-T, T], L^\infty(\mathbb{T}))$. Moreover, by Corollary 1.1.12 the spaces $C([-T, T], H^{s'}(\mathbb{T})$ and also $C([-T, T], L^\infty(\mathbb{T}))$ are Banach algebras, respectively. Because compositions of real analytic maps are real analytic, see Proposition 1.4.5, the claim follows. □

The relevance of this nonlinear transformation is explained by

Proposition 3.2.6. *Let $u, v \in C([-T, T], H^2(\mathbb{T})) \cap C^1((-T, T), L^2(\mathbb{T}))$ and $v = \mathcal{G}_0(u)$. Then, u is a solution of (3.2) if and only if v is a solution of*

$$\begin{aligned}
\partial_t v(t) - i\partial_x^2 v(t) &= -v^2 \partial_x \overline{v}(t) + \tfrac{i}{2}|v|^4 v(t) - i\mu(v)|v|^2 v(t) + i\psi(v)v(t) \\
v(0) &= \mathcal{G}_0(u_0)
\end{aligned} \tag{3.6}$$

where

$$\psi(v)(t) = \frac{1}{2\pi} \int_0^{2\pi} 2\,\mathrm{Im}(\overline{v}_x v)(t, \theta) - \frac{1}{2}|v|^4(t, \theta)\, d\theta + \mu(v)^2 \tag{3.7}$$

Proof. Suppose that u is a smooth solution to (3.2) and let us derive an equation for v

$$\begin{aligned} v_t &= \exp(-i\mathcal{I}(u))(-i\mathcal{I}(u)_t u + u_t) \\ v_{xx} &= \exp(-i\mathcal{I}(u))(-\mathcal{I}(u)_x^2 u - i\mathcal{I}(u)_x u_x - i(\mathcal{I}(u)_x u)_x + u_{xx}) \end{aligned}$$

By the L^2 conservation law we have $\|u(t)\|_{L^2(\mathbb{T})} = \|u(0)\|_{L^2(\mathbb{T})}$, see Appendix A.5. With

$$\mu = \mu(u) = \frac{1}{2\pi}\|u(0)\|^2_{L^2(\mathbb{T})}$$

we have

$$\mu(u) = \mu(v) = \frac{1}{2\pi}\|v(0)\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi}\|v(t)\|^2_{L^2(\mathbb{T})}$$

and $\mathcal{I}(u)_x = |u|^2 - \mu$. Therefore,

$$\begin{aligned} & v_t - iv_{xx} \\ &= e^{-i\mathcal{I}(u)}\left(u_t - iu_{xx} - (\mathcal{I}(u)_x u)_x + i\mathcal{I}(u)_x^2 u - \mathcal{I}(u)_x u_x - i\mathcal{I}(u)_t u\right) \\ &= e^{-i\mathcal{I}(u)}\left(-|u|^2 u_x + i(|u|^2 - \mu)^2 u - i\mathcal{I}(u)_t u\right) \end{aligned} \tag{3.8}$$

Moreover,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\theta^x |u(t,y)|^2 - \mu\, dy = \int_\theta^x u_t\overline{u}(t,y) + u\overline{u}_t(t,y)\, dy \\ &= \int_\theta^x \left(iu_{xx}\overline{u} - i\overline{u}_{xx}u + \overline{u}(|u|^2 u)_x + u(|u|^2\overline{u})_x - 2\mu\overline{u}u_x - 2\mu u\overline{u}_x\right)(t,y)\, dy \end{aligned}$$

Integration by parts yields

$$\int_\theta^x iu_{xx}\overline{u}(t,y) - i\overline{u}_{xx}u(t,y)\, dy = 2\,\mathrm{Im}(\overline{u}_x u)(t,x) - 2\,\mathrm{Im}(\overline{u}_x u)(t,\theta)$$

and

$$\int_\theta^x \overline{u}(|u|^2 u)_x(t,y) + u(|u|^2\overline{u})_x(t,y)\, dy = \frac{3}{2}|u|^4(t,x) - \frac{3}{2}|u|^4(t,\theta)$$

and

$$\int_\theta^x \overline{u}u_x(t,y) + u\overline{u}_x(t,y)\, dy = |u|^2(t,x) - |u|^2(t,\theta)$$

which shows

$$\begin{aligned} \mathcal{I}(u)_t =& 2\,\mathrm{Im}(\overline{u}_x u)(t,x) + \frac{3}{2}|u|^4(t,x) - 2\mu|u|^2(t,x) \\ & - \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Im}(\overline{u}_x u)(t,\theta) + \frac{3}{2}|u|^4(t,\theta) - 2\mu|u|^2(t,\theta)\, d\theta \end{aligned}$$

Let us define

$$\phi(u)(t) := \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Im}(\overline{u}_x u)(t,\theta) + \frac{3}{2}|u|^4(t,\theta)\,d\theta - 2\mu^2$$

Plugging this into (3.8) we arrive at

$$v_t - iv_{xx} = e^{-i\mathcal{I}(u)}\big(- u^2\overline{u}_x - \tfrac{i}{2}|u|^4 u + i\mu^2 u + i\phi(u)u\big)$$

Using $|u| = |v|$ as well as $u_x = e^{i\mathcal{I}(u)}v_x + i(|u|^2 - \mu)u$ we get

$$\begin{aligned} v_t - iv_{xx} &= - v^2\overline{v}_x + i|v|^2(|v|^2 - \mu)v - \tfrac{i}{2}|v|^4 v + i\mu^2 v + i\phi(u)v \\ &= - v^2\overline{v}_x + \tfrac{i}{2}|v|^4 v - i\mu|v|^2 v + i\mu^2 v + i\phi(u)v \end{aligned}$$

With

$$\psi(v)(t) = \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Im}(\overline{v}_x v)(t,\theta) - \frac{1}{2}|v|^4(t,\theta)\,d\theta + \mu^2(v)$$

we have

$$\psi(v) = \phi(u) + \mu^2(v)$$

and therefore obtain

$$v_t - iv_{xx} = -v^2\overline{v}_x + \tfrac{i}{2}|v|^4 v - i\mu(v)|v|^2 v + i\psi(v)v$$

which shows the first claim of the proposition. We may also perform the calculation in reverse order. □

Remark 3.2.7. We may rewrite the right hand side as

$$\Big(\int_0^{2\pi} \tfrac{i}{\pi}\,\mathrm{Im}(\overline{v}_x v)(t,\theta)\,d\theta - v\overline{v}_x\Big)v + \frac{i}{2}\big(|v|^4 - \tfrac{1}{2\pi}\|v(t)\|^4_{L^4(\mathbb{T})}\big)v - i\mu(v)(|v|^2 - \mu(v))v$$

This is remarkable because in each of the three terms certain frequency interactions cancel out. However, we will not exploit this cancellation in the sequel and use the formula from Proposition 3.2.6.

Corollary 3.2.8. *The nonlinear transformation*

$$\mathcal{G} = \mathcal{G}_0 \circ \tau^- : C([-T,T], H^s(\mathbb{T})) \to C([-T,T], H^s(\mathbb{T}))$$

is a homeomorphism. Moreover, $u \in C([-T,T], H^2(\mathbb{T})) \cap C^1((-T,T), L^2(\mathbb{T}))$ *is a solution of* (3.1)*, if and only if* $v = \mathcal{G}(u) \in C([-T,T], H^2(\mathbb{T})) \cap C^1((-T,T), L^2(\mathbb{T}))$ *is a solution of* (3.6).

3.3 Multi-linear estimates

We start with an elementary bound for the ($\mathbb{R}$)-tri-linear multiplier in the spirit of [CKS$^+$04]. We define $\lambda_0 = \tau + \xi^2, \lambda_j = \tau_j + \xi_j^2$ for $j = 1, 2$ and $\lambda_3 = \tau_3 - \xi_3^2$ as well as the subregions A_j of $\mathbb{R}^3 \times \mathbb{Z}^3$ such that

$$\langle \lambda_j \rangle = \max\{\langle \lambda_0 \rangle, \langle \lambda_1 \rangle, \langle \lambda_2 \rangle, \langle \lambda_3 \rangle\} \quad \text{in } A_j$$

for $j = 0, 1, 2, 3$. Let χ_{A_j} denote the characteristic function of the set A_j.

Proposition 3.3.1. *Let $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{Z}$ and define*

$$M(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \frac{\langle \xi \rangle^{\frac{1}{2}} i \xi_3}{\prod_{k=0}^3 \langle \lambda_k \rangle^{\frac{1}{2}} \prod_{k=1}^3 \langle \xi_k \rangle^{\frac{1}{2}}}$$

where $(\tau, \xi) = (\tau_1, \xi_1) + (\tau_2, \xi_2) + (\tau_3, \xi_3)$. Moreover, for $j \in \{0, 1, 2, 3\}$ let

$$M_j(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \chi_{A_j} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}} \prod_{k=0, k \neq j}^{3} \langle \lambda_k \rangle^{-\frac{1}{2}}$$

$$N(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) = \prod_{k=0}^{3} \langle \lambda_k \rangle^{-\frac{1}{2}}$$

Then, the estimate

$$|M| \leq 16\Big(\sum_{j=0}^{3} M_j + N\Big) \tag{3.9}$$

holds true.

Proof. The key for the proof will be the observation (cp. [Tak99], p.569) that

$$\tau + \xi^2 - \left(\tau_1 + \xi_1^2 + \tau_2 + \xi_2^2 + \tau_3 - \xi_3^2\right) = 2(\xi - \xi_1)(\xi - \xi_2) \tag{3.10}$$

which implies the estimate

$$\langle (\xi - \xi_1)(\xi - \xi_2) \rangle^{\frac{1}{2}} \leq 4 \sum_{k=0}^{3} \chi_{A_k} \langle \lambda_k \rangle^{\frac{1}{2}} \tag{3.11}$$

It is useful to distinguish 4 cases:

(i) $|\xi| > 2|\xi_1|$ and $|\xi| > 2|\xi_2|$: In this case $4\langle (\xi - \xi_1)(\xi - \xi_2) \rangle \geq \langle \xi \rangle^2$ and $|\xi_3| \leq 2|\xi|$, hence $|M| \leq 16 \sum_{j=0}^{3} M_j$.

(ii) $|\xi| \le 2|\xi_1|$ and $|\xi| \le 2|\xi_2|$: In this case $|\xi_3| \le 4\max\{|\xi_1|, |\xi_2|\}$ and $|\xi| \le 2\min\{|\xi_1|, |\xi_2|\}$, which shows $|M| \le 4N$.

(iii) $|\xi| > 2|\xi_1|$ and $|\xi| \le 2|\xi_2|$: It is $|\xi| \le 2|\xi - \xi_1|$ and $|\xi||\xi - \xi_2| \le 2\langle(\xi - \xi_1)(\xi - \xi_2)\rangle$. Because of $|\xi_3|^{\frac{1}{2}} \le |\xi - \xi_2|^{\frac{1}{2}} + |\xi_1|^{\frac{1}{2}}$ it follows

$$\langle\xi\rangle^{\frac{1}{2}}|\xi_3|^{\frac{1}{2}} \le 2|\xi|^{\frac{1}{2}}|\xi_3|^{\frac{1}{2}} \le 2|\xi|^{\frac{1}{2}}|\xi - \xi_2|^{\frac{1}{2}} + 2|\xi|^{\frac{1}{2}}|\xi_1|^{\frac{1}{2}}$$

The first term is bounded by $4\langle(\xi - \xi_1)(\xi - \xi_2)\rangle^{\frac{1}{2}}$ which in turn is controlled by (3.11). The second term is smaller than $4\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_1\rangle^{\frac{1}{2}}$ which proves $|M| \le 16\sum_{j=0}^{3} M_j + 4N$.

(iv) $|\xi| \le 2|\xi_1|$ and $|\xi| > 2|\xi_2|$: By the symmetry of M in ξ_1, ξ_2 the same estimate as in case (iii) applies.

The proof of Proposition 3.3.1 is complete. □

For the convolution $f_1 * \ldots * f_k(\tau, \xi)$ we will write

$$\int_* \sum_* \prod_{j=1}^{k} f_j(\tau_j, \xi_j) := \int_{\tau = \tau_1 + \ldots + \tau_k} \sum_{\xi = \xi_1 + \ldots + \xi_k} \prod_{j=1}^{k} f_j(\tau_j, \xi_j)$$

$$:= \int_{\mathbb{R}^{k-1}} \sum_{(\xi_1, \ldots, \xi_{k-1}) \in \mathbb{Z}^{k-1}} \prod_{j=1}^{k-1} f_j(\tau_j, \xi_j) f_k(\tau - \sum_{j=1}^{k-1} \tau_j, \xi - \sum_{j=1}^{k-1} \xi_j) d\tau_1 \ldots d\tau_{k-1}$$

Theorem 3.3.2. *There exists $c, \varepsilon > 0$, such that for $T \in (0, 1]$ and $u_j \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ with $\operatorname{supp}(u_j) \subset \{(t, x) \mid |t| \le T\}$, $j = 1, 2, 3$, we have*

$$\|u_1 u_2 \partial_x u_3\|_{X_{\frac{1}{2}, -\frac{1}{2}}} \le cT^{\varepsilon} \|u_1\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2}, \frac{1}{2}}} \|u_3\|_{X^{-}_{\frac{1}{2}, \frac{1}{2}}} \tag{3.12}$$

Proof. We define $f_j(\tau_j, \xi_j) = \langle\lambda_j\rangle^{\frac{1}{2}}\langle\xi_j\rangle^{\frac{1}{2}}\mathcal{F}u_j(\tau_j, \xi_j)$ for $j = 1, 2, 3$. With the Fourier multiplier M defined in Proposition 3.3.1 we rewrite the left hand side as

$$\|u_1 u_2 \partial_x u_3\|_{X_{\frac{1}{2}, -\frac{1}{2}}} = \left\| \int_* \sum_* M(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\|_{l^2_\xi L^2_\tau}$$

By an application of the triangle inequality we may assume $f_j, \mathcal{F}u_j \ge 0$ and $\|u_j\|_{X_{s,b}} = \|\chi_T u_j\|_{X_{s,b}}$. By the point-wise bound (3.9) on $|M|$ the left hand side is bounded by the sum over the corresponding terms with M replaced

by M_0, M_1, M_2, M_3 and N, respectively.
Estimate for M_0:

$$\left\| \int_* \sum_* M_0(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^2_\tau} = \|u_1 u_2 J^{\frac{1}{2}} u_3\|_{L^2_t L^2_x} =: m_0$$

Using Hölder's inequality we get

$$m_0 \leq \|u_1\|_{L^8_t L^8_x} \|u_2\|_{L^8_t L^8_x} \|J^{\frac{1}{2}} u_3\|_{L^4_t L^4_x} \leq c\|u_1\|_{X_{\frac{3}{8},\frac{3}{8}}} \|u_2\|_{X_{\frac{3}{8},\frac{3}{8}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}}$$

where we used Sobolev's inequality (2.14) on u_1, u_2 as well as the L^4 Strichartz inequality (2.18) on $J^{\frac{1}{2}} u_3$. By the localization in time, see Lemma 2.2.5,

$$m_0 \leq cT^\varepsilon \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}}$$

Estimate for M_1:

$$\begin{aligned} &\left\| \int_* \sum_* M_1(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^2_\tau} \\ =&\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1 u_2 J^{\frac{1}{2}} u_3\|_{X_{0,-\frac{1}{2}}} \leq \|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1 u_2 J^{\frac{1}{2}} u_3\|_{X_{0,-\frac{3}{8}}} =: m_1 \end{aligned} \tag{3.13}$$

Then, by the dual Sobolev estimate (2.15)

$$\begin{aligned} m_1 \leq& c\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1 u_2 J^{\frac{1}{2}} u_3\|_{L^{8/7}_t L^2_x} \\ \leq& c\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1\|_{L^2_t L^8_x} \|u_2 J^{\frac{1}{2}} u_3\|_{L^{8/3}_t L^{8/3}_x} \\ \leq& c\|J^{-\frac{1}{2}} \mathcal{F}^{-1} f_1\|_{L^2_t L^8_x} \|u_2\|_{L^8_t L^8_x} \|J^{\frac{1}{2}} u_3\|_{L^4_t L^4_x} \end{aligned}$$

Now we use the Sobolev inequality (2.14) on the first two factors as well as the L^4 Strichartz inequality (2.18) on $J^{\frac{1}{2}} u_3$ and obtain

$$m_1 \leq cT^\varepsilon \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}} \tag{3.14}$$

Estimate for M_2:

$$\begin{aligned} &\left\| \int_* \sum_* M_2(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^2_\tau} \\ =&\|u_1 J^{-\frac{1}{2}} \mathcal{F}^{-1} f_2 J^{\frac{1}{2}} u_3\|_{X_{0,-\frac{1}{2}}} \leq \|u_1 J^{-\frac{1}{2}} \mathcal{F}^{-1} f_2 J^{\frac{1}{2}} u_3\|_{X_{0,-\frac{3}{8}}} =: m_2 \end{aligned} \tag{3.15}$$

As for m_1, by exchanging the roles of the first two factors we obtain

$$m_2 \leq cT^{\varepsilon} \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}} \tag{3.16}$$

Estimate for M_3:

$$\begin{aligned}
&\left\| \int_* \sum_* M_3(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\|_{l^2_\xi L^2_\tau} \\
=&\|u_1 u_2 \mathcal{F}^{-1} f_3\|_{X_{0,-\frac{1}{2}}} \leq \|u_1 u_2 \mathcal{F}^{-1} f_3\|_{X_{0,-\frac{7}{16}}} =: m_3
\end{aligned} \tag{3.17}$$

We apply dual Strichartz (2.19), Hölder's inequality and Sobolev (2.14) to conclude

$$\begin{aligned}
m_3 \leq& c\|u_1 u_2 \mathcal{F}^{-1} f_3\|_{L^{4/3}_t L^{4/3}_x} \\
\leq& c\|u_1\|_{L^8_t L^8_x} \|u_2\|_{L^8_t L^8_x} \|f_3\|_{L^2_t L^2_x} \\
\leq& cT^{\varepsilon} \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}}
\end{aligned} \tag{3.18}$$

Estimate for N:

$$\begin{aligned}
&\left\| \int_* \sum_* N(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) \prod_{j=1}^{3} f_j(\tau_j, \xi_j) \right\|_{l^2_\xi L^2_\tau} \\
=&\|J^{\frac{1}{2}} u_1 J^{\frac{1}{2}} u_2 J^{\frac{1}{2}} u_3\|_{X_{0,-\frac{1}{2}}} \leq \|J^{\frac{1}{2}} u_1 J^{\frac{1}{2}} u_2 J^{\frac{1}{2}} u_3\|_{X_{0,-\frac{7}{16}}} =: n
\end{aligned} \tag{3.19}$$

Strichartz inequalities (2.18) and (2.19) yield

$$\begin{aligned}
n \leq& c\|J^{\frac{1}{2}} u_1 J^{\frac{1}{2}} u_2 J^{\frac{1}{2}} u_3\|_{L^{4/3}_t L^{4/3}_x} \\
\leq& c\|J^{\frac{1}{2}} u_1\|_{L^4_t L^4_x} \|J^{\frac{1}{2}} u_2\|_{L^4_t L^4_x} \|J^{\frac{1}{2}} u_3\|_{L^4_t L^4_x} \\
\leq& cT^{\varepsilon} \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}}
\end{aligned} \tag{3.20}$$

and the proof is complete. □

Proposition 3.3.3. *Let $\delta \in (0, \frac{1}{6})$. With the notation from Proposition 3.3.1 we define*

$$\tilde{M}(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) := \langle \lambda_0 \rangle^{-\frac{1}{2}} M(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3)$$

and

$$\tilde{M}_0(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) := \frac{\chi_{A_0}}{\langle\xi\rangle^{\frac{1}{2}-3\delta}\langle\xi_1\rangle^{\frac{1}{2}}\langle\xi_2\rangle^{\frac{1}{2}}\langle\xi_3\rangle^{\frac{1}{2}-3\delta}\prod_{k=1}^{3}\langle\lambda_k\rangle^{\frac{1}{2}+\delta}}$$

and

$$\tilde{M}_j(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) := \langle\lambda_0\rangle^{-\frac{1}{2}}M_j(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3)\ , j\in\{1,2,3\}$$
$$\tilde{N}(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) := \langle\lambda_0\rangle^{-\frac{1}{2}}N(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3)$$

Then, the estimate

$$|\tilde{M}| \le 128(\sum_{j=0}^{3}\tilde{M}_j + \tilde{N}) \tag{3.21}$$

holds true.

Proof. By Proposition 3.3.1 it suffices to consider the region A_0 and to show that

$$\langle\tau+\xi^2\rangle^{-\frac{1}{2}}M_0 \le 8\tilde{M}_0 + 4\tilde{N}$$

(i) $|\xi| > 2|\xi_1|$ and $|\xi| > 2|\xi_2|$: In this case $|\xi_3| \le 2|\xi|$. In A_0 we have $16\langle\tau+\xi^2\rangle \ge \langle\xi\rangle^2$, since $\langle\tau+\xi^2\rangle \ge \langle\tau_1+\xi_1^2\rangle, \langle\tau_2+\xi_2^2\rangle, \langle\tau_3-\xi_3^2\rangle$ which implies

$$8\langle\tau+\xi^2\rangle^{\frac{1}{2}} \ge \langle\tau_1+\xi_1^2\rangle^{\delta}\langle\tau_2+\xi_2^2\rangle^{\delta}\langle\tau_3-\xi_3^2\rangle^{\delta}\langle\xi\rangle^{\frac{1}{2}-3\delta}\langle\xi_3\rangle^{\frac{1}{2}-3\delta}$$

(ii) $|\xi| \le 2|\xi_1|$ and $|\xi| \le 2|\xi_2|$: In this case we have $|\xi_3| \le 4\max\{|\xi_1|,|\xi_2|\}$ and $|\xi| \le 2\min\{|\xi_1|,|\xi_2|\}$, which shows $|\tilde{M}| \le 4\tilde{N}$.

(iii) $|\xi| > 2|\xi_1|$ and $|\xi| \le 2|\xi_2|$: Here $\xi \ne 0$ and without loss we may assume $\xi_3 \ne 0$, since otherwise $\tilde{M} = 0$. We have

$$|\xi| \le 2|\xi-\xi_1| \text{ and } |\xi||\xi-\xi_2| \le 2\langle(\xi-\xi_1)(\xi-\xi_2)\rangle$$

In the subregion where $|\xi_1| \le |\xi-\xi_2|$ we have

$$|\xi_3| \le |\xi-\xi_2| + |\xi_1| \le 2|\xi-\xi_2|$$

and therefore

$$\langle\xi\rangle\langle\xi_3\rangle \le 2|\xi||\xi_3| \le 4|\xi||\xi-\xi_2| \le 8\langle(\xi-\xi_1)(\xi-\xi_2)\rangle$$

which is bounded by $32\langle\tau+\xi^2\rangle$, since we are in region A_0. Then,

$$8\langle\tau+\xi^2\rangle^{\frac{1}{2}} \ge \langle\xi\rangle^{\frac{1}{2}-3\delta}\langle\xi_3\rangle^{\frac{1}{2}-3\delta}\langle\tau_1+\xi_1^2\rangle^{\delta}\langle\tau_2+\xi_2^2\rangle^{\delta}\langle\tau_3-\xi_3^2\rangle^{\delta}$$

which proves $\langle\tau+\xi^2\rangle^{-\frac{1}{2}}M_0 \le 8\tilde{M}_0$. In the subregion where $|\xi_1| > |\xi-\xi_2|$ we have $|\xi_3| \le 2|\xi_1|$ and we arrive at $\tilde{M} \le 4\tilde{N}$.

(iv) $|\xi| \leq 2|\xi_1|$ and $|\xi| > 2|\xi_2|$: By the symmetry of $\tilde{M}$ in ξ_1, ξ_2 we find the same estimate as in case (iii).

The proof of Proposition 3.3.3 is complete. □

Theorem 3.3.4. *There exists $c, \varepsilon > 0$, such that for $T \in (0,1]$ and $u_j \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ with $\operatorname{supp}(u_j) \subset \{(t,x) \mid |t| \leq T\}$, $j = 1,2,3$, we have*

$$\|u_1 u_2 \partial_x u_3\|_{Y_{\frac{1}{2},-1}} \leq cT^{\varepsilon} \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}} \tag{3.22}$$

Proof. We use the notation from the proof of Theorem 3.3.2. With the Fourier multiplier $\tilde{M}$ defined in Proposition 3.3.1 we rewrite the left hand side as

$$\|u_1 u_2 \partial_x u_3\|_{Y_{\frac{1}{2},-1}} = \left\| \int_* \sum_* \tilde{M}(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^{3} f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^1_\tau}$$

By the estimate (3.21) we successively replace $\tilde{M}$ by $\tilde{M}_0, \tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ and $\tilde{N}$.

Estimate for $\tilde{M}_0$: We observe that by the Cauchy-Schwarz inequality we have for fixed ξ

$$\left\| \langle \tau + \xi^2 \rangle^{-\frac{1}{2}-\delta'} \phi(\tau,\xi) \right\|_{L^1_\tau} \leq \left(\int \langle \tau \rangle^{-1-2\delta'} \, d\tau \right)^{\frac{1}{2}} \|\phi(\cdot,\xi)\|_{L^2_\tau} \tag{3.23}$$

for $\delta' > 0$. Now, for fixed ξ_1, ξ_2, ξ_3 and $\xi = \xi_1 + \xi_2 + \xi_3$

$$\begin{aligned}
&\left\| \int_* \tilde{M}_0(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^{3} f_j(\tau_j,\xi_j) \right\|_{L^1_\tau} \\
=&\langle \xi \rangle^{-\frac{1}{2}+3\delta} \left\| \int_{\tau=\tau_1+\tau_2+\tau_3} \prod_{j=1}^{2} \frac{f_j(\tau_j,\xi_j)}{\langle \xi_j \rangle^{\frac{1}{2}} \langle \lambda_j \rangle^{\frac{1}{2}+\delta}} \; \frac{f_3(\tau_3,\xi_3)}{\langle \xi_3 \rangle^{\frac{1}{2}-3\delta} \langle \lambda_3 \rangle^{\frac{1}{2}+\delta}} \right\|_{L^1_\tau} \\
\leq& c\langle \xi \rangle^{-\frac{1}{2}+3\delta} \prod_{j=1}^{2} \left\| \frac{f_j(\tau_j,\xi_j)}{\langle \xi_j \rangle^{\frac{1}{2}} \langle \lambda_j \rangle^{\delta/2}} \right\|_{L^2_\tau} \left\| \frac{f_3(\tau_3,\xi_3)}{\langle \xi_3 \rangle^{\frac{1}{2}-3\delta} \langle \lambda_3 \rangle^{\delta/2}} \right\|_{L^2_\tau}
\end{aligned}$$

by Young's inequality and (3.23) with $\delta' = \delta/2$. With $g_j = \langle \lambda_j \rangle^{-\delta/2} f_j$ we have

$$
\begin{aligned}
&\left\| \int_* \sum_* \tilde{M}_0(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^1_\tau} \\
\leq & c \left\| \langle \xi \rangle^{-\frac{1}{2}+3\delta} \sum_{\xi=\xi_1+\xi_2+\xi_3} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}} \langle \xi_3 \rangle^{-\frac{1}{2}+3\delta} \prod_{j=1}^3 \|g_j(\cdot,\xi_j)\|_{L^2_\tau} \right\|_{l^2_\xi}
\end{aligned}
$$

An application of Hölder's and Young's inequalities, choosing $\delta = 1/24$, gives the upper bound

$$
\begin{aligned}
& c \left\| \sum_{\xi=\xi_1+\xi_2+\xi_3} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}} \langle \xi_3 \rangle^{-\frac{3}{8}} \prod_{j=1}^3 \|g_j(\cdot,\xi_j)\|_{L^2_\tau} \right\|_{l^4_\xi} \\
\leq & c \prod_{j=1}^3 \left\| \langle \xi_j \rangle^{-\frac{3}{8}} \|g_j(\cdot,\xi_j)\|_{L^2_\tau} \right\|_{l^{4/3}_\xi} \leq c \prod_{j=1}^3 \|g_j\|_{L^2_\tau l^2_\xi} \\
\leq & c \|u_1\|_{X_{\frac{1}{2},\frac{23}{48}}} \|u_2\|_{X_{\frac{1}{2},\frac{23}{48}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{23}{48}}}
\end{aligned}
$$

which finally proves that

$$
\left\| \int_* \sum_* \tilde{M}_0(\tau_1,\ldots,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^1_\tau} \leq c T^\varepsilon \|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}} \|u_3\|_{X^-_{\frac{1}{2},\frac{1}{2}}}
$$

Estimate for $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ and $\tilde{N}$: We show that the estimates from the proof of Theorem 3.3.2 are strong enough to treat these terms, too. Indeed, an application of (3.23) implies

$$
\begin{aligned}
&\left\| \int_* \sum_* \tilde{M}_1(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^1_\tau} \\
\leq & c \left\| \langle \tau+\xi^2 \rangle^{\frac{1}{8}} \int_* \sum_* M_1(\tau_1,\tau_2,\tau_3,\xi_1,\xi_2,\xi_3) \prod_{j=1}^3 f_j(\tau_j,\xi_j) \right\|_{l^2_\xi L^2_\tau} \\
= & c m_1
\end{aligned}
$$

where m_1 is defined in (3.13) and is bounded according to (3.14). The same reasoning applies to $\tilde{M}_2, \tilde{M}_3$ and $\tilde{N}$, where we use the bounds established in (3.16), (3.18) and (3.20). □

The next proposition contains estimates for the polynomial terms in the nonlinearity. Concerning $X_{s,b}$ estimates for lower values of s we refer to [Bou93]. This along with the estimate from Section A.2 also shows that one could deal with additional nonlinear terms of polynomial type (not necessarily gauge invariant), cp. Remark 3.1.4.

Proposition 3.3.5. *Let $k, k_1 \in \mathbb{N}$, $k \geq 2$ and $0 \leq k_1 \leq k$. There exists $c, \varepsilon > 0$, such that for $T \in (0,1]$ and $u_j \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ with $\operatorname{supp}(u_j) \subset \{(t,x) \mid |t| \leq T\}$, $j = 1, \ldots, k$, we have*[1]

$$\Big\| \prod_{j=1}^{k} u_j \Big\|_{L_t^2 H^{\frac{1}{2}}} \leq cT^{\varepsilon} \prod_{j=1}^{k_1} \|u_j\|_{X^-_{\frac{1}{2},\frac{1}{2}}} \prod_{j=k_1+1}^{k} \|u_j\|_{X_{\frac{1}{2},\frac{1}{2}}} \tag{3.24}$$

Proof. As in the previous proofs it suffices to consider $\mathcal{F}u_j \geq 0$. For $\xi = \sum_{l=1}^{k} \xi_l$ we have $\langle \xi \rangle^{\frac{1}{2}} \leq \sum_{l=1}^{k} \langle \xi_l \rangle^{\frac{1}{2}}$ which implies

$$\Big\| \prod_{j=1}^{k} u_j \Big\|_{L_t^2 H^{\frac{1}{2}}} \leq \sum_{l=1}^{k} \Big\| J^{\frac{1}{2}} u_l \prod_{j=1, j\neq l}^{k} u_j \Big\|_{L_t^2 L_x^2}$$

Each of the k terms can be estimated, using the Strichartz estimate (2.18) as follows

$$\begin{aligned} \Big\| J^{\frac{1}{2}} u_l \prod_{j=1, j\neq l}^{k} u_j \Big\|_{L_t^2 L_x^2} &\leq \|J^{\frac{1}{2}} u_l\|_{L_t^4 L_x^4} \prod_{j=1, j\neq l}^{k} \|u_j\|_{L_t^{4k-4} L_x^{4k-4}} \\ &\leq cT^{\varepsilon} \prod_{j=1}^{k_1} \|u_j\|_{X^-_{\frac{1}{2},\frac{1}{2}}} \prod_{j=k_1+1}^{k} \|u_j\|_{X_{\frac{1}{2},\frac{1}{2}}} \end{aligned}$$

where in the last step we used the Sobolev estimate (2.14). □

We summarize the required estimates in a slightly more general form.

Corollary 3.3.6. *Let $s \geq \frac{1}{2}$. There exists $c, \varepsilon > 0$, such that for $T \in (0,1]$ and $u_j \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$ with $\operatorname{supp}(u_j) \subset \{(t,x) \mid |t| \leq T\}$, $j = 1, \ldots, 5$*

$$\|u_1 u_2 \partial_x \overline{u}_3\|_{Y_{s,-1} \cap X_{s,-\frac{1}{2}}} \leq cT^{\varepsilon} \sum_{k=1}^{3} \|u_k\|_{X_{s,\frac{1}{2}}} \prod_{\substack{j=1 \\ j\neq k}}^{3} \|u_j\|_{X_{\frac{1}{2},\frac{1}{2}}} \tag{3.25}$$

$$\Big\| \overline{u}_1 \overline{u}_2 \prod_{j=3}^{5} u_j \Big\|_{X_{s,0}} \leq cT^{\varepsilon} \sum_{k=1}^{5} \|u_k\|_{X_{s,\frac{1}{2}}} \prod_{\substack{j=1 \\ j\neq k}}^{5} \|u_j\|_{X_{\frac{1}{2},\frac{1}{2}}} \tag{3.26}$$

[1]We use the convention $\prod_{j=j_1}^{j_2} \cdots = 1$ if $j_2 < j_1$.

$$
\begin{aligned}
&|\mu(u_1)-\mu(u_2)|\Big\|\overline{u}_3u_4u_5\Big\|_{X_{s,0}} \\
\leq & cT^\varepsilon\|u_1-u_2\|_{Z_0}(\|u_1\|_{Z_0}+\|u_2\|_{Z_0})\sum_{k=3}^{5}\|u_k\|_{X_{s,\frac{1}{2}}}\prod_{\substack{j=3\\ j\neq k}}^{5}\|u_j\|_{X_{\frac{1}{2},\frac{1}{2}}}
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
&\Big\|(\psi(u_1)-\psi(u_2))u_3\Big\|_{X_{s,0}} \\
\leq & cT^\varepsilon(1+\|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}\cap Z_0}+\|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}\cap Z_0})^3\|u_1-u_2\|_{X_{\frac{1}{2},\frac{1}{2}}\cap Z_0}\|u_3\|_{X_{s,\frac{1}{2}}}
\end{aligned}
\tag{3.28}
$$

Proof. We observe that $\|\overline{u}\|_{X_{s,b}}=\|u\|_{X^-_{s,b}}$ and

$$\langle\xi\rangle^s\leq c\sum_{k=1}^{l}\langle\xi_k\rangle^s \quad , \text{ for } \xi=\sum_{k=1}^{l}\xi_k \text{ and } s\geq 0$$

Furthermore, $\mu(u_i)=\frac{1}{2\pi}\|u_i(0)\|^2_{L^2}$ and the embedding $Z_0\hookrightarrow C(\mathbb{R},L^2(\mathbb{T}))$ gives

$$|\mu(u_1)-\mu(u_2)|\leq c\|u_1-u_2\|_{Z_0}(\|u_1\|_{Z_0}+\|u_2\|_{Z_0})$$

and by (A.5)

$$
\begin{aligned}
&\|\psi(u)-\psi(v)\|_{L^4_T} \\
\leq & cT^\varepsilon(1+\|u_1\|_{X_{\frac{1}{2},\frac{1}{2}}\cap Z_0}+\|u_2\|_{X_{\frac{1}{2},\frac{1}{2}}\cap Z_0})^3\|u_1-u_2\|_{X_{\frac{1}{2},\frac{1}{2}}\cap Z_0}
\end{aligned}
$$

Using this, the corollary follows from (3.12), (3.22), (3.24). □

3.4 Proof of the well-posedness results

The following diagram describes the structure of the proof of our main results.

$$
\begin{array}{ccccc}
u_0 & \xrightarrow{\mathcal{G}_0} & \mathcal{G}_0(u_0) & \xleftarrow{\mathcal{G}_0} & u_0 \\
S_{R,T}\big\downarrow & & \tilde{S}_{R,T}\big\downarrow & & S^0_{R,T}\big\downarrow \\
(\text{DNLS}) & \xleftarrow{\mathcal{G}^{-1}} & (\overline{\text{DNLS}}) & \xrightarrow{\mathcal{G}_0^{-1}} & (\text{DNLS}_0)
\end{array}
$$

We start with the gauge equivalent Cauchy problem, here denoted with $(\overline{\text{DNLS}})$, and derive the results for (DNLS) and (DNLS$_0$) by conjugating this flow map $\tilde{S}_{R,T}$ with the corresponding gauge transform.

3.4.1 The gauge equivalent Cauchy problem

We define for $v \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$

$$N(v) = -v^2\partial_x\overline{v} + \tfrac{i}{2}|v|^4v - i\mu(v)|v|^2v + i\psi(v)v$$

where $\mu(v)(t) = \frac{1}{2\pi}\|v(t)\|_{L^2}^2$ and

$$\psi(v)(t) = \frac{1}{2\pi}\int_0^{2\pi} 2\,\mathrm{Im}(\overline{v}_x v)(t,\theta) - \frac{1}{2}|v|^4(t,\theta)\,d\theta + \frac{1}{4\pi^2}\|v(t)\|_{L^2}^4$$

and $N_T(v) = N(\chi_T v)$ as well as

$$\Phi_T(v)(t) = \chi(t)\int_0^t W(t-t')N_T(v)(t')\,dt' \tag{3.29}$$

We recall the definition of the space

$$Z_s = X_{s,\frac{1}{2}} \cap Y_{s,0}$$

see (2.10), (2.11) and (2.12). By Corollary 3.3.6, the embedding (2.17) and the linear estimate (2.23) we may extend Φ_T uniquely to a continuous operator

$$\Phi_T : Z_s \to Z_s$$

for all $s \geq \frac{1}{2}$. Moreover,

$$\Phi_T\big|_{[-T,T]} : Z_s^T \to Z_s^T$$

is a continuous operator since it only depends on $v\big|_{[-T,T]}$. Our aim is to find a solution $v \in Z_s^T$ of

$$v(t) = W(t)v_0 + \Phi_T(v)(t) \quad , \; t \in [-T,T] \tag{3.30}$$

Theorem 3.4.1. *There exists a non-increasing function $T^* : (0,\infty) \to (0,\infty)$, such that for all $R > 0$ and $0 < T \leq T^*(R)$ there exists a Lipschitz continuous map*

$$\tilde{S}_{R,T} : B_R = \{v_0 \in H^{\frac{1}{2}}(\mathbb{T}) \mid \|v_0\|_{H^{\frac{1}{2}}(\mathbb{T})} < R\} \to C([-T,T], H^{\frac{1}{2}}(\mathbb{T}))$$

with the following properties:

(i) For all $v_0 \in B_R$ we have $v = \tilde{S}_{R,T}(v_0) \in Z_{\frac{1}{2}}^T$ and v is the unique solution of the equation (3.30) in $Z_{\frac{1}{2}}^T$.

(ii) *For all* $s \geq \frac{1}{2}$ *we have* $\tilde{S}_{R,T}(B_R \cap H^s(\mathbb{T})) \subset C([-T,T], H^s(\mathbb{T}))$ *and*

$$\tilde{S}_{R,T} : B_R \cap H^s(\mathbb{T}) \to C([-T,T], H^s(\mathbb{T}))$$

is locally Lipschitz. Moreover, it is real analytic.

Remark 3.4.2. We remark that Theorem 3.4.1 extends to nonlinear terms of the type $\overline{u}^k \partial_x \overline{u}$ by Grünrock's result [Grü00]. On the other hand, Christ [Chr03] proved a strong ill-posedness result for the nonlinearities $u^k \partial_x u$, for every $k \in \mathbb{N}$.

The proof of local well-posedness of the gauge equivalent problem will be a straightforward application of the contraction mapping principle, cp. [Bou93] and e.g. [Grü02] for a summary of the method.

Local existence and analytic dependence. For $v_0 \in H^s(\mathbb{T})$ we use again the estimates from Corollary 3.3.6 and (2.17), (2.22) and (2.23) as well as Lemma 2.2.5 to show that there exists $c, \varepsilon > 0$, such that

$$\|\chi W(\cdot)v_0 + \Phi_T(v)\|_{Z_s} \leq c\|v_0\|_{H^s} + cT^\varepsilon(1 + \|v\|_{Z_s})^3 \|v\|_{Z_s}^2$$

and

$$\|\Phi_T(v_1) - \Phi_T(v_2)\|_{Z_s} \leq cT^\varepsilon(1 + \|v_1\|_{Z_s} + \|v_2\|_{Z_s})^3 (\|v_1\|_{Z_s} + \|v_2\|_{Z_s}) \|v_1 - v_2\|_{Z_s}$$

Then, for all $v_0 \in H^s$ with $\|v_0\|_{H^s} \leq r$ and $R = 2cr$ and $T > 0$ so small that $T \leq (4c^2 r(1 + 4cr)^3)^{-\frac{1}{\varepsilon}}$ we see that

$$v \mapsto \chi W(\cdot)v_0 + \Phi_T(v)$$

maps the closed ball $\overline{B}_R \subset Z_s$ to itself and is a strict contraction. This shows the existence of a solution $v \in \overline{B}_R \subset Z_s$. By restriction to the interval $[-T, T]$ we found a solution $v \in Z_s^T \subset C([-T,T], H^s(\mathbb{T}))$ of

$$v(t) = W(t)v_0 + \Phi_T(v)(t) \quad , \; t \in [-T, T]$$

Moreover, $(v_0, v) \mapsto \chi W(\cdot)v_0 + \Phi_T(v)$ is real analytic since it is a composition of k-linear maps (over $\mathbb{R}$). Hence, by Theorem 1.4.6, the map

$$\tilde{S}_{R,T} : H^s(\mathbb{T}) \supset B_R \to C([-T,T], H^s(\mathbb{T})), \; v_0 \to v$$

is real analytic. □

Uniqueness in whole space. Assume that $v_1, v_2 \in Z^T_{\frac{1}{2}}$ are two solutions of (3.30), such that

$$T' := \sup\{t \in [0,T] \mid v_1(t) = v_2(t)\} < T$$

and we define $w_j(t) = \tilde{v}_j(T' + t)$, $j = 1, 2$ for extensions $\tilde{v}_j$ of v_j. By approximation we see

$$w_1(t) - w_2(t) = \Phi_T(w_1)(t) - \Phi_T(w_2)(t) \quad -T' \le t \le T - T'$$

Choosing $\delta > 0$ small enough, we arrive at

$$\|\chi_\delta(w_1 - w_2)\|_{Z_{\frac{1}{2}}} \le c\delta^\varepsilon (1 + \|w_1\|_{Z_{\frac{1}{2}}} + \|w_2\|_{Z_{\frac{1}{2}}})^4 \|\chi_\delta(w_1 - w_2)\|_{Z_{\frac{1}{2}}}$$

which forces $w_1(t) = w_2(t)$ for $|t| \le \delta$ and therefore contradicts the definition of T'. The same argument applies in the interval $[-T, 0]$. $\square$

Time of existence. Finally, the standard iteration argument, using the estimates from Corollary 3.3.6, shows that the maximal time of existence $T > 0$ depends only on $\|v_0\|_{H^{\frac{1}{2}}}$: We fix $s \ge \frac{1}{2}$ and

$$B_{r,r_s} = \{v_0 \in H^s(\mathbb{T}) \mid \|v_0\|_{H^{\frac{1}{2}}} \le r \ \text{ and } \ \|v_0\|_{H^s} \le r_s\}$$

and define T_s as the supremum of all $T \in [0,1]$ such that the following statement is true: There exists a real analytic map $\tilde{S} : B_{r,r_s} \to Z_s$ such that $v = \tilde{S}(v_0)$ solves

$$v = \chi W(\cdot)v_0 + \Phi_T(v)$$

and if $u \in Z^T_{\frac{1}{2}}$ is a solution of (3.30), then

$$u = \tilde{S}(v_0)\big|_{[-T,T]}$$

Part 1 and 2 of the proof show that $T_s > 0$. Let $v = \tilde{S}(v_0) \in Z_s$. If $T_s^\varepsilon \le (4c^2 r(1 + 4cr)^3)^{-1}$ we see from the proof of part 1 that $\|v\|_{Z_{\frac{1}{2}}} \le 2cr$. By Corollary 3.3.6 and (2.17) and the estimates (2.22), (2.23) and Lemma 2.2.5 we infer that

$$\begin{aligned}
\|v\|_{Z_s} \le& c_s r_s + c_s T_s^\varepsilon (1 + \|v\|_{Z_{\frac{1}{2}}})^3 \|v\|_{Z_{\frac{1}{2}}} \|v\|_{Z_s} \\
\le& c_s r_s + c_s T_s^\varepsilon (1 + 2cr)^3 2cr \|v\|_{Z_s}
\end{aligned}$$

Therefore, if additionally $T_s^\varepsilon \le (2c_s(1 + 2cr)^3 2cr)^{-1}$, we obtain

$$\sup_{|t| \le T_s} \|v(t)\|_{H^s} \le c\|v\|_{Z_s} < 2c_s r_s \tag{3.31}$$

If these assumptions about T_s were true, we could apply part 1 and 2 of the proof. We find a $\delta > 0$ and a real analytic map $G : H^s \supset B_{2c_s r_s} \to Z_s$ such that $w = G(w_0)$ satisfies

$$w = \chi W(\cdot)w_0 + \Phi_{2\delta}(w)$$

and if $u \in Z^{2\delta}_{\frac{1}{2}}$ is a solution of

$$u(t) = W(t)w_0 + \Phi_T(u)(t) \quad , \; t \in [-T, T]$$

then

$$u\big|_{[-2\delta,2\delta]} = G(w_0)\big|_{[-2\delta,2\delta]}$$

Define

$$H : \; v_0 \mapsto \eta_\delta \tilde{S}(v_0) + \eta_\delta^+ G(\tilde{S}(v_0)(T_s))(\cdot - T_s) + \eta_\delta^- G(\tilde{S}(v_0)(-T_s))(\cdot + T_s)$$

as a map from B_{r,r_s} to Z_s with smooth cutoff functions $\eta_\delta, \eta_\delta^+, \eta_\delta^-$, such that $\eta_\delta + \eta_\delta^+ + \eta_\delta^- = 1$ on $[-T_s - \delta, T_s + \delta]$ with

$$\mathrm{supp}(\eta_\delta) \subset [-T_s + \delta, T_s - \delta] \; , \; \mathrm{supp}(\eta_\delta^\pm) \subset [\pm T_s - 2\delta, \pm T_s + 2\delta]$$

Then, H is real analytic by Proposition 1.4.5 and

$$H(v_0) = \chi W(\cdot)v_0 + \Phi_{T_s+\delta}(H(v_0))$$

and if $u \in Z^{T_s+\delta}_{\frac{1}{2}}$ is a solution of

$$u(t) = W(t)v_0 + \Phi_{T_s+\delta}(u)(t) \quad , \; t \in [-T_s - \delta, T_s + \delta]$$

then part 2 of the proof also gives

$$u\big|_{[-T_s-\delta,T_s+\delta]} = H(v_0)\big|_{[-T_s-\delta,T_s+\delta]}$$

which contradicts the definition of T_s and we conclude that

$$T_s^\varepsilon \geq \min\{(4c^2 r(1 + 4cr)^3)^{-1}, (2c_s(1 + 2cr)^3 2cr)^{-1}\}$$

and if $T_s < 1$ then

$$\lim_{t \uparrow T_s} \|u(t)\|_{H^{\frac{1}{2}}} = \infty \tag{3.32}$$

for a solution u because otherwise we could iterate the argument above. $\square$

3.4.2 Proof of the main results

Results for (DNLS). Part I

We start with the proof of Theorem 3.1.1.

Existence. We fix $s \geq \frac{1}{2}$ and let $u_0 \in H^s(\mathbb{T})$ with $\mu := \frac{1}{2\pi}\|u_0\|_{L^2}^2$. Then, we define $v_0 := \mathcal{G}(u_0) \in H^s(\mathbb{T})$, see Proposition 3.2.5. According to Theorem 3.4.1, there exists a unique solution $v \in Z_s^T \subset C([-T,T], H^s(\mathbb{T}))$ of (3.30). Now, we claim that $u := \mathcal{G}^{-1}(v) \in C([-T,T], H^s(\mathbb{T}))$ solves

$$u(t) = W(t)u_0 + \int_0^t W(t-t')\partial_x(|u|^2 u)(t')\,dt', \quad t \in (-T,T) \tag{3.33}$$

For smooth functions this follows from Proposition 3.2.6. Let $u_{0,n} \in C^\infty(\mathbb{T})$ with $u_{0,n} \to u_0$ in H^s and $\|u_{0,n}\|_{L^2} = \|u_0\|_{L^2}$. Moreover, let $v_n \in Z_s^T$ be the solutions of (3.30) with initial data $\mathcal{G}(u_{0,n})$ and $u_n := \mathcal{G}^{-1}(v_n)$. Then,

$$\begin{aligned}&\sup_{t\in(-T,T)}\left\|\int_0^t W(t-t')\partial_x(|u|^2u - |u_n|^2u_n)(t')\,dt'\right\|_{H^{-1}}\\ \leq& c(\|u\|^2_{L_T^\infty H^{\frac{1}{2}}} + \|u_n\|^2_{L_T^\infty H^{\frac{1}{2}}})\|u-u_n\|_{L_T^1 L_x^2}\end{aligned}$$

Because $\mathcal{G}$ is continuous in H^s, $\mathcal{G}(u_{0,n}) \to v_0$ and due to the continuity of the flow map of (3.30) we have $v_n \to v$ in $C([-T,T], H^s(\mathbb{T}))$. Since also $\mathcal{G}^{-1}$ is continuous, the above term tends to zero. This shows that u solves (3.33) because obviously also the linear part converges in $C([-T,T], H^s(\mathbb{T}))$. □

Uniqueness. Let $u_1, u_2 \in C([-T,T], H^{\frac{1}{2}}(\mathbb{T}))$ be two solutions of (3.33) with $u_1(0) = u_2(0)$, such that $\mathcal{G}(u_1), \mathcal{G}(u_2) \in Z_{\frac{1}{2}}^T$ solve (3.30) with the same initial datum. By the uniqueness of the solutions to (3.30) we have $\mathcal{G}(u_1) = \mathcal{G}(u_2)$ and therefore $u_1 = u_2$.

Since the hypothesis here is somehow technical, we will now derive an alternative uniqueness statement: Assume that $u_0 \in H^s(\mathbb{T})$ and $s > \frac{7}{6}$. Let

$$u = S_{T,R}(u_0) \in C([-T,T], H^s(\mathbb{T}))$$

be the solution constructed above. Then, $v_x \in Z_{s-1}^T$, where $v = \mathcal{G}(u)$. This implies $J^{s-1-\varepsilon}v_x \in L^p([-T,T] \times \mathbb{T})$ due to (2.21), where we choose $0 < \varepsilon < s - \frac{7}{6}$ and $2 \leq p < 6$ with $s - 1 - \varepsilon > \frac{1}{p}$. The Sobolev embedding theorem for Bessel potential spaces, see e.g. [ST87] Section 3.5.5, applied in the x variable implies that $v_x \in L^p([-T,T], L^\infty(\mathbb{T}))$. Now, with $\mu = (2\pi)^{-1}\int_0^{2\pi}|u|^2 dx$ we have

$$v_x(t,x) = e^{-i\mathcal{I}(u)}\left(-i(|u|^2-\mu)u + u_x\right)(t, x - 2\mu t)$$

and it follows

$$u_x \in L^1([-T,T], L^\infty(\mathbb{T}))$$

i.e. our solution lies in the uniqueness class given by the energy estimate from Proposition A.4.1. In particular, solutions in $C([-T,T], H^{\frac{3}{2}+\varepsilon})$ are unique and, due to Proposition 1.3.5, the solutions constructed above are unique limits of smooth solutions. □

Continuity properties of the flow for (DNLS). Part I. Since the flow map to (3.33) $S_{R,T} : H^s(\mathbb{T}) \cap B_R \to C([-T,T], H^s(\mathbb{T}))$ results from conjugating the flow map to (3.30) $\tilde{S}_{R,T} : H^s(\mathbb{T}) \cap B_R \to C([-T,T], H^s(\mathbb{T}))$ with the gauge transformation $\mathcal{G}$, i.e. $S_{R,T} = \mathcal{G}^{-1} \circ \tilde{S}_{R,T} \circ \mathcal{G}$, its continuity properties follow from the real analyticity of $\tilde{S}_{R,T}$ and Proposition 3.2.5. □

Global existence. It suffices to prove an a priori bound for smooth solutions. By Lemma A.5.2 and the Sobolev embedding we have

$$\begin{aligned}
&\|\partial_x u(t)\|_{L^2(\mathbb{T})}^2 + \frac{3}{2} \operatorname{Im} \int_0^{2\pi} |u|^2 u \partial_x \overline{u}(t)\, dx + \frac{1}{2} \|u(t)\|_{L^6(\mathbb{T})}^6 \\
\leq &c(1 + \|u_0\|_{H^1(\mathbb{T})})^6
\end{aligned}$$

We observe

$$\operatorname{Im} \int_0^{2\pi} |u|^2 u \partial_x \overline{u}(t)\, dx \geq -\|u(t)\|_{L^6}^3 \|\partial_x u(t)\|_{L^2}$$

which shows that

$$\left(\|\partial_x u(t)\|_{L^2(\mathbb{T})} - \frac{3}{4} \|u(t)\|_{L^6}^3 \right)^2 - \frac{1}{16} \|u(t)\|_{L^6(\mathbb{T})}^6 \leq c(1 + \|u_0\|_{H^1(\mathbb{T})})^6$$

and it follows

$$\begin{aligned}
\|\partial_x u(t)\|_{L^2(\mathbb{T})} \leq &\frac{3}{4} \|u(t)\|_{L^6}^3 + \left(\frac{1}{16} \|u(t)\|_{L^6(\mathbb{T})}^6 + c(1 + \|u_0\|_{H^1(\mathbb{T})})^6 \right)^{\frac{1}{2}} \\
\leq &\|u(t)\|_{L^6}^3 + \sqrt{c}(1 + \|u_0\|_{H^1(\mathbb{T})})^3
\end{aligned} \tag{3.34}$$

Now, we use the Gagliardo-Nirenberg inequality from Appendix A.1 and find for any $\varepsilon > 0$ a $c_\varepsilon > 0$ such that

$$\begin{aligned}
\|u(t)\|_{L^6}^3 \leq &\frac{1}{2\pi} \|u(t)\|_{L^2}^2 \left(\|\partial_x u(t)\|_{L^2}^{\frac{1}{3}} + \|u(t)\|_{L^2}^{\frac{1}{3}} \right)^3 \\
\leq &\frac{1+\varepsilon}{2\pi} \|u(t)\|_{L^2}^2 \|\partial_x u(t)\|_{L^2} + c_\varepsilon \|u(t)\|_{L^2}^3
\end{aligned}$$

We fix $\delta < \sqrt{2\pi}$ and choose $\varepsilon = (2\pi - \delta^2)/(2\delta^2)$ such that $(1+\varepsilon)\delta^2 < 2\pi$. Using this in (3.34) and the L^2 conservation law we have for $\|u(t)\|_{L^2} \le \delta$

$$(1 - \frac{1+\varepsilon}{2\pi}\delta^2)\|\partial_x u(t)\|_{L^2} \le C_\delta(1 + \|u_0\|_{H^1(\mathbb{T})})^3$$

which shows that

$$\|\partial_x u(t)\|_{L^2} \le \tilde{C}_\delta(1 + \|u_0\|_{H^1(\mathbb{T})})^3$$

This estimate together with the L^2 conservation law from Lemma A.5.1 shows that for $\|u_0\|_{L^2}^2 \le \delta^2 < 2\pi$ there exists $C(\delta) > 0$ such that

$$\|u(t)\|_{H^1(\mathbb{T})} \le C(\delta)(1 + \|u_0\|_{H^1(\mathbb{T})})^3$$

as desired[2]. □

Results for (DNLS$_0$)

The proof of Theorem 3.1.5 works exactly in the same way by replacing $\mathcal{G}$ with $\mathcal{G}_0$ and using the real analyticity of $\mathcal{G}_0$. We additionally show the sharpness in terms of the regularity of the flow map, which is (essentially) based on the counterexamples from [Tak99] and the general idea from [MST01].

Proof of sharpness result for (DNLS$_0$). Let $n \in \mathbb{N}$ and $u_0^{(n)} := \langle n \rangle^{-s} e^{inx}$. Then, $\|u_0^{(n)}\|_{H^s} = 1$. Let u_n be the solution of (DNLS$_0$) with initial datum $u_0^{(n)}$, then the third derivative of $\lambda \mapsto S^0_{R,T}(\lambda u_0^{(n)})$, evaluated at $\lambda = 0$, is given by the sum of

$$12\int_0^t W(t-t')\left(|W(t')u_0^{(n)}|^2 - \frac{1}{2\pi}\int_0^{2\pi}|W(t')u_0^{(n)}|^2 dx\right)\partial_x W(t')u_0^{(n)}\,dt'$$

and

$$6\int_0^t W(t-t')\left((W(t')u_0^{(n)})^2 \partial_x \overline{W(t')u_0^{(n)}}\right)\,dt'$$

The first term vanishes because $|W(t')u_0^{(n)}|^2 \equiv \langle n \rangle^{-2s}$, and for the second term

$$\begin{aligned}
&\langle n \rangle^s \left|\mathcal{F}_x\left(\int_0^t W(t-t')\left((W(t')u_0^{(n)})^2 \partial_x \overline{W(t')u_0^{(n)}}\right)\,dt'\right)(n)\right| \\
\ge &\langle n \rangle^{-2s}\left|\int_0^t e^{it'n^2} e^{-2it'n^2}(-i)n e^{it'n^2}\,dt'\right| \\
\ge &c|t|\langle n \rangle^{1-2s}
\end{aligned}$$

[2] Observe that in this proof does not use the gauge transformation, cp. [HO94], pp. 1499-1500.

which shows that for fixed $T > 0$

$$\left\| \frac{\mathrm{d}^3}{\mathrm{d}\lambda^3}(\lambda \mapsto S^0_{R,T}(\lambda u_0^{(n)})) \right\|_{C([-T,T],H^s(\mathbb{T}))} \geq cn^{1-2s}$$

If the flow were C^3 this would be bounded uniformly in $n \in \mathbb{N}$ which is a contradiction for $s < \frac{1}{2}$. $\square$

Results for (DNLS). Part II

Here we additionally show that the flow map $S_{R,T}$ for the (DNLS) is not uniformly continuous on bounded subsets for all $s \geq \frac{1}{2}$, if the L^2 norm of the initial data is permitted to vary.

Lack of uniform continuity of the flow on balls. Assume that

$$S_{R,T} : H^s(\mathbb{T}) \supset \{u_0 \mid \|u_0\|_{H^{\frac{1}{2}}(\mathbb{T})} < R\} \to C([-T,T], H^s(\mathbb{T}))$$

for some $R > 0$ and $0 < T < T^*(R)$ is uniformly continuous a small ball around the origin. Then, for all $|t| \leq T$ also

$$S_{R,T}(t) : H^s(\mathbb{T}) \supset \{u_0 \mid \|u_0\|_{H^{\frac{1}{2}}(\mathbb{T})} < R\} \to H^s(\mathbb{T})$$

is uniformly continuous on this ball. Let $0 < t, s \leq T$ and define for $u_0 \in H^2(\mathbb{T})$, with $\|u_0\|_{H^{\frac{1}{2}}(\mathbb{T})}$ small, the function $u^{(t)}(s) = S_{R,T}(s)S^0_{R,T}(-t)u_0$. This solves

$$\begin{aligned} \partial_s u^{(t)}(s) - i\partial_x^2 u^{(t)}(s) &= \partial_x(|u^{(t)}|^2 u^{(t)})(s) \\ u^{(t)}(0) &= S^0_{R,T}(-t)u_0 \end{aligned}$$

Now, we consider $v^{(t)}(s,x) := u^{(t)}(s, x - 2s\|S^0_{R,T}(-t)u_0\|^2_{L^2})$, which solves (DNLS$_0$), see Proposition 3.2.2. By the L^2 conservation law from Lemma A.5.1 it follows $\|S^0_{R,T}(-t)u_0\|^2_{L^2} = \|u_0\|^2_{L^2}$. Due to the uniqueness of solutions $S^0_{R,T}(s)S^0_{R,T}(-t)u_0 = v^{(t)}(s)$. By a similar argument, considering the function $S^0_{R,T}(s-t)u_0$, we also show that $S^0_{R,T}(s-t)u_0 = S^0_{R,T}(s)S^0_{R,T}(-t)u_0$. Therefore, with $t = s$ we have

$$u_0(\cdot + 2t\|u_0\|^2_{L^2}) = S_{R,T}(t)S^0_{R,T}(-t)u_0$$

By continuity this also holds for general $u_0 \in H^s(\mathbb{T})$, $s \geq \frac{1}{2}$ with small $H^{\frac{1}{2}}(\mathbb{T})$ norm, which shows that

$$u_0 \mapsto u_0(\cdot + 2t\|u_0\|^2_{L^2})$$

as a map from a small ball in $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$ is uniformly continuous, because by assumption this is true for $S_{R,T}(t)S^0_{R,T}(-t)$. This is a contradiction to Proposition 3.2.2. $\square$

3.4.3 A Counterexample to tri-linear estimates

The aim of this subsection is to provide a reason for the use of the gauge transformation. We show that it is not possible to prove estimates in the Fourier restriction norm spaces from Subsection 2.2.1 for the (DNLS_0) directly. We already know that estimates which enable us to perfom the Picard iteration argument for the (DNLS) cannot be true because the flow is not uniformly continuous on balls. Let

$$N(u) = 2\left(|u|^2 - \frac{1}{2\pi}\int_0^{2\pi} |u|^2 dx\right)\partial_x u + u^2\partial_x \overline{u}$$

and

$$N_1(u) = \left(|u|^2 - \frac{1}{2\pi}\int_0^{2\pi} |u|^2 dx\right)\partial_x u\ , \quad N_2(u) = u^2\partial_x \overline{u}$$

Theorem 3.4.3. *Let $s, b \in \mathbb{R}$. For any $n \in \mathbb{N}$ we find $u_n \in X_{s,b}$ and $c_n > 0$ which satisfy*

$$\|N(u_n)\|_{X_{s,b-1}} \geq c_n\|u_n\|_{X_{s,b}} \quad , c_n \to \infty$$

Remark 3.4.4. The proof will show that we may replace $X_{s,b}$ by $X_{s,b}\cap Y_{s,b-\frac{1}{2}}$ on the right hand side.

Proof. 1. Case: $s \in \mathbb{R}, b < 1/3$. Let us define the sequence of functions

$$u_n(t,x) = (n^{-s}e^{-it(n+2)^2+inx} + e^{-it+ix} + e^{-it-ix})\chi(t)$$

where $\chi \in \mathcal{S}(\mathbb{R})$ defined by $\chi = \mathcal{F}_t^{-1}\eta$ for $\eta \in C_0^\infty(-10, 10)$ nonnegative and even with $\eta \equiv 1$ on $[-5, 5]$. Observe that u_n consists of three different frequency modes to $n, 1, -1$. Our aim is to calculate the $n + 2$ frequency mode of $N(u)$. We first observe that the only contribution of the second term $N_2(u_n)$ to the frequency $n + 2$ is

$$\begin{aligned} f_n(t) :=& \chi^3(t)2n^{-s}e^{-it(n+2)^2+inx}e^{-it+ix}\partial_x e^{it+ix} \\ =& \chi^3(t)2in^{-s}e^{-it(n+2)^2+i(n+2)x} \end{aligned}$$

Next, we calculate the contributions from the first term $2N_1(u_n)$ to the frequency $n+2$.

$$\begin{aligned}&|u_n|^2 - \frac{1}{2\pi}\int_0^{2\pi} |u_n|^2 dx\\ =&\chi^2(t)(n^{-s}e^{-it(n+2)^2+it+i(n-1)x} + n^{-s}e^{-it(n+2)^2+it+i(n+1)x}\\ &+ n^{-s}e^{it(n+2)^2-it-i(n-1)x} + n^{-s}e^{it(n+2)^2-it-i(n+1)x} + e^{2ix} + e^{-2ix})\end{aligned}$$

with frequencies $n-1, n+1, -n+1, -n-1, 2, -2$. Hence, the only contribution to the $n+2$ frequency of $2N_1(u_n)$ is

$$\begin{aligned}g_n(t) :=&2\chi^3(t)(n^{-s}e^{-it(n+2)^2+it+i(n+1)x}\partial_x e^{-it+ix} + e^{2ix}\partial_x n^{-s}e^{-it(n+2)^2+inx})\\ =&\chi^3(t)2in^{-s}(1+n)e^{-it(n+2)^2+i(n+2)x}\end{aligned}$$

Now,

$$\mathcal{F}u_n(\tau,\xi) = c\begin{cases}\eta(\tau+(n+2)^2)n^{-s} & ,\xi = n\\ \eta(\tau+1) & ,\xi = \pm 1\\ 0 & ,\text{otherwise}\end{cases}$$

which shows that

$$\begin{aligned}\|u_n\|^2_{X_{s,b}} =&cn^{-2s}\int \langle \tau+n^2\rangle^{2b}\langle n\rangle^{2s}|\eta(\tau+(n+2)^2)|^2 d\tau + c\\ \leq&c\int \langle \nu - 4n - 4\rangle^{2b}|\eta(\nu)|^2 d\nu + c \leq cn^{2b}\end{aligned}$$

because η has compact support. On the other hand,

$$\|N(u_n)\|^2_{X_{s,b-1}} \geq c\int \langle \tau+(n+2)^2\rangle^{2b-2}\langle n+2\rangle^{2s}|\mathcal{F}(f_n+g_n)(\tau)|^2 d\tau$$

Because $|\eta * \eta * \eta(\tau)| \geq 1$ for $|\tau| \leq 1$ it is

$$\begin{aligned}&\int \langle \tau+(n+2)^2\rangle^{2b-2}\langle n+2\rangle^{2s}|\mathcal{F}(f_n+g_n)(\tau)|^2 d\tau\\ \geq&cn^2\int \langle \tau+(n+2)^2\rangle^{2b-2}|\eta*\eta*\eta(\tau+(n+2)^2)|^2 d\tau\\ \geq&cn^2\int \langle \tau\rangle^{2b-2}|\eta*\eta*\eta(\tau)|^2 d\tau \geq cn^2\end{aligned}$$

which proves the claim in the case $b < 1/3$ because n^{2-6b} is unbounded.

2. Case: $s \in \mathbb{R}, b > 0$. Let us define

$$v_n(t,x) = (n^{-s}e^{-itn^2+inx} + e^{-it+ix} + e^{-it-ix})\chi(t)$$

which consists of three modes to the frequencies $n, 1, -1$. Our aim is to calculate the $n+2$ frequency mode of $N(u)$. As above the only contribution of the term $N_2(u_n)$ is

$$\begin{aligned} k_n(t) :=&\chi^3(t)2n^{-s}e^{-itn^2+inx}e^{-it+ix}\partial_x e^{it+ix} \\ =&\chi^3(t)2in^{-s}e^{-itn^2+i(n+2)x} \end{aligned}$$

Next, we calculate the contributions from the first term $2N_1(u_n)$ to the frequency $n+2$. As above

$$\begin{aligned} &|v_n|^2 - \frac{1}{2\pi}\int_0^{2\pi} |v_n|^2 dx \\ =&\chi^2(t)(n^{-s}e^{-itn^2+it+i(n-1)x} + n^{-s}e^{-itn^2+it+i(n+1)x} \\ &+ n^{-s}e^{itn^2-it-i(n-1)x} + n^{-s}e^{itn^2-it-i(n+1)x} + e^{2ix} + e^{-2ix}) \end{aligned}$$

with frequency modes $n-1, n+1, -n+1, -n-1, 2, -2$. Hence, the only contribution to the $n+2$ frequency of $2N_1(u_n)$ is

$$\begin{aligned} l_n(t) :=&2\chi^3(t)(n^{-s}e^{-itn^2+it+i(n+1)x}\partial_x e^{-it+ix} + e^{2ix}\partial_x n^{-s}e^{-itn^2+inx}) \\ =&\chi^3(t)2in^{-s}(1+n)e^{-itn^2+i(n+2)x} \end{aligned}$$

A similar calculation as above shows

$$\mathcal{F}v_n(\tau,\xi) = c\begin{cases} \eta(\tau+n^2)n^{-s} & ,\xi = n \\ \eta(\tau+1) & ,\xi = \pm 1 \\ 0 & ,\text{otherwise} \end{cases}$$

which shows that

$$\|v_n\|_{X_{s,b}}^2 = cn^{-2s}\int \langle\tau+n^2\rangle^{2b}\langle n\rangle^{2s}|\eta(\tau+n^2)|^2 d\tau + c \le c$$

On the other hand

$$\begin{aligned} &\int \langle\tau+(n+2)^2\rangle^{2b-2}\langle n+2\rangle^{2s}|\mathcal{F}(k_n+l_n)(\tau)|^2 d\tau \\ \ge&cn^2\int \langle\tau+(n+2)^2\rangle^{2b-2}|\eta*\eta*\eta(\tau+n^2)|^2 d\tau \\ \ge&cn^2\int \langle\nu+4n+4\rangle^{2b-2}|\eta*\eta*\eta(\nu)|^2 d\nu \ge cn^2 n^{2b-2} \end{aligned}$$

because $|\eta*\eta*\eta(\tau)| \ge 1$ for $|\tau| \le 1$. Now, n^{2b} is unbounded. □

3.5 Notes and References

In the case of the real line local well-posedness of the Cauchy problem for the (DNLS) in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$ was obtained by H. Takaoka [Tak99] (for slightly more general nonlinearities). This result is sharp in the sense that for $s < \frac{1}{2}$ the uniform continuity of the flow map (on balls) fails to hold due to counterexamples constructed by H.A. Biagoni - F. Linares [BL01]. Remarkably, the critical regularity for the scaling argument is L^2, which means the following: Assume u is a solution of (3.1), then also $u_\mu(t,x) = \mu^{\frac{1}{2}} u(\mu^2 t, \mu x)$ is a solution and $\|u_\mu(t)\|_{L^2} = \|u(t)\|_{L^2}$ for all $\mu > 0$.

J. Colliander - M. Keel - G. Staffiliani - H. Takaoka - T. Tao [CKS+02] proved global well-posedness for initial data in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$ which satisfies the $L^2(\mathbb{R})$ smallness condition $\|u_0\|^2_{L^2(\mathbb{R})} < 2\pi$ and improved the global result in $H^1(\mathbb{R})$ of N. Hayashi - T. Ozawa [HO94] for the (DNLS). Recently, A. Grünrock [Grü05b] obtained a local result for initial data in spaces $\widehat{H^r_s}$ defined by the norms $\|\langle \xi \rangle^s \mathcal{F} u_0\|_{L^{r'}}$ for $s \geq \frac{1}{2}$, $1 < r \leq 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$. The (DNLS) equation found application as a model in plasma physics and it satisfies infinitely many conservation laws [KN78].

The local well-posedness result of H. Takaoka [Tak99] was proved by using the gauge transform developed by N. Hayashi - T. Ozawa [HO92, Hay93, HO94] to derive a gauge equivalent equation and H. Takaoka showed that this can be treated by the Fourier restriction norm method of J. Bourgain [Bou93] as long as $s \geq \frac{1}{2}$. The proof of the main tri-linear estimate uses local smoothing and Strichartz estimates.

In the periodic case M. Tsutsumi - I. Fukuda [TF80, TF81] obtained well-posedness results for $s > \frac{3}{2}$ (also valid in the non-periodic case) as well as global weak solutions in H^1 with an additional smallness condition. It is also known that there exist global (weak) solutions in Sobolev spaces corresponding to $H^1(\mathbb{T})$ subject to Dirichlet and generalized periodic boundary conditions due to the results from Y. Chen [Che86] and T. Meškauskas [Meš98] (for initial data fulfilling a smallness condition). Moreover, these solutions are unique in some range. The results from this Chapter are significant improvements to these results and are based on the dispersive structure of the equation.

The major difference to the non-periodic case is characterized by the dispersive properties of solutions. These are weaker in the sense that there are no local smoothing estimates available which could be used to control derivatives in nonlinear terms. Moreover, above L^4 the Strichartz estimates are only known to hold with a loss of $\varepsilon > 0$ derivatives. In the nonperiodic case we expect that one can derive a local well-posedness result for small data and for $s > \frac{1}{2}$ by applying multilinear estimates directly without using the gauge transformation. In principle this should follow in the same way as for the modified Benjamin - Ono equation as in [MR04] using spaces based on the local smoothing and maximal function estimates, cp. also [KPV93b] for larger values of s. This is different in the periodic setting, cp. also Subsection 3.4.3 for the (DNLS_0) in Fourier restriction norm spaces. Nevertheless, our results in this chapter show that the general method of proof used in [Tak99] carries over to the periodic case despite of the aforementioned obstructions.

Chapter 4

Benjamin-Ono type equations

In this chapter we consider the non-periodic case and the phase function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(\xi) = |\xi|^\alpha \xi$, $\alpha > 0$, except for Section 4.3 where we also study the periodic case.

4.1 Motivation

We consider the Cauchy problem

$$\begin{aligned} \partial_t u(t) - |D|^\alpha \partial_x u(t) + \frac{1}{2}\partial_x u^2(t) = 0 \quad \text{in } (-T,T) \\ u(0) = u_0 \end{aligned} \tag{4.1}$$

for $0 < \alpha \leq 2$ and initial data belonging to some L^2-based Sobolev space.

In all cases discussed here there are at least three formally conserved quantities for real valued solutions, namely

$$\int_{\mathbb{R}} u \, dx$$

$$\int_{\mathbb{R}} u^2 \, dx \quad (L^2 \text{ norm})$$

$$\frac{1}{2}\int_{\mathbb{R}} \left||D|^{\frac{\alpha}{2}} u\right|^2 \, dx - \frac{1}{6}\int_{\mathbb{R}} u^3 \, dx \quad (\text{Hamiltonian})$$

The Korteweg-de Vries equation ($\alpha = 2$) was introduced by D.J. Korteweg - G. de Vries in [KDV95] and the Benjamin-Ono equation ($\alpha = 1$) was developed by T.B. Benjamin [Ben67] and H. Ono [Ono75]. Both are one-dimensional model equations for nonlinear dispersive waves arising in water wave theory[1] and they possess an infinite number of conservation laws. For all $0 < \alpha \leq 2$ there is a well-posedness theory [Sau79] for regular data and for $1 \leq \alpha \leq 2$ these equations admit solitary wave solutions [Wei87].

4.2 Equations of Benjamin-Ono type

In this section we focus on the Cauchy problem (4.1) in the cases $1 < \alpha < 2$ and we aim to prove well-posedness in low regularity spaces by a fixed point argument.

It was observed by L. Molinet - J.-C. Saut - N. Tzvetkov [MST01] that there is a major obstruction in the range $\alpha < 2$ for the iterative methods successfully used in the study of the KdV equation. They show that interactions of linear waves of very low frequency with linear waves of high frequency cannot be controlled by bilinear estimates based on $H^s(\mathbb{R})$ information only and that the flow map for $H^s(\mathbb{R})$ data is not C^2.

In the following, we prove a local well-posedness result in the range $1 < \alpha < 2$ for data in the spaces $H^{(s,\omega)}$ defined in Subsection 2.2.2 which almost closes the gap to the local well-posedness theory for KdV known so far. Moreover, we derive a global result for real valued data based on the L^2 conservation law. The low frequency condition in these results is shown to be sharp with respect to the C^2 continuity of the flow map. This will be made precise below. The well-posedness results will include the existence and uniqueness of solutions in the sense of distributions in a certain space, the persistence of initial regularity and the analyticity of the flow map.

Theorem 4.2.1. *Let $1 < \alpha < 2$ and $\omega = \frac{1}{\alpha} - \frac{1}{2}$. Then, for $s \geq s_0 > -\frac{3}{4}(\alpha - 1)$ there exists $b > \frac{1}{2}$ and a non-increasing function $T^* : (0,\infty) \to (0,\infty)$, such that for $R > 0$ and $T \leq T^*(R)$ there exists a continuous map*

$$S_{R,T} : B_R = \{u_0 \in H^{(s_0,\omega)} \mid \|u_0\|_{H^{(s_0,\omega)}} \leq R\} \to C\big([-T,T], H^{(s_0,\omega)}\big)$$

with the properties:

(i) For all $u_0 \in B_R$ we have $S_{R,T}(u_0) \in X^T_{s_0,\omega,b} \subset L^\infty_x(\mathbb{R}, L^2_t([-T,T]))$ and $u = S_{R,T}(u_0)$ is the unique solution in $X^T_{s_0,\omega,b}$ of the Cauchy problem (4.1) in the sense that

$$\int_{\mathbb{R}} \int_{-T}^{T} u\partial_t\varphi - u|D|^\alpha \partial_x\varphi + \frac{1}{2}u^2\partial_x\varphi dtdx = 0$$

[1]For $1 < \alpha < 2$ cp. remarks in the introductions of [KPV90, CKS03b, MR06, Sau79]

holds true for all $\varphi \in C_0^\infty((-T,T)\times\mathbb{R})$.

(ii) For every $s \geq s_0$ *we have* $S_{R,T}(B_R \cap H^{(s,\omega)}) \subset C([-T,T],H^{(s,\omega)})$ *and*

$$S_{R,T}\,|_{B_R\cap H^{(s,\omega)}}\colon B_R \cap H^{(s,\omega)} \to C\big([-T,T],H^{(s,\omega)}\big)$$

is analytic.

Remark 4.2.2. If $\alpha \to 1^+$ the lower bound for s tends to 0 and for $\alpha \to 2^-$ the bound converges to $-3/4$. For all admissible values of α our result includes the L^2 case where a conserved quantity is available.

Corollary 4.2.3. *Let* $1 < \alpha < 2$ *and* $\omega = \frac{1}{\alpha} - \frac{1}{2}$.

(i) For all $s > -\frac{3}{4}(\alpha-1)$ *the Cauchy problem* (4.1) *is locally well-posed in* $H^{(s,\omega)}$.

(ii) For all $s \geq 0$ *the Cauchy problem* (4.1) *is globally well-posed for real valued initial data in* $H^{(s,\omega)}$.

The next theorem shows that our low frequency condition is optimal with respect to the analyticity of the flow map.

Theorem 4.2.4. *Let* $1 \leq \alpha < 2$ *and assume that the Cauchy problem* (4.1) *is well-posed in* $H^{(s,\omega)}$ *for* $0 \leq \omega < \frac{1}{\alpha} - \frac{1}{2}$ *and* $s \in \mathbb{R}$. *Then, the flow map*

$$H^{(s,\omega)} \supset B_R \ni u_0 \mapsto u \in C([-T,T],H^s(\mathbb{R}))$$

is not C^2 *at the origin. In particular, the bilinear estimate corresponding to* (4.2) *fails.*

4.2.1 A bilinear estimate

This section is devoted to

Theorem 4.2.5. *Let* $1 < \alpha < 2$, $s \geq s_0 > -\frac{3}{4}(\alpha-1)$ *and* $\omega = \frac{1}{\alpha} - \frac{1}{2}$. *There exists* $b' > -\frac{1}{2}$ *and* $b \in (\frac{1}{2}, b'+1)$ *such that*

$$\|\partial_x(u_1u_2)\|_{X_{s,\omega,b'}} \leq c\|u_1\|_{X_{s,\omega,b}}\|u_2\|_{X_{s_0,\omega,b}} + \|u_1\|_{X_{s_0,\omega,b}}\|u_2\|_{X_{s,\omega,b}} \tag{4.2}$$

holds true for all $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$.

Before we start with the proof, we note an algebraic fact concerning our phase function, which we will call resonance relation.

Lemma 4.2.6. *Let* $1 < \alpha < 2$. *Define*

$$h(\xi_1, \xi_2, \xi) = \xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha - \xi_2|\xi_2|^\alpha \tag{4.3}$$

Then, for $\xi = \xi_1 + \xi_2$ *we have*

$$|h(\xi_1, \xi_2, \xi)| \geq c|\xi_{\min}||\xi_{\max}|^\alpha \tag{4.4}$$

where $|\xi_{\min}| := \min\{|\xi_1|, |\xi_2|, |\xi|\}$ *and* $|\xi_{\max}| := \max\{|\xi_1|, |\xi_2|, |\xi|\}$.

Proof. For $\beta \geq 0$ we define $f(\beta) := (1+\beta)^{1+\alpha} - \beta^{1+\alpha} - 1$. This function satisfies $f(0) = 0$ and $f'(\beta) = (1+\alpha)((1+\beta)^\alpha - \beta^\alpha) > 0$ as well as $f''(\beta) = (1+\alpha)\alpha((1+\beta)^{\alpha-1} - \beta^{\alpha-1}) > 0$ for $\beta > 0$. This implies that

$$f(\beta) \geq f'(0)\beta = (1+\alpha)\beta \quad \text{for} \quad \beta \in [0,1]$$

We observe that $f(\beta) = \beta^{1+\alpha} f(1/\beta)$ for all $\beta > 0$, which implies

$$f(\beta) = \beta^{1+\alpha} f(1/\beta) \geq (1+\alpha)\beta^\alpha \quad \text{for } \beta \geq 1.$$

Now we start we the proof of (4.4). We suppose the constraint $\xi = \xi_1 + \xi_2$ to hold and consider two cases:

Case 1: $\xi_1\xi_2 > 0$. Since h is symmetric with respect to ξ_1 and ξ_2, it suffices to consider $\xi_1 = \beta\xi_2$ with $\beta \geq 1$. Then,

$$|h(\xi_1, \xi_2, \xi)| = f(\beta)|\xi_2|^{1+\alpha} \geq c\beta^\alpha|\xi_2|^{1+\alpha} = c|\xi_1|^\alpha|\xi_2|.$$

Since $|\xi| \leq |\xi_1| + |\xi_2| \leq 2|\xi_1|$ we have $|\xi_1| \geq 1/2|\xi_{\max}|$ and this implies (4.4).

Case 2: $\xi_1\xi_2 < 0$. By symmetry we may assume $\xi_2\xi < 0$ and $\xi_1\xi > 0$. Then $\xi_1 = \beta\xi$ for some $\beta > 1$. We calculate

$$\begin{aligned} |h(\xi_1, \xi_2, \xi)| &= |1 - \beta|\beta|^\alpha - (1-\beta)|1-\beta|^\alpha||\xi|^{1+\alpha} \\ &= f(\beta - 1)|\xi|^{1+\alpha} \\ &\geq c \begin{cases} |\beta - 1||\xi|^{1+\alpha} & , 1 < \beta \leq 2 \\ |\beta - 1|^\alpha|\xi|^{1+\alpha} & , \beta > 2 \end{cases} \\ &= c \begin{cases} |\xi_2||\xi|^\alpha & , 1 < \beta \leq 2 \\ |\xi_2|^\alpha|\xi| & , \beta > 2 \end{cases} \end{aligned}$$

It is $|\xi_1| = |\xi_{\max}|$. If $1 < \beta \leq 2$ we have $2|\xi| \geq |\xi_1|$, which implies (4.4). In the case $\beta > 2$ we use $|\xi_2| = (1 - 1/\beta)|\xi_1| \geq 1/2|\xi_1|$ to conclude (4.4). □

Proof of Theorem 4.2.5. Let us fix notation. We define $\sigma = |\tau| + |\xi|^{1+\alpha}$ and $\sigma_i = |\tau_i| + |\xi_i|^{1+\alpha}$ as well as $\lambda = \tau - \xi|\xi|^\alpha$ and $\lambda_i = \tau_i - \xi_i|\xi_i|^\alpha$. Moreover, we set

$$f_i(\tau_i, \xi_i) = |\xi_i|^{-\omega} \langle \xi_i \rangle^{s-\alpha\omega} \langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega \mathcal{F} u_i(\tau_i, \xi_i)$$

and

$$\mathcal{F} v_i(\tau_i, \xi_i) := f_i(\tau_i, \xi_i) \langle \lambda_i \rangle^{-b}.$$

For brevity we write

$$\int_* g(\tau_1, \xi_1) h(\tau_2, \xi_2) := \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} g(\tau_1, \xi_1) h(\tau_2, \xi_2) \, d\tau_1 d\xi_1$$

which is nothing else but the convolution $(g * h)(\tau, \xi)$. We first consider the case $s = s_0 = -\frac{3}{4}(\alpha - 1) + \varepsilon$ for small $\varepsilon > 0$. Our goal is to bound

$$\|\partial_x(u_1 u_2)\|_{X_{s,\omega,b'}} = \left\| |\xi|^{1-\omega} \langle \xi \rangle^{s-\alpha\omega} \langle \lambda \rangle^{b'} \langle \sigma \rangle^\omega \int_* \prod_{i=1}^2 \frac{|\xi_i|^\omega \langle \xi_i \rangle^{\alpha\omega - s} f_i(\tau_i, \xi_i)}{\langle \lambda_i \rangle^b \langle \sigma_i \rangle^\omega} \right\|_{L^2_{\tau,\xi}}$$

by the product of the L^2 norms of the f_i, where we may assume that $0 \le f_i \in \mathcal{S}(\mathbb{R}^2)$.

Due to the symmetry in ξ_1, ξ_2 it suffices to consider the subregion of the domain of integration where $|\xi_1| \le |\xi_2|$. By the convolution constraint $\xi = \xi_1 + \xi_2$ we then have $|\xi| \le 2|\xi_2|$. This region is split up into

(i) Region D_1: $4|\xi_1| \le |\xi_2|$. There, $|\xi_1| \le \frac{1}{4}|\xi_2| \le \frac{1}{3}|\xi| \le \frac{2}{3}|\xi_2|$.

(ii) Region D_2: $|\xi_1| \le |\xi_2| \le 4|\xi_1|$. There, $|\xi| \le 2|\xi_2|$, $|\xi| \le 5|\xi_1|$.

Let A, A_1, A_2 be subregions of the domain of integration, such that in A we have $\langle \lambda \rangle \ge \langle \lambda_1 \rangle, \langle \lambda_2 \rangle$, in A_1 we have $\langle \lambda_1 \rangle \ge \langle \lambda \rangle, \langle \lambda_2 \rangle$ and in A_2 the inequalities $\langle \lambda_2 \rangle \ge \langle \lambda \rangle, \langle \lambda_1 \rangle$ hold.

We first consider the region D_1 and subdivide it into two parts $D_1 = D_{11} \cup D_{12}$, where in D_{11} we have $|\xi_1| \le 2$ and in D_{12} we have $|\xi_1| \ge 2$. In D_1 we see by Lemma 4.2.6

$$|\lambda - \lambda_1 - \lambda_2| = |h(\xi_1, \xi_2, \xi)| \ge c|\xi_1||\xi|^\alpha$$

because $|\xi_1| \le |\xi|, |\xi_2|$.

Now we start the analysis in the subregion D_{11}. We exploit

$$|\xi|^{1-\frac{\alpha}{2}} = |\xi|^{\alpha\omega} \le c|\xi_1|^{-\omega}(\chi_A \langle \lambda \rangle^\omega + \chi_{A_1} \langle \lambda_1 \rangle^\omega + \chi_{A_2} \langle \lambda_2 \rangle^\omega).$$

Therefore in D_{11} the bilinear estimate follows from

$$\sum_{k=0}^{2} \|J_{11,k}\|_{L^2} \leq c \prod_{i=1}^{2} \|f_i\|_{L^2} \tag{4.5}$$

where

$$J_{11,0} = \int_* \chi_{D_{11}\cap A} |\xi|^{\frac{\alpha}{2}-\omega} \langle\xi\rangle^{s-\alpha\omega} \langle\lambda\rangle^{b'+\omega} \langle\sigma\rangle^{\omega} |\xi_2|^{\omega} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)\langle\xi_i\rangle^{\alpha\omega-s}}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}}$$

and for $k = 1, 2$

$$J_{11,k} = \int_* \chi_{D_{11}\cap A_k} |\xi|^{\frac{\alpha}{2}-\omega} \langle\xi\rangle^{s-\alpha\omega} \langle\lambda\rangle^{b'} \langle\sigma\rangle^{\omega} |\xi_2|^{\omega} \langle\lambda_k\rangle^{\omega} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)\langle\xi_i\rangle^{\alpha\omega-s}}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}}$$

We observe that in D_{11}

$$\langle\xi_2\rangle^{\alpha\omega-s}\langle\xi\rangle^{s-\alpha\omega} \leq c \text{ and } \langle\xi_1\rangle^{\alpha\omega-s} \leq c \tag{4.6}$$

In addition, we use $b' + \omega \leq 0$ and $|\xi_2|^{\omega} \leq c|\xi|^{\omega}$ to show that

$$\|J_{11,0}\|_{L^2} \leq c \left\| \int_* \chi_{D_{11}\cap A} |\xi|^{\frac{\alpha}{2}} \langle\sigma\rangle^{\omega} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}} \right\|_{L^2}$$

Because of the convolution constraint $(\tau,\xi) = (\tau_1,\xi_1) + (\tau_2,\xi_2)$ we also have

$$\frac{\langle\sigma\rangle}{\langle\sigma_1\rangle\langle\sigma_2\rangle} \leq c \frac{1}{\min_{i=1,2}\langle\sigma_i\rangle} \leq c \tag{4.7}$$

which implies

$$\|J_{11,0}\|_{L^2} \leq c \left\| \int_* \chi_{D_{11}\cap A} |\xi|^{\frac{\alpha}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2}$$

In region D_{11}

$$|\xi|^{\frac{\alpha}{2}} \leq c \left| |\xi_2|^{\alpha} - |\xi_1|^{\alpha} \right|^{\frac{1}{2}}$$

such that with (2.36)

$$\begin{aligned}\|J_{11,0}\|_{L^2} &\leq c \left\| \int_* \left| |\xi_2|^{\alpha} - |\xi_1|^{\alpha} \right|^{\frac{1}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2} \\ &\leq c \left\| I_*^{\frac{\alpha}{2}}(v_1,v_2) \right\|_{L^2} \leq c \prod_{i=1}^{2} \|v_i\|_{X_{0,0,b}} = c \prod_{i=1}^{2} \|f_i\|_{L^2}\end{aligned}$$

since $b > 1/2$. For $J_{11,1}$ we use (4.6) and (4.7) again and get

$$\|J_{11,1}\|_{L^2} \le c \left\| \int_* \frac{\chi_{D_{11}\cap A_1}|\xi|^{\frac{\alpha}{2}}}{\min_{i=1,2}\langle\sigma_i\rangle^\omega} \langle\lambda\rangle^{b'} f_1(\tau_1,\xi_1)\langle\lambda_1\rangle^{\omega-b} f_2(\tau_2,\xi_2)\langle\lambda_2\rangle^{-b} \right\|_{L^2}$$

We may assume that $|\lambda_1| \ge 2|\lambda|$ because otherwise the same argument as for $J_{11,0}$ applies. If $\langle\sigma_1\rangle \le \langle\sigma_2\rangle$ we have $\langle\lambda_1\rangle^\omega \le \min_{i=1,2}\langle\sigma_i\rangle^\omega$. If we suppose that $\langle\sigma_2\rangle \le \langle\sigma_1\rangle$ we see

$$|\lambda_1| = |\tau_1 - \xi_1|\xi_1|^\alpha| = |\tau - \tau_2 - \xi|\xi|^\alpha + \xi|\xi|^\alpha - \xi_1|\xi_1|^\alpha| \le |\lambda| + 16|\sigma_2|$$

since we are in region D_{11}. This implies $\langle\lambda_1\rangle \le c\langle\sigma_2\rangle$ and we also have

$$\langle\lambda_1\rangle^\omega \le c \min_{i=1,2}\langle\sigma_i\rangle^\omega.$$

Therefore,

$$\|J_{11,1}\|_{L^2} \le c \left\| \int_* \chi_{D_{11}\cap A_1}|\xi|^{\frac{\alpha}{2}} \langle\lambda\rangle^{b'} f_1(\tau_1,\xi_1)\langle\lambda_1\rangle^{-b} f_2(\tau_2,\xi_2)\langle\lambda_2\rangle^{-b} \right\|_{L^2}$$

In D_{11} we have $|\xi|^{\frac{\alpha}{2}} \le c||\xi_2|^\alpha - |\xi_1|^\alpha|^{\frac{1}{2}}$ and by assumption $b' \le 0$ such that we may proceed as above with $J_{11,0}$ and use the estimate (2.36) to conclude

$$\|J_{11,1}\|_{L^2} \le c \left\| \int_* ||\xi_2|^\alpha - |\xi_1|^\alpha|^{\frac{1}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2} \le c \prod_{i=1}^{2} \|f_i\|_{L^2}$$

For $J_{11,2}$ we have by (4.6) and (4.7)

$$\|J_{11,2}\|_{L^2} \le c \left\| \int_* \chi_{D_{11}\cap A_2}|\xi|^{\frac{\alpha}{2}} \langle\lambda\rangle^{b'} f_1(\tau_1,\xi_1)\langle\lambda_1\rangle^{-b} f_2(\tau_2,\xi_2)\langle\lambda_2\rangle^{\omega-b} \right\|_{L^2}$$

In $D_{11} \cap A_2$ we have

$$|\xi|^{\frac{\alpha}{2}} \le c||\xi|^\alpha - |\xi_1|^\alpha|^{\frac{1}{2}} \text{ and } \langle\lambda_2\rangle^{\omega-b} \le \langle\lambda\rangle^{\omega-b}$$

such that, because of $b' + \omega \le 0$,

$$\begin{aligned} \|J_{11,2}\|_{L^2} &\le c \left\| K_*^{\frac{\alpha}{2}}(\overline{v}_1, \mathcal{F}^{-1}f_2) \right\|_{X_{0,0,-b}} \\ &\le c\|v_1\|_{X_{0,0,b}} \|\mathcal{F}^{-1}f_2\|_{L^2} = c\prod_{i=1}^{2}\|f_i\|_{L^2} \end{aligned}$$

for $b > 1/2$ by the estimate (2.37).

Let us now consider the region D_{12}. We define the contributions

$$J_{12,0} = \int_* \chi_{D_{12}\cap A} |\xi|^{1-\omega} \langle\xi\rangle^{s-\alpha\omega} \langle\lambda\rangle^{b'} \langle\sigma\rangle^{\omega} \prod_{i=1}^{2} \frac{|\xi_i|^{\omega} \langle\xi_i\rangle^{\alpha\omega - s} f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}}$$

and, for $k = 1, 2$,

$$J_{12,k} = \int_* \chi_{D_{12}\cap A_k} |\xi|^{1-\omega} \langle\xi\rangle^{s-\alpha\omega} \langle\lambda\rangle^{b'} \langle\sigma\rangle^{\omega} \prod_{i=1}^{2} \frac{|\xi_i|^{\omega} \langle\xi_i\rangle^{\alpha\omega - s} f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}}$$

In the subregion $D_{12}\cap A$ we use

$$|\xi|^{-\alpha b'} \langle\xi_1\rangle^{-b'} \le c\langle\lambda\rangle^{-b'}$$

and $\|J_{12,0}\|_{L^2}$ is bounded by

$$\left\| \int_* \chi_{D_{12}\cap A} |\xi|^{1-\omega+\alpha b'} \langle\xi\rangle^{s-\alpha\omega} \langle\sigma\rangle^{\omega} \langle\xi_1\rangle^{b'+\alpha\omega-s} \langle\xi_2\rangle^{\alpha\omega - s} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)|\xi_i|^{\omega}}{\langle\lambda_i\rangle^b \langle\sigma_i\rangle^{\omega}} \right\|_{L^2}$$

Using $\langle\xi_2\rangle^{\alpha\omega-s}\langle\xi\rangle^{s-\alpha\omega} \le c$ and (4.7) this is bounded by

$$\left\| \int_* \chi_{D_{12}\cap A} |\xi|^{1+\alpha b'} \frac{\langle\xi_1\rangle^{b'+\alpha\omega-s+\omega}}{\min_{i=1,2}\langle\sigma_i\rangle^{\omega}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2}$$

Now, for $b' + \omega \le 0$ we estimate

$$|\xi|^{1+\alpha b'} \le c|\xi|^{\frac{\alpha}{2}} \langle\xi_1\rangle^{1-\frac{\alpha}{2}+\alpha b'}$$

since $1 - \frac{\alpha}{2} + \alpha b' \le 0$. Moreover,

$$1 - \frac{\alpha}{2} + \alpha b' + b' + \alpha\omega - s + \omega - (1+\alpha)\omega = 1 - \frac{\alpha}{2} + \alpha b' + b' - s$$

which is negative for

$$s \ge \alpha(-\frac{1}{2} + b') + 1 + b'$$

Therefore, choosing $b' \le \min\{-\omega, -\frac{1}{4}\}$ we continue for $s \ge -\frac{3}{4}(\alpha - 1)$ with

$$\left\| \int_* \chi_{D_{12}\cap A} |\xi|^{\frac{\alpha}{2}} \prod_{i=1}^{2} \frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^b} \right\|_{L^2} \le c \left\| I_*^{\frac{\alpha}{2}}(v_1, v_2) \right\|_{L^2} \le c \prod_{i=1}^{2} \|f_i\|_{L^2}$$

Next, we study the contribution of $J_{12,1}$. We may assume that $\langle\lambda_1\rangle \geq 2\langle\lambda\rangle$ because otherwise we use the same argument as in $D_{12} \cap A$. In $D_{12} \cap A_1$ we exploit

$$|\xi|\langle\xi_1\rangle^{\frac{1}{\alpha}} \leq c\langle\lambda_1\rangle^{\frac{1}{\alpha}}$$

We observe that

$$|\lambda_1| = |\tau_1 - \xi_1|\xi_1|^\alpha| \leq |\lambda| + c\langle\sigma_2\rangle \Rightarrow \langle\lambda_1\rangle \leq c\langle\sigma_2\rangle$$

and therefore

$$\langle\lambda_1\rangle^\omega \leq c \min_{i=1,2}\langle\sigma_i\rangle^\omega$$

This shows

$$\|J_{12,1}\|_{L^2} \leq c\left\|\int_* \chi_{D_{12}\cap A_1}\langle\lambda\rangle^{b'}\langle\xi_1\rangle^{-\frac{1}{\alpha}+\omega+\alpha\omega-s}\langle\lambda_1\rangle^{\frac{1}{2}-b}\langle\lambda_2\rangle^{-b}\prod_{i=1}^2 f_i(\tau_i,\xi_i)\right\|_{L^2}$$

We choose $b > \frac{1}{2}$ and in D_{12} we have $|\xi_1| \leq |\xi_2|$. Since we only consider $s \leq \frac{1}{2} - \frac{\alpha}{2}$ (which means $\varepsilon \leq \frac{\alpha-1}{4}$) we have

$$\|J_{12,1}\|_{L^2} \leq c\left\|\int_* \langle\lambda\rangle^{b'}\langle\xi_2\rangle^{\frac{1}{2}-\frac{\alpha}{2}-s}\langle\lambda_2\rangle^{-b}\prod_{i=1}^2 f_i(\tau_i,\xi_i)\right\|_{L^2}$$

With $b' \leq -\frac{1}{4}$ and Sobolev in time we see

$$\begin{aligned}\|J_{12,1}\|_{L^2} &\leq c\left\|\mathcal{F}^{-1}f_1 J^{\frac{1}{2}-\frac{\alpha}{2}-s}v_2\right\|_{L_t^{4/3}L_x^2} \\ &\leq c\|f_1\|_{L_t^2L_x^2}\|J^{\frac{1}{2}-\frac{\alpha}{2}-s}v_2\|_{L_t^4L_x^\infty}\end{aligned}$$

Finally, by (2.31)

$$\|J^{\frac{1}{2}-\frac{\alpha}{2}-s}v_2\|_{L_t^4L_x^\infty} \leq c\|v_2\|_{X_{0,0,b}} = \|f_2\|_{L^2}$$

because $\frac{1}{2} - \frac{\alpha}{2} - s \leq \frac{\alpha-1}{4}$, which is equivalent to $s \geq -\frac{3}{4}(\alpha-1)$. Now, we turn to the contribution of $D_{12} \cap A_2$ where we use

$$|\xi|^{-\alpha b'}\langle\xi_1\rangle^{-b'} \leq c\langle\lambda_2\rangle^{-b'}$$

and it follows that $\|J_{12,2}\|_{L^2}$ is bounded by

$$\left\|\int_* \chi_{D_{12}\cap A_2}|\xi|^{1+\alpha b'}\frac{\langle\xi_1\rangle^{b'+\omega+\alpha\omega-s}}{\min_{i=1,2}\langle\sigma_i\rangle^\omega}\langle\lambda\rangle^{b'}\langle\lambda_2\rangle^{-b'-b}\langle\lambda_1\rangle^{-b}\prod_{i=1}^2 f_i(\tau_i,\xi_i)\right\|_{L^2}$$

We have

$$\langle\lambda\rangle^{b'}\langle\lambda_2\rangle^{-b'-b} \leq \langle\lambda\rangle^{-b}$$

and

$$\min_{i=1,2}\langle\sigma_i\rangle^{\omega} \geq \langle\xi_1\rangle^{(1+\alpha)\omega}$$

and if $b' \leq -\omega$ we have $1+\alpha b' - \frac{\alpha}{2} \leq 0$ and therefore

$$|\xi|^{1+\alpha b'} \leq c|\xi|^{\frac{\alpha}{2}}\langle\xi_1\rangle^{1+\alpha b' - \frac{\alpha}{2}}$$

If $b' \leq -\frac{1}{4}$ and $s \geq -\frac{3}{4}(\alpha-1)$ we estimate $b' - s + 1 + \alpha b' - \frac{\alpha}{2} \leq 0$ and

$$|\xi|^{\frac{\alpha}{2}} \leq c\big||\xi|^{\alpha} - |\xi_1|^{\alpha}\big|^{\frac{1}{2}}$$

and by the dual bilinear smoothing estimate (2.37)

$$\begin{aligned}
\|J_{12,2}\|_{L^2} &\leq c\left\|\int_* \big||\xi|^{\alpha} - |\xi_1|^{\alpha}\big|^{\frac{1}{2}}\langle\lambda\rangle^{-b}\langle\lambda_1\rangle^{-b}\prod_{i=1}^{2} f_i(\tau_i,\xi_i)\right\|_{L^2} \\
&\leq c\prod_{i=1}^{2}\|f_i\|_{L^2}
\end{aligned}$$

This completes the discussion of the subregion D_1.

Let us now consider the domain D_2, where $|\xi_1| \leq |\xi_2| \leq 4|\xi_1|$, $|\xi| \leq 2|\xi_2|$ and $|\xi| \leq 5|\xi_1|$. We subdivide $D_2 = D_{21} \cup D_{22}$, where in

$$D_{21}\ :\ \xi_1\xi_2 > 0 \text{ or } |\xi| \geq \frac{1}{2}|\xi_1| \text{ or } |\xi_2| \leq 1$$

and in

$$D_{22}\ :\ \xi_1\xi_2 < 0 \text{ and } |\xi| \leq \frac{1}{2}|\xi_1| \text{ and } |\xi_2| \geq 1$$

additionally hold. As above, we define for $j = 1, 2$

$$J_{2j,0} = \int_* \chi_{D_{2j}\cap A}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^{\omega}\prod_{i=1}^{2}\frac{|\xi_i|^{\omega}\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}\langle\sigma_i\rangle^{\omega}}$$

and for $k = 1, 2$

$$J_{2j,k} = \int_* \chi_{D_{2j}\cap A_k}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'}\langle\sigma\rangle^{\omega}\prod_{i=1}^{2}\frac{|\xi_i|^{\omega}\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}\langle\sigma_i\rangle^{\omega}}$$

We start with the discussion of D_{21}, where all frequencies are of comparable size or smaller then a constant, which shows that

$$\frac{|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}|\xi_1|^{\omega}|\xi_2|^{\omega}\langle\sigma\rangle^{\omega}}{\langle\xi_1\rangle^{s-\alpha\omega}\langle\xi_2\rangle^{s-\alpha\omega}\langle\sigma_1\rangle^{\omega}\langle\sigma_2\rangle^{\omega}} \leq c\langle\xi\rangle^{1-s}$$

Therefore,

$$\|J_{21,0}\|_{L^2} \leq c\left\|\int_* \chi_{D_{21}\cap A}\langle\xi\rangle^{1-s}\langle\lambda\rangle^{b'}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2}$$

In A we have

$$\langle\xi\rangle^{-b'(1+\alpha)} \leq c\langle\lambda\rangle^{-b'}$$

and we use the Strichartz estimate (2.31) to conclude

$$\begin{aligned}
\|J_{21,0}\|_{L^2} &\leq c\left\|\int_* \langle\xi\rangle^{1-s+b'(1+\alpha)-\frac{\alpha-1}{4}}\langle\xi_1\rangle^{\frac{\alpha-1}{4}}\mathcal{F}v_1(\tau_1,\xi_1)\mathcal{F}v_2(\tau_2,\xi_2)\right\|_{L^2}\\
&\leq c\|J^{\frac{\alpha-1}{4}}v_1\|_{L_t^4L_x^\infty}\|v_2\|_{L_t^4L_x^2} \leq c\prod_{i=1}^{2}\|f_i\|_{L^2}
\end{aligned}$$

since $1-s+b'(1+\alpha)-\frac{\alpha-1}{4} \leq 0$ which is equivalent to $\frac{5}{4}+b'-\frac{\alpha}{4}+\alpha b' \leq s$. This is fulfilled for $b' \leq -\frac{1}{2}+\frac{\varepsilon}{3}$. In A_1 we have

$$\langle\xi\rangle^{b(1+\alpha)} \leq c\langle\lambda_1\rangle^{b}$$

and we use Sobolev in time and the Strichartz estimate (2.31) to conclude for $b' \leq -\frac{1}{4}$

$$\begin{aligned}
\|J_{21,1}\|_{L^2} &\leq c\left\|\int_* \langle\xi\rangle^{1-s-b(1+\alpha)-\frac{\alpha-1}{4}}\langle\lambda\rangle^{b'}f_1(\tau_1,\xi_1)\langle\xi_2\rangle^{\frac{\alpha-1}{4}}\mathcal{F}v_2(\tau_2,\xi_2)\right\|_{L^2}\\
&\leq c\|\mathcal{F}^{-1}f_1J^{\frac{\alpha-1}{4}}v_2\|_{L_t^{4/3}L_x^2} \leq c\|f_1\|_{L_{tx}^2}\|J^{\frac{\alpha-1}{4}}v_2\|_{L_t^4L_x^\infty}\\
&\leq c\prod_{i=1}^{2}\|f_i\|_{L^2}
\end{aligned}$$

The same argument applies to $J_{21,2}$ by exchanging the roles of f_1, f_2.

Finally, we turn to the contributions from the region D_{22}. Here, we have $\xi_1\xi_2 < 0$. Therefore, we may write $\xi_1 = \beta\xi_2$ for $\beta \in [-1, -\frac{1}{4}]$. By the mean value theorem this shows

$$||\xi_1|^{\alpha}-|\xi_2|^{\alpha}|^{\frac{1}{2}} = ||\beta|^{\alpha}-1|^{\frac{1}{2}}|\xi_2|^{\frac{\alpha}{2}} \geq \frac{1}{2}||\beta|-1|^{\frac{1}{2}}|\xi_2|^{\frac{\alpha}{2}} = \frac{1}{2}|\xi|^{\frac{1}{2}}|\xi_2|^{\frac{\alpha-1}{2}} \quad (4.8)$$

Let us start with the subregion A. We have

$$\langle\sigma\rangle^{\omega} \leq c\langle\lambda\rangle^{\omega} + c\chi_{|\xi|\geq 1}\langle\xi\rangle^{\omega+\alpha\omega}$$

which shows

$$\|J_{22,0}\|_{L^2} \leq c\|K_{22,0}\|_{L^2} + c\|L_{22,0}\|_{L^2}$$

where

$$K_{22,0} = \int_* \chi_{D_{22}\cap A}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\lambda\rangle^{b'+\omega}\prod_{i=1}^{2}\frac{|\xi_i|^{\omega}\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}\langle\sigma_i\rangle^{\omega}}$$

and

$$L_{22,0} = \int_* \chi_{D_{22}\cap A}\chi_{|\xi|\geq 1}\langle\xi\rangle^{1+s}\langle\lambda\rangle^{b'}\prod_{i=1}^{2}\frac{|\xi_i|^{\omega}\langle\xi_i\rangle^{\alpha\omega-s}f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}\langle\sigma_i\rangle^{\omega}}$$

Using

$$|\xi|^{-b'-\omega}\langle\xi_2\rangle^{-\alpha b'-\alpha\omega} \leq c\langle\lambda\rangle^{-b'-\omega}$$

and (4.8) we see that $\|K_{22,0}\|_{L^2}$ is bounded by

$$\begin{aligned}
&\left\|\int_* \chi_{D_{22}\cap A}|\xi|^{1+b'}\langle\xi\rangle^{s-\alpha\omega}\langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2}\\
\leq &c\left\|\int_* \langle\xi\rangle^{\frac{1}{2}+b'+s-\alpha\omega}\langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega-\frac{\alpha-1}{2}}\,||\xi_1|^{\alpha}-|\xi_2|^{\alpha}|^{\frac{1}{2}}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2}
\end{aligned}$$

If $b' \leq -\frac{1}{4}$ and $\varepsilon \leq \frac{1}{4}$, then $\frac{1}{2}+b'+s-\alpha\omega \leq 0$. Moreover, for $b' \leq -\frac{1}{2}+\varepsilon$ we have $-2s+\alpha b'+\alpha\omega-\frac{\alpha-1}{2} \leq 0$. By the bilinear smoothing estimate (2.36)

$$\|K_{22,0}\|_{L^2} \leq c\left\|\int_* ||\xi_1|^{\alpha}-|\xi_2|^{\alpha}|^{\frac{1}{2}}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2} \leq c\prod_{i=1}^{2}\|f_i\|_{L^2}$$

For the second term $\|L_{22,0}\|_{L^2}$ we use

$$|\xi|^{-b'}\langle\xi_2\rangle^{-\alpha b'} \leq c\langle\lambda\rangle^{-b'}$$

and find with (4.8)

$$\begin{aligned}
\|L_{22,0}\|_{L^2} &\leq c\left\|\int_* \chi_{D_{22}\cap A}\chi_{|\xi|\geq 1}\langle\xi\rangle^{s+1+b'}\langle\xi_2\rangle^{\alpha b'-2s}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2}\\
&\leq c\left\|\int_* \langle\xi\rangle^{\frac{1}{2}+s+b'}\langle\xi_2\rangle^{\alpha b'-2s-\frac{\alpha-1}{2}}\,||\xi_1|^{\alpha}-|\xi_2|^{\alpha}|^{\frac{1}{2}}\prod_{i=1}^{2}\frac{f_i(\tau_i,\xi_i)}{\langle\lambda_i\rangle^{b}}\right\|_{L^2}
\end{aligned}$$

We only consider $\varepsilon < \frac{3}{4}(\alpha-1)$. Then, for $b' \leq -\frac{1}{2} + \frac{3}{4}(\alpha-1) - \varepsilon$ we observe that $\frac{1}{2} + s + b' \leq 0$. Moreover $\alpha b' - 2s - \frac{\alpha-1}{2} \leq 0$ for $b' \leq -\frac{1}{2} + \varepsilon$. Using the bilinear smoothing estimate (2.36) we arrive at

$$\|J_{22,0}\|_{L^2} \leq c \prod_{i=1}^{2} \|f_i\|_{L^2}$$

Next, we consider the subregion A_1. We have

$$\langle\sigma\rangle^{\omega} \leq c\langle\lambda_1\rangle^{\omega} + c\chi_{|\xi|\geq 1}\langle\xi\rangle^{\omega+\alpha\omega}$$

which shows that

$$\|J_{22,1}\|_{L^2} \leq c\|K_{22,1}\|_{L^2} + c\|L_{22,1}\|_{L^2}$$

where

$$K_{22,1} = \int_* \chi_{D_{22}\cap A_1}|\xi|^{1-\omega}\langle\xi\rangle^{s-\alpha\omega}\langle\xi_2\rangle^{-2s}\langle\lambda\rangle^{b'}\langle\lambda_1\rangle^{-b+\omega}\langle\lambda_2\rangle^{-b}\prod_{i=1}^{2} f_i(\tau_i,\xi_i)$$

$$L_{22,1} = \int_* \chi_{D_{22}\cap A_1}\chi_{|\xi|\geq 1}\langle\xi\rangle^{1+s}\langle\lambda\rangle^{b'}\langle\lambda_1\rangle^{-b}\langle\lambda_2\rangle^{-b}\langle\xi_2\rangle^{-2s}\prod_{i=1}^{2} f_i(\tau_i,\xi_i)$$

As above, by

$$|\xi|^{-b'-\omega}\langle\xi_2\rangle^{-\alpha b'-\alpha\omega} \leq c\langle\lambda_1\rangle^{-b'-\omega}$$

we see that the first term $\|K_{22,1}\|_{L^2}$ is bounded by

$$\left\|\int_* \chi_{D_{22}\cap A}|\xi|^{1+b'}\langle\xi\rangle^{s-\alpha\omega}\langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega}\langle\lambda\rangle^{-b}\langle\lambda_2\rangle^{-b}\prod_{i=1}^{2} f_i(\tau_i,\xi_i)\right\|_{L^2}$$

which in turn is controlled by

$$\left\|\int_* \langle\xi\rangle^{1+b'+s-\alpha\omega}\langle\xi_2\rangle^{-2s+\alpha b'+\alpha\omega-\frac{\alpha}{2}}\,||\xi|^{\alpha}-|\xi_2|^{\alpha}|^{\frac{1}{2}}\,\langle\lambda\rangle^{-b}\langle\lambda_2\rangle^{-b}\prod_{i=1}^{2} f_i(\tau_i,\xi_i)\right\|_{L^2}$$

Here, we used that because of $|\xi| \leq \frac{3}{4}|\xi_2|$ and $|\xi_2| \geq 1$ we have

$$||\xi|^{\alpha}-|\xi_2|^{\alpha}|^{\frac{1}{2}} \geq c\langle\xi_2\rangle^{\frac{\alpha}{2}}$$

By estimating $\langle\xi\rangle^{\frac{1}{2}} \leq \langle\xi_2\rangle^{\frac{1}{2}}$ and with the same restrictions on s, b' as above we may apply the dual bilinear smoothing estimate (2.37) and get

$$\|K_{22,1}\|_{L^2} \leq c \left\| \int_* \left||\xi|^\alpha - |\xi_2|^\alpha\right|^{\frac{1}{2}} \langle\lambda\rangle^{-b}\langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right\|_{L^2} \leq c \prod_{i=1}^{2} \|f_i\|_{L^2}$$

For the second term $\|L_{22,1}\|_{L^2}$ we use

$$|\xi|^{-b'}\langle\xi_2\rangle^{-\alpha b'} \leq c\langle\lambda_1\rangle^{-b'}$$

and find that $\|L_{22,1}\|_{L^2}$ is bounded by

$$c \left\| \int_* \langle\xi\rangle^{1+s+b'}\langle\xi_2\rangle^{\alpha b' - 2s - \frac{\alpha}{2}} \left||\xi|^\alpha - |\xi_2|^\alpha\right|^{\frac{1}{2}} \langle\lambda\rangle^{-b}\langle\lambda_2\rangle^{-b} \prod_{i=1}^{2} f_i(\tau_i, \xi_i) \right\|_{L^2}$$
$$\leq c \prod_{i=1}^{2} \|f_i\|_{L^2}$$

by the dual bilinear smoothing estimate (2.37) with the same restrictions on s, b', b as in the region A since $\langle\xi\rangle^{\frac{1}{2}} \leq \langle\xi_2\rangle^{\frac{1}{2}}$.

Finally, we turn to the region A_2. In D_{22} the frequencies ξ_1 and ξ_2 are of comparable size and due to $|\xi| \leq \frac{1}{2}|\xi_1|$ and $|\xi_1| \geq \frac{1}{4}|\xi_2| \geq \frac{1}{4}$ we have

$$\left||\xi|^\alpha - |\xi_1|^\alpha\right|^{\frac{1}{2}} \geq c\langle\xi_1\rangle^{\frac{\alpha}{2}}$$

Now we use the same argument as in A_1 with the roles of f_1, f_2 exchanged.

This finishes the proof of the bilinear estimate for $s = s_0 = -\frac{3}{4}(\alpha - 1) + \varepsilon$ and $\varepsilon \leq \frac{\alpha-1}{4}$. The restrictions on b' can be summarized to

$$b' \leq \min\{-\frac{1}{4}, -\omega, -\frac{1}{2} + \frac{\varepsilon}{3}, -\frac{1}{2} + \frac{3}{4}(\alpha - 1) - \varepsilon\}$$

For b we assumed $\frac{1}{2} < b < b' + 1$. Now we turn to the case $s > s_0 = -\frac{3}{4}(\alpha - 1) + \varepsilon$. Let $\rho = s - s_0$. Because of

$$\langle\xi\rangle^\rho \leq c\langle\xi_1\rangle^\rho + c\langle\xi_2\rangle^\rho \tag{4.9}$$

we see

$$\begin{aligned} \|\partial_x(u_1u_2)\|_{X_{s,\omega,b'}} &\leq c\|\partial_x(J^\rho u_1u_2)\|_{X_{s_0,\omega,b'}} + \|\partial_x(u_1J^\rho u_2)\|_{X_{s_0,\omega,b'}} \\ &\leq c\|u_1\|_{X_{s,\omega,b}}\|u_2\|_{X_{s_0,\omega,b}} + \|u_1\|_{X_{s_0,\omega,b}}\|u_2\|_{X_{s,\omega,b}} \end{aligned}$$

This proves that for all $s \geq s_0 > -\frac{3}{4}(\alpha - 1)$ we find suitable $b' \in (-\frac{1}{2}, 0)$ and $b \in (\frac{1}{2}, b' + 1)$ such that the bilinear estimate holds true. $\square$

Remark 4.2.7. The additional elliptic weight $\langle|\tau|+|\xi|^{1+\alpha}\rangle$ was used to control interactions of low frequency waves with Fourier transform which is localized far away from the characteristic set

$$P_\alpha := \{(\tau,\xi)\,|\,\tau = \xi|\xi|^\alpha\}$$

with essentially linear waves of high frequency which result in an essentially linear wave of high frequency. Notice that, informally speaking, for $|\xi| \geq 1$ and close to P_α the space $X_{s,\omega,b}$ corresponds to the $X_{s,b}$ space of Bourgain, whereas far away from P_α the space $X_{s,\omega,b}$ corresponds to $X_{s-(\alpha+1)\omega,b+\omega}$.

We used only a minimal portion of such a weight because we focus on a low regularity threshold, but instead of (4.9) one could also perform a similar argument with the elliptic weight in order to increase regularity in t and x simultaneously.

A similar weight was used by I. Bejenaru [Bej04] in the context of certain nonlinear Schrödinger equations involving derivatives.

4.2.2 Proof of well-posedness

This section contains the proof of Theorem 4.2.1 and Corollary 4.2.3. It will be a standard application of the methods which are well known from the literature and repeats some of the arguments applied to the Schrödinger equation in Subsection 3.4.1. Throughout this section let $1 < \alpha < 2$, $s \geq s_0 > -\frac{3}{4}(\alpha-1)$ and $\omega = 1/\alpha - 1/2$. Moreover, we fix b', b according to Theorem 4.2.5. We may restrict ourself to $0 < T \leq 1$, since the same arguments apply on any compact time interval. For $u \in \mathcal{S}(\mathbb{R}^2)$ we define

$$\Phi_T(u)(t) := -\frac{1}{2}\chi_T(t)\int_0^t W_\alpha(t-t')\partial_x(u^2)(t')\,dt'$$

An application of Proposition 2.2.15 and the bilinear estimate (4.2) allows us to extend Φ_T uniquely to

$$\Phi_T : X_{s,\omega,b} \to X_{s,\omega,b}$$

such that

$$\|\Phi_T(u) - \Phi_T(v)\|_{X_{s,\omega,b}} \leq cT^\varepsilon(\|u\|_{X_{s,\omega,b}} + \|v\|_{X_{s,\omega,b}})\|u-v\|_{X_{s,\omega,b}} \qquad (4.10)$$

holds true. We can also define

$$\Phi_T\,|_{[-T,T]} : X^T_{s,\omega,b} \to X^T_{s,\omega,b}$$

since $\Phi_T(u)\,|_{[-T,T]}$ only depends on $u\,|_{[-T,T]}$.

For the following proof we first consider solutions to an operator equation and finally show that these coincide with solutions of the Cauchy problem in the sense of distributions, see Proposition 4.2.10.

Definition 4.2.8. We say $u \in X^T_{s,\omega,b} \subset C([-T,T], H^{(s,\omega)})$ is a solution of the operator equation associated to (4.1) on $[-T,T]$, if

$$u(t) = W_\alpha(t)u_0 + \Phi_T(u)(t), \quad \text{for } t \in [-T,T]. \tag{4.11}$$

We subdivide the proof into several parts.

Proof of Theorem 4.2.1: local existence and analytic dependence. We define for $0 < T \leq 1$

$$\begin{aligned}\Lambda_T : H^{(s,\omega)} \times X_{s,\omega,b} &\to X_{s,\omega,b}\\ \Lambda_T(u_0,u) &:= \chi W_\alpha u_0 + \Phi_T(u)\end{aligned}$$

Obviously, Λ_T is an analytic map, since it is a composition of bounded linear and bilinear maps. Let $u_0 \in H^{(s,\omega)}$ with $\|u_0\|_{H^{(s,\omega)}} \leq r$ and $u \in X_{s,\omega,b}$ with $\|u\|_{X_{s,\omega,b}} \leq R$. Then, by (2.28), and the estimate (4.10)

$$\|\Lambda_T(u_0,u)\|_{X_{s,\omega,b}} \leq cr + RcT^\varepsilon \|u\|_{X_{s,\omega,b}} < R$$

for $R = 2cr$ and $T^\varepsilon = (8c^2r)^{-1}$. With these choices for R and T we restrict Λ_T to closed balls $\overline{B}_r \times \overline{B}_R \subset H^{(s,\omega)} \times X_{s,\omega,b}$ and the bilinear estimate (4.10) shows that

$$\Lambda_T(u_0,\cdot) : \overline{B}_R \to \overline{B}_R$$

is a strict contraction, uniformly in $u_0 \in \overline{B}_r$. Therefore we find

$$S_r : H^{(s,w)} \supset B_r \to B_R \subset X_{s,\omega,b}$$

with

$$\Lambda_T(u_0,u) = u \in B_R \iff u = S_r(u_0)$$

for all $u_0 \in B_r$ and an application of the Implicit Function Theorem 1.4.6 to $\mathrm{Id} - \Lambda_T$ yields the analyticity of S_r and also of $S_{r,T} := S_r\big|_{[-T,T]} : H^{(s,w)} \to X^T_{s,\omega,b}$. Moreover, the functions $S_{r,T}(u_0) \in X^T_{s,\omega,b}$ are solutions of (4.11). $\square$

Proof of Theorem 4.2.1: persistence and uniqueness. The persistence property follows from the embedding $X_{s,\omega,b} \subset C(\mathbb{R}, H^{(s,\omega)})$. Assume that $u, v \in X^T_{s_0,\omega,b}$ are two solutions of (4.11) with extensions $\tilde{u}, \tilde{v} \in X_{s_0,\omega,b}$, such that

$$T' := \sup\{t \in [0,T] \mid u(t) = v(t)\} < T.$$

Define $u^*(t) := \tilde{u}(t+T')$, $v^*(t) := \tilde{v}(t+T')$ for $-T' \le t \le T-T'$. Because both u and v are solutions of (4.11), we see by approximation with smooth functions

$$u^*(t) - v^*(t) = \Phi_T(u^*)(t) - \Phi_T(v^*)(t) \tag{4.12}$$

for $-T' \le t \le T-T'$. Therefore, for small $\delta > 0$

$$\|\chi_\delta(u^* - v^*)\|_{X_{s_0,\omega,b}} \le c\delta^\varepsilon \|\chi_\delta(u^* - v^*)\|_{X_{s_0,\omega,b}} (\|u^*\|_{X_{s_0,\omega,b}} + \|v^*\|_{X_{s_0,\omega,b}})$$

By choosing δ small enough we conclude $u^*(t) = v^*(t)$ for $|t| \le \delta$ which implies $u(t+T') = v(t+T')$ for $|t| \le \delta$. This contradicts the definition of T'. If u, v did not coincide on $[-T, 0]$, we would find a similar contradiction. □

Lemma 4.2.9. *Let $s \ge 0$. There exists $C > 0$, such that for all smooth, real valued solutions u of* (4.1)*, we have*

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^{(0,\omega)}} \le C\|u(0)\|_{H^{(0,\omega)}} + CT\|u(0)\|^2_{H^{(0,\omega)}} \tag{4.13}$$

Proof. We easily verify the conservation law

$$\|u(t)\|^2_{L^2} = \|u(0)\|^2_{L^2}, \quad t \in (-T,T)$$

Therefore it suffices to prove an a priori estimate for the low frequency part in $\dot{H}^{-\omega}$. We define

$$\mathcal{F}_x v(t)(\xi) = \chi(\xi)|\xi|^{-\omega}\mathcal{F}_x u(t)(\xi)$$

The function v solves the equation

$$\begin{aligned} v_t - |D|^\alpha v_x &= f \ \text{ in } (-T,T)\times\mathbb{R} \\ v(0) &= v_0 \end{aligned}$$

where v_0, f are defined via

$$\mathcal{F}_x v_0(\xi) = \chi(\xi)|\xi|^{-\omega}\mathcal{F}_x u(0)(\xi)$$

and

$$\mathcal{F}_x f(t)(\xi) = -\frac{i}{2}\chi(\xi)\xi|\xi|^{-\omega}\mathcal{F}_x u^2(t)(\xi)$$

respectively. For fixed t we estimate

$$\begin{aligned} \|f(t)\|_{L^2_x} &\le c\|\chi(\xi)\mathcal{F}_x u^2(t)(\xi)\|_{L^2_\xi} \le c\|\mathcal{F}_x u^2(t)\|_{L^\infty_\xi} \\ &\le c\|u^2(t)\|_{L^1_x} \le c\|u(t)\|^2_{L^2_x} \end{aligned}$$

This shows

$$\begin{aligned}\|v\|_{L^\infty_T L^2_x} &\le c\|v_0\|_{L^2_x} + c\|f\|_{L^1_T L^2_x} \le c\|u(0)\|_{H^{(0,\omega)}} + cT\|u\|^2_{L^\infty_T L^2_x}\\ &\le c\|u(0)\|_{H^{(0,\omega)}} + cT\|u(0)\|^2_{H^{(0,\omega)}}\end{aligned}$$

as desired. □

Proof of Theorem 4.2.1: Time of existence. We fix $s \ge s_0$ and a ball

$$B_{r,r_s} = \{v_0 \in H^{(s,\omega)} \mid \|v_0\|_{H^{(s_0,\omega)}} \le r \ \text{ and } \ \|v_0\|_{H^{(s,\omega)}} \le r_s\}$$

and define T_s as the supremum of all $T \in [0,1]$ such that the following statement is true: There exists an analytic map $F : B_{r,r_s} \to X_{s,\omega,b}$ such that

$$\Lambda_T(v_0, F(v_0)) = F(v_0)$$

and if $u \in X^T_{s_0,\omega,b}$ is a solution of (4.11), then

$$u\big|_{[-T,T]} = F(v_0)\big|_{[-T,T]}$$

Part 1 and 2 of the proof show that $T_s > 0$ and let $v = F(v_0) \in X_{s,\omega,b}$. If $T_s^\varepsilon \le (8c_{s_0}^2 r)^{-1} < 1$ we see from the proof of part 1 that $\|v\|_{X_{s_0,\omega,b}} \le 2c_{s_0} r$. An application of our bilinear estimate (4.2) together with (2.28), (2.30) gives

$$\|v\|_{X_{s,\omega,b}} \le c_s r_s + c_s T_s^\varepsilon \|v\|_{X_{s_0,\omega,b}} \|v\|_{X_{s,\omega,b}}$$

Therefore,

$$\|v\|_{X_{s,\omega,b}} \le c_s r_s + 2c_s c_{s_0} r T_s^\varepsilon \|v\|_{X_{s,\omega,b}}$$

and, if additionally $T_s^\varepsilon \le (4c_{s_0} c_s r)^{-1}$, we conclude

$$\sup_{|t|\le T_s} \|v(t)\|_{H^{(s,\omega)}} \le c\|v\|_{X_{s,\omega,b}} < C_s r_s \tag{4.14}$$

If these assumptions about T_s were true, we could apply part 1 and 2 of the proof. We find a $\delta > 0$ and an analytic map $G : H^{(s,\omega)} \supset B_{C_s r_s} \to X_{s,\omega,b}$ such that

$$\Lambda_{2\delta}(w_0, G(w_0)) = G(w_0)$$

and if $u \in X^{2\delta}_{s_0,\omega,b}$ is a solution of (4.11) with initial datum $w_0 \in B_{C_s r_s}$, then

$$u\big|_{[-2\delta,2\delta]} = G(w_0)\big|_{[-2\delta,2\delta]}$$

Define

$$H : v_0 \mapsto \eta_\delta F(v_0) + \eta_\delta^+ G(F(v_0)(T_s))(\cdot - T_s) + \eta_\delta^- G(F(v_0)(-T_s))(\cdot + T_s)$$

as a map from B_{r,r_s} to $X_{s,\omega,b}$ with smooth cutoff functions $\eta_\delta, \eta_\delta^+, \eta_\delta^-$, such that $\eta_\delta + \eta_\delta^+ + \eta_\delta^- = 1$ on $[-T_s - \delta, T_s + \delta]$ with

$$\operatorname{supp}(\eta_\delta) \subset [-T_s + \delta, T_s - \delta]\ ,\ \operatorname{supp}(\eta_\delta^\pm) \subset [\pm T_s - 2\delta, \pm T_s + 2\delta]$$

It is not hard to verify that H is analytic, since it is a composition of analytic maps, and

$$\Lambda_{T_s+\delta}(v_0, H(v_0)) = H(v_0)$$

and if $u \in X^{T_s+\delta}_{s_0,\omega,b}$ is a solution of (4.11), then part 2 of the proof also gives

$$u\big|_{[-T_s-\delta, T_s+\delta]} = H(v_0)\big|_{[-T_s-\delta, T_s+\delta]}$$

which contradicts the definition of T_s and we conclude that

$$T_s^\varepsilon \geq \min\{(4c_{s_0}c_s r)^{-1}, (8c_{s_0}^2 r)^{-1}\}.$$

This lower bound shows that if $T_s < 1$ we have

$$\lim_{t\uparrow T_s} \|u(t)\|_{H^{(s_0,\omega)}} = \infty \tag{4.15}$$

because otherwise we could iterate the argument above. □

Proof of Theorem 4.2.1: Global existence. The same proof as above applies in the closed subspaces of real valued functions in $H^{(s,\omega)}$, $X_{s,\omega,b}$ and $X^T_{s,\omega,b}$. We regard these as Hilbert spaces over the real numbers and the analytic flow maps as real analytic. Using (4.15) and the a priori bound (4.13) this proves $T_s = 1$ for all $s \geq 0$. As already mentioned, the same arguments may be applied to any compact time interval. □

Now, we show that the notion of solutions considered above coincides with the formulation of Theorem 4.2.1

Proposition 4.2.10. *Let $T > 0$, $s > -\frac{3}{4}(\alpha - 1)$ and $u \in X^T_{s,\omega,b}$ for some $b > \frac{1}{2}$ and $\omega = \frac{1}{\alpha} - \frac{1}{2}$. Then, u solves (4.11) if and only if u solves*

$$\int_{\mathbb{R}} \int_{-T}^{T} u\partial_t\varphi - u|D|^\alpha \partial_x \varphi + \frac{1}{2}u^2 \partial_x \varphi dtdx = 0 \tag{4.16}$$

for all $\varphi \in C_0^\infty((-T,T) \times \mathbb{R})$.

Proof. Let us first assume that $u \in X^T_{s,\omega,b}$ is a solution of (4.11). Then, there exists a sequence $u_n \in \mathcal{S}^T(\mathbb{R})$ such that $u_n \to u$ in $X^T_{s,\omega,b}$. Due to (2.32) and $\frac{3}{4}(\alpha-1) \leq \frac{\alpha}{2}$ we also have $u_n \to u$ in $L^\infty_x(\mathbb{R}, L^2_t(-T,T))$ and for $t \in [-T,T]$

$$u_n(t) = W_\alpha(t)u_n(0) - \frac{1}{2}\int_0^t W_\alpha(t-t')\partial_x(u_n(t'))^2 dt' + R(u,u_n)(t)$$

with error term

$$R(u,u_n) = u_n - u + W_\alpha(\cdot)(u(0)-u_n(0)) + \Phi_T(u) - \Phi_T(u_n)$$

such that $\|R(u,u_n)\|_{L^\infty_x(\mathbb{R},L^2_t(-T,T))} \to 0$ for $n \to \infty$ because of (4.10), (2.28) and (2.32). Now, we integrate against $\partial_t\varphi$ for a $\varphi \in C_0^\infty((-T,T)\times\mathbb{R})$ and get

$$\begin{aligned}&\int_{\mathbb{R}}\int_{-T}^T u_n\partial_t\varphi - W_\alpha(t)u_n(0)\partial_t\varphi + \frac{1}{2}\int_0^t W_\alpha(t-t')\partial_x(u_n(t'))^2 dt'\partial_t\varphi\, dtdx\\ &= \int_{\mathbb{R}}\int_{-T}^T R(u,u_n)\partial_t\varphi\, dtdx\end{aligned}$$

The right hand side tends to zero. Integration by parts and the symmetry of $|D|^\alpha$ shows that the left hand side equals

$$\begin{aligned}&\int_{\mathbb{R}}\int_{-T}^T u_n\partial_t\varphi + \frac{1}{2}(u_n)^2\partial_x\varphi\\ &-|D|^\alpha\partial_x\varphi\left(W_\alpha(t)u_n(0) - \frac{1}{2}\int_0^t W_\alpha(t-t')\partial_x(u_n(t'))^2 dt'\right) dtdx\end{aligned}$$

This expression is equal to

$$\int_{\mathbb{R}}\int_{-T}^T u_n\partial_t\varphi + \frac{1}{2}(u_n)^2\partial_x\varphi - |D|^\alpha\partial_x\varphi\,(u_n - R(u,u_n))\, dtdx$$

Now, we notice that $|D|^\alpha\partial_x\varphi \in L^1_x(\mathbb{R}, L^2_t[-T,T])$ because $\langle x\rangle|D|^\alpha\partial_x\varphi \in L^2(\mathbb{R}^2)$ which follows from $\partial_\xi\mathcal{F}_x|D|^\alpha\partial_x\varphi \in L^2(\mathbb{R}^2)$. We also have $u_n \to u$ in $L^\infty_x(\mathbb{R}, L^2_t(-T,T))$ as well as $(u_n)^2 \to u^2$ with respect to $L^\infty_x(\mathbb{R}, L^1_t(-T,T))$ and $R(u,u_n) \to 0$ in $L^\infty_x(\mathbb{R}, L^2_t(-T,T))$, such that the above expression tends to

$$\int_{\mathbb{R}}\int_{-T}^T u\partial_t\varphi - u|D|^\alpha\partial_x\varphi + \frac{1}{2}\partial_x\varphi u^2\, dtdx$$

Now, we start with a solution $u \in X^T_{s,\omega,b}$ in the sense of distributions. By a truncation argument we can show that $\varphi(t,x) = \eta_T(t)W_\alpha(t)\varphi_0(x)$ for

$\eta_T \in C_0^\infty((-T,T))$, $\varphi_0 \in C_0^\infty(\mathbb{R})$ is admissible in (4.16). To verify the operator equation, we choose a sequence $\mathcal{S}^T(\mathbb{R}) \ni u_n \to u$ in $X^T_{s,\omega,b}$ as in the first part and also define $v(t) = W_\alpha(-t)u(t)$ and $v_n(t) = W_\alpha(-t)u_n(t)$. It is

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{-T}^{T} u\partial_t\varphi - u|D|^\alpha\partial_x\varphi \, dtdx \\
&= \lim_{n\to\infty}\int_{\mathbb{R}}\int_{-T}^{T} W_\alpha(t)v_n\partial_t\varphi - W_\alpha(t)v_n|D|^\alpha\partial_x\varphi \, dtdx
\end{aligned}$$

Multiple integration by parts shows that this equals

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}}\int_{-T}^{T} -|D|^\alpha\partial_x W_\alpha(t)v_n\varphi - W_\alpha(t)\partial_t v_n\varphi - W_\alpha(t)v_n|D|^\alpha\partial_x\varphi \, dtdx \\
&= \lim_{n\to\infty}\int_{\mathbb{R}}\int_{-T}^{T} v_n\partial_t(W_\alpha(-t)\varphi) \, dtdx \\
&= \int_{-T}^{T}\langle v(t), \partial_t(W_\alpha(-t)\varphi(t))\rangle_{H^s} dt
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{H^s}$ is the pairing of H^s and H^{-s}. Integration by parts shows

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{-T}^{T} -\frac{1}{2}u^2\partial_x\varphi \, dtdx \\
&= \lim_{n\to\infty}\int_{\mathbb{R}}\int_{-T}^{T} \frac{1}{2}\partial_x(u_n)^2\varphi \, dtdx \\
&= -\lim_{n\to\infty}\int_{\mathbb{R}}\int_{-T}^{T}\int_0^t W_\alpha(-t')\frac{1}{2}\partial_x(u_n)^2 dt'\partial_t(W_\alpha(-t)\varphi) \, dtdx
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{-T}^{T} -\frac{1}{2}u^2\partial_x\varphi \, dtdx \\
&= \int_{-T}^{T}\langle W_\alpha(-t)\Phi_T(u)(t), \partial_t(W_\alpha(-t)\varphi(t))\rangle_{H^s} dt
\end{aligned}$$

Because u is a solution in the sense of distributions

$$\int_{-T}^{T}\langle v(t) - W_\alpha(-t)\Phi_T(u)(t), \partial_t(W_\alpha(-t)\varphi(t))\rangle_{H^s} dt = 0$$

and by the definition of φ we conclude that

$$\int_{-T}^{T} \eta_T'(t)\langle v(t) - W_\alpha(-t)\Phi_T(u)(t), \varphi_0)\rangle_{H^s} dt = 0$$

for all $\eta_T \in C_0^\infty(-T,T)$, $\varphi_0 \in C_0^\infty(\mathbb{R})$, which shows that

$$v(t) - W_\alpha(-t)\Phi_T(u)(t) = v(0)$$

in H^s due to [Hör83], Theorem 3.1.4 because $\Phi_T(u)(0) = 0$ and it follows

$$u(t) = W_\alpha(t) + \Phi_T(u)(t), \quad t \in [-T,T]$$

as desired. □

4.2.3 Sharpness of the low frequency condition

In this section we modify the counterexamples of L. Molinet - J.-C. Saut - N. Tzvetkov [MST01] which imply that the flow map is not C^2 without any low frequency condition in order to prove the sharpness of our choice of ω. Here, we also include the Benjamin-Ono case ($\alpha = 1$).

Proof of Theorem 4.2.4. We just give the modifications of Molinet, Saut and Tzvetkov's argument. For a more detailed calculation in the case $\omega = 0$ we refer to the original work[2] [MST01]. Define a sequence of initial data via

$$\widehat{\phi}_N := N^{(\alpha+\varepsilon)(\frac{1}{2}-\omega)}\chi_1 + N^{\frac{\alpha+\varepsilon}{2}-s}\chi_2$$

where

$$\chi_1(\xi) = \chi_{\frac{1}{2}N^{-\alpha-\varepsilon}\le\xi\le N^{-\alpha-\varepsilon}}, \quad \chi_2(\xi) = \chi_{N\le\xi\le N+N^{-\alpha-\varepsilon}}$$

Notice that $\|\phi_N\|_{H^{(s,\omega)}} \le 2$. As shown in [MST01] we have

$$\left\| \int_0^t W_\alpha(t-t')\partial_x(W_\alpha(t')\phi_N)^2\, dt' \right\|_{H^s} \ge c\|F(t)\|_{H^s}$$

where $\widehat{F(t)}(\xi)$ is given by

$$\frac{N^{\alpha+\varepsilon}\xi e^{it\xi|\xi|^\alpha}}{N^{s+(\alpha+\varepsilon)\omega}} \int (\chi_1(\xi_1)\chi_2(\xi-\xi_1) + \chi_2(\xi_1)\chi_1(\xi-\xi_1)) \frac{e^{itr(\xi_1,\xi)}-1}{r(\xi_1,\xi)}\, d\xi_1$$

with

$$r(\xi_1,\xi) = \xi_1|\xi_1|^\alpha + (\xi-\xi_1)|\xi-\xi_1|^\alpha - \xi|\xi|^\alpha$$

[2] notice that our notation slightly differs from [MST01]

which is $-h(\xi_1, \xi - \xi_1, \xi)$ for the resonance function h from Lemma 4.2.6. In the domain of integration we have $|r(\xi_1, \xi)| \leq cN^{-\alpha-\varepsilon}N^{\alpha} = cN^{-\varepsilon}$. A Taylor expansion shows that

$$\left|\frac{e^{itr(\xi_1,\xi)} - 1}{r(\xi_1, \xi)}\right| \geq |t| - ct^2 N^{-\varepsilon}$$

which implies for $t > 0$

$$\|F(t)\|_{H^s} \geq cNN^s N^{-\alpha-\varepsilon} N^{-\frac{\alpha+\varepsilon}{2}} N^{\alpha+\varepsilon} N^{-s-(\alpha+\varepsilon)\omega}$$

which tends to infinity if $1 - \frac{\alpha+\varepsilon}{2} - (\alpha+\varepsilon)\omega > 0$, which is equivalent to

$$\omega < \frac{1}{\alpha+\varepsilon} - \frac{1}{2} \to \frac{1}{\alpha} - \frac{1}{2} \quad (\varepsilon \to 0).$$

This calculation implies that the bilinear expression, which corresponds to a second derivative at the origin in direction ϕ_N, is unbounded as $N \to \infty$, but on the other hand $\|\phi_N\|_{H^{(s,\omega)}} \leq 2$. This contradicts the C^2 regularity of the flow and the bilinear estimate. □

4.3 The Benjamin-Ono equation in the periodic case

In this section we consider the periodic case and the phase function $\phi : \mathbb{Z} \to \mathbb{R}$, $\phi(\xi) = \xi|\xi|$. Here,

$$W(t) : H^s_0(\mathbb{T}) \to H^s_0(\mathbb{T}), \mathcal{F}W(t)\phi(\xi) = e^{it\xi|\xi|}\mathcal{F}\phi(\xi)$$

We are interested in the Cauchy problem for the Benjamin-Ono equation

$$\begin{aligned} \partial_t u(t) - |D|\partial_x u(t) + \frac{1}{2}\partial_x u^2(t) = 0\,, \quad t \in (-T, T) \\ u(0) = u_0 \in H^s_0(\mathbb{T}) \end{aligned} \tag{4.17}$$

where $H^s_0(\mathbb{T}) := \{u : \mathbb{R} \to \mathbb{R} \,|\, u \in H^s(\mathbb{T})\,, \mathcal{F}u(0) = 0\}$.

We restrict ourself to initial data with zero mean and remark that the mean value is a conserved quantity for real valued solutions of (4.17): Assume that u is a sufficiently smooth solution of (4.17) with $u(0) = u_0$ and $\mathcal{F}u_0(0) = 0$, then

$$\partial_t \mathcal{F}u(\xi) - i\xi|\xi|\mathcal{F}u(\xi) + \frac{i}{2}\xi\mathcal{F}u^2(\xi) = 0$$

hence, $\partial_t \mathcal{F}u(0) = 0$.

Recently, L. Molinet [Mol06] proved a global well-posedness result for the Benjamin-Ono equation in $L^2(\mathbb{T})$. His result includes the analyticity of the flow map $u_0 \mapsto u$ on balls containing initial data with prescribed mean value and his approach is based on a gauge transformation (cp. [Tao04]) and subsequent multi-linear estimates for the transformed problem. In [Mol06] it is also shown that the low regularity threshold $s = 0$ is optimal in the sense that for $s < 0$ the flow is not $C^{1+\alpha}$, $\alpha > 0$ on $H_0^s(\mathbb{T})$.

Moreover, we know that the flow map is not uniformly continuous on $B_R(0) \subset H^s(\mathbb{T})$ for $s > 0$, $R > 0$, if we permit the initial data to have arbitrary mean value, see [Mol05]. This is related to the observations from H. Koch - N. Tzvetkov [KT05b] and L. Molinet - J.-C. Saut - N. Tzvetkov [MST01] on the real line. However, in contrast to our choice of initial data, in [MST01] the interaction of two linear waves, one with frequencies near N and one with frequencies near $1/N$ serve as a counterexample to the corresponding bilinear estimates.

Here, we show that it is in fact impossible to prove reasonable (in the sense of Theorem 4.3.1) bilinear estimates directly, even if one restricts the spaces to functions with mean zero and in spite of the analyticity of the flow map shown in [Mol06]. Let $\mathcal{S}_0^T(\mathbb{T})$ be the set of all $\mathcal{S}^T(\mathbb{T})$ functions with zero mean value.

Theorem 4.3.1. *Let $s \in \mathbb{R}$, $T > 0$. There does not exist a normed space X^T with $\mathcal{S}_0^T(\mathbb{T}) \subset X^T$ and $X^T \hookrightarrow C([0,T], H_0^s(\mathbb{T}))$, such that*

$$\|W(t)u_0\|_{X^T} \leq c\|u_0\|_{H^s(\mathbb{T})}, \quad u_0 \in H_0^s(\mathbb{T}), \tag{4.18}$$

$$\left\| \int_0^t W(t-t')\partial_x(u(t'))^2 \, dt' \right\|_{X^T} \leq c\|u\|_{X^T}^2, \quad u \in \mathcal{S}_0^T(\mathbb{T}), \tag{4.19}$$

are valid[3].

This result is a statement about the failure of techniques and not a statement about the regularity of the flow. It is in some sense antithetic to the case of the real line, where A. Ionescu - C.E. Kenig [IK05] proved a bilinear estimate for initial data fulfilling an additional low frequency condition, compare also Subsection 4.2.1 for the dispersion generalized case.

Remark 4.3.2. We remark without proof that this theorem also holds in the dispersion generalized case for $1 \leq \alpha < 2$ with a similar proof by using the Taylor expansion of the phase function $\phi(\xi) = \xi|\xi|^\alpha$.

[3] It is part of the assumptions that the solution to the linear equation and the Duhamel term are elements of X^T

4.3.1 A Counterexample to bilinear estimates

The result is based on the lemma of this subsection. We construct real valued initial data with zero mean, such that a four-linear low-low-low-high interaction of the corresponding linear waves provides a suitable estimate for the Duhamel term in $H^s(\mathbb{T})$ from below. We will use the following notations. Let

$$B(u,v)(t) = \int_0^t W(t-t')\partial_x(u(t')v(t'))\,dt'$$

and for initial data ψ we define

$$\begin{aligned} &I_1 = I_1(\psi) = W(\cdot)\psi\ ,\ I_2 = I_2(\psi) = B(I_1(\psi), I_1(\psi)) \\ &I_{2,2} = I_{2,2}(\psi) = B(I_2(\psi), I_2(\psi)) \end{aligned}$$

Lemma 4.3.3. *For $s \in \mathbb{R}$, $N \in \mathbb{N}$ define*

$$\psi_N(x) := \sqrt{\frac{2}{\pi}}(N^{-s}\cos(Nx) - \cos(2x) + \cos(x))$$

Then, there exists $c > 0$ such that for all $N \geq 10$ and $t > 0$ we have

$$\|I_{2,2}(\psi_N)(t)\|_{H^s(\mathbb{T})} \geq N\sin^2(t) - c. \tag{4.20}$$

Proof of Lemma 4.3.3. The Fourier transform of ψ_N is

$$\widehat{\psi_N}(k) = \begin{cases} 1 & ,|k| = 1 \\ -1 & ,|k| = 2 \\ N^{-s} & ,|k| = N \end{cases}$$

Since $W(-t)$ is unitary

$$\|I_{2,2}(t)\|_{H^s(\mathbb{T})} = \|W(-t)I_{2,2}(t)\|_{H^s(\mathbb{T})} \geq N^s|\widehat{W(-t)I_{2,2}}(t)(N)|, \tag{4.21}$$

and we just take into account all interactions which contribute to frequency N. Since $\widehat{I_2(t)}(0) = 0$ we have to calculate $\widehat{I_2(t)}(k)$ for $|k| = 1$, $|k| = 2$, $|k-N| = 1$, $|k-N| = 2$.

$$\begin{aligned} \widehat{I_2(t)}(\pm 1) &= \pm i2\int_0^t e^{\pm i(t-t')}(-e^{\pm 4it'})e^{\mp it'}\,dt' \\ &= \mp i2e^{\pm it}\int_0^t e^{\pm 2it'}\,dt' \\ &= -e^{\pm it}(e^{\pm 2it} - 1), \end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\widehat{I_2(t)}(\pm 2) &= \pm 2i \int_0^t e^{\pm 4i(t-t')} e^{\pm it'} e^{\pm it'} dt' \\ &= \pm 2i e^{\pm 4it} \int_0^t e^{\mp 2it'} dt' \\ &= -e^{\pm 4it}(e^{\mp 2it} - 1). \end{aligned} \tag{4.23}$$

Similarly we get

$$\begin{aligned}\widehat{I_2(t)}(N+1) &= 2i(N+1) \int_0^t e^{i(t-t')(N+1)^2} N^{-s} e^{it'N^2} e^{it'} dt' \\ &= 2i(N+1) N^{-s} e^{it(N+1)^2} \int_0^t e^{it'(N^2+1-(N+1)^2)} dt' \\ &= -\frac{(N+1)}{N} N^{-s} e^{it(N+1)^2}(e^{-2Nit} - 1), \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\widehat{I_2(t)}(N+2) &= 2i(N+2) \int_0^t e^{i(t-t')(N+2)^2} N^{-s} e^{it'N^2} (-1) e^{4it'} dt' \\ &= \frac{(N+2)}{2N} N^{-s} e^{it(N+2)^2}(e^{-4Nit} - 1), \end{aligned} \tag{4.25}$$

as well as

$$\begin{aligned}\widehat{I_2(t)}(N-1) &= 2i(N-1) \int_0^t e^{i(t-t')(N-1)^2} N^{-s} e^{it'N^2} e^{-it'} dt' \\ &= 2i(N-1) N^{-s} e^{it(N-1)^2} \int_0^t e^{it'(N^2-1-(N-1)^2)} dt' \\ &= N^{-s} e^{it(N-1)^2}(e^{(2N-2)it} - 1), \end{aligned} \tag{4.26}$$

and

$$\begin{aligned}\widehat{I_2(t)}(N-2) &= 2i(N-2) \int_0^t e^{i(t-t')(N-2)^2} N^{-s} e^{it'N^2} (-1) e^{-4it'} dt' \\ &= -\frac{1}{2} N^{-s} e^{it(N-2)^2}(e^{(4N-8)it} - 1). \end{aligned} \tag{4.27}$$

Next, we calculate the contribution to $N^s \widehat{W(-t) I_{2,2}}(t)(N)$ from $\widehat{I_2(t')}(N-1)$

and $\widehat{I_2(t')}(1)$, where we use the relation $-N^2+1+(N-1)^2+2N-2=0$.

$$\begin{aligned}
J_1 &:= 2\int_0^t e^{-it'N^2} iN(-1)e^{it'}(e^{2it'}-1)e^{it'(N-1)^2}(e^{(2N-2)it'}-1)dt' \\
&= -2iN\Big(\int_0^t e^{2it'}-1dt' - \int_0^t e^{it'(2-2N)}(e^{2it'}-1)dt'\Big) \\
&= -2iN\Big(\frac{e^{2it}-1}{2i}-t\Big)+R_1, \qquad (4.28)
\end{aligned}$$

with the bounded remainder term

$$R_1 := -N\Big(\frac{e^{it(4-2N)}-1}{N-2} - \frac{e^{it(2-2N)}-1}{N-1}\Big).$$

Similarly, because of $-N^2-1+(N+1)^2-2N=0$ the contribution to $N^sW(\widehat{-t)I_{2,2}}(t)(N)$ coming from $\widehat{I_2(t')}(N+1)$ and $\widehat{I_2(t')}(-1)$ is

$$\begin{aligned}
J_2 &:= 2i(N+1)\int_0^t e^{-it'N^2}e^{-it'}(e^{-2it'}-1)e^{it'(N+1)^2}(e^{-2Nit'}-1)dt' \\
&= 2i(N+1)\left(\int_0^t e^{-2it'}-1dt' - \int_0^t e^{2Nit'}(e^{-2it'}-1)dt'\right) \\
&= 2i(N+1)\left(\frac{e^{-2it}-1}{-2i}-t\right)+R_2, \qquad (4.29)
\end{aligned}$$

with the bounded remainder term

$$R_2 := -(N+1)\left(\frac{e^{(2N-2)it}-1}{N-1} - \frac{e^{2Nit}-1}{N}\right).$$

Due to $-N^2+4+(N-2)^2+4N-8=0$ the contribution to $N^sW(\widehat{-t)I_{2,2}}(t)(N)$ coming from $\widehat{I_2(t')}(N-2)$ and $\widehat{I_2(t')}(2)$ amounts to

$$\begin{aligned}
J_3 &:= iN\int_0^t e^{-it'N^2}e^{4it'}(e^{-2it'}-1)e^{it'(N-2)^2}(e^{(4N-8)it'}-1)dt' \\
&= iN\left(\int_0^t e^{-2it'}-1dt' - \int_0^t e^{(-4N+8)Nit'}(e^{-2it'}-1)dt'\right) \\
&= iN\left(\frac{e^{-2it}-1}{-2i}-t\right)+R_3, \qquad (4.30)
\end{aligned}$$

with the bounded remainder term

$$R_3 := N\left(\frac{e^{(-4N+6)it}-1}{4N-6} - \frac{e^{(-4N+8)it}-1}{4N-8}\right).$$

The last contribution to $N^s W(\widehat{-t)I_{2,2}}(t)(N)$ comes from $\widehat{I_2(t')}(N+2)$ and $\widehat{I_2(t')}(-2)$. As above, using $-N^2-4+(N+2)^2-4N=0$, we calculate

$$\begin{aligned} J_4 &:= -i(N+2)\int_0^t e^{-it'N^2}e^{-4it'}(e^{2it'}-1)e^{it'(N+2)^2}(e^{-4Nit'}-1)dt' \\ &= -i(N+2)\left(\int_0^t e^{2it'}-1dt'-\int_0^t e^{4Nit'}(e^{2it'}-1)dt'\right) \\ &= -i(N+2)\left(\frac{e^{2it}-1}{2i}-t\right)+R_4, \end{aligned} \tag{4.31}$$

with the bounded remainder term

$$R_4 := (N+2)\left(\frac{e^{(4N+2)it}-1}{4N+2}-\frac{e^{4Nit}-1}{4N}\right).$$

Therefore, summing up all contributions to $N^s W(\widehat{-t)I_{2,2}}(t)(N)$, we arrive at

$$\|I_{2,2}(t)\|_{H^s(\mathbb{T})} \geq |J_1+J_2+J_3+J_4|-c \tag{4.32}$$

With the complex number

$$z(t) := -i\left(\frac{e^{2it}-1}{2i}-t\right)$$

we rewrite

$$J_1+J_2+J_3+J_4 = (6N+4)\operatorname{Re} z(t)$$

and we observe $\operatorname{Re} z(t) = \sin^2(t)$. □

Proof of Theorem 4.3.1. Assume that there exists a normed space $X_T \hookrightarrow C([0,T],H_0^s(\mathbb{T}))$ with the properties (4.18), (4.19) and define $u_N = I_2(\psi_N)$. Then, an application of the estimate (4.19) shows

$$\left\|\int_0^t W(t-t')\partial_x(u_N^2(t'))\,dt'\right\|_{X_T} \leq c\|u_N\|_{X_T}^2$$

which is bounded by

$$c\|W(\cdot)\psi_N\|_{X_T}^4 \leq c\|\psi_N\|_{H^s(\mathbb{T})}^4 \leq c$$

due to (4.18). On the other hand, because of the continuous embedding, this is bounded from below by

$$c\left\|\int_0^t W(t-t')\partial_x(u_N^2(t'))\,dt'\right\|_{H^s(\mathbb{T})} = c\|I_{2,2}(t)\|_{H^s(\mathbb{T})} \geq cN\sin^2(t)-c$$

for any $N \geq 10$, $t \in [0,T]$, which is a contradiction. □

4.4 Equations with weak dispersion

In this section we consider the non-periodic case and the phase function $\phi : \mathbb{R} \to \mathbb{R}, \phi(\xi) = \xi|\xi|^\alpha$ for $0 < \alpha \leq 1$. We study the Cauchy problem

$$\begin{aligned} \partial_t u(t) - |D|^\alpha \partial_x u(t) + \frac{1}{2}\partial_x u^2(t) = 0 \ , \quad t \in (-T, T) \\ u(0) = u_0 \end{aligned} \tag{4.33}$$

for any $0 < \alpha \leq 1$ and our aim is to show a local well-posedness result for real valued initial data $u_0 \in H^s(\mathbb{R})$ for certain s below $\frac{3}{2}$ by a modification of the argument from H. Koch - N. Tzvetkov [KT03b].

These equations arise as models for vorticity waves in the coastal zone [MST01, SV96]. Since the equations are even less dispersive than Benjamin-Ono or Schrödinger, we expect that iterative methods cannot work because of the arguments in [MST01, KT05b]. Moreover, there is a loss of derivatives in the Strichartz estimates [KPV91a]. Nevertheless, we show that the method used in [KT03b] is robust enough to extend to this situation. In the following, we will only highlight the necessary modifications to the arguments in [KT03b].

Notice that above $s = \frac{3}{2}$ one does not need to exploit the dispersive structure of these equations; in this range local well-posedness results can be deduced even for the Burger's type equations ($\alpha = 0$) for real valued data, cp. e.g. the remarks in the introduction of [KPV90].

We expect that the improved technique of C.E. Kenig - K.D. Koenig [KK03], using additional local smoothing and maximal function estimates may give better results, since it does so in the case $\alpha \geq 1$.

Assumption. For brevity, we denote with $H^s(\mathbb{R})$ the space of real valued functions in $H^s(\mathbb{R})$ throughout this section.

Theorem 4.4.1. *Let $0 < \alpha \leq 1$ and $s \geq s_0 > \frac{3}{2} - \frac{\alpha}{4}$. There exists a non-increasing function $T^* : (0, \infty) \to (0, \infty)$, such that for $R > 0$ and $0 < T \leq T^*(R)$ there exists a continuous map*

$$S_{R,T} : B_R = \{u_0 \in H^{s_0}(\mathbb{R}) \mid \|u_0\|_{H^{s_0}} \leq R\} \to C([-T, T], H^{s_0}(\mathbb{R}))$$

with the properties:

(i) For all $u_0 \in B_R$ we have

$$S_{R,T}(u_0) \in V^T_{s_0} := \{u \in C([-T, T], H^{s_0}) \mid u_x \in L^1([-T, T], L^\infty(\mathbb{R})\}$$

and $u = S_{R,T}(u_0)$ *is the unique solution in* $V_{s_0}^T$ *of the Cauchy problem* (4.33) *in the sense that*

$$u(t) = W_\alpha(t)u_0 - \frac{1}{2}\int_0^t W_\alpha(t-t')\partial_x u^2(t')dt',\ t \in (-T,T) \tag{4.34}$$

(ii) For every $s \geq s_0$ *we have* $S_{R,T}(B_R \cap H^s(\mathbb{R})) \subset C([-T,T], H^s(\mathbb{R}))$ *and*

$$S_{R,T}\,|_{B_R\cap H^s(\mathbb{R})}\colon B_R \cap H^s(\mathbb{R}) \to C\big([-T,T], H^s(\mathbb{R})\big)$$

is continuous.

Remark 4.4.2. The theorem extends to corresponding equations with a more general phase function $\phi : \mathbb{R} \to \mathbb{R}, \phi(-\xi) = -\phi(\xi)$, which is smooth enough, grows at most polynomially and yields the same Strichartz estimates (cf. [KPV91a]) as in Theorem 2.1.3. As the proof shows, this is the only point where the exact structure of ϕ comes in.

4.4.1 Review of a refined energy method

The next proposition follows by a parabolic approximation argument, see e.g. [Sau79], or by the abstract method of T. Kato [Kat75].

Proposition 4.4.3. *Let* $s \geq 3$. *To every* $u_0 \in H^s(\mathbb{R})$, *there exists* $T > 0$ *(such that* T *is a non-increasing function of* $\|u_0\|_{H^s}$*) and a unique* $u \in C([-T,T], H^s(\mathbb{R})) \cap C^1((-T,T), H^1(\mathbb{R}))$ *which solves* (4.33).

The energy estimate together with the commutator estimates [KP88] give

Proposition 4.4.4. *Let* $u \in C([-T,T], H^3(\mathbb{R}) \cap H^s \cap C^1((-T,T), H^1(\mathbb{R})))$ *be a solution to* (4.33). *Then, for* $s \geq 0$

$$\|u\|_{L^\infty_T H^s} \leq \|u_0\|_{H^s} e^{c\|u_x\|_{L^1_T L^\infty_x}} \tag{4.35}$$

Moreover, for $u, v \in V_1^T$ *which solve the integral equation* (4.34) *we have the bound*

$$\|u - v\|_{L^\infty_T L^2_x} \leq c\|u(0) - v(0)\|_{L^2} e^{c\|u_x\|_{L^1_T L^\infty_x} + c\|v_x\|_{L^1_T L^\infty_x}} \tag{4.36}$$

The proof for smooth solutions may be found in [KPV91b] and [KT03b], and under the more general hypothesis the estimate (4.35) may be shown in the same way. For (4.36) we use the argument as in the proof of (A.6).

Now, we prove that an a priori bound for $\|u_x\|_{L^1_T L^\infty_x}$ for solutions u determines a lower bound for the time of existence. This allows us to find a common time interval for an approximating sequence of solutions, which we need to establish the existence part of Theorem 4.4.1.

Corollary 4.4.5. *Let $s \geq 3$. Suppose there exist $C > 0$ and $T^* > 0$, such that for every solution $u \in C([-T,T], H^s(\mathbb{R})) \cap C^1((-T,T), H^1(\mathbb{R}))$ with $T \leq T^*$,*

$$\|u_x\|_{L^1_T L^\infty_x} \leq C. \tag{4.37}$$

Then, to every $u_0 \in H^s$ the maximal solution exists at least up to time T^*.

Proof. Let $u_0 \in H^s$. Define

$$T' = \sup\{T > 0 \,|\, \exists \text{ solution } u \in C([-T,T], H^s(\mathbb{R}))\}.$$

From Proposition 4.4.3 we have $T' > 0$. Suppose $T' < T^*$. By (4.35) and (4.37) there exists $C > 0$, such that for all $|t| \leq T'$ we have $\|u(t)\|_{H^s} \leq C$. Let T be the non-increasing function from Proposition 4.4.3, and set $\varepsilon = \frac{1}{2}\min\{T(C), T'\} > 0$. Thus we can re-apply Proposition 4.4.3 with $\tilde{u}_0 = u(\pm T' \mp \varepsilon)$, which contradicts the definition of T'. $\square$

Now we use the argument from H. Koch - N. Tzvetkov [KT03b] to prove Theorem 4.4.1 and we focus on the a priori estimate. The next proposition generalizes the nonlinear estimate from [KT03b], Theorem 3.1.

Proposition 4.4.6. *Let $0 < T \leq 1$, $\sigma > 1$ and $0 < \alpha \leq 1$. Let $(p,q) \neq (4,\infty)$ be an admissible pair. Then, for every solution $u \in C([-T,T], H^3(\mathbb{R})) \cap C^1((-T,T), H^1(\mathbb{R}))$ of (4.33) we have*

$$\|J^\sigma u\|_{L^p_T L^q_x} \leq c\left(1 + \|J^\sigma u\|_{L^\infty_T L^2_x}\right)\left(1 + \|u_x\|^2_{L^1_T L^\infty_x}\right)\|J^{\sigma + \frac{2-\alpha}{p}} u\|_{L^\infty_T L^2_x} \tag{4.38}$$

To prove this estimate we first apply the general Strichartz estimates (2.2), (2.3) from Theorem 2.1.3 (cp. [KPV91b]) to solutions to the linearized equation

$$w_t - |D|^\alpha w_x + V w_x = F$$

with $\operatorname{supp} \widehat{w} \subset [-\lambda, \lambda]$ to obtain

$$\|J^{\frac{\alpha-1}{p}} w\|_{L^p_T L^q_x} \leq c\lambda^{\frac{1}{p}}\left(\|w\|_{L^\infty_T L^2_x} + \|J^\sigma V\|_{L^\infty_T L^2_x}\|w\|_{L^\infty_T L^2_x} + \|F\|_{L^1_T L^2_x}\right) \tag{4.39}$$

which replaces [KT03b], Lemma 2.1 and then we proceed as in the proof of [KT03b], Theorem 3.1.

Next, we establish the a priori bound for smooth solutions.

Proposition 4.4.7. *Let $s > \frac{6-\alpha}{4}$ and $0 < \alpha \leq 1$ and $0 < T \leq 1$. There exists $C > 0, \varepsilon > 0$, such that*

$$\|u_x\|_{L^1_T L^\infty_x} \leq C \tag{4.40}$$

$$\|u\|_{L^\infty_T H^s} \leq c\|u_0\|_{H^s} \tag{4.41}$$

for solutions $u \in C([-T,T],H^3(\mathbb{R})) \cap C^1((-T,T),H^1(\mathbb{R}))$ *with* $\|u_0\|_{H^s} \leq \varepsilon$.

Proof. Let $s > \frac{6-\alpha}{4}$ and $\sigma := s - \frac{2-\alpha}{4} > 1$. Using the Sobolev embedding in x and Hölder in t, we find an admissible pair (p,q), such that

$$\|u_x\|_{L^1_T L^\infty_x} \leq c\|J^\sigma u\|_{L^p_T L^q_x}.$$

Now we use (4.38) and obtain

$$\|u_x\|_{L^1_T L^\infty_x} \leq c\left(1 + \|J^\sigma u\|_{L^\infty_T L^2_x}\right)\left(1 + \|u_x\|^2_{L^1_T L^\infty_x}\right)\|J^{\sigma + \frac{2-\alpha}{p}} u\|_{L^\infty_T L^2_x}$$

and, because $p > 4$, the energy estimate (4.35) yields

$$\|u_x\|_{L^1_T L^\infty_x} \leq c\left(1 + \|J^\sigma u\|_{L^\infty_T L^2_x}\right)\left(1 + \|u_x\|^2_{L^1_T L^\infty_x}\right)\|u_0\|_{H^s} e^{c\|u_x\|_{L^1_T L^\infty_x}}.$$

We set $N(T) := \|u_x\|_{L^1_T L^\infty_x} + \|J^\sigma u\|_{L^\infty_T L^2_x}$, $T \in [0,1]$. We proved, if $\|u_0\|_{H^s} \leq \varepsilon$, that

$$N(T) \leq \varepsilon c(1 + N(T))^3 e^{cN(T)}, \tag{4.42}$$

with an appropriate $c > 0$. Consider the continuous function

$$F : \mathbb{R} \to \mathbb{R},\ F(x) = \varepsilon c(1+x)^3 e^{cx} - x.$$

We observe that $F(x) > 0$ for $x \in [0,\varepsilon]$, $F(1) < 0$ if ε is small enough and $N(0) \leq \varepsilon$.
Assume $N(T) \geq 1$. The continuity of F and N imply the existence of a $0 < T' \leq T$ with $F(N(T')) < 0$, which contradicts (4.42). Therefore $N(T) \leq 1$ must hold, and in particular (4.40) is proved. Inserting this into the energy estimate (4.35) also yields the second claim (4.41) under the hypothesis $\|u_0\|_{H^s} \leq \varepsilon$. □

We now sketch the proof of Theorem 4.4.1. The existence of solutions in $L^\infty([-T,T],H^s(\mathbb{R}))$ to small initial data follows from Propositions 4.4.7 and 4.4.3 by a compactness argument. Then, we remove the smallness assumption, using the following scaling consideration: Let u be a solution to problem (4.33) with initial datum u_0. Then, for each $\lambda > 0$, $\tilde{u}(t,x) = \lambda^\alpha u(\lambda^{1+\alpha}t, \lambda x)$ is a solution with initial datum $\tilde{u}_0(x) = \lambda^\alpha u_0(\lambda x)$. Due to (4.36), solutions with $u_x \in L^1_T L^\infty_x$ are unique in this class. The persistence property and continuous dependence on the initial data follow from a version of the energy inequality ([KT03b], Lemma 3.6) in exactly the same way as in [KT03b].

4.5 Notes and References

There are many contributions to the well-posedness theory for case $\alpha = 2$, the Korteweg de Vries equation. The works of J. Bourgain [Bou93] and C.E. Kenig - G. Ponce - L. Vega [KPV93c, KPV96] as well as J. Colliander - M. Keel - G. Staffilani - H. Takaoka - T. Tao [CKS$^+$03a] showed that this problem is locally [globally] well-posed with locally Lipschitz continuous (even analytic) dependence on the [real valued] initial data $u_0 \in H^s(\mathbb{R})$ for $s > -3/4$. The local result is established by the contraction mapping principle. Moreover, M. Christ - J. Colliander - T. Tao [CCT03] proved the failure of uniformly continuous dependence (on balls) below $-3/4$ and proved local well-posedness for $s = -3/4$.

In the case $\alpha = 1$ of the Benjamin-Ono equation the Cauchy problem is differently behaved with respect to the smoothness properties of the flow map. In [KT05b] H. Koch - N. Tzvetkov prove that the flow map is not uniformly continuous on balls in $H^s(\mathbb{R})$ for $s > 0$ (this is related to [MST01]). On the other hand, the problem is globally well-posed with continuous dependence on the real valued data in $H^s(\mathbb{R})$ for any $s \geq 0$ due to a recent result of A.D. Ionescu - C.E. Kenig [IK05]. This is established by combining the gauge transformation introduced in the work of T. Tao [Tao04] with a new bilinear estimate. For some previous results we refer the reader to [KT03b, KK03, Tao04, BP05].

The Cauchy problems in the cases $1 < \alpha < 2$ share a property with the Benjamin-Ono case, namely that it is not possible to prove reasonable bilinear estimates in order to perform the Picard iteration in $H^s(\mathbb{R})$ due to counterexamples found by L. Molinet - J.-C. Saut - N. Tzvetkov [MST01]. Therefore, we call these equations to be of Benjamin-Ono type. Nevertheless, there are well-posedness results in $H^s(\mathbb{R})$ for real valued initial data due to C.E. Kenig - G. Ponce - L. Vega [KPV91b] ($s \geq (9 - 3\alpha)/4$), improved by C.E. Kenig - K. Koenig [KK03] ($s > 3/2 - 3\alpha/8$). The proofs are based on local smoothing, maximal function, Strichartz and energy type estimates. In [CKS03b] J. Colliander - C.E. Kenig - G. Staffilani proved a local well-posedness result for $s \geq \alpha/2$ by a contraction argument for initial data in a weighted Sobolev space in the range $1 < \alpha < 2$.

That a low frequency condition might be useful was strongly motivated by the examples found by L. Molinet - J.-C. Saut - N. Tzvetkov [MST01] and by H. Koch - N. Tzvetkov [KT05b]. K. Kato [Kat04] indicated that in the BO case a homogeneous low frequency weight might lead to well-posedness and in the range $1 < \alpha < 2$ L. Molinet - F. Ribaud [MR06] showed well-posedness for $s > \frac{1+\alpha}{4}$ (local) and $s \geq \alpha/2$ (global) with $\omega \geq \frac{1+3\alpha}{8\alpha}$ in spaces similar to $X_{s,\omega,b}$. A low frequency condition of a similar type was also used by A.D. Ionescu - C.E. Kenig [IK05] in their main bilinear estimate for the BO case $\alpha = 1$, which they combined successfully with a gauge transformation. We also remark that Mizohata's decay condition for certain Schrödinger equations is related, see e.g. Chapter VII of [Miz85].

The results remarked in Section 4.4 are improvements to previous results of J.-C. Saut [Sau79]. The arguments are a combination of two main ingredients, namely the general Strichartz estimates from [KPV91b] and the refined energy method of H. Koch - N. Tzvetkov [KT03b].

The borderline case $\alpha = 0$, which is not discussed here, corresponds to Burger's equation, which is a non dispersive model for shock waves, see e.g. L.C. Evans [Eva98], Section 3.4.

Moreover, in some range of α it is possible to construct weak solutions in $L^2(\mathbb{R})$ and $H^{\frac{\alpha}{2}}(\mathbb{R})$ by compactness arguments, see e.g. J. Ginibre - G. Velo [GV91] for the precise results and further references.

For properties of solitary waves for the cases $1 \leq \alpha \leq 2$ see [Wei87, BL97].

Appendix A

Auxiliary estimates

A.1 A Gagliardo-Nirenberg estimate in the periodic case

Lemma A.1.1. *Let $f \in H^1(\mathbb{T})$. Then,*

$$\|f\|_{L^6(\mathbb{T})} \leq \frac{1}{\sqrt[3]{2\pi}} \left(\|f\|^{\frac{2}{3}}_{L^2(\mathbb{T})} \|f'\|^{\frac{1}{3}}_{L^2(\mathbb{T})} + \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} |f| \, dx \right) \tag{A.1}$$

Proof. Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic. By approximation it suffices to consider f which are continuously differentiable.

1. Assume that there exists $\xi \in [0, 2\pi]$ such that $f(\xi) = 0$. By a translation $x \mapsto x + \xi$ we may assume that $\xi = 0$. Define

$$g(x) = \begin{cases} f(x) & , x \in [0, 2\pi] \\ 0 & , x \in \mathbb{R} \setminus [0, 2\pi] \end{cases}$$

Then,

$$g'(x) = \begin{cases} f'(x) & , x \in [0, 2\pi] \\ 0 & , x \in \mathbb{R} \setminus [0, 2\pi] \end{cases}$$

is the weak derivative and we apply the Gagliardo-Nirenberg inequality on $\mathbb{R}$ with the optimal constant, see [SN41, Wei83]:

$$\|g\|_{L^6(\mathbb{R})} \leq \frac{1}{\sqrt[3]{2\pi}} \|g\|^{\frac{2}{3}}_{L^2(\mathbb{R})} \|g'\|^{\frac{1}{3}}_{L^2(\mathbb{R})}$$

and it follows

$$\|f\|_{L^6(\mathbb{T})} \leq \frac{1}{\sqrt[3]{2\pi}} \|f\|^{\frac{2}{3}}_{L^2(\mathbb{T})} \|f'\|^{\frac{1}{3}}_{L^2(\mathbb{T})} \tag{A.2}$$

which of course also implies (A.1).

2. Assume that $f(x) \neq 0$ for all $x \in [0, 2\pi]$ and define

$$\tilde{f}(x) = |f(x)| - \frac{1}{2\pi}\int_0^{2\pi} |f(y)|dy$$

Now, $\tilde{f}$ is continuously differentiable with

$$\tilde{f}' = \frac{\mathrm{Re}(f)\,\mathrm{Re}(f') + \mathrm{Im}(f)\,\mathrm{Im}(f')}{|f|}$$

such that $|\tilde{f}'| \leq |f'|$ and there exists $\xi \in [0, 2\pi]$ such that $\tilde{f}(\xi) = 0$. By (A.2) it follows

$$\begin{aligned}
&\||f| - \frac{1}{2\pi}\int_0^{2\pi} |f(y)|dy\|_{L^6(\mathbb{T})} \\
\leq &\frac{1}{\sqrt[3]{2\pi}}\||f| - \frac{1}{2\pi}\int_0^{2\pi} |f(y)|dy\|_{L^2(\mathbb{T})}^{\frac{2}{3}}\|\tilde{f}'\|_{L^2(\mathbb{T})}^{\frac{1}{3}} \\
\leq &\frac{1}{\sqrt[3]{2\pi}}\|f\|_{L^2(\mathbb{T})}^{\frac{2}{3}}\|f'\|_{L^2(\mathbb{T})}^{\frac{1}{3}}
\end{aligned}$$

which implies (A.1). □

A.2 An estimate involving exponentials

We prove that for all $s \geq 0$ there exists $c > 0$, such that for $f, g, h \in H^s(\mathbb{T})$ we have

$$\left\|(e^{\pm i\mathcal{I}(f)} - e^{\pm i\mathcal{I}(g)})h\right\|_{H^s} \leq ce^{c\|f\|_{H^s}^2 + c\|g\|_{H^s}^2}(\|f\|_{H^s} + \|g\|_{H^s})\|f-g\|_{H^s}\|h\|_{H^s}$$

To simplify the notation we only consider the plus sign since the same argument works with the minus sign. Moreover, it suffices to consider smooth f, g, h and we start with the case $s > 0$. We write

$$(e^{i\mathcal{I}(f)} - e^{i\mathcal{I}(g)})h = ih(\mathcal{I}(f) - \mathcal{I}(g))\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k-1}(i\mathcal{I}(f))^j(i\mathcal{I}(g))^{k-1-j} \quad \text{(A.3)}$$

Let $s' = \max\{s, \frac{1}{2} + \varepsilon\}$ for some $0 < \varepsilon < \frac{1}{2}$ to be chosen later. Then, the H^s norm of the expression (A.3) is bounded by

$$\|h\|_{H^s}\|\mathcal{I}(f) - \mathcal{I}(g)\|_{H^{s'}}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k-1}(c\|\mathcal{I}(f)\|_{H^{s'}})^j(c\|\mathcal{I}(g)\|_{H^{s'}})^{k-1-j}$$

where we used the Sobolev multiplication law from Corollary 1.1.12. Now, we observe that

$$\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=0}^{k-1}(c\|\mathcal{I}(f)\|_{H^{s'}})^j(c\|\mathcal{I}(g)\|_{H^{s'}})^{k-1-j}\le e^{c\|\mathcal{I}(f)\|_{H^{s'}}+c\|\mathcal{I}(g)\|_{H^{s'}}}$$

Moreover,

$$\|\mathcal{I}(f)\|_{H^{s'}}\le \||f|^2\|_{H^{s'-1}}+\|f\|_{L^2}^2$$

In the case where $s\ge\frac{1}{2}+\varepsilon$ it follows $\||f|^2\|_{H^{s'-1}}\le\||f|^2\|_{H^s}\le c\|f\|_{H^s}^2$ and otherwise, with $p=\frac{1}{1-\varepsilon}$

$$\||f|^2\|_{H^{-\frac{1}{2}+\varepsilon}}\le c\||f|^2\|_{L^p}\le c\|f\|_{L^{2p}}^2\le c\|f\|_{H^{\frac{\varepsilon}{2}}}^2$$

by Sobolev embeddings from Proposition 1.1.11. Now, choosing $\varepsilon\le 2s$ we have

$$\|\mathcal{I}(f)\|_{H^{s'}}\le c\|f\|_{H^s}^2$$

Similarly, we get

$$\|\mathcal{I}(f)-\mathcal{I}(g)\|_{H^{s'}}\le c(\|f\|_{H^s}+\|g\|_{H^s})\|f-g\|_{H^s} \tag{A.4}$$

and the claim follows for $s>0$. Finally, for $s=0$

$$\begin{aligned}\left\|(e^{i\mathcal{I}(f)}-e^{i\mathcal{I}(g)})h\right\|_{L^2}&\le\left\|e^{i\mathcal{I}(f)}-e^{i\mathcal{I}(g)}\right\|_{L^\infty}\|h\|_{L^2}\\&\le\|\mathcal{I}(f)-\mathcal{I}(g)\|_{L^\infty}\|h\|_{L^2}\\&\le 2(\|f\|_{L^2}+\|g\|_{L^2})\|f-g\|_{L^2}\|h\|_{L^2}\end{aligned}$$

A.3 An estimate for ψ

Lemma A.3.1. *Let ψ be defined by* (3.7)*. Then,*

$$\begin{aligned}|\psi(u)(t)-\psi(v)(t)|\le&c\big(1+\|u(t)\|_{H^{\frac{1}{2}}}+\|v(t)\|_{H^{\frac{1}{2}}}\big)^3\|(u-v)(t)\|_{H^{\frac{1}{2}}}\\&+2(\|u(0)\|_{L^2}^3+\|v(0)\|_{L^2}^3)\|u(0)-v(0)\|_{L^2}\end{aligned} \tag{A.5}$$

Proof. We suppress the t dependence and just write $u=u(t)$, $v=v(t)$.

$$\left|\int_0^{2\pi}(\mathrm{Im}(\overline{u}_xu)-\mathrm{Im}(\overline{v}_xv))(x)\,dx\right|\le|(u-v,u_x)_{L^2}|+|(v,u_x-v_x)_{L^2}|$$

Since $J_x^{\frac{1}{2}}$ is formally self-adjoint with respect to $(\cdot,\cdot)_{L^2}$ we get

$$\begin{aligned}
&|(u-v,u_x)_{L^2}| + |(v,u_x-v_x)_{L^2}| \\
=&\left|\left(J_x^{\frac{1}{2}}(u-v), J_x^{-\frac{1}{2}}\partial_x u\right)_{L^2}\right| + \left|\left(J_x^{\frac{1}{2}}v, J_x^{-\frac{1}{2}}\partial_x(u-v)\right)_{L^2}\right| \\
\le& c(\|u\|_{H^{\frac{1}{2}}} + \|v\|_{H^{\frac{1}{2}}})\|u-v\|_{H^{\frac{1}{2}}}
\end{aligned}$$

Moreover, $\left|\int_0^{2\pi}(|u|^4-|v|^4)(x)\,dx\right|$ is bounded by

$$\begin{aligned}
&\int_0^{2\pi} ||u|-|v||\,(|u|^3+|u|^2|v|+|u||v|^2+|v|^3)(x)\,dx \\
\le& 2(\|u\|_{L^6}^3 + \|v\|_{L^6}^3)\|u-v\|_{L^2}
\end{aligned}$$

Finally,

$$\left|\|u(0)\|_{L^2}^4 - \|v(0)\|_{L^2}^4\right| \le 2(\|u(0)\|_{L^2}^3 + \|v(0)\|_{L^2}^3)\|u(0)-v(0)\|_{L^2}$$

These three estimates together with the Sobolev embedding $H^{\frac{1}{3}} \hookrightarrow L^6$ prove (A.5). □

A.4 Energy estimate for the DNLS

Proposition A.4.1. *For all*

$$u,v \in C([-T,T],H^1(\mathbb{T})) \quad \text{such that} \quad u_x,v_x \in L^1([-T,T],L^\infty(\mathbb{T}))$$

which solve the DNLS in integral form (3.3) *it holds*

$$\|u(t)-v(t)\|_{L^2(\mathbb{T})} \le e^{l(u,v)}\|u(0)-v(0)\|_{L^2(\mathbb{T})} \tag{A.6}$$

where

$$l(u,v) = c(\|u_x\|_{L_T^1 L^\infty} + \|v_x\|_{L_T^1 L^\infty})(\|u\|_{L_T^\infty L^\infty} + \|v\|_{L_T^\infty L^\infty})$$

In particular, solutions are unique in this class.

Proof. We observe for $\tilde{u}(t) = W(-t)u(t)$, $\tilde{v}(t) = W(-t)v(t)$ that $\tilde{u},\tilde{v} \in C^1((-T,T),L^2)$ and

$$\partial_t(\tilde{u}(t)-\tilde{v}(t)) = W(-t)\partial_x(|u|^2u)(t) - W(-t)\partial_x(|v|^2v)(t)$$

such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v(t)\|_{L^2(\mathbb{T})}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}(t)-\tilde{v}(t)\|_{L^2(\mathbb{T})}^2\\ &= 2\operatorname{Re}\int_0^{2\pi} W(-t)\partial_x\left(|u|^2u-|v|^2v\right)\overline{W(-t)(u-v)}dx\\ &= 2\operatorname{Re}\int_0^{2\pi}\partial_x\left(|u|^2u-|v|^2v\right)(\overline{u}-\overline{v})dx\end{aligned}$$

Now, we use

$$|u|^2u-|v|^2v=(|u|^2+|v|^2)(u-v)+uv(\overline{u}-\overline{v})$$

and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v(t)\|_{L^2(\mathbb{T})}^2 =&2\operatorname{Re}\int_0^{2\pi}\partial_x\left((|u|^2+|v|^2)(u-v)\right)(\overline{u}-\overline{v})dx\\ &+2\operatorname{Re}\int_0^{2\pi}\partial_x\left(uv(\overline{u}-\overline{v})\right)(\overline{u}-\overline{v})dx\\ =&\int_0^{2\pi}\partial_x(|u|^2+|v|^2)|u-v|^2dx\\ &+\operatorname{Re}\int_0^{2\pi}\partial_x(uv)(\overline{u}-\overline{v})^2dx\end{aligned}$$

using integration by parts. Then,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v(t)\|_{L^2}^2\\ \leq&2(\|u_x(t)\|_{L^\infty}+\|v_x(t)\|_{L^\infty})(\|u(t)\|_{L^\infty}+\|v(t)\|_{L^\infty})\|u(t)-v(t)\|_{L^2}^2\end{aligned}$$

and Gronwall's inequality proves

$$\|u(t)-v(t)\|_{L^2}^2\leq e^{2(\|u_x\|_{L^1_TL^\infty}+\|v_x\|_{L^1_TL^\infty})(\|u\|_{L^\infty_TL^\infty}+\|v\|_{L^\infty_TL^\infty})}\|u(0)-v(0)\|_{L^2}^2$$

□

A.5 Conservation laws for the DNLS

The results for the (DNLS) in this section are well-known in the case of the real line (cp. [CH98], Proposition 6.1.1, appendix of [HO94], or [KN78]) and formally everything transfers to the periodic setting. The results also follow from [TF81]. Nevertheless, we provide them here for completeness.

Lemma A.5.1. *If*

$$u \in C([-T,T], H^2(\mathbb{T})) \cap C^1([-T,T], L^2(\mathbb{T}))$$

is a solution of (3.1), (3.6) *or* (3.2), *we have for* $t \in (-T,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{L^2(\mathbb{T})} = 0$$

Proof. One easily shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2_{L^2(\mathbb{T})} = 2 \operatorname{Re} \int_0^{2\pi} \overline{u} N(u)(t) \, dx$$

for $N(u) = \partial_x(|u|^2 u)$ or $N(u) = -u^2 \partial_x \overline{u} + \frac{i}{2}|u|^4 u - i\mu(u)|u|^2 u + i\psi(u)u$, or $N(u) = 2(|u|^2 - \frac{1}{2\pi}\int_0^{2\pi} |u|^2 dx)\partial_x u + u^2 \partial_x \overline{u}$, respectively. Partial integration yields

$$\operatorname{Re} \int_0^{2\pi} \overline{u} \partial_x (|u|^2 u) \, dx = 0$$

and

$$\operatorname{Re} \int_0^{2\pi} \overline{u} u^2 \partial_x \overline{u} \, dx = 0$$

Obviously, the other terms also vanish. □

Lemma A.5.2. *If*

$$u \in C([-T,T], H^3(\mathbb{T})) \cap C^1([-T,T], H^1(\mathbb{T}))$$

is a solution of (3.1), *we have for* $t \in (-T,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|u_x(t)\|^2_{L^2(\mathbb{T})} + \frac{3}{2} \operatorname{Im} \int_0^{2\pi} |u|^2 u \overline{u}_x(t) \, dx + \frac{1}{2} \|u(t)\|^6_{L^6(\mathbb{T})} \right) = 0$$

Proof. Firstly, using (3.1) we verify

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_x\|^2_{L^2} = 2 \operatorname{Re} \int_0^{2\pi} (|u|^2 u)_{xx} \overline{u}_x \, dx \tag{A.7}$$

Secondly, we again exploit (3.1) and carry out all the differentiations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \int_0^{2\pi} |u|^2 u \overline{u}_x \, dx &= \operatorname{Im} \int_0^{2\pi} (|u|^2 u)_t \overline{u}_x \, dx + \operatorname{Im} \int_0^{2\pi} |u|^2 u \overline{u}_{tx} \, dx \\ &= 4 \operatorname{Re} \int_0^{2\pi} |u|^2 u_x \overline{u}_{xx} \, dx + 4 \operatorname{Im} \int_0^{2\pi} \overline{u}_x^2 u^2 |u|^2 \, dx \end{aligned} \tag{A.8}$$

Thirdly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^6}^6 = -6\,\mathrm{Im}\int_0^{2\pi} |u|^4\overline{u}u_{xx}\,dx + 6\,\mathrm{Re}\int_0^{2\pi} |u|^4\overline{u}(|u|^2u)_x\,dx$$

and

$$\mathrm{Re}\int_0^{2\pi} |u|^4\overline{u}(|u|^2u)_x\,dx = \frac{3}{8}\int_0^{2\pi}(|u|^8)_x\,dx = 0$$

Moreover, we integrate by parts and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^6}^6 = -12\,\mathrm{Im}\int_0^{2\pi} |u|^2u^2\overline{u}_x^2\,dx \tag{A.9}$$

Now, combining (A.7), (A.8) and (A.9) and integrating by parts we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \frac{3}{2}\,\mathrm{Im}\int_0^{2\pi} |u|^2u\overline{u}_x(t)\,dx + \frac{1}{2}\|u(t)\|_{L^6(\mathbb{T})}^6\right)\\
=&6\,\mathrm{Re}\int_0^{2\pi} |u|^2u_x\overline{u}_{xx}\,dx - 2\,\mathrm{Re}\int_0^{2\pi}(|u|^2u)_x\overline{u}_{xx}\,dx\\
=&2\,\mathrm{Re}\int_0^{2\pi} |u|^2u_x\overline{u}_{xx}\,dx - 2\,\mathrm{Re}\int_0^{2\pi} u^2\overline{u}_x\overline{u}_{xx}\,dx = 0
\end{aligned}$$

and the conservation law is proved. □

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