English Translation

Lower Bounds for the Immersion Dimension of Homogeneous Spaces

Doctoral thesis

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Contents

Introduction

It is an old problem of differential topology to determine the immersion dimension of a compact smooth manifold X. The immersion dimension is the smallest integer j such that X can be immersed in an Euclidean space with dimension j.

There are a lot of results concerning the projective spaces ([Ati62], [San64], [AGM65], [Fed66], [MM67], [Mil67], [Jam71], [Ste71], [SS78], [DM77], [DM79], [Cra91], [Dav93]).

In [Coh85] Cohen gave an upper bound for all compact smooth manifolds which only depends on the dimension of the manifold. This upper bound is sharp: For all integers $d > 1$ there is a compact smooth d-dimensional manifold with immersion dimension being equal to Cohen's upper bound.

For certain homogeneous space some authors established other upper bounds. In [Tor68] Tornehave calculated an upper bound for the immersion dimension of coset spaces of centralizers of tori. For many flag manifolds Lam determined lower upper bounds (see [Lam75], also [Hil82b]). The essential tool of those authors was Hirsch's immersion theorem ([Hir59]).

The integrality theorems due to Atiyah and Hirzebruch ([AH59]) and Mayer ([May65]) can be used as a tool for the determination of lower bounds for the immersion dimension. By application of these theorems Sugawara ([Sug79]), Paryjas ([Par88]) and Mayer ([May97], [May98]) found lower bounds for the immersion dimension of Grassmannian manifolds. By other methods Hoggar ([Hog71]), Oproiu ([Opr76], [Opr81]), Ilori ([Ilo79]), Hiller and Stong

([HS81]), Markl ([Mar88]) and Tang ([Tan93a], [Tan93b], [Tan95]) as well as Connell ([Con74]) proved non-immersion theorems for Grassmannian manifolds and for low dimensional complex flag manifolds, respectively.

For a compact Lie group G and a closed subgroup U of G many topological invariants of the homogeneous space G/U can be expressed by structural datas of the Lie groups G und U. Examples of homogenous spaces are given by projective spaces und more general by flag manifolds.
In 1958 many important relations between the topological invariants and those structural datas were already well known and published in the fundamental articles "Characteristic classes and homogenous spaces" by Borel and Hirzebruch ([BH58], [BH59], [BH60]). In these articles the twisted Todd genus and the untwisted A-genus are calculated and existence theorems for complex, almost complex and Spin-structures on G/U are proved.
Up to now several other results, for example about the signature ([Sha79], [HS90], [BMP90], [Slo92]), have been established.

The object of the present work is to calculate characteristic numbers which are related to the immersion dimension of G/U by Lie group invariants of G and U.
The first chapter is devoted to collect well known immersion und non-immersion theorems. Subsequently (virtual) differential operators with indices equal to the values of a Hilbert polynomial are defined.
The second chapter provides some results of the representation theory of compact Lie groups and some relations between the topological structure of a homogeneous space and the algebraic structure of the Lie groups.
The subject matter of the third chapter is to calculate the indices of the differential operators introduced in the first chapter in the case of homogeneous spaces. The result is an expression for the index by algebraic invariants of the Lie groups.
In the first section of the fourth chapter we establish some identities and

inequalities. They will be of use in the subsequent sections.

In the other five sections of the fourth chapter we calculate lower bounds for the immersion dimension of (complex, quaternionalal resp. oriented real) flag manifolds and the manifolds $Sp(n)/U(n_1) \times \cdots \times U(n_s)$, $SO(2n)/U(n_1) \times \cdots \times U(n_s)$.

In the tables of the appendix lower and upper bounds for concrete homogeneous spaces are assembled.

I want to express special thanks to Professor Dr. Karl Heinz Mayer for a lot of useful hints and numerous inspiring discussions.

Chapter 1

The Hilbert polynomial

1.1 The $\hat{A}$–class and the Hilbert polynomial

In this chapter let X be a compact connected smooth oriented manifold of even dimension $2n$ with Pontrjagin classes $p_i(X) \in H^{4i}(X;\mathbb{Z})$ and fundamental class $[X]$.

Let $K(X)$ be the K–ring of X.

If A is a commutative ring with 1, $H^*(X;A)$ stands for the singular cohomology ring of X with coefficients in A.

Moreover let $ch : K(X) \to H^*(X;\mathbb{Q})$ be the Chern character and $ch(X) \subset H^*(X;\mathbb{Q})$ the image of $K(X)$ by ch.

For an element $z = \sum_{j=0}^{\infty} z_{2j} \in H^*(X;\mathbb{Q})$ with $z_{2j} \in H^{2j}(X;\mathbb{Q})$ and a rational number $t \in \mathbb{Q}$ we set $z^{(t)} = \sum\limits_{j=0}^{\infty} z_{2j}t^j$.

Proposition 1.1

If $t \in \mathbb{Z}$ *and* $z \in ch(X)$ *then* $z^{(t)} \in ch(X)$.

Proof: [AH59], p.387. □

Definition 1.2

We set

$$\hat{\mathcal{A}}(X) = \sum_{j=1}^{\infty} \hat{A}_j \left(p_1(X), \ldots, p_j(X)\right),$$

where $\left\{\hat{A}_j\right\}$ *is the multiplicative sequence belonging to the power series* $\dfrac{\frac{1}{2}\sqrt{z}}{\sinh\left(\frac{1}{2}\sqrt{z}\right)}$.

$\hat{\mathcal{A}}(X)$ *is called the* $\hat{A}$*–class of* X.

For all $d \in H^2(X;\mathbb{Q})$ *and* $z \in H^*(X;\mathbb{Q})$ *we define*

$$\hat{A}\left(X, d, z\right) = \left(z \cdot e^d \hat{\mathcal{A}}(X)\right)[X]. \quad \square$$

Proposition and Definition 1.3

If $d \in H^2(X;\mathbb{Q})$ *and* $z \in ch(X)$ *then*

$$H(t) = \hat{A}\left(X, \frac{d}{2}, z^{(t)}\right)$$

is a polynomial in t *of degree lower or equal to* n *with rational coefficients.*

H *is called the Hilbert–Polynom of* X *associated with* d *and* z. □

Remark 1.4

If $t \in \mathbb{Z}$, $d \in H^2(X;\mathbb{Z})$ *and* $d \equiv w_2(X)$ *mod* 2 *then* $H(t)$ *is an integer.*

Proof: [AH59], p.388. □

1.2 Immersion and Non-immersion theorems

The importance of the Hilbert polynomial for the immersion problem is given by the following integrality theorem ([May65]):

Proposition 1.5 (Mayer)
Let X be a $2n$–dimensional compact oriented smooth manifold and H be the Hilbert polynomial associated with $d \in H^2(X;\mathbb{Z})$ and $z \in ch(X)$.
If X can be immersed in $\mathbb{R}^{2n+k}$ with $k \in \{2s, 2s+1\}$ then $2^{n+s}H(\frac{1}{2})$ is an integer.
Consequently X can not be immersed in an Euclidean space with dimension $-2\nu_2(\big(H(\frac{1}{2})\big) - 1$.

Thereby we use the following notation:

Notation 1.6
For $q \in \mathbb{Q}$ we write $\nu_2(q)$ for the exponent of the prime 2 as prime factor of q.

Remark 1.7
In the integrality theorem in [May65] the following non-embedding result is included: If X can be embedded in $\mathbb{R}^{2n+k}$ with $k \in \{2s, 2s+1\}$ then $2^{n+s-1}H(\frac{1}{2})$ is an integer. The theorem contains sharper results for the cases $z \in chO(X)$ and $z \in chSp(X)$. $\square$

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Upper bounds for the immersion dimension are given by the next theorems:

Theorem 1.8 (Cohen)
Let X be a d–dimensional compact smooth Mannigfaltigkeit with $d > 1$. Then X can be immersed in an Euclidean space with dimension $2d - \alpha(d)$. Thereby $\alpha(d)$ is the number of the digit 1 in the dyadic representation of d.

Proof: [Coh85].

Remark 1.9
For every integer $d > 1$ there exists a d–dimensional compact smooth manifold X with immersion dimension equal to $2d - \alpha(d)$. ([Coh85]. p.238) □

For homogeneous spaces Tornehave ([Tor68]) established other upper bounds for the immersion dimension:

Proposition 1.10
Let G be a compact Lie group and Ad the adjoint representation of G on the real Lie algebra $\mathfrak{g}_0$ of G. If U is the centralizer $Z(S)$ of a toral subgroup S of G and the dimension of the center of U is equal to s, then G/U can be immersed in an Euclidean space with dimension $dim(\mathfrak{g}_0) - s$.

Proof: [Sch86], Prop.4.

Remark 1.11

(i) For the notations see chapter 2.

(ii) In [Lam75] Lam determined more results for real und quaternional flag manifolds. For the exact statements see the remarks 4.26 and 4.34. □

The proofs of those theorems are based on the following results of Hirsch ([Hir59]):

Theorem 1.12 (Hirsch)
Let X be a d–dimensional compact smooth manifold. If there is a real k–dimensional vector bundle η over X such that $k \geq 1$ and $T(X) \oplus \eta$ is trivial then X can be immersed in an Euclidean space with dimension $d + k$.

Proof: [Tor68], p.24. □

Theorem 1.13 (Hirsch)
Let X be a d–dimensional compact smooth manifold. If X can be immersed in an Euclidean space with dimension $d + k + r$ such that the normal bundle contains a trivial r–dimensional subbundle then X can be immersed in an Euclidean space with dimension $d + k$.

Proof: [Hir59], p.269. □

1.3 Hilbert polynomials and differential operators

This section ist devoted to introduce results due to Mayer and Schwarzenberger ([May65], [MS73]). They serve as a tool for evaluating Hilbert polynomials at $\frac{1}{2}$.

Notation 1.14

For natural numbers k, n let $G(2n, 2, k) \subset Spin(2n+2+k)$ be the preimage of $SO(2n)\times SO(2)\times SO(k) \subset SO(2n+2+k)$ unter the canonical two-sheeted covering map $\lambda : Spin(2n+2+k) \to SO(2n+2+k)$.

Proposition 1.15

Let X be a $2n$–dimensional compact oriented smooth S^1–manifold. We assume the fixed point set Y of the S^1–operation to be finite.

Additionaly let E be an equivariant complex line bundle over X, D an equivariant r-dimensional complex vector bundle and F be an equivariant k-dimensional real vector bundle over X.

We suppose $c_1(E) \equiv w_2(F) + w_2(X)$ mod 2 and F to be oriented.

We understand $T(X) \oplus E \oplus F \oplus D$ to be a vector bundle with structure group $SO(2n) \times SO(2) \times SO(k) \times U(r)$ and principal bundle $\mathcal{P}$. There is an S^1-action on $\mathcal{P}$, which induces the S^1-action on $T(X) \oplus E \oplus F \oplus D$.

Additionally there is a principal bundle $\mathcal{Q}$ over X with structure group $G(2n, 2, k)$ and a two–sheeted covering map $\kappa : \mathcal{Q} \to \mathcal{P}$, such that for all $(q, g_1, g_2) \in \mathcal{Q} \times G(2n, 2, k) \times U(m)$ the identity $\kappa(q \cdot (g_1, g_2)) = \kappa(q) \cdot (\lambda(g_1), g_2)$ holds.

If there is moreover an S^1–action on $\mathcal{Q}$ which induces the S^1–action on $\mathcal{P}$ (we quote this property by (*)) then there is an equivariant elliptic differential operator of first order on X such that the index $\Gamma(X,E,F,D) \in R(S^1)$ has the following properties:

(i) $\Gamma(X,E,F,D)(1)$

$$= (-1)^n 2^{\lfloor \frac{k}{2} \rfloor} \left(e^{\frac{1}{2}c_1(E)} ch(D) \left(\prod_i \cosh\left(\frac{y_i}{2}\right)\right) \hat{\mathcal{A}}(X) \right) \Big[X\Big].$$

Thereby $p(F) = \prod_i (1+y_i^2)$ is the total Pontrjagin class of F.

(ii) For all elements g of a dedicated dense subset of S^1 the following identity holds:

$$\Gamma(X,E,F,D)(g)$$
$$= \sum_{y\in Y} \Bigg(2^{l(y)} g^{\frac{1}{2}\gamma(y)} \cdot \sum_{\rho=1}^{r} g^{\mu_\rho(y)} \cdot \prod_{\nu=1}^{n} \left(g^{-\frac{1}{2}m_\nu(y)} - g^{\frac{1}{2}m_\nu(y)} \right)^{-1}$$
$$\cdot \prod_{\sigma=1}^{s} \left(g^{\frac{1}{2}\beta_\sigma(y)} + g^{-\frac{1}{2}\beta_\sigma(y)} \right) \hat{\mathcal{A}}(\{y\}) \Bigg) \Big[\{y\}\Big].$$

Thereby for a fixed point $y \in Y$ we denote the rotation number of the complex representation E_y of S^1 by $\gamma(y)$, the rotation numbers of the complex representation D_y of S^1 with $\mu_1(y), \ldots, \mu_r(y)$, the positive rotation numbers of the real representation $T_y(X)$ of S^1 by $m_1(y), \ldots, m_n(y)$ and the positive rotation numbers of the real representation F_y of S^1 by $\beta_1(y), \ldots, \beta_s(y)$. Additionally the trivial one–dimensional representation appears with multiplicitiy $2l(y)$ or $2l(y)+1$ as subrepresentation of F_y. All representation numbers have to be counted concerning their multiplicities. If the orientation of $T_y(X)$ with all rotation numbers positive is equal to the orientation induced by the manifold X then we understand the singleton $\{y\}$ to be oriented positive else negative.

We pay attention to the fact that the representations $T_y(X)$ have no trivial subrepresentations. We notice that $T_y(X)$ has a complex structure, such that all rotation numbers belonging to this complex structure are positive. Let the orientation of $\{y\}$ be induced by this complex structure.

Remark 1.16

(i) The assumption $(*)$ garuantees that the term on the the right hand side of the formula in (ii) is a meromorphic function in g. Due the continuity in 1 of the term on the left hand side 1 is a removable singularity of this meromorphic function. So the term $\Gamma(X, E, F, D)(1)$ can be calculated by determination of a limit.

(ii) If the extra assumption $(*)$ fails to be satisfied then there are S^1–actions on X, E, F, G such that the assumption $(*)$ is satisfied and all rotation numbers are doubled. ([AH70], Prop.2.1 or [Sch72], Satz (2.6)).

Also in this case $\Gamma(X, E, F, D)(1)$ can be calculated as limit of the term in (ii) (with the datas coming from the original S^1–action).

Remark 1.17

(i) Virtual equivariant bundles E, F, D satisfying the prerequisites of the theorem yield an equivariant "virtual" differential operator. The statements of the theorem remain valid for its formal index.

(ii) If we set $F = 0$ and substitute D by $\psi_t(D)$ with an integer t and ψ_t the Adams operation then the folowing identities hold:

$$\begin{aligned}
\Gamma(X, E, F, \psi_t(D))(1) &= (-1)^n \left(e^{\frac{1}{2}c_1(E)} ch(\psi_t(D)) \hat{\mathcal{A}}(X) \right) \Big[X\Big] \\
&= (-1)^n \left(e^{\frac{1}{2}c_1(E)} ch(D)^{(t)} \hat{\mathcal{A}}(X) \right) \Big[X\Big] \\
&= (-1)^n \hat{A} \left(X, \frac{c_1(E)}{2}, ch(D)^{(t)} \right) \\
&= (-1)^n H(t).
\end{aligned}$$

Thereby H *is the Hilbert polynomial associated with* $c_1(E)$ *and* $ch(D)$. $\square$

Chapter 2

Homogeneous spaces

2.1 Basic definitions

Proposition and Definition 2.1

Let G be a compact connected Lie group and U a connected closed subgroup of G.

We denote the set of left cosets of G modulo U by $G/U = \{gU \mid g \in G\}$.

We furnish G/U with the quotient topology and the $\mathcal{C}^\infty$–structure characterized by the fact that the canonical projection $\pi : G \to G/U$ is smooth and G/U is a quotient manifold with respect to π.

A manifold constructed in this way is called a homogeneous space. ([BD85], I(4.3)) □

Proposition 2.2

$(G, G/U, \pi)$ is a principal bundle with structure group U. ([BD85], I(4.3)) □

2.2 Lie groups

There is a deep coherence between the topological structure of a homogeneous space and the algebraic properties of the defining Lie groups. Hence we are going to rephrase important concepts and results of the representation theory of compact Lie groups. They can be looked up in most textbooks about representation theory (e.g. [Ada69], [BD85], [FH96] or [Kna96]).

In this section we understand G to be a compact connected Lie group with neutral element e.

Proposition and Definition 2.3

$T_e(G)$ has the structure of a real Lie algebra and is referred to be the Lie algebra $\mathfrak{g}_0$ of G ([Kna96], p.3). Its complexification $\mathfrak{g}_0 \otimes \mathbb{C}$ is denoted by $\mathfrak{g}$. There is a natural $\mathcal{C}^\infty$–mapping $\exp : \mathfrak{g}_0 \to G$ with $\exp(0) = e$ and $T_0(\exp) = id : T_0(\mathfrak{g}_0) = \mathfrak{g}_0 \to \mathfrak{g}_0$. $\exp$ is called the exponential map of G. ([Kna96], p.49) $\square$

Proposition 2.4

If H is another (not necessarily connected) Lie group and $\theta : G \to H$ a homomorphism of Lie groups then $T_e(\theta)$ is a homorphism of Lie algebras. θ is determined by $T_e(\theta)$.

Proof: [Ada69], 1.7 and 2.17.

Definition 2.5

A finite dimensional complex representation of G is a pair (V, Φ) consisting of a finite dimensional complex vector space V and a continuous homomorphism $\Phi : G \to Aut(V)$. V is called the representation space.

For the sake of convenience we often denote the representation by V *and the element* $\Phi(g)(v)$ *by* $g(v)$ *or* gv*.*

In a similar manner the concept of a real or quaternional representation of G *is defined.* □

Definition 2.6

A finite dimensional complex representation of a complex Lie algebra $\mathfrak{a}$ *is a pair* (V, φ) *consisting of a finite dimensional complex vector space* V *and a homomomorphism of Lie algebras* $\varphi : \mathfrak{a} \to End(V)$*.* V *is called the representation space.*

For the sake of convenience we often denote the representation by V *and the element* $\varphi(g)(v)$ *by* $g(v)$ *or* gv*.*

In a similar manner the concept of a real or quaternional representation of $\mathfrak{a}$ *is defined.* □

Remark 2.7

In a natural way concepts like "unitary representation", "irreducibility of representations" and "invariance of subspaces" can be introduced. Furthermore, most functorial constructions known from linear algebra can be transferred to representations. □

Example 2.8

The conjugation mapping $A : G \to Aut(G)$ *with* $A(g)(h) = g^{-1}hg$ *induces real representations* Ad *of* G *and* ad *of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_0$ *and a complex representation* ad *of* $\mathfrak{g}$ *on* $\mathfrak{g}$*. These representations are referred as adjoint representations of* G*. ([Ada69], 1.10)* □

Due to the compactness of G the following statements hold:

Proposition 2.9

(i) *If (V, Φ) is a finite dimensional complex or real representation of G then there is an Euclidean structure on V such that (V, Φ) is Euclidean.*

(ii) *Let V be a finite dimensional complex representation on G. Then there are invariant subspaces $V_1, \ldots, V_s$ of V such that $V = V_1 \oplus \cdots \oplus V_s$ and the representations $V_1, \ldots, V_s$ are irreducible.*

Proof: [Ada69], 3.20. $\square$

Definition 2.10
Let $R_{\mathbb{R}}(G)$ and $R(G) = R_{\mathbb{C}}(G)$ be the free abelian groups generated by the set of irreducible representations of G. The tensor product induces a ring structure on these groups. $R_{\mathbb{R}}(G)$ and $R(G) = R_{\mathbb{C}}(G)$ are called the real or complex representation ring of G, repectively. $\square$

Proposition and Definition 2.11
Let (V, Φ) be a finite dimensional complex representation of G. We associate a mapping $\chi_V = \chi_\Phi : G \to \mathbb{C}$ by $\chi_V(g) = trace(\Phi(g))$. χ_V is called the character of (V, Φ). It has the following properties:

(i) $\chi_V(e) = dim_{\mathbb{C}} V$.

(ii) *χ_V is continuous and constant on the conjugation classes of G. Such a map is called a class function.*

(iii) $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ *for all* $g \in G$.

(iv) χ_V *defines an injective homomorphism of rings*

$$\chi : R(G) \to \mathcal{CL}(G) = \{f \in \mathcal{C}(G, \mathbb{C}) \mid f \text{ is class function}\}.$$

The image χ *is called the character ring of* G*. The character ring will be denoted by* $R(G)$*, too.*

Proof: [Ada69], 3.32. □

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The representation theory of toral groups is very easy:

Proposition 2.12
Let $T^k = \mathbb{R}^k/\mathbb{Z}^k$ *the* k*–dimensional standard torus. Then the following statements hold:*

(i) T^k *ist monogenic, i.e.* T^k *has a generating element.*

(ii) If V *is a irreducible complex representation of* T^k *then* V *has dimension one.*

(iii) If $(\mathbb{C}, \Phi)$ *is a complex representation of* T^k *then* Φ *has the form* $\Phi([x_1, \ldots, x_k])(z) = e^{2\pi i(n_1x_1+\ldots+n_kx_k)}z$ *with integers* $n_1, \ldots n_k$*.*

(iv) Let ρ_j *be the one-dimensional complex representation of* T^k *with* $\rho_j([x_1, \ldots, x_k])(z) = e^{2\pi i(x_j)}z$*.* $R(T^k)$ *is the ring consisting of the finite Laurent series in* $\rho_1, \ldots, \rho_k$*.*

(v) If V *is a irreducible real representation of* T^k *then either* V *is one-dimensional and trivial or the realization of a non-trivial complex irreducible representation.*

Proof: [Ada69], 4.3, 3.71, 3.76, 3.77, 3.78. □

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In order to classify the representations of a compact Lie group one makes use of the knowledge about the representations of a maximal abelean subgroup of G. Those are toral due to the following proposition:

Proposition 2.13

A compact connected abelean Lie group is a torus.

Proof: [Ada69], 2.32. □

Definition 2.14

A maximal torus in G is a toral subgroup T such that there is no toral subgroup S of G containing T as a proper subgroup. □

The next proposition gives a survey of the properties of maximal tori:

Proposition and Definition 2.15

(i) There is a maximal torus in G. Each toral subgroup is contained in a maximal torus.

(ii) Two maximal tori of G are conjugated. Consequently they have the same dimension. This dimension is referred to as rank of G.

(iii) Let T be a maximal torus in G and $N_G(T)$ ist normalizer in G. Then $N_G(T)/T$ is a finite group and is called the (analytic) Weyl group of G (belonging to T).

(iv) The canonic homomorphism $i^* : R(G) \to R(T)$ is an isomorphism onto the subring $R(T)^{W(G)}$ consisting of the $W(G)$–invariant elements.

Proof: [Ada69], 4.8, 2.23 and [BD85], IV(1.4), VI(2.1) □

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In the next propositions we assume T to be a fixed maximal torus in G. Let $\mathfrak{t}_0$ be the Lie algebra of T, $\mathfrak{t} = \mathfrak{t}_o \otimes \mathbb{C}$ be the complexified Lie algebra of T.

Remark 2.16
We can understand the elements of $W(G)$ in an algebraic sense, i.e. as self mappings of $\mathfrak{t}$ or $\mathfrak{t}_0$. ([Kna96], 4.54) □

Definition 2.17

(i) A multiplicative character of T is a continuous homomorphism $\xi : T \to S^1$. ([Kna96], 4.32)

(ii) An element $\mu \in \mathfrak{t}^*$ is called analytically integral if there ist a multiplicative character ξ_μ of T with $\xi_\mu(\exp H) = e^{\mu(H)}$ for all $H \in \mathfrak{t}_0$. ([Kna96], 4.58)

Remark 2.18
An element $\mu \in \mathfrak{t}^*$ is analytically integral iff $\mu(H) \in 2\pi i\mathbb{Z}$ for all $H \in \mathfrak{t}_0$ with $\exp H = 1$. *([Kna96], 4.58)* □

Proposition 2.19
Let $\mu \in \mathfrak{t}^*$ be analytically integral. For all $w \in W(G)$ the element $\mu \circ w$ is analytically integral. Furthermore, there is an element ρ of the representation ring of G with

$$\chi_\rho(\exp H) = \sum_{\mu' \in \mu W(G)} e^{\mu'(H)} \text{ for all } H \in \mathfrak{t}_0.$$

Proof: The term on the right hand side is $W(G)$-invariant. (Prop. 2.15(iv)) □

Proposition and Definition 2.20

(i) Let V be a complex s–dimensional representation of G. As a complex representation of T V decomposes in one–dimensional subrepresentations $V_{\beta_1}, \dots, V_{\beta_s}$ with $\{\beta_1, \dots, \beta_s\}$ a $W(G)$–invariant set of analytically integral elements and T acting on V_{β_j} by $g(v) = e^{\beta_j(g)} \cdot v$ for all

$g \in T$ and $v \in V_{\beta_j}$. The elements $\beta_1, \dots, \beta_s$ are called the weights of the representation V.

(ii) *Let V be a real s–dimensional representation of G. As a complex representation of T V decomposes in an r–dimensional trivial subrepresentation V_0 and two–dimensional subrepresentations $V_{\beta_1}, \dots, V_{\beta_d}$ with $s - r = 2d$ even, $\{\pm\beta_1, \dots, \pm\beta_d\}$ a $W(G)$–invariant set of analytically integral elements and T acting on V_{β_j} by the realization of the complex representation given by $g(v) = e^{\beta_j(g)} \cdot v$ for all $g \in T$ and $v \in V_{\beta_j}$. The elements $\pm\beta_1, \dots, \pm\beta_d$ are called the weights of the representation V.*

(iii) *Let $V = \mathfrak{g}_0$ be the adjoint representation of G. Then $V_0 = \mathfrak{t}_0$. The weights of the adjoint representation $\mathfrak{g}_0$ are called the roots of G.*

All roots are purely imaginary on $\mathfrak{t}_0$. ([Kna96], 4.58) □

Definition 2.21

Let (L_i) be a base of $\mathfrak{t}_0^*$. A total ordering on $\mathfrak{t}_0^*$ is given by

$$\sum \lambda_i L_i > \sum \mu_i L_i \iff \lambda_1 = \mu_1, \dots, \lambda_{r-1} = \mu_{r-1}, \lambda_r > \mu_r \text{ for a } r \geq 1. \square$$

Definition 2.22

A positive root is called simple if it is not representable as the sum of two positive roots. □

Notation 2.23

(i) *The root system of G is denoted by $\Sigma(G)$.*

(ii) *$\Sigma^+(G) = \{\alpha \in \Sigma(G) \mid \alpha > 0\}$ is referred to as the system of the positive roots of G with respect to the given ordering.* □

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There is a close relation between the structure theory and representation theory of G and the corresponding theories of the Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}$ of G. Hence we are going to collect results of the theory of Lie algebras.

For the sake of simplicity we define all concepts for the complex case.

$\mathfrak{g}$ and $\mathfrak{g}_0$ being the Lie algebras of the compact Lie group G we do not need the theory of Lie algebras in its full generality.

So we may introduce some objects by properties which are more convenient than the properties which have to be used in the general context.

Definition 2.24

(i) For subsets $\mathfrak{a}, \mathfrak{b}$ *of* $\mathfrak{g}$ *we define*

$$[\mathfrak{a}, \mathfrak{b}] = \{[A, B] \,|\, A \in \mathfrak{a}, B \in \mathfrak{b}\}.$$

In a similar way $\mathfrak{a} + \mathfrak{b}$ *is defined.*

(ii) A vector subspace $\mathfrak{a}$ *of* $\mathfrak{g}$ *with* $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ *is called a Lie subalgebra.*

(iii) A Lie subalgebra $\mathfrak{a}$ *of* $\mathfrak{g}$ *with* $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ *is called an ideal of* $\mathfrak{g}$. □

Example 2.25

(i) If $\mathfrak{a}, \mathfrak{b}$ *are ideals of* $\mathfrak{g}$ *then* $\mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a} + \mathfrak{b}$ *and* $[\mathfrak{a}, \mathfrak{b}]$ *are ideals of* $\mathfrak{g}$*, too.*

(ii) The ideal $[\mathfrak{g}, \mathfrak{g}]$ *is called the commutator ideal of* $\mathfrak{g}$.

(iii) $\mathfrak{z}_{\mathfrak{g}} = \{H_1 \in \mathfrak{g} \,|\, [H_1, H_2] = 0 \text{ for all } H_2 \in \mathfrak{g}\}$

is an ideal of $\mathfrak{g}$ *and called the center of* $\mathfrak{g}$.

Proof: [Kna96], 1.7. □

Proposition and Definition 2.26

Under our assumptions it holds $\mathfrak{g} = \mathfrak{z}_\mathfrak{g} \oplus [\mathfrak{g}, \mathfrak{g}]$.

$[\mathfrak{g}, \mathfrak{g}]$ *is semisimple in the sense of Lie algebra theory and is called the semisimple part of* $\mathfrak{g}$.

G, $\mathfrak{g}$ *and* $\mathfrak{g}_0$ *are called semisimple if* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. *This is equivalent to the finiteness of* $Z(G)$ *and to the triviality of* $\mathfrak{z}_\mathfrak{g}$. *([Kna96], 4.25, 4.29)* □

Proposition and Definition 2.27

(i) $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ *with*

$$B(H_1, H_2) = trace(ad(H_1) \circ ad(H_2))$$

is a symmetric bilinear form on $\mathfrak{g}$. B *is called the Killing–Form of* G.

(ii) The restriction of B *to the semisimple part* $[\mathfrak{g}, \mathfrak{g}]$ *is non-singular. ([Kna96], 1.42)*

(iii) $\mathfrak{t}' = \mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$ *is a Cartan algebra of* $[\mathfrak{g}, \mathfrak{g}]$. *([Kna96], 2.13)*

(iv) $\mathfrak{t}'^*$ *can be understood as subset of* $\mathfrak{t}^*$. *Elements of* $\mathfrak{t}'^*$ *map elements of* $\mathfrak{z}_\mathfrak{g}$ *to* 0. *([Kna96], p.200)* □

Proposition and Definition 2.28

Let B *be the Killing form of* G. *The restriction of* B *to* $\mathfrak{t}'$ *is non-singular. The induced bilinear form on* $\mathfrak{t}'^*$ *is denoted by* $\langle\ ,\ \rangle$. *The restriction of* $\langle\ ,\ \rangle$ *to the real subspace* $\mathfrak{t}_0 \cap \mathfrak{t}'$ *is negative definite; the restriction to the real subspace* $i(\mathfrak{t}_0 \cap \mathfrak{t}')$ *is positive definite. ([Kna96], p.207)* □

Definition 2.29

(i) An element $\mu \in \mathfrak{t}'^*$ *is callled algebraically integral with respect to* G*, if the following condition holds:*

$$\frac{2\,\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z} \text{ for all } \alpha \in \Sigma(G).$$

([Kna96], 4.59)

(ii) An element $\mu \in \mathfrak{t}'^*$ *is called algebraically semiintegral with respect to* G *if* 2μ *is algebraically integral.*

Remark 2.30

(i) Analytically integral elements of $\mathfrak{t}'^*$ *are algebraically integral. ([Kna96], 4.59)*

(ii) If G *is semisimple with trivial center then each analytically integral element is an integral linear combination of the roots. ([Kna96], 4.68)* □

Proposition and Definition 2.31

(i) $w \in W(G)$ *permutes the roots of* G*. ([Ada69], 4.37)*

(ii) For an element $w \in W(G)$ *the identity* $\det(w) = (-1)^{\left|\{\alpha\in\Sigma^+(G)\,|\,\alpha w<0\}\right|}$ *is valid. We denote* $\det(w)$ *with sign*(w)*. sign:* $W(G) \to \{\pm 1\}$ *is a homomorphism of groups. ([Kna96], II.12.21–23 or [Hil82a], (1.5) and the remark before (3.2))*

(iii) The bilinear form $\langle\, ,\, \rangle$ *on* $\mathfrak{t}'^*$ *is invariant with respect to the operation of* $W(G)$*. ([Kna96], 2.62)* □

Proposition and Definition 2.32

We define

$$\delta = \frac{1}{2} \sum_{\alpha\in\Sigma^+(G)} \alpha.$$

δ *is algebraically integral with respect to* G*.*

Proof: [Kna96], 2.69 und 4.62. □

Proposition 2.33

$$\sum_{w\in W(G)} sign(w)e^{\delta w(H)} = \prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}\right)$$

for all $H \in \mathfrak{t}'^*$. ([Kna96], 5.111) □

Definition 2.34
Let $Q^+ = \{H \in \mathfrak{t}_0 \,|\, \alpha(H) > 0$ for all $\alpha \in \Sigma^+(G)\}$. Q^+ is a maximal convex subset of $Q = \{H \in \mathfrak{t}_0 \,|\, \alpha(H) \neq 0$ for all $\alpha \in \Sigma^+(G)\}$. We refer to it as the positive Weyl chamber or fundamental chamber of G. □

Proposition 2.35
Let μ be an algebraically semiintegral element with respect to G and $\langle\, ,\, \rangle$ be the bilinear form on $\mathfrak{t}^*$ induced by the Killing form. Then the following identity holds:

$$\lim_{\substack{H\to 0\\ H\in\mathfrak{t}'}} \frac{\sum\limits_{w\in W(G)} sign(w)e^{\mu(w(H))}}{\prod\limits_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}\right)} = \prod_{\alpha\in\Sigma^+(G)} \frac{\langle\mu,\alpha\rangle}{\langle\delta,\alpha\rangle}. \qquad (*)$$

Remark 2.36
For a simple root α the equation $2\langle\delta,\alpha\rangle = \langle\alpha,\alpha\rangle > 0$ holds. Another positive root is a sum of simple roots. Hence the denominator on the right hand side of $(*)$ is different from 0. ([Kna96], 2.69)

Proof of Proposition 2.35:
Case 1: μ is algebraically integral and an element of the closure of the positive Weyl chamber.
The statement is a corollary of the Weyl dimension formula ([BH58], sect. 3.4.).

Case 2: μ is algebraically integral.

By [BH58], sect. 2.7 there is an element $w_0 \in W(G)$, such that μw_0 is an element of the closed positive Weyl chamber. Case 1 yields:

$$\lim_{H\to 0} \frac{\sum_{w\in W(G)} \operatorname{sign}(w) e^{\mu(w(H))}}{\prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}\right)}$$

$$\overset{2.31(ii)}{=} \lim_{H\to 0} \operatorname{sign}(w_0) \frac{\sum_{w\in W(G)} \operatorname{sign}(w) e^{\mu(w_0(w(H)))}}{\prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}\right)}$$

$$\overset{\text{Case 1}}{=} \operatorname{sign}(w_0) \prod_{\alpha\in\Sigma^+(G)} \frac{\langle \mu w_0, \alpha\rangle}{\langle \delta, \alpha\rangle}$$

$$\overset{2.31(iii)}{=} \operatorname{sign}(w_0) \prod_{\alpha\in\Sigma^+(G)} \frac{\left\langle \mu, \alpha w_0^{-1}\right\rangle}{\langle \delta, \alpha\rangle}$$

$$\overset{2.31(ii)}{=} \prod_{\alpha\in\Sigma^+(G)} \frac{\langle \mu, \alpha\rangle}{\langle \delta, \alpha\rangle}.$$

Case 3: μ ist algebraically semiintegral.

$$\lim_{H\to 0} \frac{\sum_{w\in W(G)} \operatorname{sign}(w) e^{\mu(w(H))}}{\prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}\right)}$$

$$= \lim_{H\to 0} \frac{\sum_{w\in W(G)} \operatorname{sign}(w) e^{\mu(w(2H))}}{\prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(2H)} - e^{-\frac{1}{2}\alpha(2H)}\right)}$$

$$= \lim_{H\to 0} \frac{\sum_{w\in W(G)} \operatorname{sign}(w) e^{2\mu(w(H))}}{\prod_{\alpha\in\Sigma^+(G)} \left(e^{\alpha(H)} - e^{-\alpha(H)}\right)}$$

$$
\begin{aligned}
&= \lim_{H\to 0} \frac{\displaystyle\sum_{w\in W(G)} \operatorname{sign}(w) e^{2\mu(w(H))}}{\displaystyle\prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} + e^{-\frac{1}{2}\alpha(H)}\right) \prod_{\alpha\in\Sigma^+(G)} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(2H)}\right)} \\
&\overset{\text{Case 2}}{=} \prod_{\alpha\in\Sigma^+(G)} \frac{\langle 2\mu, \alpha\rangle}{2\langle \delta, \alpha\rangle} \\
&= \prod_{\alpha\in\Sigma^+(G)} \frac{\langle \mu, \alpha\rangle}{\langle \delta, \alpha\rangle}. \qquad \square
\end{aligned}
$$

2.3 The topological structure of homogeneous spaces

We assume G to be a compact connected Lie group and U to be a closed subgroup of G with maximal rank. Let T be a maximal torus of U.

Notation 2.37

(i) *We denote the Lie algebra of G by $\mathfrak{g}_0$, its complexification $\mathfrak{g}_0 \otimes \mathbb{C}$ by $\mathfrak{g}$.*

In the same way let $\mathfrak{t}_0$ be the Lie algebra of T and $\mathfrak{t}$ its complexification. We define $\mathfrak{t}'$ by $\mathfrak{t}' = \mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$.

(ii) *Let $\Sigma(G)$ be the root system of G, $\Sigma^+(G)$ be a system of positive roots of G.*

Let $\Sigma(U) \subset \Sigma(G)$ be the root system of U and $\Sigma^+(U) = \Sigma(U) \cap \Sigma^+(G)$.

(iii) *The elements of $\Psi = \Sigma^+(G) \setminus \Sigma^+(U)$ are called the positive complementary roots of G with respect to U.*

(iv) *We refer to the Weyl group of G by $W(G)$ and to the Weyl group of U by $W(U)$.*

Remark 2.38

(i) *$\Sigma^+(U)$ is a system of positive roots of U.*

(ii) *We can understand $W(U)$ as a subset of $W(G)$.* $\square$

Definition 2.39

U operates by adjunction on the tangent space $T_U(G/U)$ with weights $\alpha \in \Sigma^+(G) \setminus \Sigma^+(U) = \Psi$. This representation is called the isotropy representation $\iota : U \to Aut^+(T_U(G/U))$. $\square$

Proposition 2.40

The tangent bundle of G/U has an U-structure via ι.

Proof: [BH58] (Prop. 7.5) and the subsequent remark. □

★

Proposition 2.41

(i) G/U is a simply connected manifold with dimension $2|\Psi|$. An orientation of G/U is given by an orientation of $T_U(G/U)$.

(ii) The maps $\tilde{g} : G/U \to G/U$ with $g \in G$ and $xU \mapsto gxU$ are orientation preserving diffeomorphisms of G/U.

Proof:

(i) We consider the homotopy sequence to the principal bundle $G \to G/U$:

$$\cdots \to \pi_1(U) \to \pi_1(G) \to \pi_1(G/U) \to \pi_0(U) \to \pi_0(G) \to \cdots$$

Due to the connectedness of G and U statement (i) is equivalent to the surjectivity of $\pi_1(U) \to \pi_1(G)$. $\pi_1(T) \to \pi_1(G)$ is surjective since G/T is simply connected ([Ada69], Lemma 5.54). Therefore $\pi_1(U) \to \pi_1(G)$ is surjective.

(ii) If g_t is a path from e to g then $\tilde{g}_t$ is an isotopy from id to $\tilde{g}$. □

The root space decomposition of G is given by

$$T_e(G) = \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \bigoplus_{\alpha \in \Sigma^+(G)} \mathfrak{g}_{0,\alpha},$$

whereby the real representation of T on $\mathfrak{g}_{0,\alpha}$ is equal to the realization of the complex one–dimensional representation of T given by the root α.

The root space decomposition of U is given by

$$T_e(U) = \mathfrak{u}_0 = \mathfrak{t}_0 \oplus \bigoplus_{\alpha \in \Sigma^+(U)} \mathfrak{g}_{0,\alpha}. \quad \square$$

Definition 2.42

We orient $T_U(G/U)$ and therefore G/U by identifying the root spaces $\mathfrak{g}_{0,\alpha}$, $\alpha \in \Psi$, with copies of $\mathbb{C}$. $\square$

★

Proposition and Definition 2.43

We define

$$\delta = \frac{1}{2} \sum_{\alpha \in \Sigma^+(G)} \alpha,$$

$$\delta' = \frac{1}{2} \sum_{\alpha \in \Sigma^+(U)} \alpha,$$

$$\tilde{\delta} = \frac{1}{2} \sum_{\alpha \in \Psi} \alpha.$$

If G is simply connected then the following statements are equivalent:

(i) G/U has a Spin structure.

(ii) δ' is integral with respect to G.

(iii) $\tilde{\delta}$ is integral with respect to G.

Proof: [HS90], p.327. $\square$

Chapter 3

Hilbert polynomials of homogeneous spaces

We want to apply the results of section 1.3 and Proposition 2.35 to determine Hilbert polynomials of homogeneous spaces.

3.1 An S^1-action on G/U

We choose a regular one-parameter subgroup $\lambda : S^1 \to T$ in G within the positive Weyl chamber, i.e. the differential of λ at 1 is a linear map $d_1\lambda : \mathbb{R} = \mathfrak{s}^1 \to \mathfrak{t}_0$ with $d_1\lambda(1)$ being a member of the positive Weyl chamber.

Since we can eliminate any possibility of misunderstandings we denote both the homomorphism $S^1 \to T$ and its differential $\mathbb{R} \to \mathfrak{t}_0$ by λ.

By means of λ every T-action can be restricted to an S^1-action.

In particular, there is a canonical S^1-action on G/U induced by λ.

Proposition 3.1

The fixed point set of this S^1-action on G/U is given by

$$(G/U)^\lambda = \{gU \in G/U \mid g \in N_G(T)\}.$$

Remark 3.2

Given $g_1, g_2 \in N_G(T)$ the identity $g_1U = g_2U$ holds iff $g_1 \in g_2N_U(T)$. Hence $(G/U)^\lambda$ is a finite set in bijection to the sets $N_G(T)/N_U(T)$ and $(N_G(T)/T)/(N_U(T)/T) \cong W(G)/W(U)$.
Furthermore, given $g \in N_G(T)$ the left coset gU depends only on the left coset in $W(G)/W(U)$ represented by g.
So the expressions wU and $[w]U$ for $w \in W(G)$ and $[w] \in W(G)/W(U)$ are well defined.

With the notations intoroduced above we can reformulate Proposition 3.1:

Corollary 3.3

The fixed points of the S^1-action on G/U are the distinct points $[w]U \in G/U$ with $[w] \in W(G)/W(U)$.

Proof of 3.1 and 3.3: [HS90], sect. 2.5 □

3.2 Equivariant vector bundles over homogeneous spaces

Let (V, ρ) be a real or complex representation of U. Via the canonical U-principal bundle $G \to G/U$ this representation induces a vector bundle $G \times_\rho V$. G acts equivariantly on the canonical principal bundle and consequently on the associated vector bundle. The same is true for any closed subgroup of G.

Proposition 3.4

The T-action on the fibre $(G \times_\rho V)_{wU}$ with $w \in W(G)$ is equivalent to $\rho \circ w^{-1}$ ([HS90]). The same is true for all closed subgroups of T.

In particular, the weights of the T-action on the tangent space in the fixed point wU are given by $\alpha \circ w^{-1}$ with $\alpha \in \Psi$. Here we understand w^{-1} to act on the Lie algebra $\mathfrak{t}_0$.

Proof: Let $g \in N_G(T)$ represent the element $w \in W(G)$. All elements of $G \times_\rho V$ being in the fibre over the fixed point $[g] \in G/U$ is representable in the form $[g, v]$ with $v \in V$ uniquely determined. For $t \in T$ and $v \in V$ we have:

$$\begin{aligned} t[g,v] &= [tg, v] \\ &= [g\,\underbrace{g^{-1}tg}_{\in T}, v] \\ &= [g, \rho(g^{-1}tg)v] \\ &= [g, \rho(w^{-1}(t))(v))] \quad \square \end{aligned}$$

★

Proposition 3.5

Let $G \supset U$ be connected Lie groups with same rank. Moreover let (L, η) be a complex one-dimensional representation of U with weight γ and (K, ζ) be an complex r-dimensional representation of U with weights $\mu_1, \ldots, \mu_r$.

In addition let (V, φ) be a k-dimensional real representation of U with positive weights $\beta_1, \ldots, \beta_s$. We assume that the trivial one-dimensional representation of U appears as subrepresentation of V with multiplicity $2l$ or $2l+1$.

All weights have to be counted according to their multiplicity.

Furthermore we assume

$$c_1(G \times_\eta L) \equiv w_2(G \times_\varphi V) + w_2(G/U) \ mod\ 2 \qquad (*).$$

Then we have the identity

$$2^{\lfloor \frac{k}{2} \rfloor} \left(e^{\frac{1}{2} c_1(G \times_\eta L)} ch(G \times_\zeta K) \left(\prod_i \cosh\left(\frac{y_i}{2}\right) \right) \hat{\mathcal{A}}(G/U) \right) \left[G/U \right]$$
$$= \quad 2^l \cdot \sum_\rho \sum_{\varepsilon:\{1,\ldots,s\} \to \{\pm 1\}} \frac{\prod_{\alpha \in \Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta' + \frac{1}{2} \sum_\sigma \varepsilon(\sigma)\beta_\sigma, \alpha \right\rangle}{\prod_{\alpha \in \Sigma^+(G)} \langle \delta, \alpha \rangle}.$$

In this formula we use the notations:

$p(G \times_\varphi V) = \prod_i (1 + y_i^2)$ is the total Pontrjagin class of $G \times_\varphi V$.

$\langle\, ,\, \rangle$ is the bilinear form on $\mathfrak{t}'^*$ induced by the Killing form on G. δ is the half sum of the positive weights of G, δ' is the half sum of the positive weights of U.

$\Sigma^+(G)$ is the set of positive weights of G, Ψ the set of positive complementary weights of G with respect to U.

Remark 3.6

(i) The fulfillment of condition $(*)$ *can be read off the weights of the representations. ([BH58], sect. 11). For the sake of simplicity we want to give the criterion just in the case of* $V = 0$*:*

$$c_1(G \times_\eta L) \equiv w_2(G/U) \ \mathit{mod}\ 2$$

$$\iff \frac{1}{2}\left(\gamma + \sum_{\alpha \in \Psi} \alpha\right) \ \text{is analytically integral with respect to } G.$$

(ii) If U *is the centralizer of a toral subgroup of* G *then a theorem due to Wang says that* G/U *possesses a homogeneous complex structure. ([Wan54].)*

In the case $L = \Lambda^{|\Psi|}(T_U(G/U))$, K *one-dimensional and* μ_1 *being positive and orthogonal to the roots of* U *the formular in 3.5 coincides with the formula given in [Sug79], sect.2. (see also [BH59], sect. 24.7.)*

Proof of Satz 3.5:

We define $E = G \times_\eta L$ and $F = G \times_\varphi V$ and $D = G \times_\zeta K$. Furthermore, let $T(G/U)$ be the tangent bundle of G/U. S^1 acts on these spaces as in section 3.1. $T(G/U)$ is equivariant isomorphic zu $G \times_\iota T_U(G/U)$. Due to the connectedness of U, F ist orientable.

The positive weights of $T_U(G/U))$ are the complementary roots $\alpha \in \Psi$.

Table of notations

representation	filed	dim.	weights	ass.bundle
(L, η)	$\mathbb{C}$	1	γ	E
(V, φ)	$\mathbb{R}$	$2s + 2l$ or $2s + 2l + 1$	$\pm\beta_1, \ldots, \pm\beta_s, 0, 0, \ldots, 0$	F
(K, ζ)	$\mathbb{C}$	r	$\mu_1, \ldots, \mu_r$	D
$T_U(G/U)$	$\mathbb{R}$	$2\|\Psi\|$	complementary roots	T(G/U)

Due to 3.4 the fibres of those bundles over the fixed point wU are representations of T with weights $\gamma w^{-1}, \pm\beta_j w^{-1}, \mu_j w^{-1}, \alpha w^{-1}$.
The results of section 1.3 cause the next identity being valid for all members x of a dense subset of $\mathbb{R}$:

$$\Gamma(G/U, E, F, D)(e^{2\pi ix}) \quad = \quad \frac{1}{|W(U)|} \sum_{w \in W(G)} \gamma(wU, E, F, K)(e^{2\pi ix}) \qquad (**)$$

with

$$\begin{aligned} &\gamma(wU, E, F, D)\left(e^{2\pi ix}\right) \\ &\quad = \quad e^{\frac{1}{2}(\gamma w^{-1}\lambda(x))} \cdot \left(\sum_{\rho} e^{\mu_\rho w^{-1}\lambda(x)}\right) \\ &\qquad \cdot 2^l \prod_{\sigma} \left(e^{\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))} + e^{-\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}\right) \\ &\qquad \cdot \prod_{\alpha \in \Psi} \left(e^{-\frac{1}{2}(\alpha w^{-1}\lambda(x))} - e^{\frac{1}{2}(\alpha w^{-1}\lambda(x))}\right)^{-1}. \end{aligned}$$

By means of the results in section 3.2 $(**)$ is equal to the formula in section 1.3. We just have to take care about the sign coming from the orientations of the fixed points. We have to orient them in such a way that all rotation numbers of the tangent space are positive. So each α with negative αw^{-1} gives a change of the orientation.

This fact was taken into account because for each such α the denominator above has another sign than the corresponding term in section 1.3.

In order to apply Proposition 2.35 we transform the term:

$$
\begin{aligned}
&\Gamma(G/U,E,F,D)(e^{2\pi ix})\\
= \quad & \frac{1}{|W(U)|}\sum_{w\in W(G)}\gamma(wU,E,F,K)(e^{2\pi ix})\\
= \quad & \frac{1}{|W(U)|}\sum_{w\in W(G)} e^{\frac{1}{2}(\gamma w^{-1}\lambda(x))}\cdot\left(\sum_\rho e^{\mu_\rho w^{-1}\lambda(x)}\right)\\
&\cdot 2^l\prod_\sigma\left(e^{\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}+e^{-\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}\right)\\
&\cdot\prod_{\alpha\in\Psi}\left(e^{-\frac{1}{2}(\alpha w^{-1}\lambda(x))}-e^{\frac{1}{2}(\alpha w^{-1}\lambda(x))}\right)^{-1}\\
= \quad & (-1)^{|\Psi|}\frac{1}{|W(U)|}\sum_{w\in W(G)} e^{\frac{1}{2}(\gamma w^{-1}\lambda(x))}\cdot\left(\sum_\rho e^{\mu_\rho w^{-1}\lambda(x)}\right)\\
&\cdot 2^l\prod_\sigma\left(e^{\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}+e^{-\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}\right)\\
&\cdot\prod_{\alpha\in\Psi}\left(e^{\frac{1}{2}(\alpha w^{-1}\lambda(x))}-e^{-\frac{1}{2}(\alpha w^{-1}\lambda(x))}\right)^{-1}\\
= \quad & (-1)^{|\Psi|}\frac{1}{|W(U)|}\sum_{w\in W(G)} e^{\frac{1}{2}(\gamma w^{-1}\lambda(x))}\cdot\left(\sum_\rho e^{\mu_\rho w^{-1}\lambda(x)}\right)\\
&\cdot 2^l\prod_\sigma\left(e^{\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}+e^{-\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))}\right)\\
&\cdot\prod_{\alpha\in\Sigma^+(U)}\left(e^{\frac{1}{2}(\alpha w^{-1}\lambda(x))}-e^{-\frac{1}{2}(\alpha w^{-1}\lambda(x))}\right)\\
&\cdot\prod_{\alpha\in\Sigma^+(G)}\left(e^{\frac{1}{2}(\alpha w^{-1}\lambda(x))}-e^{-\frac{1}{2}(\alpha w^{-1}\lambda(x))}\right)^{-1}\\
\overset{2.31(ii)}{=} \quad & (-1)^{|\Psi|}\frac{1}{|W(U)|}\sum_{w\in W(G)}\operatorname{sign}(w)e^{\frac{1}{2}(\gamma w^{-1}\lambda(x))}\cdot\left(\sum_\rho e^{(\mu_\rho w^{-1}\lambda(x))}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \cdot 2^l \prod_{\sigma} \left(e^{\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))} + e^{-\frac{1}{2}(\beta_\sigma w^{-1}\lambda(x))} \right) \\
& \cdot \prod_{\alpha \in \Sigma^+(U)} \left(e^{\frac{1}{2}(\alpha w^{-1}\lambda(x))} - e^{-\frac{1}{2}(\alpha w^{-1}\lambda(x))} \right) \\
& \cdot \prod_{\alpha \in \Sigma^+(G)} \left(e^{\frac{1}{2}(\alpha\lambda(x))} - e^{\frac{1}{2}(-\alpha\lambda(x))} \right)^{-1} \\
\overset{2.33}{=} \quad & (-1)^{|\Psi|} \frac{1}{|W(U)|} \sum_{w \in W(G)} \operatorname{sign}(w) e^{\frac{1}{2}(\gamma w^{-1}\lambda(x))} \cdot \left(\sum_{\rho} e^{\mu_\rho w^{-1}\lambda(x)} \right) \\
& \cdot 2^l \sum_{\varepsilon:\{1,\dots,s\}\to\{\pm 1\}} \left(\prod_{\sigma} e^{\frac{1}{2}\varepsilon(\sigma)(\beta_\sigma w^{-1}\lambda(x))} \right) \\
& \cdot \sum_{w' \in W(U)} \left(\operatorname{sign}(w') e^{\delta' w'^{-1} w^{-1}\lambda(x)} \right) \\
& \cdot \prod_{\alpha \in \Sigma^+(G)} \left(e^{\frac{1}{2}(\alpha\lambda(x))} - e^{-\frac{1}{2}(\alpha\lambda(x))} \right)^{-1} \\
= \quad & (-1)^{|\Psi|} \frac{2^l}{|W(U)|} \sum_{w' \in W(U)} \operatorname{sign}(w') \sum_{\rho} \prod_{\alpha \in \Sigma^+(G)} \left(e^{\frac{1}{2}(\alpha\lambda(x))} - e^{-\frac{1}{2}(\alpha\lambda(x))} \right)^{-1} \\
& \cdot \sum_{\varepsilon} \sum_{w \in W(G)} \operatorname{sign}(w) e^{\left(\frac{1}{2}\gamma + \mu_\rho + \delta' w'^{-1} + \frac{1}{2}\sum_\sigma \varepsilon(\sigma)\beta_\sigma\right) w^{-1}\lambda(x)}
\end{aligned}
$$

$x \to 0$ causes $\lambda(x) \to 0$. So Proposition 2.35 gives:

$$
\begin{aligned}
& 2^{\lfloor \frac{k}{2} \rfloor} \left(e^{\frac{1}{2}c_1(G\times_\eta L)} ch(G \times_\zeta K) \left(\prod_i \cosh\left(\frac{y_i}{2}\right) \right) \hat{\mathcal{A}}(G/U) \right) \Big[G/U \Big] \\
= \quad & (-1)^{|\Psi|} \Gamma(G/U, E, F, D)(1) \\
= \quad & \frac{2^l}{|W(U)|} \cdot \prod_{\alpha \in \Sigma^+(G)} \langle \delta, \alpha \rangle^{-1} \\
& \cdot \sum_{w' \in W(U)} \operatorname{sign}(w') \sum_{\rho} \sum_{\varepsilon} \Bigg[\\
& \qquad \prod_{\alpha \in \Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta' w'^{-1} + \frac{1}{2} \sum_{\sigma} \varepsilon(\sigma)\beta_\sigma, \alpha \right\rangle \Bigg] \\
\overset{2.31(ii)}{=} \quad & \frac{2^l}{|W(U)|} \cdot \prod_{\alpha \in \Sigma^+(G)} \langle \delta, \alpha \rangle^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \sum_{w' \in W(U)} \sum_{\rho} \sum_{\varepsilon} \prod_{\alpha \in \Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta' w'^{-1} + \frac{1}{2}\sum_{\sigma} \varepsilon(\sigma)\beta_\sigma, \alpha w'^{-1} \right\rangle \\
\overset{2.31(iii)}{=} \quad & \frac{2^l}{|W(U)|} \cdot \prod_{\alpha \in \Sigma^+(G)} \langle \delta, \alpha \rangle^{-1} \\
& \cdot \sum_{w' \in W(U)} \sum_{\rho} \sum_{\varepsilon} \prod_{\alpha \in \Sigma^+(G)} \left\langle \frac{1}{2}\gamma w' + \mu_\rho w' + \delta' + \frac{1}{2}\sum_{\sigma} \varepsilon(\sigma)\beta_\sigma w', \alpha \right\rangle \\
= \quad & \frac{2^l}{|W(U)|} \cdot \prod_{\alpha \in \Sigma^+(G)} \langle \delta, \alpha \rangle^{-1} \\
& \cdot \sum_{w' \in W(U)} \sum_{\rho} \sum_{\varepsilon} \prod_{\alpha \in \Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta' + \frac{1}{2}\sum_{\sigma} \varepsilon(\sigma)\beta_\sigma, \alpha \right\rangle \\
= \quad & 2^l \cdot \sum_{\rho} \sum_{\varepsilon:\{1,\dots,s\}\to\{\pm 1\}} \frac{\prod_{\alpha \in \Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta' + \frac{1}{2}\sum_{\sigma} \varepsilon(\sigma)\beta_\sigma, \alpha \right\rangle}{\prod_{\alpha \in \Sigma^+(G)} \langle \delta, \alpha \rangle}
\end{aligned}
$$

In the penultimate step we made use of the fact that the sets $\{\gamma\}$, $\{\pm\beta_1, \dots, \pm\beta_s\}$ and $\{\mu_1, \dots, \mu_r\}$ are $W(U)$-invariant. They are weights of representations of U. □

3.3 Non-immersion theorems for homogeneous spaces

We apply the results of the preceding section to a special situation: We set $V = 0$ and substitute K by $K(t) = \psi_t(K)$ with K a virtual complex representation of U and t an integer. We associate to ψ_t the Adams operation.

Proposition 3.7

Let $G \supset U$ be connected Lie groups with same rank. Moreover, let (L, η) be a complex one-dimensional representation of U with weight γ and $(\mu_1, \ldots, \mu_r)$ be a $W(U)$–invariant family of analytically integral elements.

Due to 2.15(iv) there are complex representations $(K_1, \zeta_1), \ldots, (K_s, \zeta_s)$ and integers $n_1, \ldots, n_s$ with $\sum_{\rho=1}^{r} e^{\mu_\rho} = \sum_{\sigma=1}^{s} n_\sigma \chi_{K_\sigma}$.

We assume $\frac{1}{2}\left(\gamma + \sum_{\alpha\in\Psi} \alpha\right)$ to be analytically integral with respect to G. This implies:

(i) $$\hat{A}\left(G/U, \frac{c_1(G \times_\eta L)}{2}, z\right) = \frac{\sum_{\rho=1}^{r} \prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha \rangle}.$$

(ii) $$H(t) = \frac{\sum_{\rho=1}^{r} \prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + t\mu_\rho + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha \rangle}.$$

Here we use the following notations:

By H we denote the Hilbert polynomial of G/U associated with $c_1(G \times_\eta L)$ and $z = \sum_{\sigma=1}^{s} n_\sigma ch\,(G \times_{\zeta_\sigma} K_\sigma)$.

$\langle\, ,\, \rangle$ is the bilinear form on $\mathfrak{t}'^*$ induced by the Killing form of G. By δ we denote the half sum of the positive weights of G, by δ' the half sum of the positive weights of U.
$\Sigma^+(G)$ is the set of positive weights of G, Ψ the set of positive complementary weights of G with respect to U.

Remark 3.8

(i) If G/U is an almost complex homogeneous space with invariant almost complexe structure then L may be chosen as bundle of determinant forms of $T_H(G/U)$ (considered as a complex vector space). The weight of this bundle is given by the sum of the positive complementary weights. Hence we get $\frac{1}{2}\left(\gamma + \sum_{\alpha\in\Psi} \alpha\right) = \sum_{\alpha\in\Psi} \alpha$ and $\frac{1}{2}\gamma + \delta' = \delta$.

(ii) If G/U is a homogeneous Spin-manifold we may choose L as trivial bundle. In this situation we have $\frac{1}{2}\left(\gamma + \sum_{\alpha\in\Psi} \alpha\right) = \frac{1}{2}\sum_{\alpha\in\Psi} \alpha$ and $\frac{1}{2}\gamma + \delta' = \delta'$.

(iii) Linked with Proposition 1.5 the proposition above yields a non-immersion theorem for homogeneous spaces. In the case $r = 1$ we have $H(t)$ in a factorized form. So the value $\nu_2\left(H\left(\frac{1}{2}\right)\right)$ can be calculated in an easy way. In the case $r > 1$ we have to make more efforts. □

Proof of 3.7:

For all $\sigma \in \{1, \ldots, s\}$ let $\mu_1^{(\sigma)}, \ldots, \mu_{r(\sigma)}^{(\sigma)}$ be the set of weights of K_σ. For all analytically integral elements μ we have

$$|\{\rho \in \{1, \ldots, r\} \,|\, \mu_\rho = \mu\}| = \sum_{\sigma=1}^{s} \sum_{\substack{\rho=1 \\ \mu_\rho^{(\sigma)}=\mu}}^{r(\sigma)} n_\sigma.$$

For all $\sigma \in \{1, \ldots, s\}$ Proposition 3.5 causes

$$\hat{A}\left(G/U, \frac{c_1(G \times_\eta L)}{2}, ch\left(G \times_{\zeta_\sigma} K_\sigma\right)\right) = \frac{\sum_{\rho=1}^{r(\sigma)} \prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho^{(\sigma)} + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha \rangle}.$$

Using $z = \sum_\sigma n_\sigma ch\,(G \times_{\zeta_\sigma} K_\sigma)$ this leads to

$$\begin{aligned}
&\hat{A}\left(G/U, \frac{c_1(G\times_\eta L)}{2}, z\right) \\
&= \sum_{\sigma=1}^{s} n_\sigma \hat{A}\left(G/U, \frac{c_1(G\times_\eta L)}{2}, ch\,(G\times_{\zeta_\sigma} K_\sigma)\right) \\
&= \sum_{\sigma=1}^{s} n_\sigma \sum_{\rho=1}^{r(\sigma)} \frac{\prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho^{(\sigma)} + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha\rangle} \\
&= \sum_{\mu\, analyt.int.} \sum_{\sigma=1}^{s} \sum_{\substack{\rho=1 \\ \mu_\rho^{(\sigma)}=\mu}}^{r(\sigma)} n_\sigma \frac{\prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha\rangle} \\
&= \sum_{\mu\, analyt.int.} \sum_{\substack{\rho=1 \\ \mu_\rho=\mu}}^{r} \frac{\prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha\rangle} \\
&= \sum_{\rho=1}^{r} \frac{\prod_{\alpha\in\Sigma^+(G)} \left\langle \frac{1}{2}\gamma + \mu_\rho + \delta', \alpha \right\rangle}{\prod_{\alpha\in\Sigma^+(G)} \langle \delta, \alpha\rangle}.
\end{aligned}$$

This establishes part (i) of the statement. For integers t part (ii) is implied by part (i) and the additivity of the Adams operation. □

Chapter 4

Applications

4.1 Preliminaries

The essential tool for the determination of the numbers $\nu_2(q)$ is the next lemma. First we need some notations:

Notation 4.1
For $n \in \mathbb{N}$ we define:

$\alpha(n) =$ *number of the digit* 1 *in the dyadic representation of* n;

$$\alpha_1(n) = \sum_{\kappa=0}^{n-1} \alpha(\kappa).$$

Lemma 4.2
$\nu_2(n!) = n - \alpha(n)$. $\square$

For the purpose of applications we note the subsequent identitities:

Proposition 4.3

$$\prod_{i=1}^{n+1}(2i-1) = \frac{(2n+1)!}{2^n \cdot n!}$$

$$\prod_{i=1}^{n}(2i) = 2^n \cdot n!$$

$$\prod_{1\le i<j\le n}(j-i) = \prod_{j=1}^{n}(j-1)!$$

$$\prod_{1\le i<j\le n}(j+i) = \prod_{j=1}^{n}\frac{(2j-1)!}{j!}$$

$$\prod_{1\le i<j\le n}(j+i+1) = \prod_{j=1}^{n}\frac{(2j)!}{(j+1)!}$$

$$\prod_{1\le i<j\le n}(j+i-1) = \prod_{j=1}^{n}\frac{(2j-2)!}{(j-1)!}$$

$$\prod_{1\le i<j\le n}(j+i-2) = \prod_{j=1}^{n-1}\frac{(2j-1)!}{(j-1)!}$$

$$\alpha(n) = \begin{cases} \alpha\left(\frac{n}{2}\right), & n \text{ even} \\ \alpha(n-1)+1, & n \text{ odd} \end{cases}$$

$$\alpha_1(n) = \begin{cases} 2\alpha_1\left(\frac{n}{2}\right)+\frac{n}{2}, & n \text{ even} \\ \alpha_1(n-1)+\alpha(n-1), & n \text{ odd} \end{cases}$$

$$\nu_2\left(\prod_{1\le i<j\le n}(j-i)\right) = \frac{n(n-1)}{2} - \alpha_1(n)$$

$$\nu_2\left(\prod_{1\le i<j\le n}(j+i)\right) = \frac{n(n-3)}{2} + \alpha(n)$$

$$\nu_2\left(\prod_{1\le i<j\le n}(j+i+1)\right) = \frac{(n-2)(n+1)}{2} + \alpha(n+1)$$

$$\nu_2\left(\prod_{1\le i<j\le n}(j+i-1)\right) = \frac{n(n-1)}{2}$$

$$\nu_2\left(\prod_{1\le i<j\le n}(j+i-2)\right) = \frac{(n-2)(n-1)}{2}$$

Remark 4.4

We will use the formulas above without an explicit reference.

Remark 4.5

(i) α_1 *is monotonely increasing. If* n *is a power of* 2 *than we have* $\alpha_1(n) = \frac{n}{2}\log_2(n)$.

(ii) By means of the identities above we can calculate $\alpha(n)$ *and* $\alpha_1(n)$ *with time exposure* $O(\log n)$ *and* $O((\log n)^2)$*, respectively.* □

Values of α_1 will be used only in the form $\alpha_1(n) - \alpha_1(k) - \alpha_1(n-k)$ with $0 \le k \le n$. So the following propositions are of interest:

Proposition 4.6

Let n *be a natural number and* $k \in \{0, \dots, n\}$*. Then:*

$$0 \le \alpha_1(n) - \alpha_1(k) - \alpha_1(n-k) \le \min\{2^\rho \mid \rho \in \mathbb{N} \text{ and } 2^\rho \ge n\} - 1 < 2n - 1.$$

Proof:

For natural numbers p and ρ we define

$$\alpha^{(\rho)}(p) = \text{ digit with value } 2^\rho \text{ in the dyadic representation of } p;$$

$$\alpha_1^{(\rho)}(p) = \sum_{\kappa=0}^{p-1} \alpha^{(\rho)}(\kappa).$$

Claim 1: Let $\rho \in \mathbb{N}$. For all natural numbers p there are uniquely determined natural numbers $s(p), r(p)$ with $p = s(p) \cdot 2^{\rho+1} + r(p)$ and $0 \le r(p) < 2^{\rho+1}$. With these notations we have

$$\alpha_1^{(\rho)}(p) = s(p) \cdot 2^\rho + \max\{0, r(p) - 2^\rho\}.$$

Proof of Claim 1:

For $p = 0$ the statement is trivial.

Let $p > 0$.

Case 1: $0 \le r(p-1) < 2^\rho$.

This gives $\alpha^{(\rho)}(p-1) = 0$, $s(p) = s(p-1)$ and $r(p) = r(p-1)+1$. We obtain

$$\begin{aligned}
&s(p) \cdot 2^\rho + \max\{0, r(p) - 2^\rho\} \\
&\quad = \quad s(p) \cdot 2^\rho \\
&\quad = \quad s(p-1) \cdot 2^\rho + \max\{0, r(p-1) - 2^\rho\} \\
&\quad = \quad \alpha_1^{(\rho)}(p-1) \\
&\quad = \quad \alpha_1^{(\rho)}(p).
\end{aligned}$$

Case 2: $2^\rho \le r(p-1) < 2^{\rho+1} - 1$.

This gives $\alpha^{(\rho)}(p-1) = 1$, $s(p) = s(p-1)$ and $r(p) = r(p-1)+1$. We obtain

$$\begin{aligned}
&s(p) \cdot 2^\rho + \max\{0, r(p) - 2^\rho\} \\
&\quad = \quad s(p) \cdot 2^\rho + r(p) - 2^\rho \\
&\quad = \quad s(p-1) \cdot 2^\rho + \max\{0, r(p-1) - 2^\rho\} + 1 \\
&\quad = \quad \alpha_1^{(\rho)}(p-1) + 1 \\
&\quad = \quad \alpha_1^{(\rho)}(p).
\end{aligned}$$

Case 3: $r(p-1) = 2^{\rho+1} - 1$.

This gives $\alpha^{(\rho)}(p-1) = 1$, $s(p) = s(p-1) + 1$ and $r(p) = 0$. We obtain

$$\begin{aligned}
&s(p) \cdot 2^\rho + \max\{0, r(p) - 2^\rho\} \\
&\quad = \quad s(p-1) \cdot 2^\rho + 2^\rho \\
&\quad = \quad s(p-1) \cdot 2^\rho + \max\{0, r(p-1) - 2^\rho\} + 1 \\
&\quad = \quad \alpha_1^{(\rho)}(p-1) + 1 \\
&\quad = \quad \alpha_1^{(\rho)}(p).
\end{aligned}$$

So Claim 1 is proved.

<u>Claim 2:</u> Let $\rho \in \mathbb{N}$. This causes

$$0 \leq \alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \leq 2^\rho.$$

<u>Proof of Claim 2:</u>

We have

$$n = (s(k) + s(n-k)) \cdot 2^{\rho+1} + r(k) + r(n-k).$$

Case 1: $0 \leq r(k) < 2^\rho$ and $0 \leq r(n-k) < 2^\rho$.
This gives $0 \leq r(k) + r(n-k) < 2^{\rho+1}$, hence $s(n) = s(k) + s(n-k)$ and $r(n) = r(k) + r(n-k)$. We obtain

$$\begin{aligned} &\alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \\ &= \max\{r(n) - 2^\rho, 0\} \\ &\in \{0, \ldots, 2^\rho - 1\}. \end{aligned}$$

Case 2: $2^\rho \leq r(k) < 2^{\rho+1}$ and $0 \leq r(n-k) < 2^\rho$.
This gives $2^\rho \leq r(k) + r(n-k) < 2^{\rho+1}$ or $2^{\rho+1} \leq r(k) + r(n-k) < 3 \cdot 2^\rho$.
In the first subcase we have $s(n) = s(k) + s(n-k)$ and $r(n) = r(k) + r(n-k)$. We obtain

$$\begin{aligned} &\alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \\ &= r(n) - r(k) \\ &= r(n-k) \\ &\in \{0, \ldots, 2^\rho - 1\}. \end{aligned}$$

In the second subcase we have $s(n) = s(k) + s(n-k) + 1$ and $r(n) = r(k) + r(n-k) - 2^{\rho+1}$. We obtain

$$\begin{aligned} &\alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \\ &= 2^\rho - (r(k) - 2^\rho) = 2^{\rho+1} - r(k) \\ &\in \{1, \ldots, 2^\rho\}. \end{aligned}$$

Case 3: $2^\rho \leq r(n-k) < 2^{\rho+1}$ and $0 \leq r(k) < 2^\rho$.

The statement is analogous to Case 2.

Case 4: $2^\rho \leq r(k) < 2^{\rho+1}$ and $2^\rho \leq r(n-k) < 2^{\rho+1}$.

This gives $2^{\rho+1} \leq r(k)+r(n-k) < 3\cdot 2^\rho$ or $3\cdot 2^\rho \leq r(k)+r(n-k) < 2^{\rho+2}$.

In both subcases we have $s(n) = s(k)+s(n-k)+1$

and $r(n) = r(k)+r(n-k)-2^{\rho+1}$.

In the first subcase we obtain

$$\begin{aligned}
&\alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \\
&= 2^\rho - (r(k)-2^\rho) - (r(n-k)-2^\rho) \\
&= 3\cdot 2^\rho - r(k) - r(n-k) \\
&\in \{1,\ldots,2^\rho\}.
\end{aligned}$$

Im the second subcase we obtain

$$\begin{aligned}
&\alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \\
&= 2^\rho + (r(n)-2^\rho) - (r(k)-2^\rho) - (r(n-k)-2^\rho) \\
&= 0.
\end{aligned}$$

So Claim 2 is proved.

<u>Claim 3:</u> Let $\rho \in \mathbb{N}$ and $\rho \geq \log_2(n)$. Then for all $\kappa \in \{0,\ldots,n-1\}$ $\alpha^{(\rho)}(\kappa)$ is vanishing. In particular, for all $\rho \geq \log_2(n)$ the term $\alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k)$ is vanishing.

<u>Proof of Claim 3:</u>

Else we got $n > \kappa \geq 2^\rho \geq 2^{\log_2(n)} = n$.

With $\rho_0 = \max\{\rho \in \mathbb{Z} \,|\, \rho < \log_2(n)\}$ we obtain $\rho_0 + 1 = \min\{\rho \in \mathbb{Z} \,|\, \rho \geq \log_2(n)\}$. This leads to

$$
\begin{aligned}
&\alpha_1(n) - \alpha_1(k) - \alpha_1(n-k) \\
&= \sum_{\rho=0}^{\rho_0} \alpha_1^{(\rho)}(n) - \alpha_1^{(\rho)}(k) - \alpha_1^{(\rho)}(n-k) \\
&\leq \sum_{\rho=0}^{\rho_0} 2^\rho \\
&= 2^{\rho_0+1} - 1 \\
&= \min\{2^\rho \mid \rho \in \mathbb{Z} \text{ and } \rho \geq \log_2(n)\} - 1 \\
&< 2n - 1 \quad \square
\end{aligned}
$$

Lemma 4.7

$0 \leq \alpha(p) \leq \log_2(p) + 1$ *for all* $p \in \mathbb{N}$.

Proof:
We assume $p = \sum_{\rho=0}^{R} a_\rho 2^\rho$ with $a_\rho \in \{0,1\}$ for all $\rho \in \{0, \ldots, R-1\}$ and $a_R = 1$. This leads to $\alpha(p) \leq R + 1 = \log_2(2^R) + 1 \leq \log_2(p) + 1$. $\square$

Proposition 4.8
Let $g, h : \{0, 1, \ldots, n\} \to \mathbb{Z}$ *be given by*

$g(k) = 4k(n-k) + 2\alpha_1(k) + 2\alpha_1(n-k)$ *and*

$h(k) = 8k(n-k) - 2k + 2\alpha_1(k) + 2\alpha_1(n-k).$

Then the following statements are true:

(i) $g(k) = g(n-k)$ *if* $k \leq \frac{n}{2}$*;*

(ii) $g(k) \geq g(k-1)$ *if* $0 < k \leq \frac{n}{2} - \frac{\log_2(n) - 1}{4}$*;*

(iii) $h(k) \geq h(n-k)$ *if* $k \leq \frac{n}{2}$*;*

(iv) $h(k) \geq h(k-1)$ *if* $0 < k \leq \frac{n}{2} - \frac{\log_2(n)-2}{8}$.

Proof:

$$\begin{aligned} & g(k) - g(k-1) \\ &= -8k + 4n + 4 + 2\alpha(k-1) - 2\alpha(n-k) \\ &\geq -8k + 4n + 4 - 2\log_2(n) - 2 \\ &= -8k + 4n + 2 - 2\log_2(n); \end{aligned}$$

$$\begin{aligned} & h(k) - h(k-1) \\ &= -16k + 8n + 6 + 2\alpha(k-1) - 2\alpha(n-k) \\ &\geq -16k + 8n + 6 - 2\log_2(n) - 2 \\ &= -16k + 8n + 4 - 2\log_2(n). \quad \square \end{aligned}$$

★

In order to calculate favourable parameters while dealing with the immersion problem of complex flag manifolds we have to solve the following minimizing problem:

Lemma 4.9

Le $n \geq 2$ *be an integer and the mapping*

$R_n : \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_\kappa \neq x_\lambda \, fuer \, \kappa \neq \lambda\} \to \mathbb{N}_0$

given by $R_n(x_1, \ldots, x_n) = \sum_{1 \leq \kappa < \lambda \leq n} \nu_2(x_\lambda - x_\kappa)$.
R_n *has a minimum in* $(1, \ldots, n)$.

Proof:

First we note that for any permutation $\sigma \in \mathfrak{S}_n$ the values $R_n(x_1,\dots,x_n)$, $R_n(x_1+1,x_2+1,\dots,x_n+1)$ and $R_n(x_{\sigma(1)},\dots,x_{\sigma(n)})$ coincide.

We perform the proof by induction. If $n=2$ then the statement is trivial.

We assume $n\geq 3$ and $(x_1,\dots,x_n)$ to be a minimum of R_n. Without loss of generality we assume $x_1,\dots,x_m$ to be odd and $x_{m+1},\dots,x_n$ to be even. Furthermore we may suppose $m\geq \frac{n}{2}$. Else we could increase all components by 1.

Let $x_\kappa = 2y_\kappa - 1$ for $\kappa \leq m$ and $x_\kappa = 2y_\kappa$ for $\kappa > m$.

We have $m\neq n$, else we would obtain the contradiction

$$\begin{aligned}
&R_n(x_1,\dots,x_n)\\
&= \sum_{1\leq\kappa<\lambda\leq n} \nu_2(2y_\lambda - 1 - 2y_\kappa + 1)\\
&= \binom{n}{2} + \sum_{1\leq\kappa<\lambda\leq n} \nu_2(y_\lambda - y_\kappa)\\
&= \binom{n}{2} + R_n(y_1,\dots,y_n)\\
&> R_n(y_1,\dots,y_n).
\end{aligned}$$

We have

$$\begin{aligned}
&R_n(x_1,\dots,x_n)\\
&= \sum_{1\leq\kappa<\lambda\leq m} \nu_2(x_\lambda - x_\kappa) + \sum_{m+1\leq\kappa<\lambda\leq n} \nu_2(x_\lambda - x_\kappa)\\
&= \sum_{1\leq\kappa<\lambda\leq m} \nu_2(2y_\lambda - 2y_\kappa) + \sum_{m+1\leq\kappa<\lambda\leq n} \nu_2(2y_\lambda - 2y_\kappa)\\
&= \binom{m}{2} + \sum_{1\leq\kappa<\lambda\leq m} \nu_2(y_\lambda - y_\kappa) + \binom{n-m}{2} + \sum_{m+1\leq\kappa<\lambda\leq n} \nu_2(y_\lambda - y_\kappa)
\end{aligned}$$

$$
\begin{aligned}
&= \binom{m}{2} + R_m(y_1, \ldots, y_m) + \binom{n-m}{2} + R_{n-m}(y_{m+1}, \ldots, y_n) \\
&\geq \binom{m}{2} + R_m(1, \ldots, m) + \binom{n-m}{2} + R_{n-m}(1, \ldots, n-m) \\
&= R_n(1, 3, \ldots, 2m-1, 2, 4, \ldots, 2(n-m))
\end{aligned}
$$

Consequently $(1, 3, \ldots, 2m-1, 2, 4, \ldots, 2(n-m))$ is a minimum, too.

If we assume $m > \frac{n+1}{2}$ we would get $2m-3 > n-2$ and $n-m < m-1$. This would lead to the contradiction

$$
\begin{aligned}
&R_n(1, 3, \ldots, 2m-3, 2m-1, 2, 4, \ldots, 2(n-m)) \\
-&R_n(1, 3, \ldots, 2m-3, 2(n-m+1), 2, 4, \ldots, 2(n-m)) \\
&\quad = \sum_{1 \leq \kappa \leq m-1} \nu_2(2m-1-2\kappa+1) - \sum_{1 \leq \kappa \leq n-m} \nu_2(2(n-m+1)-2\kappa) \\
&\quad = \sum_{1 \leq \kappa \leq m-1} \nu_2(2(m-\kappa)) - \sum_{1 \leq \kappa \leq n-m} \nu_2(2(n-m+1-\kappa)) \\
&\quad = \sum_{1 \leq \kappa \leq m-1} \nu_2(2\kappa) - \sum_{1 \leq \kappa \leq n-m} \nu_2(2\kappa) \\
&\quad = \sum_{n-m+1 \leq \kappa \leq m-1} \nu_2(2\kappa) \\
&\quad = \sum_{n-m+1 \leq \kappa \leq m-1} (\nu_2(\kappa)+1) \\
&\quad > 0.
\end{aligned}
$$

Therefore $m = \lfloor \frac{n+1}{2} \rfloor$ and $(1, 3, \ldots, 2\lfloor \frac{n+1}{2} \rfloor - 1, 2, 4, \ldots, 2\lfloor \frac{n}{2} \rfloor)$ and $(1, 2, 3, 4, \ldots, n)$ are minima. $\square$

★

While dealing with the immersion problem for the quaternional flag manifolds we have to consider a determinant of a matrix of the form:

Proposition 4.10

For all $n \in \mathbb{N}$ the following equation is true:

$$\begin{aligned}
&\det\left((1-x_\kappa)^{2\lambda-1}+(1+x_\kappa)^{2\lambda-1}+(1-y_\kappa)^{2\lambda-1}+(1+y_\kappa)^{2\lambda-1}\right)_{1\le\kappa,\lambda\le n}\\
&= 2\frac{(2n-1)!}{(n-1)!}\cdot\det\left(x_\kappa^{2\lambda-2}+y_\kappa^{2\lambda-2}\right)_{1\le\kappa,\lambda\le n}.
\end{aligned}$$

Proof: We prove by induction for $l \in \{1, \ldots, n\}$:

$$\begin{aligned}
&\det\left((1-x_\kappa)^{2\lambda-1}+(1+x_\kappa)^{2\lambda-1}+(1-y_\kappa)^{2\lambda-1}+(1+y_\kappa)^{2\lambda-1}\right)_{1\le\kappa,\lambda\le n}\\
&= 2^l\prod_{\nu=1}^{l}(2\nu-1)\\
&\cdot\det\begin{pmatrix}\overbrace{x_\kappa^{2\lambda-2}+y_\kappa^{2\lambda-2}}^{\lambda\in\{1,\ldots,l\}}\\ \underbrace{(1-x_\kappa)^{2\lambda-1}+(1+x_\kappa)^{2\lambda-1}+(1-y_\kappa)^{2\lambda-1}+(1+y_\kappa)^{2\lambda-1}}_{\lambda\in\{l+1,\ldots,n\}}\end{pmatrix}_{1\le\kappa\le n}.
\end{aligned}$$

In the case $l = 1$ the statement is trivial. We assume the statement to be true for all $l \in \{1, \ldots, n-1\}$. This leads to

$$\begin{aligned}
&\det\left((1-x_\kappa)^{2\lambda-1}+(1+x_\kappa)^{2\lambda-1}+(1-y_\kappa)^{2\lambda-1}+(1+y_\kappa)^{2\lambda-1}\right)_{1\le\kappa,\lambda\le n}\\
&= 2^l\prod_{\nu=1}^{l}(2\nu-1)\\
&\cdot\det\begin{pmatrix}\overbrace{x_\kappa^{2\lambda-2}+y_\kappa^{2\lambda-2}}^{\lambda\in\{1,\ldots,l\}}\\ (1-x_\kappa)^{2l+1}+(1+x_\kappa)^{2l+1}+(1-y_\kappa)^{2l+1}+(1+y_\kappa)^{2l+1}\\ \underbrace{(1-x_\kappa)^{2\lambda-1}+(1+x_\kappa)^{2\lambda-1}+(1-y_\kappa)^{2\lambda-1}+(1+y_\kappa)^{2\lambda-1}}_{\lambda\in\{l+2,\ldots,n\}}\end{pmatrix}_{1\le\kappa\le n}
\end{aligned}$$

$$\overset{(*)}{=} \quad 2^l \prod_{\nu=1}^{l} (2\nu - 1) \cdot \det \begin{pmatrix} \overbrace{x_\kappa^{2\lambda-2} + y_\kappa^{2\lambda-2}}^{\lambda\in\{1,\dots,l\}} \\ \sum_{j=0}^{2l+1} \binom{2l+1}{j} \left((-1)^j + 1\right) \left(x_\kappa^j + y_\kappa^j\right) \\ \underbrace{(1-x_\kappa)^{2\lambda-1} + (1+x_\kappa)^{2\lambda-1} + (1-y_\kappa)^{2\lambda-1} + (1+y_\kappa)^{2\lambda-1}}_{\lambda\in\{l+2,\dots,n\}} \end{pmatrix}_{1\le\kappa\le n}$$

$$= \quad 2^l \prod_{\nu=1}^{l} (2\nu - 1) \cdot \det \begin{pmatrix} \overbrace{x_\kappa^{2\lambda-2} + y_\kappa^{2\lambda-2}}^{\lambda\in\{1,\dots,l\}} \\ \sum_{k=0}^{l} \binom{2l+1}{2k} 2 \left(x_\kappa^{2k} + y_\kappa^{2k}\right) \\ \underbrace{(1-x_\kappa)^{2\lambda-1} + (1+x_\kappa)^{2\lambda-1} + (1-y_\kappa)^{2\lambda-1} + (1+y_\kappa)^{2\lambda-1}}_{\lambda\in\{l+2,\dots,n\}} \end{pmatrix}_{1\le\kappa\le n}$$

$$\overset{(**)}{=} \quad 2^{l+1} \prod_{\nu=1}^{l+1} (2\nu - 1) \cdot \det \begin{pmatrix} \overbrace{x_\kappa^{2\lambda-2} + y_\kappa^{2\lambda-2}}^{\lambda\in\{1,\dots,l+1\}} \\ \underbrace{(1-x_\kappa)^{2\lambda-1} + (1+x_\kappa)^{2\lambda-1} + (1-y_\kappa)^{2\lambda-1} + (1+y_\kappa)^{2\lambda-1}}_{\lambda\in\{l+2,\dots,n\}} \end{pmatrix}_{1\le\kappa\le n}$$

The identity $(*)$ is implied by the binomic formula. The identity $(**)$ is due to the invariance of the determinant to elementary transformations.

The statement for $l = n$ yields the proposition. $\square$

4.2 Non-immersion theorems for complex flag manifolds

Notation 4.11

(i) *Let* $n_1, \ldots, n_s$ *be positive integers.*

(ii) $n = \sum_{\sigma=1}^{s} n_\sigma$, $l_\sigma = 1 + \sum_{j=1}^{\sigma-1} n_j$, $m_\sigma = \sum_{j=1}^{\sigma} n_j$.

(iii) *Let* $\tau : \{1, \ldots, n\} \to \{1, \ldots, s\}$ *be given by*
$\tau(\lambda) = \sigma \iff l_\sigma \le \lambda \le m_\sigma$.

(iv) $G = U(n)$, $U = U(n_1) \times \cdots \times U(n_s)$.

(v) $T = U(1) \times U(1) \times \cdots \times U(1)$.

Example 4.12
For $s = 3$, $n_1 = 1$, $n_2 = 4$ *and* $n_3 = 3$ *we obtain:*

σ	n_σ	l_σ	m_σ	$\tau^{-1}(\sigma)$
1	1	1	1	$\{1\}$
2	4	2	5	$\{2,3,4,5\}$
3	3	6	8	$\{6,7,8\}$

Proposition 4.13
$U(n_1) \times \cdots \times U(n_s)$ *is the centralizer* $Z(S)$ *of the toral subgroup* $S = \left\{ diag\left(e^{ir_{\tau(1)}}, \ldots, e^{ir_{\tau(n)}}\right) \middle| r_1, \ldots, r_s \in \mathbb{R} \right\}$ *in* $U(n)$.

Proof: [Tor68], p.25 ff. $\square$

Proposition 4.14

(i) *Due to Remark 3.6(ii) G and U fulfill the prerequisites of Remark 3.8(i). T is a maximal torus of G and U.*

(ii) *In a canonical way $\mathfrak{g}_0$ can be understood way as the Lie algebra $\mathfrak{u}(n)$ of skew hermitian complex $n \times n$-matrices. ([Ada69], 5.17(i).)*

Due to [Kna96], I.15.4 $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ can be identified with the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of complex $n \times n$-matrices.

Hence $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}(n, \mathbb{C})$.

$\mathfrak{t}'^*$ is the span of the linear maps L_λ ($\lambda = 1, \ldots, n$) given by $L_\lambda(D) = D_{\lambda\lambda}$ modulo the relation $L_1 + \cdots + L_n = 0$.

(iii) *The Weyl group $W(G)$ ist isomorphic to $\mathfrak{S}_n$. The Weyl group $W(U)$ ist isomorphic to $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_s}$. ([Ada69], 5.17(i).)*

(iv) *The Killing form of G induces the bilinear form $\langle\, , \rangle$ on $\mathfrak{t}'^*$ given by*

$$\left\langle \sum_{1\le\lambda\le n} a_\lambda L_\lambda, \sum_{1\le\kappa\le n} b_\kappa L_\kappa \right\rangle = \frac{1}{2n}\left(\sum_{1\le\kappa\le n} a_\kappa b_\kappa - \frac{1}{n}\left(\sum_{1\le\lambda\le n} a_\lambda\right)\left(\sum_{1\le\kappa\le n} b_\kappa\right)\right).$$

([FH96], p.213)

(v) *A system of positive roots of G ist given by*

$$\Sigma^+(G) = \{L_\lambda - L_\kappa \,|\, 1 \le \lambda < \kappa \le n\}.$$

([Ada69], 5.28)

The half sum of the positive roots is equal to

$$\delta = \frac{1}{2}\sum_{\lambda=1}^{n}(n-2\lambda+1)L_\lambda.$$

(vi) For integers $\mu_1, \dots, \mu_s$ the element $\sum\limits_{\lambda=1}^{n} \mu_{\tau(\lambda)} L_\lambda \in \mathfrak{t}'^*$ is $W(U)$-invariant and analytically integral. ([Kna96], IV,9.17) $\square$

Given these data the corresponding Hilbert polynomial of G/U is equal to

$$\begin{aligned} H(t) & \\ = \; & \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle \sum\limits_{\nu=1}^{n}\left(t\mu_{\tau(\nu)}+\frac{1}{2}(n-2\nu+1)\right)L_\nu, L_\kappa - L_\lambda\right\rangle}{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle \sum\limits_{\nu=1}^{n}\left(\frac{1}{2}(n-2\nu+1)\right)L_\nu, L_\kappa - L_\lambda\right\rangle} \\ = \; & \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left(\left(t\mu_{\tau(\kappa)}+\frac{1}{2}(n-2\kappa+1)\right)-\left(t\mu_{\tau(\lambda)}+\frac{1}{2}(n-2\lambda+1)\right)\right)}{\prod\limits_{1\le\kappa<\lambda\le n}\left(\frac{1}{2}(n-2\kappa+1)-\frac{1}{2}(n-2\lambda+1)\right)} \\ = \; & \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left(\left(t\mu_{\tau(\kappa)}-\kappa\right)-\left(t\mu_{\tau(\lambda)}-\lambda\right)\right)}{\prod\limits_{1\le\kappa<\lambda\le n}(-\kappa+\lambda)}. \end{aligned}$$

So we obtain

Proposition 4.15

If $\mu_1, \dots, \mu_s$ are integers then there exists an element $z \in ch(G/U)$, such that the Hilbert polynomial associated with $c_1(G/U)$ and z is given by

$$H(t) = \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left(\left(t\mu_{\tau(\kappa)}-\kappa\right)-\left(t\mu_{\tau(\lambda)}-\lambda\right)\right)}{\prod\limits_{1\le\kappa\le n}(\kappa-1)!}.$$

In order to get non immersion results we are interested in those integers $\mu_1, \ldots, \mu_s$ with minimal $\nu_2\left(H\left(\frac{1}{2}\right)\right)$.
Therefore we choose a subset $S \subset \{1, \ldots, s\}$. Let k be the sum $k = \sum_{\sigma \in S} n_\sigma$.
This causes $n - k = \sum_{\sigma \notin S} n_\sigma$.
We determine the minimal value of $\nu_2(H(\frac{1}{2}))$ under the assumption that μ_σ is even iff $\sigma \in S$.
We introduce integers $\gamma_1, \ldots, \gamma_s$ by $\mu_\sigma = 2\gamma_\sigma$ for $\sigma \in S$ and $\mu_\sigma = 2\gamma_\sigma - 1$ for $\sigma \notin S$.

$$
\begin{aligned}
&\nu_2\left(H\left(\frac{1}{2}\right)\right) \\
&= \sum_{\substack{1 \le \kappa < \lambda \le n \\ \tau(\kappa) \in S, \tau(\lambda) \in S}} \nu_2\left(\frac{1}{2}\left(\mu_{\tau(\kappa)} - \mu_{\tau(\lambda)}\right) + \lambda - \kappa\right) \\
&\quad + \sum_{\substack{1 \le \kappa < \lambda \le n \\ \tau(\kappa) \notin S, \tau(\lambda) \notin S}} \nu_2\left(\frac{1}{2}\left(\mu_{\tau(\kappa)} - \mu_{\tau(\lambda)}\right) + \lambda - \kappa\right) \\
&\quad + \sum_{\substack{1 \le \kappa, \lambda \le n \\ \tau(\kappa) \in S, \tau(\lambda) \notin S}} \nu_2\left(\frac{1}{2}\left(\mu_{\tau(\kappa)} - \mu_{\tau(\lambda)}\right) + \lambda - \kappa\right) \\
&\quad - \sum_{1 \le \kappa < \lambda \le n} \nu_2\left(\lambda - \kappa\right) \\
&= \sum_{\substack{1 \le \kappa < \lambda \le n \\ \tau(\kappa) \in S, \tau(\lambda) \in S}} \nu_2\left(\left(\gamma_{\tau(\kappa)} - \kappa\right) - \left(\gamma_{\tau(\lambda)} - \lambda\right)\right) \\
&\quad + \sum_{\substack{1 \le \kappa < \lambda \le n \\ \tau(\kappa) \notin S, \tau(\lambda) \notin S}} \nu_2\left(\left(\gamma_{\tau(\kappa)} - \kappa\right) - \left(\gamma_{\tau(\lambda)} - \lambda\right)\right) \\
&\quad + \sum_{\substack{1 \le \kappa, \lambda \le n \\ \tau(\kappa) \in S, \tau(\lambda) \notin S}} \nu_2\left(-\frac{1}{2} + \left(\gamma_{\tau(\kappa)} - \kappa\right) - \left(\gamma_{\tau(\lambda)} - \lambda\right)\right) \\
&\quad - \frac{n(n-1)}{2} + \alpha_1(n) \\
&= \sum_{\substack{1 \le \kappa < \lambda \le n \\ \tau(\kappa) \in S, \tau(\lambda) \in S}} \nu_2\left(\left(\gamma_{\tau(\kappa)} - \kappa\right) - \left(\gamma_{\tau(\lambda)} - \lambda\right)\right)
\end{aligned}
$$

$$+ \sum_{\substack{1\le\kappa<\lambda\le n \\ \tau(\kappa)\notin S,\tau(\lambda)\notin S}} \nu_2\left(\left(\gamma_{\tau(\kappa)}-\kappa\right)-\left(\gamma_{\tau(\lambda)}-\lambda\right)\right)$$
$$-k(n-k)-\frac{n(n-1)}{2}+\alpha_1(n)$$

Lemma 4.16

In the situation above $\nu_2\left(H\left(\frac{1}{2}\right)\right)$ is minimal for

$$\gamma_\sigma = 1 + m_\sigma + \sum_{\substack{1\le\vartheta<\sigma \\ \vartheta\in S}} n_\vartheta \text{ for } \sigma \in S \text{ and}$$

$$\gamma_\sigma = 1 + m_\sigma + \sum_{\substack{1\le\vartheta<\sigma \\ \vartheta\notin S}} n_\vartheta \text{ for } \sigma \notin S.$$

Proof: Due to lemma 4.9 the sum is minimal if

$\{\gamma_{\tau(\kappa)} - \kappa \,|\, 1 \le \kappa \le n, \tau(\kappa) \in S\} = \{1, 2, \ldots, k\}$ and

$\{\gamma_{\tau(\kappa)} - \kappa \,|\, 1 \le \kappa \le n, \tau(\kappa) \notin S\} = \{1, 2, \ldots, n-k\}$.

By the definion of γ_σ given above this is fulfilled. We show this by induction over the cardinality of S: If $S = \emptyset$, then the statement is trivial. If $S \neq \emptyset$, let $\sigma = \max(S)$ and $S_1 = S \setminus \{\sigma\}$. Thies gives

$$\begin{aligned}
&\{\gamma_{\tau(\kappa)} - \kappa \,|\, 1 \le \kappa \le n, \tau(\kappa) \in S\} \\
&= \{\gamma_{\tau(\kappa)} - \kappa \,|\, 1 \le \kappa \le n, \tau(\kappa) \in S_1\} \cup \{\gamma_\sigma - \kappa \,|\, l_\sigma \le \kappa \le m_\sigma,\} \\
&= \{1, \ldots, k-n_\sigma\} \cup \{1 + m_\sigma + (k-n_\sigma) - \kappa \,|\, l_\sigma \le \kappa \le m_\sigma,\} \\
&= \{1, \ldots, k-n_\sigma\} \\
&\qquad \cup \{1 + m_\sigma + (k-n_\sigma) - m_\sigma, \ldots, 1 + m_\sigma + (k-n_\sigma) - l_\sigma\} \\
&= \{1, \ldots, k-n_\sigma\} \cup \{1 + k - n_\sigma, \ldots, k\}
\end{aligned}$$

For the second sum we apply the same technique. □

Lemma 4.17

Let $n_1, \ldots, n_s$ be integers and S be a subset of $\{1, \ldots, s\}$. Using the notations $n = \sum_{\sigma=1}^{s} n_\sigma$ and $k = \sum_{\substack{1 \le \sigma \le s \\ \sigma \in S}} n_\sigma$ we obtain:
The minimal value of all values $\nu_2\left(H\left(\frac{1}{2}\right)\right)$ constructed under the restriction μ_σ is even iff $\sigma \in S$ is given by

$$-2k(n-k) + \alpha_1(n) - \alpha_1(k) - \alpha_1(n-k).$$

Proof: The minimal value is equal to

$$\begin{aligned}
& \sum_{1 \le \kappa < \lambda \le k} \nu_2(\lambda - \kappa) + \sum_{1 \le \kappa < \lambda \le n-k} \nu_2(\lambda - \kappa) \\
& -k(n-k) - \frac{n(n-1)}{2} + \alpha_1(n) \\
= \; & \frac{k(k-1)}{2} - \alpha_1(k) + \frac{(n-k)(n-k-1)}{2} - \alpha_1(n-k) \\
& -\frac{n(n-1)}{2} + \alpha_1(n) - k(n-k) \\
= \; & \alpha_1(n) - \alpha_1(k) - \alpha_1(n-k) - 2k(n-k). \quad \square
\end{aligned}$$

Proposition 4.18

The complex flag manifold $U(n)/U(n_1) \times \cdots \times U(n_s)$ has real dimension $n^2 - \sum_{\sigma=1}^{s} n_\sigma^2$ and can not be immersed in an Euclidean space with dimension

$$4k(n-k) - 2\alpha_1(n) + 2\alpha_1(k) + 2\alpha_1(n-k) - 1.$$

Thereby k is an arbitrary integer which can be represented as sum $\sum_{\sigma \in S} n_\sigma$ with $S \subset \{1, \ldots, s\}$.

Remark 4.19

(i) In the case of complex Grassmannians ($s = 2$, $n_1 = k$, $n_2 = n - k$) the results coincide with the results given in in [Sug79] and [May97].

(ii) *In [Lam75] there is given a positive result: The complex flag manifold* $U(n)/U(n_1) \times \cdots \times U(n_s)$ *is a* π*–manifold or can be immersed in an Euclidean space with dimension* $n^2 - s$.

(iii) *We obtain good results if we choose* k *to be close to* $\frac{n}{2}$. *(see Proposition 4.8)* $\square$

4.3 Non-immersion theorems for quaternional flag manifolds

Notation 4.20

(i) *Let* $n_1, \ldots, n_s$ *be positive integers.*

(ii) $n = \sum_{\sigma=1}^{s} n_\sigma$, $l_\sigma = 1 + \sum_{j=1}^{\sigma-1} n_j$, $m_\sigma = \sum_{j=1}^{\sigma} n_j$.

(iii) *Let* $\tau : \{1, \ldots, n\} \to \{1, \ldots, s\}$ *be given by*
$\tau(\lambda) = \sigma \iff l_\sigma \leq \lambda \leq m_\sigma$.

(iv) $G = Sp(n)$, $U = Sp(n_1) \times \cdots \times Sp(n_s)$.

(v) $T = U(1) \times U(1) \times \cdots \times U(1)$.

Proposition 4.21

(i) G *and* U *fulfill the prerequisites of Remark 3.8(ii).* T *is a maximal torus of* G *and* U*.*

(ii) $\mathfrak{g}_0$ *can be understood as the Lie algebra* $\mathfrak{sp}(n)$ *of skew Hermitian quaternional* $n \times n$*-matrices. ([BD85], I.2.19) Due to [Kna96], pp.35-36,* $\mathfrak{g}_0$ *can be identified with the Lie agebra* $\mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(2n)$*.*

$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ *can be identified with the Lie algebra* $\mathfrak{sp}(n, \mathbb{C})$*.*

This yields the simplicity of $\mathfrak{g}$ *and*

$$\mathfrak{t} = \mathfrak{t}' = \left\{ \begin{pmatrix} D_1 & 0 \\ 0 & -D_1 \end{pmatrix} \middle| D_1 \text{ is a diagonal matrix in } \mathfrak{gl}(n, \mathbb{C}) \right\}.$$

$\mathfrak{t}^*$ *is the span of the linear maps* L_λ $(\lambda = 1, \ldots, n)$ *given by* $L_\lambda(D) = D_{\lambda\lambda}$.

(iii) *The Weyl-group $W(G)$ is isomorphic to the semidirect product of $\mathfrak{S}_n$ and $\{\pm 1\}^n$. The operation of $\mathfrak{S}_n$ on $\{\pm 1\}^n$ is given by the standard operation on the set of indices.*

The Weyl group $W(U)$ is isomorphic to the semidirect product of $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_s}$ and $\{\pm 1\}^n$. ([BD85], IV.3.8)

(iv) *The Killing form of G induces the bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{t}'^*$ given by*

$$\left\langle \sum_{1 \leq \lambda \leq n} a_\lambda L_\lambda, \sum_{1 \leq \kappa \leq n} b_\kappa L_\kappa \right\rangle = \frac{1}{4n+4} \left(\sum_{1 \leq \kappa \leq n} a_\kappa b_\kappa \right).$$

([FH96], p.241)

(v) *A system of positive roots of G is given by*

$$\Sigma^+(G) = \{L_\lambda - L_\kappa,\, L_\lambda + L_\kappa,\, 2L_\lambda \,|\, 1 \leq \lambda < \kappa \leq n\}.$$

([Ada69], 5.28) The half sum is equal to

$$\delta = \sum_{\lambda=1}^{n} (n - \lambda + 1) L_\lambda.$$

(vi) *A system of positive roots of U is given by*

$$\Sigma^+(U) = \bigcup_{\sigma=1}^{s} \{L_\lambda - L_\kappa,\, L_\lambda + L_\kappa,\, 2L_\lambda \,|\, l_\sigma \leq \lambda < \kappa \leq m_\sigma\}.$$

The half sum is equal to

$$\delta' = \sum_{\lambda=1}^{n} (m_{\tau(\lambda)} - \lambda + 1) L_\lambda.$$

(vii) For given integers $\mu_1, \ldots, \mu_s$ the set of analytically integral elements

$$\left\{ \sum_{1 \le \lambda \le n} \varepsilon_\lambda \mu_{\tau(\lambda)} L_\lambda \,\middle|\, \varepsilon_\kappa \in \left\{ \begin{array}{ll} \{\pm 1\}, & \textit{falls } \mu_{\tau(\kappa)} \neq 0 \\ \{1\}, & \textit{falls } \mu_{\tau(\kappa)} = 0 \end{array} \right. \right\} (*)$$

is $W(U)$-invariant. ([Kna96], IV.9.19)

Remark 4.22

In the family

$$\left(\sum_{1 \le \lambda \le n} \varepsilon_\lambda \mu_{\tau(\lambda)} L_\lambda \,\middle|\, \varepsilon_\kappa \in \{\pm 1\},\ \textit{falls } 1 \le \kappa \le n \right)$$

each member of $(*)$ appears exactly $\left(2^{\sum_{\substack{1 \le \sigma \le s \\ \mu_\sigma = 0}} n_\sigma} \right)$*-times.* □

★

Given these data we obtain a Hilbert polynomial of G/U by:

$$2^{\sum_{\substack{1 \le \sigma \le s \\ \mu_\sigma = 0}} n_\sigma} H(t)$$

$$= \pm \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \left[\frac{\prod_{1 \le \kappa \le n} \left\langle \sum_{\nu=1}^{n} (\varepsilon_\nu t \mu_{\tau(\nu)} + m_{\tau(\nu)} - \nu + 1) L_\nu, 2L_\kappa \right\rangle}{\prod_{1 \le \kappa \le n} \left\langle \sum_{\nu=1}^{n} (n - \nu + 1) L_\nu, 2L_\kappa \right\rangle} \right.$$

$$\cdot \frac{\prod_{1 \le \kappa < \lambda \le n} \left\langle \sum_{\nu=1}^{n} (\varepsilon_\nu t \mu_{\tau(\nu)} + m_{\tau(\nu)} - \nu + 1) L_\nu, L_\kappa - L_\lambda \right\rangle}{\prod_{1 \le \kappa < \lambda \le n} \left\langle \sum_{\nu=1}^{n} (n - \nu + 1) L_\nu, L_\kappa - L_\lambda \right\rangle}$$

$$\left. \cdot \frac{\prod_{1\le\kappa<\lambda\le n} \left\langle \sum_{\nu=1}^{n} (\varepsilon_\nu t\mu_{\tau(\nu)} + m_{\tau(\nu)} - \nu + 1) L_\nu, L_\kappa + L_\lambda \right\rangle}{\prod_{1\le\kappa<\lambda\le n} \left\langle \sum_{\nu=1}^{n} (n-\nu+1) L_\nu, L_\kappa + L_\lambda \right\rangle} \right]$$

$$= \pm \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \left[\frac{\prod_{1\le\kappa\le n} 2\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right)}{\prod_{1\le\kappa\le n} 2\,(n-\kappa+1)} \right.$$

$$\cdot \frac{\prod_{1\le\kappa<\lambda\le n} \left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right) + \left(\varepsilon_\lambda t\mu_{\tau(\lambda)} + m_{\tau(\lambda)} - \lambda + 1\right)\right)}{\prod_{1\le\kappa<\lambda\le n} ((n-\kappa+1) + (n-\lambda+1))}$$

$$\left. \cdot \frac{\prod_{1\le\kappa<\lambda\le n} \left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right) - \left(\varepsilon_\lambda t\mu_{\tau(\lambda)} + m_{\tau(\lambda)} - \lambda + 1\right)\right)}{\prod_{1\le\kappa<\lambda\le n} ((n-\kappa+1) - (n-\lambda+1))} \right]$$

$$= \pm \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \left[\prod_{1\le\kappa\le n} \left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right) \right.$$

$$\left. \cdot \prod_{1\le\kappa<\lambda\le n} \left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right)^2 - \left(\varepsilon_\lambda t\mu_{\tau(\lambda)} + m_{\tau(\lambda)} - \lambda + 1\right)^2\right) \right]$$

$$\cdot \prod_{1\le\kappa<\lambda\le n} ((n-\kappa+1) - (n-\lambda+1))^{-1}$$

$$\cdot \prod_{1\le\kappa<\lambda\le n} ((n-\kappa+1) + (n-\lambda+1))^{-1}$$

$$\cdot \prod_{1\le\kappa\le n} (n-\kappa+1)^{-1}$$

$$= \pm \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \left[\det\left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right)^{2\lambda-2} \right)_{1\le\kappa,\lambda\le n} \right.$$

$$\left. \cdot \prod_{1\le\kappa\le n} \left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1\right) \right]$$

$$\cdot \prod_{1\le\kappa<\lambda\le n} (\lambda-\kappa)^{-1} \cdot \prod_{1\le\kappa<\lambda\le n} (2n-\kappa-\lambda+2)^{-1} \cdot \prod_{1\le\kappa\le n} (n-\kappa+1)^{-1}$$

$$= \pm \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \left[\det\left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} \right)_{1\le\kappa,\lambda\le n} \right]$$

$$\cdot \prod_{1\le\kappa<\lambda\le n} (\lambda-\kappa)^{-1} \cdot \prod_{1\le\kappa<\lambda\le n} (\kappa+\lambda)^{-1} \cdot \prod_{1\le\kappa\le n} \kappa^{-1}$$

$$= \pm \prod_{1\le\kappa\le n} (\kappa-1)!^{-1} \cdot \prod_{1\le\kappa\le n} \kappa! \cdot \prod_{1\le\kappa\le n} (2\kappa-1)!^{-1} \cdot n!^{-1}$$

$$\cdot \det\left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} \right)_{1\le\kappa,\lambda\le n}$$

$$= \pm \prod_{1\le\kappa\le n} (2\kappa-1)!^{-1}$$

$$\cdot \det\left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} \right)_{1\le\kappa,\lambda\le n}$$

At this point we are going to perform elementary row transformations within the "σ-th" block for all $\sigma \in \{1, \dots, s\}$, i.e. for $\kappa \in \{l_\sigma, \dots, m_\sigma\}$.

Notation 4.23

For all $r \in \mathbb{Z}$ we introduce a relation $\sim_r$ on the set of $n_\sigma \times n$-matrices:

$A_1 \sim_r A_2 \iff$

For all $(n - n_\sigma) \times n$-matrices B: $\det \begin{pmatrix} A_1 \\ B \end{pmatrix} = \pm 2^r \det \begin{pmatrix} A_2 \\ B \end{pmatrix}$.

This leads to:

$$\left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa + 1 \right)^{2\lambda-1} \right)_{\substack{l_\sigma\le\kappa\le m_\sigma \\ 1\le\lambda\le n}}$$

$$\sim_0 \left((-t\mu_\sigma + \kappa)^{2\lambda-1} + (t\mu_\sigma + \kappa)^{2\lambda-1} \right)_{\substack{1\le\kappa\le n_\sigma \\ 1\le\lambda\le n}}$$

$$\sim_0 \begin{pmatrix} (-t\mu_\sigma + 1)^{2\lambda-1} + (t\mu_\sigma + 1)^{2\lambda-1} \\ (-t\mu_\sigma + 2)^{2\lambda-1} + (t\mu_\sigma + 2)^{2\lambda-1} \\ \underbrace{(-t\mu_\sigma + \kappa)^{2\lambda-1} + (t\mu_\sigma + \kappa)^{2\lambda-1}}_{3\le\kappa\le n_\sigma} \end{pmatrix}_{1\le\lambda\le n}$$

$$\simeq_1 \left(\begin{array}{c} (-t\mu_\sigma + 1)^{2\lambda-1} + (t\mu_\sigma + 1)^{2\lambda-1} + (-t\mu_\sigma + 1)^{2\lambda-1} + (t\mu_\sigma + 1)^{2\lambda-1} \\ (-t\mu_\sigma + 2)^{2\lambda-1} + (t\mu_\sigma)^{2\lambda-1} + (-t\mu_\sigma)^{2\lambda-1} + (t\mu_\sigma + 2)^{2\lambda-1} \\ \underbrace{(-t\mu_\sigma + \kappa)^{2\lambda-1} - (-t\mu_\sigma + \kappa - 2)^{2\lambda-1} - (t\mu_\sigma + \kappa - 2)^{2\lambda-1} + (t\mu_\sigma + \kappa)^{2\lambda-1}}_{3\leq\kappa\leq n_\sigma \qquad\qquad 1\leq\lambda\leq n} \end{array} \right)$$

$$\sim_0 \Big((-t\mu_\sigma + \kappa)^{2\lambda-1} + (t\mu_\sigma - \kappa + 2)^{2\lambda-1} + (-t\mu_\sigma - \kappa + 2)^{2\lambda-1} + (t\mu_\sigma + \kappa)^{2\lambda-1} \Big)_{\substack{1\leq\kappa\leq n_\sigma \\ 1\leq\lambda\leq n}}$$

$$\sim_0 \Big(\left(-t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)} + 1\right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} - \kappa + l_{\tau(\kappa)} + 1\right)^{2\lambda-1} + \left(-t\mu_{\tau(\kappa)} - \kappa + l_{\tau(\kappa)} + 1\right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)} + 1\right)^{2\lambda-1} \Big)_{\substack{l_\sigma\leq\kappa\leq m_\sigma \\ 1\leq\lambda\leq n}}$$

In step $\sim_{-1}$ we added the negative of the $(\kappa-2)$-th row to the κ-th row $(\kappa = n_\sigma, \dots, 3)$, doubled the first row and added a null row to the second row. We observe the admissibility of that tranformation in the case of $n_\sigma \in \{1, 2\}$. Consequently the Hilbert polynomial is given by

$$\begin{aligned} & 2^{\sum_{\substack{1\leq\sigma\leq s \\ \mu_\sigma=0}} n_\sigma} H(t) \\ = \;& \pm 2^{-s} \prod_{1\leq\kappa\leq n} (2\kappa-1)!^{-1} \\ & \cdot \det \Big(\left(-t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)} + 1\right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} - \kappa + l_{\tau(\kappa)} + 1\right)^{2\lambda-1} \\ & \quad + \left(-t\mu_{\tau(\kappa)} - \kappa + l_{\tau(\kappa)} + 1\right)^{2\lambda-1} + \left(t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)} + 1\right)^{2\lambda-1} \Big)_{1\leq\kappa,\lambda\leq n} \\ \overset{4.10}{=} \;& \pm 2^{1-s} \det \Big(\left(-t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)}\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)}\right)^{2\lambda-2} \Big)_{1\leq\kappa,\lambda\leq n} \\ & \cdot (n-1)!^{-1} \cdot \prod_{1\leq\kappa\leq n-1} (2\kappa-1)!^{-1} \end{aligned}$$

We summarize the outcomes:

Proposition 4.24

Let $\mu_1, \ldots, \mu_s$ be integers. Then there exists an element $z \in ch(G/U)$ such that the Hilbert polynomial associated with $0 \in H^2(G/U)$ and z is given by

$$H(t) \;=\; \pm \det\left(\left(-t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)}\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)}\right)^{2\lambda-2}\right)_{1\leq\kappa,\lambda\leq n} \cdot (n-1)!^{-1} \cdot \prod_{1\leq\kappa\leq n-1} (2\kappa-1)!^{-1} \cdot 2^{1-s-\sum_{\substack{1\leq\sigma\leq s\\ \mu_\sigma=0}} n_\sigma} . \quad \square$$

★

In order to obtain non immersion results we set $t = \frac{1}{2}$. Provoked by the results in the complex case we choose the integers $\mu_1, \ldots, \mu_s$ in a similar manner.

Let $S \subset \{1, \ldots, s\}$, $k = \sum_{\sigma\in S} n_\sigma$, $l = \sum_{\sigma\notin S} n_\sigma = n - k$ and

$$\mu_\sigma = \begin{cases} 1 + 2\sum_{\substack{\theta<\sigma\\ \sigma\in S}} n_\theta, & \text{if } \sigma \in S \\ 2\sum_{\substack{\theta<\sigma\\ \sigma\notin S}} n_\theta, & \text{if } \sigma \notin S \end{cases} .$$

W.l.o.g. we assume S to be of the form $S = \{1, \ldots, p\}$ with $p \in \{0, \ldots, s\}$. This causes $k = m_p$, $l = n - m_p$ and

$$\mu_\sigma = \begin{cases} 2l_\sigma - 1, & \text{if } 1 \leq \sigma \leq p \\ 2(l_\sigma - k - 1), & \text{if } p+1 \leq \sigma \leq s \end{cases} .$$

★

We perform elementary row transformations within the upper k rows:

$$\begin{aligned}
&\left(\left(-\frac{1}{2}\mu_{\tau(\kappa)}+\kappa-l_{\tau(\kappa)}\right)^{2\lambda-2}+\left(\frac{1}{2}\mu_{\tau(\kappa)}+\kappa-l_{\tau(\kappa)}\right)^{2\lambda-2}\right)_{\substack{1\le\kappa\le k\\1\le\lambda\le n}}\\
&= \left(\left(\frac{1}{2}-2l_{\tau(\kappa)}+\kappa\right)^{2\lambda-2}+\left(-\frac{1}{2}+\kappa\right)^{2\lambda-2}\right)_{\substack{1\le\kappa\le k\\1\le\lambda\le n}}\\
&= \left(\left(-\frac{1}{2}+2l_{\tau(\kappa)}-\kappa\right)^{2\lambda-2}+\left(-\frac{1}{2}+\kappa\right)^{2\lambda-2}\right)_{\substack{1\le\kappa\le k\\1\le\lambda\le n}}\\
&\sim_p \left(\left(-\frac{1}{2}+\kappa\right)^{2\lambda-2}\right)_{\substack{1\le\kappa\le k\\1\le\lambda\le n}}.
\end{aligned}$$

The last relation is implied by the fact that for all $\sigma\in\{1,\dots,p\}$ and all $\kappa\in\{l_\sigma,\dots,m_\sigma\}$ we have:

- If $\kappa=l_\sigma$ then
  $-\frac{1}{2}+2l_{\tau(\kappa)}-\kappa=-\frac{1}{2}+\kappa.$
- If $l_\sigma+1\le\kappa\le 2l_\sigma-1$ then
  $\frac{1}{2}\le-\frac{1}{2}+2l_{\tau(\kappa)}-\kappa<-\frac{1}{2}+\kappa.$
- If $\kappa\ge 2l_\sigma$ then
  $\frac{1}{2}\le-\left(-\frac{1}{2}+2l_{\tau(\kappa)}-\kappa\right)<-\frac{1}{2}+\kappa.$

So

$$\left(\left(-\frac{1}{2}+2l_{\tau(\kappa)}-\kappa\right)^{2\lambda-2}+\left(-\frac{1}{2}+\kappa\right)^{2\lambda-2}\right)_{\substack{1\le\kappa\le k\\1\le\lambda\le n}}$$

is a matrix of the form

$$\left(x_\kappa^{2\lambda-2}+y_\kappa^{2\lambda-2}\right)_{\substack{1\le\kappa\le k\\1\le\lambda\le n}},$$

such that for all $\kappa\in\{1,\dots,k\}$ there is an element $\nu\in\{1,\dots,\kappa\}$ with $x_\kappa=y_\nu$. So the matrix can be simplified by elementary row transformations step by step.

We perform elementary row transformations within the lower $n-k$ rows:

$$\begin{aligned}
&\left(\left(-\frac{1}{2}\mu_{\tau(\kappa)}+\kappa-l_{\tau(\kappa)}\right)^{2\lambda-2}+\left(\frac{1}{2}\mu_{\tau(\kappa)}+\kappa-l_{\tau(\kappa)}\right)^{2\lambda-2}\right)_{\substack{k+1\le\kappa\le n\\1\le\lambda\le n}}\\
&= \quad \left(\left(k+1+\kappa-2l_{\tau(\kappa)}\right)^{2\lambda-2}+(-k-1+\kappa)^{2\lambda-2}\right)_{\substack{k+1\le\kappa\le n\\1\le\lambda\le n}}\\
&= \quad \left(\left(-k-1-\kappa+2l_{\tau(\kappa)}\right)^{2\lambda-2}+(-k-1+\kappa)^{2\lambda-2}\right)_{\substack{k+1\le\kappa\le n\\1\le\lambda\le n}}\\
&\sim_{s-p-1+n_{p+1}} \quad \left((-k-1+\kappa)^{2\lambda-2}\right)_{\substack{k+1\le\kappa\le n\\1\le\lambda\le n}}\\
&= \quad \left((-1+\kappa)^{2\lambda-2}\right)_{\substack{1\le\kappa\le n-k\\1\le\lambda\le n}}
\end{aligned}$$

The penultimate relation is implied by the fact that for all $\sigma\in\{p+1,\ldots,s\}$ and all $\kappa\in\{l_\sigma,\ldots,m_\sigma\}$ we have:

- If $\sigma=p+1$ then
  $k+1+\kappa-2l_{\tau(\kappa)}=-k-1+\kappa.$
- If $\sigma\in\{p+2,\ldots,s\}$ and $\kappa=l_\sigma$ then
  $-k-1-\kappa+2l_{\tau(\kappa)}=-k-1+\kappa.$
- If $\sigma\in\{p+2,\ldots,s\}$ and $l_\sigma+1\le\kappa\le 2l_\sigma-k-1$ then
  $0\le -k-1-\kappa+2l_{\tau(\kappa)}<-1-k+\kappa.$
- If $\sigma\in\{p+2,\ldots,s\}$ and $\kappa\ge 2l_\sigma-k$ then
  $1\le k+1+\kappa-2l_{\tau(\kappa)}<-1-k+\kappa.$

★

Given these data the corresponding Hilbert polynomial of G/U is given by

$$\begin{aligned}
&H\left(\frac{1}{2}\right)\\
&= \quad \pm\det\begin{pmatrix}\left(\left(-\frac{1}{2}+\kappa\right)^{2\lambda-2}\right)_{1\le\kappa\le k}\\ \left((-1+\kappa)^{2\lambda-2}\right)_{1\le\kappa\le n-k}\end{pmatrix}_{1\le\lambda\le n}
\end{aligned}$$

$$\cdot(n-1)!^{-1}\cdot\prod_{1\le\kappa\le n-1}(2\kappa-1)!^{-1}$$

$$= \pm\prod_{1\le\kappa<\lambda\le k}(\lambda-\kappa)\prod_{1\le\kappa<\lambda\le k}(-1+\lambda+\kappa)\prod_{1\le\kappa<\lambda\le n-k}(\lambda-\kappa)$$

$$\cdot\prod_{1\le\kappa<\lambda\le n-k}(-2+\lambda+\kappa)\prod_{\substack{1\le\kappa\le k\\1\le\lambda\le n-k}}\left(-\frac{1}{2}+\lambda-\kappa\right)\prod_{\substack{1\le\kappa\le k\\1\le\lambda\le n-k}}\left(-\frac{3}{2}+\lambda+\kappa\right)$$

$$\cdot(n-1)!^{-1}\cdot\prod_{1\le\kappa\le n}(2\kappa-1)!^{-1}$$

$$= \pm\prod_{1\le\kappa\le k}(\kappa-1)!\prod_{1\le\kappa\le k}\frac{(2\kappa-2)!}{(\kappa-1)!}\prod_{1\le\kappa\le n-k}(\kappa-1)!$$

$$\cdot\prod_{1\le\kappa\le n-k-1}\frac{(2\kappa-1)!}{(\kappa-1)!}\prod_{\substack{1\le\kappa\le k\\1\le\lambda\le n-k}}\left(-\frac{1}{2}+\lambda-\kappa\right)\prod_{\substack{1\le\kappa\le k\\1\le\lambda\le n-k}}\left(-\frac{3}{2}+\lambda+\kappa\right)$$

$$\cdot(n-1)!^{-1}\cdot\prod_{1\le\kappa\le n-1}(2\kappa-1)!^{-1}$$

$$= \pm\prod_{1\le\kappa\le k}(2\kappa-2)!\cdot\prod_{1\le\kappa\le n-k-1}(2\kappa-1)!\cdot(n-k-1)!$$

$$\cdot\prod_{\substack{1\le\kappa\le k\\1\le\lambda\le n-k}}\left(-\frac{1}{2}+\lambda-\kappa\right)\prod_{\substack{1\le\kappa\le k\\1\le\lambda\le n-k}}\left(-\frac{3}{2}+\lambda+\kappa\right)$$

$$\cdot(n-1)!^{-1}\cdot\prod_{1\le\kappa\le n-1}(2\kappa-1)!^{-1}$$

★

This leads to

$$\nu_2\left(H\left(\frac{1}{2}\right)\right)$$

$$= \sum_{1\le\kappa\le k}(2\kappa-2)-\sum_{1\le\kappa\le k}\alpha(2\kappa-2)$$

$$+\sum_{1\le\kappa\le n-k-1}(2\kappa-1)-\sum_{1\le\kappa\le n-k-1}\alpha(2\kappa-1)$$

$$+(n-k-1)-\alpha(n-k-1)$$

$$-2k(n-k)$$

$$
\begin{aligned}
& - \sum_{1\le\kappa\le n-1} (2\kappa-1) - (n-1) + \sum_{1\le\kappa\le n-1} \alpha(2\kappa-1) + \alpha(n-1) \\
= \; & k(k-1) - \sum_{1\le\kappa\le k} \alpha(\kappa-1) \\
& +(n-k-1)^2 - (n-k-1) - \sum_{1\le\kappa\le n-k-1} \alpha(\kappa-1) \\
& +(n-k-1) - \alpha(n-k-1) \\
& -2k(n-k) \\
& -(n-1)^2 - (n-1) + (n-1) + \sum_{1\le\kappa\le n-1} \alpha(\kappa-1) + \alpha(n-1) \\
= \; & k - 4k(n-k) - \alpha_1(k) - \alpha_1(n-k) + \alpha_1(n)
\end{aligned}
$$

Proposition 4.25

The quaternional flag manifold $Sp(n)/Sp(n_1)\times\cdots\times Sp(n_s)$ has real dimension $2\left(n^2 - \sum_{\sigma=1}^{s} n_\sigma^2\right)$ and can not be immersed in an Euclidean space with dimension

$$8k(n-k) - 2k + 2\alpha_1(k) + 2\alpha_1(n-k) - 2\alpha_1(n) - 1.$$

Thereby k is an arbitrary integer which can be represented as sum $\sum_{\sigma\in S} n_\sigma$ with $S \subset \{1,\ldots,s\}$.

Remark 4.26

(i) *In the case of the quaternional Grassmannian ($s = 2$, $n_1 = k$, $n_2 = n-k$) the results coincide with the results given in [May97].*

(ii) *In [Lam75] a positive result was proved: The quaternional flag manifold $Sp(n)/Sp(n_1)\times\cdots\times Sp(n_s)$ is a π–manifold or can be immersed in an Euclidean space with dimension $2n^2 - n - s$.* □

4.4 Non-immersion theorems for real flag manifolds

Notation 4.27

(i) *Let* $n_1, \ldots, n_s$ *be positive integers.*

(ii) $n = \sum_{\sigma=1}^{s} n_\sigma$, $l_\sigma = 1 + \sum_{j=1}^{\sigma-1} n_j$, $m_\sigma = \sum_{j=1}^{\sigma} n_j$.

(iii) *Let* $\tau : \{1, \ldots, n\} \to \{1, \ldots, s\}$ *be given by*
$\tau(\lambda) = \sigma \iff l_\sigma \le \lambda \le m_\sigma$.

(iv) $G = SO(2n)$, $U = SO(2n_1) \times \cdots \times SO(2n_s)$.

(v) $T = SO(2) \times SO(2) \times \cdots \times SO(2)$.

Proposition 4.28

(i) G *and* U *fulfill the prerequisites of Remark 3.8(ii).* T *is a maximal torus of* G *and* U.

(ii) $\mathfrak{g}_0$ *can be understood as the Lie algebra* $\mathfrak{so}(2n)$ *of skew symmetric real* $(2n) \times (2n)$*-matrices. ([BD85], I.2.15)*

$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ *can be identified with the Lie algebra* $\mathfrak{so}(2n, \mathbb{C})$ *of skew symmetric complex* $(2n) \times (2n)$*-matrices. ([Kna96], I.15.4)*

This yields the simplicity of $\mathfrak{g}$ *and*

$$\mathfrak{t} = \mathfrak{t}' = \left\{ diag\left(\left(\begin{array}{cc} 0 & z_1 \\ -z_1 & 0 \end{array} \right), \ldots, \left(\begin{array}{cc} 0 & z_n \\ -z_n & 0 \end{array} \right) \middle| \; z_1, \ldots, z_n \in \mathbb{C} \right) \right\}.$$

$\mathfrak{t}^*$ *is the span of the linear maps* L_λ *(*$\lambda = 1, \ldots, n$*) given by*

$$L_\lambda \left(diag \left(\left(\begin{array}{cc} 0 & z_1 \\ -z_1 & 0 \end{array} \right), \ldots, \left(\begin{array}{cc} 0 & z_n \\ -z_n & 0 \end{array} \right) \right) \right) = z_\lambda.$$

(iii) The Weyl group $W(G)$ *is isomorphic to the semidirect product of* $\mathfrak{S}_n$ *and* $\left\{ (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n \;\middle|\; \prod_{\kappa=1}^{n} \varepsilon_\kappa = 1 \right\}$*. The operation of* $\mathfrak{S}_n$ *on* $\left\{ (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n \;\middle|\; \prod_{\kappa=1}^{n} \varepsilon_\kappa = 1 \right\}$ *is the standard operation on the set of indices.*

The Weyl group $W(U)$ *is isomorphic to the direct product of accordant semidirect products of* $\mathfrak{S}_{n_\sigma}$ *and* $\left\{ (\varepsilon_{l_\sigma}, \ldots, \varepsilon_{m_\sigma}) \in \{\pm 1\}^{n_\sigma} \;\middle|\; \prod_{\kappa=l_\sigma}^{m_\sigma} \varepsilon_\kappa = 1 \right\}$*. ([FH96], p.271) ([BD85], IV.3.6)*

(iv) The Killing form of G *induces the bilinear form* $\langle\, , \rangle$ *on* $\mathfrak{t}'^*$ *given by*

$$\left\langle \sum_{1 \le \lambda \le n} a_\lambda L_\lambda, \sum_{1 \le \kappa \le n} b_\kappa L_\kappa \right\rangle = \frac{1}{4n-4} \left(\sum_{1 \le \kappa \le n} a_\kappa b_\kappa \right).$$

([FH96], p.272)

(v) A system of positive roots of G *is given by*

$$\Sigma^+(G) = \{L_\lambda - L_\kappa,\, L_\lambda + L_\kappa \,|\, 1 \le \lambda < \kappa \le n\}.$$

([Ada69], 5.28)

The half sum is equal to

$$\delta = \sum_{\lambda=1}^{n} (n - \lambda) L_\lambda.$$

(vi) A system of positive roots of U *is given by*

$$\Sigma^+(U) = \bigcup_{\sigma=1}^{s} \{L_\lambda - L_\kappa,\, L_\lambda + L_\kappa,\, | \, l_\sigma \leq \lambda < \kappa \leq m_\sigma\}.$$

The half sum is equal to

$$\delta' = \sum_{\lambda=1}^{n} (m_{\tau(\lambda)} - \lambda) L_\lambda.$$

(vii) For integers $\mu_1, \ldots, \mu_s$ *the set of analytically integral elements*

$$\left\{ \sum_{1 \leq \lambda \leq n} \varepsilon_\lambda \mu_{\tau(\lambda)} L_\lambda \;\middle|\; \begin{array}{l} \varepsilon_\kappa \in \begin{cases} \{\pm 1\}, & \text{if } \mu_{\tau(\kappa)} \neq 0 \\ \{1\}, & \text{if } \mu_{\tau(\kappa)} = 0 \end{cases} \\ \prod_{\kappa=l_\sigma}^{m_\sigma} \varepsilon_\kappa = 1 \text{ fuer alle } \sigma \in \{1, \ldots, s\} \end{array} \right\} (*)$$

is $W(U)$*-invariant. ([Kna96], IV.9.20)*

Remark 4.29

In the family

$$\left(\sum_{1 \leq \lambda \leq n} \varepsilon_\lambda \mu_{\tau(\lambda)} L_\lambda \;\middle|\; \begin{array}{l} \varepsilon_\kappa \in \{\pm 1\}, \text{ if } 1 \leq \kappa \leq n \\ \prod_{\kappa=l_\sigma}^{m_\sigma} \varepsilon_\kappa = 1 \text{ fuer alle } \sigma \in \{1, \ldots, s\} \end{array} \right)$$

each member of $(*)$ *appears exactly* $\left(2^{\sum_{\substack{1 \leq \sigma \leq s \\ \mu_\sigma = 0}} (n_\sigma - 1)} \right)$*-times.*

Remark 4.30

G/U *is a Spin manifold because of* $w_2(G/U) = 0$.

Proof:

We denote the canonic bundles over G/U by $\xi_1, \ldots, \xi_s$. The tangent bundle of G/U is equivariant isomorphic to the vector bundle $\bigoplus_{1 \leq \sigma < \theta \leq s} \xi_\sigma \otimes \xi_\theta$.

$\xi_1, \ldots, \xi_s$ are orientable, so $w_2(G/U) = \prod_{1\le\sigma<\theta\le s} (2n_\sigma w_2(\xi_\theta) + 2n_\theta w_2(\xi_\sigma)) = 0$.
$\square$

★

Given those data the corresponding Hilbert polynomial of G/U is given by

$$2^{\sum_{\substack{1\le\sigma\le s\\ \mu_\sigma=0}} (n_\sigma-1)} H(t)$$

$$= \pm \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n=\pm1\\ \prod_{\kappa=l_\sigma}^{m_\sigma}\varepsilon_\kappa=1\,\forall\,\sigma}} \left[\frac{\prod_{1\le\kappa<\lambda\le n} \left\langle \sum_{\nu=1}^{n} (\varepsilon_\nu t\mu_{\tau(\nu)} + m_{\tau(\nu)} - \nu) L_\nu, L_\kappa - L_\lambda \right\rangle}{\prod_{1\le\kappa<\lambda\le n} \left\langle \sum_{\nu=1}^{n} (n-\nu) L_\nu, L_\kappa - L_\lambda \right\rangle} \right.$$

$$\left. \cdot \frac{\prod_{1\le\kappa<\lambda\le n} \left\langle \sum_{\nu=1}^{n} (\varepsilon_\nu t\mu_{\tau(\nu)} + m_{\tau(\nu)} - \nu) L_\nu, L_\kappa + L_\lambda \right\rangle}{\prod_{1\le\kappa<\lambda\le n} \left\langle \sum_{\nu=1}^{n} (n-\nu) L_\nu, L_\kappa + L_\lambda \right\rangle} \right]$$

$$= \pm \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n=\pm1\\ \prod_{\kappa=l_\sigma}^{m_\sigma}\varepsilon_\kappa=1\,\forall\,\sigma}} \left[\frac{\prod_{1\le\kappa<\lambda\le n} \left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right) - \left(\varepsilon_\lambda t\mu_{\tau(\lambda)} + m_{\tau(\lambda)} - \lambda\right)\right)}{\prod_{1\le\kappa<\lambda\le n} ((n-\kappa) - (n-\lambda))} \right.$$

$$\left. \cdot \frac{\prod_{1\le\kappa<\lambda\le n} \left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right) + \left(\varepsilon_\lambda t\mu_{\tau(\lambda)} + m_{\tau(\lambda)} - \lambda\right)\right)}{\prod_{1\le\kappa<\lambda\le n} ((n-\kappa) + (n-\lambda))} \right]$$

$$= \pm \prod_{1\le\kappa<\lambda\le n} ((n-\kappa) - (n-\lambda))^{-1} \cdot \prod_{1\le\kappa<\lambda\le n} ((n-\kappa) + (n-\lambda))^{-1}$$

$$\cdot \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n=\pm1\\ \prod_{\kappa=l_\sigma}^{m_\sigma}\varepsilon_\kappa=1\,\forall\,\sigma}} \prod_{1\le\kappa<\lambda\le n} \left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^2 - \left(\varepsilon_\lambda t\mu_{\tau(\lambda)} + m_{\tau(\lambda)} - \lambda\right)^2\right)$$

$$= \pm \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_n=\pm1\\ \prod_{\kappa=l_\sigma}^{m_\sigma}\varepsilon_\kappa=1\,\forall\,\sigma}} \prod_{1\le\kappa<\lambda\le n} (\lambda-\kappa)^{-1} \cdot \prod_{1\le\kappa<\lambda\le n} (2n-\kappa-\lambda)^{-1}$$

$$
\begin{aligned}
& \cdot \det\left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2}\right)_{1\le\kappa,\lambda\le n} \\
\overset{(**)}{=}\; & \pm 2^{-s} \prod_{1\le\kappa<\lambda\le n} (\lambda-\kappa)^{-1} \cdot \prod_{1\le\kappa<\lambda\le n} (2n-\kappa-\lambda)^{-1} \\
& \cdot \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm1} \det\left(\left(\varepsilon_\kappa t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2}\right)_{1\le\kappa,\lambda\le n} \\
=\; & \pm 2^{-s} \prod_{1\le\kappa<\lambda\le n} (\lambda-\kappa)^{-1} \cdot \prod_{1\le\kappa<\lambda\le n} (\kappa+\lambda-2)^{-1} \\
& \cdot \det\left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2}\right)_{1\le\kappa,\lambda\le n} \\
=\; & \pm 2^{-s} \prod_{1\le\kappa\le n} (\kappa-1)!^{-1} \cdot \prod_{1\le\kappa\le n-1} (\kappa-1)! \cdot \prod_{1\le\kappa\le n-1} (2\kappa-1)!^{-1} \\
& \det\left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2}\right)_{1\le\kappa,\lambda\le n} \\
=\; & \pm 2^{-s} (n-1)!^{-1} \prod_{1\le\kappa\le n-1} (2\kappa-1)!^{-1} \\
& \cdot \det\left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2}\right)_{1\le\kappa,\lambda\le n}
\end{aligned}
$$

Step $(**)$ is true because for all $\sigma \in \{1,\ldots,s\}$ the m_σ-th row is independent of the sign of $\varepsilon_{m_\sigma} = \pm 1$.

We perform elementary row transformations within the "σ-th block", i.e. for $\kappa \in \{l_\sigma,\ldots,m_\sigma\}$:

With the notations of 4.23 we obtain:

$$
\begin{aligned}
& \left(\left(-t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + m_{\tau(\kappa)} - \kappa\right)^{2\lambda-2}\right)_{\substack{l_\sigma\le\kappa\le m_\sigma \\ 1\le\lambda\le n}} \\
\sim_0\;& \left(\left(-t\mu_\sigma + \kappa - 1\right)^{2\lambda-2} + \left(t\mu_\sigma + \kappa - 1\right)^{2\lambda-2}\right)_{\substack{1\le\kappa\le n_\sigma \\ 1\le\lambda\le n}} \\
\sim_0\;& \left(\left(-t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)}\right)^{2\lambda-2} + \left(t\mu_{\tau(\kappa)} + \kappa - l_{\tau(\kappa)}\right)^{2\lambda-2}\right)_{\substack{l_\sigma\le\kappa\le m_\sigma \\ 1\le\lambda\le n}}
\end{aligned}
$$

So the corresponding Hilbert polynomial of G/U is given by

$$\begin{aligned} 2^{\sum\limits_{\substack{1\leq\sigma\leq s\\ \mu_\sigma=0}}(n_\sigma-1)} H(t) &= \pm\det\left(\left(-t\mu_{\tau(\kappa)}+\kappa-l_{\tau(\kappa)}\right)^{2\lambda-2}+\left(t\mu_{\tau(\kappa)}+\kappa-l_{\tau(\kappa)}\right)^{2\lambda-2}\right)_{1\leq\kappa,\lambda\leq n} \\ &\quad\cdot\prod_{1\leq\kappa\leq n-1}(2\kappa-1)!^{-1}\cdot(n-1)!^{-1} \end{aligned}$$

We summerize the outcome:

Proposition 4.31

Let $\mu_1,\ldots,\mu_s$ be integers. Then there exists an Element $z\in ch(G/U)$ such that the Hilbert polynom associated with $0\in H^2(G/U)$ snd z is equal to

$$\begin{aligned} H(t) &= \pm\det\left(\left(-t\mu_\sigma+\kappa-l_{\tau(\sigma)}\right)^{2\lambda-2}+\left(t\mu_\sigma+\kappa-l_{\tau(\sigma)}\right)^{2\lambda-2}\right)_{1\leq\kappa,\lambda\leq n} \\ &\qquad\cdot 2^{-s-\sum\limits_{\substack{1\leq\sigma\leq s\\ \mu_\sigma=0}}(n_\sigma-1)}\cdot(n-1)!^{-1}\cdot\prod_{1\leq\kappa\leq n-1}(2\kappa-1)!^{-1}.\quad\Box \end{aligned}$$

★

If we denote the corresponding Hilbert polynomial of the quaternional flag manifold $Sp(n)/Sp(n_1)\times\cdots\times Sp(n_s)$ by $A(t)$ we obtain

$$H(t)=A(t)\cdot 2^{|\{\sigma\,|\,\mu_\sigma=0\}|-1}$$

and

$$\nu_2\left(H\left(\frac{1}{2}\right)\right)=\nu_2\left(A\left(\frac{1}{2}\right)\right)+|\{\sigma\,|\,\mu_\sigma=0\}|-1.$$

(see Proposition 4.24.)

In particular, by choosing the integers $\mu_1,\ldots,\mu_s$ as in section 4.3 we get a non-immersion theorem:

Proposition 4.32

The real oriented flag manifold $SO(2n)/SO(2n_1) \times \cdots \times SO(2n_s)$ *has real dimension* $2\left(n^2 - \sum_{\sigma=1}^{s} n_\sigma^2\right)$ *and can not be immersed in an Euclidean space with dimension*

$$8k(n-k) - 2k + 2\alpha_1(k) + 2\alpha_1(n-k) - 2\alpha_1(n) - 1.$$

Thereby k *is an arbitrary integer which can be represented as sum* $\sum_{\sigma \in S} n_\sigma$ *with* $S \subset \{1, \ldots, s\}$.

Corollary 4.33

The real flag manifold $O(2n)/O(2n_1) \times \cdots \times O(2n_s)$ *has real dimension* $2\left(n^2 - \sum_{\sigma=1}^{s} n_\sigma^2\right)$ *and can not be immersed in an Euclidean space with dimension*

$$8k(n-k) - 2k + 2\alpha_1(k) + 2\alpha_1(n-k) - 2\alpha_1(n) - 1.$$

Thereby k *is an arbitrary integer which can be represented as sum* $\sum_{\sigma \in S} n_\sigma$ *with* $S \subset \{1, \ldots, s\}$.

Proof: The canonical projection $SO(2n)/SO(2n_1) \times \cdots \times SO(2n_s) \to O(2n)/O(2n_1) \times \cdots \times O(2n_s)$ is a covering map and therefore an immersion.

Remark 4.34

(i) *In the case of real Grassmannians* ($s = 2$, $n_1 = k$, $n_2 = n - k$) *the results coincide with the results given in [May97].*

(ii) *If we substitute the set* $(*)$ *by the* $W(U)$*-invariant set of analytically integral elements*

$$\left\{ \sum_{1 \le \lambda \le n} \varepsilon_\lambda \mu_{\tau(\lambda)} L_\lambda \;\middle|\; \varepsilon_\kappa \in \begin{cases} \{\pm 1\}, & \text{if } \mu_{\tau(\kappa)} \neq 0 \\ \{1\}, & \text{if } \mu_{\tau(\kappa)} = 0 \end{cases} \right\} (*')$$

we obtain a Hilbert polynomial

$$H(t) = A(t) \cdot 2^{s-1}.$$

This generalizes the results in [May98].

(iii) *There is a positive result given in [Lam75]: The real flag manifold* $O(2n)/O(2n_1) \times \cdots \times O(2n_s)$ *is a* π*–manifold or can be immersed in an Euclidean space with dimension* $2n^2 - n$. □

4.5 Non-immersion theorems for the manifolds $Sp(n)/U(n_1) \times \cdots \times U(n_s)$

Notation 4.35

(i) *Let $n_1, \ldots, n_s$ be positive integers.*

(ii) $n = \sum_{\sigma=1}^{s} n_\sigma,\ l_\sigma = 1 + \sum_{j=1}^{\sigma-1} n_j,\ m_\sigma = \sum_{j=1}^{\sigma} n_j.$

(iii) *Let $\tau : \{1, \ldots, n\} \to \{1, \ldots, s\}$ be given by*

$\tau(\lambda) = \sigma \iff l_\sigma \leq \lambda \leq m_\sigma.$

(iv) $G = Sp(n),\ U = U(n_1) \times \cdots \times U(n_s).$

(v) $T = U(1) \times U(1) \times \cdots \times U(1).$

Proposition 4.36

$U(n_1) \times \cdots \times U(n_s)$ is the centralizer $Z(S)$ of the toral subgroup $S = \left\{ \mathrm{diag}\left(e^{ir_{\tau(1)}}, \ldots, e^{ir_{\tau(n)}}\right) \middle| r_1, \ldots, r_s \in \mathbb{R}\right\}$ in $Sp(n)$.

Proof: Let $A \in Z(S)$. A commutes with the matrix $\mathrm{diag}\,(i, \ldots, i)$, so all entries of A commute with i. Consequently all entries of A are complex and the centralizer of S in $Sp(n)$ is equal to the centralizer of S in $U(n)$. The statement follows from Proposition 4.13. □

Proposition 4.37

(i) *Due to Remark 3.6(ii) G and U fulfill the prerequisites of Remark 3.8(i). T is a maximal torus of G and U.*

$\mathfrak{t}^*$*, $W(G)$, $W(U)$, $\langle\, ,\, \rangle$, $\Sigma^+(G)$ and δ are decribed in Proposition 4.14 and Proposition 4.21.*

(ii) *For integers $\mu_1, \ldots, \mu_s$ the analytically integral element $\sum_{\nu=1}^{n} \mu_{\tau(\nu)} L_\nu$ is $W(U)$-invariant.* □

Given these data the corresponding Hilbert polynomial of G/U is equal to

$$
\begin{aligned}
&H(t)\\
&= \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(t\mu_{\tau(\nu)}+(n-\nu+1))L_\nu, L_\kappa-L_\lambda\right\rangle}{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(n-\nu+1)L_\nu, L_\kappa-L_\lambda\right\rangle}\\
&\quad\cdot\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(t\mu_{\tau(\nu)}+(n-\nu+1))L_\nu, L_\kappa+L_\lambda\right\rangle}{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(n-\nu+1)L_\nu, L_\kappa+L_\lambda\right\rangle}\\
&\quad\cdot\frac{\prod\limits_{1\le\kappa\le n}\left\langle\sum\limits_{\nu=1}^{n}(t\mu_{\tau(\nu)}+(n-\nu+1))L_\nu, 2L_\kappa\right\rangle}{\prod\limits_{1\le\kappa\le n}\left\langle\sum\limits_{\nu=1}^{n}(n-\nu+1)L_\nu, 2L_\kappa\right\rangle}\\
&= \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa+1)-(t\mu_{\tau(\lambda)}+n-\lambda+1)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}((n-\kappa+1)-(n-\lambda+1))}\\
&\quad\cdot\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa+1)+(t\mu_{\tau(\lambda)}+n-\lambda+1)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}((n-\kappa+1)+(n-\lambda+1))}\\
&\quad\cdot\frac{\prod\limits_{1\le\kappa\le n}\big(t\mu_{\tau(\kappa)}+n-\kappa+1\big)}{\prod\limits_{1\le\kappa\le n}(n-\kappa+1)}\\
&= \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa+1)-(t\mu_{\tau(\lambda)}+n-\lambda+1)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}(-\kappa+\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \frac{\prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)+(t\mu_{\tau(\lambda)}+n-\lambda+1)\big)}{\prod_{1\le\kappa<\lambda\le n}(2n-\kappa-\lambda+2)} \\
& \cdot \frac{\prod_{1\le\kappa\le n}\big(t\mu_{\tau(\kappa)}+n-\kappa+1\big)}{\prod_{1\le\kappa\le n}(n-\kappa+1)} \\
= \; & \pm \frac{\prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)-(t\mu_{\tau(\lambda)}+n-\lambda+1)\big)}{\prod_{1\le\kappa<\lambda\le n}(-\kappa+\lambda)} \\
& \cdot \frac{\prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)+(t\mu_{\tau(\lambda)}+n-\lambda+1)\big)}{\prod_{1\le\kappa<\lambda\le n}(\kappa+\lambda)} \\
& \cdot \frac{\prod_{1\le\kappa\le n}\big(t\mu_{\tau(\kappa)}+n-\kappa+1\big)}{\prod_{1\le\kappa\le n}\kappa} \\
= \; & \pm \prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)-(t\mu_{\tau(\lambda)}+n-\lambda+1)\big) \\
& \cdot \prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)+(t\mu_{\tau(\lambda)}+n-\lambda+1)\big) \\
& \cdot \prod_{1\le\kappa\le n}\big(t\mu_{\tau(\kappa)}+n-\kappa+1\big) \\
& \cdot n!^{-1} \prod_{1\le\kappa\le n}\kappa! \prod_{1\le\kappa\le n}(\kappa-1)!^{-1} \prod_{1\le\kappa\le n}(2\kappa-1)!^{-1} \\
= \; & \pm \prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)-(t\mu_{\tau(\lambda)}+n-\lambda+1)\big) \\
& \cdot \prod_{1\le\kappa<\lambda\le n} \big((t\mu_{\tau(\kappa)}+n-\kappa+1)+(t\mu_{\tau(\lambda)}+n-\lambda+1)\big) \\
& \cdot \prod_{1\le\kappa\le n}\big(t\mu_{\tau(\kappa)}+n-\kappa+1\big) \cdot \prod_{1\le\kappa\le n}(2\kappa-1)!^{-1}
\end{aligned}
$$

We summarize the outcome:

Proposition 4.38

Let $\mu_1, \ldots, \mu_s$ be integers. Then there exists an element $z \in ch(G/U)$ such that the Hilbert polynomial associated with $c_1(G/U) \in H^2(G/U)$ and z is given by

$$\begin{aligned} H(t) \;=\; \pm \Bigg(& \prod_{1 \le \kappa < \lambda \le n} (t\mu_{\tau(\kappa)} + n - \kappa + 1) - (t\mu_{\tau(\lambda)} + n - \lambda + 1) \Bigg) \\ & \prod_{1 \le \kappa < \lambda \le n} \big((t\mu_{\tau(\kappa)} + n - \kappa + 1) + (t\mu_{\tau(\lambda)} + n - \lambda + 1) \big) \\ & \prod_{1 \le \kappa \le n} \big(t\mu_{\tau(\kappa)} + n - \kappa + 1 \big) \prod_{1 \le \kappa \le n} (2\kappa - 1)!^{-1}. \quad \square \end{aligned}$$

★

The results in 4.2 induce to set the integers $\mu_1, \ldots, \mu_s$ in a similar way. We choose a subset $S \subset \{1, \ldots, s\}$ and define

$$\gamma_\sigma = -n + m_\sigma + \sum_{\substack{1 \le \vartheta < \sigma \\ \vartheta \in S}} n_\vartheta, \text{ if } \sigma \in S,$$

$$\gamma_\sigma = -n + m_\sigma + \sum_{\substack{1 \le \vartheta < \sigma \\ \vartheta \notin S}} n_\vartheta, \text{ if } \sigma \notin S,$$

$$\mu_\sigma = 2\gamma_\sigma, \text{ if } \sigma \in S,$$

$$\mu_\sigma = 2\gamma_\sigma - 1, \text{ if } \sigma \notin S \text{ and}$$

$$k = \sum_{\sigma \in S} n_\sigma.$$

Then the sets $\{\gamma_{\tau(\kappa)} + n - \kappa + 1 \,|\, 1 \le \kappa \le n, \tau(\kappa) \in S\}$ and $\{1, 2, \ldots, k\}$ are equal. Also the sets $\{\gamma_{\tau(\kappa)} + n - \kappa + 1 \,|\, 1 \le \kappa \le n, \tau(\kappa) \notin S\}$ and $\{1, 2, \ldots, n - k\}$ are equal.

This leads to

$$\begin{aligned}
&\nu_2\left(H\left(\frac{1}{2}\right)\right)\\
&= \sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\in S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)-\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda+1\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\notin S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)-\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda+1\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa,\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)-\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda+1\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\in S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)+\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda+1\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\notin S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)+\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda+1\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa,\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)+\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda+1\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa\le n\\ \tau(\kappa)\in S}} \nu_2\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa+1\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\notin S}} \nu_2\left(\frac{1}{2}\mu_{\tau(\kappa)}+(n-\kappa+1)\right)\\
&\quad-\sum_{1\le\kappa\le n} \nu_2\left((2\kappa-1)!\right)\\
&= \sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\in S}} \nu_2\left((\gamma_{\tau(\kappa)}+n-\kappa+1)-(\gamma_{\tau(\lambda)}+n-\lambda+1)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\notin S,\tau(\lambda)\notin S}} \nu_2\left((\gamma_{\tau(\kappa)}+n-\kappa+1)-(\gamma_{\tau(\lambda)}+n-\lambda+1)\right)\\
&\quad+\sum_{\substack{1\le\kappa,\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\notin S}} \nu_2\left(-\frac{1}{2}+(\gamma_{\tau(\kappa)}+n-\kappa+1)-(\gamma_{\tau(\lambda)}+n-\lambda+1)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\in S}} \nu_2\left((\gamma_{\tau(\kappa)}+n-\kappa+1)+(\gamma_{\tau(\lambda)}+n-\lambda+1)\right)
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{\substack{1\leq\kappa<\lambda\leq n \\ \tau(\kappa)\notin S, \tau(\lambda)\notin S}} \nu_2\left(-1 + (\gamma_{\tau(\kappa)} + n - \kappa + 1) + (\gamma_{\tau(\lambda)} + n - \lambda + 1)\right) \\
& + \sum_{\substack{1\leq\kappa,\lambda\leq n \\ \tau(\kappa)\in S, \tau(\lambda)\notin S}} \nu_2\left(-\frac{1}{2} + (\gamma_{\tau(\kappa)} + n - \kappa + 1) + (\gamma_{\tau(\lambda)} + n - \lambda + 1)\right) \\
& + \sum_{\substack{1\leq\kappa\leq n \\ \tau(\kappa)\in S}} \nu_2\left(\gamma_{\tau(\kappa)} + n - \kappa + 1\right) \\
& + \sum_{\substack{1\leq\kappa<\lambda\leq n \\ \tau(\kappa)\notin S}} \nu_2\left(-\frac{1}{2} + \gamma_{\tau(\kappa)} + n - \kappa + 1\right) \\
& - \sum_{1\leq\kappa\leq n} \nu_2\left((2\kappa-1)!\right) \\
= \; & \sum_{1\leq\kappa<\lambda\leq k} \nu_2(\kappa-\lambda) + \sum_{1\leq\kappa<\lambda\leq n-k} \nu_2(\kappa-\lambda) - k(n-k) \\
& + \sum_{1\leq\kappa<\lambda\leq k} \nu_2(\kappa+\lambda) + \sum_{1\leq\kappa<\lambda\leq n-k} \nu_2(-1+\kappa+\lambda) - k(n-k) \\
& + \sum_{1\leq\kappa\leq k} \nu_2(\kappa) - (n-k) \\
& - \sum_{1\leq\kappa\leq n} \nu_2\left((2\kappa-1)!)\right) \\
= \; & \frac{k(k-1)}{2} - \alpha_1(k) + \frac{(n-k)(n-k-1)}{2} - \alpha_1(n-k) - k(n-k) \\
& + \frac{k(k-3)}{2} + \alpha(k) + \frac{(n-k)(n-k-1)}{2} - k(n-k) \\
& + k - \alpha(k) - (n-k) - n^2 + n + \alpha_1(n) \\
= \; & -(n-k) - 4k(n-k) - \alpha_1(k) - \alpha_1(n-k) + \alpha_1(n).
\end{aligned}
$$

We obtain as non-immersion result:

Proposition 4.39

The manifold $Sp(n)/U(n_1)\times\cdots\times U(n_s)$ has real dimension $n(2n+1)-\sum_{\sigma=1}^{s} n_\sigma^2$ and can not be immersed in an Euclidean space with dimension

$$8k(n-k) + 2(n-k) - 2\alpha_1(n) + 2\alpha_1(k) + 2\alpha_1(n-k) - 1.$$

Thereby k is an arbitrary integer which can be represented as sum $\sum_{\sigma \in S} n_\sigma$ with $S \subset \{1, \ldots, s\}$. $\square$

Proposition 4.40

$Sp(n)/U(n_1) \times \cdots \times U(n_s)$ is a π–manifold or can be immersed in an Euclidean space with dimension $2n^2 + n - s$.

Proof: The statement is caused by Proposition 1.10 and Proposition 4.36. $\square$

4.6 Non-immersion theorems for the manifolds $SO(2n)/U(n_1)\times\cdots\times U(n_s)$

Notation 4.41

(i) *Let* $n_1,\dots,n_s$ *be positive integers.*

(ii) $n=\sum_{\sigma=1}^{s} n_\sigma$, $l_\sigma = 1+\sum_{j=1}^{\sigma-1} n_j$, $m_\sigma=\sum_{j=1}^{\sigma} n_j$.

(iii) *Let* $\tau:\{1,\dots,n\}\to\{1,\dots,s\}$ *be given by*
$\tau(\lambda)=\sigma \iff l_\sigma\le\lambda\le m_\sigma$.

(iv) $G=SO(2n)$, $U=U(n_1)\times\cdots\times U(n_s)$.

(v) $T=U(1)\times U(1)\times\cdots\times U(1)=SO(2)\times\cdots\times SO(2)$.

Proposition 4.42
$U(n_1)\times\cdots\times U(n_s)$ is the centralizer $Z(S)$ of the toral subgroup

$$S=\left\{ diag\left(\begin{pmatrix} \cos r_{\tau(1)} & \sin r_{\tau(1)} \\ -\sin r_{\tau(1)} & \cos r_{\tau(1)}\end{pmatrix},\dots,\begin{pmatrix} \cos r_{\tau(n)} & \sin r_{\tau(n)} \\ -\sin r_{\tau(n)} & \cos r_{\tau(n)}\end{pmatrix}\right)\middle|\, r_1,\dots,r_s\in\mathbb{R}\right\}.$$

in $SO(2n)$.

Proof: Let $A\in Z(S)$. We understand A to be an $\mathbb{R}$-linear endomorphism of $\mathbb{C}^n$. A commutes with the matrix $\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},\dots,\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)$. So A is $\mathbb{C}$-linear. Consequently the centralizer of S in $SO(2n)$ es equal to the centralizer of S in $U(n)$. The staement is caused by Proposition 4.13. □

Proposition 4.43

(i) *Due to Remark 3.6(ii) G and U fulfill the prerequisites of Remark 3.8(i). T is a maximal torus of G and U.*

(ii) $\mathfrak{t}^*$, $W(G)$, $W(U)$, $\langle\,,\,\rangle$, $\Sigma^+(G)$ *und* δ *are described in Proposition 4.28 and Proposition 4.14.*

(iii) For integers $\mu_1,\ldots,\mu_s$ the analytically integral element $\sum\limits_{\nu=1}^{n}\mu_{\tau(\nu)}L_\nu$ is $W(U)$-invariant. □

★

Given these data we obtain a Hilbert polynomial of G/U equal to

$$
\begin{aligned}
H(t) & \\
= \; & \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(t\mu_{\tau(\nu)}+(n-\nu))L_\nu, L_\kappa-L_\lambda\right\rangle}{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(n-\nu)L_\nu, L_\kappa-L_\lambda\right\rangle} \\
& \cdot\frac{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(t\mu_{\tau(\nu)}+(n-\nu))L_\nu, L_\kappa+L_\lambda\right\rangle}{\prod\limits_{1\le\kappa<\lambda\le n}\left\langle\sum\limits_{\nu=1}^{n}(n-\nu)L_\nu, L_\kappa+L_\lambda\right\rangle} \\
= \; & \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)-(t\mu_{\tau(\lambda)}+n-\lambda)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}((n-\kappa)-(n-\lambda))} \\
& \cdot\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)+(t\mu_{\tau(\lambda)}+n-\lambda)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}((n-\kappa)+(n-\lambda))} \\
= \; & \pm\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)-(t\mu_{\tau(\lambda)}+n-\lambda)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}(-\kappa+\lambda)} \\
& \cdot\frac{\prod\limits_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)+(t\mu_{\tau(\lambda)}+n-\lambda)\big)}{\prod\limits_{1\le\kappa<\lambda\le n}(2n-\kappa-\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
&= \pm \frac{\displaystyle\prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)-(t\mu_{\tau(\lambda)}+n-\lambda)\big)}{\displaystyle\prod_{1\le\kappa<\lambda\le n}(-\kappa+\lambda)} \\
&\quad \cdot \frac{\displaystyle\prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)+(t\mu_{\tau(\lambda)}+n-\lambda)\big)}{\displaystyle\prod_{1\le\kappa<\lambda\le n}(\kappa+\lambda-2)} \\
&= \pm \prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)-(t\mu_{\tau(\lambda)}+n-\lambda)\big) \\
&\quad \cdot \prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)+(t\mu_{\tau(\lambda)}+n-\lambda)\big) \\
&\quad \cdot \prod_{1\le\kappa\le n-1}(\kappa-1)! \prod_{1\le\kappa\le n-1}(2\kappa-1)!^{-1} \prod_{1\le\kappa\le n}(\kappa-1)!^{-1} \\
&= \pm \prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)-(t\mu_{\tau(\lambda)}+n-\lambda)\big) \\
&\quad \cdot \prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)+(t\mu_{\tau(\lambda)}+n-\lambda)\big) \\
&\quad \cdot (n-1)!^{-1} \prod_{1\le\kappa\le n-1}(2\kappa-1)!^{-1}
\end{aligned}
$$

We summarize the result:

Proposition 4.44

Let $\mu_1, \ldots, \mu_s$ be integers. Then there exists an element $z \in ch(G/U)$ such that the Hilbert polynomial associated with $c_1(G/U) \in H^2(G/U)$ and z is given by

$$
\begin{aligned}
H(t) &= \pm \prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)-(t\mu_{\tau(\lambda)}+n-\lambda)\big) \\
&\quad \cdot \prod_{1\le\kappa<\lambda\le n}\big((t\mu_{\tau(\kappa)}+n-\kappa)+(t\mu_{\tau(\lambda)}+n-\lambda)\big) \\
&\quad (n-1)!^{-1} \prod_{1\le\kappa\le n-1}(2\kappa-1)!^{-1} \quad \Box
\end{aligned}
$$

★

In a similar way to the preceding section we set the integers $\mu_1,\dots,\mu_s$:
We choose a subset $S\subset\{1,\dots,s\}$ and define

$$\gamma_\sigma = -n + m_\sigma + \sum_{\substack{1\le\vartheta<\sigma\\ \vartheta\in S}} n_\vartheta,\ \text{if } \sigma\in S,$$

$$\gamma_\sigma = -n + 1 + m_\sigma + \sum_{\substack{1\le\vartheta<\sigma\\ \vartheta\notin S}} n_\vartheta,\ \text{if } \sigma\notin S,$$

$$\mu_\sigma = 2\gamma_\sigma,\ \text{if } \sigma\in S,$$

$$\mu_\sigma = 2\gamma_\sigma - 1,\ \text{if } \sigma\notin S \text{ and}$$

$$k=\sum_{\sigma\in S} n_\sigma.$$

Then the sets $\{\gamma_{\tau(\kappa)}+n-\kappa \,|\, 1\le\kappa\le n, \tau(\kappa)\in S\}$ and $\{0,1,\dots,k-1\}$ are equal and the sets $\{\gamma_{\tau(\kappa)}+n-\kappa \,|\, 1\le\kappa\le n, \tau(\kappa)\notin S\}$ and $\{1,2,\dots,n-k\}$ are equal.

$$\begin{aligned}
&\nu_2\left(H\left(\frac{1}{2}\right)\right)\\
&= \sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\in S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa\right)-\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\notin S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa\right)-\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa,\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa\right)-\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\in S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa\right)+\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda\right)\right)\\
&\quad+\sum_{\substack{1\le\kappa<\lambda\le n\\ \tau(\kappa)\notin S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa\right)+\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{\substack{1\le\kappa,\lambda\le n\\ \tau(\kappa)\in S,\tau(\lambda)\notin S}} \nu_2\left(\left(\frac{1}{2}\mu_{\tau(\kappa)}+n-\kappa\right)+\left(\frac{1}{2}\mu_{\tau(\lambda)}+n-\lambda\right)\right)\\
& - \sum_{1\le\kappa\le n-1} \nu_2\left((2\kappa-1)!\right)-\nu_2((n-1)!)\\
= & \sum_{1\le\kappa<\lambda\le k} \nu_2\left(\kappa-\lambda\right)+\sum_{1\le\kappa<\lambda\le n-k} \nu_2\left(\kappa-\lambda\right)-k(n-k)\\
& + \sum_{1\le\kappa<\lambda\le k} \nu_2\left(\kappa+\lambda-2\right)+\sum_{1\le\kappa<\lambda\le n-k} \nu_2\left(-1+\kappa+\lambda\right)-k(n-k)\\
& - \sum_{1\le\kappa\le n-1} \nu_2\left((2\kappa-1)!\right)-\nu_2((n-1)!)\\
= & \frac{k(k-1)}{2}-\alpha_1(k)+\frac{(n-k)(n-k-1)}{2}-\alpha_1(n-k)-k(n-k)\\
& +\frac{(k-2)(k-1)}{2}+\frac{(n-k)(n-k-1)}{2}-k(n-k)\\
& -(n-1)^2+(n-1)+\alpha_1(n-1)-(n-1)+\alpha(n-1)\\
= & (n-k)-4k(n-k)-\alpha_1(k)-\alpha_1(n-k)+\alpha_1(n)
\end{aligned}
$$

Intertwining $k \leftrightarrow n-k$ leads to:

Proposition 4.45

The manifold $SO(2n)/U(n_1)\times\cdots\times U(n_s)$ has real dimension $n(2n-1)-\sum_{\sigma=1}^{s} n_\sigma^2$ and can not be immersed in an Euclidean space with dimension
$8k(n-k)-2k-2\alpha_1(n)+2\alpha_1(n-k)+\alpha_1(k)-1.$

Thereby k is an arbitrary integer which can be represented as sum $\sum_{\sigma\in S} n_\sigma$ with $S\subset\{1,\ldots,s\}$. □

Proposition 4.46

$SO(2n)/U(n_1)\times\cdots\times U(n_s)$ is a π–manifold or can be immersed in an Euclidean space with dimension $2n^2-n-s$.

Proof: The statement is caused by Proposition 1.10 and Proposition 4.42.
□

Appendix

Each of the five tables in this appendix is concerned with one of the five types of homogenous spaces considered in the last chapter. Each table contains the dimensions and bounds of 25 homogeneous spaces chosen by random.

For the sake of convenience we denote groups like $U(n_1) \times \cdots \times U(n_s)$ by $U(n_1, \ldots, n_s)$.

Complex flag manifolds

Manifold	dim	l.b.	u.b.
U(128)/U(14,16,7,9,25,5,24,21,7)	14086	16256	16375
U(99)/U(13,2,7,9,6,21,20,21)	8180	9702	9793
U(46)/U(16,10,3,17)	1462	2036	2112
U(51)/U(4,20,4,23)	1640	2544	2597
U(33)/U(4,19,8,2)	644	1026	1085
U(32)/U(25,7)	350	666	694
U(15)/U(3,4,8)	136	210	222
U(66)/U(3,6,9,17,12,16,1,2)	3536	4290	4348
U(127)/U(10,24,8,9,16,23,13,16,8)	14034	16002	16120
U(91)/U(23,24,17,4,23)	6342	8190	8276
U(89)/U(24,25,24,16)	5888	7790	7917
U(45)/U(23,2,18,2)	1164	1980	2021
U(30)/U(21,5,4)	418	724	832
U(53)/U(10,18,14,11)	2068	2750	2805
U(154)/U(23,25,10,20,21,23,4,22,2,4)	20572	23562	23706
U(71)/U(15,24,18,11,3)	3786	4970	5036
U(145)/U(16,24,19,9,25,14,21,14,3)	18284	20880	21016
U(59)/U(8,9,15,1,4,22)	2610	3422	3475
U(56)/U(6,11,14,6,19)	2386	3064	3131
U(65)/U(22,7,3,13,15,5)	3264	4160	4219
U(67)/U(25,16,12,14)	3268	4368	4485
U(52)/U(10,21,17,1,3)	1864	2648	2699
U(65)/U(3,10,6,21,1,11,13)	3348	4160	4218
U(128)/U(23,13,15,12,16,13,2,16,10,8)	14468	16256	16374
U(38)/U(23,15)	690	1346	1375

Quaternional flag manifolds

Manifold	dim	l.b.	u.b.
Sp(85)/Sp(6,4,18,17,13,3,24)	11612	14280	14358
Sp(61)/Sp(9,22,7,20,3)	5396	7320	7376
Sp(121)/Sp(16,20,22,4,23,20,16)	24600	29040	29154
Sp(113)/Sp(5,9,1,22,6,23,25,1,21)	21092	25312	25416
Sp(74)/Sp(15,16,3,11,25,4)	8448	10800	10872
Sp(125)/Sp(6,24,25,21,22,21,6)	25972	30988	31118
Sp(121)/Sp(25,22,10,7,24,15,18)	24516	29040	29154
Sp(54)/Sp(15,14,15,10)	4340	5698	5774
Sp(63)/Sp(5,19,21,3,15)	5816	7758	7870
Sp(28)/Sp(2,21,5)	628	1136	1251
Sp(134)/Sp(9,25,9,23,20,17,5,1,25)	30600	35644	35769
Sp(105)/Sp(22,1,17,25,21,19)	17648	21696	21939
Sp(66)/Sp(11,19,8,15,13)	6832	8576	8641
Sp(97)/Sp(1,9,10,7,11,4,13,19,10,13)	16484	18624	18711
Sp(105)/Sp(22,18,11,25,13,14,2)	18204	21840	21938
Sp(35)/Sp(9,10,16)	1576	2368	2412
Sp(46)/Sp(7,18,21)	2604	4112	4183
Sp(109)/Sp(24,18,22,25,6,7,1,4,2)	19532	23544	23644
Sp(18)/Sp(3,15)	180	344	356
Sp(19)/Sp(1,18)	72	138	142
Sp(94)/Sp(18,22,21,3,8,8,14)	14508	17484	17571
Sp(58)/Sp(8,15,21,10,3,1)	5048	6612	6664
Sp(149)/Sp(23,9,9,3,23,7,9,24,24,18)	38732	44104	44243
Sp(93)/Sp(10,21,16,20,11,2,13)	14316	17112	17198
Sp(115)/Sp(1,5,14,24,18,1,15,24,13)	22264	26220	26326

Real oriented flag manifolds

Manifold	dim	l.b.	u.b.
SO(206)/SO(36,50,32,16,16,24,24,8)	17944	21012	21115
SO(76)/SO(26,4,12,6,28)	2060	2812	2850
SO(90)/SO(36,38,16)	2552	3872	4005
SO(66)/SO(26,22,12,6)	1508	2112	2145
SO(312)/SO(48,20,20,12,46,34,44,38,6,44)	42736	48360	48516
SO(218)/SO(28,32,16,18,34,12,32,46)	20348	23544	23653
SO(70)/SO(12,28,30)	1536	2336	2415
SO(98)/SO(4,44,2,12,12,24)	3392	4704	4753
SO(260)/SO(28,22,46,34,46,42,20,22)	29148	33540	33670
SO(98)/SO(22,6,32,38)	3308	4658	4753
SO(108)/SO(18,8,26,16,8,16,16)	4884	5720	5778
SO(82)/SO(46,32,4)	1784	3238	3321
SO(42)/SO(18,16,8)	560	828	861
SO(256)/SO(10,20,42,20,18,36,28,38,2,42)	28628	32512	32640
SO(268)/SO(38,32,30,48,30,50,4,36)	30720	35644	35778
SO(62)/SO(50,12)	600	1160	1196
SO(30)/SO(22,8)	176	330	349
SO(232)/SO(28,12,34,50,36,22,50)	22480	26674	26796
SO(54)/SO(20,34)	680	1316	1356
SO(176)/SO(38,22,16,20,46,32,2)	12624	15312	15400
SO(178)/SO(24,16,10,30,24,50,24)	13100	15664	15753
SO(192)/SO(26,50,34,10,14,32,4,22)	15356	18240	18336
SO(302)/SO(50,18,48,48,48,30,32,28)	39380	45286	45451
SO(278)/SO(38,8,36,6,48,34,18,50,40)	33280	38364	38503
SO(230)/SO(12,34,30,44,20,48,30,12)	22508	26220	26335

The manifolds

$Sp(n)/U(n_1) \times \cdots \times U(n_s)$

Manifold	dim	l.b.	u.b.
Sp(45)/U(7,1,13,6,18)	3516	4038	4090
Sp(64)/U(11,10,23,20)	7106	8178	8252
Sp(68)/U(11,19,21,8,9)	8248	9224	9311
Sp(110)/U(16,16,8,4,11,25,8,17,5)	22594	24200	24301
Sp(47)/U(13,12,22)	3668	4402	4462
Sp(129)/U(13,20,14,20,18,6,9,15,6,8)	31480	33282	33401
Sp(61)/U(16,24,19,2)	6306	7282	7499
Sp(70)/U(8,21,18,23)	8512	9682	9866
Sp(86)/U(6,7,11,5,10,22,6,17,2)	13734	14792	14869
Sp(112)/U(6,1,22,16,3,17,19,14,14)	23372	25088	25191
Sp(51)/U(22,2,9,18)	4360	5190	5249
Sp(47)/U(4,21,1,21)	3566	4402	4461
Sp(114)/U(19,25,10,16,13,23,2,6)	24026	25992	26098
Sp(43)/U(2,16,10,9,6)	3264	3698	3736
Sp(50)/U(9,11,18,2,4,6)	4468	4996	5044
Sp(59)/U(23,2,17,16,1)	5942	6872	7016
Sp(79)/U(12,19,9,4,6,10,19)	11462	12482	12554
Sp(73)/U(7,19,24,15,8)	9456	10616	10726
Sp(84)/U(24,11,11,15,9,14)	12876	14088	14190
Sp(56)/U(25,9,22)	5138	6208	6325
Sp(85)/U(4,10,11,14,21,15,7,3)	13378	14450	14527
Sp(118)/U(12,19,12,3,10,8,22,22,10)	26076	27848	27957
Sp(23)/U(22,1)	596	597	1079
Sp(53)/U(16,24,13)	4670	5578	5668
Sp(114)/U(2,6,24,13,19,2,13,16,19)	24170	25992	26097

The manifolds

$SO(2n)/U(n_1) \times \cdots \times U(n_s)$

Manifold	dim	l.b.	u.b.
SO(282)/U(24,18,23,2,21,18,10,15,10)	36998	39480	39612
SO(130)/U(5,14,20,15,5,6)	7478	8300	8379
SO(136)/U(22,13,11,9,13)	8156	9106	9175
SO(166)/U(18,10,6,5,15,2,19,8)	12556	13612	13687
SO(248)/U(4,8,19,21,10,25,2,16,15,4)	28520	30504	30618
SO(66)/U(24,6,3)	1524	1678	2142
SO(86)/U(3,25,14,1)	2824	3526	3651
SO(160)/U(20,16,22,4,4,14)	11352	12640	12714
SO(96)/U(12,5,8,21,2)	3882	4502	4555
SO(88)/U(13,23,8)	3066	3778	3825
SO(104)/U(24,22,6)	4260	5280	5353
SO(104)/U(9,1,14,19,9)	4636	5280	5351
SO(270)/U(15,9,24,17,24,25,10,11)	33722	36180	36307
SO(90)/U(20,25)	2980	3918	4003
SO(198)/U(14,14,17,18,16,9,10,1)	18060	19404	19495
SO(92)/U(4,19,5,1,17)	3494	4140	4181
SO(124)/U(6,14,11,4,15,10,2)	6928	7564	7619
SO(194)/U(17,7,2,18,6,25,22)	16910	18624	18714
SO(106)/U(18,1,8,20,3,3)	4758	5512	5559
SO(30)/U(3,3,5,4)	376	420	431
SO(38)/U(3,7,2,7)	592	684	699
SO(186)/U(13,6,8,7,20,2,18,12,7)	15966	17112	17196
SO(24)/U(2,10)	172	173	274
SO(274)/U(11,22,21,19,2,23,4,16,19)	34828	37264	37392
SO(56)/U(3,25)	906	907	1538

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