# Farthest points on convex surfaces

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Doctoral Thesis

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University of Dortmund

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# Zusammenfassung

Sei $S$ eine konvexe Fläche in $I\!R^3$ versehen mit der induzierten Metrik $\rho$. Sei $F$ bzw. $M$ die mengenwertige Abbildung, die einem Punkte $x \in S$ die Menge aller globalen, bzw. lokalen Maxima der Distanzfunktion $\rho_x$ zum Punkte $x$ zuordnet.

Im ersten Teil dieser Arbeit werden Injektivitäts- und (Nicht)surjektivitätseigenschaften von $F$ und $M$ untersucht, auch in Beziehung zu den kritischen Punkten der Distanzfunktionen. Beispielsweise ist $F$ injektiv, sobald $S$ glatt ist.

Einige unserer Ergebnisse bleiben richtig, wenn wir zur Sphäre homöomorphe Riemann'sche Flächen anstelle von konvexen Flächen betrachten.

Anschließend finden wir Kriterien dafür, dass ein Punkt lokales Maximum für irgendeine Distanzfunktion sei. Außerdem werden die Punkte charakterisiert, die lokale Maxima für mehr als nur eine Distanzfunktion sind.

Ein Hauptergebnis des Arbeit ist die Implikationskette *a)* $\rightarrow$ *b)* $\rightarrow$ *c)* $\rightarrow$ *d)* $\rightarrow$ *e)* bezüglich der Aussagen: *a)* Es gibt einen Punkt $x \in S$ mit unzusammenhängender Menge $F_x$; *b)* Es gibt einen Punkt $x \in S$ mit unzusammenhängender Menge $M_x$; *c)* Es gibt an einem Punkt $x \in S$ eine Schlinge deren Länge kleiner als $2\rho(x, F_x)$ ist; *d)* Die Abbildung $F$ ist nicht surjektiv; *e)* Es gibt einen Punkt $x \in S$ mit mehrwertigem $F_x$.

Als weiteres Hauptergebnis beweisen wir die Äquivalenz der Aussagen *a)* - *e)* im bezüglich der Baire'schen Kategorien typischen Falle. Des weiteren beweisen wir auch im polyedralen Falle die Äquivalenz der Aussagen *a)* - *e)*.

Sei $\mathcal{S}_2$ die Familie aller Flächen $S$, die die obige Eigenschaft *b)* besitzen. Wir verwenden Satz 17, um den Rand von $\mathcal{S}_2$ zu charakterisieren. Außerdem finden wir Kriterien für die Existenz eines Punktes $x \in S$ mit mehrwertigem $F_x$.

Im zweiten Teil der Arbeit wenden wir einige der erzielten Ergebnisse auf spezielle Klassen von Flächen an.

Wir finden Klassen von Flächen mit bijektivem und (nicht)involutivem $F$ in Verbindung zu einer alten Vermutung von H. Steinhaus ([10], S. 44).

Dann untersuchen wir den Schnittort und die Maxima der Distanzfunktionen auf doppelten konvexen Polygonen und auf typischen entarteten konvexen Fächen. Wir berechnen explizit die Schnittorte und die Maxima der Distanzfunktionen auch in einem höher-dimensionalen Fall, dem des doppelten beliebigen $d$-Simplizes.

Die Arbeit endet mit einer Reihe von offenen Fragen und Problemen.

# Contents

# Introduction

> For centuries, mathematicians and non-mathematicians alike have been fascinated by geometrical problems, particularly problems that are "intuitive", in the sense of being easy to state.
>
> Croft, Falconer and Guy [10]

The convex geometry particularly provides "intuitive" problems, the solutions of which require ingenious or sophisticated mathematical ideas and techniques. The reader can find an overview of most branches of convex geometry, and also numerous references, in the handbook [17].

This study is about "geometry", in the originary sense of the word, "$\gamma\hat{\eta}$ $\mu\epsilon\tau\rho\acute{\epsilon}\omega$": measurements on the surface of the world.

A *convex surface* in $I\!R^d$ is the boundary of an open bounded convex set in $I\!R^d$, $d \geq 2$.

We find local and global properties of distance functions on convex surfaces in the 3-dimensional Euclidean space. The results also hold for *degenerate* convex surfaces, which can be seen as doubly covered planar convex bodies (a precise definition is given in the end of the Introduction). Sometimes it was possible to replace the condition of convexity with the assumption of smoothness of the metric on a surface homeomorphic to the sphere.

Denote by $\mathcal{S}$ the space of all convex surfaces (degenerate or not) in $I\!R^3$, and by $\mathcal{G}_0$ the set of all Riemannian surfaces homeomorphic to the 2-dimensional sphere.

Unless otherwise stated, we shall implicitly assume that a surface $S$ belongs to $\mathcal{S} \cup \mathcal{G}_0$.

For two points $x, y \in S$, $\rho(x, y)$ is the *geodesic (intrinsic) distance* between them induced by the Euclidean distance, and $\rho_x$ is the distance function from $x$: $\rho_x(y) = \rho(x, y)$. For $x \in S$ denote by $F_x$ the set of all *farthest points from* $x$ (i.e., global maxima of $\rho_x$) and by $F$ the multivalued mapping associating to any point $x \in S$ the set $F_x$. Let $M_x$ be the set of all local maxima of $\rho_x$, and $M$ the corresponding mapping. For simplicity, we shall not distinguish between a point and the set containing exactly that point.

In the first part of this study we obtain several properties of $F$ and $M$, in order to investigate their injectivity and (non-) surjectivity, and the relationship to critical points of distance functions. Some consequences and examples are given in the second part.

For $S \in \mathcal{S}$, it is a known result that the mapping $F$ is upper semi-continuous. In the last years, several questions about farthest points proposed by H. Steinhaus (see the chapter A35 of the book [10] of H. T. Croft, K. J. Falconer and R. K. Guy) have been answered by T. Zamfirescu ([47], [48], [49], [51], [53]). J. Rouyer [30], [32] showed that some of his results are also true in the framework of Riemannian geometry.

We say that $F$ is *injective* on $S$ if $F_x \cap F_y = \emptyset$ for any pair of distinct points $x, y \in S$. Also, we call $F$ *surjective* if every point of $S$ belongs to $F_x$ for some point $x \in S$. When we say that $F$ is bijective or a homeomorphism, we implicitly state that $F$ is single-valued. The definitions for $M$ are similar.

In the first section we establish some basic properties of the sets $M_y^{-1} = \{x \in S;\ y \in M_x\}$ and $F_y^{-1} = \{x \in S;\ y \in F_x\}$, and show that if the total angle at the point $y \in S$ is equal to $2\pi$ then $y$ is a local maximum for at most one distance function. Consequently, for any $S$ and for all points $y \in S$, except possibly those in a set at most countable, $|F_y^{-1}| \leq 1$; and $F$ is injective on any smooth $S$, hence on nearly all $S \in \mathcal{S}$ (in a sense to be rendered precise later). Here, $|A|$ denotes the cardinality of the set $A$.

Next we give some criteria for a point to be a local maximum for some distance function, and characterize those points which are local maxima for more than one distance function.

We also obtain some topological properties of sets related to the mapping $F$. For example, on most convex surfaces there exists no point $x$ with an arc in $F_x$ (Theorem 12).

The union of two *segments* (i.e., shortest paths) from $x \in S$ to some point $y \in S$, which make an angle equal to $\pi$ at $y$, will be called a *loop at $x$*; necessarily, these segments have equal lengths.

One main result of the first part of this thesis (Theorem 17) shows, for a polytopal or a typical (the definition is given in the end of the Introduction) convex surface $S$, the equivalence of the following statements: *a)* there is

$x \in S$ with a disconnected set of relative maxima of $\rho_x$; *b)* there is $x \in S$ with a disconnected set of absolute maxima of $\rho_x$; *c)* the mapping $M$ is not surjective; *d)* the mapping $F$ is not surjective; *e)* there exists a loop $\Lambda$ at some point $x$, of length $l(\Lambda)$ less than twice the radius of $S$ at $x$.

Denote by $\mathcal{S}_2$ the set (introduced by T. Zamfirescu in [51]) of all convex surfaces in $\mathcal{S}$ which satisfy the statement *b)* of Theorem 16.

Making use of Theorem 17, we characterize the boundary of $\mathcal{S}_2$, thus giving a partial answer to a question proposed in [51]. We also find several criteria for a surface to contain points with multiple farthest points.

The problem of determining the sets of farthest points on some particular surface by direct computation via the equations of the geodesics, may be very difficult (see the case of ellipsoids).

In the second part of this thesis we restrict the study, and apply some of the obtained results, to special classes of surfaces, which also provide interesting examples (from the theoretical point of view).

Our main results are related to a conjecture of H. Steinhaus, saying that if the mapping $F$ is single-valued and involutive on $S \in \mathcal{S}$ then $S$ is a sphere (see [10], p. 44). We study Steinhaus' conditions assuming central symmetry, and present (Theorem 35) a class $\mathcal{I}$ of convex surfaces disproving this conjecture. The class $\mathcal{I}$ contains the ellipsoids of revolution with semi-axes $a = b > c$ (Theorem 34).

We also show (Theorem 37) that the mapping $F$ is a homeomorphism but not an involution on ellipsoids of revolution with semi-axes $a = b < c < 2a$.

For a point $x \in S$, let $C_x$ be the set of all points joined to $x$ by at least two segments, and $C(x)$ the *cut locus of* $x$, i.e. the set of all endpoints (different from $x$) of maximal (by inclusion) segments starting at $x$. Clearly, $C_x \subset C(x)$, and $C(x)$ is a tree (see [22] or [33] for basic properties of the cut locus).

Recall that a *tree* is a set $T$ any two points of which can be joined by a unique arc included in $T$.

A point $y \in T$ is called a *ramification point* of $T$ if $T \setminus \{y\}$ has at least three components, and a tree is called *non-degenerate* if it has at least one ramification point. The *degree* of a ramification point $y$ of $T$ is the number of components the removal of $y$ produces. A $Y$-tree is a tree with a unique ramification point of degree 3.

An extremity of $T$ is a point whose removal does not disconnect $T$. Of course, a tree may have no extremities.

Next we study cut loci and maxima of distance functions on degenerate convex surfaces: doubly covered polygons and typical degenerate convex surfaces. For example, in the latter case, if $x$ is a point interior to a face then the set $C(x) \setminus C_x$ is dense and of second Baire category in the opposite face (Theorem 45).

Since the doubly covered convex polygons can be viewed as degenerate polyhedral surfaces, our study completes previous results of J. Rouyer [29], [31]. The problem of determining shortest paths, cut loci and farthest points on polyhedral convex surfaces was also studied from the viewpoint of computational geometry, in connection to computer science (see, for example, [1], [5], [26], [34]).

We determine explicitly the sets of farthest points on a higher dimensional example, the double of an arbitrary $d$-dimensional simplex.

We conclude with some open questions and final remarks; for example, we show that there is no special relationship between the notion of an endpoint and that of a farthest point. An *endpoint* of the surface $S \in \mathcal{S}$ is a point not interior to any segment of $S$ [42].

We reserve this last part of the Introduction for additional notation and definitions. We refer to [3] or [8] for the notions not defined here.

Put $\rho_x(A) = \rho(x, A) = \inf_{y \in A} \rho_x(y)$. The *radius* $\mathrm{rad}(S)$ of the surface $S$ is defined by $\mathrm{rad}(S) = \inf_{x \in S} \rho_x(F_x)$, its *diameter* $\mathrm{diam}(S)$ is defined by $\mathrm{diam}(S) = \sup_{x \in S} \rho_x(F_x)$ and the *injectivity radius* $\mathrm{inj}(S)$ by $\mathrm{inj}(S) = \inf_{x \in S} \rho(x, C_x)$.

A *domain* of $S$ is an open connected subset of $S$.

Define $F_S = \cup_{x \in S} F_x$, and $F_A^{-1} = \cup_{y \in A} F_y^{-1}$ $(A \subset S)$.

For $r > 0$, let $B(A, r) = \{x \in S; \rho(x, A) < r\}$.

$\mathrm{bd}(A)$ denotes the boundary of the set $A$, $\mathrm{int}(A)$ its interior, and $\mathrm{cl}(A)$ its closure. aff(A) stands for the affine subspace spanned by $A$, and $l(\Gamma)$ for the length of the curve $\Gamma$.

For distinct points $x, y \in \mathbb{R}^d$, let $[xy]$ denote the line-segment from $x$ to $y$; $|| \cdot ||$ denotes the Euclidean norm.

We consider the space $\mathcal{S}$ of all convex surfaces endowed with the topology induced by the usual Pompeiu-Hausdorff metric

$\delta(S_1, S_2) = \max\{\sup_{x \in S_1} \inf_{y \in S_2} ||x - y||, \sup_{x \in S_2} \inf_{y \in S_1} ||x - y||\}$.

Thus, $\mathcal{S}$ is a Baire space. Each convex surface, taken with its intrinsic metric, is itself a Baire space. In any Baire space *most* (or *typical*) elements means “all except those in a set of first category”.

Let $d \geq 2$; let $H$ be a hyperplane in $I\!\!R^{d+1}$, $C$ a convex surface in $H$ and $\mathcal{S}_C$ the set of all convex surfaces in $I\!\!R^{d+1}$ which contain $C$. For $S \in \mathcal{S}_C$, we shall use the notation $S = S_1 \bowtie S_2$, where $S_1$ and $S_2$ are the intersections of $S$ with the two (closed) halfspaces of $I\!\!R^{d+1}$ determined by $H$.

$D$ is a *$d$-dimensional degenerate convex surface* if there exists a compact convex set $B_D \subset I\!\!R^d$ with non-empty interior, such that $D = B \cup B'$, where $B$ and $B'$ are isometric copies of $B_D$, and the image of $\mathrm{bd}(B_D)$ via each of these isometries is $B \cap B'$. Thus, $D$ can be viewed as a limit in $I\!\!R^{d+1}$ of non-degenerate convex surfaces from $\mathcal{S}_{\mathrm{bd}(B_D)}$. We write $D = B \bowtie B'$ and say that $D$ is *the double of* $B$; $B$ and $B'$ are the (isometric) *faces* of $D$; *the border of* $D$ is the image of $\mathrm{bd}(B_D)$ through the above mentioned isometries. Each degenerate convex surface $D$ is considered with the resulting metric $\rho$.

# Chapter 1

# General properties

## 1.1 Maxima and injectivity

The set $F_x$ cannot have Hausdorff dimension larger than 1 (see [19] and [47]), and the same is true for $M_x$, too (see [27]). In contrast to them, the set $F_y^{-1}$ can be very large: in the first example given in [40], $F_{v_i}^{-1}$ is a half-surface ($i = 1, 2$); moreover, $M_y^{-1} \cup \{y\}$ can be the whole surface (see Theorem 3).

In this section we partially describe the sets $F_y^{-1}$ and $M_y^{-1}$ (a detailed description will be given later), and afterwards obtain the general injectivity of the mappings $F$ and $M$.

We first state some lemmas which will be needed later on. The arguments given in [49] for the case $S \in \mathcal{S}$ also hold for $S \in \mathcal{G}_0$:

**Lemma 1** *Let $S \in \mathcal{S} \cup \mathcal{G}_0$, $x \in S$ and $y, z \in C(x)$ be distinct. Suppose $\Gamma_y$, $\Gamma'_y$ are (possibly coinciding) segments from $x$ to $y$ and $\Gamma_z$, $\Gamma'_z$ are (possibly coinciding) segments from $x$ to $z$. Then there is a domain $\Delta$ with boundary $\Gamma_y \cup \Gamma'_y \cup \Gamma_z \cup \Gamma'_z$ and a Jordan arc $J_{yz}$ in $C_x \cup \{y, z\}$ joining $y$ to $z$. Moreover, every point in $J_{yz} \setminus \{y, z\}$ belongs to $\Delta$ and can be joined to $x$ by two segments the union of which separates $y$ from $z$.*

The following result appears implicitly in [40]; roughly speaking, it says that, in a quadrilateral, the sum of lengths of two opposite edges is less than the sum of lengths of the diagonals.

**Lemma 2 (quadrilateral inequality)** *Let $(X, \rho)$ be a metric space whose distance between any two points is realized by at least one segment. Suppose that the segments do not branch, and let $a, b, c, d \in X$. Let $\Gamma_{ac}$ and $\Gamma_{bd}$ be segments joining the points $a$ and $c$, and respectively $b$ and $d$. If $\Gamma_{ac} \cap \Gamma_{bd} = \{e\}$ and $\rho(a,b)+\rho(c,d) \geq \rho(a,c)+\rho(b,d)$, then $a = d$ or $b = c$ or $a = c \in \Gamma_{bd}$ or $b = d \in \Gamma_{ac}$.*

*Proof:* By applying twice the triangle inequality, we get

$$\rho(a,b) \leq \rho(a,e) + \rho(e,b), \quad \rho(c,d) \leq \rho(c,e) + \rho(e,d).$$

Thus,

$$\rho(a,b) + \rho(c,d) \leq \rho(a,e) + \rho(e,b) + \rho(c,e) + \rho(e,d) = \rho(a,c) + \rho(b,d).$$

But we know that

$$\rho(a,b) + \rho(c,d) \geq \rho(a,c) + \rho(b,d),$$

whence

$$\rho(a,b) = \rho(a,e) + \rho(e,b) \text{ and } \rho(c,d) = \rho(c,e) + \rho(e,d).$$

Since the segments do not admit bifurcations, this ends the proof. □

**Lemma 3** *The multivalued mapping $F^{-1}$, associating to any point $y \in F_S$ the set $F_y^{-1}$, is upper semi-continuous.*

*Proof:* Consider a sequence of points $y_n \in F_S$ converging to $y$, and a sequence of points $x_n \in F_{y_n}^{-1}$ converging in $S$ to some point $x$.

The upper semi-continuity of the mapping $F$ yields now $\lim_{n\to\infty} F_{x_n} \subset F_x$, whence $\lim_{n\to\infty} y_n = y \in F_x$. □

The set $F_y^{-1}$ is clearly closed, but $M_y^{-1}$ may not be closed; Theorem 3 provides some suitable examples. The next result gives basic properties of these sets.

**Proposition 1** *Let $y$ be a point in a surface $S \in \mathcal{S} \cup \mathcal{G}_0$ such that $M_y^{-1} \neq \emptyset$. Then $M_y^{-1}$ is arcwise connected, and for any two distinct points $x_1$, $x_2$ in $M_y^{-1}$ there exists a Jordan arc $J_{x_1x_2} \subset M_y^{-1}$ joining $x_1$ to $x_2$, such that $J_{x_1x_2} \setminus C(y)$ is the union of at most two segments. Moreover, $S \setminus M_y^{-1}$ is connected.*

*The above statements also hold if $M^{-1}$ is replaced by $F^{-1}$. Moreover, if $N \subset F_S$ is connected then $F_N^{-1}$ is connected.*

*Proof:* Suppose, for the point $y \in S$, that there are two distinct points $x_1, x_2$ in $M_y^{-1}$. Let $V_i$ be neighbourhoods of $y$ such that $\rho(x_i, v) \leq \rho(x_i, y)$ for all points $v$ in $V_i$ $(i = 1, 2)$.

Let $\Gamma_{x_i}$ be a segment from $y$ to $x_i$, and let $z_i$ be the cut point of $y$ in the direction of $\Gamma_{x_i}$ $(i = 1, 2)$. If $\Gamma_{z_i}$ is a segment from $y$ to $z_i$ including $\Gamma_{x_i}$, and $u \in \Gamma_{z_i} \setminus \Gamma_{x_i}$, then for any $u' \in V_1 \cap V_2$ we have

$$\rho(u, y) = \rho(u, x_i) + \rho(x_i, y) \geq \rho(u, x_i) + \rho(x_i, u') \geq \rho(u, u').$$

Therefore, $y \in M_u$.

Thus, for any segment $\Gamma_x$ joining $y$ to some point $x \in M_y^{-1}$, the cut point $z$ of $y$ along $\Gamma_x$ also belongs to $M_y^{-1}$, whence $M_y^{-1} \cap C(y) \neq \emptyset$.

Now, for the cut points $z_1, z_2$, we have (by Lemma 1) a Jordan arc $J_{z_1 z_2} \subset C_y \cup \{z_1, z_2\}$ joining $z_1$ to $z_2$. Moreover, every point $z \in J_{z_1 z_2} \setminus \{z_1, z_2\}$ can be joined to $y$ by two segments, say $\Gamma_z$ and $\Gamma'_z$, the union of which separates $z_1$ from $z_2$.

We claim that $M_z = y$ holds for all points $z$ in $J_{z_1 z_2} \setminus \{z_1, z_2\}$.

To prove the claim, suppose there exists $y' \in V_1 \cap V_2$ such that $\rho(z, y') \geq \rho(z, y)$. Since $\Gamma_z \cup \Gamma'_z$ separates $z_1$ from $z_2$, it also separates $y'$ either from $z_1$ or from $z_2$. Assume that $\Gamma_z \cup \Gamma'_z$ separates $y'$ from $z_2$, and let $\Gamma_{z_2 y'}$ be a segment from $z_2$ to $y'$. Put $\Gamma_{z_2 y'} \cap (\Gamma_z \cup \Gamma'_z) = \{e\}$; we may assume that $e \in \Gamma_z$. Summing up the inequalities

$$\rho(z, y') \geq \rho(z, y), \quad \rho(z_2, y) \geq \rho(z_2, y'),$$

we obtain

$$\rho(z, y') + \rho(z_2, y) \geq \rho(z_2, y') + \rho(z, y).$$

Since $z$, $z_2$ and $y$ are distinct, as well as $z$, $z_2$ and $y$, from Lemma 2 we get $y' = y$, and therefore, $y \in M_z$.

In conclusion, for any two distinct points $x_1$, $x_2$ in $M_y^{-1}$ there exists a Jordan arc $J_{x_1 x_2} \subset M_y^{-1}$ joining $x_1$ to $x_2$, such that $J_{x_1 x_2} \setminus C(y)$ is the union of at most two subsegments of segments starting at $y$.

Suppose now that $S \setminus M_y^{-1}$ is disconnected. Denote by $S'$ the component of $S \setminus M_y^{-1}$ containig $y$, and take a point $u$ in a component $S''$ of $S \setminus M_y^{-1}$ different from $S'$. Consider a segment $\Gamma_{yu}$ from $y$ to $u$. It follows that the set $\Gamma_{yu} \setminus (S' \cup S'')$ meets $M_y^{-1}$.

Take a point $x$ in $M_y^{-1} \cap \Gamma_{yu}$; then $y \in M_x$ and, as we have proved above, all points of $\Gamma_{yu}$ from $x$ to $u$ also belong to $M_y^{-1}$, in particular $u \in M_y^{-1}$, a contradiction.

All proofs are similar if $M^{-1}$ is replaced by $F^{-1}$.

For the last part, let $N$ be a connected subset of $F_S$, and assume $F_N^{-1} = A \cup B$, with $A \cap B = \emptyset$. Set $A' = F_A \cap N$ and $B' = F_B \cap N$, hence $N = A' \cup B'$. Since $N$ is connected, we have either $\mathrm{cl}(A') \cap B' \neq \emptyset$, or $\mathrm{cl}(B') \cap A' \neq \emptyset$. Assume that $\mathrm{cl}(A') \cap B' \neq \emptyset$, whence there exists a point $y \in B'$, and a sequence of points $y_n \in A'$ which converges to $y$.

Since $y \in B' = F_B \cap N$, there exists $x \in B$ such that $y \in F_x$. If $F_y^{-1} \cap A \neq \emptyset$ then there exist a point $z \in F_y^{-1} \cap A$, and an arc $J \subset F_y^{-1} \subset A \cup B$ joining $z$ to $x$, hence $F_N^{-1}$ is connected.

Similarly, since $y_n \in A' = F_A \cap N$, there exists $x_n \in A$ such that $y_n \in F_{x_n}$, for $n \geq 1$. If there exists a subscript $n$ such that $F_{y_n}^{-1} \cap B \neq \emptyset$, then there exist a point $z_n \in F_{y_n}^{-1} \cap B$, and an arc $J \subset F_{y_n}^{-1} \subset A \cup B$ joining $z_n$ to $x_n$, hence $F_N^{-1}$ is connected.

Assume now that $F_{y_n}^{-1} \subset A$ and $F_y^{-1} \subset B$. Since $y_n \to y$, the upper semi-continuity of $F^{-1}$ implies the existence of a set $L \subset F_y^{-1}$ such that, possibly passing to a subsequence, $\lim_{n\to\infty} F_{y_n}^{-1} = L \subset B$. We obtain $\mathrm{cl}(A) \cap B \neq \emptyset$, whence $F_N^{-1}$ is connected. □

The space $T_y$ of all unit tangent directions at $y \in S$ is a closed Jordan curve in the unit 2-sphere. Denote by $\theta_y$ the *total angle at* $y$, i.e. the length of $T_y$. The *total curvature at* $y$ is defined by $\omega_y = 2\pi - \theta_y$. A point $y \in S$ is called *conical* if $\theta_y < 2\pi$.

The following result can be found in [47].

**Lemma 4** *If $S \in \mathcal{S}$, $x \in S$ and $y \in M_x$, then each arc in $T_y$ of length $\pi$ contains the tangent direction of a segment from $x$ to $y$. Thus, if $\omega_y < \pi$, then there are at least two segments from $x$ to $y$, and if $S$ is differentiable at $y$ and there are only two segments from $x$ to $y$ then these have opposite tangent directions at $y$.*

For $S \in \mathcal{G}_0$, the well known (see [14], for example) Berger Lemma asserts a similar thing.

The point $y \in S$ is called a *critical point for* $\rho_x$ if for any vector $v$ tangent to $S$ at $y$ there exists a segment from $y$ to $x$ whose direction at $y$ makes an angle at most $\pi/2$ with $v$.

**Lemma 5** *For $S \in \mathcal{G}_0$ and $x \in S$, any local maximum for $\rho_x$ is a critical point for $\rho_x$.*

Recall [45] that in a complete metric space $(\mathcal{X}, \delta)$, a set $M$ is *porous* if for each $x \in M$ there are a number $a > 0$ and points $y$ arbitrarily closed to $x$ such that $B(y, a\delta(x, y)) \cap M = \emptyset$. We say that the complement of any countable union of porous sets in $\mathcal{X}$ contains *nearly all* elements of $\mathcal{X}$. In Euclidean spaces, by a version of Lebesgue's density theorem, a set containing nearly all elements is large from both the measure and the Baire category points of view.

T. Zamfirescu [43] improved via porosity a result independently obtained by V. L. Klee [21] and P. Gruber [16], by showing that:

**Lemma 6** *Nearly all convex surfaces in $\mathbb{R}^d$ are smooth and strictly convex.*

**Theorem 1** *A point $y$ in $S$ with $\theta_y = 2\pi$ is a local maximum of $\rho_x$ for at most one point $x$ in $S$. Thus, the mappings $F$ and $M$ are injective on any smooth surface $S \in \mathcal{S} \cup \mathcal{G}_0$, hence on nearly all $S \in \mathcal{S}$.*

*Proof:* Let $y$ be a point in $S$ with $\theta_y = 2\pi$. Then, by Lemmas 4 and 5, there exist at least two segments from each $x \in M_y^{-1}$ to $y$, hence $M_y^{-1} \subset C_y$. Since $C_y$ is a tree and $M_y^{-1}$ is arcwise connected by Proposition 1, $M_y^{-1}$ is also a tree. Assume that this tree contains a non-degenerate arc. Since the set of points joined with $y$ by at least three segments is at most countable, there are distinct points $x_1, x_2 \in M_y^{-1}$ joined with $y$ by precisely two segments. Let $\Gamma_i$ and $\Gamma_i'$ be the segments from $x_i$ to $y$ $(i = 1, 2)$. Then, since $\theta_y = 2\pi$, the angle at $y$ between the direction $\tau_i$ of $\Gamma_i$ and $\tau_i'$ of $\Gamma_i'$ is equal to $\pi$ $(i = 1, 2)$, again by Lemmas 4 and 5. Let $S'$ and $S''$ be the components of $S \setminus (\Gamma_1 \cup \Gamma_1')$, and assume that $x_2 \in S''$.

Since geodesics do not branch, it follows that $\Gamma_2 \cup \Gamma_2' \subset S'' \cup \{y\}$, $\tau_1$, $\tau_2$, $\tau_1'$, $\tau_2'$ are distinct and $\tau_1$, $\tau_1'$ do not separate $\tau_2$, $\tau_2'$ on $T_y$. This contradicts the fact that all arcs $\tau_i\tau_i'$ on $T_y$ have length $\pi$.

The last part is a consequence of Lemma 6 and the first assertion. $\square$

One can see on a standard torus that above we cannot drop the assumption $S \in \mathcal{G}_0$.

## 1.2 Some criteria for maxima

**Theorem 2** *On any convex surface $S$, the set of all points $y$ in $S$ with $|M_y^{-1}| > 1$ is at most countable.*

*Proof:* A point $y$ in $S$ with $|M_y^{-1}| > 1$ is a conical point, by Theorem 1; but a convex surface has at most countably many conical points (see [35] for a short proof). □

Two segments joining the points $x$ and $y$ are called *consecutive* if their union bounds a domain no point of which is interior to a segment from $x$ to $y$.

We shall implicitly use the following:

**Lemma 7** *Let $x, y$ be two points in $S$. The set $\{\alpha_i\}_{i \in I}$, consisting of all angles made at $y$ by pairs of consecutive segments from $x$ to $y$, is at most countable, and if $I \neq \emptyset$ then $\sup_{i \in I} \alpha_i$ is attained.*

*Proof:* The space $T_y$ of all unit tangent directions at $y$ is homeomorphic to the unit circle $S^1$. Fix a homeomorphism $f : T_y \to S^1$.

The interiors of any two angles, determined by distinct pairs of consecutive segments, are disjoint. Therefore, the set $\{\alpha_i\}_{i \in I}$ is at most countable.

Suppose that $\sup_{i \in I} \alpha_i$ is not attained. Then there exists a sequence of angles $\{\alpha_{i_n}\}_{n \geq 1}$ converging to $\sup_{i \in I} \alpha_i > 0$, whence $\sum_{n=1}^{\infty} \alpha_{i_n} = \infty$. But this contradicts the fact that $\Sigma_{i \in I} \alpha_i \leq \theta_y \leq 2\pi$. □

Lemmas 4 and 5 give necessary conditions for a point $y$ to be a local maximum for some distance function $\rho_x$. Next we prove a converse theorem.

For a point $x \in S$, denote by $M_x^*$ the set of strict local maxima of $\rho_x$.

**Theorem 3** *Let $x, y$ be points in a convex surface $S$, and denote by $\alpha_i$ the angles at $y$ between consecutive segments from $x$, $i \in I \neq \emptyset$. If $\max_{i \in I} \alpha_i < \pi$ then $y$ is a strict maximum for the restriction of the distance function $\rho_x$ to $B(y, 2\rho(x, y) \cos(\max_{i \in I} \alpha_i / 2))$.*

*In particular, if $\theta_y < \pi$ then $y \in M_z^*$ for all points $z$ in $S \setminus \{y\}$.*

*Proof:* To prove that $y$ is a strict local maximum for $\rho_x$, it suffices to show that, for some neighbourhood $V_y$ of $y$, $\rho_x(y) > \rho_x(z)$ holds for all $z \in V_y \cap C(x)$.

Let $l = \rho(x, y)$, and take $V_y = B(y, 2l\cos(\max_{i\in I}\alpha_i/2))$.

For $z \in C(x) \setminus \{y\}$, there exists a digon $\Delta$, bounded by two consecutive segments from $x$ to $y$, such that $z \in \Delta$. Let $\alpha_1$ be the angle of $\Delta$ at $y$, hence $\alpha_1 \leq \max_{i\in I}\alpha_i < \pi$, and we get

$$0 < \rho(y, z) < 2l\cos(\max_{i\in I}\alpha_i/2)) \leq 2l\cos(\alpha_1/2).$$

One of the two angles made at $y$ by a segment from $z$ to $y$, with the two segments bounding $\Delta$, is at most $\alpha_1/2$; denote it by $\beta$.

Consider the planar triangle $\bar{x}\bar{y}\bar{z}$ with $||\bar{x} - \bar{y}|| = l$, $||\bar{y} - \bar{z}|| = \rho(y, z)$ and the angle at $\bar{y}$ equal to $\beta$. Since $\beta \leq \alpha_1/2 < \pi/2$, we have

$$||\bar{y} - \bar{z}|| = \rho(y, z) < 2l\cos(\alpha_1/2) \leq 2l\cos\beta,$$

hence the angle at $\bar{z}$ is larger than $\beta$ and $||\bar{x} - \bar{z}|| < ||\bar{x} - \bar{y}||$.

By the convexity of the metric of $S$ (see [3] or [8]), we also have

$$\rho(x, z) \leq ||\bar{x} - \bar{z}||,$$

so we obtain $\rho(x, z) < \rho(x, y)$, i.e. $y$ is a strict local maximum for $\rho_x$. □

**Theorem 4** *Let $S$ be a convex surface and $x, y \in S$ such that $\omega_y > 0$, and denote by $\alpha_i$ the angles at $y$ between consecutive segments from $x$, $i \in I \neq \emptyset$. If $\Sigma_{i\in I}\alpha_i = \theta_y$ and $\omega_y + \max_{i\in I}\alpha_i \leq \pi$ then $M_y^{-1} = \{x\}$.*

*Proof:* From $\omega_y + \max_{i\in I}\alpha_i \leq \pi$ and $\omega_y > 0$, we obtain $\max_{i\in I}\alpha_i < \pi$, hence $y \in M_x^*$, by Theorem 3.

Since $2\pi - \theta_y = \omega_y < \pi$, we have $\theta_y > \pi$, hence (by Lemma 4) there exist at least two segments from $x$ to $y$, i.e. $x \in C_y$.

Suppose there exists $z \in M_y^{-1} \setminus \{x\}$. Then $z \in C_y$, too; since the segments do not branch, $z$ is interior to no segment from $x$ to $y$, whence it belongs to a domain $\Delta$ of $S$ bounded by two consecutive segments from $x$ to $y$. It is clear that all segments joining $z$ to $y$ remain in $\Delta$. Denote by $\alpha_1$ the angle made at $y$ by the segments bounding $\Delta$, hence $\alpha_1 \leq \max_{i\in I}\alpha_i$.

The same arguments as above show that the point $x$ belongs to a domain $\Delta'$ of $S$ bounded by two consecutive segments from $z$ to $y$. Denote by $\delta$ the angle made at $y$ by these consecutive segments hence, by Lemma 4 applied to $y \in M_z$, $\delta \leq \pi$. Thus, we obtain $\theta_y - \alpha_1 < \delta \leq \pi$. We have

$$\omega_y + \alpha_1 \leq \omega_y + \max_{i\in I}\alpha_i \leq \pi$$

and

$$\omega_y + \alpha_1 + \theta_y - \alpha_1 = 2\pi,$$

whence we obtain $\theta_y - \alpha_1 \geq \pi$, a contradiction. □

**Theorem 5** *If $y$ is a point in a convex surface $S$ such that $\theta_y < \pi$, and there exists $x \in F_y^{-1}$ with $F_x = y$, then $\mathrm{int}(F_y^{-1}) \neq \emptyset$ and $y$ is an isolated point of $F_S$. In particular, if $\theta_y < \pi/2$ then the conclusion holds.*

*Proof:* Since $S$ is not smooth at $y$, we have $y \in C(x)$ for all $x \in S$. Take a point $x$ in $F_y^{-1}$ with $F_x = y$, a positive number $\varepsilon_1 < \cos(\theta_y/2)$, and let

$$\varepsilon_2 = (\cos(\theta_y/2) + \varepsilon_1)^{-1}(\cos(\theta_y/2) - \varepsilon_1).$$

By the upper semi-continuity of $F$, since $F_x = y$, there is a neighbourhood $U$ of $x$ such that $F_U \subset B(y, \varepsilon_1 \rho(x, y))$. Possibly passing to a subset, we may assume that $U \subset B(x, \varepsilon_2 \rho(x, y))$. For an arbitrary point $v$ in $U$, we have

$$(\cos(\theta_y/2) + \varepsilon_1)\rho(x, v) < (\cos(\theta_y/2) - \varepsilon_1)\rho(x, y),$$

whence

$$(\cos(\theta_y/2) + \varepsilon_1)\rho(x, v) + \varepsilon_1 \rho(x, y) < \cos(\theta_y/2)\rho(x, y).$$

Since $\rho(x, y) \leq \rho(x, v) + \rho(v, y)$, we obtain

$$(\cos(\theta_y/2) + \varepsilon_1)\rho(x, v) + \varepsilon_1 \rho(x, y) < \cos(\theta_y/2)(\rho(x, v) + \rho(v, y)),$$

and

$$\varepsilon_1 \rho(x, y) < \varepsilon_1(\rho(x, v) + \rho(x, y)) < \rho(v, y)\cos(\theta_y/2).$$

Therefore,

$$F_v \subset B(y, \varepsilon_1 \rho(x, y)) \subset B(y, \rho(v, y)\cos(\theta_y/2)).$$

Since the maximal angle between two consecutive segments from $v$ to $y$ is less than $\theta_y < \pi$, and $\cos|_{[0,\pi]}$ is a strictly decreasing function, Theorem 3 applied to $v$ yields $\rho(v, y) > \rho(v, w)$, for any point $w \in B(y, \rho(v, y)\cos(\theta_y/2))$.

Thus $F_v = y$ and, consequently, $F_y^{-1} \supset U$.

To see that $\{y\}$ is an isolated point of $F_S$, assume that there is a sequence of points $y_n \in F_S$ converging to $y$. By the upper semi-continuity of $F^{-1}$

(Lemma 3), the sequence of sets $F_{y_n}^{-1}$ converges to a subset of $F_y^{-1}$. Consider a sequence of points $x_n \in F_{y_n}^{-1}$, convergent to the point $x' \in F_y^{-1}$. For $n$ large enough, we have

$$x_n \in B(x', \varepsilon_2 \rho(x', y)), \quad y_n \in B(y, \varepsilon_1 \rho(x', y)).$$

By replacing $x$ with $x'$ and $v$ with $x_n$ in previous inequalities, we get $\varepsilon_1 \rho(x', y) < \rho(x_n, y) \cos(\theta_y/2)$, i.e.

$$y_n \in B(y, \rho(x_n, y) \cos(\theta_y/2)).$$

Now, Theorem 3 applied to $x_n$ yields $\rho(x_n, y_n) < \rho(x_n, y)$, a contradiction.

For the second part, it suffices to prove that if $x \in F_y$ then $F_x = y$.

Suppose not, and take a point $z$ in $F_x \setminus \{y\}$. For any segments $\Gamma_{yx}$ and $\Gamma_{yz}$, joining $y$ to $x$ and respectively to $z$, the angle they make at $y$, say $\alpha$, satisfies $\alpha < \pi/4$.

Consider the planar triangle $\bar{x}\bar{y}\bar{z}$ with $||\bar{x}-\bar{y}|| = \rho(x, y)$, $||\bar{y}-\bar{z}|| = \rho(y, z)$ and the angle at $\bar{y}$ equal to $\alpha$. By the convexity of the metric $\rho$, we have

$$||\bar{x} - \bar{z}|| \geq \rho(x, z) \geq \rho(x, y),$$

hence the angle at $\bar{z}$ is smaller than or equal to $\alpha < \pi/4$. It follows that the angle at $\bar{x}$ is larger than $\pi/2$, whence

$$\rho(y, z) = ||\bar{y} - \bar{z}|| > ||\bar{x} - \bar{y}|| = \rho(x, y),$$

a contradiction to the fact that $x \in F_y$. □

The following criterion follows immediately from Theorem 1; we give it separately for its own interest and for later use.

**Theorem 6** *If $S$ is a smooth surface and $y$ a point in $M_x \setminus F_x$ for some point $x$ in $S$ then $y \notin F_S$.*

The following statement can be found in [50] and [51].

**Lemma 8** *Let $S \in \mathcal{S} \cup \mathcal{G}_0$, $x \in S$ and $y, z \in C(x)$. Let $J$ be the arc joining $y$ to $z$ in $C(x)$. If $u \in J$ is a relative minimum of $\rho_x|_{\text{int}(J)}$ then $u$ is the mid-point of a loop $\Lambda$ at $x$ and, except for the two subarcs of $\Lambda$, no segments connect $x$ to $u$.*

**Theorem 7** *Suppose that $y$ is the mid-point of a loop $\Lambda$ at some point $x$ in $S$, and the length of $\Lambda$ is less than $2\rho(x, F_x)$. Then $y \notin F_S$.*

*Moreover, a strict local minimum for the restriction $\rho_x|_{\mathrm{int}(J)}$ of some distance function $\rho_x$ to the interior of an arc $J \subset C(x)$ is a local maximum for no other distance function.*

*Proof:* Let $y$ be the mid-point of a loop $\Lambda$ at some point $x$ in $S$, with the length of $\Lambda$ less than $2\rho(x, F_x)$, hence $\theta_y = 2\pi$ and $y \notin F_x$.

Suppose that $y \in F_z$ for some point $z$ in $S$. Then, by Lemmas 4 and 5, we have $y \in C_z$.

Denote by $S'$ and $S''$ the components of $S \setminus \Lambda$. Since the segments do not branch, and $z \in C_y \setminus \{x\}$ ($y \notin F_x$), we have $z \notin \Lambda$, whence we may assume that $z \in S''$.

Let $\{\Lambda_i\}_{i \in I}$ be the family of all segments joining $z$ to $y$.

Since the segments from $x$ to $y$ included in $\Lambda$ make an angle of $\pi$ at $y$, by Lemmas 4 and 5 again, at least one of $\Lambda_i$, say $\Lambda_1$, intersects $S'$. Since $\Lambda_1$ ends at $z \in S''$ and intersects $S'$, it also intersects $\Lambda$. Therefore $z = x$, which is a contradiction.

Suppose now $y$ is a strict local minimum for the restriction $\rho_x|_J$ of $\rho_x$ to an arc $J \subset C(x)$. Then, by Lemma 8, $y$ be the mid-point of a loop $\Lambda$ at $x$. If $y \in M_z$ for some point $z \in S$ then the same arguments as above yield $z = x$, which is impossible. □

The intrinsic diameter of $S$ is the largest distance from a point in $S$ to its set of global maxima.

**Theorem 8** *On any smooth surface $S$, hence on nearly all $S \in \mathcal{S}$, the diameter is realized by points $x, y$ with $F_x = y$.*

*Proof:* Suppose there exist points $y_1 \neq y_2$ in $F_x$ such that $\rho(y_i, x) = \mathrm{diam}(S) = \rho(y_i, F_{y_i})$ $(i = 1, 2)$; then $x \in F_{y_1} \cap F_{y_2}$ and $|F_x^{-1}| \geq 2$, in contradiction to Theorem 1. □

With suplimentary assumptions, one can say more about the diameter; see Theorems 32 and 33.

## 1.3 Distance functions with a common maximum

We study next sets of points in a convex surface, the corresponding distance functions of which have a common local maximum on the surface, and obtain properties of this maximum.

The following theorem, together with Theorem 11 and Lemma 11 (see Section 1.4), underlines the contrast between properties of the mappings $F$ and $F^{-1}$.

We shall use the following result, proved in [36] and [51]; here, $\mu_1$ denotes the 1-dimensional Hausdorff measure.

**Lemma 9** *Let $x \in S$ and $A \subset C(x)$ be a Jordan arc with an endpoint at $y$. Then $A$ has a definite direction $\tau$ at $y$, and no segment from $y$ to $x$ has direction $\tau$ at $y$. Further, if the arc $\tau_1\tau_2 \subset T_y$ is minimal such that $\tau \in \tau_1\tau_2$, and in each of the directions $\tau_1, \tau_2$ there is a segment from $y$ to $x$, then $\mu_1(\tau\tau_1) = \mu_1(\tau\tau_2)$.*

We call a tree *sun-like* if it has a unique ramification point. A tree is called *finite* if it has finitely many extremities.

**Theorem 9** *Any combinatorial type of finite tree can be realized as the set $F_y^{-1}$ for some point $y$ on some convex surface.*

*Proof:* We first find a surface $S \in \mathcal{S}$, and a point $y \in S$, such that the set $F_y^{-1}$ is a sun-like tree with arbitrarily (but finitely) many extremities.

For a natural number $m \geq 3$, consider the regular pyramid $P_m$ with a polygon with $m$ vertices, say $a_1, ...a_m$, as basis, and the $(m+1)^{st}$ vertex $y$ such that $\theta_y = \pi$. Denote by $o$ the center of the basis of $P_m$. Then, because of the symmetry of $P_m$, $C(y)$ is precisely $\cup_{j=1,...,m}[oa_j]$, i.e. a sun-like tree with $m$ extremities.

We next prove that $F_y^{-1}$ is also a sun-like tree with $m$ extremities.

Notice that $o$ belongs to $F_y^{-1}$. To see this, notice that the union of all full geodesic triangles with the vertices at $o$, $y$ and $a_j$ is precisely $P_m$ ($j = 1, ..., m$). Take a vertex of the basis of $P_m$, say $a_1$; then each geodesic triangle $oya_1$ has a planar unfolding (to which we implicitly refer in the following); its angle at $o$ is equal to $\pi/m \leq \pi/3$, while the angle at $y$ is equal to $\pi/(2m) \leq$

$\pi/6$. Thus, its angle at $a_1$ is equal to $\pi - [\pi/m + \pi/(2m)]$, hence at least $\pi/2$, and $\rho(o,y) > \rho(o,z)$ for all points $z$ in $oya_1$. We obtain $F_o = y$.

We claim that any point $x$ in $C(y) \setminus \{o\}$ close enough to $o$ also belongs to $F_y^{-1}$.

Indeed, because of the symmetry of $P_m$, $x$ is joined to $y$ by precisely two segments, which make at $y$ an angle $\alpha_x < \pi/m$, since $\theta_y = \pi$. By Theorem 4, $y \in M_x^*$ and $\rho(x,y) > \rho(x,z)$ for all points $z \in B(y, \rho(x,y)\cos(\frac{\pi-\alpha_x}{2})) \setminus \{y\}$.

If we have $x, x' \in C(y) \setminus \{o\}$ in a small neighbourhood of $o$, such that $x$ lies between $o$ and $x'$ on an arc of $C(y)$, then elementary arguments show that $\rho(x,y) > \rho(x',y)$ and $\alpha_{x'} < \alpha_x < \pi/m$. Consequently, we have

$$B(y, \rho(x',y)\cos(\frac{\pi - \alpha_{x'}}{2})) \subset B(y, \rho(x,y)\cos(\frac{\pi - \alpha_x}{2})).$$

By the upper semi-continuity of $F$, if $x$ is close to $o$ then $F_x$ is close to $y = F_o$. Thus, for any point $x$ in $C(y)$ close enough to $o$ we obtain

$$F_x \subset B(y, \rho(x,y)\cos(\frac{\pi - \alpha_x}{2})),$$

whence $F_x = y$, and the claim is proved.

A point $z \notin C(y)$ is joined to $y$ by a unique segment, say $\Gamma_{zy}$; since the metric of $P_m$ is piecewise linear and $\theta_y = \pi$, if $z$ is close enough to $o$ then arbitrarily close to $y$ in the direction (at $y$) orthogonal to that of $\Gamma_{zy}$, there are points in $P_m$ at larger distance to $z$ than $y$; it follows that $y$ is not a local maximum for $\rho_z$, hence $z \notin F_y^{-1}$. Thus, $F_y^{-1}$ is a sun-like tree with $m$ extremities.

Next we shall prove by induction over $n$ the following assertion: for any tree $T$ with $n$ extremities, there exist $n$ points $b_1$,..., $b_n$ in a plane $\Pi$, and a point $y \notin T$ such that, on the surface

$$S_n = \text{bd}(\text{conv}\{b_1, ...b_n, y\})$$

$F_y^{-1} \subset C(y)$, the ramification points of $F_y^{-1}$ are arbitrarily close to $o$, and $F_y^{-1}$ is homeomorphic to $T$.

For $n = 3$ there is only one combinatorial type of tree, the $Y$-tree, and the construction given in the first part of the proof shows that it is realized.

We make first some remarks, which we shall implicitely use later, and afterwords treat the general case of induction.

In the following we denote by $C^S(y)$ the cut locus, on the surface $S$, of the point $y \in S$.

Consider distinct points $b_1, ...b_m \in \Pi$. A continuity argument shows imediately that for any half-line $L$ orthogonal to $\Pi$ at its extremity $o$, there exists a unique point $y \in L$ such that, on the surface $S_m = \mathrm{bd}(\mathrm{conv}\{b_1, ...b_m, y\})$, we have $\theta_y = \pi$. Indeed, while a variable point $u$ is moving with constant speed on $L$ from $o$ to infinity, the total angle at $u$ of the surface $\mathrm{bd}(\mathrm{conv}\{u, b_1, ...b_m\})$ is decreasing continuously from $2\pi$ to 0.

The cut locus $C^{S_m}(y)$ of $y$ is a tree with $m$ extremities. Moreover (see [5]), it is the union of line-segments.

There exists some $\varepsilon$ small enough such that the set $B([yb_1], \varepsilon) \setminus \{y, b_1\}$ does not contain vertices of $S_m$. Cutting along the line-segment $[yb_1]$ and unfolding $B([yb_1], \varepsilon)$, one can see that the direction of the arc of $C^{S_m}(y)$ at $b_1$ is the bisector of the resulting angle, by Lemma 9.

Now, arguments similar to those given in the first part of the proof show that $F_y^{-1}$ is a subtree of $C^{S_m}(y)$ and, moreover, all points of $C^{S_m}(y)$ close enough to $o$ belong to $F_y^{-1}$. We shall not repeat them here. Therefore, if the ramification points of $C^{S_m}(y)$ are close enough to $o$, then the tree $F_y^{-1}$ has also $m$ extremities, hence it has the combinatorial type of $C^{S_m}(y)$.

For the general case of induction, assume now that $n > 3$.

Consider an arbitrary tree $T_n$ with $n$ extremities, and denote by $v_1$,..., $v_n$ its extremities and by $T_{n-1}$ the graph obtained from $T_n$ after deleting the extremity $v_n$ and its corresponding edge.

Suppose that the combinatorial type of $T_{n-1}$ is obtained as the set $F_y^{-1}$, on a pyramid $S_{n-1} = \mathrm{bd}(\mathrm{conv}\{y', b_1, ...b_{n-1}\})$ as in the induction's assumption. Hence, on $S_{n-1}$, the ramification points of $F_{y'}^{-1}$ are arbitrarily close to the point $o$ in $\Pi$ given by $yo \perp \Pi$. Moreover, $F_{y'}^{-1}$ and $C^{S_{n-1}}(y')$ have both the combinatorial type of $T_{n-1}$.

For $j = 1, .., n-1$, consider an arc $A_j$ of a circle in $\Pi$ through $b_{j-1}$, $b_j$ and of arbitrarily large radius, so that $A_j$ is exterior to $\mathrm{conv}\{b_1, ...b_{n-1}\}$; here, $b_0 = b_{n-1}$. For radii arbitrarily large, the arcs $A_j$ are arbitrarily close to the line-segments $[b_{j-1}b_j]$.

Consider a point $b$ moving continuously on $\cup_{j=1,..,n-1} A_j$, from $b_1$ to $b_2$, from $b_2$ to $b_3$, and so on, until it reaches again $b_1$. We saw that for each position of $b$ there exists a unique point $y$ in the half-line $L = [oy'$ such that the total angle at $y$ on the surface $S_b = \mathrm{bd}(\mathrm{conv}\{b_1, ...b_{n-1}, b, y\})$ equals $\pi$.

We claim that the point $y$ varies continuously with the point $b$.

To prove the claim, suppose that the point $b$ is moving continuously on $A_{n-1}$ from $b_{n-1}$ to $b_1$. The total angle at $y$ is given by

$$\theta_y = \Sigma_{j=1}^{n-2} \angle b_j y b_{j+1} + \angle b_{n-1} y b + \angle b y b_1.$$

Since the expression $\Sigma_{j=1}^{n-2} \angle b_j y b_{j+1}$ does not depend on the position of the point $b$ on $C$ between $b_{n-1}$ and $b_1$, we can define a constant $K$ by

$$K = \pi - \Sigma_{j=1}^{n-2} \angle b_j y b_{j+1}.$$

Therefore, the assumption $\theta_y = \pi$ is equivalent to $\angle b_{n-1} y b + \angle b y b_1 = K$.

Consider the function $h : A_{n-1} \times L \to I\!R$, given by

$$h(b, y) = \angle b_{n-1} y b + \angle b y b_1 - K.$$

It is clearly continuous, and for any point $b$ there exists a unique point $y$ such that $h(b, y) = 0$. Thus, $y$ depends continuously on $b$ and the claim is proved.

Consequently, $S_b$ varies continuously with the position of $b$ and, in the special positions $b = b_j$, $S_b$ coincides to $S_{n-1}$ $(j = 1, ..., (n-1))$.

Moreover, when $b$ is close to $b_j$, the total angle of $S_b$ at $b_j$ is close to the total angle of $S_{n-1}$ at $b_j$.

Thus, $C^{S_b}(y)$ varies continuously with the position of $b$ and, in the special positions $b = b_j$, it coincides to $C^{S_{n-1}}(y')$ $(j = 1, ..., (n-1))$. Therefore, the combinatorial type of $C^{S_b}(y)$ is that of $C^{S_{n-1}}(y)$ plus an edge.

Since $C^{S_b}(y)$ is an union of line-segments, and it depends continuously on $b$, there is a position of $b$ in $\cup_{j=1,..,n-1} A_j$, say $b_n$, such that $C^{S_b}(y)$ has (on $S_b = S_n$) the combinatorial type of $T_n$, and its ramification points are close to $o$ (by Lemma 9). Therefore, $F_y^{-1}$ also has the combinatorial type of $T_n$, and the proof is ended. □

We say that the set $M_y^{-1}$ is *of type* $T_m$, and write $M_y^{-1} \sim T_m$, if it is a tree with $m$ extremities. We conclude with the following characterization result ($M_S = \cup_{x \in S} M_x$).

**Theorem 10** *Let $S$ be a convex surface and $y$ a point in $M_S$.*

*a) If $\theta_y = 2\pi$ then $M_y^{-1}$ is a point.*

*b) If $M_y^{-1}$ is of type $T_m$ with $m \geq 2$ then $\pi \leq \theta_y < \frac{m}{m-1}\pi$; there are at most countably many points $y$ such that $M_y^{-1} \sim T_2$, and at most 7 points $y$ such that $M_y^{-1} \sim T_m$ with $m \geq 3$.*

*c) If $\mathrm{int}(M_y^{-1}) \neq \emptyset$ then $\theta_y \leq \pi$, and there are at most 3 points $y \in S$ with $\mathrm{int}(M_y^{-1}) \neq \emptyset$.*

*Proof:* The assertion *a)* is the first part of Theorem 1.

*b)* Each point $y \in S$ with $|M_y^{-1}| > 1$ is a conical point, again by Theorem 1. Denote by $x_i$ the extremities of the graph $M_y^{-1}$ $(i = 1, ..., m)$.

Because $\mathrm{int}(M_y^{-1}) = \emptyset$, it follows, from Proposition 1, that $M_y^{-1}$ is included in the arcwise connected union of $C(y)$ with some subsegments of segments from $y$, hence $M_y^{-1}$ is a tree. Since the segments do not branch, the ramification points of $M_y^{-1}$ belong to $C(y)$, hence in the set of ramification points of $C(y)$.

Therefore, there are points $x_i' \in M_y^{-1} \cap C(y)$ which are joined to $y$ by precisely two segments, say $\Gamma_i$ and $\Gamma_i'$; this, because the set of ramification points of the tree $C(y)$ is at most countable for any point $y$ on any convex surface.

Because the tree $M_y^{-1}$ has finitely many extremities, if we take the point $x_i'$ close enough to $x_i$ then $\Gamma_i \cup \Gamma_i'$ separates $x_i$ from all ramification points of $M_y^{-1} \setminus \{x_i\}$. Denote by $\alpha_i$ the angle at $y$ of the domain of $S$ bounded by $\Gamma_i \cup \Gamma_i'$ and containing $x_i$.

By Lemma 4, we have $\theta_y - \alpha_i < \pi$. Since $\Sigma_{i=1}^m \alpha_i < \theta_y$, we get

$$m\pi > \Sigma_{i=1}^m (\theta_y - \alpha_i) = m\theta_y - \Sigma_{i=1}^m \alpha_i > (m-1)\theta_y,$$

whence $\theta_y < \frac{m}{m-1}\pi$.

The inequality $\theta_y \geq \pi$ follows from Theorem 3.

Let $m \geq 3$; the total curvature at $y$ is

$$\omega_y = 2\pi - \theta_y > 2\pi - \frac{m}{m-1}\pi = \pi - \frac{1}{m-1}\pi \geq \pi/2.$$

Since the total curvature of $S$ is equal to $4\pi$, there are at most 7 points $y \in S$ such that $M_y^{-1} \sim T_m$ with $m \geq 3$. The rest is precisely Theorem 2.

*c)* If $M_y^{-1}$ contains interior points then there is a segment $\Gamma_y$ and distinct points $x, z \in \Gamma_y$ such that $y \in M_x \cap M_z$. Suppose $\rho(x, y) > \rho(z, y)$. It follows that the subsegment of $\Gamma_y$ from $z$ to $y$ is the only segment joining $z$ to $y$, because geodesics do not branch. By Lemma 4, $\theta_y \leq \pi$ holds.

Since $\theta_y \le \pi$, the total curvature at $y$ is $\omega_y = 2\pi - \theta_y \ge \pi$, and there are at most $k \le 4$ points $y \in S$ with $\mathrm{int}(M_y^{-1}) \ne \emptyset$. Suppose $k = 4$; then $S$ is linear everywhere except for the points $y$ where $\omega_y = \theta_y = \pi$, hence it is either a doubly covered rectangle, or a tetrahedron with the curvatures equal to $\pi$ at all of its vertices. In the first case one can easily check, and for the second case it follows from the proof of Theorem 9, that all vertices $y$ have $M_y^{-1} \sim T_m$, with $m \le 3$. Thus, $k \le 3$. □

Proposition 1 and Theorems 5 and 10 immediately yield the following:

**Corollary** *If $\theta_y \ne \pi$ then $M_y^{-1}$ is either the empty set, or a (possibly degenerate) tree, or $S \setminus \{y\}$.*

**Remark** More precisely, Case *b)* of Theorem 10 implies that on any convex surface there are at most 7 points $y$ such that $M_y^{-1} \sim T_3$; or at most 5 points $y$ such that $M_y^{-1} \sim T_m$ with $4 \le m \le 6$; or at most 4 points $y$ such that $M_y^{-1} \sim T_m$ with $m \ge 7$.

**Remark** One can see examples for Case *a)* of Theorem 10 in Sections 2.1 - 2.3.

For Case *b)* and $m > 2$, see Theorem 9; in particular, the tetrahedra with curvature equal to $\pi$ at all vertices provide examples of four points $y_i$, on the same surface, with $M_{y_i}^{-1} \sim T_3$.

For $m = 2$, we have the doubly covered polygons (see Section 2.4), and also the following example, where $F_{y_i}^{-1}$ is an arc for any natural number $i$.

**An example** *In the following we construct a convex surface $S$ which has a countable set of points $x_n$ with $F_{x_n}$ an arc, and also a countable set of points $y_n$ with $F_{y_n}^{-1}$ an arc.*

Take in a plane a quarter of circle with the centre at $o$, bounded by the radii $[ox]$ and $[oy_0]$; denote it by $J_0$. Let $x_1$ be the mid-point of the segment $[ox]$, and $y_1$ the point of the bisector of the angle $\angle ox_1y_0$, determined by $||x_1 - y_0|| = ||x_1 - y_1||$. Let $J_1$ be the arc of the circle centered at $x_1$, between $y_0$ and $y_1$, and of smallest length. Inductively, let $x_n$ be the mid-point of the segment $[xx_{n-1}]$, and $y_n$ the point on the bisector of the angle $\angle ox_ny_{n-1}$, such that $||x_n - y_{n-1}|| = ||x_n - y_n||$. Denote by $J_n$ the arc of the circle centered at $x_n$, between $y_{n-1}$ and $y_n$, and of smallest length.

Then the sequence $\{y_n\}_{n\geq 0}$ converges to a point $y$ which belongs to the line $ox$, and we also have $\lim_{n\to\infty} x_n = x$.

Let $S$ be the doubly-covered compact planar region bounded by

$$\cup_{n\geq 0} J_n \cup [xy].$$

One can easily check, on $S$, that $F_x = y$, and for all integers $n \geq 1$ we have $F_{x_n} = J_n$ and $F_{y_n}^{-1} = [x_n x_{n+1}]$. □

## 1.4 Arcs of farthest points

T. Zamfirescu [47] constructed examples of sets $F_x$ homeomorphic to any compact subsets of the line and proved that:

**Lemma 10** *For any convex surface $S$ and any point $x$ in $S$, each component of $F_x$ is either a point or a Jordan arc.*

We shall see that the same result is also true for $S \in \mathcal{G}_0$ (Theorem 11). Notice that if $S \in \mathcal{G}_0$ and $x \in S$ then $F_x \subset C_x$, by Lemma 5.

For the following two results, see [49], [51] and [53].

A *$Y$-tree* is a tree with precisely one ramification point and three extremities. A tree is called *finite* if it has finitely many extremities.

**Lemma 11** *Suppose that the set $F_x$ contains more than one point for some point $x$ in $S$. Then $F_x$ is contained in a minimal (by inclusion) finite tree $J_x \subset C(x)$.*

*Let us restrict ourselves to $S \in \mathcal{S}$; if $S$ is a doubly covered acute triangle and $x$ is the centre of its circumscribed circle then $J_x$ is a $Y$-tree; otherwise, $J_x$ is an arc. All points of $J_x$ excepting possibly its endpoints lie in $C_x$.*

**Lemma 12** *All critical points of the surface $S \in \mathcal{S} \cup \mathcal{G}_0$, with respect to $x \in S$, belong to some finite tree lying in $C(x)$, a $Y$-tree if $S$ is convex.*

**Theorem 11** *For any surface $S$ in $\mathcal{S} \cup \mathcal{G}_0$ and any point $x$ in $S$, each component of $M_x$ is either a point or a Jordan arc. Consequently, this is also true for $F_x$.*

*Proof:* Let $x \in S$ and take a component $M^1$ of $M_x$. Then $M^1$ is connected and, since it is included in a finite tree (Lemma 12), $M^1$ is also a (not necessarily closed) tree.

Consequently, since the restriction of $\rho_x$ to $M^1$ is continuous, it is a constant function, and has a local minimum at each point interior to $M^1$.

Suppose we have a component $M^1$ of $M_x$ which is a non-degenerate tree. Then a ramification point $y$ of $M^1$ is joined to $x$ by at least three segments, in contradiction to Lemma 8. □

We shall refer to the following result several times. The arguments used to prove it can be found in [3], p. 60 or 214, in a slightly different form.

**Lemma 13** *Let $J \subset C(x)$ be an arc, each point of which is joined to $x$ by precisely two segments. Let $y_1, y_2$ be the endpoints of $J$, and let $\Delta$ denote the domain which is bounded by the segments from $x$ to $y_1, y_2$, and contains $J \setminus \{y_1, y_2\}$. Then $\Delta \cap C(x) \subset J$.*

*Proof:* Suppose there is a point $z$ in $\Delta \cap C(x) \setminus J$. Since $C(x)$ is a tree, there is a minimal (by inclusion) subarc $A$ of $C(x)$ containing $z$ and meeting $J$. Let $\{u\} = A \cap J$ .

Assume first that $u \in \Delta$. Then $u \in \text{int}(J)$, so it is a ramification point of $C(x)$, whence it is joined to $x$ by at least three segments, a contradiction.

Assume now that $u \in C(x) \setminus \Delta$. Take two points $y_1', y_2'$ in $\text{int}(J)$ such that the connected domain $\Delta'$, bounded by the segments $\Gamma_{y_i'}, \Gamma'_{y_i'}$ from $x$ to $y_i'$ $(i = 1, 2)$, still contains the point $z$ (see Lemma 1). Clearly, $u \notin \Delta'$.

Then, since $z \in \Delta'$ and $x \notin C(x)$, the arc $A$ intersects $\text{bd}(\Delta')$ in $y_1'$ or $y_2'$, say $y_1'$. Hence $y_1' = u$ and $u$ is a ramification point of $C(x)$, whence it is joined to $x$ by at least three segments, a contradiction. □

The following result was discovered by T. Zamfirescu [42].

**Lemma 14** *On most convex surfaces, most points are endpoints.*

Lemmas 13 and 14 imply the following:

**Theorem 12** *For a typical convex surface $S$, there exist no $x \in S$ and $a > 0$ such that $\rho_x^{-1}(a) \cap C(x)$ contains a non-degenerate arc. In particular, there exists no point $x$ with a non-degenerate arc in $F_x$.*

*Proof:* Let $S$ be a typical surface in $\mathcal{S}$, and suppose there exist a point $x \in S$ and $a > 0$ such that $\rho_x^{-1}(a) \cap C(x)$ contains a non-degenerate arc $J'$.

Let $y_1, y_2$ be two interior points of $J'$, and denote by $J$ the subarc of $J'$ joining them. Since each point $z$ interior to $J'$ is a relative minimum for $\rho_x|_J$, we obtain, by Lemma 8, that $z$ is the mid-point of a loop $\Lambda_z$ at $x$, and no segments connect $x$ to $z$ excepting those in $\Lambda_z$. Now, denote by $\Delta$ the domain bounded by $\Lambda_{y_1} \cup \Lambda_{y_2}$. By Lemma 13, we have $\Delta \cap C(x) \subset J$, whence each point of $\Delta$ is interior to a geodesic, a fact which contradicts Lemma 14. □

## 1.5 On points with multiple farthest points

Recently, T. Zamfirescu [51] established the following result for $S \in \mathcal{S}$, result adapted for $S \in \mathcal{G}_0$ by J. Rouyer [30]:

**Lemma 15** *On any surface $S \in \mathcal{S} \cup \mathcal{G}_0$, for nearly all points $x \in S$, $F_x$ contains a single point.*

The proof of Lemma 15 contains the following fact, which we give separately for future use. Note that, by Lemma 11, the minimal tree $J_x$ included in $C(x)$ which contains $F_x$, has finitely many extremities.

**Lemma 16** *Suppose that $x \in S$ and $|F_x| \geq 2$. Let $z$ be an extremity of $J_x$. For $\varepsilon > 0$, take $z' \in J_x$ such that $0 < \operatorname{diam}(J^x_{zz'}) < \varepsilon$, where $J^x_{zz'} \subset J_x$ connects $z$ and $z'$.*

*Then there exist an arc $A$ starting at $x$, and a number $k > 0$, such that for any $v \in A$ and any $u \in B(v, k\rho(v,x))$, we have $F_u \subset B(J^x_{zz'}, \varepsilon)$; a fortiori, $F_u \subset B(z, 2\varepsilon)$.*

The set $A_2(S) = \{x \in S; |F_x| \geq 2\}$ was introduced by J. Rouyer [29].

**Theorem 13** $\operatorname{Int}(F_y^{-1}) \cap A_2(S) = \emptyset$.

*Proof:* Suppose there exists a point $x \in \operatorname{int}(F_y^{-1}) \cap A_2(S)$. Let $z \neq y$ be an extremity of the tree $J_x$ defined by Lemma 11, and let $V$ be a neighbourhood of $x$ in $F_y^{-1}$.

We can find, by Lemma 16, an open set $V' \subset V$ such that $F_u$ is sufficiently close to $z$ not to contain $y$, for all $u \in V'$. So, $F_{V'} \cap \{y\} = \emptyset$, a contradiction to $V' \subset F_y^{-1}$. □

**Theorem 14** *Assume $y$ is an isolated point of $F_S$ in $S$. Then either* $\mathrm{int}(F_y^{-1}) \neq \emptyset$*, or $F_y^{-1}$ consists of a single point $x$ with $|F_x| \geq 3$.*

*Proof:* Consider a neighbourhood $V$ of $y$ such that $V \cap F_S \setminus \{y\} = \emptyset$, and a point $x$ in $F_y^{-1}$.

If $F_x = y$ then, since $y$ is an isolated point of $F_S$, we get, from the upper semi-continuity of $F$, a neighbourhood $U$ of $x$ such that $F_U$ is included in $V$, hence $F_U = \{y\}$.

Suppose $F_x = \{y, z\}$. Then there exists, by Lemma 16, an open set $U$ arbitrarily close to $x$, such that $F_U \subset V$, hence $F_U = \{y\}$. □

**Remark** A sufficient condition for a point to be isolated in $F_S$ is given by Theorem 5. An example for the second case of Theorem 14 is the centre of a doubly covered regular triangle (see also Theorem 49).

**Theorem 15** *If $F_S$ has finitely many components, and $F_S^1$ is one of them, then* $\mathrm{bd}(F^{-1}(F_S^1))$ *consists of points $x$ with disconnected set $F_x$.*

*Proof:* Note that each component $F_S^i$ of $F_S$ is closed, by the upper semi-continuity of $F$ $(i \in I)$. Moreover, by Lemma 15, we have for any $i \neq j$

$$\mathrm{int}(F^{-1}(F_S^i) \cap F^{-1}(F_S^j)) = \emptyset.$$

Suppose that $F_S$ has at least two, but finitely many components.

The sets $F^{-1}(F_S^i)$ are all closed (by Lemma 3), as well as $\cup_{j \neq i} F^{-1}(F_S^j)$, for any subscript $i \in I$.

Because $S = \cup_i F^{-1}(F_S^i)$, for each subscript $i$ we have

$$S \setminus F^{-1}(F_S^i) \subset \cup_{j \neq i} F^{-1}(F_S^j),$$

whence

$$\mathrm{bd}(S \setminus F^{-1}(F_S^1)) \subset \cup_{j \neq 1} F^{-1}(F_S^j).$$

Thus, for each point $x$ in $\mathrm{bd}(F^{-1}(F_S^1))$, there exists an index $j \neq 1$ such that $x \in F^{-1}(F_S^1) \cap F^{-1}(F_S^j)$. Therefore, $F_x \cap F_S^1 \neq \emptyset$ and $F_x \cap F_S^j \neq \emptyset$. □

## 1.6 Maxima and non-surjectivity

Theorem 6 states that, on a smooth surface, a strictly local maximum of some distance function is a global maximum for no distance function; thus, if $M \neq F$ then $F$ is not surjective. We can complete now the picture.

The set $\mathcal{S}_2 = \{S \in \mathcal{S} \cup \mathcal{G}_0$; there exists $x \in S$ with disconnected $M_x\}$ is open in $\mathcal{S}$; it was introduced by T. Zamfirescu [51], who showed that on most $S \in \mathcal{S}_2 \cap \mathcal{S}$ there exist a point $x$ and a Jordan arc in $C(x)$ containing infinitely many points of $M_x$.

The simple example of a long and thin surface $S \in \mathcal{S} \cup \mathcal{G}_0$ shows that $S \setminus F_S$ is a non-empty open set (since $F_S$ is closed), and this happens on a whole neighbourhood of $S$.

We shall use the following topological result of L. E. J. Brouwer, [25] p. 52; see also [20] for the definition of the degree of a continuous map.

**Lemma 17** *Any continuous map $S^d \to S^d$ with degree different from $(-1)^{d+1}$ has a fixed point.*

**Theorem 16** *For a surface $S \in \mathcal{S} \cup \mathcal{G}_0$, consider the following statements:*
*a) there exists $x \in S$ with disconnected set of global maxima for $\rho_x$;*
*b) there exists $x \in S$ with disconnected set of local maxima for $\rho_x$;*
*c) there exists a loop at some point $x \in S$, of length less than $2\rho(x, F_x)$;*
*d) $F$ is not surjective;*
*e) there exists $x \in S$ with $|F_x| > 1$.*
*Then the following implications are true: a) $\to$ b) $\to$ c) $\to$ d) $\to$ e).*

*Proof:* The assertion *a)* $\to$ *b)* is easy.

*b)* $\to$ *c)* If the set $M_x$ is disconnected, take a point $y$ in $F_x$, and a point $z$ in $M_x$ so that $y$ and $z$ do not lie in the same component of $M_x$.

If $z \in F_x$ then there exists an arc $J \subset C(x)$ joining $y$ to $z$, and a local minimum $u \in J \setminus F_x$ of $\rho_x|_{\mathrm{int}(J)}$, which is the mid-point of a loop at $x$ (see Lemma 8) of length less than twice the radius of $S$ at $x$.

Suppose $z \in M_x \setminus F_x$. Each component of $M_x$ is either a point or an arc, by Theorem 11.

If the component $M_x^1$ of $M_x$ containing $z$ has more than one point then it contains an arc $J$ each interior point of which is a local minimum of $\rho_x|_{\mathrm{int}(J)}$. Now, Lemma 8 provides a loop at $x$ of length equal to $2\rho(x, z) < 2\rho(x, F_x)$.

If $M_x^1 = \{z\}$ then there exists an arc $J \subset C(x)$ and a strict local minimum of $\rho_x|_{\mathrm{int}(J)}$; Lemma 8 provides now a loop at $x$ of length less than $2\rho(x, F_x)$.

Theorem 7 yields *c)* $\rightarrow$ *d)*.

To see that *d)* $\rightarrow$ *e)*, suppose that $F$ is single-valued. Then it is continuous. Since $F$ is not surjective, it has degree zero, and Brouwer's fixed point theorem (Lemma 17) shows that the map $F : S \rightarrow S$ has a fixed point, impossible. □

**Theorem 17** *Let $S$ be a surface of $\mathcal{S} \cup \mathcal{G}_0$ such that for any point $x$ in $S$, the set $F_x$ contains no arc. Then the statements* a) - e) *given in Theorem 16 are equivalent to each other, and equivalent to "f) $M$ is not surjective".*

*In particular, this is true for typical and also for polyhedral convex surfaces.*

*Proof:* By Theorem 11, for any surface $S \in \mathcal{S} \cup \mathcal{G}_0$ and any point $x \in S$, each component of $F_x$ is either a point or a Jordan arc.

Thus, if there exists a point $x$ in $S$ with $|F_x| > 1$ then $F_x$ is disconnected, and *e)* $\rightarrow$ a*)* holds.

If the set $F_x$ is disconnected then there exists an arc $J \subset C(x)$ and a local minimum of $\rho_x|_{\mathrm{int}(J)}$ (see Lemma 8), which is not in the image $M_S$ of the mapping $M$, by Theorem 7. Thus, a*)* $\rightarrow$ *f)*

Clearly, $F_S \subset M_S$ and $M_S \neq S$ yield *f)* $\rightarrow$ *e)*.

On a typical convex surface, there is no point $x$ with an arc in $F_x$, by Theorem 12.

For a polyhedral convex surface $P$, one can easily see by unfolding that for any point $x$ in $P$, the only points in $F_x$ which can be joined to $x$ by precisely two segments are among the vertices of $P$, so $F_x$ does not contain arcs (see [31] or [54] for a complete proof). □

Denote by $Q_x$ the set of critical points of the distance function $\rho_x$.

**Theorem 18** *If there exists a point $x$ in a convex surface $S$ with disconnected set $Q_x$ then there exists a loop at $x$ of length less than $2\rho(x, F_x)$.*

*Proof:* Consider a point $x$ in $S$ with $Q_x$ disconnected; take a point $y$ in $F_x$, and a point $z$ in $Q_x$, such that $y$ and $z$ do not lie in the same component of $Q_x$.

If $z \in F_x$ then there exists an arc $J \subset C(x)$ joining $y$ to $z$, and a local minimum $u \in J \setminus F_x$ of $\rho_x|_{\text{int}(J)}$, which is the mid-point of a loop at $x$ (see Lemma 8) of length less than twice the radius of $S$ at $x$.

Suppose $z \notin F_x$. If there are two segments joining $z$ to $x$, making an angle equal to $\pi$ at $z$, then their union is the loop we are searching for. If not, then the maximum angle between consecutive segments from $z$ to $x$ is strictly less than $\pi$, so $y$ is a strict local maximum for $\rho_x$, by Theorem 3. Therefore, the set $M_x$ is disconnected and there exists an arc $J \subset C(x)$ and a local minimum $u$ of $\rho_x|_{\text{int}(J)}$ which is not in $F_x$. The conclusion follows now from Lemma 8. □

**Remark** Suppose the surface $S$ has no conical points; if it contains no loop then $F$ is bijective. Indeed, $F$ is surjective by Theorem 17, and therefore for all $x \in S$ the set $F_x$ is connected. By Lemma 8, there exist infinitely many loops at a point $x$ with an arc in $F_x$, so $F$ is also single-valued.

**Theorem 19** *For any surface in $\mathcal{S} \cup \mathcal{G}_0$, if $F$ is continuous then it is surjective.*

*Proof:* $F$ is continuous if and only if it is single-valued. Indeed, if $F$ is single-valued then it is clearly continuous; conversely, suppose $F$ is continuous, and there exists $x \in S$ such that $|F_x| > 1$. By the proof of Lemma 16, there is a sequence of points $x_n \in S \setminus \{x\}$ convergent to $x$ such that the sequence of sets $F_{x_n}$ converges to a strict subset of $F_x$, a contradiction.

The surjectivity of $F$ follows now from Theorem 17. □

## 1.7 The boundary of $\mathcal{S}_2$

In the following we characterize in $\mathcal{S}$ the closure $\text{cl}(\mathcal{S}_2)$ and the boundary $\text{bd}(\mathcal{S}_2) = \text{cl}(\mathcal{S}_2) \setminus \mathcal{S}_2$ of the open set $\mathcal{S}_2$; this yields a partial answer to the last question proposed in [51].

We shall need Alexandrov's gluing theorem ([3], p. 362) or its polyhedral variant ([3], p. 317), which we state here.

**Lemma 18** *If a 2-manifold $M$ results from gluing together several polygons with metrics of positive curvature such that the sum of the angles at vertices glued together is not larger than $2\pi$, then the metric of $M$ is also of positive curvature.*

We shall implicitly use Pogorelov's well-known rigidity theorem ([28] p. 167), which says that:

**Lemma 19** *Any two isometric convex surfaces are congruent.*

**Theorem 20** *The convex surface $S$ belongs to* $\mathrm{cl}(\mathcal{S}_2)$ *if and only if there exist points $x \in S$ and $x' \in C(x)$, and two (possibly coinciding) segments from $x'$ to $x$ of directions $\tau_1$ and $\tau_2$ at $x'$, such that the two angles determined by $\tau_1, \tau_2$ on the tangent cone $T_{x'}$ at $x'$ are not larger than $\pi$.*

*Moreover, if the convex surface $S$ belongs to* $\mathrm{bd}(\mathcal{S}_2)$ *then one can choose the point $x$ such that $x'$ belongs to $F_x$.*

*Proof:* Suppose first that $S \in \mathrm{cl}(\mathcal{S}_2)$.

If $S \in \mathcal{S}_2$ then there is a point $x$ in $S$ with $|F_x| > 1$, by Theorem 16. Consider two points $y, y'$ in $F_x$. There exists, in the arc $J_x \subset C(x)$ joinig them (see Lemma 1 or Lemma 11), a local minimum $x'$ of $\rho_x$. Lemma 8 now ends this part of the proof.

Take now $S \in \mathrm{bd}(\mathcal{S}_2) = \mathrm{cl}(\mathcal{S}_2) \setminus \mathcal{S}_2$; then there exist a sequence of surfaces $S_n \in \mathcal{S}_2$ converging to $S$, and points $x_n \in S_n$ with disconnected set $F_{x_n}$ (by Theorem 17). We may take $x'_n$ as a local minimum for the restriction of $\rho_{x_n}$ to the interior of a subarc $J_{x_n}$ of $C(x_n)$, joining points $y_n, z_n$ in different components of $F_{x_n}$.

Possibly passing to a subsequence, we may assume that $x_n \to x \in S$, and also $y_n \to y \in S$, $z_n \to x \in S$. Clearly, $y, z \in F_x$, by the upper semi-continuity of $F$.

If $F_x$ is disconnected, then we apply Lemma 8. Suppose $F_x$ is connected.

Since all arcs $J_{x_n}$ are connected, their limit is also connected, hence either $\lim_{n\to\infty} J_{x_n}$ is an arc included in $F_x$ (and now Lemma 8 ends the proof), or it is a point $x' = F_x$. In the latter case, we obtain $y_n \to x'$ and $z_n \to x'$, whence $x'_n \to x'$ too.

By Lemma 8, there are precisely two segments from $x_n$ to $x'_n$, which make an angle equal to $\pi$ at $x'_n$. Passing to the limit, by the lower semi-continuity of the angles (see [8], p. 100), we obtain that the angles determined at the tangent cone $T_{x'}$ by the limit segments are both less than or equal to $\pi$.

Conversely, suppose there exist points $x \in S$ and $x' \in C(x)$, and two (possibly coinciding) segments $\Gamma$ and $\Gamma'$ from $x'$ to $x$ with directions $\tau_1$ and

$\tau_2$ at $x'$, such that the two angles determined by $\tau_1, \tau_2$ on the tangent cone $T_{x'}$ at $x'$, are not larger than $\pi$.

Two points as above are said to have the *property* $P$.

In the following we shall repeatedly use Lemma 19.

Suppose $x$ is a smooth point in $S$.

Assume first that there is an extremity $y$ of the tree $C(x)$, different from $x'$. Then, by Lemma 9, $x$ is joined to $y$ by either *i)* a unique segment $\Sigma$, or *ii)* a continuous family $\mathcal{F}$ of segments.

In Case *i)*, consider a minimal by inclusion subarc $J$ of $C(x)$ joining $x'$ to $y$, and a sequence of points $y_n \in J \setminus \{y, x'\}$ converging to $y$, such that each $y_n$ is joined to $x$ by only two segments, say $\Sigma_n$ and $\Sigma'_n$. (The points $y_n$ are among the points in the tree $C(x)$ which are nor extremities, neither ramifications.) We cut $S$ along $\Sigma_n \cup \Sigma'_n$ and get two pieces, $S_n$ and $S'_n$; suppose that $y \in S'_n$. By taking the points $y_n$ close enough to $y$, we may assume $\Gamma \cup \Gamma' \subset S_n$.

By Lemma 18, and with the use of an auxiliary geodesic triangulation of $S_n$, we may glue $\Gamma$ with $\Gamma'$ and obtain from $S_n$ a new convex surface, which will be also denoted by $S_n$.

Notice that $x_n$ is a conical point.

We claim that the points $x_n$, $x'_n$ corresponding to $x$ and respectively $x'$, have the property $P$. Indeed, since any point $y_n$ lie in $J$ between $x'$ and $y$, it follows from Lemma 1 that $\Sigma_n \cup \Sigma'_n$ separates $y$ from $x'$. Therefore, $\mathrm{cl}(S'_n) \cap (\Gamma \cup \Gamma') = \{x\}$, so the metric of $S_n$ is precisely the restriction to $S_n$ of the metric of $S$, and this proves the claim.

Clearly, if $y_n$ tends to $y$ then $\Sigma_n$ and $\Sigma'_n$ converge to $\Sigma$, hence $S'_n$ tends to $\Sigma$, and $S_n$ to $S$.

In Case *ii)*, consider two sequences of distinct interior elements in $\mathcal{F}$, say $\Sigma_n$ and $\Sigma'_n$, with a common limit $\Sigma$. We cut $S$ along $\Sigma_n \cup \Sigma'_n$ and get two pieces, denoted by $S_n$ and $S'_n$; assume that $\Gamma \cup \Gamma' \subset S_n$.

By Lemma 18, and with the use of an auxiliary geodesic triangulation of $S_n$, we may glue the piece $S_n$ such that to obtain a convex surface, also denoted by $S_n$. There, the correspondents $x_n$, $x'_n$ of $x$ and respectively $x'$, have the property $P$. Moreover, $x_n$ is a conical point.

Clearly, if $\Sigma_n$ and $\Sigma'_n$ converge to $\Sigma$ then $S'_n$ tends to $\Sigma$ and $S_n$ to $S$.

Assume now that $C(x) = x'$. Then the family $\mathcal{F}$ of all segments from $x$ to $x'$ is continuous and the union of its elements equals $S$. Again, we may consider two sequences of distinct interior elements in $\mathcal{F}$, say $\Sigma_n$ and $\Sigma'_n$, with a common limit $\Sigma$. By cutting and glueing as above, we find convex surfaces $S_n$ arbitrarily close to $S$, and points $x_n, x'_n \in S_n$ with the property $P$. Moreover, $x_n$ is a conical point.

Thus, it remains to consider the case of points $x, x' \in S$ with the property $P$, such that $x$ is a conical point.

Suppose first that $\Gamma = \Gamma'$.

Consider a sequence of numbers $\eta_n \in ]0, \min\{\omega_x, \omega_{x'}\}[$, convergent to 0.

Also consider a planar convex arc $A$ of length $l = \rho(x, x')$, and the surface of revolution $S'$ generated by the rotation of $A$ around the line through its endpoints.

Cut from $S'$ a piece $S'_n$ determined by two half-meridians which make angles less than $\eta_n$ at their intersecting points. Also cut $S$ along $\Gamma$. By Lemma 18 and with the use of an auxiliary geodesic triangulation of $S$, we can glue together the pieces $S$ and $S'_n$, such that to obtain a convex surface, denoted by $S_n$. On $S_n$, the points $x_n$, $x'_n$, corresponding to $x$ and respectively $x'$, also have the property $P$, relative to two distinct segments joining them. Moreover, $\omega_{x_n} > 0$. Clearly, if $\eta_n$ converges to 0 then $S_n$ tends to $S$.

Suppose now that $\Gamma \neq \Gamma'$.

Cut $S$ along $\Gamma \cup \Gamma'$ and get two pieces $S_1$, $S_2$. Let $x_i, x'_i \in S_i$ and $\Gamma_i, \Gamma'_i \subset S_i$ correspond to $x, x'$ and $\Gamma, \Gamma'$, respectively ($i = 1, 2$).

Take $0 < \varepsilon < \omega_x/2$, and let $vab$ be an isosceles triangle with the sides $[va]$, $[vb]$ of length $\rho(x, x')$ and with the angle $\varepsilon$ between them.

By Lemma 18, we can glue together $S_1$, $S_2$, and two copies $v_i a_i b_i$ of $vab$ as it follows:
- the points $x_1$, $x_2$, $v_1$ and $v_2$ will coincide;
- the points $x'_1$, $a_1$ and $a_2$ will coincide;
- the points $x'_2$, $b_1$ and $b_2$ will coincide.

Denote by $S_\varepsilon$ the resulting convex surface and by $\rho^\varepsilon$ its metric. We have, by Lemma 19, an isometry

$$i : \cup_{i=1,2}(v_i a_i b_i \cup S_i) \to S_\varepsilon.$$

On $S_\varepsilon$, the distance function $\rho^\varepsilon_{i(x)}$ has a local minimum in the arc of $C(i(x))$ from $i(x'_1)$ to $i(x'_2)$, whence $S_\varepsilon \in \mathcal{S}_2$.

By taking $\varepsilon$ arbitrarily small, we obtain surfaces $S_\varepsilon$ arbitrarily close to $S$. Thus $S \in \mathrm{cl}(\mathcal{S}_2)$, and the result is proven. □

**Corollary** *Any convex surface of revolution is in* $\mathrm{cl}(\mathcal{S}_2)$.

*Proof:* The points of the surface determining the axis of revolution verify the condition of Theorem 20. □

## 1.8 Some criteria for the existence of multiple farthest points

As further applications of Theorem 17, we give in the following sufficient conditions for a surface $S$ to contain a point $x$ with $|F_x| > 1$; the first one is immediate.

**Theorem 21** *If there is a loop on* $S$ *of length less than* $2\mathrm{rad}(S)$ *then there exists a point* $x$ *in* $S$ *with* $|F_x| > 1$.

**Lemma 20** *Let* $y, z$ *be two points on a surface* $S$. *Then the set* $E(y,z)$ *of all points in* $S$ *at equal distance to* $y$ *and to* $z$, *is a closed curve which separates* $y$ *from* $z$.

*Proof:* Define the sets

$$S_y = \{u \in S; \rho(u,y) < \rho(u,z)\}, \ \ S_z = \{u \in S; \rho(u,y) > \rho(u,z)\}.$$

Then $S_y$ and $S_z$ are open disjoint subsets of $S$.

Since any point $u \in S_y$ is joined to $y$ by (at least) one segment $\Gamma_{uy}$, and $\Gamma_{uy} \subset S_y$, it follows that $S_y$ (and similarly $S_z$) is arcwise connected. Therefore, the set

$$E(y,z) = S \setminus (S_y \cup S_z)$$

separates them, so it separates $y \in S_y$ from $z \in S_z$.

All indices below are to be taken modulo 2.

For $i = 0, 1$, consider distinct points $v_i \in E(y,z)$, and $\Gamma_{v_iy}$, $\Gamma_{v_iz}$ segments joining $v_i$ to $y$, and respectively to $z$. Clearly,

$$(\Gamma_{v_0y} \cup \Gamma_{v_1y}) \setminus \{v_0, v_1\} \subset S_y, \ \ (\Gamma_{v_0z} \cup \Gamma_{v_1z}) \setminus \{v_0, v_1\} \subset S_z.$$

Therefore $\Gamma_{v_iy} \cap \Gamma_{v_{i+1}z} = \emptyset$, so we obtain

$$(\Gamma_{v_0y} \cup \Gamma_{v_0z}) \cap (\Gamma_{v_1y} \cup \Gamma_{v_1z}) = \{y, z\}.$$

It follows that the closed curve $\Lambda = \cup_{i=0,1}(\Gamma_{v_i y} \cup \Gamma_{v_i z})$ bounds two open, connected and disjoint subsets of $S$, say $S'$ and $S''$.

Now, arguments similar to those used to prove Theorem 1 in [47] show that $E(y,z) \cap S'$ and $E(y,z) \cap S'$ are arcs joining $v_0$ to $v_1$. Therefore, $E(y,z)$ is a closed curve (see [41]). $\square$

**Theorem 22** *If* $\mathrm{rad}(S) = \mathrm{diam}(S)/2$ *then* $S \in \mathcal{S}_2$*, or there exists a closed geodesic* $\Lambda \subset S$ *of length equal to* $\mathrm{diam}(S)$ *such that* $F_x$ *is an arc,* $\rho(x, F_x) = \mathrm{rad}(S)$ *for any point* $x$ *in* $\Lambda$*, and* $\cup_{x \in \Lambda} F_x = S$*.*

*Proof:* Suppose $S \notin \mathcal{S}_2$, hence each point $x$ of $S$ has connected set of local maxima, which is precisely $F_x$.

Let $x, y, z \in S$ such that $\mathrm{rad}(S) = \rho(x, F_x)$ and $\mathrm{diam}(S) = \rho(y,z)$. Then

$$2\rho(x, F_x) \geq \rho(x,y) + \rho(x,z) \geq \rho(y,z) = \mathrm{diam}(S),$$

whence $y, z \in F_x$ and $x$ belongs to a segment $\Gamma_{yz}$ from $y$ to $z$. Thus, because $\Gamma_{yz}$ does not branch at $x$, we have $\{y,z\} \subset C(x) \setminus C_x$, whence $\theta_y, \theta_z \leq \pi$, by Lemma 4.

Since $S \notin \mathcal{S}_2$, the set $F_x$ is connected (Theorem 16), hence $F_x$ contains an arc joining $y$ and $z$ (Lemma 10). But there are unique segments joining $x$ to $y$ and respectively to $z$; since each point $v$ interior to $F_x$ is a local minimum for $\rho_x|_{F_x}$, hence the mid-point of a loop $\Lambda_v$ at $x$, by Lemma 8, we obtain that $y$ and $z$ are the endpoints of the arc $F_x$.

Now, the domain $\Delta$ provided by Lemma 1 applied to $y, z \in F_x$, verifies $\Delta = S \setminus \Gamma_{yz}$. Moreover, by previous remarks and Lemma 13, $\Delta = \cup_{v \in \mathrm{int} F_x} \Lambda_v$ and, because the geodesics do not pass beyond conical points, $S$ is smooth everywhere except at $y, z$.

Consider now the set $E(y,z)$ of all points in $S$ at equal distance to $y$ and to $z$. By Lemma 20, $|E(y,z) \cap F_x| \geq 1$.

For an arbitrary point $w \in E(y,z) \setminus \{x, F_x\}$, denote by $\Lambda_w$ the loop at $x$ through $w$, so $\Lambda_w$ separates $y$ from $z$. Since $y, z$ are points of the tree $C(w)$, there exists a (minimal by inclusion) arc $J_w \subset C(w)$ joining them. Consider a point $w' \in J_w \cap \Lambda_w \neq \emptyset$, hence

$$\rho(w, w') \leq l(\Lambda_w)/2 = \mathrm{rad}(S).$$

We also have

$$2\mathrm{rad}(S) = \mathrm{diam}(S) = \rho(y,z) \leq \rho(w,y) + \rho(w,z) = 2\rho(w,y),$$

so we obtain

$$\rho(w, w') \leq l(\Lambda_w)/2 \leq \rho(w, y).$$

If $\rho(w, w') < \rho(w, y) = \rho(w, z)$ then $\rho_w|_{\mathrm{int}J_w}$ has disconnected set of local minima and therefore $S \in \mathcal{S}_2$, which, as we assumed, is not the case.

Thus, we have

$$\rho(w, w') = l(\Lambda_w)/2 = \mathrm{rad}(S) = \rho(w, y) = \rho(w, z).$$

From

$$\mathrm{diam}(S) = l(\Lambda_w) = \rho(w, y) + \rho(w, z),$$

we obtain that $w$ is the mid-point of a segment from $y$ to $z$, and no other segments join $w$ to $y$ or $z$.

Consider now the equalities $\rho(w, w') = \rho(w, y) = \rho(w, z)$. Since the distance function $\rho_w|_{\mathrm{int}J_w}$ cannot have disconnected set of local maxima (we assumed $S \notin \mathcal{S}_2$), it is a constant function, and therefore $F_w$ is an arc containing $J_w$. The same arguments used for $F_x$ show that, actually, $F_w = J_w$.

The equality $\rho(w, w') = l(\Lambda_w)/2$ implies that $\Lambda_w$ is a closed geodesic, in particular its directions at $x$ make an angle of $\pi$.

Suppose there exists $w_1 \in E(y, z)\setminus(\{x, F_x\}\cup\Lambda_w)$. Then, since $\Lambda_w\cap\Lambda_{w_1} = \{x\}$, the directions of $\Lambda_{w_1}$ at $x$ make an angle less than $\pi$, hence $\Lambda_{w_1}$ is not a closed geodesic, and consequently we obtain $\rho(w_1, w_1') < \mathrm{rad}(S)$, hence the distance function $\rho_{w_1}$ has disconnected set of local maxima and $S \in \mathcal{S}_2$, case excluded. Thus,

$$E(y, z) \setminus (\{x, F_x\} \cup \Lambda_w) = \emptyset,$$

so

$$E(y, z) \subset \{x, F_x\} \cup \Lambda_w.$$

Therefore, since $x \in E(y, z) \cap \Lambda_w$, we obtain $E(y, z) = \Lambda_w$, because both $E(y, z)$ and $\Lambda_w$ are closed curves and $F_x$ is an arc.

Now, the considerations produced by the above inequalities show that for all points $w \in E(y, z) \setminus \{x, F_x\}$, we have $F_w = J_w$.

Consider the point $\{u\} = F_x \cap E(y, z)$.

By the upper semi-continuity of $F$, if the sequence of points $w_n \in E(y, z)$ is convergent to $u$ then, possibly passing to a subsequence, the sets $F_{w_n}$ converge to a subset of $F_u$. Thus $\{y, z\} \subset F_u$ and, since $F_u$ is assumed to be connected, it follows (as above) that it is an arc joining $y$ and $z$.

In the following we shall use the notations $\Lambda = E(y, z)$, and $\mathcal{P}(S)$ for the set of compact subsets of $S$. We have also obtained that the restriction $F|_\Lambda : \Lambda \to \mathcal{P}(S)$ is continuous.

Finally, we have to prove that we have $\cup_{w\in\Lambda} F_w = S$.

To see this, notice first that, from Lemma 13, we have $C(w) = F_w$, for all points $w \in \Lambda$.

Observe, for $w_1, w_2 \in \Lambda$, $w_1 \neq w_2$, that $F_{w_1} \cap F_{w_2} = \{y, z\}$. Indeed, a point $v$ interior to the arc $F_{w_1}$ is a smooth point, hence a maximum for precisely one distance function, by Theorem 1.

Suppose now $S \setminus \cup_{w\in\Lambda} F_w \neq \emptyset$. We shall consider two cases.

*Case a)* $\Lambda \setminus \cup_{w\in\Lambda} F_w \neq \emptyset$.

Define the map $f : \Lambda \to \Lambda$ by $f(w) = F_w \cap \Lambda \neq \emptyset$. Then, since $F|_\Lambda$ is continuous, $f$ is also continuous. Clearly, it is not surjective, hence its degree is zero and, by Lemma 17, it has a fixed point, which is impossible.

*Case b)* $\Lambda \setminus \cup_{w\in\Lambda} F_w = \emptyset$.

The set $\cup_{w\in\Lambda} F_w$ is clearly closed. Consider a connected component $D$ of the open set $S \setminus \cup_{w\in\Lambda} F_w \neq \emptyset$, hence $\mathrm{bd}(D) \subset \cup_{w\in\Lambda} F_w$. Since the arcs $F_w$ are mutually disjoint, excepting their endpoints $\{y, z\}$, there exist $w_1, w_2 \in \Lambda$, $w_1 \neq w_2$, such that $\mathrm{bd}(D) = F_{w_1} \cup F_{w_2}$.

We also have $\Lambda \cap D = \emptyset$, hence $\Lambda$ separates $D$ either from $y$ or from $z$. Suppose that $\Lambda$ separates $D$ from $z$, hence $F_{w_1} \cap F_{w_2} \cap \{z\} = \emptyset$, a contradiction which ends the proof. □

Notice that one cannot replace in Theorem 22 the radius and the diameter of $S$ with the extrinsic ones; this can be seen by choosing convenient surfaces in the class $\mathcal{R}$ of convex surfaces defined in the next part (see Theorem 34).

The following well-known result, Toponogov's Comparison Theorem, can be found, for example, in [9]. All indices below are to be taken modulo 3.

A *geodesic triangle* in the Riemannian manifold $M$ is a collection of three segments $\gamma_1, \gamma_2, \gamma_3$ of lengths $l_1, l_2, l_3$, parametrized by arc length, such that $\gamma_i(l_i) = \gamma_{i+1}(0) = a_{i+2}$ and $l_i + l_{i+1} \geq l_{i+2}$. We shall denote the triangle by $(\gamma_1, \gamma_2, \gamma_3)$ or $a_1a_2a_3$. Let $\alpha_i = \angle(-\gamma'_{i+1}(l_{i+1}), \gamma'_{i+2}(0))$ be the angle between $-\gamma'_{i+1}(l_{i+1})$ and $\gamma'_{i+2}(0)$, $0 \leq \alpha_i \leq \pi$.

A *hinge* is a configuration $(\gamma_1, \gamma_2, \alpha)$ of an angle $\alpha$ and two segments $\gamma_1, \gamma_2$, such that $\gamma_1(l_1) = \gamma_2(0)$ and $\angle(-\gamma'_1(l_1), \gamma'_2(0)) = \alpha$.

Let $K$ denote the sectional curvature of a given Riemannian manifold, and $M_H$ the simply connected 2-dimensional space of constant curvature $H$.

**Lemma 21** *(A) Let $M$ be a complete manifold with $K \geq H$, and $(\gamma_1, \gamma_2, \gamma_3)$ a geodesic triangle in $M$. If $H > 0$, suppose $l_i \leq \pi/\sqrt{H}$ for all $i$. Then there exists in $M_H$ a geodesic triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ such that $l(\gamma_i) = l(\overline{\gamma}_i)$ and the corresponding angles $\overline{\alpha}_i$ satisfy $\overline{\alpha}_i \leq \alpha_i$. Except for the case $H > 0$ and $l_i = \pi/\sqrt{H}$ for some $i$, the triangle in $M_H$ is uniquely determined.*

*If the inequality $K \leq H$ is assumed, then the conclusion $\overline{\alpha}_i \geq \alpha_i$ follows.*

*(B) Let $(\gamma_1, \gamma_2, \alpha)$ be a hinge in $M$. If $H > 0$, suppose $l(\gamma_2) \leq \pi/\sqrt{H}$. Let $\overline{\gamma}_1, \overline{\gamma}_2 \subset M_H$ such that $l(\gamma_i) = l(\overline{\gamma}_i) = l_i$ and $\angle(-\overline{\gamma}_1'(l_1), \overline{\gamma}_2'(0)) = \alpha$. Then $\rho(\gamma_1(0), \gamma_2(l_2)) \leq \rho(\overline{\gamma}_1(0), \overline{\gamma}_2(l_2))$.*

*If we assume $K \leq H$ then $\rho(\gamma_1(0), \gamma_2(l_2)) \geq \rho(\overline{\gamma}_1(0), \overline{\gamma}_2(l_2))$.*

**Theorem 23** *Suppose the surface $S$ is of class $\mathcal{C}^2$ on a neighbourhood of a loop $\Lambda$ of length $l$. If the curvature satisfies $K < \pi^2/l^2$ along $\Lambda$, then there exists a point $x$ in $S$ with $|F_x| > 1$. In particular, this is true if $K$ is at most 0 along a loop.*

*Proof:* Notice that, since $\Lambda$ is the union of two segments issuing from a point $x$, its mid-point $y$ is the cut point of $x$ along each of these segments.

Denote by $V$ a neighbourhood $V \subset S$ of class $\mathcal{C}^2$ of $\Lambda$. Then the curvature is continuous on $V$ and, since $K < \pi^2/l^2$ along $\Lambda$, we may assume that $K < \pi^2/l^2$ on $V$.

Let $\Gamma^\perp$ be a segment orthogonal to $\Lambda$ at $y$, and take a point $z \in \Gamma^\perp \cap V$ close enough to $\Lambda$ so that $y \notin C(z)$.

Let $\bar{x}, \bar{y}, \bar{z}$ be points on the sphere of curvature $\pi^2/l^2$, such that $\rho_0(\bar{x}, \bar{y}) = \rho(x, y) = l/2$, $\rho_0(\bar{y}, \bar{z}) = \rho(y, z)$ and the angles at $\bar{y}$ and $y$ are equal (to $\pi/2$), see *(B)* in Lemma 21; here, $\rho_0$ is the standard metric of the considered sphere. It follows that $\rho_0(\bar{x}, \bar{z}) = \rho_0(\bar{x}, \bar{y})$. By Toponogov's comparison theorem, the hinge variant, we have $\rho_0(\bar{x}, \bar{z}) < \rho(x, z)$, hence $\rho(x, z) > \rho(x, y)$ and $y \notin F_x$. By Theorem 16, there exists a point $x$ in $S$ with $|F_x| > 1$. □

**Remark** The above estimate is sharp, in the sense that it may happen that $K = \pi^2/l^2$ on $\Lambda$ and still $F_x = y$, as it shown by the corollary of Theorem 36.

**Remark** If the surface $S \in \mathcal{S}$ is smooth enough then there exist at least three closed geodesics on $S$, by the classical result of L. A. Lusternik and L. G. Schnirelman [24]. More recently, a combined result of J. Franks and V. Bangert (see [13] and [6]) shows that for every Riemannian metric on the 2-sphere there exist infinitely many closed geodesics. In contrast to these, by a result of P. Gruber [18], most convex surfaces have no closed geodesic. A. V. Pogorelov [28] proved that any $S \in \mathcal{S}$ has at least three closed *quasigeodesics*, each of which may (not necessarily) be a loop. These are good candidates to verify the condition of Theorem 23.

**Examples** As particular cases of Theorem 23, we have many compact surfaces of revolution, for example those with a radius of curvature $R$ along a circle-loop $\Lambda$ of radius $r < R/4$. More particularly, the ellipsoids with semi-axes $a = b < 2a < c$, or some cylinders of revolution.

One can also find, on the boundaries of (rectangular) boxes and on the doubly covered polygons, a loop $\Lambda$ and a neighbourhood of $\Lambda$ isometric to an open subset of some cylinder, where the curvature is zero.

**Theorem 24** *If there exists a point $y$ in $S$ such that $\theta_y < \pi$ then there exists a point $x$ in $S$ with $|F_x| > 1$.*

*Proof:* By Theorem 3, we have $M_y^{-1} = S \setminus \{y\}$, hence there are points $x$ close enough to $y$ so that $y \in M_x \setminus F_x$, i.e. the set $M_x$ is disconnected. By Theorem 16, there exists a point $x$ in $S$ with $|F_x| > 1$. □

**Theorem 25** *If $M$ or $F$ is not injective on the surface $S$ then $S \in \mathrm{cl}(\mathcal{S}_2)$.*

*Proof:* Let $y \in M_{x_1} \cap M_{x_2}$. We know, from Proposition 1, that $M_y^{-1}$ is arcwise connected and $M_y^{-1} \cap C(y) \neq \emptyset$, hence we have one of the following two cases.

*Case a)* There exist two distinct points in $M_y^{-1} \cap C(y)$, and therefore an arc included in $M_y^{-1} \cap C(y)$ joining them (by Proposition 1). In this case, because the set of all ramifications points of $C(y)$ is at most countable, there exists $x \in M_y^{-1}$ joined to $y$ by only two segments. By Lemmas 4 and 5, and by Theorem 20, we obtain $S \in \mathrm{cl}(\mathcal{S}_2)$.

*Case b)* The set $M_y^{-1} \cap C(y)$ consists of a single point, hence $x_1$, $x_2$ and $y$ belong all to some segment $\Gamma$. Since the segments do not branch, from Lemma 4 we get $\theta_y \leq \pi$, and Theorem 20 yields $S \in \mathrm{cl}(\mathcal{S}_2)$. □

**Theorem 26** *Let $x, y$ be two points on the convex surface $S$, $y \in M_x$, and denote by $\alpha_i$ the angles at $y$ between consecutive segments from $x$ ($i \in I$). If $\omega_y + \max_{i \in I} \alpha_i > \pi$ then $S \in \mathrm{cl}(\mathcal{S}_2)$.*

*Proof:* We may assume, by Lemma 7, that $\max_{i \in I} \alpha_i = \alpha_1$.

Denote by $\mathcal{F}_j$ a maximal (by inclusion) continuous family of segments from $y$ to $x$, and denote by $\beta_j$ the angle on the tangent cone at $y$ determined by $\mathcal{F}_j$ ($j \in J$). We clearly have

$$\Sigma_{i \in I} \alpha_i + \Sigma_{j \in J} \beta_j = \theta_y.$$

Put $\beta = \Sigma_{j \in J} \beta_j$. From

$$2\pi = \omega_y + \beta + \Sigma_{i \in I} \alpha_i = \omega_y + \alpha_1 + \beta + \Sigma_{i \neq 1} \alpha_i$$

and $\omega_y + \alpha_1 > \pi$ we obtain $\beta + \Sigma_{i \neq 1} \alpha_i < \pi$.

By Lemma 4, because $y \in M_x$, we get $\alpha_1 \leq \pi$.

Thus, since both $\alpha_1$ and $\theta_y - \alpha_1$ are at most $\pi$, we can apply Theorem 20 and obtain $S \in \mathrm{cl}(\mathcal{S}_2)$. □

Together, Theorems 17 and 26 do provide an indirect proof for the following result discovered by J. Rouyer [31]:

**Lemma 22** *Let $P$ be a polyhedral convex surface, $x \in P$ and $y \in F_x$ such that $|F_y^{-1}| = 1$ and the total curvature at $y$ satisfies $\omega_y > \frac{N-2}{N-1}\pi$, where $N > 1$ is the number of segments from $x$ to $y$. Then $P \setminus F_P \neq \emptyset$.*

## 1.9 Semi-continuity of the cut loci and local maxima of $\rho_x$

In this section we first show a basic property of the cut loci, their lower semi-continuous dependence on the point and the surface (Theorem 27). This result provides an alternative proof to Theorem 28.

For the case of a given Riemannian manifold, it is known that the cut locus depends continuously on the point (see [7], for example). However, without smoothness (i.e. on Alexandrov spaces), the cut locus is only semi-continuous [23].

We say that the cut locus $C^S(x)$ depends *lower semi-continuously* on the point $x$ and on the surface $S$ in $\mathcal{S}$ if for any point $y$ in $C^S(x)$, any sequence of surfaces $S_n$ in $\mathcal{S}$ convergent to $S$, and any sequence of points $x_n$ in $S_n$ converging to $x$, there exist a sequence of points $y_n$ in $C^{S_n}(x_n)$ with $y$ as a limit point.

The dependence is called *continuous* if, moreover, any convergent sequence of points $z_n$ in $C^{S_n}(x_n)$ has the limit in $C^S(x)$.

**Theorem 27** *The cut locus $C^S(x)$ depends lower semi-continuously on the surface $S \in \mathcal{S}$ and on the point $x \in S$.*

*Proof:* Consider a subsequence of surfaces $S_n$ in $\mathcal{S}$ convergent to the surface $S \in \mathcal{S}$, and let $x_n \in S_n$, $x \in S$ such that $x_n \to x$. Take $y \in C^S(x)$.

Let $z_n \in S_n$ be a nearest point from $y$, let $\Gamma_{z_n}$ be a segment of $S_n$ joining $x_n$ to $z_n$, and denote by $y_n \in C^{S_n}(x_n)$ the cut point of $x_n$ in the direction of $\Gamma_{z_n}$. Let $\Gamma_{y_n} \supset \Gamma_{z_n}$ be the segment from $x_n$ to $y_n$

We take the limit over $n$ and find $z_n \to y$.

Some subsequence of the bounded sequence $\{\Gamma_{y_n}\}_{n \in \mathbb{N}}$ converges to a segment $\Gamma_{y'}$ joining $x$ to some point $y'$. Since $z_n \in \Gamma_{y_n}$, we have $y \in \Gamma_{y'}$. Since $y \in C^S(x)$, we obtain $y' = y$. □

**Remark.** One cannot obtain continuity in the above result. Indeed, we can approximate the standard sphere with typical convex surfaces, most points of which are endpoints, hence in all cut loci. But the cut locus of any point on the sphere is precisely its antipodal point.

We cannot obtain continuity even restricting ourselves to a fixed surface $S$. To see this, consider a convex surface of revolution $S$ symmetric with respect to the origin, and such that the poles are endpoints (e. g. conical points). Denote by $v_i$ the poles of $S$ $(i = 1, 2)$, and take arbitrary points $x_n \in S \setminus \{v_1, v_2\}$, all of them in the same half-meridian $\mathcal{M}$. Since the poles are endpoints, they belong to $C(x_n)$. Because any cut locus is a (possibly degenerate) tree, and by the symmetry of $S$, we find $-\mathcal{M} \subset C(x_n)$. Now take the sequence $\{x_n\}_{n \in \mathbb{N}}$ convergent to $v_1$, and observe that the cut locus of $v_1$ is precisely $v_2$.

One can also consider a polyhedral convex surface (with finitely many vertices), and a sequence of smooth points convergent to a vertex.

**Remark.** On typical convex surfaces, the non-continuity of $C_S$ is a simple consequence of the fact that most points are endpoints, hence in any cut locus (Lemma 14), while the Hausdorff measure of any cut locus equals 0 (see [27]).

Next result, together with Theorem 17, completes the last theorem in [46] and further describes surfaces in $\mathcal{S}_2$.

Denote by $B_2(S)$ the set of all points $x$ in $S$ with disconnected set $M_x$.

**Theorem 28** *For any surface $S$ in $\mathcal{S} \cup \mathcal{G}_0$, any point whose distance function has at least two strict local maxima is interior to $B_2(S)$.*

*Proof:* For a point $x$ in $S$, let $y_1, y_2 \in S$ be strict local maxima for $\rho_x$.

Consider an intrinsic circle $C_i$ centered at $y_i$, small enough so that it is homeomorphic to the unit circle $S^1$, and we have $(i = 1, 2)$

$$\rho(x, y_i) > \sup_{v \in C_i} \rho(x, v).$$

For any point $w$ in $B(x, 2^{-1}(\rho(x, y_i) - sup_{v \in C_i} \rho(x, v)))$ and any point $v$ in $C_i$, we have

$$2\rho(x, w) < \rho(x, y_i) - \rho(x, v),$$

hence

$$\rho(w, y_i) \geq \rho(x, y_i) - \rho(x, w) > \rho(x, w) + \rho(x, v) \geq \rho(w, v),$$

and $\rho_w$ has a local maximum interior to each $C_i$. □

**Remark.** There exists an alternative proof for Theorem 28 which makes use of Theorem 27.

# Chapter 2

# Applications and examples

## 2.1 A surface with $F$ bijective

We saw how Theorem 17 helps to decide that, on a surface, the mapping $F$ is not single-valued. The aim of this section is to show how Theorem 17 can be applied for an explicit example, to prove that $F$ is bijective.

Let $oXYZ$ be the usual system of coordinates for $\mathbb{R}^3$.

Let $S_1$ be the the convex surface obtained by union of a unit half-sphere with a unit disk, hence we may assume that

$$S_1 = S^+ \bowtie D,$$

with

$$S^+ = \{X^2 + Y^2 + Z^2 = 1; Z \geq 0\},$$
$$D = \{X^2 + Y^2 \leq 1; Z = 0\}.$$

We shall make use of Lemma 23, the classical relation of Clairaut (see [9] p. 257), where we implicitely assume the following fact (see [2] for the proof of a more general result). Let $P$ be a parallel of singular points, and $\Gamma$ a shortest path crossing it at the point $z$. Then the semi-directions of $\Gamma$ at $z$ make equal angles with $P$.

**Lemma 23** *Let $S$ be a surface of revolution. For a variable point $\Gamma(t)$ on a geodesic $\Gamma$ of $S$, denote by $r_t$ the distance from $\Gamma(t)$ to the axis of revolution, and by $\theta_t$ the angle made at $\Gamma(t)$ by $\Gamma$ with the parallel through $\Gamma(t)$. Then the expression $r_t \cos\theta_t$ does not depend on $t$.*

**Theorem 29** *The mapping $F$ is a homeomorphism on the surface $S_1$.*

*Proof:* $F$ is injective on $S_1$, by Theorem 1, because $\theta_y = 2\pi$ for all $y \in S_1$.

Notice that $S_1$ is a surface of revolution, a meridian of which has the length equal to $2 + \pi$.

Recall that for two points on the unit sphere, the distance between them is given by the angle they determine at the origin.

We shall prove that $|M_x| = 1$ for all points $x$ in $S_1$, whence $M_x = F_x$ and $F$ is bijective. Then it follows immediately that $F$ is a homeomorphism.

Assume that there exists a point $x$ in $S_1$ with $|M_x| > 1$.

We consider first the case $x \in D$. Clearly, $D \cap C(x) = \emptyset$.

If $x = o$ then all half-meridians are segments, hence $|M_o| = 1$.

Suppose $x \neq o$. If the only loop at $x$ is the meridian $x$ determines then necessarily, by Theorem 16 and Lemma 8, $F_x$ is precisely the mid-point of this loop.

So we may assume that there exists a non-meridian loop $\Lambda$ at $x$, whose mid-point $y$ belongs to $S^+$. Then, since the two subsegments of $\Lambda$ from $x$ to $y$ make at $y$ an angle equal to $\pi$, $\Lambda \cap S^+$ is a half of a great circle; consequently, the points $u, v \in S_1$ given by $\Lambda \cap S^+ \cap D$ are diametrally opposite on $D$. We have

$$2\rho(x, y) = \rho(x, u) + \rho(u, y) + \rho(y, v) + \rho(v, x) =$$

$$= \rho(x, u) + \rho(x, v) + \pi \geq 2 + \pi,$$

with equality if and only if $x \in [uv]$.

For $x \in [ou] \setminus \{o\}$, the segment $[uv]$ is orthogonal at $u$ to the boundary of $D$, hence the half of a great circle through $u$, $v$ and $y$ is also, by Lemma 23, hence it is a meridian.

Thus, all points $x \in D$ have $|M_x| = 1$.

Let now $x \in S^+$; we may assume that $X(x) < 0$ and $Y(x) = 0$.

Since we assumed that $|M_x| > 1$, there exists a local minimum $y$ for the restriction of the distance function $\rho_x$ to the interior of some subarc of $C(x)$. By Lemma 8, $y$ is the mid-point of a loop $\Lambda$ at $x$ and, except for the two subarcs of $\Lambda$, no segments connect $x$ to $y$.

Suppose that $y$ belongs to $D$ (the case $y \in S^+$ is clearly impossible). Let

$$\{u, v\} = \Lambda \cap S^+ \cap D.$$

Let $\Lambda'$ be the symmetric set of $\Lambda$ with respect to the plane $oXZ$, and suppose that $\Lambda \neq \Lambda'$. Because $S_1$ is symmetric with respect to the plane $oXZ$, $\Lambda'$ is another loop at $x$.

Suppose first that $\Lambda \cap \Lambda' \setminus \{x\} = \emptyset$; without loss of generality, we may assume that $\rho(x,u) > \rho(x,v)$; in this case, the arc of circles included in $\Lambda$ which joints $x$ to $u$ and respectively to $v$, and the line-segment (included in $\Lambda$) which joints $u$ to $v$, do not verify at $u$ or at $v$ the condition given by Lemma 23, a contradiction.

If $\Lambda \cap \Lambda' \setminus \{x\} \neq \emptyset$ then, since $\Lambda$ and $\Lambda'$ can meet only at $x$ and $y$, we get $y \in \Lambda \cap \Lambda' \cap C(x)$, whence $y \in oXZ$. Thus, $y$ is joined to $x$ by at least 4 segments, a contradiction which shows that $|M_x| = 1$.

So, if $|M_x| > 1$ then necessarily $\Lambda = \Lambda'$ and $uv \perp oX$.

Assume now that $\Lambda = \Lambda'$ and $uv \perp oX$.

Consider first the case $X(y) < 0$.

Since $y$ is the projection of $u$ onto the $XZ$-plane, we have $\angle xou > \angle xoy$.

For $0 \leq \beta \leq \pi/2$, the triangle inequality implies $\sin\beta + \cos\beta \geq 1$. This yields, for $\beta = \angle you$, the inequality

$$1 - ||y|| \leq ||u - y||.$$

We obtain

$$\rho(x,y) = \rho(x,u) + \rho(u,y) = \angle xou + ||u - y|| \geq \angle xoy + 1 - ||y||.$$

Let $\mathcal{M}_x$ be the meridian through $x$ (and $y$); then $\angle xoy + 1 - ||y||$ is precisely the length of one part of $\mathcal{M}_x$ joining $x$ and $y$, whence we have obtained a contradiction to the assumption $\{y\} = C(x) \cap \Lambda$.

Consider now the case $X(y) > 0$.

Let $w = -u$, and denote by $\beta$ the angle made at $u$ by the circle through $u, x, w$, with the circle $C$ through $u, w, v$, hence $\beta \in ]0, \pi/2[$. We obtain, from Clairaut's relation (Lemma 23), that $vu$ makes at $u$ an angle $\beta$ with $C$, whence $\angle uwv = \beta$.

Consider the points $a \in oXY$ and $b \in uw$ such that $xa \perp oXY$ and $ab \perp uw$, hence $xb \perp uw$ too, and consequently $\angle xba = \beta$. With $\alpha = \angle xow$, since $\angle boa = \beta$, we have

$$||x - b|| = \sin\alpha, \quad ||a - b|| = \sin\alpha\cos\beta,$$

and

$$||o - b|| = \cos\alpha = \sin\alpha \cos\beta (\tan\beta)^{-1},$$

whence we obtain

$$\alpha = \arctan \frac{\sin\beta}{\cos^2\beta}.$$

Next we prove that $\pi + 2 < l(\Lambda)$, inequality which is equivalent to

$$\angle xow + \angle xou + 2 < \angle xou + \angle xov + ||v - u||.$$

Since $\angle xov = \angle xou$, it suffices to prove that

$$\angle xow + 2 < \angle xou + ||v - u||,$$

i.e.

$$\alpha + 2 < \pi - \alpha + 2\sin\beta,$$

or

$$\pi - 2 + 2\sin\beta - 2\arctan \frac{\sin\beta}{1 - \sin^2\beta} > 0.$$

Define $f : ]0, 1[ \to I\!R$ by

$$f(s) = \pi - 2 + 2s - 2\arctan \frac{s}{1 - s^2}.$$

Its derivative is

$$f'(s) = 2 - 2\frac{1 + s^2}{(1 - s^2)^2 + s^2} < 0,$$

hence $f$ is a decreasing function. Since we quickly obtain $\lim_{s\to 1-} f(s) = 0$, it follows that $f(s) > 0$ for all $s \in ]0, 1[$, whence the inequality $l(\Lambda) > 2 + \pi$ holds, and $\Lambda$ is not a loop, a contradiction.

Thus, all points $x \in S^+$ have $|M_x| = 1$, too. □

## 2.2 On a conjecture of Steinhaus

The results presented in this section improve and complete the last part in [40]. In the interesting book [10] of Croft, Falconer and Guy, we find (p. 44

ii)) the following conjecture of H. Steinhaus: *a convex surface $S$ is a sphere if the following two properties are verified for all points $x \in S$*

$$(SC) : \begin{cases} |F_x| = 1 \\ F_{F_x} = x \end{cases}$$

Next we shall study Steinhaus' conditions $(SC)$ assuming central symmetry, and discover a whole class $\mathcal{I}$ of convex surfaces which verify $(SC)$, including the ellipsoids of revolution with semi-axes $a = b > c$. Thus, two new problems arise naturally:

*Problem A.* Characterize the family $\mathcal{H}$ of all convex surfaces which verify Steinhaus' conditions $(SC)$.

*Problem B.* Sharpen $(SC)$ to obtain a characterization of the sphere.

In the last part of this section we give a partial answer to *Problem A*.

### 2.2.1 Steinhaus' conditions and symmetry

In this section, the image of a point or a set through a given central symmetry will be denoted by a "prime".

**Theorem 30** *Let $S$ be a centrally symmetric surface ($S = S'$) and let $x \in S$. The following statements are equivalent:*
*i) $F_y = x$ for all $y \in F_x$;*
*ii) $F_x = x'$.*
*Moreover, both of them imply $|F_x| = 1$.*

*Proof:* The part *ii)* → *i)* is immediate.
We next show that *i)* → *ii)* holds. Suppose there exists a point $y \in F_x \setminus x'$. By the hypothesis and by symmetry, we have $F_y = x \neq y'$, $F_{x'} = y'$, $F_{y'} = x'$ and $\rho(x, y) = \rho(x', y')$.
Denote by $\Sigma$ a segment from $x$ to $x'$. By symmetry, $\Sigma'$ is a segment joining $x'$ to $x$, and the curve $\Sigma \cup \Sigma'$ divides the surface $S$ into two (topologically open) half-surfaces $S_1$ and $S_1'$, symmetric to each other. Since $y' \neq x$ and $\rho(x, y) \geq \rho(x, x')$, we have $y \notin \Sigma \cup \Sigma'$.
Suppose that $y \in S_1$, hence $y' \in S_1'$, and let $\Lambda$ be a segment from $y$ to $y'$. We have $(\Lambda \cap \Sigma) \cup (\Lambda \cap \Sigma') \neq \emptyset$; assume that $\Lambda \cap \Sigma = \{z\}$. Since

$\rho(y,x) \geq \rho(y,y')$, Lemma 2 applied to the quadrilateral $xyx'y'$ ends the proof. □

The following question seems natural: *is it true that on any centrally symmetric convex surface $S$, if $|F_x| = 1$ for all $x \in S$ then $F_x = x'$?*

We shall answer this question in Section 2.3.

For a given convex surface $S$, endow the space $\mathcal{P}(S)$ of all compact subsets of $S$ with the induced Pompeiu-Hausdorff metric $\rho^{\mathcal{P}(S)}$.

**Theorem 31** *$S \in \mathcal{S}$ is a centrally symmetric surface of $\mathcal{H}$ if and only if the associated mapping $F$ is an isometry.*

*Proof:* Let $S$ be a centrally symmetric surface of $\mathcal{H}$. By Theorem 30, $F_x = x'$ for all $x$ on $S$, hence $F$ is the restriction at $S$ of the symmetry (of the whole space) with respect to the centre of $S$, and therefore an isometry of $S$.

If there is a point $x \in S$ with $|F_x| > 1$, then (see Lemma 16) we can find sequences $x_n^1$ and $x_n^2$ tending to $x$ such that (for $i = 1, 2$) $|F_{x_n^i}| = 1$ and

$$\lim_{n\to\infty} F_{x_n^1} \neq \lim_{n\to\infty} F_{x_n^2}.$$

In this case, we have

$$\lim_{n\to\infty} \rho(x_n^1, x_n^2) = 0$$

and

$$\lim_{n\to\infty} \rho^{\mathcal{P}(S)}(F_{x_n^1}, F_{x_n^2}) > 0.$$

Thus, $F$ is an isometry between the metric spaces $(S, \rho)$ and $(\mathcal{P}(S), \rho^{\mathcal{P}(S)})$ if and only if it is an isometry of $(S, \rho)$.

If $F$ is an isometry then it is single-valued and $\rho(x, F_x) = \rho(F_x, F_{F_x})$, hence $F \circ F = \mathrm{id}_S$ and $S \in \mathcal{H}$. By Pogorelov's Rigidity Theorem (Lemma 19), the isometric convex surfaces $S$ and $F(S)$ are congruent via an extension $f$ of the isometry $F$ to the whole space. Since $f$ leaves $S$ invariant and has no fixed points on $S$, it must be the symmetry with respect to the mid-point $o$ of some line-segment joining a point $x$ to its (unique) farthest point. □

**Theorem 32** *If two points determine the diameter of a centrally symmetric surface, then they are symmetric to each other.*

*Proof:* The following arguments are similar to those used to prove Theorem 30. Suppose there is a point $y \in F_x$ such that $y \neq x'$. By the hypothesis and by symmetry, we have $x \in F_y$, $x \neq y'$, $F_{x'} = y'$, $x' \in F_{y'}$ and $\rho(x, y) = \rho(x', y')$.

Denote by $\Sigma$ a segment from $x$ to $x'$. By symmetry, $\Sigma'$ is a segment from $x'$ to $x$, and the curve $\Sigma \cup \Sigma'$ divides the surface $S$ into two symmetric (topologically open) half-surfaces, say $S_1$ and $S_1'$.

Since $y' \neq x$ and $\rho(x, y) \geq \rho(x, x')$, we get $y \notin \Sigma \cup \Sigma'$. Suppose that $y \in S_1$, hence $y' \in S_1'$, and let $\Lambda$ be a segment from $y$ to $y'$. Because $(\Lambda \cap \Sigma) \cup (\Lambda \cap \Sigma') \neq \emptyset$, we may assume that $\Lambda \cap \Sigma = \{z\}$. Since $\rho(y, x) \geq \rho(y, y')$, Lemma 2 now provides a contradiction. □

**Theorem 33** *The diameter of a centrally symmetric surface of revolution is equal to the half-length of a meridian.*

*Proof:* Let $x, y \in S$ such that $\rho(x, y) = \mathrm{diam}(S)$. By Theorem 32, we have $y = F_x = x'$.

Since all meridians are geodesics and any two centrally symmetric points determine a meridian $\mathcal{M}$ containing them, we get $\rho(x, x') \leq l(\mathcal{M})/2.$. But the only segments between the two poles of the surface are the meridians, and therefore $\rho(x, y) = l(\mathcal{M})/2$. □

### 2.2.2 Surfaces with involutive $F$

Next we provide examples of surfaces with the associated mapping $F$ an involutive homeomorphism.

Let $\varphi : [0, a] \to I\!R$ be a convex function differentiable at 0, such that $\varphi'(0) = (1, 0)$, $-a < \varphi(0) < 0 = \varphi(a)$, and the function $\psi : [0, a] \to I\!R$ given by $\psi(u) = \sqrt{u^2 + \varphi(u)^2}$ is increasing.

Let $\mathcal{R}$ be the set of all surfaces obtained by rotating the curves $C$ given by $\pm\varphi$; i.e., for $u \in [0, a]$ and $\beta \in [0, 2\pi]$, they are described by

$$\begin{cases} X = u \cos \beta \\ Y = u \sin \beta \\ Z = \pm\varphi(u) \end{cases}$$

Each surface in $\mathcal{R}$ is convex (because $\varphi$ is a convex function), is differentiable at its points with $X = Y = 0$ (because $\varphi'(0) = (1, 0)$), and is symmetric with respect to the $XY$-plane.

Notice that $\mathcal{R}$ contains the ellipsoids of revolution with semi-axes $a = b > c$.

Next we intrinsically define a set $\mathcal{I}$ which contains $\mathcal{R}$, by

$$\mathcal{I} = \{S \in \mathcal{S}; \mathrm{rad}(S) = \mathrm{diam}(S)\}.$$

The following result can be found in [8], p. 80.

**Lemma 24** *The length of a curve $\Gamma \subset I\!R^3$ exterior to a convex surface $S$ is at least as long as its metric projection onto $S$, with equality if and only if $\Gamma$ coincides with its projection.*

**Theorem 34** *$\mathcal{R} \subset \mathcal{I}$.*

*Proof:* Let $S$ be a surface in $\mathcal{R}$.

First, let $u \in S$ be an equatorial point and let $\mathcal{M}$ be the meridian of $S$ through $u$ and $-u$. Let $S_{\mathcal{M}}$ be the surface obtained by the rotation of $\mathcal{M}$ around the line through $u$ and $-u$. Because $\psi$ is an increasing function, the surface $S_{\mathcal{M}}$ lies in the convex hull of $S$, and we have $S \cap S_{\mathcal{M}} = \mathcal{M}$.

Thus, by Lemma 24, the only segments of $S$ from $u$ to $-u$ are the half-meridians, and therefore $F_u = -u$, by Theorems 33 and 32.

Let $x \in S$ be now an arbitrary (but not equatorial) point. Let $\Gamma$ be segment joining $x$ with $-x$, and let $u$ be the intersection point of $\Gamma$ with the equator.

By the symmetry of $S$, we get $\rho(-x, u) = \rho(x, -u)$. Therefore,

$$l(\Gamma) = \rho(u, x) + \rho(x, -u) \geq l(\mathcal{M})/2.$$

Thus, $\rho(x, -x) = l(\mathcal{M})/2$ and $\mathcal{R} \subset \mathcal{I}$. □

Recall that we denoted by $\mathcal{H} \subset \mathcal{S}$ the subset of all convex surfaces which verify the condition of Steinhaus: $F$ is a single-valued involution.

**Theorem 35** *$\mathcal{I} \subset \mathcal{H}$.*

*Proof:* Let $S$ be a surface in $\mathcal{I}$. Notice that $\rho(x, F_x) = \text{diam}(S)$ for all $x \in S$. Therefore, for any points $x \in S$, $y \in F_x$, we have $x \in F_y$, which implies that $F$ is surjective. By Theorem 17, for every $x \in S$, $F_x$ is connected, and by Lemma 10 it must be an arc or a point.

We claim that $F_x$ actually reduces to a point, for all $x \in S$.

Suppose that there is $x \in S$ with $F_x$ a non-degenerate arc. For all $y \in F_x$ we have $x \in F_y$, hence $|F_x^{-1}| > 1$ and $x$ is a conical point, by Theorem 1.

Let $z_1, z_2$ be the endpoints of $F_x$. Let $\Delta \subset S$ be the closure of the maximal open connected set not meeting any segment from $x$ to $z_1$ or $z_2$, but meeting $F_x$ (see Lemma 1).

For each point $z \in F_x$ there is a loop $\Lambda_z$ at $x$ through $z$.

Let $S_1, S_2$ be the two components of $S \setminus \Delta$ so that $z_i \in \text{bd}S_i$. For any loop $\Lambda$ at $x$ such that $\Lambda \cap F_x \neq \emptyset$, define the "left side" such that $S_1$ is at its left side and (consequently) $S_2$ is at its right side. Take $y_1 \in S_1$ close to $z_1$ such that $|F_{y_1}| = 1$ and there is a loop $\Lambda_1$ at $x$ which separates $y_1$ and $F_{y_1} = x_1$ in such a way that $y_1$ is at its left side. Also take $y_2 \in S_2$ close to $z_2$ such that $|F_{y_2}| = 1$ and there is a loop $\Lambda_2$ at $x$ which separates $y_2$ and $F_{y_2} = x_2$ in such a way that $y_2$ is at its right side. Such points exist, by the upper semi-continuity of $F$ and by Lemma 15.

It follows that there is a continuous family of loops $\{\Lambda_t\}$ at $x$ such that each $\Lambda_t$ separates $y_1$ and $x_2$ (being at its left side) from $y_2$ and $x_1$ (being at its right side), and $\Lambda \cap F_x \neq \emptyset$.

Let $A$ be an arc on $S$ from $x_1$ to $x_2$ such that $x \notin A$.

Then $F_A = \cup_{z \in A} F_z$ is connected. Indeed, assume $B$ is a component of $F_A$ different from $F_A$. Then $F_B^{-1}$ and $F_{F_A \setminus B}^{-1}$ must have a common limit point $z \in A$ since $A$ is connected. By the upper semi-continuity of $F$, $F_z$ must meet both $B$ and $F_A \setminus B$. Since $F_z$ is connected, this contradicts the fact that $B$ is a component of $F_A$.

Now move continuously from $x_1$ to $x_2$, with points $z'$ on the arc $A$.

Since $A$ and $F_A$ are connected, and also is $F_{z'}$ for all $z' \in A$, and by the choice of the family $\{\Lambda_t\}$, there exist a loop $\Lambda_{t_0} \in \{\Lambda_t\}$ and a point $z \in \Lambda_t \setminus \{x\}$ such that $F_z \cap \Lambda_t \neq \emptyset$. Since $x$ is a conical point, no segment from $z$ to $F_z$ passes through $x$, so

$$\rho(z, F_z) < \rho(x, F_x) = l(\Lambda_t)/2,$$

a contradiction to $\text{rad}(S) = \text{diam}(S)$ which proves the claim.

Thus, for all $x \in S$ we have $|F_x| = 1$ and $F_{F_x} = x$, whence $S \in \mathcal{H}$. $\square$

**Remark** Notice that $\mathcal{R}$ intersects all neighbourhoods of the unit sphere $S^2$, but also contains surfaces far from $S^2$. For example, the distance between the doubly covered unit disk (who belongs to $\mathcal{R}$) and $S^2$ is 1, and both these surfaces have the extrinsic diameter equal to 2 (see also Theorem 22).

**Remark** We have also shown, in the proof of Theorem 34, that for any surface $S \in \mathcal{R}$, the only segments from an arbitrary point $x \in S$ to $x' = F_x$ are the half-meridians. Thus, by Theorem 20, we get $\mathcal{R} \subset \text{bd}(\mathcal{S}_2)$.

**Remark** Notice that if $F$ is an involution then it is surjective, because $y \in S \setminus F_S$ implies $F_{F_y} \neq y$. Applying Theorem 17, we obtain the topological criterion

$$\mathcal{H} \subset \mathcal{S} \setminus \mathcal{S}_2.$$

### 2.2.3 Towards a generic solution

We consider here the following generic variant of Steinhaus' problem treated in Section 2.2. For most points $x$ on most convex surfaces, we cannot have simultaneously: $|F_x| = 1$ and $F_{F_x} = x$.

The example of a doubly-covered regular triangle, where all points $x$ have $|F_{F_x}| \geq 2$, shows that the preceding problem cannot be affirmatively solved if it is enounced for all $S$ in $\mathcal{S}$.

Notice that the upper semi-continuity of $F$ easily implies (see the proof of Proposition 2) the closeness of the set $E = \{x \in S; F_{F_x} = x\}$, so if $E$ is dense in $S$ then it equals the whole surface $S$. Thus, by Lemma 15 which solves the first part, the preceding generic problem actually reduces to the following one.

Let $\mathcal{S}^{\neq} = \{S \in \mathcal{S};\ \exists x \in S \text{ with } x \notin F_{F_x}\}$; is it true that $\mathcal{S}^{\neq}$ contains most elements of $\mathcal{S}$?

We make here a first step towards its solution.

**Proposition 2** *The set $\mathcal{S}^{\neq}$ is open in $\mathcal{S}$ and it contains $\mathcal{S}_2$.*

*Proof:* Clearly, by Theorem 17, for any surface $S \in \mathcal{S}_2$ there exists a point $y \in S \setminus F_S$, hence $y \notin F_{F_y}$. Thus, $\mathcal{S}_2 \subset \mathcal{S}^{\neq}$.

Next we prove that the complement of $\mathcal{S}^{\neq}$ in $\mathcal{S}$ is closed. To see this, consider a sequence of surfaces $S_n$ in $\mathcal{S} \setminus \mathcal{S}^{\neq}$, convergent to some surface $S \in \mathcal{S}$.

Let $x$ be an arbitrary point in $S$, and consider a sequence of points $x_n$ in $S_n$ convergent to $x$. The upper semi-continuity of $F$ implies that any convergent sequence of points $y_n$ in $F_{x_n}$ has the limit in $F_x$, and consequently that any convergent sequence of points in $F_{F_{x_n}}$ has the limit in $F_{F_x}$. Therefore, since $x_n \in F_{F_{x_n}}$, the point $x = \lim_{n\to\infty} x_n$ belongs to $F_{F_x}$. Since this is true for all points $x$ in $S$, we are done. □

## 2.3 Non-involutive homeomorphisms

The ideas of Prof. T. Zamfirescu essentially improved this section.

The following problem naturally arose in the preceding section: *is it true that on any centrally symmetric convex surface $S$, if $F$ is a bijection then it is an involution?*

In this section we see that suitable bounds on curvature and radius guarantee $F$ to be a homeomorphism. Nevertheless, the answer to the preceding question will be shown to be negative.

The classical Sphere Theorem (see [9], [37], or [15]) gives sufficient conditions for a complete, simply connected manifold $M$ to be homeomorphic to the unit sphere: $1/4 < K < 1$.

In fact, this 1/4-pinching of the curvature proves sufficient for $F$ to be a homeomorphism.

**Theorem 36** *Let $S$ be a convex surface with the curvature $K \geq 1$. If $\mathrm{rad}(S) > \pi/2$ then $F$ is a homeomorphism.*

*Proof:* It is implicitly assumed that $S$ is of class $C^2$. Thus, the injectivity of $F$ follows from Theorem 1.

The single-valuedness of $F$ and its surjectivity can in fact be found inside the proof of Theorem 3 in [15]. For the reader's convenience, we give here a short direct proof.

Suppose that there exists $x \in S$ with $|F_x| > 1$. Then there are $y, z \in F_x$, an arc $J$ joining $y$ to $z$ in $C_x$ and $u \in J$ a relative minimum of $\rho_x|_J$. By

Lemma 8, $u$ is the mid-point of a loop $\Lambda$ at $x$, so both subarcs from $x$ to $u$ are segments and make an angle equal to $\pi$ at $u$.

Let $uy$ be a segment from $u$ to $y$. Then one of the two angles determined at $y$ by $\Lambda$ and $uy$ is at most $\pi/2$. Consider the triangle $xuy \subset S$ containing that angle, and a triangle $\bar{x}\bar{u}\bar{y} \subset S^2$ isometric to $xuy$.

Comparing the triangles $xuy$ and $\bar{x}\bar{u}\bar{y}$, we have $\angle\bar{x}\bar{u}\bar{y} \leq \pi/2$. Let $\rho_0$ denote the metric of the unit sphere $S^2$. Since

$$\rho_0(\bar{x}, \bar{y}) = \rho(x, y) \geq \mathrm{rad}(S) > \pi/2,$$

$\bar{y}$ lies in the open half-sphere of $S^2$ opposite to $\bar{x}$. Then the inequality

$$\rho_0(\bar{x}, \bar{u}) \leq \rho_0(\bar{x}, \bar{y})$$

implies $\angle\bar{x}\bar{u}\bar{y} > \pi/2$, and a contradiction is obtained.

Thus, $F$ is single-valued on $S$ and therefore continuous. By Theorem 19, $F$ is also surjective.

Since $S$ is compact and $F$ is bijective and continuous, its inverse is also continuous. □

**Remark** By simple rescaling, Theorem 36 says that, if $S \in \mathcal{S}$, $K \geq k_0 > 0$ and $\mathrm{rad}(S) > \pi/(2\sqrt{k_0})$, then $F$ is a homeomorphism.

Next result is well-known Klingenberg's inequality ([9], p. 98)

**Lemma 25** *Let $M$ be an even-dimensional and orientable Riemannian manifold. If $K \geq K_M > 0$ then* $\mathrm{inj}(M) \geq \pi/\sqrt{K}$.

**Corollary** *If $1/4 \leq K < 1$ then $F$ is a homeomorphism.*

*Proof:* By Klingenberg's inequality, $K < 1$ implies $\mathrm{inj}(S) > \pi$, whence $\mathrm{rad}(S) > \pi$. Thus, by the preceding Remark with $k_0 = 1/4$, $K \geq 1/4$ implies that $F$ is a homeomorphism. □

**Remark** In other words, the corollary says that, if $S \in \mathcal{S}_2$ has everywhere positive Gauss curvature and $K_{\min}, K_{\max}$ denote its minimum and maximum respectively, then $K_{\max} \geq 4K_{\min}$.

The converse, however, is not true, as we can easily see on some surfaces in the class $\mathcal{R}$ defined in Subsection 2.2.2.

We treat now a special case, which will give us the answer to the problem previously mentioned.

**Theorem 37** *If $E$ is an ellipsoid of revolution with semi-axes $a = b < c < 2a$ then the mapping $F$ is a homeomorphism on $E$, but not an involution.*

*Proof:* Since the minimum of the Gauss curvature on $E$ is $k_0 = 1/c^2$, and $c < 2a$, we have

$$\pi/(2\sqrt{k_0}) = \pi c/2 < \pi a = \mathrm{rad}(E).$$

Hence $F$ is a homeomorphism, by a previous remark.

It remains to prove that $F \circ F \neq 1_E$. Assume, for simplicity, that the origin $o$ is the centre of $E$ and the $oZ$-axis is its axis of symmetry.

Clearly, by the symmetry of $E$, $x$ and $F_x$ are on the same meridian.

Let $y$ belong to the equator, i.e. $\angle yoZ = \pi/2$, and let $x \in S \setminus \{y\}$ lie on the same meridian. If $x$ is close enough to $y$, then there is an unique segment $xy$ joining $x$ to $y$. We show that, for $\rho(x, y)$ small enough, $F_x$ does not belong to the segment from $-x$ to $-y$, in particular $F_x \neq -x$. This will imply - use the continuity of $F$ - that $F$ is not an involution.

Assume $F_x$ belongs to the segment from $-x$ to $-y$. On $E$, $x$ and and $-x$ are joined by either: *i)* a meridian segment, or *ii)* a non-meridian segment.

Case *i)* is ruled out by the assumption $a < c$, for $x$ close enough to $y$.

Consider now Case *ii)*. By Lemma 4, some segment $\sigma$ from $x$ to $-x$ makes an angle $\alpha \geq \pi/2$ with $xy$. Let $z$ be the equatorial point of $\sigma$.

Subcase *a*: $\alpha = \pi/2$. In this case, by Lemma 23, the symmetry of the whole geodesic $\Gamma \supset \sigma$ with respect to the $Z$-axis implies that $z$ is the mid-point of $\sigma$.

Since the curvature of $E$ at $y$ is

$$K = 1/c^2 < 1/a^2,$$

it also remains less than $1/a^2$ in a whole neighbourhood $N$ of the equator $Q$.

Because $\sigma$ converges to a half-equator if $x$ tends to $y$, $\sigma$ lies in $N$ if $xy$ is small enough.

Therefore, on the sphere $M_{1/a^2}$ of curvature $1/a^2$, the triangle $\bar{x}\bar{y}\bar{z}$ isometric to $xyz$ has its angles at $\bar{x}$ and $\bar{y}$ larger than $\pi/2$, by Toponogov's comparison theorem. But this together with $\rho(y, z) = \pi a/2$ implies $\rho(x, y) > \pi a/2$, in contradiction with the choice of $x$.

Subcase *b*: $\alpha > \pi/2$. Now $\sigma$ has some point $x' \neq x$ as a farthest point from $Q$. Let $y'$ be the equatorial point closest to $x'$. By the classical result of Clairaut (see Lemma 23), and by the symmetry of $E$ and of $\Gamma$, there is a unique point $x'' \in \sigma$ between $x$ and $z$ at distance $\rho(x, y)$ from $Q$, and the point $-x$ is either symmetric to $x$ or to $x''$ with respect to the line $oz$. In the first case $\rho(z, y') < \pi a/2$. In the second, $\Gamma$ reaches $-x'$ beyond $-x$, $\rho(-x, -x') = \rho(x'', x') = \rho(x, x')$, and $\rho(z, y') = \pi a/2$.

We obtain the geodesic triangle $x'y'z$, to which we apply the argument from Subcase *a)*, and obtain a contradiction.

We claim that in $N \setminus Q$ the distance to $Q$ is either strictly increasing or it is strictly decreasing through $F$. Indeed, if for two points $x, x' \in N \setminus Q$, both on the same side of $Q$ and on the same meridian as the equator point $y$, we have $\rho(x, y) > \rho(F_x, -y)$ and $\rho(x', -y) < \rho(F_{x'}, -y)$, then by continuity there must be a point $x''$ between $x$ and $x'$ with $\rho(x'', y) = \rho(F_{x''}, -y)$, whence $F_{x''} = x''$ and a contradiction is obtained. □

## 2.4 Doubly covered convex polygons

The ideas of Prof. J. Itoh essentially improved this section.

Next we consider doubly covered convex polygons. We first treat the problem of the existence of a point on such a degenerate surface with a given cut locus, and give a condition necessary and sufficient for the existence of a solution to this problem. Afterwards we prove some characterization and existence results on farthest points.

In the following, we shall denote by $Q$ the double of an arbitrary planar $n$-gon $P$ with vertices $a_1, ..., a_n$ ($n \geq 3$).

All considered trees will be non-degenerate.

We call the tree $T \subset \mathbb{R}^3$ a *segments-tree* if it is a union of line-segments.

A polygon $P$ is called *circumscribed to a segments-tree* $T$ if the vertices of $P$ coincide with the vertices of degree one of $T$. In this case, we alternatively say that $T$ is *inscribed in* $P$.

On polyhedral convex surfaces, all cut loci are segments-trees, as follows from a result of B. Aronov and J. O'Rourke [5]; for completeness, we sketch here the proof in the degenerate case.

When we write "the vertices of $C(x)$", we refer to the vertices of the (geometrically realized with line-segments as edges) graph $C(x)$.

Notice that the cut locus of a point $x$ in the bord $\mathrm{bd}(P)$ of a doubly covered convex polygon is the union of the edges of $P$ not containing $x$.

**Theorem 38** *Let $P$ be a convex $n$-gon and $Q = P \bowtie P'$. For any point $x$ in* $\text{int}(P)$*, the cut locus $C(x)$ is a segments-tree without vertices of degree two, and the local maxima of $\rho_x$ are among the vertices of $C(x)$.*

*Proof:* For $x \in \text{int}(P)$, one can easily see, by cutting along the segments from $x$ to the vertices of $Q$ and unfolding, that $C(x)$ is a non-degenerate tree (on the face of $Q$ opposite to $x$) and has precisely $n$ vertices of degree one, namely $a_1, ..., a_n$, and also that $C(x)$ is a segments-tree without vertices of degree two.

Since the metric of $Q$ is locally Euclidean at all points except for the vertices, the preceding procedure of unfolding $Q$ shows that each local maximum of $\rho_x$ which is not a vertex of $P$ is joined to $x$ by at least three segments. □

**Remark** The (metric) converse of Theorem 38 is not true, namely a segments-tree is not necessarily (isometric to) the cut locus of some point on some doubly covered convex polygon.

For a suitable example, consider five points $b_j$ on the cercle $C$ centered at $o$ ($j = 1, ...5$). Let $c_j \in [ob_j]$ such that

$$2||c_j - o|| < ||b_j - o||,$$

and consider line-segments $[c_j d_j]$ orthogonal to $ob_j$ at $c_j$ such that

$$2||d_j - c_j|| < ||b_j - o||.$$

Define

$$T = \cup_{j=1,...5}([ob_j] \cup [c_j d_j]).$$

Then there exists no doubly covered convex polygon $Q$ such that $T$ is the cut locus of some point of $Q$.

Indeed, suppose that there exists a convex polygon $P$ circumscribed to the segments-tree $T$.

Assume that the points $b_j$ are consecutive on $C$, for $j = 1, ...5$. Then, since $P$ is convex, the points $d_j$ are exterior to the pentagon $b_1 b_2 b_3 b_4 b_5$.

Because of the symmetry, we may assume that the points $d_1, d_2$ are interior to $\angle b_1 o b_2$, the points $d_3, d_4$ are interior to $\angle b_3 o b_4$, and the point $d_5$ is interior to $\angle b_1 o b_5$ (all other cases are similar).

Because $2||d_j - c_j|| < ||b_j - o||$, we get

$$\min\{\angle b_1 o b_2, \angle b_3 o b_4, \angle b_5 o b_1\} > 2\pi/3,$$

and therefore the sum of all angles at $o$ were larger than $2\pi$, impossible. □

In order to describe the trees which can be realized as cut loci on doubly covered convex polygons, next we give more definitions.

Denote by $V(A)$ and $E(A)$ the set of vertices, and respectively of edges of $A$, where $A$ is either a tree or a polygon.

For a polygon $P$, let $\Delta_e$ be the line containing the edge $e \in E(P)$.

Let $\mathrm{Pr}(P)$ be the projective plane obtained as the union of $\mathrm{aff}(P)$ with its points at infinity, and denote by $\overline{P} \subset \mathrm{Pr}(P)$ the *complete polygon determined by* $P$, i.e. determined by

$$V(\overline{P}) = \{\Delta_e \cap \Delta_{e'} \ ; \ e, e' \in E(P)\}, \quad E(\overline{P}) \subset \cup_{e \in E(P)} \Delta_e.$$

Let $P$ be a convex polygon circumscribed to a segments-tree $T$.

Label $E(P)$ with $0i$, $1 \leq i \leq n$, and $E(T)$ with $ij \in \{kl;\ 1 \leq k < l \leq n\}$. We call such a labeling *good* if, for all labels, the following hold in $\mathrm{Pr}(P)$:

*i)* $\Delta_{e_{0i}} \cap \Delta_{e_{0j}} \cap \Delta_{e_{ij}} \neq \emptyset$;

*ii)* if $e_{ij} \cap e_{kl} \neq \emptyset$, $e_{ij} \neq e_{kl}$, then there is a set of $q \geq 3$ distinct subscripts $1 \leq r_1 < ... < r_{q-1} < r_q \leq n$, such that $i, j, k, l \in \{r_1, ..., r_{q-1}, r_q\}$ and $e_{r_1 r_2} \cap ... \cap e_{r_{q-1} r_q} \cap e_{r_q r_1} \neq \emptyset$.

**Remark** If $e_{0i}, e_{0j}$ are two consecutive edges of $P$ then, since the vertex of $P$ given by $e_{0i} \cap e_{0j}$ is a vertex of degree one in $T$, there is a unique edge of $T$ meeting both $e_{0i}$ and $e_{0j}$, namely $e_{ij}$, by assertions *i)* and *ii)*.

We shall use the following known result.

**Lemma 26** *For any finite tree $T$, there exists $z \in V(T)$ such that all vertices of $T$ which are neighbours of $z$, with at most one exception, are of degree one.*

*Proof* : To show the result, assume that each edge of $T$ has length one and consider a longest path in $T$ joining two vertices of degree one, say $x$ and $y$. Then the neighbour of $x$ (or of $y$) on the path $T$ fulfills the conclusion. □

The bisecting half-line of the planar angle $\angle xoy$ is, by a result in elementary geometry, the locus of all points $z$ in plane at equal distances to the half-lines $ox$ and $oy$.

**Theorem 39** *Let $Q$ be the double of a convex polygon $P$ circumscribed to a segments-tree $T$. There exists a point $x \in Q$ such that $T = C(x)$ if and only if there exists a good labeling of $P$ and $T$ such that $\cap_{e_{ij} \in E(T)} l_{e_{ij}} \neq \emptyset$, where $l_{e_{ij}}$ is the line symmetric to $\Delta_{e_{ij}}$ with respect to the line containing the bisector of the angle of $\overline{P}$ at $\Delta_{e_i} \cap \Delta_{e_j}$.*

*Proof:* Suppose $T = C(x)$, with $x$ a point on the double $Q$ of the convex polygon $P$; then, since $T$ is non-degenerate, $x$ is a point interior to $P$. For $e \in E(T)$, there are precisely two segments joining $x$ with each $z \in e \setminus V(T)$; let $f_1, f_2 \in V(Q)$ be the edges crossed by these segments, and $x_1$, $x_2$ the symmetric points of $x$ with respect to $f_1, f_2$. Then $\Delta_e$ is orthogonal to the segment $[x_1 x_2]$ through its mid-point; therefore, if $\{a_e\} = \Delta_{f_1} \cap \Delta_{f_2}$ then $a_e \in \Delta_e$ and $x a_e$ is the line symmetric to $\Delta_e$ with respect to the bisector of the angle of $\overline{P}$ at $a_e$.

For the converse, denote by $P$ the face of $Q$ which does not contain $\cap_{e_{ij} \in E(T)} l_{e_{ij}} \neq \emptyset$, and let $x$ be the point of $P$ opposite to the intersection point $\cap_{e_{ij} \in E(T)} l_{e_{ij}}$.

Next we show by induction over the number $n$ of edges of $P$ that

$$E(T) = E(C(x)).$$

Cut along the segments from $x$ to the vertices of $P$, and unfold $Q$. The following arguments will implicitly refer to this unfolding.

If $n = 3$ then any Y-tree is isometric to the cut locus of a point on a doubly covered triangle, by a classical result in elementary geometry. (Notice that, in this case, the labeling is obvious.)

Suppose $n > 3$, and consider a point $z$ in $V(T)$ such that all vertices of $T$ adjacent to $z$, possibly excepting one, are of degree one; such a point exists, by Lemma 26.

Assume that the degree of $z$ in $T$ is equal to $(m + 1) \geq 3$, and that there is a neighbour $v$ of $z$ in $V(T)$ of degree larger than one, so we have $m \leq n-2$. (If all neighbours of $z$ in $V(T)$ are of degree one then the above remark ends the proof.)

Let $a_{ij} \in V(Q)$ denote the neighbour of $z$ joined to $z$ by the edge of $T$ which has the label $ij$.

The hypothesis assure us that each point $u$ in the segment $[z a_{ij}] \subset T$, except for $a_{ij}$ and $z$, is joined to $x$ by precisely two segments. Thus, we have $[z a_{ij}] \subset C(x)$.

Since we have a good labeling, the edges of $P$ starting at $a_{ij}$ are precisely $e_i$ and $e_j$.

Let $L$ be the polygonal line which contains all edges of $P$ starting at the vertices $a_{ij}$. Assume that, for some direction on $L$, $e_l$ is the first edge of $L$, and $e_k$ is the last edge of $L$.

Then, since we have a good labeling, the edge of $T$ from $z$ to $v$ has the label $lk$. Moreover, if the point $b_{lk}$ in $\mathrm{Pr}(P)$ is given by $\{b_{lk}\} = \Delta_{e_l} \cap \Delta_{e_k}$, then $b_{lk} \in \Delta_{e_{lk}}$.

Assume first that, on the line $\Delta_{e_{lk}}$, $z$ lies between $v$ and $b_{lk}$.

Denote by $T'$ the segments-tree obtained from $T$ as it follows: remove all edges of $T$ from $z$ to vertices of $Q$; replace the segment from $z$ to $v$ with the segment from $v$ to $b_{lk}$.

Denote by $Q'$ the doubly covered polygon circumscribed to $T'$.

Then, by labeling the edge of $T'$ from $v$ to $b_{lk}$ with $lk$, we have a good labeling of $Q'$ and $T'$, naturally induced by the good labeling of $Q$ and $T$.

By the induction's assumtion, since (a face of) $Q'$ has fewer edges than $P$, it follows that the cut locus of $x$ on $Q'$ is precisely $T'$.

Since the edges of $T$ starting at $z$ are clearly included in $C(x)$, we get $T = C(x)$, because all edges of $T$ are included in $C(x)$.

Assume now that, on the line $\Delta_{e_{lk}}$, the point $v$ lies between $z$ and $b_{lk}$.

Denote by $w$ the point in $V(T) \setminus V(Q)$ with maximal number of edges in the subarc of $T$ joining it to $z$. Then all vertices of $V(T)$ adjacent to $w$, except for one, also belong to $V(Q)$.

Let $a_{i'j'} \in V(Q)$ denote the neighbour of $w$ joined to $w$ by the edge of $T$ which has the label $i'j'$.

The hypothesis assure us that each point $u$ in the segment $[wa_{i'j'}] \subset T$, except for $a_{i'j'}$ and $w$, is joined to $x$ by precisely two segments. Thus, we have $[wa_{i'j'}] \subset C(x)$.

Since we have a good labeling, the edges of $P$ starting at $a_{i'j'}$ are precisely $e_{i'}$ and $e_{j'}$.

Let $L'$ be the polygonal line which contains all edges of $P$ starting at the vertices $a_{i'j'}$. Assume that, for some direction on $L'$, $e_{l'}$ is the first edge of $L'$, and $e_{k'}$ is the last edge of $L'$.

Let $y$ be the neighbour of $w$ which belongs to $V(T)\setminus V(Q)$. Then, since we have a good labeling, the edge of $T$ from $w$ to $y$ has the label $l'k'$. Moreover, if the point $b_{l'k'} \in \mathrm{Pr}(P)$ is given by $\{b_{l'k'}\} = \Delta_{e_{l'}} \cap \Delta_{e_{k'}}$, then $b_{l'k'} \in \Delta_{e_{l'k'}}$.

Since the point $v$ lies, on the line $\Delta_{e_{lk}}$, between $z$ and $b_{lk}$, we obtain that the point $w$ lies, on the line $\Delta_{e_{l'k'}}$, between $y$ and $b_{l'k'}$.

Denote by $T'$ the segments-tree obtained from $T$ as it follows: remove all edges of $T$ from $w$ to vertices of $Q$; replace the segment from $y$ to $w$ with the segment from $y$ to $b_{l'k'}$.

Denote by $Q'$ the doubly covered polygon circumscribed to $T'$.

Then, by labeling the edge of $T'$ from $y$ to $b_{l'k'}$ with $l'k'$, we have a good labeling of $Q'$ and $T'$, naturally induced by the good labeling of $Q$ and $T$.

By the induction's assumtion, since (a face of) $Q'$ has fewer edges than $P$, it follows that the cut locus of $x$ on $Q'$ is precisely $T'$.

Since the edges of $T$ starting at $w$ are clearly included in $C(x)$, we get $T = C(x)$, because all edges of $T$ are included in $C(x)$. □

**Corollary** *Let $Q$ be the double of a convex $n$-gon $P$ circumscribed to a segments-tree $T$ with $m$ vertices. If $T = C(x)$ for some $x \in Q$ then $m \leq 2n - 2$.*

*Proof:* Because $T$ has no vertex of degree two, an easy result in graph theory says that the number $r$ of its ramification points plus 1 is less than the number of its vertices of degree one, i.e. $m = n + r \leq 2n - 2$. □

**Theorem 40** *Any combinatorial type of finite tree without vertices of degree two can be realized as the cut locus of some point on some doubly covered convex polygon.*

*Proof:* Let $T$ be a finite tree without vertices of degree two. We shall prove the result by induction over the number $n$ of vertices of degree one in $T$.

If $n = 3$ then $T$ is a $Y$-tree, and is isometric to the cut locus of a point on a doubly covered triangle, by a classical result in elementary geometry. Suppose $n > 3$.

Denote by $Z \subset V(T)$ the set of points given by Lemma 26, and take $z \in Z$; we have $\deg z = m \geq 3$. Denote by $T'$ the tree obtained from $T$ by deleting in $T$ the neighbours of $z$ of degree one, together with the edges joining them to $z$.

If $T'$ is precisely $\{z\}$ then $T$ has a unique ramification point, namely $z$, and the conclusion follows by considering a regular polygon. Suppose $T' \neq \{z\}$, hence it has at least one ramification point.

By the induction assumption, there exists a doubly covered polygon $Q'$ and $x \in Q'$ such that $C(x)$ is isomorphic to $T'$.

Denote by $a_1$ the vertex of $Q'$ corresponding to $z$; let $Ell$ be an ellipse with a focus at $x$ and tangent to the edges of $Q'$ issuing from $a_1$ at some points $b_1, b_{m+1}$ close to $a_1$. Consider $m-1$ points $b_2, ..., b_m$ on the arc of $Ell$ bounded by $b_1, b_{m+1}$ and closest to $a_1$. Denote by $l_k$ the lines tangent to $Ell$ at $b_k$ and by $Q$ the minimal (by inclusion) doubly covered convex polygon obtained by removing from $Q'$ the segments $[b_1 a_1]$ and $[a_1 b_{m+1}]$, and replacing them with the segments determined by the intersection points $l_k \cap l_{k+1}$ ($k = 1, ..., m$; here, $m + 1 = 1$). By Theorem 39, it follows that the cut locus of $x$ on $Q$ is isomorphic to $T$. □

**Remark** For any double $Q$ of a convex $n$-gon $P$ ($n \geq 3$), there exist points $x \in \text{int}(P)$ such that the distance function $\rho_x$ has at least one local maximum interior to the face opposite to $x$. For example, take $x$ the centre of the circle inscribed in $P$ (if it is not unique, consider an inscribed ellipse); in this case, the angle at $x$ between any two consecutive radii through tangent points is less than $\pi$, hence the point opposite to $x$ is a local maximum for $\rho_x$ (by Theorem 3).

It follows immediately that the set of points $x$ as above has non-empty interior. However, the existence of such points is not clear if one asks for global maximum or, moreover, for multiple global maxima.

For example, if $\alpha_i$ denotes the angle of $P$ at $a_i$ ($i = 1, ..., n$), we have the following result:

**Theorem 41** *Let $Q$ be the double of a convex polygon $P$ with distinct vertices $a_k, a_l$ such that $\max\{\alpha_k, \alpha_l\} \leq \pi/4$. Then $F_Q = \{a_k, a_l\}$, except if $x$ is the centre of an isosceles rectangular triangle.*

*Proof:* If the vertices are consecutive then the result follows immediately from Theorem 49. Suppose we don't have consecutive vertices; it suffices to treat the case $n = 4$.

Let $V(P) = \{a, b, c, d\}$, with $\max\{\alpha_a, \alpha_c\} \leq \pi/4$. The assertion is clear for boundary points of $P$, so consider $x \in \text{int}(P)$. Since the proof is similar for all cases, we may assume (see Theorem 39) that

$$C(x) = [ay] \cup [by] \cup [yz] \cup [zc] \cup [zd]$$

and

$$yz \cap ab \cap cd = \{e\}.$$

Suppose that $y \in F_x$, hence

$$\rho_x(y) \geq \max\{\rho_x(a), \rho_x(c)\}.$$

Cut $Q$ along $C(x)$ and unfold such that to obtain a planar figure; on it, denote by $y_j, z_j$ $(j = 1, 3)$ the symmetric points of $y$ and $z$ with respect to the edges $ab$ and respectively $cd$, and by $y_4, z_2$ the symmetric points of $y$ and $z$ with respect to the edges $ad$ and respectively $bc$. We have, by Theorem 39

$$\angle xay_1 = \angle xay_4, \quad \angle y_1ay_4 = 2\alpha_a,$$

$$\angle xcz_2 = \angle xcz_3, \quad \angle z_2cz_3 = 2\alpha_c.$$

Since $\rho_x(y_i) = \rho_x(y) \geq \max\{\rho_x(a), \rho_x(c)\}$, we obtain $(i = 1, 3, 4)$

$$\angle ay_1x \leq \angle xay_1 \leq \pi/4, \;\; \angle ay_4x \leq \angle xay_4 \leq \pi/4,$$

$$\angle cy_3x \leq \angle xcy_3 < \angle xcz_3 \leq \pi/4, \quad \angle cy_1x \leq \angle xcy_1 < \angle xcz_2 \leq \pi/4.$$

Thus,

$$\min\{\angle axy_1, \angle y_1xc, \angle cxy_3, \angle y_4xa\} \geq \pi/2.$$

Summing up, we get

$$2\pi \leq \angle axy_1 + \angle y_1xc + \angle cxy_3 + \angle y_4xa < 2\pi,$$

a contradiction. □

**Theorem 42** *Let $Q = P \bowtie P'$ be a doubly covered convex $n$-gon.*

*a) If $x \in \mathrm{int}(P)$ then $|M_x \cap V(Q)| \leq 3$ and if, moreover, $n > 3$ then $|F_x \cap V(Q)| \leq 2$. If $x \in \mathrm{bd}(Q)$ then $M_x \subset V(Q)$.*

*b) There exist points $x$ in $Q$ with $|F_x| > 1$.*

*Proof:* a) For $x \in \mathrm{bd}(Q)$ and $z \in E(Q) \setminus V(Q)$, there are only two segments joining them; since the total angle at $z$ is equal to $2\pi$, in any neighbourhood of $z$ there are points $y \in \mathrm{bd}(Q)$ with $\rho(x, y) > \rho(x, z)$.

If $x \in \mathrm{int}(P)$ and $y \in V(Q)$ then there is a unique shortest path joining them; thus, the total angle $\theta_y$ of $Q_n$ at some $y \in V(Q) \cap M_x$ is less than $\pi$, hence the total curvature at $y$ is $\omega_y = 2\pi - \theta_y > \pi$. Since $Q$ is homeomorphic to the sphere $S^2$, its total curvature is equal to $4\pi$, and therefore there are at most 3 vertices of curvature larger that $\pi$.

Suppose that $x \in \text{int}(P)$ such that $|F_x \cap V(Q)| \geq 3$. Then the points of $F_x \cap V(Q)$ lie on a non-degenerate tree included in $C(x)$, in contradiction to Lemma 11.

b) Let $a_1, a_2 \in V(Q)$ such that $\rho(a_1, a_2) = \text{diam}(Q) = \max_{x,y \in Q} \rho(x, y)$. If $|F_{a_1}| > 1$, we are done; suppose not.

Take $z \in E(Q) \setminus V(Q)$; let $l$ be the line orthogonal at $z$ to an edge of $Q$ and $v = (l \cap \text{bd}(Q)) \setminus \{z\}$. Let $\Gamma_1, \Gamma_2$ be the segments from $v$ to $z$ and denote by $Q^1$ and $Q^2$ the open subsets of $Q$ bounded by $\Gamma_1 \cup \Gamma_2$. We may take $z$ such that $a_1 \in Q^1$ and $a_2 \in Q^2$.

Clearly, $z$ is a local minimum for $\rho_{v|C(v)}$, hence there exist $y_1 \in Q^1$ and $y_2 \in Q^2$ local maxima for $\rho_v$. We may assume that $y_1 \in F_v$.

Now move continuously the point $u$, on $\text{bd}(Q)$, from $v$ to $a_1$. By the convexity of $Q$, $\rho_u$ continue to have a local maxima in $Q^1$. Since $F_{a_1} \subset Q^1$ and the mapping $F$ is upper semi-continuous, there is a point $x \in \text{bd}(Q) \cap Q^1$ with $F_x \cap Q^1 \neq \emptyset$ and $F_x \cap Q^2 \neq \emptyset$, hence $|F_x| > 1$. □

**Theorem 43** *For any natural number $m$ there exists a convex $n$-gon $P$ and a point $x \in \text{int}(P)$ such that, on the double of $P$, we have $|F_x| = m$ and $F_x \cap \{a_1, ..., a_n\} = \emptyset$.*

*Proof:* Fix a natural number $m \geq 2$, and consider a system of coordinates $oXY$ in the plane, two real numbers $a > c > 0$, and the points $x = (-c, 0)$, $y_k = (c - 2k\varepsilon, 0)$, with $k = 0, ..., m-1$ and $0 < \varepsilon < \frac{c}{2m}$.

Denote by $Ell_k$ the ellipse with the foci at $x$ and $y_k = (c - 2k\varepsilon, 0)$ and the sum of its focal radii equal to $2a$, hence

$$Ell_k = \{z; ||z - x|| + ||z - y_k|| = 2a\}$$

and its equation is

$$Ell_k : \frac{(X + k\varepsilon)^2}{a^2} + \frac{Y^2}{a^2 - (c - k\varepsilon)^2} = 1.$$

Let $l_k$ be the lines through $y_k$ given by $X = c - 2k\varepsilon$, and let $\{z_k^1, z_k^2\} = Ell_k \cap l_k$. We may assume that $Y(z_k^1) > 0$ for all $k$; a short calculus gives us

$$Y(z_k^1) = a^{-1}[a^2 - (c - k\varepsilon)^2].$$

The line $d_k^1$ tangent to $Ell_k$ at $z_k^1$ has the equation

$$d_k^1: \ (X + k\varepsilon)(c - k\varepsilon) + aY = a^2.$$

Easy calculations yield, for $j = 1, ..., m-1$,

$$X(Ell_j \cap Ell_{j-1}) = X(d_j^1 \cap d_{j-1}^1) = c - 2j\varepsilon + \varepsilon,$$

which is precisely the mid-point of the interval $[c - 2j\varepsilon, c - 2(j-1)\varepsilon]$.

Consider the real functions

$$Y_{Ell_j}(s) = [a^2 - (c - j\varepsilon)^2]^{\frac{1}{2}} \, |1 - a^{-2}(s + j\varepsilon)^2|^{\frac{1}{2}}$$

and

$$Y_{d_j^1}(s) = a^{-1}[a^2 - (s + j\varepsilon)(c - j\varepsilon)].$$

One can easily check that

$$Y_{d_{j-1}^1}(X(z_j^1)) > Y(z_j^1), \quad Y_{d_j^1}(X(z_{j-1}^1)) > Y(z_{j-1}^1),$$

hence

$$Y_{d_{j-1}^1}(s) > Y_{d_j^1}(s) \geq Y_{Ell_j}(s) > Y_{Ell_{j-1}}(s) \quad \text{as} \quad s < c - 2j\varepsilon + \varepsilon$$

and

$$Y_{d_j^1}(s) > Y_{d_{j-1}^1}(s) \geq Y_{Ell_{j-1}}(s) > Y_{Ell_j}(s) \quad \text{as} \quad s > c - 2j\varepsilon + \varepsilon.$$

These imply that $d_j^1 \cap Ell_{j-1} = \emptyset$, and also $d_{j-1}^1 \cap Ell_j = \emptyset$.

Thus, for each $j = 1, ..., m-1$, there exist points $a_0^2$ in $Ell_0$ and $a_j^1, a_j^2$ in $Ell_j$ such that:

*i)* $X(z_0^1) > X(a_0^2)$, $X(a_j^1) > X(z_j^1) > X(a_j^2)$; and

*ii)* $a_{j-1}^2 a_j^1$ is tangent to both $Ell_{j-1}$ and $Ell_j$.

The slope of the line $d_k^1$ is equal to $-a^{-1}(c - k\varepsilon)$ $(k = 0, ..., m-1)$; the one of $a_{j-1}^2 a_j^1$ belongs to the interval $]-a^{-1}(c - (j-1)\varepsilon), -a^{-1}(c - j\varepsilon)[$, whence we obtain

$$a_{j-1}^2 a_j^1 \cap (\cup_{k=0,\dots,m-1} Ell_k) = \{a_{j-1}^2, a_j^1\}.$$

Thus, the lines $a_{j-1}^2 a_j^1$ $(j = 1, ..., m-1)$ together with the lines tangent to $Ell_k$ at $z_k^2$ $(k = 0, ..., m-1)$ determine a convex polygonal domain which contains $\cup_{k=0,\dots,m-1} Ell_k$. We conveniently complete it with edges (see next remarks). It follows, for the resulting convex polygon $P$, that $x \in \text{int}(P)$. Moreover, the angles made at $y_j$ by $y_j a_j^1$ and $y_j a_j^2$, by $y_j a_j^2$ and $y_j z_j^2$, and respectively by $y_j z_j^2$ and $y_j a_j^1$, are all less than $\pi$, by Condition *i)*. Thus, $y_0, \dots, y_{m-1}$ are local maxima for the distance function $\rho_x$ on $D = P \bowtie P'$ (by Theorem 3) and $\rho(x, y_k) = 2a$ for all $k = 0, \dots, m-1$, because all ellipses $Ell_k$ have the same sum of focal radii.

With conveniently added edges, the points $y_j$ ($j = 1, \dots, m$) are, on the resulting doubly covered polygon $P \bowtie P'$, the only ramification points of $C(x)$ with the first coordinate at least $c - 2m\varepsilon$, by Condition *ii)* and by Theorem 39, and therefore $F_x = \{y_0, \dots, y_{m-1}\}$. □

Combining the constructions of Theorems 40 and 43, one can obtain:

**Theorem 44** *For any tree $T$ and any set $V$ of vertices of a path in $T$, there is a double $Q$ of a convex polygon and a point $x \in Q$ such that $C(x)$ is isomorphic to $T$ and $F_x$ corresponds to $V$.*

*Proof:* We can construct, as in the proof of Theorem 43, a doubly covered convex polygon $Q'$ and a point $x \in Q'$ such that $F_x$ corresponds to $V$. Completing with edges as in the proof of Theorem 40, we obtain that, on the resulting doubly covered convex polygon $Q$, $C(x)$ is isomorphic to $T$, and $F_x$ still corresponds to $V$. □

## 2.5 Typical degenerate convex surfaces

T. Zamfirescu [42] discovered that, on most convex surfaces, most points are endpoints; thus, the cut locus of any point on such a surface is residual. But typical convex surfaces in $\mathbb{R}^{d+1}$ are smooth and strictly convex ([16], [21]), hence his result does not refer to $d$-dimensional degenerate convex surfaces. In contrast to the general case, we shall see that on typical doubles we have relatively few endpoints and still, for points interior to a face, very large cut loci (residual in the opposite face). Other properties concerning the distance functions are also obtained.

The first two results of this section are proved for arbitrary dimension $d$, while the last one for $d = 2$.

Denote by $\mathcal{B}^d$ the space of all compact convex bodies in $I\!R^d$ with non-empty interior, by $\mathcal{S}^d$ the set of their boundaries and by $\mathcal{D}^d$ the space of all $d$-dimensional degenerate convex surfaces. Endowed with the Pompeiu-Hausdorff metric, $\mathcal{B}^d$, $\mathcal{S}^d$ and $\mathcal{D}^d$ are Baire spaces.

We shall make use of several lemmas. The next result was obtained by V. A. Alexandrov [4].

**Lemma 27** *Let $D$ and $D'$ be open convex sets on $I\!R^d$ ($d \geq 2$), whose boundaries $S$ and $S'$ are piecewise of class $\mathcal{C}^1$. If $S$ and $S'$ are isometric in the induced intrinsic metrics then $D$ and $D'$ are isometric.*

Let $x$ be a smooth point on the convex surface $S \subset I\!R^d$. Consider at $x$ a tangent direction $\tau$, the normal 2-dimensional section of $S$ in the direction $\tau$ and the lower and upper radii of curvature, $r_i^\tau(x)$ and $r_s^\tau(x)$, of this normal section (see [8], p. 14). Denote by $\gamma_i^\tau(x)$ and $\gamma_s^\tau(x)$ the *lower* and *upper curvatures* of $S$ at $x$ in the tangent direction $\tau$ [45]; we have $\gamma_i^\tau(x) = r_s^\tau(x)^{-1}$ and $\gamma_s^\tau(x) = r_i^\tau(x)^{-1}$. If they are equal, the common value $\gamma^\tau(x)$ is the curvature of $S$ at $x$ in the direction $\tau$.

T. Zamfirescu (see [45] or [44]) proved that:

**Lemma 28** *For most convex surfaces $S \in I\!R^d$,*

*(i) at each point $x \in S$, $\gamma_i^\tau(x) = 0$ or $\gamma_s^\tau(x) = \infty$ for any tangent direction $\tau$ at $x$;*

*(ii) at most points $x \in S$, $\gamma_i^\tau(x) = 0$ and $\gamma_s^\tau(x) = \infty$ for any tangent direction $\tau$ at $x$.*

The following result will also be useful.

**Proposition 3** *The set of all convex surfaces not isometric to a given one is open and dense in $\mathcal{S}^d$.*

*Proof:* For a convex surface $S$ in $\mathcal{S}^d$, denote by $\mathcal{I} \subset \mathcal{S}^d$ the set of convex surfaces isometric to $S$. Clearly, $\mathcal{I}$ is closed in $\mathcal{S}^d$.

Consider a sequence of points $x_n \in I\!R^d \setminus \mathrm{conv}(S)$ which converges to some point $x \in S$, and let $S_n = \mathrm{bd}(\mathrm{conv}(S \cup \{x_n\}))$. It follows that $S_n \to S$ and $S_n \notin \mathcal{I}$, whence $\mathcal{S} \setminus \mathcal{I}$ is dense in $\mathcal{S}^d$. □

Let $D = B \bowtie B'$ be an element of $\mathcal{D}^d$ and $x$ a point in $D$; in this section, a *ramification point of* $C(x)$ is a point in $C(x)$ no neighbourhood of which in $C(x)$ is homeomorphic to a closed disk of $I\!R^d$.

**Theorem 45** *If* $D = B \bowtie B'$ *is a typical element of* $\mathcal{D}^d$ *then:*
*a) most points in* $\mathrm{bd}(B)$ *are endpoints of* $D$*;*
*b) for any point* $x \in \mathrm{int}(B)$*,* $C(x) \setminus C_x$ *is residual in* $B'$*;*
*c) any segment from a point* $x \in \mathrm{int}(B)$ *to a point* $y \in \mathrm{int}(B')$ *is crossing the border at a point with* $\gamma_i = 0$ *and* $\gamma_s < \infty$ *on* $\mathrm{bd}(B)$*.*

*Proof:* Note that, by Lemmas 6 and 27, and by Proposition 3, a typical convex surface in $\mathcal{S}^d$ is the boundary of a typical convex body in $\mathcal{B}^d$ which corresponds to a typical double in $\mathcal{D}^d$, and conversely.

Denote by $B_D$ a convex body in $I\!R^d$ isometric to $B$ and let $x_D$ and $z_D$ be the correspondents on $B_D$ of $x \in \mathrm{int}(B)$ and $z \in C(x)$. Let $E_x \subset I\!R^d$ be the hyperellipsoid of revolution with the foci at $x_D$ and $z_D$, and the sum of focal radii equal to $\rho(x, z)$. Then $E_x \cap \mathrm{bd}(B_D) \neq \emptyset$ and $E_x \subset B_D$, since otherwise one can find points in $\mathrm{int}(\mathrm{conv}E_x) \cap B_D$, the correspondents of which give paths on $D$ from $x$ to $z$ shorter than $\rho(x, z)$, impossible.

Since $E_x \subset B_D$, $\mathrm{bd}(B_D)$ clearly has finite upper curvatures at each point $y \in E_x \cap \mathrm{bd}(B_D)$ in all tangent directions. It follows that for any tangent direction $\tau$ at $y$, $\gamma_i^\tau(y) = 0$ (by *i)* of Lemma 28).

Consequently, by *ii)* of Lemma 28, most points of $\mathrm{bd}(B)$ are not interior to any segment joining points on different faces of $D$. Since $\mathrm{bd}(B)$ is strictly convex (Lemma 6), no other segments pass through points of the border, hence most points of $\mathrm{bd}(B)$ are endpoints.

Next we show that, for any point $x \in \mathrm{int}(B)$, the set of ramification points of $C(x)$ is dense in $B'$. Suppose there exists, for some $x \in \mathrm{int}(B)$, a non-empty, open (in the induced topology of $C(x)$) and connected subset $U \subset C(x)$ without ramification points. Thus, $U \cap \mathrm{bd}(B) = \emptyset$.

Therefore, the set $\mathcal{U}_{x,U}$ of tangent directions at $x$ of all segments joining $x$ to points in $U$ is non-empty and open in the unit tangent cone $T_x^1$ at $x$. Denote by $\mathcal{U}$ a component of $\mathcal{U}_{x,U}$, and by $\mathcal{F}$ the set of all segments from $x$ to $U$ whose tangent directions at $x$ belong to $\mathcal{U}$. Set

$$V = \cup_{\Gamma \in \mathcal{F}} (\Gamma \cap \mathrm{bd}(B)).$$

Since $U \cap \mathrm{bd}(B) = \emptyset$ and $U \subset C(x)$, we have $V \cap C(x) = \emptyset$, hence $V$ is homeomorphic to $\mathcal{U}$, a component of the set $T_x^1$ homeomorphic to $S^{d-1}$, so $V$ is an open subset of $\mathrm{bd}(B)$.

Concluding, we have found a non-empty open subset $V$ of $\mathrm{bd}(B)$ each point of which is interior to a segment, hence of finite upper curvatures at every point of it, in contradiction to Lemma 28.

Thus, the set of ramifications points of $C(x)$ is dense in $B'$.

Next, an immediate adaptation of the proof of Theorem 2 in [52] shows that $C(x) \setminus C_x$ is residual in $B'$.

Let $E_m$ be the set of those points $z \in B'$ interior to a segment from $x$ to $C(x)$, whose length from $z$ to $C(x)$ is at least $1/m$. We show that $E_m$ is nowhere dense in $B'$.

Indeed, let $V$ be an open set of $S$, and $y$ a point in $C_x \cap V$. Suppose that there exists a sequence of points $z_n \in E_m$ converging to $y$. Let $V$ be a compact neighbourhood of $y$, containing some ball $B(y, \varepsilon)$. Then, for integers $m_0 > m$ such that $1/m_0 < \varepsilon/3$, and $n_0$ such that $\rho(z_n, y) < \varepsilon/3$ for each $n \geq n_0$, we have $z_n \in E_{m_0} \cap V$ for all $n \geq n_0$.

Denote by $y_n$ the cut point of $x$ along the segment joining it to $z_n$. Possibly passing to a subsequence, we may assume that $\{y_n\}_{n \geq n_0}$ converges; there exists a subsequence of the corresponding sequence of segments from $x$ to $y_n$, which converges to a segment from $x$ to $y$, of length equal to

$$\lim_{n\to\infty} (\rho(x, z_n) + \rho(z_n, y_n)) \geq \lim_{n\to\infty} (\rho(x, z_n) + m_0^{-1}) = \rho(x, y) + m_0^{-1},$$

which is impossible.

Thus, there exists an open neighbourhood of $y$ in $V$ whose points are not in $E_m$, and so $E_m$ is nowhere dense in $B'$.

Hence $C(x) = B' \setminus \cup_{m \geq 1} E_m$ contains most points of $B'$.

Let now $G_n$ be the set of points in $C(x)$ joined to $x$ by two segments at Pompeiu-Hausdorff distance at least $1/n$. We show that $G_n$ is nowhere dense in $B'$.

Indeed, let again $V$ be an open set of $S$, and let $y \in V$. Assume that there exists a sequence of points $y_k \in G_n$ which converges to $y$. Then, the two sequences of segments from $x$ to $y_k$ converge to two segments from $x$ to $y$, at Pompeiu-Hausdorff distance at least $1/n$, impossible. Thus, a whole neighbourhood of $y$ in $V$ is disjoint from $G_n$, wherefore $G_n$ is nowhere dense in $B'$, and $C_x = \cup_{m \geq 1} G_n$ is of first category in $B'$. □

**Remark** Let $D = B \bowtie B'$ be typical in $\mathcal{D}^d$. If $x \in \mathrm{bd}(B)$ then $F_x \subset \mathrm{bd}(B)$. If $x \in \mathrm{int}(B)$ then $F_x \subset \mathrm{int}(B')$, by Lemmas 5 and 6, because otherwise the

$d$-dimensional measure of the unit tangent cone at a point $y \in F_x \cap \mathrm{bd}(B)$ were less than the measure of a half-sphere.

The next result completes Theorem 12 and shows, via Theorem 17, that a typical degenerate convex surfaces belongs to $\mathcal{S}_2$; see also Lemma 10.

**Theorem 46** *If $d = 2$ and $D$ is a typical element of $\mathcal{D}^2$ then:*
*a) there exists no point $x \in D$ with an arc in $F_x$;*
*b) there exists points $x \in D$ with $|F_x| > 1$.*

*Proof*: Suppose there were a point $x \in D$ with $F_x$ containing an arc $J'$; take $y_1, y_2$ two interior points of $J'$, and denote by $J$ the subarc of $J'$ joining them. Since each point interior to $J$ is a relative minimum for $\rho_x|_J$, we obtain, by Lemma 8, that $y_i$ is the mid-point of a loop $\Lambda_{y_i}$ at $x$, and no segments connect $x$ to $y_i$ excepting those in $\Lambda_{y_i}$. Now, denote by $\Delta$ the domain bounded by $\Lambda_{y_1} \cup \Lambda_{y_2}$; by Lemma 13, $\Delta \cap C(x) \subset J$, whence each point of $\Delta$ is interior to a geodesic, in contradiction to Theorem 45 *a*).

For $D = B \bowtie B'$ typical, since $\mathrm{bd}(B)$ is of class $\mathcal{C}^1$, the tangent cone at a point of its border is isometric to (but it is not) a plane in $\mathbb{R}^3$. Thus, if such a point $y \in \mathrm{bd}(B)$ is a farthest point for some $x \in D$, hence a critical point for $\rho_x$, then there are at least two segments from $x$ to $y$. But there are unique segments from $x \in \mathrm{int}(B)$ to the points of $\mathrm{bd}(B)$, hence $F_x \subset D \backslash B = \mathrm{int}(B')$ if $x \in \mathrm{int}(B)$.

We claim that there exists $x \in \mathrm{bd}(B)$ with $|F_x| > 1$.

To prove the claim, let $a_1, a_2$ be two points in $\mathrm{bd}(B)$ be such that

$$\rho(a_1, a_2) = \max_{x,y \in \mathrm{bd}(B)} \rho(x, y).$$

If $|F_{a_1}| > 1$ then we are done, so we may assume that this is not the case, hence $F_{a_1} = a_2$.

Notice that the line $a_1 a_2$ is orthogonal to $\mathrm{bd}(B)$ at $a_1$, and similarly at $a_2$. Indeed, if this is not the case then, since $\mathrm{bd}(B)$ is smooth, the line orthogonal to $a_1 a_2$ at $a_1$ intersects $\mathrm{int}(B)$. Thus, there exists points $w$ in $\mathrm{bd}(B)$ exterior to the closed disk of radius $\rho(a_1, a_2)$ centered at $a_2$, so $\rho(a_2, w) > \rho(a_1, a_2)$, a contradiction.

Take a typical point $z$ in $\mathrm{bd}(B) \setminus \{a_1, a_2\}$, hence (by above Lemmas) $z$ is a smooth point of the curve $\mathrm{bd}(B)$, and $\gamma_i(z) = 0$, $\gamma_s(z) = \infty$.

Let $v$ be the point of bd$(B)$ different from $z$ on the normal to bd$(B)$ at $z$; denote by $C_1$ and $C_2$ the (relatively) open subsets of bd$(B)$ determined by the line $zv$, and by $D_1$ and $D_2$ the corresponding open subsets of $D$.

Because the line $a_1a_2$ is orthogonal to bd$(B)$ at $a_1$ and at $a_2$, a simple continuity argument shows that we can choose the point $z$ such that $a_1 \in D_1$ and $a_2 \in D_2$.

Since $\gamma_i(z) = 0$ and $\gamma_s(z) = \infty$, the distance function $\rho_v$ has local maxima on both $C_1$ and $C_2$, and $z \notin M_v$.

If $|F_v| > 1$ then the claim is proved. Suppose $|F_v| = 1$, and assume that $y_1 = F_v \in D_1$.

Since the total angle at $z$ is equal to $2\pi$, it follows (by Theorem 1 and its proof) that $z$ can be a farthest point only for $v$, which is not the case because of the curvature properties at $z$.

Now move continuously a point $u$ on bd$(D)$ from $v$ to $a_1$. Since $F_{a_1} \in C_2$ and $F_v \in C_1$, by the upper semi-continuity of $F$, there exists a point $x \in$ bd$(D) \cap C_1$ with $F_x \cap C_1 \neq \emptyset$ and $F_x \cap C_2 \neq \emptyset$, hence $|F_x| > 1$. □

The example of a doubly covered disk, which belongs to the set $\mathcal{I}$ (see Theorems 34 and 35), shows that one cannot prove the above results for all degenerate convex surfaces.

The arguments used for Theorem 45 can be translated such that to justify:

**A non-degenerate example** Consider the non-degenerate convex surface $S = S_1 \bowtie S_2 \subset I\!R^{d+1}$, a symmetric (with respect to the hyperplane $H$) double cone over a typical convex surface $C \subset H \cong I\!R^d$ (see the definition given at the end of the Introduction).

Then the statements of Theorem 45 are also true if we replace $D = B \bowtie B'$ by $S = S_1 \bowtie S_2$, $B$ and $B'$ by $S_1$ and respectively $S_2$, and bd$(B)$ by $C$.

## 2.6 A higher-dimensional example

In this section we determine explicitly the structure of the cut loci on the double $D$ of an arbitrary $d$-dimensional simplex, namely any cut locus is a union of lower dimensional simplices, either all of them faces of $D$, or all with a common point. For distance functions, we describe the extreme points and show that only the vertices may be farthest points; this contrasts to the 2-dimensional case (see [49]): for a doubly covered acute triangle, if $o$ is the centre of the circle circumscribed of one triangular face then $F_o$ consists of the vertices and the orthocenter of the other face.

We shall make no distinction between a simplex and its corresponding boundary. Denote by $D_d$ the double of a $d$-dimensional simplex $S_d$, $D_d = S_d \bowtie S'_d$; $S_d$ and $S'_d$ are the $d$-faces of $D_d$.

Let $a_1, a_2, ..., a_{d+1}$ be the vertices of $D_d$; for a point $y$ interior to one of its $d$-faces, denote by $S_{ij}(y)$ the $(d-1)$-dimensional simplex determined by the points $y, a_1, ..., \breve{a}_i, ..., \breve{a}_j, ..., a_{d+1}$, where $\breve{a}_i$ means that the term $a_i$ is missing in the sequence $a_1, ..., a_{d+1}$.

A *$k$-dimensional face of the set* $\cup_{i,j} S_{ij}(y)$ is, by definition, a $k$-dimensional face of some $S_{kl}(y) \subset \cup_{i,j} S_{ij}(y)$.

A point $z \in \cup_{i,j} S_{ij}(y)$ is *of order* $r_z$ if the minimal number $k$, such that a $k$-dimensional face of $\cup_{i,j} S_{ij}(y)$ is containing $z$, verifies $k = r_z$.

Unless otherwise stated, all considerations will be implicitly done for simplices in $I\!\!R^d$ (i.e., on the $d-$faces of $D_d$).

The starting point of the next results is the following statement of T. Zamfirescu (Theorem 0 in [49]):

**Lemma 29** *Let $S$ be a convex surface or a doubly covered 2-dimensional convex set in $I\!\!R^3$, and let $x, z$ be distinct points in $S$. Assume that $x$ and $z$ are joined by $n$ segments, and in the interior of each of the $n$ resulting digons there is a point at distance at least $\rho(x,z)$ from $x$ ($n \geq 3$). Then $S$ is a doubly covered acute triangle, $x$ is the centre of its circumcircle and $z$ is its orthocenter.*

Notice that, on a doubly covered triangle, the cut locus of any point is a (possibly degenerate) $Y$-tree. When passing to higher dimensions, the cut loci keep the simplest possible structure.

**Theorem 47** *Let $x$ be a point in $D_d = S_d \bowtie S'_d$.*

*a) If $x \in \text{int}(S_d)$ then there is $y_x \in \text{int}(S'_d)$ such that $C(x) = \cup_{i,j} S_{ij}(y_x)$. A point $z \in C(x) \setminus \{y_x\}$ is of order $r_z$ if and only if there are $d+1-r_z$ segments from $x$ to $z$; thus, $y_x$ is the only point of $D_d$ joined to $x$ by $d+1$ segments.*

*b) If $x \in \mathrm{bd}(S_d)$ then $C(x)$ is the union of all $(d-1)$-dimensional faces of $S_d$ not containing $x$, and all points of $C(x)$ are joined to $x$ by at most two segments.*

*Proof:* a) For $x \in \mathrm{int}(S_d)$ and $i \neq j$ running from 1 to $d+1$, set

$$E_{ij} = S_d \cap \mathrm{aff}(a_1, ..., \breve{a}_i, ..., \breve{a}_j, ..., a_{d+1}),$$

$$F_j = \mathrm{aff}(a_i, E_{ij}), \quad H^x_{ij} = \mathrm{aff}(x, E_{ij}).$$

Clearly, $F_j$ and $H^x_{ij}$ are hyperplanes of $\mathbb{R}^d$.

For $i \neq j$, let $H_{ij}$ be the unique hyperplane of $\mathbb{R}^d$ such that the angles made by $F_i$ and $H_{ij}$, and respectively by $H^x_{ij}$ and $F_j$, are equal. Then $\cap_{k=1,...,d} H_{k,d+1}$ is a point, denoted by $y_x$.

Let $y_i$ be the point symmetric to $y_x$ with respect to $F_i$ $(i = 1, ..., d+1)$. By the construction, the angles made by $\mathrm{aff}(y_i, E_{ij})$ and $H^x_{ij}$, and respectively by $H^x_{ij}$ and $\mathrm{aff}(y_j, E_{ij})$, are equal.

Let $S_{ij}(y_i)$ be the $(d-1)$-dimensional simplex determined by the points $y_i, a_1, ..., \breve{a}_i, ..., \breve{a}_j, ..., a_{d+1}$. Clearly, $S_{ij}(y_i) \subset \mathrm{aff}(y_i, E_{ij})$ and, because of the symmetry, $S_{ij}(y_i)$ and $S_{ij}(y_j)$ are congruent for any two distinct indices $i, j$. It follows that we have equal distances from $x$ to $\mathrm{aff}(y_i, E_{ij})$ and to $\mathrm{aff}(y_j, E_{ij})$, and also from $x$ to $y_i$ and to $y_j$, hence to all symmetric points of $y_x$.

We claim that

$$\cap_{i,j=1,...,d+1} H_{ij} = \{y_x\}.$$

To prove the claim, it suffices to show that $H_{ij}$ contains $y_x$, for arbitrary $i, j \neq d+1$. Take such a hyperplane, say $H_{23}$. It is enough to prove that the angle made by $F_3$ and $H'_{23} = \mathrm{aff}(y, E_{23})$ is equal to the angle made by $H^x_{23}$ and $F_2$; equivalently, that the angles made by $\mathrm{aff}(y_3, E_{23})$ and $H^x_{23}$, and respectively by $H^x_{23}$ and $\mathrm{aff}(y_2, E_{23})$, are equal. But this follows from the facts that $S_{23}(y_2)$ and $S_{23}(y_3)$ are congruent, and we have $||x - y_2|| = ||x - y_3||$, whence equal distances from $x$ to $\mathrm{aff}(y_2, E_{23})$ and to $\mathrm{aff}(y_3, E_{23})$.

Denote by $S_{ij}(y_x)$ the $(d-1)$-dimensional simplex determined by the points $y_x, a_1, ..., \breve{a}_i, ..., \breve{a}_j, ..., a_{d+1}$. Clearly, $\cap_{i,j} S_{ij}(y_x) = \{y_x\}$.

The second claim is that, on $D_d$, we have

$$C(x) = \cup_{i,j} S_{ij}(y_x).$$

For $z \in \cup_{i,j} S_{ij}(y_x)$, say $z \in S_{12}(y_x)$, let $z_k$ be its symmetric point with respect to $F_k$ $(k = 1,2)$, hence $z_1 \in S_{12}(y_x)$ and $z_2 \in S_{12}(y_x)$. Set $z'_k = xz_k \cap F_k$ $(k = 1,2)$. Then $||x - z_1|| = ||x - z_2||$ and, on $D_d$, there are two segments from $x$ to $z$, each of which is the union of two line-segments: $[xz'_k] \cup [z'_k z]$. Thus, $\cup_{i,j} S_{ij}(y_x) \subset C(x)$.

To see that $C(x) \subset \cup_{i,j} S_{ij}(y_x)$, denote by $S_i(y_x)$ the $d$-dimensional simplex detemined by the points $y_i, a_1, ..., \check{a}_i, ..., a_{d+1}$. Clearly,

$$S_i(y_x) \cap S_j(y_x) = S_{ij}(y_x), \quad \cup_{i=1,...,d+1} S_i(y_x) = S'_d.$$

Consider now a point $w$ in $D_d \setminus \cup_{i,j} S_{ij}(y_x)$. If $w \in S_d \setminus \{a_1, ..., a_{d+1}\}$ then the line-segment $[xw]$ can be prolonged beyond $w$ as geodesic segment, hence $w \in D_d \setminus C(x)$. Suppose now that $w \in \mathrm{int}(S'_d) \setminus \cup_{i,j} S_{ij}(y_x)$; without loss of generality, we can assume that $w \in \mathrm{int}(S_1(y_x)) \cup_{i,j} S_{ij}(y_x)$. Then a geodesic segment from $x$ to $w$ is composed by the union of two line-segments (intersecting at a point in $F_1$), and it can be prolonged beyond $w$ as line-segment until it intersects $\cup_{i,j} S_{ij}(y_x)$, say at the point $z$. It follows that this path from $x$ to $z$ is a geodesic segment, hence $w \notin C(x)$.

For $\alpha \in \{1, ..., d+1\}$, let $A_\alpha = \mathrm{conv}(\{y_\alpha\} \cup F_\alpha)$. Now, an immediate induction shows that the point $z \in E \subset C(x)$ is of order $r_z$, where $E$ is an $r_z$-dimensional face of $C(x)$, if and only if there are precisely $d + 1 - r_z$ distinct subscripts $\alpha$ such that $E = \cap_\alpha A_\alpha$.

For each such subscript $\alpha$ there exists some $j \in \{1, ..., d+1\}$ such that the point $z_\alpha$, symmetric to $z$ with respect to the face $F_\alpha$, belongs to $S_{j\alpha}(y_\alpha)$. It follows, by the construction procedure, that $||x - z_\alpha||$ is constant. If $z'_\alpha = xz_\alpha \cap F_\alpha$ then each value of $\alpha$ provides a segment from $x$ to $z$, namely $[xz'_\alpha] \cup [z'_\alpha z]$.

In particular, $y_x$ is the only point in $D_d$ joined to $x$ by precisely $d+1$ segments.

b) Take $x \in \mathrm{bd}(S_d)$ and let $i \in \{1, ..., d+1\}$ be such that $x \in F_j \neq F_i$. Then each point $y \in F_i$ is joined to $x$ (on $D_d$) by at most two segments, distinct if and only if $y \notin F_j \cap F_i$. Thus, $C(x)$ includes the union of all $(d-1)$-dimensional faces of $S_d$ not containing $x$, and all points of $D_d$ outside this union are joined to $x$ by only one segment. □

The next result completes the first one. A point $z \in D_d$ is called *flat* if it is not a vertex.

**Theorem 48** *If $x$ is a point in* $\mathrm{int}(S_d) \subset D_d$ *then* $M_x \subset \{y_x, a_1, ..., a_{d+1}\}$ *and* $\rho_x|_{C(x)}$ *has at most one local minimum on each* $(d-1)$*-dimensional face of* $C(x)$.

*Proof:* A flat point $z \in D_d$ is a local maximum for $\rho_x$ if and only if any open half-sphere of unit tangent directions at $z$ contains the tangent direction of a segment from $x$. Necessarily, there are at least $d+1$ segments from $x$ to a flat local maximum of $\rho_x$; for $z \in C(x) \setminus \{y_x\}$, there are precisely $d+1-r_z \leq n$ segments to $x$, by Theorem 47, hence

$$M_x \subset \{y_x, a_1, ..., a_{n+1}\}.$$

Let $E$ be an $(n-k)$-dimensional face of $C(x)$ and $z \in E$ a flat point; then $z$ is joined to $x$ by precisely $k+1$ segments, by Theorem 47.

A flat point $z \in C(x)$ is a local minimum for $\rho_x|_{C(x)}$ if and only if there are precisely two segments from $z$ to $x$, with opposite tangent direction at $z$ (when considering a hyperplane unfolding of a neighbourhood of $z$). Indeed, if not, there were two segments from $x$ to $z$, say $\Gamma_i$ and $\Gamma_j$, whose angle $\alpha_z$ at $z$ is less than $\pi$. We may assume that $\Gamma_i \cap F_i \neq \emptyset$ and $\Gamma_j \cap F_j \neq \emptyset$. If we denote by $\Pi$ the 2-plane spanned by $z$ and the tangent directions of $\Gamma_i$ and $\Gamma_j$ at $z$, then $\Pi \cap S_{ij}(y_x)$ contains a line-segment, the points of which are joined to $x$ by only two segments. Now, since $\alpha_z < \pi$, we can find arbitrarily close to $z$ points of $\Pi \cap S_{ij}(y_x)$ at smaller distance to $x$.

Take an $(d-1)$-dimensional face of $C(x)$, say $S_{12}(y_x)$, and denote by $x_1$ and $x_2$ the points symmetric to $x$ with respect to the faces $F_1$ and respectively $F_2$. It follows that $x_1$ and $x_2$ are symmetric to each other with respect to $S_{12}(y_x)$. The local minimum of $\rho_x|_{S_{12}(y_x)}$ (if exists) is precisely the point given by $x_1x_2 \cap S_{12}(y_x)$. □

**Remark** One can see on suitable doubly covered triangles that there exist points $x$ such that $\rho_x|_{C(x)}$ has no local minimum on some 1-dimensional face of $C(x)$ and $y_x \notin M_x$.

For the particular case $d=2$ (doubly covered triangle), we can say more, completing Lemma 29. For a triangle $S_2$ with the vertices $a_1, a_2, a_3$, and $D_2 = S_2 \bowtie S_2'$, denote by $o$ the centre of the circle circumscribed to $S_2$ and by $h$ the orthocenter on the face $S_2'$ opposite to $o$.

**Theorem 49** *If the triangle* $a_1a_2a_3$ *is not acute or* $x \in \mathrm{int}(a_1a_2a_3)\setminus\{o\}$ *then* $F_x \subset \{a_1, a_2, a_3\}$. *If the triangle is acute and* $x = o$ *then* $F_x = \{h, a_1, a_2, a_3\}$.

*Proof:* The second assertion is a part of Lemma 29.

For the first one, we shall see that

$$\max_{i=1,2,3}||x - a_i|| \geq \rho(x, y_x) \geq \min_{i=1,2,3}||x - a_i||,$$

with equality if and only if the triangle is acute and $x = o$.

As in the proof of Theorem 47, consider $y_x \in S_2$ such that the lines $xa_i$ and $y_xa_i$ make equal angles with the edges starting at $a_i$ ($i = 1, 2, 3$). Let $y_1$, $y_2$ and $y_3$ be the symmetric points of $y_x$ with respect to the corresponding edges of the triangle.

Assume that

$$\max_{i=1,2,3}||x - a_i|| < \rho(x, y_x).$$

Then, by the triangle inequality, we have $\angle xy_2a_1 < \angle xa_1y_2$, $\angle xy_2a_3 < \angle xa_3y_2$, and the corresponding cyclic inequalities. Summing up, we obtain

$$\angle a_1y_2a_3 + \angle a_3y_1a_2 + \angle a_2y_3a_1 < \angle y_3a_1y_2 + \angle y_2a_3y_1 + \angle y_1a_2y_3.$$

Because of symmetry, the sum in the left side is equal to the total angle at $y_x$, i.e. to $2\pi$, while the sum in the right side equals

$$2(\angle a_1a_2a_3 + \angle a_2a_3a_1 + \angle a_3a_1a_2) = 2\pi.$$

This yields $2\pi < 2\pi$, impossible.

The proof for the second inequality is similar.

If the above inequalities are equalities then, again by Lemma 29, we have $x = o$, $y_x = h$, and the triangle is acute. □

The following result will be used to prove Theorem 51. It is a nice consequence of Theorem 47 and, in some sense, it generalizes the existence of the sphere inscribed to a $d$-dimensional simplex.

**Theorem 50** *For any point* $x$ *in* $\mathrm{int}(S_d)$, *there exists a unique hyperellipsoid of revolution with a focus at* $x$ *and tangent to all* $(d-1)$*-dimensional faces of* $S_d$. *Its second focus is at* $y_x$.

*Proof:* Take a point $x \in \text{int}(S_d)$. With the notations used to prove Theorem 47, there are $d+1$ segments on $D_d$ from $x$ to $y_x$, say $\Gamma_i$, each of which intersects the corresponding $(d-1)$-dimensional face $F_i$ at $y'_i = xy_i \cap F_i$ $(i = 1, ..., d+1)$.

Let $Ell(x, y_x)$ be the hyperellipsoid of revolution with the foci at $x$ and $y_x$, and the sum of the focal radii equal to $\rho(x, y_x)$. Then $Ell(x, y_x)$ contains all points $y'_i$, and it is tangent to each $F_i$, because the normal $n_i$ to $F_i$ at $y'_i$ makes equal angles with the segments $[xy'_k]$ and $[z'_k z]$, hence $n_i$ is also normal to the hyperellipsoid. □

**Theorem 51** *If $d > 3$, for any $D_d$ and any point $x \in D_d$, $F_x \subset \{a_1, ..., a_{d+1}\}$.*

*Proof:* If $x \in \text{bd}(S_d)$ then this is a part of Theorem 48. Suppose $x \in \text{int}(S_d)$, hence $F_x \subset \{y_x, a_1, ..., a_{d+1}\}$, again by Theorem 48. We also have $y_x \in \text{int}(S_d)$, from the proof of Theorem 47. Suppose there exists some simplex $S_d$ and some point $x \in \text{int}(S_d)$ such that $y_x \in F_x$; set $R' = \rho(x, y_x)$.

Denote by $Ell(x, y_x)$ the hyperellipsoid given by Theorem 50, and let

$$\Pi_i = \text{aff}(a_i, x, y_x).$$

Then $T_i = \Pi_i \cap S_d$ is a triangle, and

$$Ell_i = \Pi_i \cap Ell(x, y_x)$$

is an ellipse interior or tangent to $T_i$, with its foci at $x, y_x$ and with the length of an axis equal to $R'$. By Theorem 49, the maximal distance from $x$ to the vertices of $T_i$ is larger than or equal to $R'$, with equality if and only if $Ell_i$ is tangent to $T_i$ and $x, y_x$ are some particular points of $T_i$.

Let $F_j, F_k$ be two faces of $S_d$ each of which contains one of the tangent points of $Ell_i \cap T_i$ $(j, k \neq i)$. Let $a_l \in F_j \cap F_k \setminus \{a_i\}$. Then previous arguments applied to $\Pi_l$ and $T_l$ show that $\Pi_l$ also contains the tangent points $(F_j \cup F_k) \cap Ell(x, y_x)$, hence these two points belong to $xy_x$. But this yields $F_j \parallel F_k$, a contradiction. □

## 2.7 Some remarks and open questions

We conclude this thesis by presenting some conclusions, and several problems which remain open and meanwhile provide future directions for study.

### 2.7.1 Some remarks

**Remark** Since there exists (see Theorems 29, 35 or 36) surfaces $S$ with $F_S = S$, and it is obviously impossible to have a domain of endpoints, it follows that there exist farthest points which are not endpoints.

Conversely, since $S \setminus F_S$ is non-empty and open on each $S \in \mathcal{S}_2$, and $\mathcal{S}_2$ is open in $\mathcal{S}$, it follows (by Lemma 14) that there exist endpoints which are not farthest points.

**Remark** Notice that, since $S$ is compact, if $F$ is continuous and bijective then it is a homeomorphism. Summing up, by Theorems 16, 25 and 37, we obtain the following logical implications:
$(S \in \mathcal{H}) \to (F$ is a homeomorphism$) \leftrightarrow (F$ is a bijection$) \to (F$ is continuous$)$ $\leftrightarrow (F$ is single-valued$) \to (F$ is a surjection$) \leftrightarrow (S \notin \mathcal{S}_2)$; and:
$(S \notin \mathrm{cl}(\mathcal{S}_2)) \to (F$ is a homeomorphism$)$.

**Remark** There are (sometimes surprising) connections between convex geometry and elementary geometry.

Indeed, known elementary results may help prove facts about convex surfaces: see the case of an $Y$-tree in the proof of Theorem 39, or Lemma 29 or Theorem 49.

On the other hand, convex metrical considerations may lead to elementary results: see Theorem 50, but also the following nice fact in planar geometry [39] proved by folding -with Alexandrov's gluing theorem- a planar figure to obtain a tetrahedron. All indices below are to be taken modulo 3.

**Theorem 52** *Let $a_1a_2a_3$ be a triangle, let $m_i$ be the line orthogonal to $a_{i+1}a_{i+2}$ through the mid-point of the segment $[a_{i+1}a_{i+2}]$, and take arbitrary points $x_i \in m_i$ $(i = 1, 2, 3)$. If $l_i$ is the line orthogonal to $x_{i+1}x_{i+2}$ through $a_i$, then $l_1$, $l_2$ and $l_3$ are all either parallel or concurrent.*

### 2.7.2 Open questions

**Question** Is it true or not that $F_y^{-1}$ is either a point, a tree or the closure of an open subset of $S$? (See Section 1.3.)

**Question** Consider a surface $S \in \mathcal{S}$ such that $F_S \neq S$. Is the set $E = \{y \in \mathrm{bd}(F_S); \exists x \in F_y^{-1} \cap A_2(S)\}$ non-empty? (See Sections 1.5 and 1.6.)

Let $A_n(S)$ be the set of points in the surface $S$ with at least $n$ farthest points.

**Question** For the case of a convex polyhedral surface $P$, J. Rouyer [31] proved that $A_3(P)$ is a finite set. Is there a number $N$ such that for any doubly covered convex polygon $Q$, $|A_3(Q)| < N$? (See Section 2.4.)

**Questions** Characterize the sets $A_2(S)$, $A_3(S)$,..., for $S \in \mathcal{S}_2$ typical. (See Section 1.4.)

**Question** Any convex surface of revolution is in $\mathrm{cl}(\mathcal{S}_2)$, by the corollary of Theorem 20; characterize those which belong to $\mathrm{bd}(\mathcal{S}_2)$. (See Theorems 22, 29, 34 and 35, 37.)

**Question** Is the set $\mathcal{S}^{\neq}$ dense in $\mathcal{S}$? (See Subsection 2.2.3.)

**Question** Do polyhedral convex surfaces with bijective $F$ exist? (See Section 2.2.)

**Question** Do surfaces on which $F$ is continuous, but not bijective exist? (See Subsection 2.7.1.)

**Question** Characterize the set $\mathcal{H}$. (See Section 2.2.)

**Question** Sharpen $(SC)$ to obtain a characterization of the sphere. (See Section 2.2.)

**Question** Characterize the segments-tree which can be inscribed in convex polygons. (See the Remark following Theorem 38, and also Theorems 39 and 40.)

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