´´Introducing quantum mechanics with the help of the Stern-Gerlach experiment at secondary high school level´´

(AN ADVANCED COURSE AT THE HIGH SCHOOL)

Dissertation

zur Erlangung des akademischen Grades
eines Doktors der Pädagogik
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Preface

This work is a suggestion of teaching of quantum physics at high school. It consists of three parts. **The first part** is the main part which is an introduction to quantum mechanics. Beginning from the Stern-Gerlach experiment and some other << gedanken>> experiments we introduce quantum mechanical concepts which explain the experiments. With the help of the simplest Hilbert space in two dimensions, which describes a spin ½ particle ,we introduce the most important concepts of quantum physics in a simple and understandable way. In addition the mathematical description is algebraic so that the mathematical difficulties associated with differential equations and integrals are overcame , since they are replaced by the algebra of vector states . Furthermore an extension to some interesting and advanced subjects are considered. **The second part** contains a series of questionnaires which were given to pupils together with the corresponding results, i.e. conclusions and remarks. **The third part** consists of three appendices, the first one contains the mathematics which must be known to the teacher in order to teach the lesson, the second one is referred to the density matrix and the third one to entanglements.

Also I would like to thank my Professors A. Pflug and E. Paschos for their unlimited help during the preparation and development of this dissertation.

N. G. Diamantis

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AN INTRODUCTION TO QUANTUM MECHANICS

An introduction to quantum mechanics

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PART A

states in quantum mechanics
operators-time evolution

1. States In Quantum Mechanics

1.1 States in Classical Mechanics

The description of a particle in the frame of classical mechanics is supposed to be complete if we know exactly the position and the velocity of the particle at any time. That means that we know exactly the functions x(t) and v(t). Therefore the state of a particle in classical mechanics is determined by the position x(t) and the velocity v(t).

Also if we know the initial values of position and velocity at t=0 that means, that we know x(0) and v(0), then applying Newton law we calculate the values x(t) and v(t) at any time t.

$$(x_{0,}\ v_0) \xrightarrow{\text{Newton law}} (x(t),\ v(t))$$

initial State → the state at any time

For a system that consists of many particles finding the position and the velocity for any member of the system (x_1, v_1), (x_2, v_2),… we determine the state of the system.

If we examine a thermodynamical system, for example a can which contains a gas then we know the state of it if we know the value of the physical quantities pressure (P), volume (V) and temperature (T). That is (P, V, T) is an equilibrium state of a thermodynamical system.

1.2 States in Quantum Mechanics

If we try to find the state of a microscopic particle i.e. electron , in the frame of classical physics, that is we try to define the position x and the velocity v (or the momentum $p = mv$) we will see that it is impossible. The simultaneous measurement of position x and momentum p is impossible. We conclude that from the Heisenberg uncertainly relation

$$\Delta x \cdot \Delta p \geq \hbar$$

That is the multiplication of the uncertainty of position Δx and momentum Δp is always greater or equal to $\hbar = \dfrac{h}{2\pi}$, where h is the Plank constant. Consequently we put the question: What is the state in quantum mechanics?

The state in quantum mechanics is absolutely connected with the measurement!! All the physical quantities, which we can measure, will be referred as observables for example position, momentum, energy, angular momentum,…. It is very reasonable to define or label a state of a particle or system by the value of an observable.

Let us suppose that we measure the Energy E and we find the value E_1. Then we can say that the particle (or system) is in state E_1 which we remark as $|E_1\rangle$ (ket). If we found the value E_2 then the particle would be in state $|E_2\rangle$ and so on.

There are observables that can be simultaneously measurable and other that can not, it depends on their dependence on momentum and position. Let two observables A and B be simultaneously measurable. Then if measuring the A and B we find the values a_i and β_j respectively, the state of the system is $\left|ai, \beta j\right\rangle$. **Repeating the measurement of the A or B we find again the values a_i or β_j respectively.** It is significant to have more and more observables simultaneously measurable to define a state.
Let us suppose that two observables A and Γ are not simultaneously measurable and let us measure the A and find the value a_1. Obviously the state is $\left|a_1\right\rangle$. Then measuring the Γ the result is γ_1. Can we say that the stat is $\left|a_1,\gamma_1\right\rangle$? The answer is no, because if we measure again the A we will no absolutely find the value a_1. Simply the state was $\left|\gamma_1\right\rangle$. As we will see later in this case the state $\left|a_i\right\rangle$ can be written as a linear combination of $\left|\gamma_j\right\rangle$ and vice-versa.

We will examine the most simple system that of a particle with spin ½ and we will try to explain more on it.

1.3 Spin

As we know the earth revolves around the sun and rotates about its axis. For the first motion the angular momentum is $L = m \cdot v \cdot r$ and for the second one the spin angular momentum is $S = I \cdot \omega$.

Where I: moment of inertia of the earth around an axis

and ω: angular velocity

It is experimentally confirmed that many microscopic particles have an intrinsic angular momentum called spin S. A pedagogic way to explain the spin is to suppose that the particles rotate about their axes, as the earth rotates about its axis. But attempts to explain the spin of microscopic particles in this way have as results peculiar conclusions such as velocities greater that the velocity of light. Therefore the spin is a clear quantum number without classical analogy but on the other side it is a kind of angular momentum.

1.4 Stern-Gerlach Apparatus

The device which is described below and we will mention it as SG apparatus (Stern-Gerlach apparatus) can be used to measure the components of the spin of a particle.

Fig 1(a) shows a schematic diagram of a SG apparatus. From a hot oven particles come out and pass through a series of narrow slits. Then the beam is directed between the poles of a magnet. One of the pole piece is flat and the other one has a sharp tip. An inhomogeneous magnetic field is produced as in fig 1 (b).

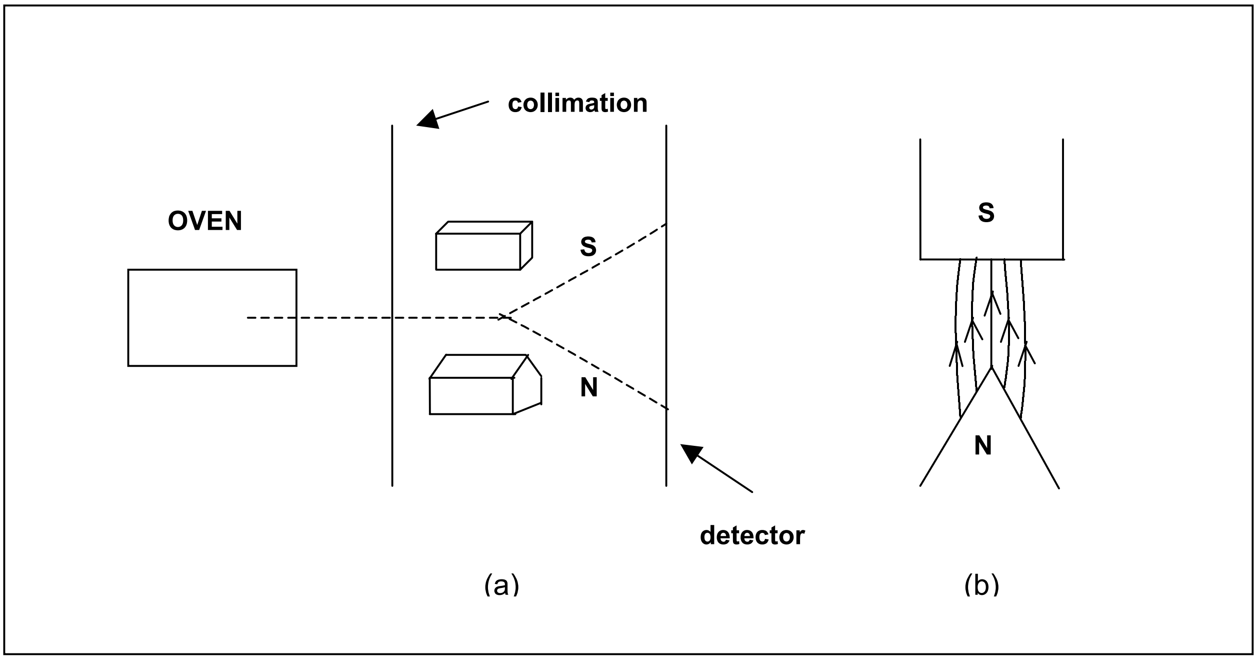


fig 1

If a particle which has a magnetic moment enters the magnetic field it is deflected and its deflection depends on the component of the spin which is parallel to the direction of magnetic field $\boldsymbol{B}$. The deflection is proportional to magnetic moment which is proportional to the spin. Consequently the deflection is proportional to the spin. Therefore the measurement of deflection gives directly the value of the component of the spin.

1.5 Measurement of spin components

a) Measurement of Sz - Well defined State

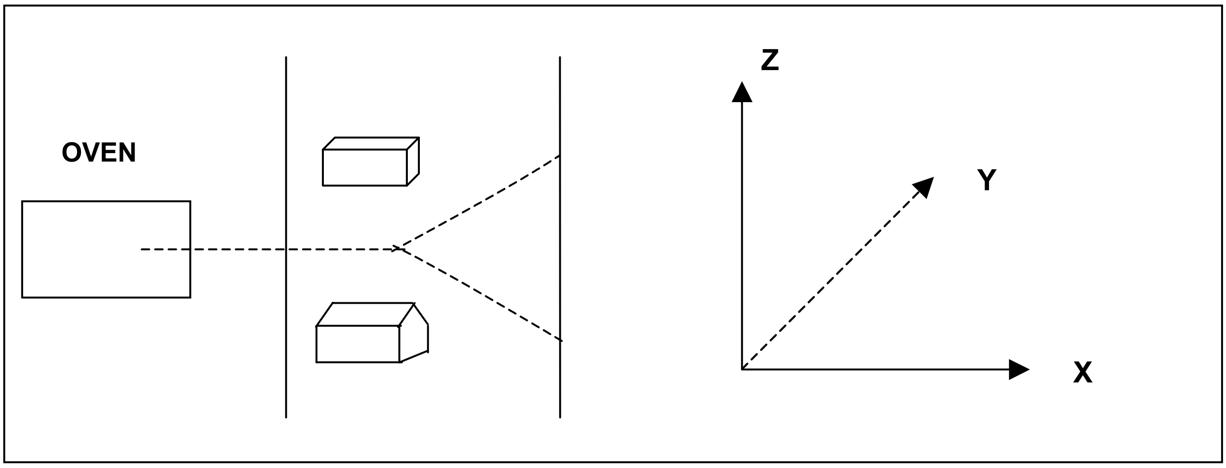


fig 2

We use the SG apparatus where $\boldsymbol{B}$ is parallel to the z-axis. It will be mentioned as SG_z apparatus (SG_x, SG_y respectively). Passing the particles with spin ½ through a SG_z apparatus we expect, in the frame of classical physics, to find for Sz any value from $+S$ to $-S$. It is happened because S can make any angle ϑ with z-axis

$0 \le \vartheta \le \pi$ (fig 3), consequently

$$Sz = S\cos\vartheta \Rightarrow -S \le Sz \le S$$

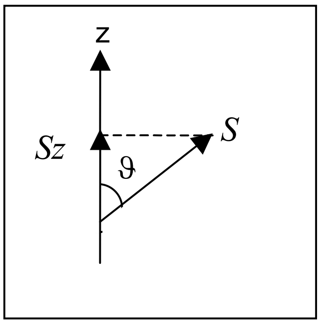


fig 3

However the experiment shows that the beam splits only in two beams. We find only two values for Sz,

$$Sz = +\frac{1}{2}\hbar \qquad \text{(spin up)}$$

and

$$Sz = -\frac{1}{2}\hbar \qquad \text{(spin down)}$$

If we use other particles perhaps we will find three and more values for Sz. But we consider only particles which their components take only two values $\pm\frac{\hbar}{2}$.

These particles are called spin ½ particles (i. e. electrons, protons). We work only with this type of particles for the rest of the text. Figure 4 shows schematically the experiment.

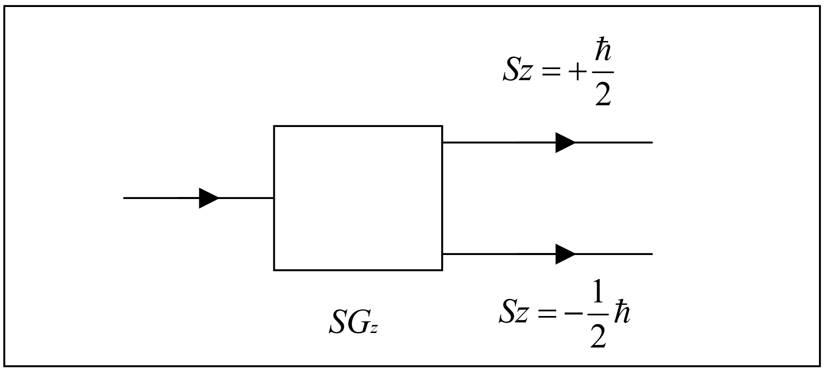


fig 4

If we block the beam below and pass the higher beam through a SG_z again we will take only one beam that with $Sz = +\frac{\hbar}{2}$ (fig 5)

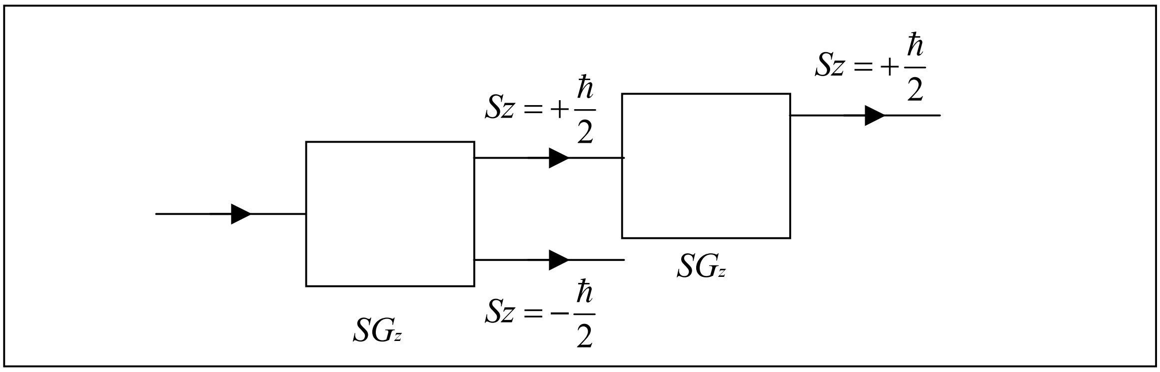


fig 5

Consequently the beam with $Sz=+\frac{\hbar}{2}$ is a **well defined state** and we write it as $|z\uparrow\rangle$. It is called a pure state

$$\left|Sz=+\frac{\hbar}{2}\right\rangle=|z\uparrow\rangle.$$

Similarly if we block the higher beam and pass the lower one through a SG_z we take only one beam with $Sz=-\frac{\hbar}{2}$ (fig 6). This is also a well define state, it is the state $|z\downarrow\rangle$. It is also a pure state

$$\left|Sz=-\frac{\hbar}{2}\right\rangle=|z\downarrow\rangle$$

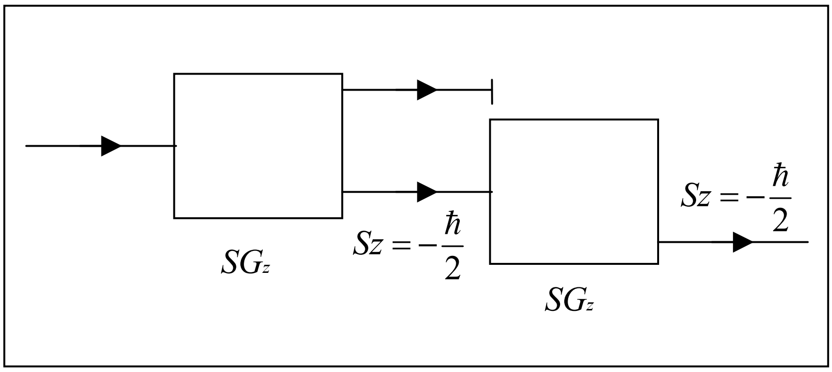


fig 6

β) Measurement of S_x, S_y

If we pass the beam from oven trough a SG_x we also take two discrete beams with

$S_x=+\frac{\hbar}{2}$ and $S_x=-\frac{\hbar}{2}$ (fig7)

which correspond to states $|x\uparrow\rangle$ and $|x\downarrow\rangle$

Therefore we write

$$\left|S_x=+\frac{\hbar}{2}\right\rangle=|x\uparrow\rangle$$

$$\left|S_x=-\frac{\hbar}{2}\right\rangle=|x\downarrow\rangle$$

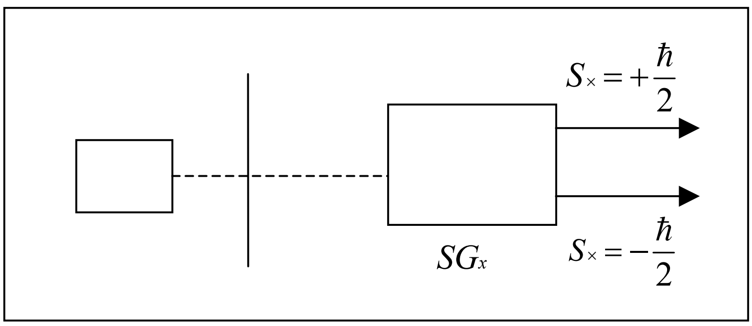


fig 7

Repeating the experiment using SG_y apparatus we define the state $|\mathbf{y}\downarrow\rangle$ and $|\mathbf{y}\uparrow\rangle$.
Where

$$|S_y = +\frac{\hbar}{2}\rangle = |\mathbf{y}\uparrow\rangle$$

and

$$|S_y = -\frac{\hbar}{2}\rangle = |\mathbf{y}\downarrow\rangle.$$

γ) Measurement of S_z and $S_\times$ simultaneously

Let us pass the beam from oven trough a SG_z apparatus and let the produced state $|\mathbf{z}\uparrow\rangle$ to pass trough a $SG_\times$ apparatus. We block the state below and examine the state above (fig 8).

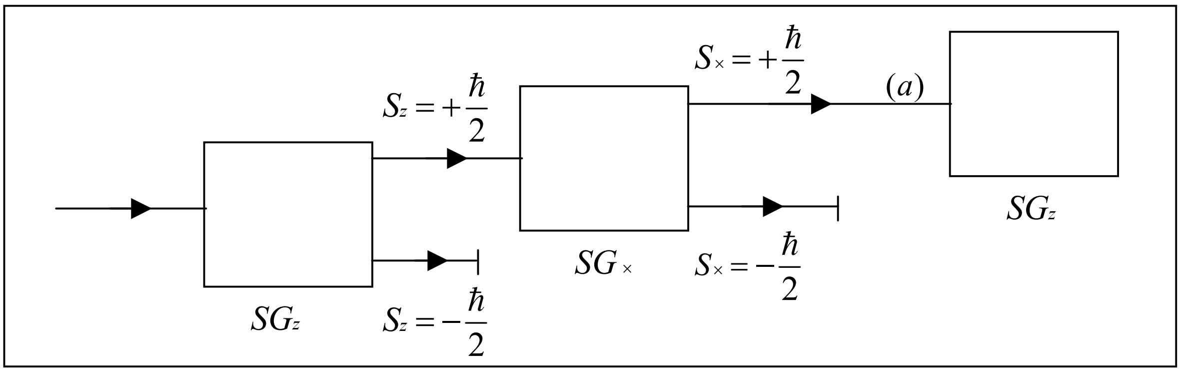


fig 8

Finally our beam is the beam (a). Can we say that the beam (a) is a state

$$|\alpha\rangle = |Sz = +\frac{\hbar}{2}, S\times = +\frac{\hbar}{2}\rangle ?$$

According to classical mechanics it is correct but now the answer is no.
If we pass the (a) trough a SG_z it splits to two beams $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$, consequently the $|\alpha\rangle$ it was not only in $|\mathbf{z}\uparrow\rangle$ (Fig. 9).

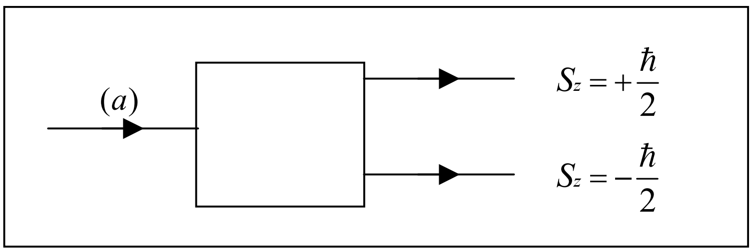


fig 9

Consequently the components S_z and S_x are not simultaneously measurable (repeating the measurement of the S_z we did find only the value $+\frac{\hbar}{2}$). Similarly we can show that the pairs (S_z , S_y), (S_x , S_y) are not also simultaneously measurable.
We can put the question: let the state be the $|\mathbf{x}\uparrow\rangle$. The experiments show that it contains $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$. Can we write the $|\mathbf{x}\uparrow\rangle$ as a function of $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$?

Let us examine the problem quantitively. If we have N (large number) particles in state $|\mathbf{x}\uparrow\rangle$ and pass them through a SG_z then we take $\frac{N}{2}$ particles in state $|\mathbf{z}\uparrow\rangle$ and $\frac{N}{2}$ in state $|\mathbf{z}\downarrow\rangle$ (fig 10).

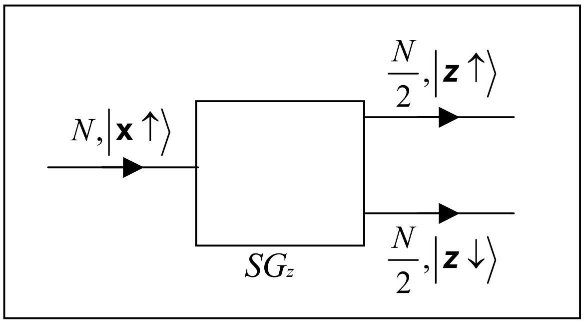


fig 10

Simply thinking someone could write the relation

$$|\mathbf{x}\uparrow\rangle = \frac{1}{2}|\mathbf{z}\uparrow\rangle + \frac{1}{2}|\mathbf{z}\downarrow\rangle$$

explaining the coefficients as the probability to find the state $|\mathbf{x}\uparrow\rangle$ in state $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$.

In the same way we could write

$$|\mathbf{x}\downarrow\rangle = \frac{1}{2}|\mathbf{z}\uparrow\rangle + \frac{1}{2}|\mathbf{z}\downarrow\rangle$$

$$|\mathbf{z}\uparrow\rangle = \frac{1}{2}|\mathbf{x}\uparrow\rangle + \frac{1}{2}|\mathbf{x}\downarrow\rangle$$

$$|\mathbf{z}\downarrow\rangle = \frac{1}{2}|\mathbf{x}\uparrow\rangle + \frac{1}{2}|\mathbf{x}\downarrow\rangle$$

(As we will see later these relations are false).

1.6 Modified SG Apparatus

The device shown in fig 11 is called modified SG apparatus (MSG). It consists of a sequence of three SG apparatus. The first and the last are the usual SG but the second one has the magnetic field in opposite direction and is twice as long.

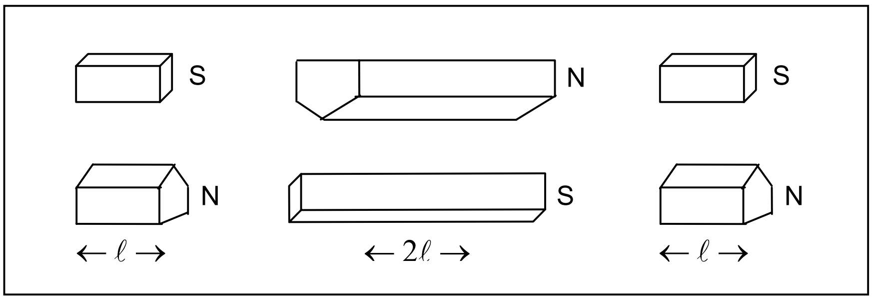


fig 11

A particle passing through a $MSG_{\times}$ follows one of the paths of fig 12, the higher path if $S_{\times} = +\frac{\hbar}{2}$ and the lower path if $S_{\times} = -\frac{\hbar}{2}$, and it comes out moving along the initial direction.

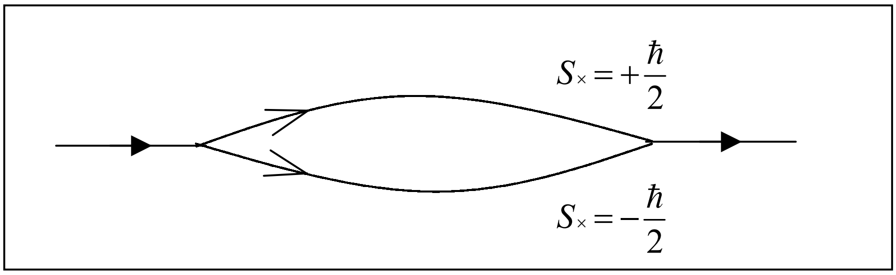


fig 12

1.7 A very crazy result

We assume that N particle in state $|\mathbf{z}\uparrow\rangle$ pass through a MSG_x apparatus and we have blocked the lower path ($S_x = -\frac{\hbar}{2}$). Finally the beam passes trough a SG_z (fig 13).

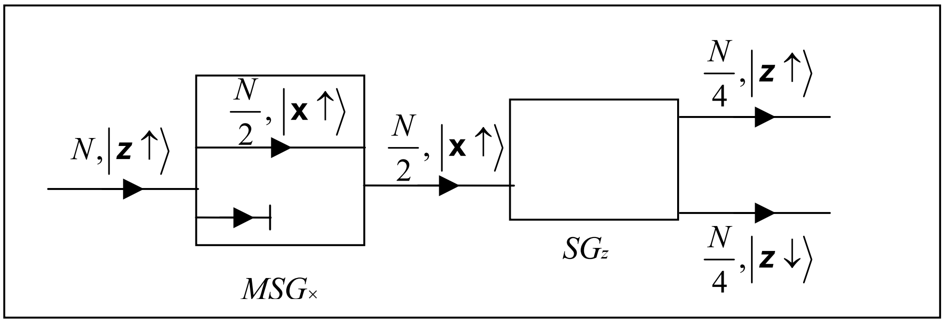


fig 13

Obviously the beam after passing the MSG_x apparatus is in state $|\mathbf{x}\uparrow\rangle$ and passing the SG_z splits as follows:

$\frac{N}{4}$ particle in state $|\mathbf{z}\uparrow\rangle$

and

$\frac{N}{4}$ particle in state $|\mathbf{z}\downarrow\rangle$.

We repeat the experiment blocking the higher path. The result is drawn below (fig 14)

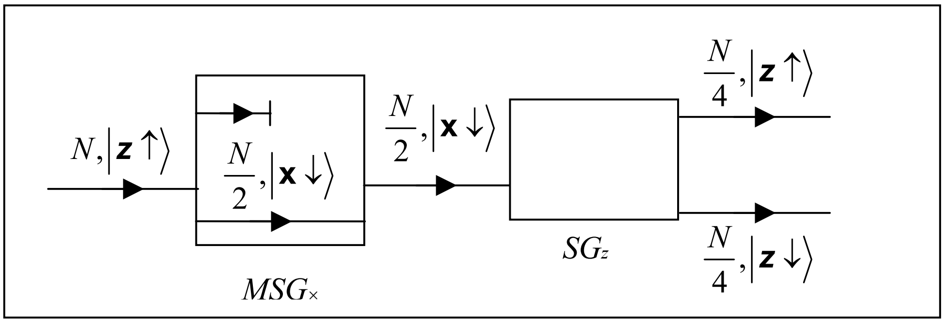


fig 14

That is we find $\frac{N}{4}$ particles in state $|\mathbf{z}\uparrow\rangle$

and

$\frac{N}{4}$ particles in state $|\mathbf{z}\downarrow\rangle$

Suppose now that as we have been doing the last experiment at once we open the higher channel. What do we expect to come out from SG_z apparatus? (fig 15).

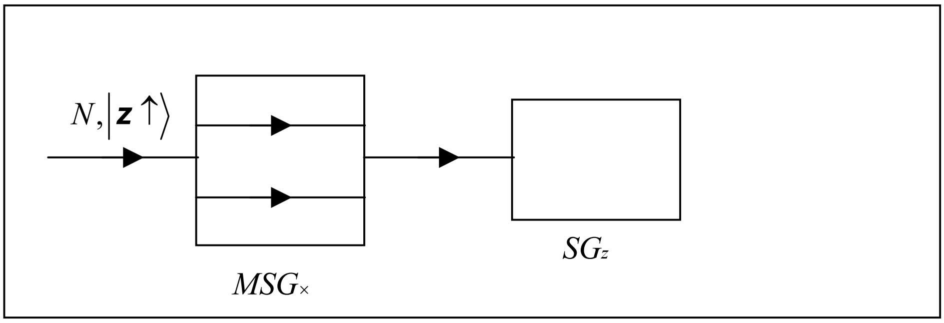


fig 15

Obviously and the remaining $\frac{N}{2}$ particles in state $|\mathbf{x}\uparrow\rangle$ will give $\frac{N}{4}$ particles in state $|\mathbf{z}\uparrow\rangle$ and $\frac{N}{4}$ particles in state $|\mathbf{z}\downarrow\rangle$. Consequently we expect to find $\frac{N}{2}$ particle in state $|\mathbf{z}\uparrow\rangle$ and $\frac{N}{2}$ particles in state $|\mathbf{z}\downarrow\rangle$ as in fig 16.

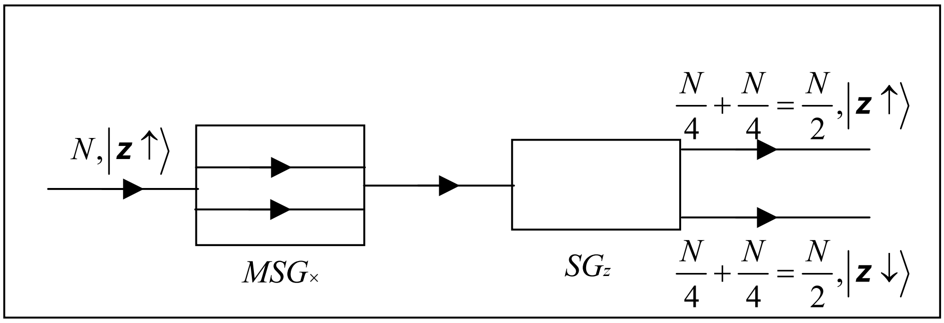


fig 16

No!! the result is different. We find all of N particles in state $|\mathbf{z}\uparrow\rangle$!!! That is opening and the other channel, the number of particles in state $|\mathbf{z}\downarrow\rangle$ becomes zero and in state $|\mathbf{z}\uparrow\rangle$ becomes N (fig 17).

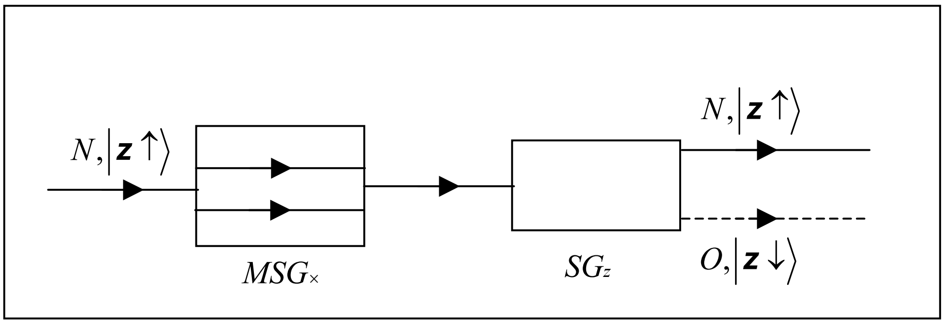


fig 17

Consequently the old view for $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ can not be applied because we are led to following inconsistency.

We have

$$\left|\boldsymbol{z}\uparrow\right\rangle=\frac{1}{2}\left|\mathbf{x}\uparrow\right\rangle+\frac{1}{2}\left|\mathbf{x}\downarrow\right\rangle=$$

$$=\frac{1}{2}\left(\frac{1}{2}\left|\boldsymbol{z}\uparrow\right\rangle+\frac{1}{2}\left|\boldsymbol{z}\downarrow\right\rangle\right)+\frac{1}{2}\left(\frac{1}{2}\left|\boldsymbol{z}\uparrow\right\rangle+\frac{1}{2}\left|\boldsymbol{z}\downarrow\right\rangle\right)=$$

$$=\frac{1}{2}\left|\boldsymbol{z}\uparrow\right\rangle+\frac{1}{2}\left|\boldsymbol{z}\downarrow\right\rangle \quad \text{(contradiction)}$$

However the experiment shows that the $\left|\boldsymbol{z}\uparrow\right\rangle$ is only $\left|\boldsymbol{z}\uparrow\right\rangle$. Therefore we cannot interpret the coefficients as the probability to find the corresponding state.

From all experiments until now we can make two crucial remarks.

i) **the values of S_z, S_x, S_y are discrete**
ii) **the contribution of the two beams $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ in the last experiment gave two results one was the zero and the other the N (maximum).**

But the above effects are wave effects. We observe discrete values in the case of standing waves and in the case of interference of two waves we find out maximum and minimum contribution.

1.8 Vector State-Hilbert Space

As we know if we want to examine an one dimensional problem of classical mechanics it is enough to work with scalar quantities. However for problems in two or three dimensions some quantities such as velocity, force, acceleration,…. must be represented by vectors. So we need different mathematics to describe the problem. Also computers work using matrix algebra.

After huge effort physicists found the convenient mathematical structure to describe the behaviour of microscopic particles. This mathematical structure is the quantum mechanics. We will try to make an introduction to this structure:

We shall restrict the discussion to spin $\frac{1}{2}$ particles. An observable quantity is the component S_z. The experiments show that S_z has two values ($\pm\frac{\hbar}{2}$). We assigned the state $\left|\boldsymbol{z}\uparrow\right\rangle$ to value $+\frac{\hbar}{2}$ and the state $\left|\boldsymbol{z}\downarrow\right\rangle$ to value $-\frac{\hbar}{2}$. The states $\left|\boldsymbol{z}\uparrow\right\rangle$ and $\left|\boldsymbol{z}\downarrow\right\rangle$ constitute a basis of the Hilbert space for our problem.

Any state of the particle is a vector in Hilbert space and can be written as a linear combination (superposition) of two vectors $\left|\boldsymbol{z}\uparrow\right\rangle$ and $\left|\boldsymbol{z}\downarrow\right\rangle$.

That is

$$\left|\Psi\right\rangle = c_1\left|\boldsymbol{z}\uparrow\right\rangle + c_2\left|\boldsymbol{z}\downarrow\right\rangle$$

Where c_1 and c_2 are complex numbers! This is required as we will see in order to be able to account interference phenomena.

We can explain the superposition, of course not strictly supposing that a beam in state $\left|\Psi\right\rangle$ pass through a MSG_z (fig 18).

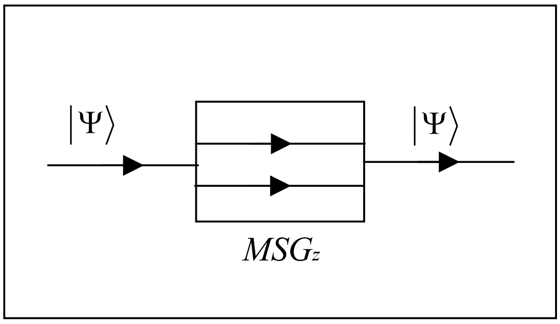


fig 18

and as we know the emerging beam is also in state $\left|\Psi\right\rangle$.

For any vector $\left|\Psi\right\rangle$ (ket) we define a $\left\langle\Psi\right|$ (bra) as follows:

$$\left\langle\Psi\right| = \overline{c}_1\left\langle\boldsymbol{z}\uparrow\right| + \overline{c}_2\left\langle\boldsymbol{z}\downarrow\right|$$

Where $\left\langle\boldsymbol{z}\uparrow\right|$ and $\left\langle\boldsymbol{z}\downarrow\right|$ are the bras corresponding to kets $\left|\boldsymbol{z}\uparrow\right\rangle$ and $\left|\boldsymbol{z}\downarrow\right\rangle$ and $\overline{c}_1$ and $\overline{c}_2$ the complex conjugates of c_1 and c_2 respectively.

We can also construct the Hilbert space of the problem by choosing as basis the <<eigenstates>> of some other observable such as S_x, S_y or S_n (n any direction). In this case we will be measuring the projection of spin in the new direction and we will find either $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.

1.9 Probability amplitude-Probability

A crucial question is the following: If a particle is in state $\left|\Psi\right\rangle$, what is the probability to find it in some state $\left|\Phi\right\rangle$? The answer is that:

We define as probability amplitude for a state $\left|\Psi\right\rangle$ to be in state $\left|\Phi\right\rangle$:

probability amplitude = $\langle \Phi | \Psi \rangle$ (inner product), (it is generally a complex number) and the corresponding probability is

$$P = \left| \langle \Phi | \Psi \rangle \right|^2$$

We have seen that when a beam of particles is in state $|\boldsymbol{z}\uparrow\rangle$ and passes through a SG_z apparatus all particles outcome in state $|\boldsymbol{z}\uparrow\rangle$. Similarly if the initial beam is in state $|\boldsymbol{z}\downarrow\rangle$ we take the particles only in state $|\boldsymbol{z}\downarrow\rangle$. Therefore we can make the following table

probability to be in state	particle in state $\lvert\boldsymbol{z}\uparrow\rangle$	particle in state $\lvert\boldsymbol{z}\downarrow\rangle$
$\lvert\boldsymbol{z}\uparrow\rangle$	$\lvert\langle\boldsymbol{z}\uparrow\lvert\boldsymbol{z}\uparrow\rangle\rvert^2 = 1$	$\lvert\langle\boldsymbol{z}\uparrow\lvert\boldsymbol{z}\downarrow\rangle\rvert^2 = 0$
$\lvert\boldsymbol{z}\downarrow\rangle$	$\lvert\langle\boldsymbol{z}\downarrow\lvert\boldsymbol{z}\uparrow\rangle\rvert^2 = 0$	$\lvert\langle\boldsymbol{z}\downarrow\lvert\boldsymbol{z}\downarrow\rangle\rvert^2 = 1$

From above table we conclude

$$\langle \boldsymbol{z}\uparrow | \boldsymbol{z}\uparrow \rangle = 1$$
$$\langle \boldsymbol{z}\uparrow | \boldsymbol{z}\downarrow \rangle = 0$$
$$\langle \boldsymbol{z}\downarrow | \boldsymbol{z}\uparrow \rangle = 0$$
$$\langle \boldsymbol{z}\downarrow | \boldsymbol{z}\downarrow \rangle = 1$$

Obviously when a particle is in state $|\Psi\rangle$ the probability to be in that state is equal to one.

Consequently:

$$\left| \langle \Psi | \Psi \rangle \right|^2 = 1 \Rightarrow \langle \Psi | \Psi \rangle = 1$$

Let state $|\Psi\rangle$ be

$$|\Psi\rangle = c_1 |\boldsymbol{z}\uparrow\rangle + c_2 |\boldsymbol{z}\downarrow\rangle$$

Then from $\langle \Psi | \Psi \rangle = 1$ we find

$|c_1|^2 + |c_2|^2 = 1$

where $|c_1|^2 = \bar{c}_1 \cdot c_1$ and $|c_2|^2 = \bar{c}_2 \cdot c_2$

Examples

We will try to write the state $|\mathbf{x}\uparrow\rangle$, $|\mathbf{x}\downarrow\rangle$, $|\mathbf{y}\uparrow\rangle$, $|\mathbf{y}\downarrow\rangle$ as a linear combination of the states $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$.

Let $|\mathbf{x}\uparrow\rangle$ be

$$|\mathbf{x}\uparrow\rangle = c_1|\mathbf{z}\uparrow\rangle + c_2|\mathbf{z}\downarrow\rangle$$

We know that

$$|\langle \mathbf{z}\uparrow|\mathbf{x}\uparrow\rangle|^2 = \frac{1}{2}, \qquad |\langle \mathbf{z}\downarrow|\mathbf{x}\uparrow\rangle|^2 = \frac{1}{2}$$

and

$$\langle \mathbf{x}\uparrow|\mathbf{x}\uparrow\rangle = 1$$

From above relations we conclude (after some amount of work) that

$c_1 = \frac{1}{\sqrt{2}}$ and $c_2 = \frac{e^{ia}}{\sqrt{2}}$.

Similarly we find that

$$|\mathbf{y}\uparrow\rangle = \frac{1}{\sqrt{2}}|\mathbf{z}\uparrow\rangle + \frac{e^{i\beta}}{\sqrt{2}}|\mathbf{z}\downarrow\rangle$$

And with the help of the relation

$$|\langle \mathbf{x}\uparrow|\mathbf{y}\uparrow\rangle|^2 = \frac{1}{2}$$

finally we find

$$|\mathbf{x}\uparrow\rangle = \frac{1}{\sqrt{2}}|\mathbf{z}\uparrow\rangle + \frac{1}{\sqrt{2}}|\mathbf{z}\downarrow\rangle$$
$$|\mathbf{y}\uparrow\rangle = \frac{1}{\sqrt{2}}|\mathbf{z}\uparrow\rangle + \frac{i}{\sqrt{2}}|\mathbf{z}\downarrow\rangle$$

Working in the same way we also find

$$|\mathbf{x}\downarrow\rangle=\frac{1}{\sqrt{2}}|\mathbf{z}\uparrow\rangle-\frac{1}{\sqrt{2}}|\mathbf{z}\downarrow\rangle$$
$$|\mathbf{y}\downarrow\rangle=\frac{1}{\sqrt{2}}|\mathbf{z}\uparrow\rangle-\frac{i}{\sqrt{2}}|\mathbf{z}\downarrow\rangle$$

Let us explain the subject more explicitly using the experiments. We pass N particles in state $|\mathbf{z}\uparrow\rangle$ trough a MSG_x where we have blocked the $|\mathbf{x}\downarrow\rangle$. Then we pass the state $|\mathbf{x}\uparrow\rangle$ trough a SG_z (fig 19).

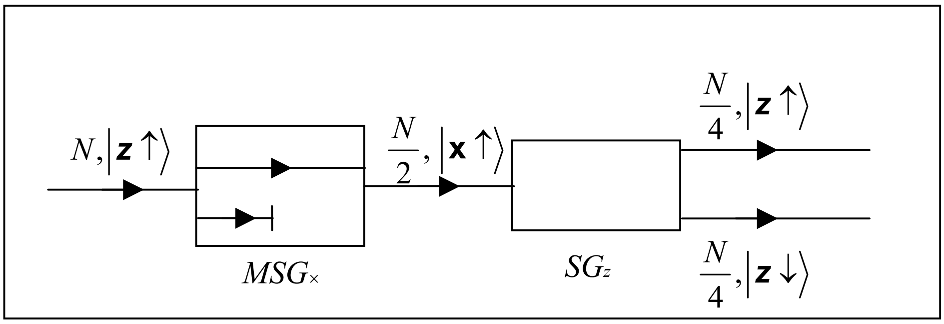


fig 19

The probability amplitude for the state $|\mathbf{z}\uparrow\rangle$ to be in state $|\mathbf{x}\uparrow\rangle$ is $\langle\mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle$ and the state

$|\mathbf{x}\uparrow\rangle$ to be in state $|\mathbf{z}\uparrow\rangle$ is $\langle\mathbf{z}\uparrow|\mathbf{x}\uparrow\rangle$. Then the probability amplitude for the state $|\mathbf{z}\uparrow\rangle$ to give $|\mathbf{z}\uparrow\rangle$ is

$$\langle\mathbf{z}\uparrow|\mathbf{x}\uparrow\rangle\cdot\langle\mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle=\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=\frac{1}{2}$$

and the probability is

$$P=\left|\langle\mathbf{z}\uparrow|\mathbf{x}\uparrow\rangle\cdot\langle\mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle\right|^2=\frac{1}{4}$$

Consequently $\frac{N}{4}$ particles are outcoming in state $|\mathbf{z}\uparrow\rangle$. Similarly the probability to emerge out in state $|\mathbf{z}\downarrow\rangle$ is

$$P=\left|\langle\mathbf{z}\downarrow|\mathbf{x}\uparrow\rangle\cdot\langle\mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle\right|^2=\frac{1}{4}$$

We repeat the experiment leaving open both of the paths (fig 20)

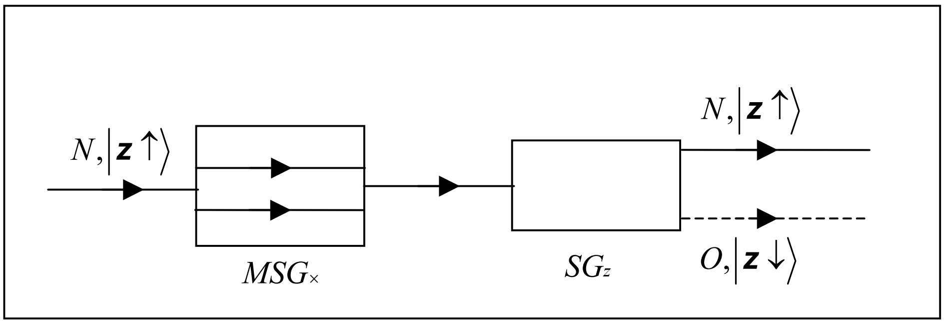


fig 20

The probability amplitude beginning as a state $|\mathbf{z}\uparrow\rangle$ to end in state $|\mathbf{z}\uparrow\rangle$ is

$$\langle \mathbf{z}\uparrow|\mathbf{x}\uparrow\rangle\langle \mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle + \langle \mathbf{z}\uparrow|\mathbf{x}\downarrow\rangle\langle \mathbf{x}\downarrow|\mathbf{z}\uparrow\rangle$$

and the probability is

$$P = \left|\langle \mathbf{z}\uparrow|\mathbf{x}\uparrow\rangle\langle \mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle + \langle \mathbf{z}\uparrow|\mathbf{x}\downarrow\rangle\langle \mathbf{x}\downarrow|\mathbf{z}\uparrow\rangle\right|^2 =$$
$$= \left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right|^2 = 1$$

However the probability amplitude for an initial state $|\mathbf{z}\uparrow\rangle$ to end up in state $|\mathbf{z}\downarrow\rangle$ is

$$\langle \mathbf{z}\downarrow|\mathbf{x}\uparrow\rangle\langle \mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle + \langle \mathbf{z}\downarrow|\mathbf{x}\downarrow\rangle\langle \mathbf{x}\downarrow|\mathbf{z}\uparrow\rangle$$

and the corresponding probability is

$$P = \left|\langle \mathbf{z}\downarrow|\mathbf{x}\uparrow\rangle\langle \mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle + \langle \mathbf{z}\downarrow|\mathbf{x}\downarrow\rangle\langle \mathbf{x}\downarrow|\mathbf{z}\uparrow\rangle\right|^2 =$$
$$= \left|\langle \mathbf{z}\downarrow|\mathbf{x}\uparrow\rangle\langle \mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle\right|^2 + \left|\langle \mathbf{z}\downarrow|\mathbf{x}\downarrow\rangle\langle \mathbf{x}\downarrow|\mathbf{z}\uparrow\rangle\right|^2$$
$$+2\cdot\langle \mathbf{z}\downarrow|\mathbf{x}\uparrow\rangle\langle \mathbf{x}\uparrow|\mathbf{z}\uparrow\rangle\cdot\langle \mathbf{z}\downarrow|\mathbf{x}\downarrow\rangle\langle \mathbf{x}\downarrow|\mathbf{z}\uparrow\rangle =$$
$$= \frac{1}{4} + \frac{1}{4} - 2\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

The last term is called interference term.

<u>Conclusion</u>: We realize that in order to explain the results of the experiments we treat the probability amplitudes as the classical probabilities (addition, multiplication) and the square of them gives the normal probability. As above we work with waves. If two waves Φ_1 and Φ_2 arrive at the same point then the resulting wave oscillates as the sum of waves $\Phi = \Phi_1 + \Phi_2$ with its intensity given by $P = |\Phi_1 + \Phi_2|^2$.

2. OPERATORS - TIME EVOLUTION

2.1 Operators - Expectation Value of an Observable

Let us assume that a beam of N particles in state

$$|\Psi\rangle = c_1|\boldsymbol{z}\uparrow\rangle + c_2|\boldsymbol{z}\downarrow\rangle$$

passes through a MSG_z apparatus with the lower path closed and the outcoming beam passes through a SG_z apparatus. The results are drawn in fig 21.

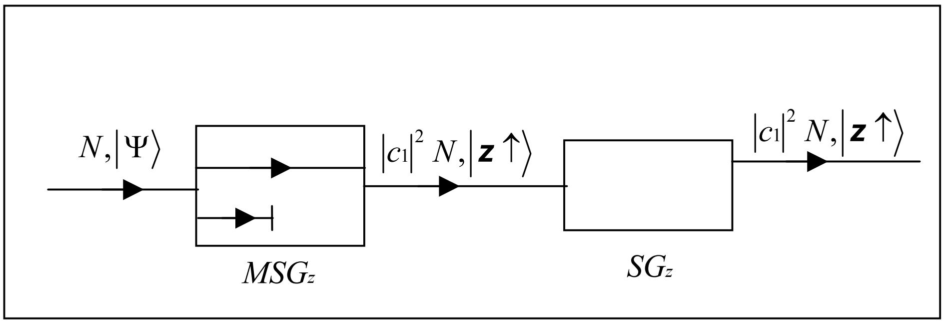


fig 21

We observe that the first apparatus projects the state $|\Psi\rangle$ onto the state $|\boldsymbol{z}\uparrow\rangle$

$$|\Psi\rangle \rightarrow |\boldsymbol{z}\uparrow\rangle$$

Mathematically it is described by an operator $\hat{P}_{z\uparrow}$ which is called projection operator and its action is:

$$\hat{P}_{z\uparrow}|\Psi\rangle = c_1|\boldsymbol{z}\uparrow\rangle = |\boldsymbol{z}\uparrow\rangle\,\langle\boldsymbol{z}\uparrow|\Psi\rangle$$

It is convenient to write $\hat{P}_{z\uparrow}$ as

$$\hat{P}_{z\uparrow} = |\boldsymbol{z}\uparrow\rangle\langle\boldsymbol{z}\uparrow|$$

Also blocking the higher path we could mathematically write the action:

$$|\Psi\rangle \rightarrow |\boldsymbol{z}\downarrow\rangle$$

as an operator $\hat{P}_{z\downarrow}$ where

$$\hat{P}_{z\downarrow}|\Psi\rangle = c_2|\boldsymbol{z}\downarrow\rangle = |\boldsymbol{z}\downarrow\rangle\,\langle\boldsymbol{z}\downarrow|\Psi\rangle$$

and it could also be written as

$$\hat{P}_{z\downarrow} = |\mathbf{z}\downarrow\rangle\langle\mathbf{z}\downarrow|$$

Let us suppose that we open both of paths. Obviously the $|\Psi\rangle$ emerges unchanged

$$|\Psi\rangle \to |\Psi\rangle$$

Mathematically this action is described by an operator called unit operator $\hat{I}$.

$$\hat{I}|\Psi\rangle = |\Psi\rangle$$

We can also write the $\hat{I}$ as follows

$$\hat{I} = \hat{P}_{z\uparrow} + \hat{P}_{z\downarrow} \qquad \text{or}$$

$$\hat{I} = |\mathbf{z}\uparrow\rangle\langle\mathbf{z}\uparrow| + |\mathbf{z}\downarrow\rangle\langle\mathbf{z}\downarrow|$$

Computing the quantity $\langle\Psi|\hat{P}_{z\uparrow}|\Psi\rangle$ we find

$$\langle\Psi|\hat{P}_{z\uparrow}|\Psi\rangle = \left(\overline{c}_1\langle\mathbf{z}\uparrow| + \overline{c}_2\langle\mathbf{z}\downarrow|\right)\cdot\left(|\mathbf{z}\uparrow\rangle\langle\mathbf{z}\uparrow|\right)\cdot\left(c_1|\mathbf{z}\uparrow\rangle + c_2|\mathbf{z}\downarrow\rangle\right)$$
$$= \left(\overline{c}_1\langle\mathbf{z}\uparrow| + \overline{c}_2\langle\mathbf{z}\downarrow|\right)\cdot\left(c_1|\mathbf{z}\uparrow\rangle\right) =$$
$$= \overline{c}_1 \cdot c_1 = |c_1|^2$$

We notice that the quantity $\langle\Psi|\hat{P}_{z\uparrow}|\Psi\rangle$ is the probability of the state $|\Psi\rangle$ to be in the state $|\mathbf{z}\uparrow\rangle$.

In the same way

$$\langle\Psi|\hat{P}_{z\downarrow}|\Psi\rangle = |c_2|^2$$

is the probability of the state $|\Psi\rangle$ to be in the state $|\mathbf{z}\downarrow\rangle$.

We want to measure the component S_z for a beam of N particles in state $|\Psi\rangle = c_1|\mathbf{z}\uparrow\rangle + c_2|\mathbf{z}\downarrow\rangle$.

Passing it through a SG_z apparatus we measure for $|c_1|^2 N$ particle the value $+\frac{\hbar}{2}$ and for $|c_2|^2 N$ particles the value $-\frac{\hbar}{2}$ (fig 22)

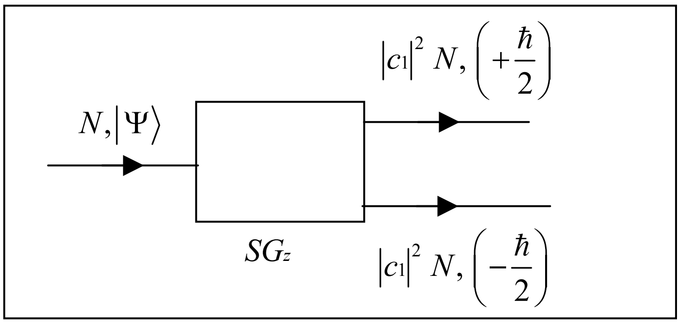


fig 22

Consequently the expectation value of the quantity S_z is

$$\langle S_z \rangle = \frac{|c_1|^2 N\left(+\frac{\hbar}{2}\right) + |c_2|^2 N\left(-\frac{\hbar}{2}\right)}{N} =$$

$$= |c_1|^2 \left(+\frac{\hbar}{2}\right) + |c_2|^2 \left(-\frac{\hbar}{2}\right) \qquad (\alpha)$$

If for the quantity S_z we define a corresponding operator $\hat{S}_z$ as follows

$$\hat{S}_z = \frac{\hbar}{2}\hat{P}_{z\uparrow} + \left(-\frac{\hbar}{2}\right)\hat{P}_{z\downarrow}$$

We have that

$$\langle \Psi | \hat{S}_z | \Psi \rangle = \langle \Psi | \left(\frac{\hbar}{2}\hat{P}_{z\uparrow} - \frac{\hbar}{2}\hat{P}_{z\downarrow} \right) | \Psi \rangle =$$

$$= \frac{\hbar}{2}\langle \Psi | \hat{P}_{z\uparrow} | \Psi \rangle - \frac{\hbar}{2}\langle \Psi | \hat{P}_{z\downarrow} | \Psi \rangle = |c_1|^2 \frac{\hbar}{2} + |c_2|^2 \left(-\frac{\hbar}{2}\right) \qquad (\beta)$$

From (α) and (β) we conclude that

$$\boxed{\langle S_z \rangle = \langle \Psi | \hat{S}_z | \Psi \rangle}$$

In addition, the action of the $\hat{S}_z$ on $|\boldsymbol{z}\uparrow\rangle$ and $|\boldsymbol{z}\downarrow\rangle$ gives

$$\hat{S}_z |\boldsymbol{z}\uparrow\rangle = +\frac{\hbar}{2}|\boldsymbol{z}\uparrow\rangle$$
$$\hat{S}_z |\boldsymbol{z}\downarrow\rangle = -\frac{\hbar}{2}|\boldsymbol{z}\downarrow\rangle$$

The states $|\boldsymbol{z}\uparrow\rangle$ and $|\boldsymbol{z}\downarrow\rangle$ are called eigenstates of the $\hat{S}_z$ with eigenvalues $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, respectively.

Substituting in the definitions of the $\hat{S}_z$ the operators $\hat{P}_{z\uparrow}$ and $\hat{P}_{z\downarrow}$ we find

$$\hat{S}_z = \frac{\hbar}{2}|\boldsymbol{z}\uparrow\rangle\langle\boldsymbol{z}\uparrow| + \left(-\frac{\hbar}{2}\right)|\boldsymbol{z}\downarrow\rangle\langle\boldsymbol{z}\downarrow|$$

<u>General conclusion:</u>

Any physical quantity which can be measured is known as observable. In quantum mechanics an observable is represented by an operator. Let A be an observable and its eigenstate are $|a_1\rangle, |a_2\rangle$ and the corresponding eigenvalues a_1, a_2. Then A is represented by the operator.

$$\hat{A} = a_1 |a_1\rangle\langle a_1| + a_2 |a_2\rangle\langle a_2|$$

The expectation value of A is given by the relation

$$\langle A \rangle = \langle \Psi | \hat{A} | \Psi \rangle$$

Where $|\Psi\rangle$ is the state of the system.

It is also valid

$$\hat{I} = |a_1\rangle\langle a_1| + |a_2\rangle\langle a_2|$$

where $\hat{I}$ is the unit operator. This relation is known as the completeness condition.

2.2 Matrix representation of operators

Generally for an operator $\hat{A}$ it is valid

$$\hat{A}|\Psi\rangle = |\Phi\rangle \qquad (\gamma)$$

where $|\Psi\rangle$ and $|\Phi\rangle$ are in general different kets.
We write $|\Psi\rangle$ and $|\Phi\rangle$ as follows:

$$|\Psi\rangle = \hat{I}|\Psi\rangle = \left(|\boldsymbol{z}\uparrow\rangle\langle\boldsymbol{z}\uparrow| + |\boldsymbol{z}\downarrow\rangle\langle\boldsymbol{z}\downarrow|\right)|\Psi\rangle =$$
$$= |\boldsymbol{z}\uparrow\rangle\langle\boldsymbol{z}\uparrow|\Psi\rangle + |\boldsymbol{z}\downarrow\rangle\langle\boldsymbol{z}\downarrow|\Psi\rangle$$

and

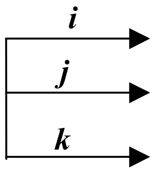
$$|\Phi\rangle = |\boldsymbol{z}\uparrow\rangle\langle\boldsymbol{z}\uparrow|\Phi\rangle + |\boldsymbol{z}\downarrow\rangle\langle\boldsymbol{z}\downarrow|\Phi\rangle$$

Then the (γ) takes the form

$$\hat{A}\left(|\mathbf{z}\uparrow\rangle\langle\mathbf{z}\uparrow|\Psi\rangle+|\mathbf{z}\downarrow\rangle\langle\mathbf{z}\downarrow|\Psi\rangle\right)=|\mathbf{z}\uparrow\rangle\langle\mathbf{z}\uparrow|\Phi\rangle+|\mathbf{z}\downarrow\rangle\langle\mathbf{z}\downarrow|\Phi\rangle \quad (\delta)$$

We will do the same as in the case where we write the vector form of Newton law $\boldsymbol{F} = m\boldsymbol{a}$ equivalently with three scalar equations. As we know taking the dot product of the vector equation with each one of the basis vector $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ we obtain these three equations.

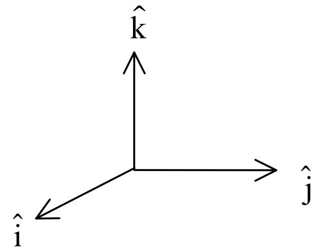
$\boldsymbol{F} = \boldsymbol{m} \cdot \boldsymbol{a}$



$F_x = m \cdot a_x$

$F_y = m \cdot a_y$

$F_z = m \cdot a_y$



In the same way taking the inner product of equation (δ) successively with the bras $\langle\mathbf{z}\uparrow|$ and $\langle\mathbf{z}\downarrow|$ we find the two equations:

$$\langle\mathbf{z}\uparrow|\hat{A}|\mathbf{z}\uparrow\rangle\langle\mathbf{z}\uparrow|\Psi\rangle+\langle\mathbf{z}\uparrow|\hat{A}|\mathbf{z}\downarrow\rangle\langle\mathbf{z}\downarrow|\Psi\rangle=\langle\mathbf{z}\uparrow|\Phi\rangle$$

and

$$\langle\mathbf{z}\downarrow|\hat{A}|\mathbf{z}\uparrow\rangle\langle\mathbf{z}\uparrow|\Psi\rangle+\langle\mathbf{z}\downarrow|\hat{A}|\mathbf{z}\downarrow\rangle\langle\mathbf{z}\downarrow|\Psi\rangle=\langle\mathbf{z}\downarrow|\Phi\rangle$$

These two equations can be conveniently cast in matrix form

$$\begin{pmatrix}\langle\mathbf{z}\uparrow|\hat{A}|\mathbf{z}\uparrow\rangle & \langle\mathbf{z}\uparrow|\hat{A}|\mathbf{z}\downarrow\rangle\\ \langle\mathbf{z}\downarrow|\hat{A}|\mathbf{z}\uparrow\rangle & \langle\mathbf{z}\downarrow|\hat{A}|\mathbf{z}\downarrow\rangle\end{pmatrix}\cdot\begin{pmatrix}\langle\mathbf{z}\uparrow|\Psi\rangle\\ \langle\mathbf{z}\downarrow|\Psi\rangle\end{pmatrix}=\begin{pmatrix}\langle\mathbf{z}\uparrow|\Phi\rangle\\ \langle\mathbf{z}\downarrow|\Phi\rangle\end{pmatrix}$$

Consequently we can represent the kets $|\Psi\rangle$ and $|\Phi\rangle$ by the column matrices:

$$|\Psi\rangle\to\begin{pmatrix}\langle\mathbf{z}\uparrow|\Psi\rangle\\ \langle\mathbf{z}\downarrow|\Psi\rangle\end{pmatrix}$$

$$|\Phi\rangle\to\begin{pmatrix}\langle\mathbf{z}\uparrow|\Phi\rangle\\ \langle\mathbf{z}\downarrow|\Phi\rangle\end{pmatrix},$$

which means that these are the components of the state vectors and the matrix elements of the operator $\hat{A}$ by the 2x2 matrix:

$$\hat{A} \to \begin{pmatrix} \langle \mathbf{z}\uparrow|\hat{A}|\mathbf{z}\uparrow\rangle & \langle \mathbf{z}\uparrow|\hat{A}|\mathbf{z}\downarrow\rangle \\ \langle \mathbf{z}\downarrow|\hat{A}|\mathbf{z}\uparrow\rangle & \langle \mathbf{z}\downarrow|\hat{A}|\mathbf{z}\downarrow\rangle \end{pmatrix}$$

Also a bra $\langle\Psi|$ is represent by the row

$$\langle\Psi| \to \left(\langle\Psi|\mathbf{z}\uparrow\rangle, \langle\Psi|\mathbf{z}\downarrow\rangle\right)$$

Obviously the representation depends on the basis which we choose. The above representations are in the basis which consists of the eigenstates of the S_z.

Remark: Any relation between operators and kets is also valid if we replace all quantities by their corresponding matrices.

Examples:

It is easy to show that the representations of $\hat{P}_{z\uparrow}$, $\hat{P}_{z\downarrow}$, and $\hat{I} = \hat{P}_{z\uparrow} + \hat{P}_{z\downarrow}$ are

$$\hat{P}_{z\uparrow} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \hat{P}_{z\downarrow} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \hat{I} = \hat{P}_{z\uparrow} + \hat{P}_{z\downarrow} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Also we can confirm that the relations

$$\hat{P}_{z\uparrow}|\mathbf{z}\uparrow\rangle = |\mathbf{z}\uparrow\rangle \qquad \text{and} \qquad \hat{I} = \hat{P}_{z\uparrow} + \hat{P}_{z\uparrow}$$

are satisfied by matrices

$$\hat{P}_{z\uparrow}|\mathbf{z}\uparrow\rangle = |\mathbf{z}\uparrow\rangle \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{P}_{z\uparrow}|\mathbf{z}\downarrow\rangle = 0 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The representations of $\hat{S}_z, \hat{S}_x$ and $\hat{S}_y$ in the $\{|\mathbf{z}\uparrow\rangle, |\mathbf{z}\downarrow\rangle\}$ basis are:

$$\hat{S}_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

2.3 Time Evolution

We will examine the energy of a system. The corresponding operator of energy E is denoted as $\hat{\mathrm{H}}$ and is called Hamiltonian. Let the eigenstates of H be $|\mathrm{E}_1\rangle$ and $|\mathrm{E}_2\rangle$ and the corresponding eigenvalues E_1 and E_2. We put the question: If a particle is in state $|\mathrm{E}_1,0\rangle = |\mathrm{E}_1\rangle$ at $t=0$ (initial state) how does it evolve with time?

As we know the states have wave features so it is reasonable to assume that its time evolution would be that of waves. Therefore it must contain terms of the form $\sin\omega t$. Thinking in this frame we suggest the form

$$|\mathrm{E}_1,\mathrm{t}\rangle = e^{-i\omega_1 t}|\mathrm{E}_1\rangle$$

What is the value of ω_1? We borrow the ideas of photons. The photon simultaneously behaves as wave and as particle and the connection between energy and frequency is given by the relation

$\mathrm{E} = h\cdot f$ or $\omega = \dfrac{\mathrm{E}}{\hbar}$.

Therefore it is reasonable to suggest

$$\omega_1 = \frac{\mathrm{E}_1}{\hbar}$$

Consequently

$$|\mathrm{E}_1,\mathrm{t}\rangle = e^{-i\frac{\mathrm{E}_1}{\hbar}\cdot t}|\mathrm{E}_1\rangle$$

the inner product is

$$\langle \mathrm{E}_1,\mathrm{t}|\mathrm{E}_1,\mathrm{t}\rangle = \langle \mathrm{E}_1|e^{+i\frac{\mathrm{E}_1}{\hbar}\cdot t}\cdot e^{-i\frac{\mathrm{E}_1}{\hbar}\cdot t}|\mathrm{E}_1\rangle = \langle \mathrm{E}_1|\mathrm{E}_1\rangle = 1$$

That means that the probability is conserved.

Generally if the initial state is

$$|\Psi(0)\rangle = c_1(0)|\mathrm{E}_1\rangle + c_2(0)|\mathrm{E}_2\rangle$$

then the state $|\Psi(t)\rangle$ (its time evolution) is given by the relation

$$|\Psi(t)\rangle = c_1(0)\cdot e^{-i\frac{\mathrm{E}_1}{\hbar}\cdot t}|\mathrm{E}_1\rangle + c_2(0)\cdot e^{-i\frac{\mathrm{E}_2}{\hbar}\cdot t}|\mathrm{E}_2\rangle$$

Remark: If $\hat{A}$ is an operator where $\hat{A}|a_i\rangle = a_i|a_i\rangle$ then any operator $f(\hat{A})$ is defined as $f(\hat{A})|a_i\rangle = f(a_i)|a_i\rangle$.

That is any function $f(\hat{A})$ of $\hat{A}$ has the same eigenstates $|a_i\rangle$ as $\hat{A}$ and its corresponding eigenvalue is $f(a_i)$.
Using the above remark we find

$$|\Psi(t)\rangle = c_1(0)\cdot e^{-i\frac{\hat{H}}{\hbar}\cdot t}|E_1\rangle + c_2(0)\cdot e^{-i\frac{\hat{H}}{\hbar}\cdot t}|E_2\rangle =$$

$$= e^{-i\frac{\hat{H}t}{\hbar}}\left(c_1(0)|E_1\rangle + c_2(0)|E_2\rangle\right) \qquad \text{or}$$

$$\boxed{|\Psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\Psi(0)\rangle} \qquad (\varepsilon)$$

2.4 The Schrödinger equation

Let a particle be at time t in state $|\Psi(t)\rangle$ and after a small time interval Δt its state is $|\Psi(t+\Delta t)\rangle$. From equation (ε) we find

$$|\Psi(t+\Delta t)\rangle = e^{-i\frac{\hat{H}\Delta t}{\hbar}}|\Psi(t)\rangle.$$

It can be proved that if $\Delta t \to 0$ then

$$e^{-i\frac{\hat{H}\Delta t}{\hbar}} \simeq (1 - i\frac{\hat{H}}{\hbar}\Delta t)$$

Consequently

$$|\Psi(t+\Delta t)\rangle = (1 - i\frac{\hat{H}}{\hbar}\Delta t)|\Psi(t)\rangle$$

or

$$|\Psi(t+\Delta t)\rangle - |\Psi(t)\rangle = -i\frac{\hat{H}\cdot\Delta t}{\hbar}|\Psi(t)\rangle$$

or

$$\frac{|\Psi(t+\Delta t)\rangle - |\Psi(t)\rangle}{\Delta t} = -i\frac{\hat{H}}{\hbar}|\Psi(t)\rangle$$

If $\Delta t \to 0$ then we take the equation

$$\hat{H}|\Psi(t)\rangle = i\hbar\frac{\mathrm{d}|\Psi(t)\rangle}{dt}$$

The last equation is the famous Schrödinger equation and it shows the time evolution of a state.

2.5 The Larmor Precession

Let an electron be in a magnetic field B parallel to z-axis. It is proved that it has energy

$$\mathrm{E} = -\mu_z \cdot B$$

where μ_z is the z-component of the spin magnetic moment. It is given by the relation

$$\mu_z = -\frac{e}{m}\cdot S_z$$

where e : the absolute value of the charge of electron
m : electron mass
S_z : the spin component on z-axis

We know that S_z takes two values $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ and therefore the energy also takes two values

$$\mathrm{E}_1 = \frac{e}{m}B\frac{\hbar}{2} = \frac{1}{2}\omega_0\hbar$$

and

$$\mathrm{E}_2 = -\frac{e}{m}B\cdot\frac{\hbar}{2} = -\frac{1}{2}\omega_0\hbar$$

where

$$\omega_0 = \frac{eB}{m}.$$

Obviously the state $|\boldsymbol{z}\uparrow\rangle$ and $|\boldsymbol{z}\downarrow\rangle$ are also eigenstates of the operator $\hat{H}$ with corresponding eigenvalues E_1 and E_2. Consequently if the state of electron at $t=0$ is

$$|\Psi(0)\rangle = c_1(0)|\boldsymbol{z}\uparrow\rangle + c_2(0)|\boldsymbol{z}\downarrow\rangle$$

after time t it is

$$\left|\Psi(t)\right\rangle = e^{-i\frac{E_1 t}{\hbar}} c_1(0)\left|\mathbf{z}\uparrow\right\rangle + e^{-i\frac{E_2 t}{\hbar}} c_2(0)\left|\mathbf{z}\downarrow\right\rangle$$

or

$$\left|\Psi(t)\right\rangle = e^{-i\frac{\omega_0 t}{2}} c_1(0)\left|\mathbf{z}\uparrow\right\rangle + e^{i\frac{\omega_0 t}{2}} c_2(0)\left|\mathbf{z}\downarrow\right\rangle$$

<u>Example:</u>
We assume that

$$\left|\Psi(0)\right\rangle = \left|\mathbf{x}\uparrow\right\rangle$$

or

$$\left|\Psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left|\mathbf{z}\uparrow\right\rangle + \frac{1}{\sqrt{2}}\left|\mathbf{z}\downarrow\right\rangle$$

Then

$$\left|\Psi(t)\right\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{\omega_0 t}{2}}\left|\mathbf{z}\uparrow\right\rangle + \frac{1}{\sqrt{2}} e^{i\frac{\omega_0 t}{2}}\left|\mathbf{z}\downarrow\right\rangle$$

The mean value of S_x is

$$\langle S_x\rangle = \left\langle\Psi(t)\right|\hat{S}_x\left|\Psi(t)\right\rangle =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\frac{\omega_0 t}{2}} & \frac{1}{\sqrt{2}} e^{-i\frac{\omega_0 t}{2}} \end{pmatrix} \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\frac{\omega_0 t}{2}} \\ \frac{1}{\sqrt{2}} e^{i\frac{\omega_0 t}{2}} \end{pmatrix} =$$

$$= \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t}) \cdot \frac{\hbar}{2}$$

or $\langle S_x\rangle = \frac{\hbar}{2}\cos\omega_0 t$

Similarly we find

$$\langle S_y\rangle = \frac{\hbar}{2}\sin\omega_0 t$$

and $\langle S_z\rangle = 0$

The above equations imply that the expectation value of the spin angular momentum vector lies down on x-y plane and rotates about z-axis with angular velocity ω_0. This picture (fig 23) is called Larmor precession.

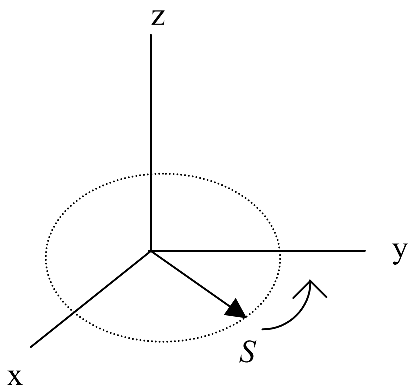


fig 23

SUMMARY

Experiments with microscopic particles show a new behaviour of nature. We observe that the particles have features of waves. The mathematical model that describes this behaviour
is the quantum mechanics. The significant points of this theory are the followings:

1. Measuring an observable A we find a set of discrete values a_1, a_2. To each value corresponds a state $|a_1\rangle, |a_2\rangle$. These states constitute a basis of Hilbert space.

2. After a measurement of A the particle (or system) is in a well defined state (pure state). If we repeat the measurement of A we will find the same value.

3. Any pure state $|\Psi\rangle$ is a vector belonging to Hilbert space
   $|\Psi\rangle = c_1 |a_1\rangle + c_2 |a_2\rangle$

4. If the particle is in state $|\Psi\rangle$ then the probability to be in state $|\Phi\rangle$ is the square of probability amplitude.

   probability amplitude $= \langle\Phi|\Psi\rangle$

   probability $= |\langle\Phi|\Psi\rangle|^2$

5. Any observable A is represented by an operator $\hat{A}$
   $\hat{A} = a_1 |a_1\rangle\langle a_1| + a_2 |a_2\rangle\langle a_2|$
   The expectation value of an observable A is given by the relation
   $\langle A\rangle = \langle\Psi|\hat{A}|\Psi\rangle$

6. The time evolution of a state $|\Psi\rangle$ is given as follows

   $$|\Psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}} |\Psi(0)\rangle$$

   where $\hat{H}$ is the Hamiltonian operator and it is governed by the Schrödinger equation

   $$\hat{H} |\Psi(t)\rangle = i\hbar \frac{d|\Psi(t)\rangle}{dt}$$

QUESTION EXERCISES

1) The mass of a body is $m = 2kg$ and a constant force $F = 4N$ is exerted on it. At time $t = 0$ it is at position $x_0 = 2\,meter$ and its velocity is $v_0 = 0$. Find the state of the body for any time t.

2) Which from the following pairs could define the state of a particle?

α) (position-momentum)   β) (force-mass)
γ) (position-velocity)   δ) (position-force)

3) How is the thermal equilibrium state defined in thermodynamics?

4) Why can we not define the state for a microparticle with the pair position-momentum?

5) Two observables A and B are simultaneously measurable. That means:

i) There is a measuring method by which we can measure the two observables A and B at the same time.

ii) Measuring the A we find the value a_1, then measuring the B we find the value β_1 and if we measure again the A we find the value a_1.

iii) The two observables must be measured at the same time.

iv) Measuring the A we find the value a and measuring the B we find the value β which is independent on the value a.

Choose the correct answer.

6) <<If we had ideal experimental devices we could simultaneously measure the position and the momentum of a particle without error.>>

Comment the above statement.

7) Two observables A and B are not simultaneously measurable. That means:

i) There is no method by which we can measure A and B at the same time.

ii) Measuring the A we find the value a, then measuring the B we find β. Repeating the measurement of A we do not find the value a with probability equal to unity.

iii) The result from the measurement of A does not depends on if we first measure the B.

Choose the correct answer.

8) What is the ket $\left(|\ \rangle \right)$?

9) The results of the measurement of the energy of the atom of Hydrogen (H) are

$-13,6\ eV$, $-\frac{13,6}{4}eV$, $-\frac{13,6}{9}eV$, $-\frac{13,6}{16}eV$, ….

i) Write the general relation which gives the values of energy.
ii) Find the states of the system.
iii) Write the states as you like.

9a) When the Energy of the atom of Hydrogen is $En = -\frac{13,6}{n^2}eV$ then the amplitude of the vector of angular momentum $\boldsymbol{L}$ takes the values $L = \sqrt{\ell(\ell+1)} \cdot \hbar$ where $\ell = 0,1,2,...,n-1$. Also the component L_z takes the values $L_z = m_e \cdot \hbar$ where $m_e = -\ell,...,+\ell$. The observables E, L, L_z are simultaneously measurable. Find the states of the system as $\left|n,\ell,m_e\right\rangle$ in the cases

α) $n=1$ β) $n=2$

10) How do we define the spin of a rigid body?

11) Describe the Stern-Gerlach device.

12) What values of S_z component of spin we expect to find as result of a Stern-Gerlach experiment?

13) Why do we say that every beam separately, emerging from of a SG_z device is a well defined state?

14) *Write T or F whether you think the statements are true or false.*

For a well defined state of a system:

i) The value for any observable concerning the system is defined.
ii) The value for at least one observable concerning the system is defined.
iii) It is unique for the system (there is no any other well defined state).
iv) It is always connecting with at least one observable.

15) Are the particles emerging from the oven of the SG experiment in a well defined state?

16) *Write T or F whether you think the statement are true or false.*

i) the state $\left|\boldsymbol{z}\uparrow\right\rangle$ is well defined state for the observable S_z but not for the observable S_x.

ii) There is no state $\left|\boldsymbol{z}\uparrow, \boldsymbol{y}\downarrow\right\rangle$ because the observable S_z and S_y are not simultaneously measurable.

iii) The state $\left|\boldsymbol{z}\uparrow\right\rangle$ can not be written as combination of $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ because the last two states concern other observable.

17) After what thoughts we wrote the relation

$$\left|\boldsymbol{z}\uparrow\right\rangle = \frac{1}{2}\left|\boldsymbol{y}\uparrow\right\rangle + \frac{1}{2}\left|\boldsymbol{y}\downarrow\right\rangle ?$$

18) Which experiment shows that the philosophy of writing the relation

$$\left|\boldsymbol{z}\uparrow\right\rangle = \frac{1}{2}\left|\mathbf{x}\uparrow\right\rangle + \frac{1}{2}\left|\mathbf{x}\downarrow\right\rangle$$

is false?

19) Which events of the experiments with the SG and the MSG devices lead us to suspect that particles have wave behaviour?

20) *Write T or F whether you think the statements are true or false.*

The Hilbert space concerning a particle with spin $\frac{1}{2}$.

i) This is two dimension, that is the basis of it consists of two vectors because we find two values for each observable S_x, S_y or S_n, where $\hat{\boldsymbol{n}}$ is an arbitrary direction.

ii) Its dimension would have been greater if we had found more values for the component S_x.

iii) Any vector of it can be written as superposition of vectors $\left|\boldsymbol{z}\uparrow\right\rangle$ and $\left|\boldsymbol{z}\downarrow\right\rangle$ or $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$.

iv) Every vector of it describes a realizable state.

21) Why the coefficients c_1 and c_2 in the relation

$$\left|\Psi\right\rangle = c_1\left|\boldsymbol{z}\uparrow\right\rangle + c_2\left|\boldsymbol{z}\downarrow\right\rangle$$

must be generally complex numbers?

22) Find the corresponding bra $\left\langle\Psi\right|$ of the ket

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left|\boldsymbol{z}\uparrow\right\rangle + \frac{i}{\sqrt{2}}\left|\boldsymbol{z}\downarrow\right\rangle .$$

23) Consider the vector

$$\left|\Psi\right\rangle = \frac{1}{2}\left|\boldsymbol{z}\uparrow\right\rangle + i\frac{\sqrt{3}}{2}\left|\boldsymbol{z}\downarrow\right\rangle .$$

We know that

$$\left|\mathbf{z}\uparrow\right\rangle = \frac{1}{\sqrt{2}}\left|\mathbf{x}\uparrow\right\rangle + \frac{1}{\sqrt{2}}\left|\mathbf{x}\downarrow\right\rangle$$

and

$$\left|\mathbf{z}\downarrow\right\rangle = \frac{1}{\sqrt{2}}\left|\mathbf{x}\uparrow\right\rangle - \frac{1}{\sqrt{2}}\left|\mathbf{x}\downarrow\right\rangle$$

Write the ket $\left|\Psi\right\rangle$ as linear combination of $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$.

24) For a particle with spin 1 we find experimentally for the component S_z three values $-1\hbar$, $0\hbar$ and $+1\hbar$. Define the corresponding Hilbert space of the particle and write any state as superposition of the vectors which constitute the basis of the space.

25) How is the probability amplitude a state $\left|\Psi\right\rangle$ to be in $\left|\Phi\right\rangle$ defined and how is the corresponding probability defined?

26) Explain why we write the relations

i) $\left\langle \mathbf{z}\uparrow | \mathbf{z}\uparrow\right\rangle = 1$ **ii)** $\left\langle \mathbf{z}\uparrow | \mathbf{z}\downarrow\right\rangle = 0$

iii) $\left|\left\langle \mathbf{z}\uparrow | \mathbf{x}\downarrow\right\rangle\right|^2 = \frac{1}{2}$ **iv)** $\left\langle \mathbf{x}\uparrow | \mathbf{x}\downarrow\right\rangle = 0$

27) A particle is in the state

$$\left|\Psi\right\rangle = c_1\left|E_1\right\rangle + c_2\left|E_2\right\rangle + c_3\left|E_3\right\rangle$$

where $\left|E_1\right\rangle$, $\left|E_2\right\rangle$ and $\left|E_3\right\rangle$ are different eigenstates of the energy. What relation do the coefficients c_1, c_2 and c_3 satisfy and why?

28) A particle is in state

$$\left|\Psi\right\rangle = \frac{3i}{5}\left|\mathbf{z}\uparrow\right\rangle + \frac{4}{5}\left|\mathbf{z}\downarrow\right\rangle$$

Compute the probability for this state to be in the state

$$\left|\Phi\right\rangle = \sqrt{\frac{2}{7}}\left|\mathbf{z}\uparrow\right\rangle - i\sqrt{\frac{5}{7}}\left|\mathbf{z}\downarrow\right\rangle$$

29) A particle with spin $\frac{1}{2}$ is in state

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{3}}\left|\mathbf{z}\uparrow\right\rangle + i\sqrt{\frac{2}{3}}\left|\mathbf{z}\downarrow\right\rangle$$

i) Passing the particle through a SG_z device, find the probability to end in the state $\left|\mathbf{z}\uparrow\right\rangle$ and similarly to end in the state $\left|\mathbf{z}\downarrow\right\rangle$.

ii) If we pass a huge numbers N of particles which are in the same state $\left|\Psi\right\rangle$, through a SG_z apparatus, how many particles will follow

α) the higher path
β) the lower path

iii) If we pass the preceding ensemble of N particles through a SG_x device, find the number of particles which will follow

α) the higher path
β) the lower path

30) Show that

i) $\hat{P}_{z\uparrow}\hat{P}_{z\uparrow}=\hat{P}_{z\uparrow}$ **ii)** $\hat{P}_{z\uparrow}\hat{P}_{z\downarrow}=0$ **iii)** $\hat{P}_{z\downarrow}\hat{P}_{z\downarrow}=\hat{P}_{z\downarrow}$

where $\hat{P}_{z\uparrow},\hat{P}_{z\downarrow}$ projection operators onto the states $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$ respectively.

31) Write the completeness condition using the state $|\mathbf{x}\uparrow\rangle$ and $|\mathbf{x}\downarrow\rangle$. Similarly for the states $|\mathbf{y}\uparrow\rangle$ and $|\mathbf{y}\downarrow\rangle$.

32) Let a particle be in the state

$$|\Psi\rangle=\frac{6}{7}|\mathbf{z}\uparrow\rangle+i\frac{\sqrt{13}}{7}|\mathbf{z}\downarrow\rangle$$

Write T or F whether you think the following statements are true or false.

i) If the particle pass through a SG_z device the probability to follow the higher path is $\frac{6}{7}$.
ii) If it pass through a SG_z the probability to follow the lower path is $^{13}/_{49}$.
iii) The expectation value of the component S_z is

$$\langle S_z\rangle=\frac{23}{49}\hbar$$

iv) If N particles which are in the same state $|\Psi\rangle$ pass through a SG_z device they will follow an intermediate path corresponding to the expectation value of the component S_z.

33) The state of a particle is

$$|\Psi\rangle=\frac{3}{5}|\mathbf{z}\uparrow\rangle+i\frac{4}{5}|\mathbf{z}\downarrow\rangle$$

i) Calculate the expectation value of the component S_z.
ii) If we measure the component S_z for one particle we will find:

α) The expectation value of S_z

β) $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$

What is the correct answer?

34) Measuring the energy for a system we find three values

$E_1 = 3\mu J$, $E_2 = 5\mu J$, $E_3 = 6\mu J$

i) Find a basis for the corresponding Hilbert space and write the general form for any vector $|\Psi\rangle$ belonging to Hilbert space.

ii) The system is in the state

$$|\Psi\rangle = \frac{\sqrt{11}}{6}|E_1\rangle + \frac{i}{2}|E_2\rangle + \frac{2}{3}|E_3\rangle$$

α) Find the most probable value for the energy.

β) Write the corresponding operator $\hat{E}$ of the energy and calculate its expectation value in the state $|\Psi\rangle$.

γ) Calculate the probability to find the value $< E >$ if we measure for one time the energy.

35) The state of a particle is

$$|\Psi\rangle = \frac{2}{3}|\boldsymbol{z}\uparrow\rangle + i\frac{\sqrt{5}}{3}|\boldsymbol{z}\downarrow\rangle$$

i) Calculate the mean value of the component S_x.

ii) Similarly for the component S_y.

36) Show that the representation of the operators $\hat{S}_z$, $\hat{S}_x$, $\hat{S}_y$ in the basis which consists of the vectors $|\boldsymbol{z}\uparrow\rangle$ and $|\boldsymbol{z}\downarrow\rangle$ are

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

37) Find the representation of the operators $\hat{S}_z$, $\hat{S}_x$, $\hat{S}_y$ in the basis $\{|\boldsymbol{x}\uparrow\rangle, |\boldsymbol{x}\downarrow\rangle\}$.

38) The state of a particle is

$$|\Psi\rangle = \frac{12}{13}|\boldsymbol{z}\uparrow\rangle + i\frac{5}{13}|\boldsymbol{z}\downarrow\rangle,$$

calculate the $\langle S_x\rangle$ and $\langle S_y\rangle$ using the corresponding matrices of representation.

39) Why do we write the relation

$$|E_1(t)\rangle = e^{-i\omega_1 t}|E_1\rangle \qquad \text{where } \omega_1 = \frac{E_1}{\hbar}?$$

40) A system has only two eigenstates of the energy, the $|E_1\rangle$ and $|E_2\rangle$. If the initial state is the state $|\Psi\rangle = |E_1\rangle$, show that it remains for ever.

41) A particle with spin $\frac{1}{2}$ is in a homogeneous magnetic field B with $\vec{B}$ parallel to z-axis. If at the time $t = 0$ it is in the state $|\Psi(0)\rangle = |\boldsymbol{z}\uparrow\rangle$, find the state $|\Psi(t)\rangle$. Then calculate the quantities $\langle S_x(t)\rangle$, $\langle S_y(t)\rangle$ and $\langle S_z(t)\rangle$.

42) Repeat the preceding problem assuming that the initial state is $|\Psi(0)\rangle = |\boldsymbol{y}\downarrow\rangle$.

PART B.

ensembles
tensor product
entanglements

3. ENSEMBLES

3.1 Measurement of Observable S_n

Before we define the pure and mixed ensemble we examine the measurement of the component S_n (where $\hat{\boldsymbol{n}}$ is arbitrary direction) of the spin S for a particle with spin $\frac{1}{2}$.

Let a beam of N particles be in state $|\boldsymbol{z}\uparrow\rangle$. We pass the beam trough a SG_n apparatus with its magnetic field to be parallel with $\boldsymbol{n}$ axis. The $\boldsymbol{n}$ axis belongs to x, z-plane and forms with z-axis an angle ϑ (see fig 1).

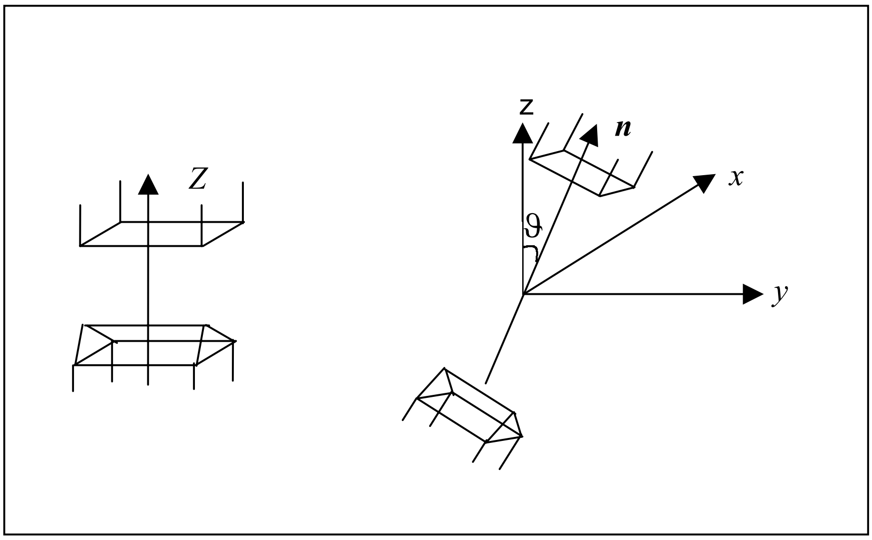


fig 1

Schematically the experiment is the following (fig 2)

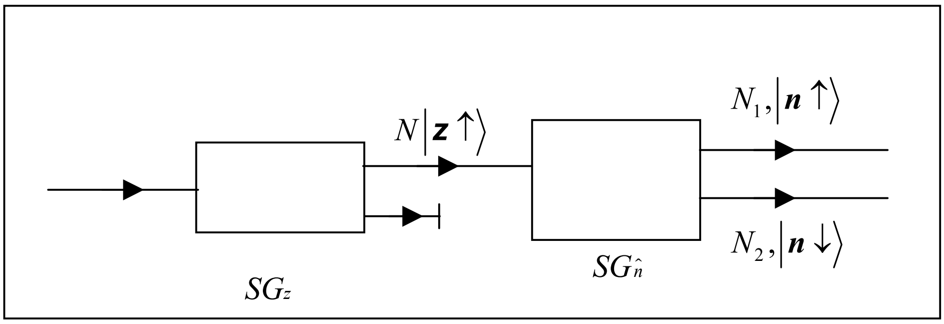


fig2

Obviously, assuming the results of previous chapters, the result of the measurement is either $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ and the corresponding eigenstates are $|\boldsymbol{n}\uparrow\rangle$ and $|\boldsymbol{n}\downarrow\rangle$.

It can be proved that the vectors $\left|\boldsymbol{n}\uparrow\right\rangle$ and $\left|\boldsymbol{n}\downarrow\right\rangle$ are the following superposition of $\left|\boldsymbol{z}\uparrow\right\rangle$ and $\left|\boldsymbol{z}\downarrow\right\rangle$.

$$\left|\boldsymbol{n}\uparrow\right\rangle=\cos\frac{\vartheta}{2}\left|\boldsymbol{z}\uparrow\right\rangle+\sin\frac{\vartheta}{2}\left|\boldsymbol{z}\downarrow\right\rangle$$

and $\left|\boldsymbol{n}\downarrow\right\rangle=\sin\frac{\vartheta}{2}\left|\boldsymbol{z}\uparrow\right\rangle-\cos\frac{\vartheta}{2}\left|\boldsymbol{z}\downarrow\right\rangle$

Consequently the probability the initial state $\left|\boldsymbol{z}\uparrow\right\rangle$ to behave as $\left|\boldsymbol{n}\uparrow\right\rangle$ is

$$\left|\left\langle\boldsymbol{n}\uparrow\middle|\boldsymbol{z}\uparrow\right\rangle\right|^2=\cos^2\frac{\vartheta}{2}$$

and so $N_1=N\cos^2\frac{\vartheta}{2}$ particles outcome with $S_n=+\frac{\hbar}{2}$.

The probability the initial state $\left|\boldsymbol{z}\uparrow\right\rangle$ to be the $\left|\boldsymbol{n}\downarrow\right\rangle$

is $\left|\left\langle\boldsymbol{n}\downarrow\middle|\boldsymbol{z}\uparrow\right\rangle\right|^2=\sin^2\frac{\vartheta}{2}$

and so $N_2=N\cdot\sin^2\frac{\vartheta}{2}$ particles emerge with $S_n=-\frac{\hbar}{2}$.

The mean value of S_n is

$$\left\langle S_n\right\rangle=\frac{N_1\left(+\frac{\hbar}{2}\right)+N_2\left(-\frac{\hbar}{2}\right)}{N}=\frac{\hbar}{2}(\cos^2\frac{\vartheta}{2}-\sin^2\frac{\vartheta}{2})=\frac{\hbar}{2}\cos\vartheta\ .$$

We also find the same result if we use the rules of quantum mechanics

$$\left\langle S_n\right\rangle=\left\langle\boldsymbol{z}\uparrow\middle|\hat{S}_n\middle|\boldsymbol{z}\uparrow\right\rangle \rightarrow \left\langle S_n\right\rangle=\left\langle\boldsymbol{z}\uparrow\middle|\hat{S}_n\middle|\boldsymbol{z}\uparrow\right\rangle$$

however $\hat{S}_n=\frac{\hbar}{2}\left|\boldsymbol{n}\uparrow\right\rangle\left\langle\boldsymbol{n}\uparrow\right|-\frac{\hbar}{2}\left|\boldsymbol{n}\downarrow\right\rangle\left\langle\boldsymbol{n}\downarrow\right|$

so

$$\left\langle S_n\right\rangle=\left\langle\boldsymbol{z}\uparrow\right|\left(\frac{\hbar}{2}\left|\boldsymbol{n}\uparrow\right\rangle\left\langle\boldsymbol{n}\uparrow\right|-\frac{\hbar}{2}\left|\boldsymbol{n}\downarrow\right\rangle\left\langle\boldsymbol{n}\downarrow\right|\right)\left|\boldsymbol{z}\uparrow\right\rangle=$$

$$=\frac{\hbar}{2}\cos\vartheta$$

3.2 Pure Ensemble

The state of a particle with spin $\frac{1}{2}$ is a vector in the Hilbert space. Let us assume that the state is

$$|\Psi\rangle = c_1 |\mathbf{z}\uparrow\rangle + c_2 |\mathbf{z}\downarrow\rangle$$

where $|\mathbf{z}\uparrow\rangle$ and $|\mathbf{z}\downarrow\rangle$ are the eigenvectors of the S_z component of the spin S.
What information does this state give us? It gives us the probability the particle to be in the state $|\Phi\rangle$ and it is given through the relation

probability = $|\langle\Phi|\Psi\rangle|^2$

More particularly that means that if we measure an observable A where the state $|\Phi\rangle$ is an its eigenstates with corresponding eigenvalue a, then the probability of the result of a measurement to be a is $|\langle\Phi|\Psi\rangle|^2$.
For example if we measure the S_z component of spin, the probability to find the value $+\frac{\hbar}{2}$ is $|\langle\mathbf{z}\uparrow|\Psi\rangle|^2 = |c_1|^2$ and the probability to find the value $-\frac{\hbar}{2}$ is $|\langle\mathbf{z}\downarrow|\Psi\rangle|^2 = |c_2|^2$.
If we have a large number N of particles and all particles are in the same state $|\Psi\rangle$ (vector of Hilbert space) then we say that we have a **pure ensemble**.
For a pure ensemble the mean value of an observable A is given through the relation (rule of quantum mechanics, see previous).

$$\langle \mathrm{A}\rangle = \langle\Psi|\hat{\mathrm{A}}|\Psi\rangle$$

where $\hat{\mathrm{A}}$ is the corresponding operator.

When do we have a pure ensemble? or how can we construct a pure ensemble?

a) After a measurement of an observable A if we select all particles with the same eigenvalue then our ensemble is a pure ensemble because all particles are in the same state. For example each beam separately which is outcoming from the SG_x apparatus is a pure ensemble (see fig 3).

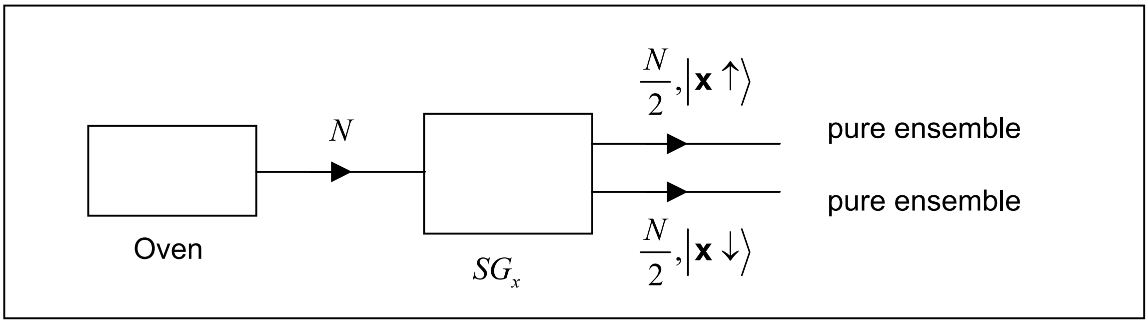


fig 3

β) If we want to construct a pure ensemble in state $|\Psi\rangle$ then we find the corresponding operator $\hat{A}$ whose state $|\Psi\rangle$ is its eigenstates with eigenvalue a. Then we measure the observable A and select all the particles with the same result of measurement equal to a. This ensemble is a pure ensemble in state $|\Psi\rangle$. However there are vectors in Hilbert space which is not constructed because there is no corresponding observables.

3.3 Mixed Ensemble

Let us construct a beam which consists of N_1 particles emerging from the upper path of a SG_x apparatus and N_2 particles outcoming from the upper path of a SG_y apparatus. That is N_1 particles are in state $|\mathbf{x}\uparrow\rangle$ and N_2 particles are in state $|\mathbf{y}\uparrow\rangle$ (see fig 4).

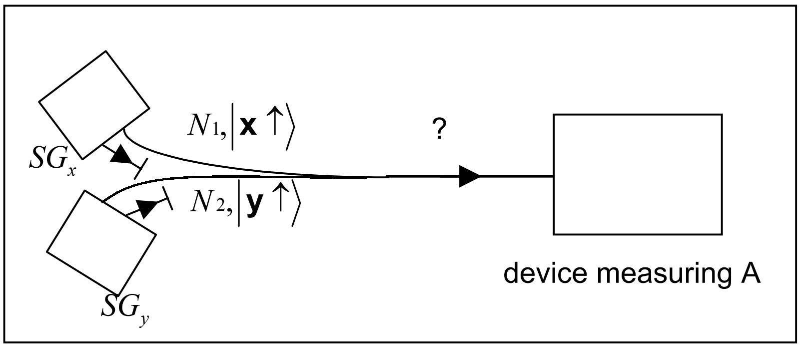


fig 4

The new ensemble is not a pure ensemble because it does not consist of particles where all of them are in the same state. In this case we say that our ensemble is a <u>mixed ensemble</u>.

Let us try to calculate the expectation value of an observable A for the new ensemble.

N_1 particles are in state $|\mathbf{x}\uparrow\rangle$. From the rules of quantum mechanics we find

$$\langle A\rangle_1 = \langle \mathbf{x}\uparrow|\hat{A}|\mathbf{x}\uparrow\rangle$$

N_2 particles are in state $|\mathbf{y}\uparrow\rangle$. Similarly we have

$$\langle A\rangle_2 = \langle \mathbf{y}\uparrow|\hat{A}|\mathbf{y}\uparrow\rangle$$

Consequently using the rules of statistics we find

$$\langle A\rangle = \frac{N_1\langle A\rangle_1 + N_2\langle A\rangle_2}{N} \Rightarrow$$

$$\langle A\rangle = \frac{N_1}{N}\langle A\rangle_1 + \frac{N_2}{N}\langle A\rangle_2 \Rightarrow$$

$$\langle A\rangle = p_1\langle A\rangle_1 + p_2\langle A\rangle_2$$

where $p_1 = \frac{N_1}{N}$ the ratio of particles in state $|\mathbf{x}\uparrow\rangle$ to all particles

and $p_2 = \frac{N_2}{N}$ the ratio of particles in state $|\mathbf{y}\uparrow\rangle$ to all particles

Obviously $p_1 + p_2 = 1$.

Generally if for a mixed ensemble the particles are in state $|\Psi_1\rangle$ with fraction p_1 and in state $|\Psi_2\rangle$ with fraction p_2 then the expectation value of an observable A is given through the relation

$$\langle A\rangle = p_1\langle\Psi_1|\hat{A}|\Psi_1\rangle + p_2\langle\Psi_2|\hat{A}|\Psi_2\rangle.$$

It is possible the states of a mixed ensemble to be more than two. That is $|\Psi_1\rangle$ with ratio p_1, $|\Psi_2\rangle$ with ratio p_2, $|\Psi_3\rangle$ with ratio p_3,…. Then we have

$$\langle A\rangle = p_1\langle\Psi_1|\hat{A}|\Psi_1\rangle + p_2\langle\Psi_2|\hat{A}|\Psi_2\rangle + p_3\langle\Psi_3|\hat{A}|\Psi_3\rangle + ...$$

where $p_1 + p_2 + p_3 + ... = 1$

3.4 Superposition and Mixed Ensemble

Let us return to impressive experiment with the MSG_x apparatus. As we saw if a beam is in state $|\mathbf{z}\uparrow\rangle$ and passes through a MSG_x apparatus then leaving both channels open the beam in state $|\mathbf{z}\uparrow\rangle$ outcomes unchanged (see fig 5)

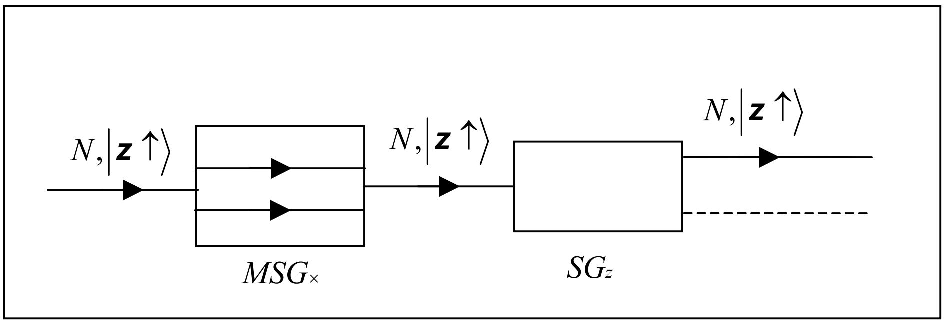


fig 5

Also we know that the state $|\mathbf{z}\uparrow\rangle$ is written as follows:

$$\left|\mathbf{z}\uparrow\right\rangle = \frac{1}{\sqrt{2}}\left|\mathbf{x}\uparrow\right\rangle + \frac{1}{\sqrt{2}}\left|\mathbf{x}\downarrow\right\rangle$$

Let us repeat the same experiment but now we use a technique in order to control the two paths (see fig 6).

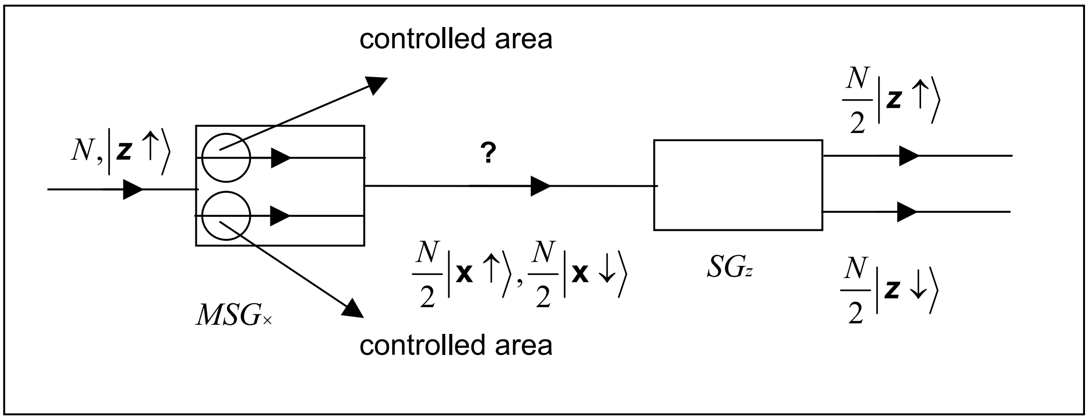


fig 6

That means that we know the path which is followed by each particle. Obviously the result is now $\frac{N}{2}$ particles in state $\left|\mathbf{x}\uparrow\right\rangle$ and $\frac{N}{2}$ particles in state $\left|\mathbf{x}\downarrow\right\rangle$. Consequently the beam is no more a pure ensemble it is now a mixed ensemble. We pass it through a SG_z apparatus. Then from the $\frac{N}{2}$ particles in state $\left|\mathbf{x}\uparrow\right\rangle$, $\frac{N}{4}$ outcome in state $\left|\mathbf{z}\uparrow\right\rangle$ and $\frac{N}{4}$ in state $\left|\mathbf{z}\downarrow\right\rangle$. Also from the $\frac{N}{2}$ particles in state $\left|\mathbf{x}\downarrow\right\rangle$, $\frac{N}{4}$ emerge in state $\left|\mathbf{z}\uparrow\right\rangle$ and $\frac{N}{4}$ in state $\left|\mathbf{z}\downarrow\right\rangle$. Really the experiment shows that $\frac{N}{4}+\frac{N}{4}=\frac{N}{2}$ particles outcome in state $\left|\mathbf{z}\uparrow\right\rangle$
and $\frac{N}{4}+\frac{N}{4}=\frac{N}{2}$ particles emerge in state $\left|\mathbf{z}\downarrow\right\rangle$.

From the last two experiments we conclude that the ensemble where all particles are in state $\left|\mathbf{z}\uparrow\right\rangle$ (vector in Hilbert space) which is a superposition of the states $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ and the ensemble which consists of 50% of particles in state $\left|\mathbf{x}\uparrow\right\rangle$ and 50% of particles in $\left|\mathbf{x}\downarrow\right\rangle$ are two different things

$$\frac{1}{\sqrt{2}}\left|\mathbf{x}\uparrow\right\rangle + \frac{1}{\sqrt{2}}\left|\mathbf{x}\downarrow\right\rangle \neq 50\%\left|\mathbf{x}\uparrow\right\rangle \text{and } 50\%\left|\mathbf{x}\downarrow\right\rangle$$

An important point in the previous process is the change of the pure ensemble to mixed ensemble. The first time the two states $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ make interference and have a wave behaviour, they describe together the same particle. However the second time when we know exactly the path which is followed by each particle we do not observe the interference effect, the two states $|\mathbf{x}\uparrow\rangle$ and $|\mathbf{x}\downarrow\rangle$ become foreign between themselves and each one describes a different particle.

<u>Conclusions:</u>

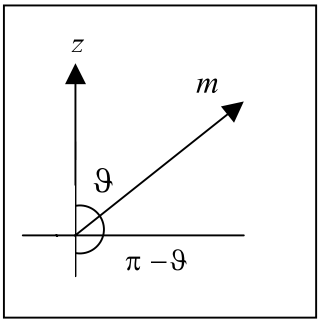
a) The particle is in a state which is a vector in Hilbert space (? we will come to this point later).

β) An ensemble is pure if all its members are in the same state $|\Psi\rangle$. In order to calculate the expectation value of an observable we use the rules of quantum mechanics.

γ) An ensemble is mixed if its members are not all in the same state $|\Psi\rangle$. In order to calculate the expectation value we use the rules of statistics.

3.5 Unpolarized Beam

If a mixed ensemble from N particles has $\frac{N}{2}$ particles in state $|\boldsymbol{z}\uparrow\rangle$ and the remaining $\frac{N}{2}$ in state $|\boldsymbol{z}\downarrow\rangle$, then for any spin-component S_m ($\boldsymbol{m}$ arbitrary) it is valid



$$\langle S_m\rangle = \frac{1}{2}\langle \boldsymbol{z}\uparrow|\hat{S}_m|\boldsymbol{z}\uparrow\rangle + \frac{1}{2}\langle \boldsymbol{z}\downarrow|\hat{S}_m|\boldsymbol{z}\downarrow\rangle =$$
$$= \frac{\hbar}{2}(\cos\vartheta + \cos(\pi - \vartheta)) = 0 .$$

Where ϑ is the angle forming with $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{m}}$.

Such a beam is called unpolarized beam and is indistinguishable from any other beam with half particles in state $|\boldsymbol{n}\uparrow\rangle$ and remaining half particles in state $|\boldsymbol{n}\downarrow\rangle$, where $\boldsymbol{n}$ is an arbitrary direction.

Generally there is no mechanism to distinguish two mixed ensemble, the first one with half particles in state $|\boldsymbol{n}\uparrow\rangle$ and remaining half is state $|\boldsymbol{n}\downarrow\rangle$ and the second one with half particles in state $|\boldsymbol{m}\uparrow\rangle$ and remaining half in state $|\boldsymbol{m}\downarrow\rangle$, where $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{m}}$ arbitrary directions.

TENSOR PRODUCT

4.1 Definition of Tensor Product

As we saw for a particle with spin $\frac{1}{2}$ it is defined a two dimensional Hilbert space. A basis of this space can be the pairs of eigenstates of any of the observables S_x or S_y or S_z. We denote this space by H_1.

Let A be an observable which is a function of $S_x, S_y, S_z,$ for example the square of the total spin $S^2 = S_x{}^2 + S_y{}^2 + S_z{}^2$, the energy when the particle is in a magnetic field B_z (it is $H = +\frac{e}{m} S_z B_z$) and so on. Then the eigenstates of A are also vectors in Hilbert Space H_1. Particularly if the observable A is a function of one observable, let it be the S_x, that means that $A = f\left(S_x\right)$, then the eigenstates of A are the same with these of S_x. That is the eigenstates of A are the $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ with corresponding eigenvalues $a_1 = f(+\frac{\hbar}{2})$ and $a_2 = f(-\frac{\hbar}{2})$.

Also for a vector state $\left|\Psi\right\rangle$ in Hilbert space there is a corresponding observable which has the state $\left|\Psi\right\rangle$ as eigenstate (this is not true for all states $\left|\Psi\right\rangle$).

Let B be an observable which is unrelated with any of the observables $S_x, S_y, S_z.$ That is the observables B and S_x or S_y or S_z are foreign between themselves. In this case an eigenstate of B does not belong to Hilbert space H_1. If the state of particle is the $\left|\Psi\right\rangle$ a vector in Hilbert space H_1, the measurement of the observable B does not change the state $\left|\Psi\right\rangle$. For example, the momentum of a particle and its spin are unrelated observables. Another example is the energy of an electron due to an electric field and the spin.

For simplicity we assume that B has also two eigenstates $\left|\beta_1\right\rangle$ and $\left|\beta_2\right\rangle$ with corresponding eigenvalues β_1 and β_2. These obviously constitute a basis of a new Hilbert space H_2. Observables which are not simultaneously measured with B and observables which are a function of B constitute a new set of observables whose eigenstates are vectors in the Hilbert space H_2.

We measure an observable A which is related with the Hilbert space H_1 and let us find the value a_1. The state of our particle after the measurement is $\left|a_1\right\rangle$. Then we measure an observable B which is related with the Hilbert space H_2 and let us find the value β_1. Obviously the state of our particle after the measurement is $\left|\beta_1\right\rangle$. If we repeat the measurement of A and B, so many times as we like, every time we will find for A and B the values a_1 and β_1 respectively. Consequently the state of the particle (or system) is a well defined state. A simple way to write this state is writing it as follows:

$$\left|\Psi_1\right\rangle = \left|a_1, \beta_1\right\rangle$$

That means that, if we measure the A we will find the value a_1 and if we measure the observable B we will find the value β_1.
From the eigenvalues of A and B we define the following four states :

$$|\Psi_1\rangle = |a_1, \beta_1\rangle$$
$$|\Psi_2\rangle = |a_1, \beta_2\rangle$$
$$|\Psi_3\rangle = |a_2, \beta_1\rangle$$
$$|\Psi_4\rangle = |a_2, \beta_2\rangle$$

For example the state $|\Psi_3\rangle$ is the state whose the results of the measurements of the observables A and B are a_2 and β_1 respectively and so on.

We put the question **"What is mathematically the new state?"** A new Hilbert space H is defined. It is called the tensor product of the spaces H_1 and H_2 and it is denoted as

$$H = H_1 \otimes H_2$$

A basis of the new space consists of the vectors

$$|\Psi_1\rangle = |a_1\rangle \otimes |\beta_1\rangle$$
$$|\Psi_2\rangle = |a_1\rangle \otimes |\beta_2\rangle$$
$$|\Psi_3\rangle = |a_2\rangle \otimes |\beta_1\rangle$$
$$|\Psi_4\rangle = |a_2\rangle \otimes |\beta_2\rangle$$

The state $|\Psi\rangle$ of the particle (or system) is now a vector in the Hilbert space H.

$$|\Psi\rangle = c_1|a_1\rangle \otimes |\beta_1\rangle + c_2|a_1\rangle \otimes |\beta_2\rangle + c_3|a_2\rangle \otimes |\beta_1\rangle + c_4|a_2\rangle \otimes |\beta_2\rangle$$

where $|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = 1$

The probability of finding the values a_1, β_1 is $|c_1|^2$, to find the values a_1, β_2 is $|c_2|^2$ and so on.
If we ask for the probability to find the value β_1 for the observable B the answer is

$$P(\beta_1) = |c_1|^2 + |c_3|^2$$

Let us have measured the observable B and the result is β_1. If after that we measure the A what is the probability to find the value a_2? The answer is

$$P\left(\frac{a_2}{\beta_1}\right) = \frac{|c_3|^2}{|c_1|^2 + |c_3|^2}$$

4.2 Operators in New Space

Every observable A which is related with the Hilbert space H_1 corresponds to an operator

$$\hat{A}_H = \hat{A} \otimes \hat{I}$$

and it acts as follows:

$$\hat{A}_H\left(|a_i\rangle \otimes |\beta_j\rangle\right) = \left(\hat{A} \otimes \hat{I}\right)\cdot\left(|a_2\rangle \otimes |\beta_j\rangle\right) = \left(\hat{A}|a_i\rangle\right)\otimes\left(\hat{I}\cdot|\beta_j\rangle\right)$$

Similarly every operator $\hat{B}$ which is related with the Hilbert space H_2 takes the form

$$\hat{B}_H = \hat{I} \otimes \hat{B}$$

and it acts as follows:

$$\hat{B}_H\left(|a_i\rangle \otimes |\beta_j\rangle\right) = \left(\hat{I} \otimes \hat{B}\right)\cdot\left(|a_i\rangle \otimes |\beta_j\rangle\right) = \left(\hat{I}|a_i\rangle\right)\otimes\left(\hat{B}\cdot|\beta_j\rangle\right)$$

The inner product is defined as

$$\left[\left(\langle a_i| \otimes \langle \beta_j|\right)\cdot\left(|a_\kappa\rangle \otimes |\beta_\lambda\rangle\right)\right]_H = $$
$$= \langle a_i|a_\kappa\rangle_{H_1} \cdot \langle \beta_j|\beta_\lambda\rangle_{H_2}$$

Where the $\langle a_i|a_\kappa\rangle_{H_1}$ and $\langle \beta_j|\beta_\lambda\rangle_{H_2}$ are the inner products as they are defined in H_1 and H_2 respectively. From this definition we find that

$$\langle\Psi_1|\Psi_1\rangle = \langle\Psi_2|\Psi_2\rangle = \langle\Psi_3|\Psi_3\rangle = \langle\Psi_4|\Psi_4\rangle = 1$$

and

$$\langle\Psi_1|\Psi_2\rangle = \langle\Psi_1|\Psi_3\rangle = \langle\Psi_1|\Psi_4\rangle = \langle\Psi_2|\Psi_3\rangle = ... = 0 .$$

That is the basis $|\Psi_1\rangle$, $|\Psi_2\rangle$, $|\Psi_3\rangle$, $|\Psi_4\rangle$ is an orthonormal basis of the Hilbert space H.

The expectation value of any observable A is calculated through the relation

$$\langle A\rangle = \langle\Psi|\hat{A}_H|\Psi\rangle$$

where $|\Psi\rangle$ is a vector in the Hilbert space $H = H_1 \otimes H_2$ and is also the state of the system.

Some important remarks

Remark I

Let us assume that the state of the system in the Hilbert space H_1 is $|\Theta\rangle = \kappa_1|a_1\rangle + \kappa_2|a_2\rangle$ and in the Hilbert space H_2 is $|\Phi\rangle = \lambda_1|\beta_1\rangle + \lambda_2|\beta_2\rangle$.

Then the probability measuring the observable A to find the value a_1 is $|\kappa_1|^2$ and to find the value a_2 is $|\kappa_2|^2$. Also the probability measuring the observable B is $|\lambda_1|^2$ and $|\lambda_2|^2$ to find the values β_1 and β_2 respectively. Measuring simultaneously the observables A and B we find the values.

a_1, β_1	with probability	$|\kappa_1|^2 |\lambda_1|^2 = |\kappa_1\lambda_1|^2$
a_1, β_2	$\Rightarrow$	$|\kappa_1|^2 |\lambda_2|^2 = |\kappa_1\lambda_2|^2$
a_2, β_1	$\Rightarrow$	$|\kappa_2|^2 |\lambda_1|^2 = |\kappa_2\lambda_1|^2$
a_2, β_2	$\Rightarrow$	$|\kappa_2|^2 |\lambda_2|^2 = |\kappa_2\lambda_2|^2$

Consequently we can conclude that for the tensor product the distributive principle it is valid.

$$\begin{aligned}
&|\Psi\rangle = |\Theta\rangle \otimes |\Phi\rangle = \\
&= \left[\kappa_1|a_1\rangle + \kappa_2|a_2\rangle + \right] \otimes \left[\lambda_1|\beta_1\rangle + \lambda_2|\beta_2\rangle\right] \\
&= \kappa_1\lambda_1|a_1\rangle \otimes |\beta_1\rangle + \kappa_1\lambda_2|a_1\rangle \otimes |\beta_2\rangle \\
&+ \kappa_2\lambda_1|a_2\rangle \otimes |\beta_1\rangle + \kappa_2\lambda_2|a_2\rangle \otimes |\beta_2\rangle .
\end{aligned}$$

From the last relation the corresponding probabilities for any combinated result can be explained.

Remark II:

From the preceding remark and from the way through we constructed the tensor product someone could be misleaded to think that for any vector $|\Psi\rangle$ in the Hilbert space $H = H_1 \otimes H_2$ we can find $|\Theta\rangle$ and $|\Phi\rangle$ from H_1 and H_2 respectively such as

$$|\Psi\rangle = |\Theta\rangle \otimes |\Phi\rangle .$$

This is not true. For example the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|a_1\rangle \otimes |\beta_2\rangle + \frac{1}{\sqrt{2}}|a_2\rangle \otimes |\beta_1\rangle$$

is not written in the form $|\Psi\rangle = |\Theta\rangle \otimes |\Phi\rangle$.

That implies that the tensor product is something more than a simple cartesian product. We will see that later and we will try to explain more on it.

4.3 Hilbert Space of Two Particles with Spin $\frac{1}{2}$

An useful example of tensor product is the Hilbert space of two particles where each one has spin $\frac{1}{2}$.Let us consider two particles, the particle A and the particle B. Every one is particle with spin $\frac{1}{2}$. The Hilbert space related to particle A is the space H_A and one basis of it are the eigenvectors of the observable $S_{Z(\mathrm{A})}$, this is the set $\left\{\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A},\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\right\}$. Similarly the Hilbert space related to particle B is the space H_B which is generated by the set $\left\{\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B},\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}\right\}$, namely the eigenvectors of observable $S_{Z(\mathrm{B})}$. The Hilbert space for the system of two particles A and B is the tensor product

$$\mathrm{H}=\mathrm{H}_\mathrm{A}\otimes\mathrm{H}_\mathrm{B}$$

and one basis is the following set.

$$\left|\Psi_1\right\rangle=\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A}\otimes\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B}=\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B}$$
$$\left|\Psi_2\right\rangle=\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A}\otimes\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}=\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}$$
$$\left|\Psi_3\right\rangle=\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\otimes\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B}=\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B}$$
$$\left|\Psi_4\right\rangle=\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\otimes\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}=\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}$$

Usually for simplicity we drop out the symbol $\otimes$.

Any state $\left|\Psi\right\rangle$ of H is written as a superposition of $\left|\Psi_1\right\rangle$, $\left|\Psi_2\right\rangle$, $\left|\Psi_3\right\rangle$ and $\left|\Psi_4\right\rangle$.

$$\left|\Psi\right\rangle=c_1\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B}+c_2\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}+c_3\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B}+c_4\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}$$

5. ENTANGLEMENT

5.1 Some Questions on Tensor Products

We tried to introduce the concept of tensor product beginning from the simple case of systems (or particles) where the corresponding Hilbert space of each of them is two dimensional. For simplicity, for the rest, we shall restrict the discussion to the system which consists of two particles A and B with spin $\frac{1}{2}$.

The A particle is described by a Hilbert space H_A where a basis is the set $\left\{\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{A},\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{A}\right\}$ and the B particle is described by a Hilbert space H_B with corresponding basis the set $\left\{\left|\boldsymbol{z}\uparrow\right\rangle_\mathrm{B},\left|\boldsymbol{z}\downarrow\right\rangle_\mathrm{B}\right\}$.

We constructed the Hilbert space H as the tensor product of H_A and H_B, namely

$$H = H_A \otimes H_B$$

and we said that this describes the system of the particles A and B. A basis of H consists of the four vectors

$$\{|z\uparrow\rangle_A|z\uparrow\rangle_B, |z\uparrow\rangle_A|z\downarrow\rangle_B, |z\downarrow\rangle_A|z\uparrow\rangle_B, |z\downarrow\rangle_A|z\downarrow\rangle_B\}$$

(for simplicity we drop out the symbol $\otimes$).

Let the particle A be in state $|\Theta_0\rangle_A$, a vector in the Hilbert space H_A and the particle B be in state $|\Phi_0\rangle_B$, a vector in the Hilbert space H_B. Then the state of system A, B is the vector

$$|\Psi\rangle_{in} = |\Theta_0\rangle_A \otimes |\Phi_0\rangle_B$$

which belongs to Hilbert space H.

If we perform an experiment with the A particle in order to measure an observable which is related to Hilbert space H_A, after the experiment the state of the particle A is a vector in the H_A and it can be written as

$$|\Theta\rangle_A = c_1|z\uparrow\rangle_A + c_2|z\downarrow\rangle_A$$

Similarly performing an experiment on the particle B its state will be a vector $|\Phi\rangle_B$ in the H_B where

$$|\Phi\rangle_B = d_1|z\uparrow\rangle_B + d_2|z\downarrow\rangle_B$$

Consequently the new state of the system A, B is

$$|\Psi\rangle_{AB} = |\Theta\rangle_A \otimes |\Phi\rangle_B =$$

$$= c_1 d_1|z\uparrow\rangle_A|z\uparrow\rangle_B + c_1 d_2|z\uparrow\rangle_A|z\downarrow\rangle_B$$
$$+ c_2 d_1|z\downarrow\rangle_A|z\uparrow\rangle_B + c_2 d_2|z\downarrow\rangle_A|z\downarrow\rangle_B.$$

Let us call these actions which are performed separately on A and B local performances. We observe that local performances lead to states which can be written in the form $|\Psi\rangle_{AB} = |\Theta\rangle_A \otimes |\Phi\rangle_B$ if the initial state was in the same form $|\Psi\rangle_{in} = |\Theta_0\rangle_A \otimes |\Phi_0\rangle_B$ (see fig 7).

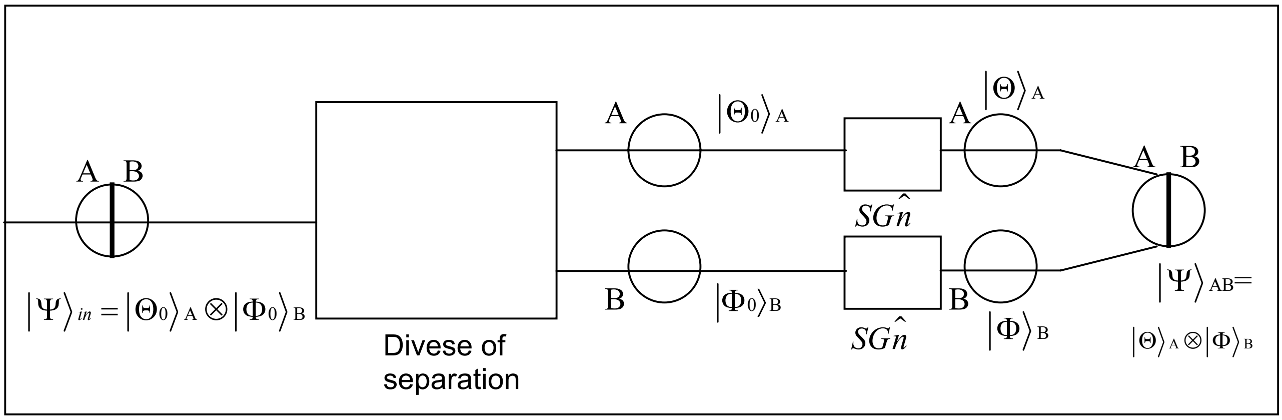


fig 7

However the new space $H = H_A \otimes H_B$ which is supposed to describe our system $A\ B$, contains also states which can not be written in the form $|\Theta\rangle_A \otimes |\Phi\rangle_B$, for example the vector

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\mathbf{z}\uparrow\rangle_A|\mathbf{z}\downarrow\rangle_B - |\mathbf{z}\downarrow\rangle_A|\mathbf{z}\uparrow\rangle_B\right).$$

Obviously we must explain the physical meaning of these states. Some one could assert the following: <<We have the pair AB. The A is a particle which is described by H_A, therefore its state is a vector of H_A. Similarly the state of the particle B is a vector in the H_B. Consequently the state of pair AB is a vector in the Hilbert space $H = H_A \otimes H_B$ and it must be in the form $|\Theta\rangle_A \otimes |\Phi\rangle_B$. Therefore vectors of the form like $|\Psi^-\rangle$ are simply mathematical constructions without physical significance.>> Is it true? We have put a question and we will try to answer it.

5.2 Definition of Entanglement

We consider two particles A, B each of them with spin $\frac{1}{2}$ and in addition, we suppose that these are very near to each other so that they form a pair because of a strong attractive

force between them which is independent of the spin. A large number N of these pairs pass through a SG_z apparatus. We suppose that the attractive force is strong enough such so that the particles A and B are not separated and emerge from the SG_z apparatus as a pair.
Then the results are the followings (see fig 8):

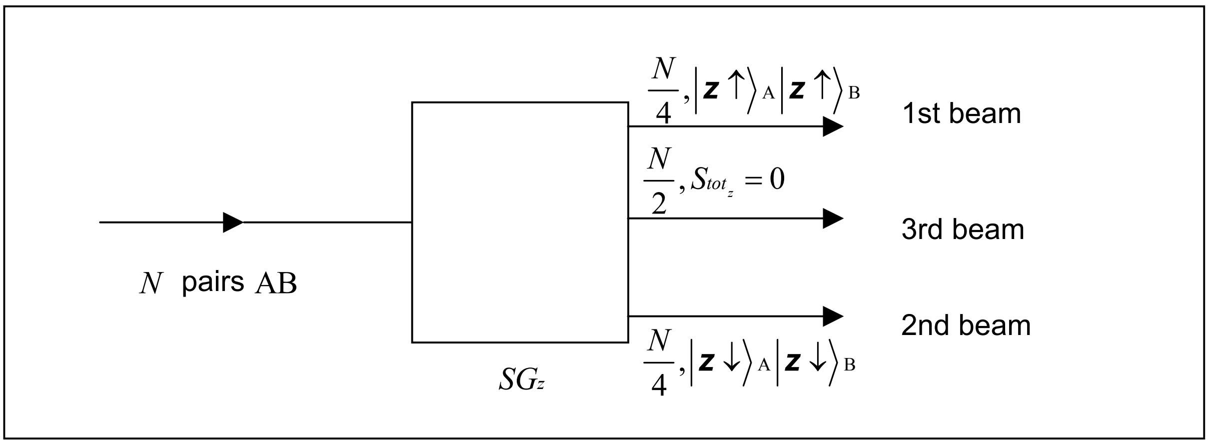


fig 8

1st beam: $\frac{N}{4}$ pairs outcome with $S_{tot_z} = +\hbar$ (upper path) and are in state $|\mathbf{z}\uparrow\rangle_A |\mathbf{z}\uparrow\rangle_B$.

2nd beam: $\frac{N}{4}$ pairs outcome with $S_{tot_z} = -\hbar$ (lower path) and are in state $|\mathbf{z}\downarrow\rangle_A |\mathbf{z}\downarrow\rangle_B$.

3rd beam: $\frac{N}{2}$ pairs pass without deflection (middle path). They have $S_{tot_{(z)}} = 0$.

Let us examine each beam separately.

<u>1st beam:</u> Each pair is in state $|\mathbf{z}\uparrow\rangle_A |\mathbf{z}\uparrow\rangle_B$ which is a vector in the Hilbert space $H = H_A \otimes H_B$.
Therefore the ensemble of pairs is a pure ensemble. If we pass the beam through a device <u>D of separation</u> which does not disturb the spin but it separates the pair into particle A and B counteracting the attractive force, we take all the particles A in state $|\mathbf{z}\uparrow\rangle_A$ and all the particles B in state $|\mathbf{z}\uparrow\rangle_B$ (see fig 9)

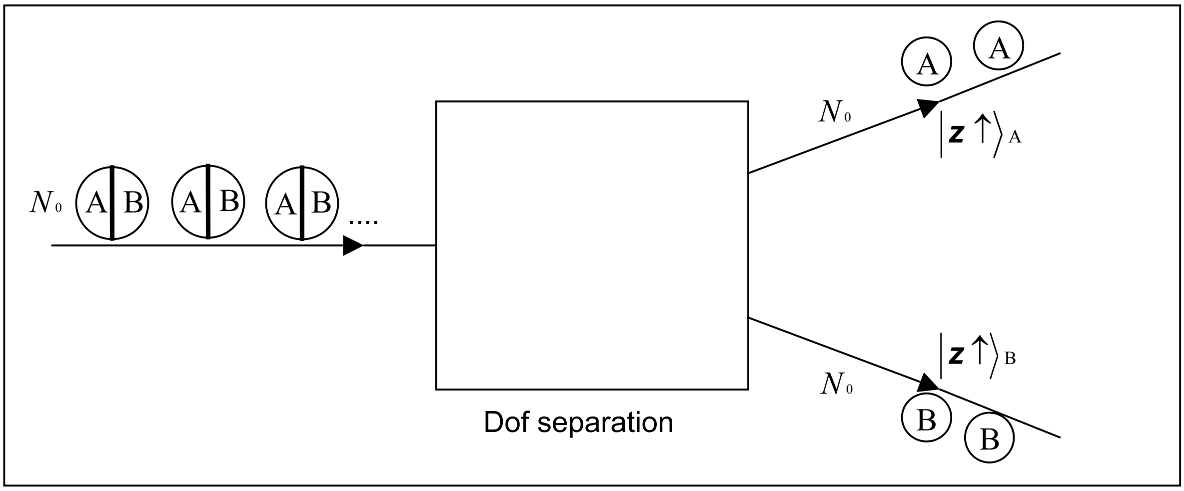


fig 9

That is examining separately the ensembles $\{A\}$ and $\{B\}$ we observe that both of them are pure ensembles.

<u>2nd beam:</u> The ensemble of pairs is also a pure ensemble and separating the pairs into the particle A and B passing the beam through a <<D of separation>> outcome two new pure subensembles. All the particles A are in state $|\mathbf{z}\downarrow\rangle_A$ and all the particles B are in state $|\mathbf{z}\downarrow\rangle_B$. Generally if the state of the pair has the form

$$|\Psi\rangle = |\Theta\rangle_A \otimes |\Phi\rangle_B ,$$

after the separation all the particles A are in the vector state $|\Theta\rangle_A$ in the Hilbert space H_A and all the particles B are in the state $|\Phi\rangle_B$, vector in the Hilbert space H_B. Consequently both subensembles $\{A\}$ and $\{B\}$ are pure ensembles.

3rd beam: For this beam we know that all pairs have $S_{tot_{(Z)}} = 0$. Obviously it is a vector belonging to the Hilbert space where a basis is the set of the vectors $|S_{tot_{(z)}} = -\hbar\rangle, |S_{tot_{(z)}} = 0\rangle$ and $|S_{tot_{(z)}} = +\hbar\rangle$. This is a space corresponding exclusively to observable $S_{tot_{(z)}}$. However the states of all pairs of this beam are not in the same vector state in the Hilbert space $H_A \otimes H_B$. It is due to the fact that the states $|\mathbf{z}\uparrow\rangle_A |\mathbf{z}\downarrow\rangle_B$, $|\mathbf{z}\downarrow\rangle_A |\mathbf{z}\uparrow\rangle_B$ and any linear combination $|\Psi\rangle = c_1|\mathbf{z}\uparrow\rangle_A |\mathbf{z}\downarrow\rangle_B + c_2|\mathbf{z}\downarrow\rangle_A |\mathbf{z}\uparrow\rangle_B$ have $S_{tot_{(z)}} = 0$ (this effect is called degeneracy). Consequently this beam is not a pure ensemble, it is a mixed one if we examine it in the frame of Hilbert space $H = H_A \otimes H_B$.

Let us pass the 3rd beam through an apparatus which measures the observable

$$S^2{}_{tot} = S^2{}_{tot_{(x)}} + S^2{}_{tot_{(y)}} + S^2{}_{tot_{(z)}}$$

where $S_{tot_{(x)}} = S_{A(x)} + S_{B(x)}$

$S_{tot_{(y)}} = S_{A(y)} + S_{B(y)}$

and $S_{tot_{(z)}} = S_{A(z)} + S_{B(z)}$

Then it is separated into two beams, the a' one with $S^2 = 2\hbar^2$ and β' one with $S^2 = 0\hbar^2$ (see fig 10).

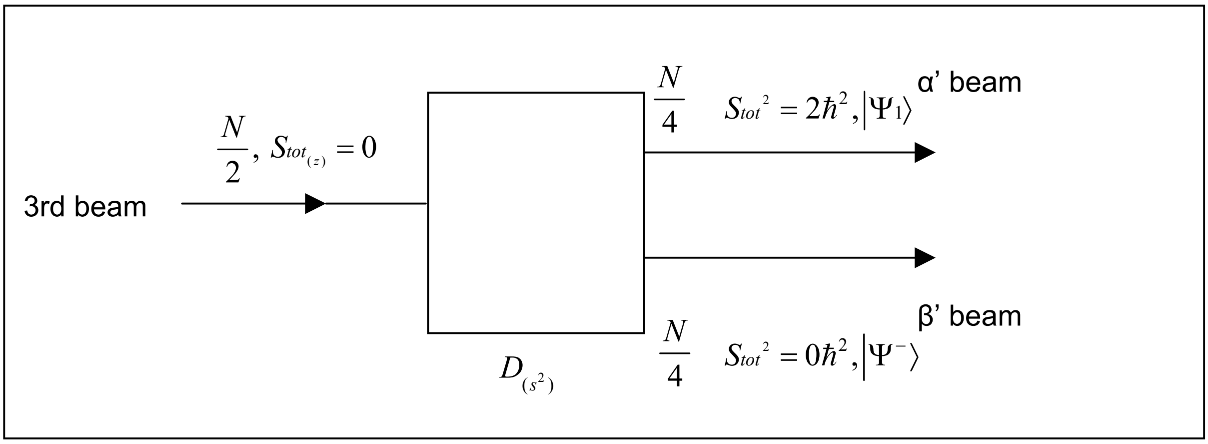


fig 10

α′ beam: It is proved that a' beam is a pure ensemble and the state of each pair is a vector

$|\Psi^+\rangle$ belonging to the Hilbert space $H = H_A \otimes H_B$ where

$$|\Psi^+\rangle = |S^2{}_{tot} = 2\hbar^2, S_{tot_{(z)}} = 0\rangle$$

Trying to write the vector $\left|\Psi^{+}\right\rangle$ as a linear compination we find that

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\downarrow\right\rangle_{\mathrm{B}} + \left|\mathbf{z}\downarrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{B}}\right)$$

Really for this form it is valid

$$\hat{S}_{tot_{(z)}}\left|\Psi^{+}\right\rangle = 0\left|\Psi^{+}\right\rangle$$

and $\hat{S}^{2}_{tot}\left|\Psi^{+}\right\rangle = 2\hbar^{2}\left|\Psi^{+}\right\rangle$

where $\hat{S}_{tot_{(z)}}$ and $\hat{S}^{2}_{t0t}$ the corresponding operators of the observables $S_{tot_{(z)}}$ and S^{2}_{tot}.

<u>*β′* beam</u>: This beam is also a pure ensemble and the state of any pair is the vector $\left|\Psi^{-}\right\rangle$ in the H. Its linear combination is

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\downarrow\right\rangle_{\mathrm{B}} - \left|\mathbf{z}\downarrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{B}}\right)$$

We can also prove that

$$\hat{S}_{tot_{(z)}}\left|\Psi^{-}\right\rangle = 0\left|\Psi^{-}\right\rangle$$

and $\hat{S}^{2}_{tot}\left|\Psi^{-}\right\rangle = 0\left|\Psi^{-}\right\rangle$

Therefore $\left|\Psi^{-}\right\rangle = \left|S^{2}_{tot} = 0, S_{tot_{(z)}} = 0\right\rangle$.

For both beam a and β the corresponding states $\left|\Psi^{+}\right\rangle$ and $\left|\Psi^{-}\right\rangle$ can not be written as $\left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$, however they are well defined states and have physical meaning.

Definition of entanglement

Entanglements are the vector states in the Hilbert space $\mathrm{H} = \mathrm{H_A} \otimes \mathrm{H_B}$ which can not be written in the form $\left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$. Namely

- If $\left|\Psi\right\rangle$ is entangled $\Leftrightarrow \nexists \left|\Theta\right\rangle_{\mathrm{A}} \in \mathrm{H_A}$ and $\left|\Phi\right\rangle_{\mathrm{B}} \in \mathrm{H_B}$ such that $\left|\Psi\right\rangle = \left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$.
- If $\left|\Psi\right\rangle$ is not entangled $\Leftrightarrow \exists \left|\Theta\right\rangle_{\mathrm{A}} \in \mathrm{H_A}$ and $\left|\Phi\right\rangle_{\mathrm{B}} \in \mathrm{H_B}$ such that $\left|\Psi\right\rangle = \left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$.

How can we have an entanglement? If we have two particles A and B where the A is in state $|\Theta\rangle_A$ and the B is in state $|\Phi\rangle_B$ then in order to the state $|\Theta\rangle_A \otimes |\Phi\rangle_B$ end in an entangled state, two particles must come near each other and interact. We cannot produce an entanglement performing any local performance on A and B.

<u>Conclusions:</u>

a) We must abandon our assertion that any particle A (or B) is in any case in a state vector in the Hilbert space H_A (or H_B). We found that states of the form $|\Psi\rangle \neq |\Theta\rangle_A |\Phi\rangle_B$ correspond to realizable situations.

β) We have two kinds of vectors in tensor product space

1st kind: not entangled states, $|\Psi\rangle = |\Theta\rangle_A |\Phi\rangle_B$
2nd kind: entangled states, $|\Psi\rangle \neq |\Theta\rangle_A |\Phi\rangle_B$

γ) To have entangled states particles must interact each other.

5.3 Maximal Entanglements

The states

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\boldsymbol{z}\uparrow\rangle_A |\boldsymbol{z}\uparrow\rangle_B \pm |\boldsymbol{z}\downarrow\rangle_A |\boldsymbol{z}\downarrow\rangle_B\right)$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\boldsymbol{z}\uparrow\rangle_A |\boldsymbol{z}\downarrow\rangle_B \pm |\boldsymbol{z}\downarrow\rangle_A |\boldsymbol{z}\uparrow\rangle_B\right)$$

constitute an orthonormal basis of Hilbert space $H = H_A \otimes H_B$ and are usually called Bell's states. Bell's states have the property that they keep their form invariant if we replace the $\hat{\boldsymbol{z}}$ direction by any arbitrary $\hat{\boldsymbol{n}}$ one. That is

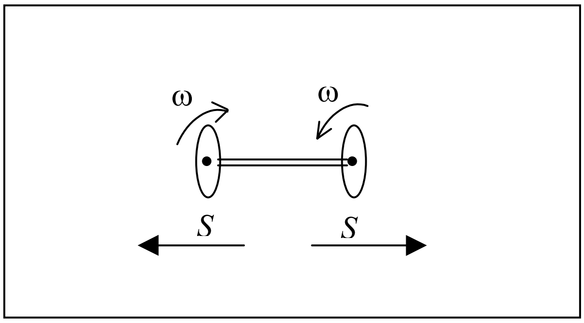
$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\boldsymbol{z}\uparrow\rangle_A |\boldsymbol{z}\uparrow\rangle_B \pm |\boldsymbol{z}\downarrow\rangle_A |\boldsymbol{z}\downarrow\rangle_B\right) = \frac{1}{\sqrt{2}}\left(|\boldsymbol{n}\uparrow\rangle_A |\boldsymbol{n}\uparrow\rangle_B \pm |\boldsymbol{n}\downarrow\rangle_A |\boldsymbol{n}\downarrow\rangle_B\right)$$

and $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\boldsymbol{z}\uparrow\rangle_A |\boldsymbol{z}\downarrow\rangle_B \pm |\boldsymbol{z}\downarrow\rangle_A |\boldsymbol{z}\uparrow\rangle_B\right) = \frac{1}{\sqrt{2}}\left(|\boldsymbol{n}\uparrow\rangle_A |\boldsymbol{n}\downarrow\rangle_B \pm |\boldsymbol{n}\downarrow\rangle_A |\boldsymbol{n}\uparrow\rangle_B\right)$

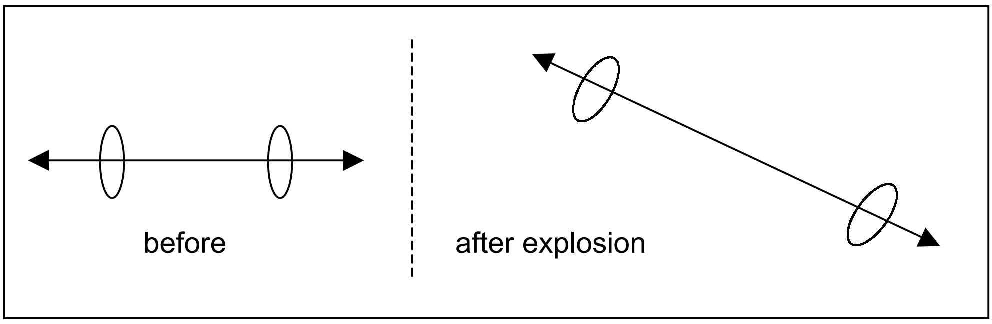
We can easily prove the above property starting from the right hand side part and replacing the $|\boldsymbol{n}\downarrow\rangle$ by a linear combination of $|\boldsymbol{z}\uparrow\rangle$ and $|\boldsymbol{z}\downarrow\rangle$.

Especially the state $|\Psi^{-}\rangle$ is very important and we will work with it at the rest of this chapter. This state $|\Psi^{-}\rangle$ is realizable for example by the decay of a spin 0 particle into two particles spin $\frac{1}{2}$ under conservation of the internal angular momentum. Then the two spins of the emerging particles are opposite and the pair is described by $\left|\Psi^-\right\rangle$.

The classical analogon with this state is the system of a pair of two similar disks which have the same axle and are rotating in the opposite direction with the same angular velocity ω. The total angular momentum is $S_{tot} = S - S = 0$ and their component in any direction are equal and opposite. For this system any direction of S is allowed and is a solution of the problem.



Also, if an explosion takes place in the middle of this system and the two disks separate then their spins S remain opposite.



5.4 Pure and Mixed State of a Particle

Let a pair AB be in state

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{z}\uparrow\right\rangle_A\left|\boldsymbol{z}\downarrow\right\rangle_B - \left|\boldsymbol{z}\downarrow\right\rangle_A\left|\boldsymbol{z}\uparrow\right\rangle_B\right)$$

which is a vector in the Hilbert space $H = H_A \otimes H_B$.

We separate the pair into particle A and particle B without disturbing the spins (see fig 11)

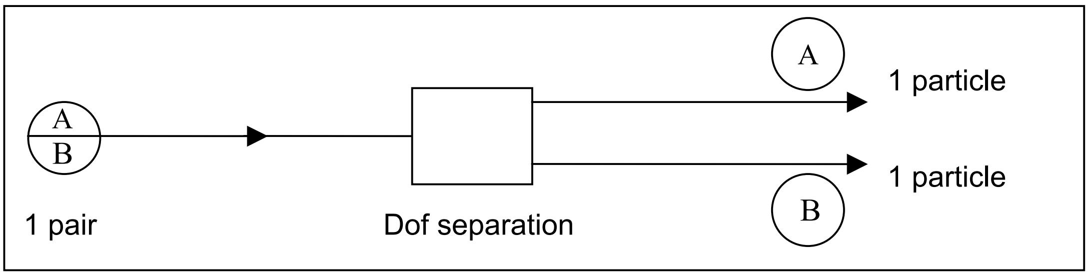


fig 11
Now we have the particle A and the particle B.

Question:

Is the particle A (or B) in some state which is a vector in the Hilbert space H_A (or H_B)?

The answer is unfortunately NO! The assertion that one particle is in any case in a state, vector in the Hilbert space is equivalent to locality and is not true!!

Proof:

We know that the way to write the $|\Psi^-\rangle$ is not unique. Let us decide to measure the component $S_{\hat{n}(A)}$ of the particle A and we find the value $+\frac{\hbar}{2}$. The corresponding projection operator of this measurement is

$$\hat{P}_{A\hat{n}\uparrow} = |\boldsymbol{n}\uparrow\rangle_{A\,A}\langle\boldsymbol{n}\uparrow| \otimes \hat{I}_B$$

Therefore the system end in state $|\Psi\rangle$ where

$$\hat{P}_{A\hat{n}\uparrow}|\Psi\rangle = |\boldsymbol{n}\uparrow\rangle_{A\,A}\langle\boldsymbol{n}\uparrow| \otimes \hat{I}_B\left[\frac{1}{\sqrt{2}}\left(|\boldsymbol{n}\uparrow\rangle_A|\boldsymbol{n}\downarrow\rangle_B - |\boldsymbol{n}\downarrow\rangle_A|\boldsymbol{n}\uparrow\rangle_B\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left(|\boldsymbol{n}\uparrow\rangle_A|\boldsymbol{n}\downarrow\rangle_B\right) \xrightarrow{\text{normalitation}} |\Psi'\rangle = |\boldsymbol{n}\uparrow\rangle_A|\boldsymbol{n}\downarrow\rangle_B\,.$$

Consequently the states of the particle A and the particle B are the vectors $|\boldsymbol{n}\uparrow\rangle_A$ in the space H_A and $|\boldsymbol{n}\downarrow\rangle_B$ in the space H_B respectively.

However as we were measuring the observable $S_{\hat{n}(A)}$ concerning the particle A we did not disturb the particle B. Therefore some one could assert that the state of particle B was also the vector $|\boldsymbol{n}\downarrow\rangle_B$ in the H_B before the measurement. But all measurements show that the particle A and the particle B have antiparallel spins. So if before the measurement they were in states which were vectors in the H_A and H_B respectively their states must have been the vectors $|\boldsymbol{n}\uparrow\rangle_A$ and $|\boldsymbol{n}\downarrow\rangle_B$. Consequently if we decided to measure on some direction $\hat{\boldsymbol{m}} \neq \hat{\boldsymbol{n}}$ there were possibility to find parallel spins because

$$\left|\left({}_A\langle\boldsymbol{m}\uparrow|_B\langle\boldsymbol{m}\uparrow|\right)\cdot\left(|\boldsymbol{n}\uparrow\rangle_A|\boldsymbol{n}\downarrow\rangle_B\right)\right|^2 \neq 0$$

However experiments show that this probability is zero. Consequently neither particle A nor particle B were in states, which are vectors in the H_A or H_B respectively.

Definition:

If a particle is not in a state which is a vector in the Hilbert space then this state is called mixed state or incoherent state. In the opposite case, when the state is a vector of Hilbert space it is called pure state or coherent state.

Remarks:

a) The entangled states are vectors which are coherent states, vectors in the Hilbert space $\mathrm{H} = \mathrm{H}_{\mathrm{A}} \otimes \mathrm{H}_{\mathrm{B}}$. However each of particle A and B is in an incoherent state.

β) A mixed ensemble can consist of particles whose each one is in a pure state, but not all in the same one. However we saw that a mixed ensemble can also consist of

particles whose each one is in incoherent state. There is no mechanism to distinguish the two cases and there is no need.

5.5 Faster than Light?

Let us assume that we have a pure ensemble of N pairs in state

$$|\Psi^{-}\rangle = \frac{1}{\sqrt{2}}\left(|\boldsymbol{n}\uparrow\rangle_{\mathrm{A}}|\boldsymbol{n}\downarrow\rangle_{\mathrm{B}} - |\boldsymbol{n}\downarrow\rangle_{\mathrm{A}}|\boldsymbol{n}\uparrow\rangle_{\mathrm{B}}\right).$$

We separate each pair and give the particles A to Alice who lives on earth and particles B to Bob who lives on some star of Andromeda galaxy. Let Alice want to send a message to Bob. She thinks this trick. She measures the first $\frac{N}{k}$ particles A in direction $\hat{\boldsymbol{x}}$, and so the first $\frac{N}{k}$ particles B are prepared in a mixed ensemble E_1 where the half particles are in state $|\mathbf{x}\uparrow\rangle_{\mathrm{B}}$ and the remaining half in state $|\mathbf{x}\downarrow\rangle_{\mathrm{B}}$.

Then Alice measures the next $\frac{N}{k}$ particles in direction $\hat{\boldsymbol{z}}$. Consequently the next $\frac{N}{k}$ particles B are prepared in a mixed ensemble E_2 where the half particles are in state $|\boldsymbol{z}\uparrow\rangle_{\mathrm{B}}$ and the remaining half in state $|\boldsymbol{z}\downarrow\rangle_{\mathrm{B}}$ …. and so on.

Some one could think that Alice has sent a message to Bob faster than light. It would be true only if Bob could distinguish the ensemble E_1 from ensemble E_2. But we know that it is impossible because the two ensembles are indistinguishable and therefore the message is unreadable.

5.6 Einstein Locality

As we saw quantum mechanics is an indeterministic theory. The complete knowledge of the state and the knowledge of the time evolution of it does not assure the complete prediction of the result of a measurement. We know only the probability to find a result. On the other side in classical physics, if we know the initial conditions and the dynamics of a system we can predict the result of any measurement with accuracy. The indeterministic features of quantum mechanics annoyed and annoy many people. Also the following sentence is very interesting.

<<Let two particles A and B constitute a system. Suppose that we separate them and move them along distance apart such as there is no any interaction between them. Then in a complete description of physical reality any action performed on A must not modify the description of particle B>>.

This criterion is known as Einstein locality. Einstein considered that a theory is complete and describes the physical reality only if it satisfies this criterion. Quantum mechanics as we saw in case of entanglements does not satisfy it. Einstein believed that there are and others hidden variables which have not been controlled with present-day experimental technique. If we achieved to control them, quantum mechanics would become a deterministic theory. More particularly, when a particle is prepared in a state $\left|\boldsymbol{z}\uparrow\right\rangle$, in reality it is prepared in state $\left|\boldsymbol{z}\uparrow,\lambda\right\rangle$ where $0\le\lambda\le1$ and λ takes any value of the interval $\left[0,1\right]$ with the same probability.

Now suppose that we measure the component $S_{\hat{n}}$ where direction $\hat{\boldsymbol{n}}$ forms an angle ϑ with axis $\hat{\boldsymbol{z}}$.

Then the outcome will be

$$\left|\boldsymbol{n}\uparrow\right\rangle \qquad \text{for} \qquad 0\le\lambda\le\cos^2\frac{\vartheta}{2}$$

$$\left|\boldsymbol{n}\downarrow\right\rangle \qquad \text{for} \qquad \cos^2\frac{\vartheta}{2}\le\lambda\le1$$

If we known the value of λ we could predict exactly the result. But λ is completely unknown and so the probability to find the state $\left|\boldsymbol{n}\uparrow\right\rangle$ or $\left|\boldsymbol{n}\downarrow\right\rangle$ agrees with the predictions of quantum mechanics.

A class of theories with hidden variables which satisfy Einstein locality are called local hidden variables theories. They are deterministic theories but the ignorance of hidden variables leads to quantum mechanics. The validity of local theories can be tested by Bell inequalities. Experiments have shown that these theories are not correct.

SUMMARY

1. Pure ensemble is an ensemble where all its members are in the same state $|\Psi\rangle$. The mean value of an observable A is given through the relation

$$\langle A\rangle = \langle\Psi|\hat{A}|\Psi\rangle$$

2. When the ratio p_1 of particles are in state $|\Psi_1\rangle$, the ratio p_2 in state $|\Psi_2\rangle$ and so on then the ensemble is a mixed ensemble and the expectation mean value of an observable A is given through the relation

$$\langle A\rangle = p_1\langle\Psi_1|A|\Psi_1\rangle + p_2\langle\Psi_2|A|\Psi_2\rangle + ..$$

3. Unpolarized beam is an ensemble with $\langle S_n\rangle = 0$ for any direction $\hat{\boldsymbol{n}}$.

4. If a particle A is described by a Hilbert space H_A and a B one by the H_B then the system A, B is described by the tensor product space $H = H_A \otimes H_B$.

5. If a vector $|\Psi\rangle$ in H space can not be written as $|\Psi\rangle = |\Theta\rangle_A \otimes |\Phi\rangle_B$ then it describes an entangled state.

6. Einstein supposed the existence of the hidden variables in order to make the Quantum mechanics a deterministic theory.

QUESTION-EXERCISES

43) Using the relations

$$\left|n\uparrow\right\rangle = \cos\frac{\vartheta}{2}\left|\boldsymbol{z}\uparrow\right\rangle + \sin\frac{\vartheta}{2}\left|\boldsymbol{z}\downarrow\right\rangle$$
$$\left|n\downarrow\right\rangle = \sin\frac{\vartheta}{2}\left|\boldsymbol{z}\uparrow\right\rangle - \cos\frac{\vartheta}{2}\left|\boldsymbol{z}\downarrow\right\rangle$$

find the relations which give the vectors $\left|\mathbf{x}\uparrow\right\rangle$ and $\left|\mathbf{x}\downarrow\right\rangle$ as a function of $\left|\boldsymbol{z}\uparrow\right\rangle$ and $\left|\boldsymbol{z}\downarrow\right\rangle$.

44) Prove the relation

$$\left\langle S_n\right\rangle = \frac{\hbar}{2}\cos\vartheta$$

by using for $\hat{S}n$ the following form

$$\hat{S}_n = \frac{\hbar}{2}\left|n\uparrow\right\rangle\left\langle n\uparrow\right| - \frac{\hbar}{2}\left|n\downarrow\right\rangle\left\langle n\downarrow\right|$$

45) A large number N of particles which are in the state $\left|\boldsymbol{z}\uparrow\right\rangle$, pass through a SG_n where $\hat{\boldsymbol{n}}$ belongs to plane xz and form an angle $\vartheta = 60^\circ$ with z-axis.

i) calculate the number of particles which will outcome out in state $\left|n\uparrow\right\rangle$ and the number in state $\left|n\downarrow\right\rangle$.

ii) Calculate the expectation value of the component S_n by two ways.

46) The quantum rule which gives the mean value is

i) $\left\langle S_n\right\rangle = \dfrac{N_1\frac{\hbar}{2} + N_2\left(-\frac{\hbar}{2}\right)}{N}$

ii) $\left\langle S_n\right\rangle = \left\langle \boldsymbol{z}\uparrow\right|\hat{S}_n\left|\boldsymbol{z}\uparrow\right\rangle$

iii) Both of them

iv) None of them

Choose the correct answer.

47) *Write T or F whether you think the following statements are true or false.*

For a pure ensemble.

i) All its members are in the same state $|\Psi\rangle$.
ii) All the members are in states which are vectors in the Hilbert space.
iii) In order to calculate of the mean value of an observable we use the quantum rules.
iv) Both statistical and quantum rules give the same results.

48) *Write T or F whether you think the statements are true or false.*

A large number N of particles pass through a device which measures an observable A. We find three values a_1, a_2 and a_3.

i) The particles N emerging from the device constitute a pure ensemble.
ii) If we select all particles with the value a_1, then we have a pure ensemble.
iii) The maximum number of pure ensembles that we can have is three.
iv) If we select the half particles with the value a_2 then they do not constitute a pure ensemble.
v) The particles with corresponding value a_3 are in the same state $|\Psi\rangle$.

49) For a mixed ensemble

i) All the particles have the same value a for some observable A.
ii) The particles are not in the same state $|\Psi\rangle$.
iii) For the calculation of the mean value of an observable we use only quantum rules.
iv) There is SG_n apparatus through which if we pass the mixed ensemble, all particles follow the same path.

Choose the correct answer.

50) The particles emerging from the oven in Stern Gerlach experiment

i) They constitute a mixed ensemble
ii) They constitute an unpolarized beam
iii) Passing through a SG_n they separate into two ensembles with the same number of members.
iv) All above are correct.

Choose the correct answer.

51) A mixed ensemble consists of 40 % particles in state $|\boldsymbol{z}\uparrow\rangle$ and 60 % in state $|\mathbf{x}\downarrow\rangle$. Calculate the quantities

i) $\langle S_z\rangle$ **ii)** $\langle S_x\rangle$ **iii)** $\langle S_y\rangle$

52) *Write T or F whether you think the statements are true or false.*

For an unpolarized beam:

i) The expectation value for any component S_n is zero.
ii) The expectation value for any observable is zero.
iii) It can be produced by several ways.
iv) It is a mixed ensemble.
v) We can distinguish if it is 50 % $|\boldsymbol{x}\uparrow\rangle$ and 50 % $|\boldsymbol{x}\downarrow\rangle$ or 50 % $|\boldsymbol{z}\uparrow\rangle$ and 50 % $|\boldsymbol{z}\downarrow\rangle$.

53) Comment the relation

$$\frac{1}{\sqrt{2}}|\mathbf{x}\uparrow\rangle+\frac{1}{\sqrt{2}}|\mathbf{x}\downarrow\rangle \neq 50\%|\mathbf{x}\uparrow\rangle+50\%|\mathbf{x}\downarrow\rangle$$

54) *Write T or F whether you think the statements are true or false.*

The Hilbert space H_1 is related to S_x, S_y and S_z.

i) The observable $A = 3S_x + S_z$ is related to H_1.

ii) The space H_1 describe the particle completely.

iii) The momentum p of the particle is related to H_1.

v) The eigenstates of the observable B are the states $|\boldsymbol{x}\uparrow\rangle$ and $|\boldsymbol{x}\downarrow\rangle$, that means vectors of H_1, consequently $B = f(S_x)$.

55) *Write T or F whether you think the statements are true or false.*

An observable B is unrelated to S_x, S_y and S_z and H_1 is the corresponding space which is connected to S_x, S_y and S_z.

i) The B is a function of S_x, S_y or S_z.
ii) The B is simultaneously measurable with S_x.
iii) There are eigenstates of B which belong to H_1.

iv) Measuring the B the state of the system in space H_1 does not change.

v) The B is related to a space which differs from H_1.

56) A basis of the space H_1 is the set $\{|a_1\rangle, |a_2\rangle\}$ and a basis of the space H_2 is the set $\{|\beta_1\rangle, |\beta_2\rangle, |\beta_3\rangle\}$. Find a basis of the space $H = H_1 \otimes H_2$.

57) A pair of particles A , B is in state $\left|\mathbf{x}\uparrow\right\rangle_{\mathrm{A}}\left|\mathbf{x}\uparrow\right\rangle_{\mathrm{B}}$.

i) Write the state as a linear combination of the vectors of the basis $\left\{\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{B}}, \left|\mathbf{z}\uparrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\downarrow\right\rangle_{\mathrm{B}}, \left|\mathbf{z}\downarrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{B}}, \left|\mathbf{z}\downarrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\downarrow\right\rangle_{\mathrm{B}}\right\}$.

ii) Calculate the expectation value of the observable $S_{tot(\mathrm{x})}$, where $S_{tot(\mathrm{x})} = S_{\mathrm{A(x)}} + S_{\mathrm{B(x)}}$

58) When we say local performance on a system of two particles we mean:

i) A measurement which is a performance on two particles at the same time.
ii) We move the two particles near each to other and so they interact.
iii) A measurement concerning only one particle.
iv) A measurement of an observable which is related to both particles.

Choose the correct answer.

59) *Write T or F whether you think the statements are true or false.*

The initial state of a system is $\left|\Psi\right\rangle_{in} = \left|\Theta_0\right\rangle_{\mathrm{A}} \otimes \left|\Phi_0\right\rangle_{\mathrm{B}}$.

After a local performance:

i) The final state can not be written as product $\left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$.
ii) The state is again a vector in the tensor product space.
iii) Maybe some of $\left|\Theta\right\rangle_{\mathrm{A}}$, $\left|\Phi\right\rangle_{\mathrm{B}}$ changes but the final state has the form $\left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$.
iv) Each particle is in a state which is a vector in the $\mathrm{H_A}$ and $\mathrm{H_B}$ space respectively.

60) Define the entangled state.

61) Why the state

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\downarrow\right\rangle_{\mathrm{B}} - \left|\mathbf{z}\downarrow\right\rangle_{\mathrm{A}}\left|\mathbf{z}\uparrow\right\rangle_{\mathrm{B}}\right)$$

is an entangled state?

62) If we choose the convenient bases of the space $\mathrm{H_1}$ and $\mathrm{H_2}$, could we write the state $\left|\Psi^-\right\rangle$ as a product $\left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$?

63) Which is the physical system that can be described by the state $\left|\Psi^-\right\rangle$?

64) Prove the relations

$$\hat{S}_{tot(z)}\left|\Psi^{+}\right\rangle = 0\left|\Psi^{+}\right\rangle$$
$$\hat{S}^{2}_{tot}\left|\Psi^{+}\right\rangle = 2\hbar^{2}\left|\Psi^{+}\right\rangle$$

65) Prove that the state

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{B}} - \left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{B}}\right)$$

can be written as

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{n}\uparrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{n}\downarrow\right\rangle_{\mathrm{B}} - \left|\boldsymbol{n}\downarrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{n}\uparrow\right\rangle_{\mathrm{B}}\right)$$

where $\hat{\boldsymbol{n}}$ is an arbitrary direction on the plane xz.

66) Show that the vectors $\left|\Psi^{+}\right\rangle$ and $\left|\Psi^{-}\right\rangle$ are orthogonal.

67) Show that for the states $\left|\Psi^{+}\right\rangle$ and $\left|\Psi^{-}\right\rangle$ it is valid $\left\langle S_{tot(n)}\right\rangle = 0$ for any $\hat{\boldsymbol{n}}$.

68) *Write T or F whether you think the following statements are true or false.*

Consider a pair of the particles A, B which is described by the state
$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{B}} - \left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{B}}\right)$$
We separate the pair and give the particle A to Alice and particle B to Bob. They go away so far that the particles do not interact any longer.

i) The particle A is in a state $\left|\Theta\right\rangle_{\mathrm{A}}$, vector in the space H_{A} which is a linear combination of $\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{A}}$ and $\left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{A}}$.

ii) Every particle A and B examining separately is in an incoherent state.

iii) Alice measures the $S_{\mathrm{A(y)}}$ and find the value $+\frac{\hbar}{2}$, then automatically the particle B <<goes>> to the state $\left|\boldsymbol{y}\downarrow\right\rangle_{\mathrm{B}}$ even though it was not disturbed by any way.

iv) The state $\left|\Psi^{-}\right\rangle$ is a well defined vector state in the space $\mathrm{H} = \mathrm{H}_{\mathrm{A}} \otimes \mathrm{H}_{\mathrm{B}}$.

v) Alice measured the $S_{\mathrm{A(x)}}$ and found the value $-\frac{\hbar}{2}$. After that if Bob measures the $S_{\mathrm{B(z)}}$ he will find the value $+\frac{\hbar}{2}$ with probability equal to unity.

69) Explain why Bob can not read the message which is sent by Alice by the way described in the text of § 5.5.

70) *Write T or F whether you think the statements are true or false .*

Quantum theory is an indeterministic theory because of:

- **i)** Knowing completely the state of a system we can not predict with accuracy the exactly result of a measurement, we are speaking only for probability of several results.
- **ii)** Knowing completely the state of the system we do not know its time evolution, so we are not able to predict with accuracy.
- **iii)** It is a non complete theory.

71) Which is the criterion of Einstein's locality?

72) *Write F or T whether you think the statements are true or false.*

According to Einstein the hidden variables:

- **i)** do not play any rule to definition of the state of the system.
- **ii)** are not under control because of the deficient experimental ability.
- **iii)** If we knew them, the result of a measurement would be different than now.
- **iv)** If they were under control, the quantum mechanics would be a deterministic theory.

PART II

QUESTIONNAIRES

The Greek high school has three classes the A', B' and C'. After the A' class the pupils follow one of the two different directions ,the classical direction which leads to classical studies (jurisprudence, ancient Greek, Latin,…) or the practical direction which leads to exact studies (polytechnic schools, mathematics, physics,…) .

The lesson of quantum physics was taught to pupils of C' class of practical direction . The number of pupils was 32 and the duration of the course was about 16 hours.

Before the beginning of the course the questionnaire A was given to the pupils and during the course the pupils answered the questionnaires 1,2,3 and 4. Two weeks after the end of the lessons they answered the questionnaire B' and completed the form <<comparisons>>.

The questionnaires and the corresponding results are following . Also there is a comparative study on the relation between quantum physics and physics/mathematics. Moreover there is a comparative study between quantum physics and the other chapters of physics which are taught to C' class.

The part of this work which corresponds to questionnaire 1 was taught to another class from another teacher and we compared the results.

At the end of this part there is the final conclusion of the research.

QUESTIONNAIRES

Questionnaire A'

1) *Write T or F whether you think the statements are true or false.*

We are observing a body, which is on the move forming a track.

☒ i) The same track is observed by every observer.
☒ ii) If we did not observe it, this would follow a different path.
☒ iii) If we did not observe it, its track would not make sense, consequently there would not be the track.
☒ iv) The body follows a path independently of any observer. If someone observes it, he will see this track (the same to everyone)

- *Choose the strongest statement.* ☒

2) *Write T or F whether you think the statements are true or false.*

A body is launched from a point on the ground with a certain velocity , forming a certain angle with the horizon, and following a track it lands at certain point on the ground. We repeat the experiment with exactly the same conditions (same velocity, angle, …).

☒ i) The second time the body will also land at the same point as the first time.
☒ ii) The second time the body will land at a different point.
☒ iii) The second time the body will follows exactly the same track as the first time, it will be moving for the same time and it will land at the same point.
☒ iv) There is a case we find that it lands at a different point but it happens because of experimental error. If the experimental apparatus was ideal we would find that it lands at the same point

- *Choose the strongest statement.* ☒

3) *Write T or F whether you think the statements are true or false.*

We have two electrons, which are under the same initial conditions and are moving exactly in the same way in a magnetic field. We measure the component of spin on the x-axis, the S_x, and for the first electron we find the value $+\frac{\hbar}{2}$. If we measure the same quantity for the other electron in the same way

☒ i) We will find the same value $+\frac{\hbar}{2}$
☒ ii) We cannot predict the result of the measurement.
☒ iii) It is possible to find the value $-\frac{\hbar}{2}$.

☒ iv) If we found the value $-\frac{\hbar}{2}$ we would have made an experimental error.

☒ v) In any case we must find the value $+\frac{\hbar}{2}$ because the measurement is exactly the same on the same systems consequently the results must be the same.

- *Choose the strongest statement.* ☒

4) *Write T or F whether you think the statements are true or false.*

A body with mass m has a certain velocity v and it is moving on the x-axis. At the moment $t = 0$ it is at the point $x_0 = 0$. The total force acting on the body is equal to zero and so its motion is rectilinear and smooth. For this body

☒ i) We can predict with certainty the position and the its velocity at any time.
☒ ii) We can predict its kinetic energy at any time.
☒ iii) When its mass is very small we cannot predict with certainty the position and the velocity.
☒ iv) The position and the velocity are two quantities , which if we know them, the description of the body is complete.

- *Choose the strongest statement.* ☒

5) *Write T or F whether you think the statements are true or false.*

A body with mass m at the moment $t = 0$ has a certain velocity v_0 and it is moving under the influence of a certain constant force F (motion in one dimension).

☒ i) The reason for the change of the motion is the force and the result is the acceleration.
☒ ii) Newton's law $F = m \cdot a$ connects the cause and the result.
☒ iii) Through Newton's law we can predict with certainty the value of any observable related with the body at any time.
☒ iv) There are some quantities whose value we cannot predict with accuracy.

- *Choose the strongest statement.* ☒

6) *Write T or F whether you think the statements are true or false.*

We measure the component S_z of an electron and we find the value $+\frac{\hbar}{2}$. Then we measure the component S_x and we find $-\frac{\hbar}{2}$. If we measure the component S_z again we will find

☒ i) Again the value $+\frac{\hbar}{2}$.

☒ ii) We can not predict, it is possible to find any value because the second measurement disturbed the particle.

☒ iii) Either $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$.

☒ iv) The physical quantities define the system and any measurement of one of them does not change the value of the others.

- *Choose the strongest statement.* ☒

7) *Write T or F whether you think the statements are true or false.*

For some reason two electrons are produced at a point of the space. They are initially at rest and because of the repulsive force between them they begin to go move apart . The distance between them becomes infinite. As we know the total momentum is conserved and its value is equal to zero. Let us measure the momentum of the first of them and find the value $+a\,kg\frac{m}{s}$. Then

☒ i) We know at the same time that the value of the momentum of the second electron is $-a\,kg\frac{m}{s}$.

☒ ii) The momentum of the second electron was also $-a\,kg\frac{m}{s}$ before the measurement of the momentum of the first one.

☒ iii) The momentum of the second electron was not $-a\,kg\frac{m}{s}$ before the measurement of the momentum of the first one, but it became $-a\,kg\frac{m}{s}$ exactly at the moment when we measured the first one.

☒ iv) It is impossible, the measurement of the momentum of the first electron influences the second one (infinite distance), consequently before the measurement the momentum of the second electron was also $-a\,kg\frac{m}{s}$.

- *Choose the strongest statement.* ☒

8) *Write T or F whether you think the statements are true or false.*

n moles of an ideal gas consist of N very small particles, the atoms which move freely and collide elastically with each other and with the inner sides of the can .

☒ i) If for any particle we knew at a moment its velocity and position we could predict with accuracy the motion of every one separately for any time.

☒ ii) We could not predict the motion of the particle even though we knew the initial conditions for each of them because after the collisions all particles together constitute one system and so we cannot examine each particle separately.

☒ iii) It is not valid Newton's law for each atom separately.

☒ iv) The calculation of the mean values of pressure is achieved with the help of Newton's law and the rules of statistic.

- *Choose the strongest statement.* ☒

9) *Write T or F whether you think the statements are true or false.*

Two particles come from large distance, collide and move apart until the distance becomes infinite so that they do not interact. Suppose that the total momentum is equal to zero.

☒ i) Can we examine each particle separately defining velocity, position, energy, momentum for every one.
☒ ii) The two particles constitute a system and must not be examined separately.
☒ iii) Measuring the momentum of one of them we know the momentum of the other.
☒ iv) The measurement of the momentum of the first particle changes the momentum of the second one , that means the measurement of the first one influences the second one instantaneously.

- *Choose the strongest statement.* ☒

10) When you hear the word quantum physics the first thing, which you remember is

☒ material waves
☒ quantum of energy
☒ Bohr's atomic theory
☒ Schroedinger equation
☒ None of them

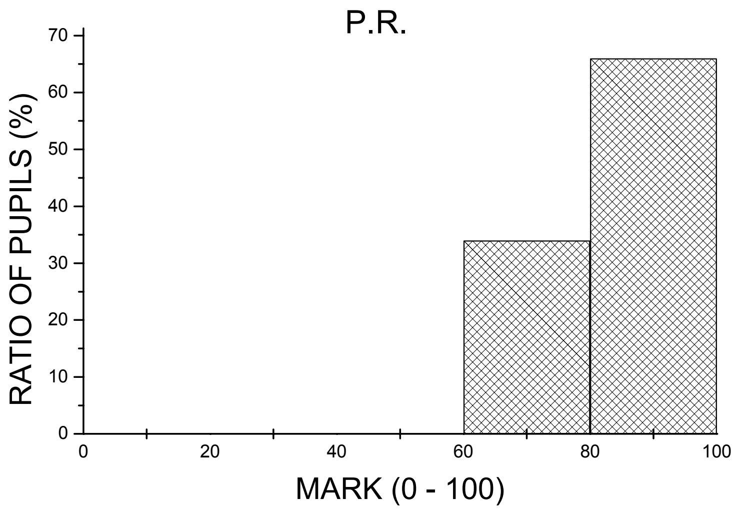
RESULTS - CONCLUSIONS OF QUESTIONNAIRE A

The aim of this questionnaire is to investigate how easily the pupils can show the concepts of classical physics which are in their mind . The objects which are investigated and the corresponding results are the following:

I. Physical reality (P.R.)

<<The world exists independently of any observer. Any effect takes place in the same way irrespective of whether we observe it or no. The corresponding question is the number 1. The plot of the ration of pupils (%) versus marks (0-100) is the following:

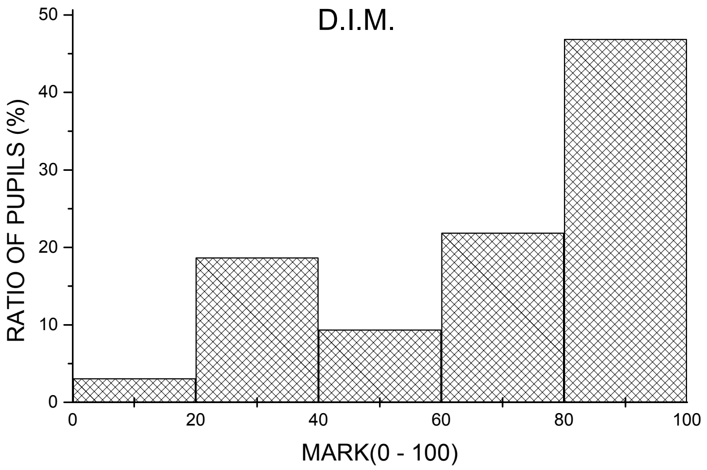
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	0	0	34	66



II. Deterministic theory – Identical measurements (D.I.M.)

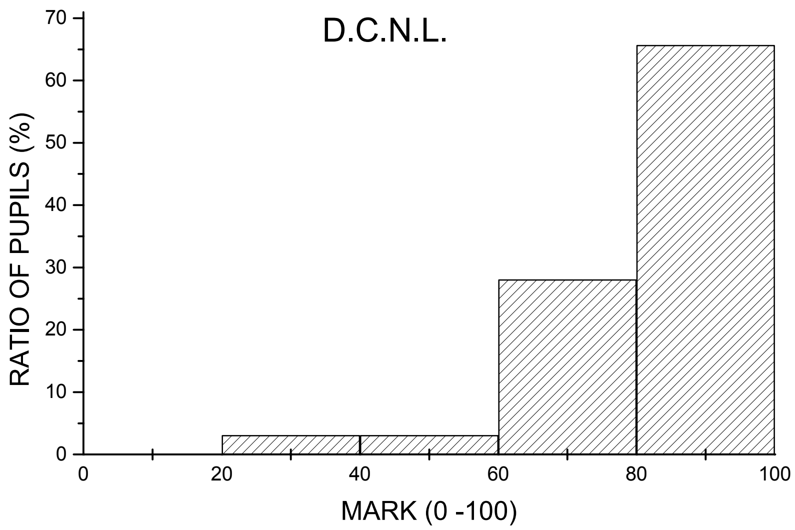
Classical theory is a deterministic theory and because of this ,identical measurements made on identical systems give identical results The corresponding questions are the numbers 2 and 3. The corresponding diagram is the following:

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	3,1	18,7	9,4	21,9	46,9



III. Deterministic theory – Causality (Newton law) (D.C.N.L.)
We investigate if the pupils understand that through the Newton law we can predict with certainty the time evolution of a system , when we know its initial condition . The questions 4 , 5 , 8i , 8ii concern this object. The results are the following:

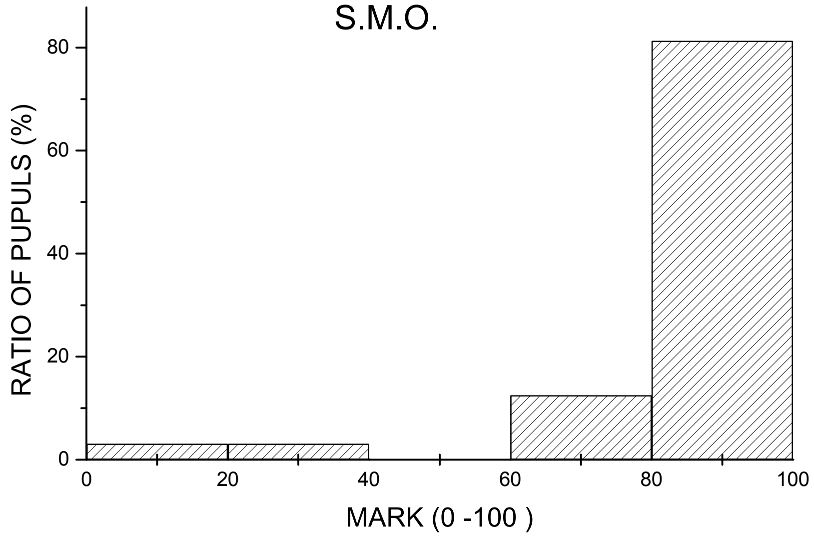
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	3,1	3,1	28,1	65,7



IV. Simultaneously measurable observable (S.M.O.)

In the frame of classical physics all observables are simultaneously measurable. The corresponding question is the number 6. The results are the following:

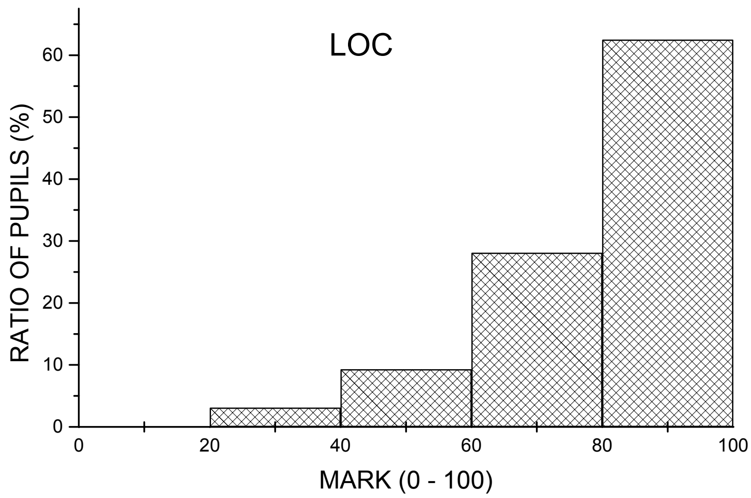
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	3,1	3,1	0	12,5	81,3



V. Locality (LOC)

What happens in a place does not immediately affect what happens in another place. Especially if two particles are in infinite distance , then any action on one of them does not influence the other one. The corresponding questions are number 7 , 9iii) , and 9vi). The results are the following:

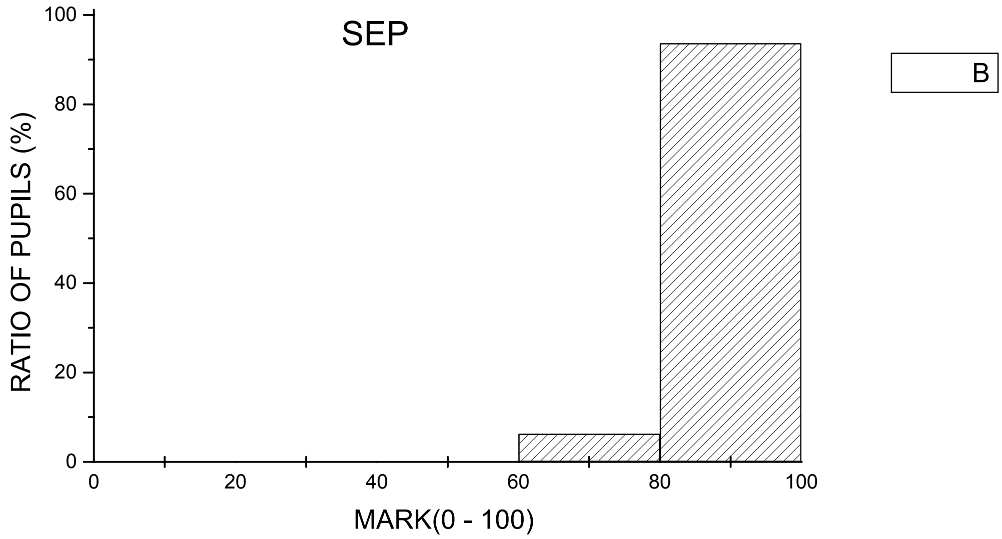
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	3,1	9,3	28,1	62,5



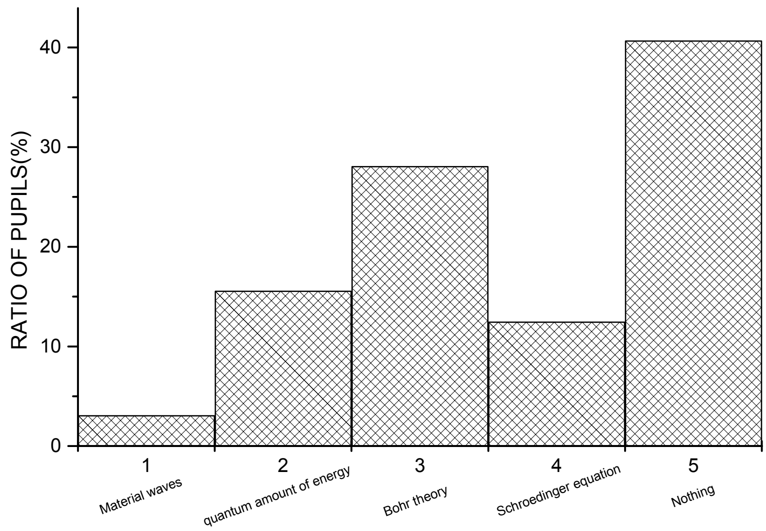
VI. Separability (SEP)

In any case we can examine a particle as a system apart from its surrounding. The corresponding questions are 8iii) , 8iv) , 9i) and 9ii) . The results are the following:

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	0	0	6,3	93,7



VII. Contact with quantum concepts.



<u>Conclusion:</u>

The results are the ones which we expected. We can see a little divergence in the case II. It happens because pupils did not take the measurements as ideal.

Questionnaire 1.

1) Which of the following pairs defines the state of a particle in the frame of classical mechanics.

i) (position, velocity) ii) (position, force)
iii) (velocity, force) iv) (velocity, acceleration)

Choose the correct answer.

2) With the ket $(|\ \rangle)$ we define the quantum state of a particle. Also

i) It defines the position and the momentum of the particle.
ii) It is connected with the value at least of one observable.
iii) It defines the value of two observables, which are not simultaneously measurable.
vi) It is always connected only with the value of one observable.

Choose the correct answer.

3) Two observables A and B of a system are simultaneously measurable if

i) We can measure both of them at the same time.
ii) The measurement of the B changes the value which we found when we first measured the A.
iii) Measuring the A and B successively and for many times we always find the same value for each of them.
iv) Measuring the A and the B the system comes to a state $|a, \beta\rangle$ which changes to state $|a^{1}, \beta\rangle$ if we measure the A again.

Choose the correct answer.

4) *Write T or F whether you think the statements are true or false.*

Measuring the component of spin S_z of a particle with spin $\frac{1}{2}$

☒ i) We find two discrete values, $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.
☒ ii) We find several values between $-S$ and $+S$.
☒ iii) Classical physics predicts values between $-S$ and $+S$ but the experiments give two discrete values $\pm\frac{\hbar}{2}$.
☒ iv) The values, which we find, depend on the experimental method which we use.

5) *Write T or F whether you think the statements are true or false.*

For a particle with spin $\frac{1}{2}$

☒ i) We can not write the state $\left|S_x = +\frac{\hbar}{2}, S_y - \frac{\hbar}{2}\right\rangle$ because the observables S_x and S_y are not simultaneously measurable.

☒ ii) There is the state $\left|\boldsymbol{z}\uparrow, \boldsymbol{x}\downarrow\right\rangle$.

☒ iii) If we measure the component S_z and find the value $+\frac{\hbar}{2}$ and then we measure the component S_y and find the value $-\frac{\hbar}{2}$, then repeating the measurement of S_z we will find the value $+\frac{\hbar}{2}$ with probability equal to unity.

☒ iv) For the state $\left|\boldsymbol{z}\uparrow\right\rangle$ we know that the value of the component S_x is $+\frac{\hbar}{2}$.

6) A particle is in the state

$$\left|\Psi\right\rangle = \frac{3}{5}\left|\boldsymbol{z}\uparrow\right\rangle + i\frac{4}{5}\left|\boldsymbol{z}\downarrow\right\rangle,$$

calculate the probability to be in the state

$$\left|\Phi\right\rangle = \frac{4}{5}\left|\boldsymbol{z}\uparrow\right\rangle + i\frac{3}{5}\left|\boldsymbol{z}\downarrow\right\rangle$$

Solution

7) The measurements of the energy of a system always give the values E_1, E_2 and E_3.

Find a basis of the corresponding Hilbert space.
If the state of the system is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}|E_3\rangle$$

find the probability the system to be in the state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}|E_2\rangle$$

Solution

8) A particle is in the state

$$|\Psi\rangle = \frac{1}{2}|\boldsymbol{z}\uparrow\rangle + i\frac{\sqrt{3}}{2}|\boldsymbol{z}\downarrow\rangle$$

i) Passing the particle through a SG_z apparatus, find the probability for it to come out in state $|\boldsymbol{z}\uparrow\rangle$ and also to come out in state $|\boldsymbol{z}\downarrow\rangle$.

ii) If $N = 4.10^{23}$ particles which are in the same state $|\Psi\rangle$ pass through a SG_z, how many will follow the higher path and how many will follow the lower path.

Solution

9) *Write T or F whether you think the statements are true or false.*

For the quantum physics

☒ i) If two particles with spin $\frac{1}{2}$ are in the same state, for example the $|z\uparrow\rangle$, measuring for each one the same quantity S_x and in the same way, it is possible to find different results.

☒ ii) The state $|z\uparrow\rangle$ allows us to predict with accuracy the value of the component $S_z\left(S_z = +\frac{\hbar}{2}\right)$ but does not allows us to predict with accuracy the value of the component S_x

☒ iii) On the other side, classical physics allows us to predict the value of any observable with certainty if we know the state of a system.

☒ iv) If we know the state of a system we also know with certainty the value of every observable concerning the system.

10) *Write T or F whether you think the statements are true or false.*

☒ i) According to quantum physics, a measurement of an observable can change the initial state of the system.

☒ ii) Classical theory is a deterministic theory but quantum physics is not.

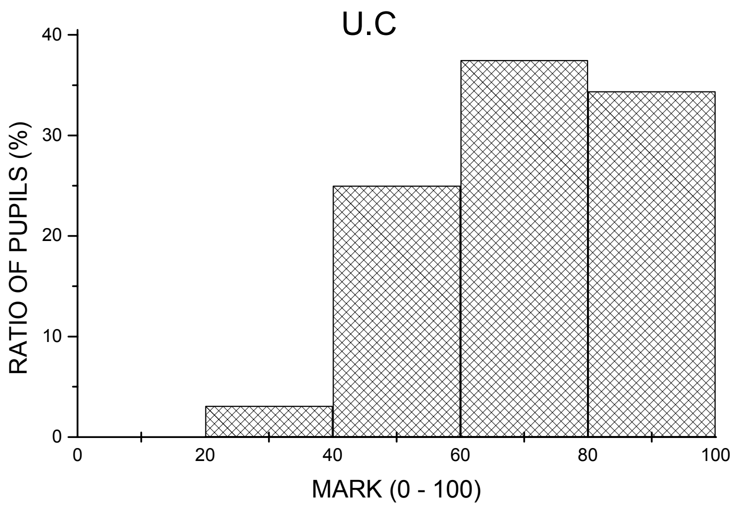
☒ iii) Quantum theory is a probabilistic theory.

RESULTS – CONCLUSIONS OF QUESTIONNAIRE 1.

The paragraphs §1 . 1 until §1 . 9 were taught. The duration of teaching was 4 hours. After that this questionnaire which concerns these paragraphs was given to pupils. The aims and the results are the following:

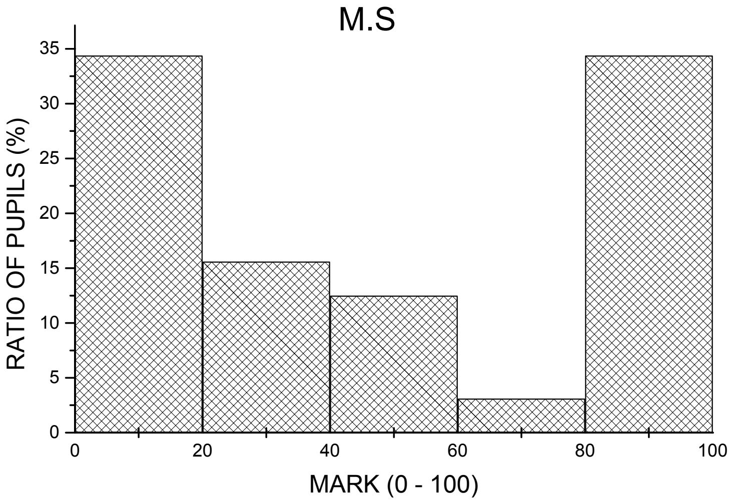
I. Understanding of new concepts.(U.C.)
The aim is to investigate the level of understanding of the new concepts like quantum state, simultaneously measurable observables and discrete values. The corresponding questions are the numbers 1,2,3,4 and 5. The plot of the ratio of pupils (%) versus marks (0- 100) is the following:

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	3,1	25	37,5	34,5



II. Mathematical skills. (M. S)
The aim is to find out the abilities of the pupils to manipulate the new mathematics as to find the bra of a ket , to calculate an inner product and the probability as a state $|\Psi>$to be in an other state $|\Phi>$. The results are the following:

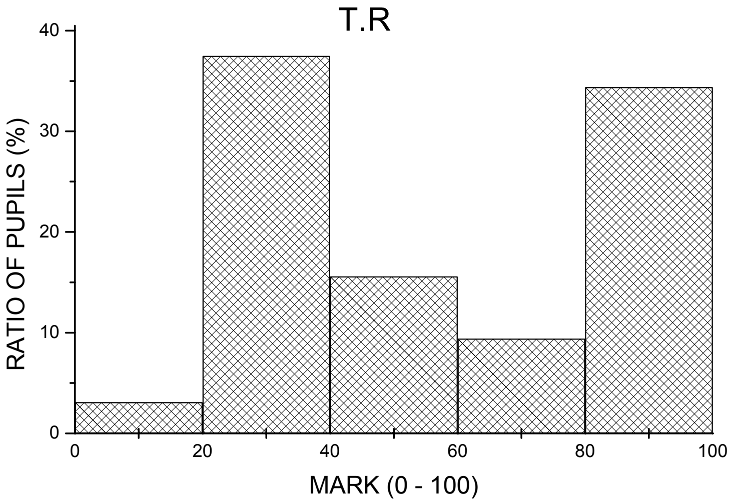
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	34,4	15,6	12,5	3,1	34,4



I. Total result. (T. R.)

This is the total result of the cases I. and II. and we can see the performance of the pupils to a normal test which is given for the lesson of Physics. The contribution of the I. and II. is with weight 14/40 and 26/40 respectively. Obviously the corresponding questions are the numbers 1 until 8. the corresponding diagram is the following:

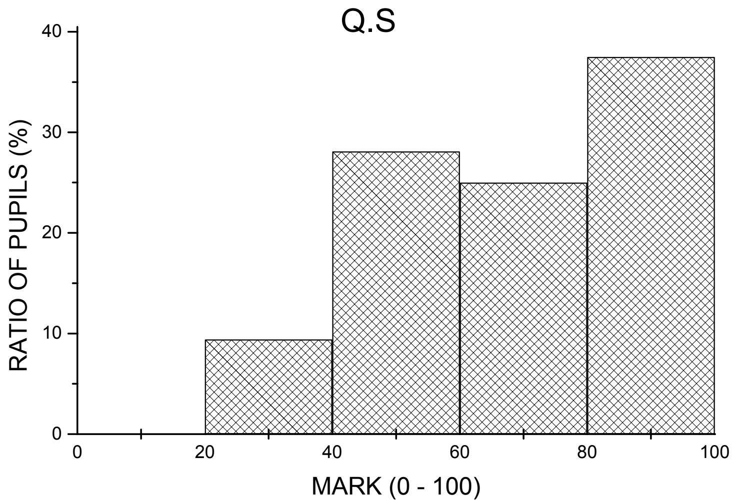
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	3,1	37,5	15,6	9,4	34,4



IV. Introduction to Quantum Spirit. (Q. S.)

The questions 9 and 10 are an expansion of the questionnaire A. We want to investigate how much the pupils are introduced to the philosophy of quantum physics. The results are the following :

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	9,4	28,1	25	37,5



Questionnaire 2.

1) The state of a particle with spin $\frac{1}{2}$ is

$$\left|\Psi\right\rangle = \frac{2}{3}\left|\boldsymbol{x}\uparrow\right\rangle + i\frac{\sqrt{5}}{3}\left|\boldsymbol{x}\downarrow\right\rangle$$

i) Write the bra $\left\langle\Psi\right|$

Answer

ii) Write the projection operators $\hat{P}_{x\uparrow}$ and $\hat{P}_{x\downarrow}$

Answer

iii) Calculate the probability the result of the measurement of S_x to be $+\frac{\hbar}{2}$. Similarly to be $-\frac{\hbar}{2}$.

Solution

iv) Find the expectation value of the component S_x.

Solution

v) Write the Operator $\hat{S}_x$.

Answer

vi) Using the relation $\langle \Psi | \hat{S}_x | \Psi \rangle$, calculate the expectation value of the component S_x.

Solution

2) For a particle with spin $\frac{1}{2}$

i) Write the operator $\hat{S}_z$

Answer

ii) Find the matrix representation of the operator $\hat{S}_z$ in the basis $\{|\boldsymbol{x}\uparrow\rangle, |\boldsymbol{x}\downarrow\rangle\}$ using the relations

$$|\boldsymbol{x}\uparrow\rangle = \frac{1}{\sqrt{2}}|\boldsymbol{z}\uparrow\rangle + \frac{1}{\sqrt{2}}|\boldsymbol{z}\downarrow\rangle, \quad \langle \boldsymbol{x}\uparrow| = \frac{1}{\sqrt{2}}\langle \boldsymbol{z}\uparrow| + \frac{1}{\sqrt{2}}\langle \boldsymbol{z}\downarrow|$$
$$|\boldsymbol{x}\downarrow\rangle = \frac{1}{\sqrt{2}}|\boldsymbol{z}\uparrow\rangle - \frac{1}{\sqrt{2}}|\boldsymbol{z}\downarrow\rangle, \quad \langle \boldsymbol{x}\downarrow| = \frac{1}{\sqrt{2}}\langle \boldsymbol{z}\uparrow| - \frac{1}{\sqrt{2}}\langle \boldsymbol{z}\downarrow|$$

Solution

iii) The state of N particles is

$$|\Psi\rangle = \frac{3}{5}|\boldsymbol{x}\uparrow\rangle + \frac{4}{5}|\boldsymbol{x}\downarrow\rangle$$

Find the matrix representations of the ket $|\Psi\rangle$ and bra $\langle\Psi|$ in the basis $\{|\boldsymbol{x}\uparrow\rangle, |\boldsymbol{x}\downarrow\rangle\}$

Solution

iv) For the N particles in state $|\Psi\rangle$, find the expectation value of the component S_z using the
relation between the matrices. The matrix representation of S_z in the basis $\{|\boldsymbol{x}\uparrow\rangle, |\boldsymbol{x}\downarrow\rangle\}$ is the following

$$S_z = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

Solution

3) The eigenvalues of the energy are $E_1 = E$ and $E_2 = -E$. The corresponding eigenstates are noticed as $|E_1\rangle$ and $|E_2\rangle$ and constitutes an orthonormal basis of the Hilbert space. The initial state of the system is

$$|\Psi(0)\rangle = \frac{12}{13}|E_1\rangle - \frac{5}{13}|E_2\rangle$$

i) Write the state $|\Psi(t)\rangle$

Answer

ii) Find the matrix representation of the $|\Psi(t)\rangle$ and $\langle\Psi(t)|$ in the basis $\{|E_1\rangle, |E_2\rangle\}$.

Solution

iii) Write the operator of energy $\hat{\mathrm{H}}$.

Answer

iv) Find the matrix representation of the operator $\hat{\mathrm{H}}$ in the basis $\left\{\left|E_1\right\rangle,\left|E_2\right\rangle\right\}$.

Solution

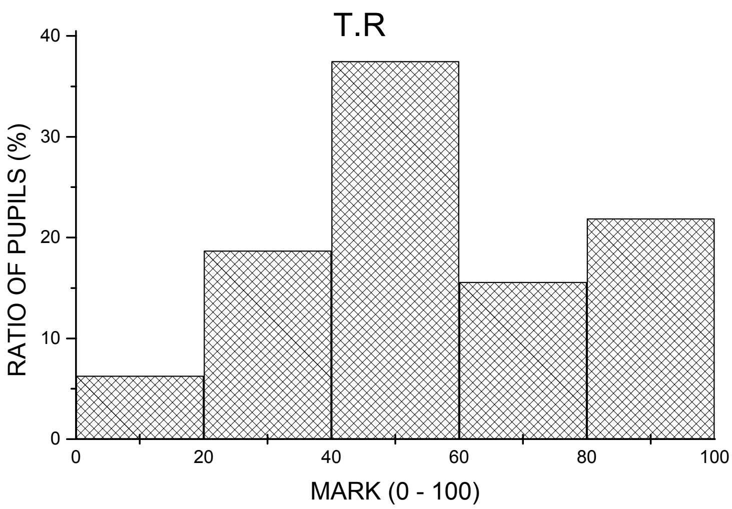
v) Calculate the expectation value of the energy for any time (with any way you prefer).

Solution

RESULTS – CONCLUSIONS OF QUESTIONNAIRE 2.

The paragraphs §2.1 until §2.5 were taught. The duration was four hours , but we taught two hours more because there were problems with new mathematics . The aim of this questionnaire is to investigate the mathematical skills of pupils and how easily they can manipulate the new mathematical objects like the calculation of mean values using the operators , finding the representation of an operator and doing calculations with matrices . The results are the following :

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	6,3	18,7	37,5	15,6	21,9



Remark : The result is more normal than that from the questionnaire 1. ,as we can observe there is some accumulation on the mark 50 . There are tow reasons for this:

1) The pupils are used to working with new things .
2) The two additional hours proved helpful.

Questionnaire 3.

1) A pure ensemble consists of

i) particles of the same type.
ii) particles which are in the same state $|\Psi\rangle$.
iii) particles of different type.
iv) particles of the same type which are not in the same state $|\Psi\rangle$.

Choose the correct answer.

2) A mixed ensemble consists of

i) same particles
ii) particles where some of them are in the $|\Psi_1\rangle$ others are in the state $|\Psi_2\rangle$ etc.
iii) particles which are in the state $|\Psi\rangle = c_1 |\boldsymbol{x}\uparrow\rangle + c_2 |\boldsymbol{x}\downarrow\rangle$
iv) particles which are selected after the measurement of an observable A with the same value a.

Choose the correct answer.

3) The dimension of the space H_1 is 2 and the dimension of the space H_2 is 3, then the dimension of the tensor product $\mathrm{H}_1 \otimes \mathrm{H}_2$ is

i) 3 ii) 2 iii) 6 iv) 5

Choose the correct answer.

4) *Write T or F whether you think the statements are true or false.*

☒ i) The particles which are emerging from the higher path of a $SG_{\hat{n}}$ apparatus constitute a pure ensemble.
☒ ii) If N particles which constitute a mixed ensemble pass through a convenient SG_n device they will follow the same path.
☒ iii) The particles of an unpolarized beam are in the same state $|\Psi\rangle$.
☒ iv) If two observables are not simultaneously measurable then they are related to the same Hilbert space.
☒ v) Irrelated observables are not related to the same Hilbert space.

5) *Write T or F whether you think the statements are true or false.*

For a mixed ensemble, the ratio of particle in state $|\Psi_1\rangle$ is p_1 and the ratio of particle in state $|\Psi_2\rangle$ is p_2 $(p_1 + p_2 = 1)$.

☒ i) The state $|\Psi\rangle = p_1|\Psi_1\rangle + p_2|\Psi_2\rangle$ corresponds to this ensemble

☒ ii) The expectation value of an observable A is given through the relation $\langle A\rangle = p_1\langle\Psi_1|A|\Psi_1\rangle + p_2\langle\Psi_2|A|\Psi_2\rangle$.

☒ iii) If we pass this ensemble through a SG_n device it splits to two pure ensembles.

☒ iv) The expectation value of an observable A can be calculated through the relation $\langle A\rangle = \langle\Psi|A|\Psi\rangle$.

6) Two observables A and B of a system are not related. The measurement of A gives three values a_1, a_2 and a_3.

i) Define the Hilbert space H_A which is related to A

Solution

ii) The measurement of B gives the values β_1 and β_2. Define the Hilbert space H_B.

Solution

iii) Then define the tensor product $H_A \otimes H_B$

Solution

7) A huge number of particles with spin $\frac{1}{2}$ pass through a SG_n device where the $\hat{n}$ is on the xz plane and form angle $\theta = 120^\circ$ with Oz axis. We select all the particles, which follow the higher path (number N).

i) Write the state of the N selected particles. Do they constitute a pure ensemble?

Solution

ii) If the ensemble of N particles passes through a SG_z device, how many particles will follow the higher path and how many the lower one? Calculate the mean value of the component S_z.

Solution

8) A mixed ensemble consists of 30 % particles in the state $|\boldsymbol{z}\uparrow\rangle$ and 70 % particles in the state $|\boldsymbol{x}\uparrow\rangle$. Find the expectation value of the component S_x.

Solution

9) *Write T or F whether you think the statements are true or false.*

☒ i) Quantum physics makes sense only in the case of a pure ensemble.
☒ ii) The result of the measurement of an observable is equal to the mean value of the observable for each member of the ensemble.
☒ iii) The measurement of an observable for each member of an ensemble gives discrete values, which are repeated.
☒ iv) For a pure ensemble we can predict the mean value of any observable.

10) *Write T or F whether you think the statements are true or false.*

☒ i) A particle always is in a state $|\Psi\rangle$, vector in the Hilbert space.
☒ ii) A state $|\Psi\rangle$ can define an ensemble if all members are in the same state $|\Psi\rangle$.
☒ iii) The quantum rules are applied only for pure ensembles.
☒ iv) The mean value of an observable makes sense in the case of a particle.

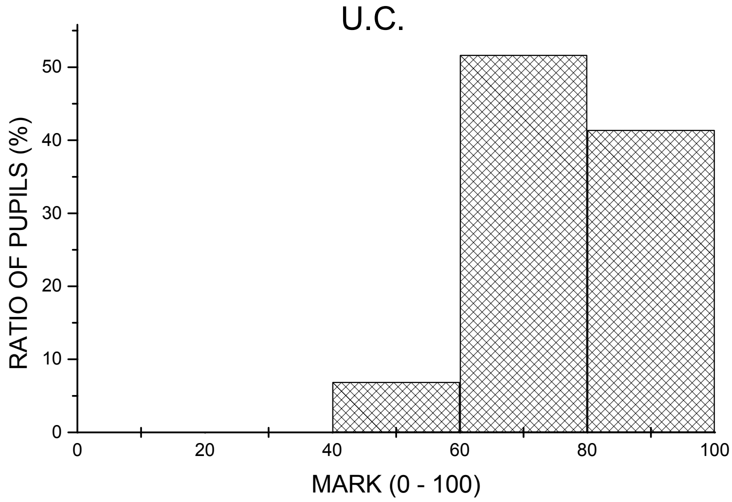
RESULTS – CONCLUSIONS OF QUESTIONNAIRE 3.

After having taught the paragraphs §3.1 until §4.3 we gave this questionnaire to pupils. The aims and the results are the following:

I. Understanding of new concepts.(U.C.)

The aim is to investigate the level of understanding of the concepts like pure ensemble , mixed ensemble and tensor product .The corresponding questions are the numbers 1,2,3,4,5,9 and 10 . The results are the following:

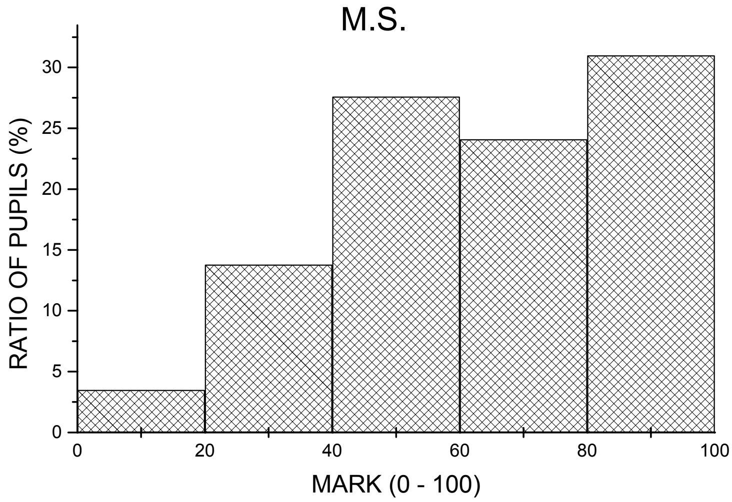
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	0	6,9	51,7	41,1



II. Mathematical skills. (M. S.)

The aim is to find out the abilities of the pupils to calculate the mean value of an observable for a pure and a mixed ensemble and to define the tensor product space of two spaces . The corresponding questions are the numbers 6,7 and 8. The results are the following:

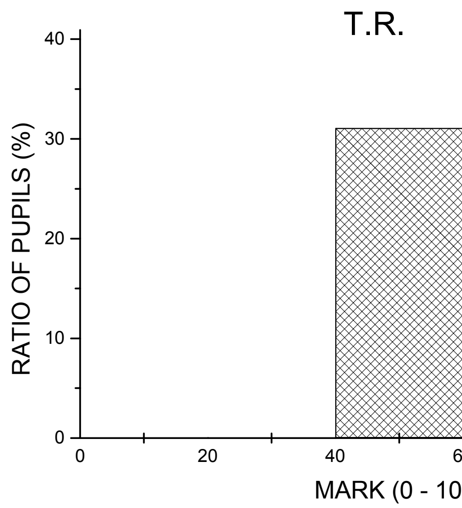
marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	3,5	13,5	27,6	24,1	31



III .Total result. (T. R.)

This is the total result of the cases I. and II. Through this we can see the performance of the pupils to a usual test such as is given for the subject of physics . The contribution of the I. and II. is with the same weight (50/100).

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	0	31,1	37,9	31



<u>Remarks:</u> We observe some improvement in the performance of the pupils. This happens

because there is a repetition of the same mathematical objects and so they became more familiar with these.

Questionnaire 4.

1) Local performance for a system of two particles A and B is

i) an action which is performed separately on A and B .
ii) an action which is performed simultaneously on A and B .
iii) a measurement concerning an observable, which is related to both particles.
iv) an action where we bring the two particles so near that they interact.

Choose the correct answer.

2) A vector in a tensor product space $H = H_A \otimes H_B$ concerns entangled state if

i) it is written as $|\Theta\rangle_A \otimes |\Phi\rangle_B$.
ii) it cannot be written as $|\Theta\rangle_A \otimes |\Phi\rangle_B$.
iii) Changing the basis we could write it as $|\Theta\rangle_A \otimes |\Phi\rangle_B$.
iv) Any vector in the H concerns entangled state.

Choose the correct answer.

3) According to Einstein locality, if two particles A,B do not interact that is they are isolated.

i) An action (measurement) on A does not modify the state of B particle.
ii) An action on B instantaneously modifies the state of A .
iii) A and B must be faced as a unified system.
iv) Any measurement of A gives information for the state of B .

Choose the correct answer.

4) *Write T or F whether you think the statements are true or false.*

A pair described by the state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\boldsymbol{z}\uparrow\rangle_A |\boldsymbol{z}\downarrow\rangle_B - |\boldsymbol{z}\downarrow\rangle_A |\boldsymbol{z}\uparrow\rangle_B\right)$$

that is an entangled state. We separate the two particles and move a long distance apart so that they do not interact.

☒ i) We can examine each particle independently from the other.

☒ ii) We measure the component $S_{z(A)}$ and find the value $+\frac{\hbar}{2}$. The state of B instantaneously becomes the $|\boldsymbol{z}\downarrow\rangle_B$.

☒ iii) We measure the component $S_{x(\mathrm{A})}$ and find the value $-\frac{\hbar}{2}$, consequently the state of B is the vector $\left|\boldsymbol{x}\uparrow\right\rangle_{\mathrm{B}}$. The state of B was also $\left|\boldsymbol{x}\uparrow\right\rangle_{\mathrm{B}}$ before the measurement because the particles were a great distance apart and the measurement of $S_{x(\mathrm{A})}$ did not disturb the particle B .

☒ iv) We can consider the two particles as two different separated systems.

5) *Write T or F whether you think the statements are true or false.*

Einstein assumed the existence of <<hidden variables>>.

☒ i) If we knew their value we could predict with accuracy the results of any measurement.

☒ ii) Their existence has been proved by experiments.

☒ iii) Some experiments show their existence of and other do not.

☒ iv) If they existed , then quantum theory would be a causal theory, consequently identical measurements made on identical systems would give identical results.

6) The message sent by Alice to Bob is unreadable because

i) An ensemble 50 % $\left|\boldsymbol{n}\uparrow\right\rangle$ and 50 % $\left|\boldsymbol{n}\downarrow\right\rangle$ is identical to another one which is 50 % $\left|\boldsymbol{m}\uparrow\right\rangle$ and 50 % $\left|\boldsymbol{m}\downarrow\right\rangle$.

ii) It could be readable if they had arranged the directions on which Alice would have measured.

iii) If it were readable it would be a message, which would not be sent instantaneously.

iv) It could be read if Bob took the particles A and Alice took the particles B .

Choose the correct answer.

7) Is the state

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{2}}\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{B}} + \frac{1}{\sqrt{2}}\left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{B}}$$

an entangled state?

Solution

8) Prove the relation

$$\left|\Psi^{-}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{B}} - \left|\boldsymbol{z}\downarrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{z}\uparrow\right\rangle_{\mathrm{B}}\right) = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{x}\uparrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{x}\downarrow\right\rangle_{\mathrm{B}} - \left|\boldsymbol{x}\downarrow\right\rangle_{\mathrm{A}}\left|\boldsymbol{x}\uparrow\right\rangle_{\mathrm{B}}\right)$$

Using the relations

$$\left|\boldsymbol{x}\uparrow\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{z}\uparrow\right\rangle + \left|\boldsymbol{z}\downarrow\right\rangle\right)$$

$$\left|\boldsymbol{x}\downarrow\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\boldsymbol{z}\uparrow\right\rangle - \left|\boldsymbol{z}\downarrow\right\rangle\right)$$

Proof

9) *Write T or F whether you think the statements are true or false.*

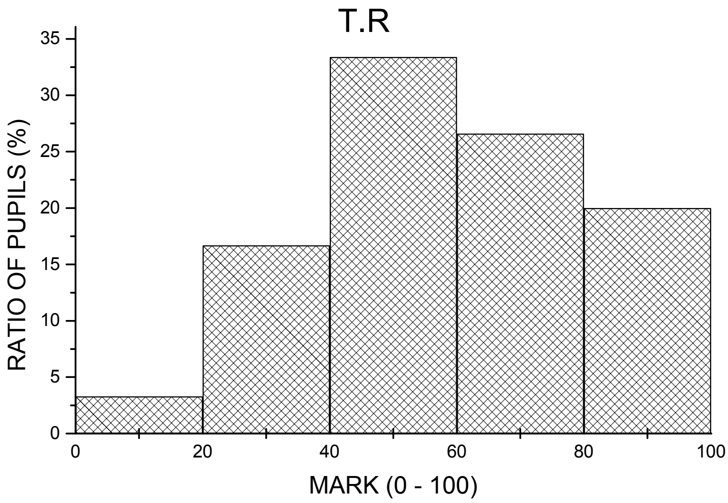
☒ i) In any case a particle is in a state which is a vector in the Hilbert space.
☒ ii) The vectors in a tensor product which are not in the form $\left|\Theta\right\rangle_{\mathrm{A}} \otimes \left|\Phi\right\rangle_{\mathrm{B}}$ do not have physical significance.
☒ iii) If we want to produce an entangled state the two particles must interact.
☒ iv) When A and B are in an entangled state , the measurement of the component $S_{x(\mathrm{A})}$ leads the system to an unentangled state.

10) Comment on the object which causes the greatest impression on you .

RESULTS – CONCLUSIONS OF QUESTIONNAIRE 4.

We taught the paragraphs §5.1 until §5.6.The duration of teaching was four hours . The aim of this questionnaire is to research the degree of understanding of the concepts entanglement , local action , locality , separability and hidden variables . Also the mathematical skills are checked by the questions 7 and 8 . The contribution of the questions 7 and 8 to the total result is 1/3 . The total result is the following:

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	3,3	16,7	33,4	26,6	20



Remarks: **1)** The results are normal without any surprises.
2) For the question number 10 the most of the pupils answered that the impressive object was that when two particles are in an entangled state a measurement on one of them defines the state of the other one , in spite of the fact that they are at an infinite distance.

Questionnaire B'

1) We have seen that quantum mechanics describes nature in a different way than the classical theory. What do you think (one answer)?

i) Two theories describe different things (microcosmos - macrocosmos) but both of them are correct.
ii) The correct theory is the quantum theory; the classical theory is an approximation of the quantum mechanics and its predictions for macro objects are very satisfactory.
iii) The correct theory is the classical one and the quantum theory is an approximation of the classical theory.
iv) If we apply the quantum rules to macrocosmos the predictions will be very different from those of classical physics.

2) *Write T or F whether you think the statements are true or false.*

A body is following a track.

☒ i) There is the track because we observe it.
☒ ii) The track is independent from any observer.
☒ iii) The observation modifies the track.
☒ iv) The track does not make sense when we do not observe it.

3) *Write T or F whether you think the statements are true or false.*

Quantum theory.

☒ i) This is a deterministic theory.
☒ ii) Identical measurements made on identical systems give identical results.
☒ iii) The measurement modifies or defines the state of the system.
☒ iv) Identical systems after the identical measurement could be in different states.

4) *Write T or F whether you think the statements are true or false.*

In quantum mechanics

☒ i) The definition of the state is incomplete because some observables are not simultaneously measurable.
☒ ii) The definition of the state is as complete as the natural laws permit it .
☒ iii) The definition of the state is incomplete because the experimental devices are not developed enough.
☒ iv) The definition of the state is incomplete because we do not know the value of the hidden variables.

5) *Write T or F whether you think the statements are true or false.*

In the case of an entangled state of two particles , when the two particles are moved a long distance apart .

- ☒ i) The two particles could be examined as separated systems (separability).
- ☒ ii) Independently of the distance they remain as a unified system (non separability).
- ☒ iii) A measurement on one of these particles does not modify the state of the other because they are an infinite distance apart.
- ☒ iv) A measurement on one of them defines the state of the other.

6) *Write T or F whether you think the statements are true or false.*

The quantum theory

- ☒ i) This is a probabilistic theory.
- ☒ ii) This does not contain a law through which we could find the time evolution of a state.
- ☒ iii) It concerns only ensembles and its significance for only one system is not so great.
- ☒ iv) This is not a completely causal theory.

7) The time evolution of a state in the frame of classical mechanics is estimated through the Newton's law. Which is the corresponding law of quantum mechanics?

Answer:

8) *Write T or F whether you think the statements are true or false.*

Elements of physical reality for a system are these quantities, which can be predicted with accuracy.

- ☒ i) Earth has spin. According to classical mechanics we know the three components of its spin but quantum mechanics allows us to know only one.
- ☒ ii) According to quantum mechanics we can know the three components S_x , S_y , S_z of the spin of an electron.
- ☒ iii) For the same system, according to classical mechanics the elements of physical reality are more than those of the quantum mechanics.
- ☒ iv) The elements of physical reality are these which are described by quantum

mechanics. But in classical mechanics the quantities are large so the theoretical errors are smaller than the experimental errors.

9) *Write T or F whether you think the statements are true or false.*

⊠ i) According to classical mechanics the results of a measurement are discrete values.
⊠ ii) The discrete values of some observables arouse suspicions that particles have wave features.
⊠ iii) The Hilbert space is a mathematical construction which is helpful to describe t nature in the frame of quantum physics.
⊠ iv) Schroedinger equation is provable.

10) Comment the Einstein saying. <<God does not play dice>>.

Answer

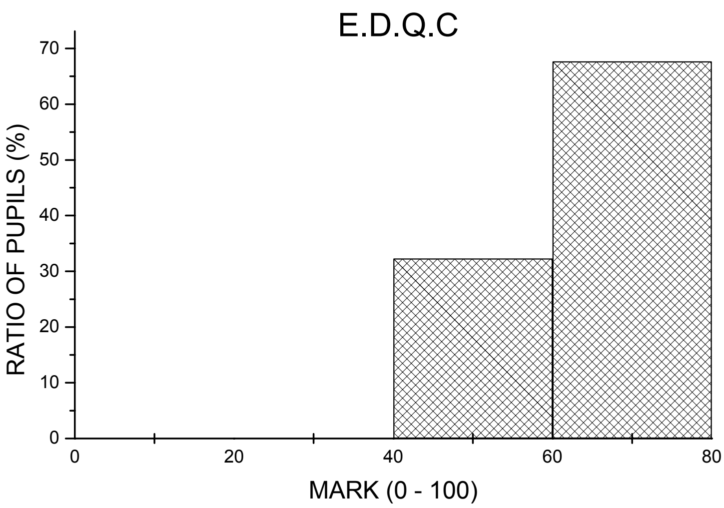
RESULTS - CONCLUSIONS OF QUESTIONNAIRE B

This questionnaire was given about two weeks after the end of the lessons. The aims and the corresponding results are the following:

1) Essential differences between quantum and classical physics(E.D.Q.C.):

By the questions 1,3,4,6,7 and 9 we tried to find out how much the pupils understood about some important differences between quantum and classical physics such as determinisms, definition of state, causality , discrete values, and so on .The results are the following:

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	0	32,3	67,7	0

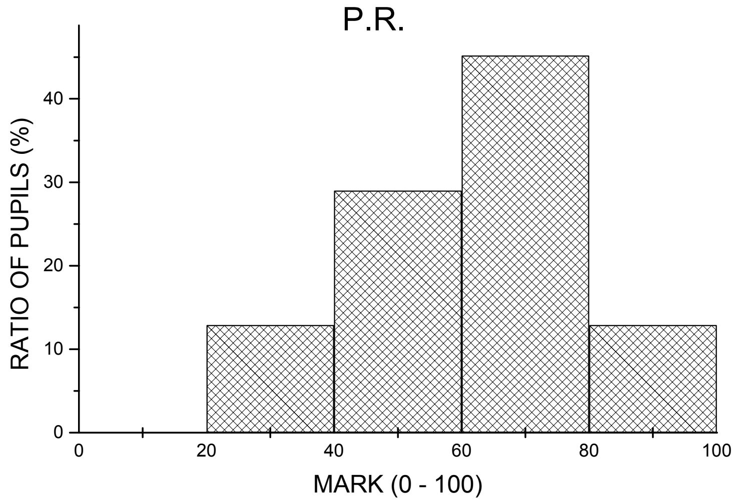


Remark: None above the 80!!!!!

2) Physical reality (P.R.)

Questions 2 and 8 investigated whether the initial views of the pupils which we had seen in questionnaire A have changed according to the new frame of quantum physics. The results are the following:

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	0	12,9	29	45,2	12,2



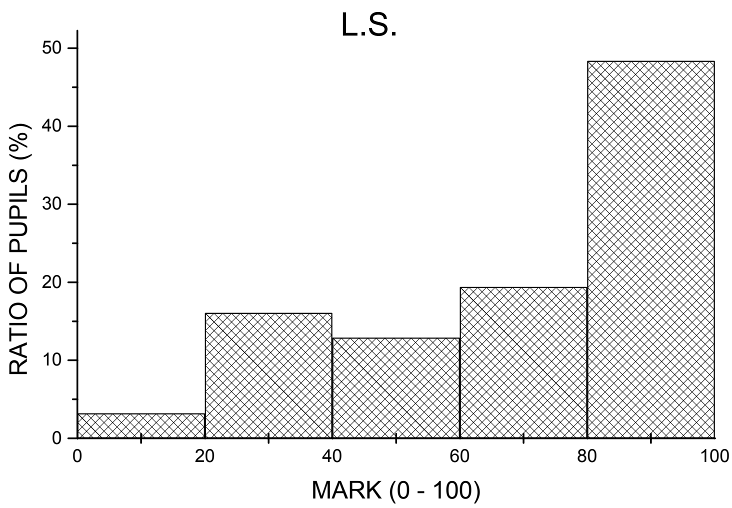
Remark: We observe a change but it is not so massive.

3) Locality , Separability(L.S.)

In question 5 we investigate if the pupils understood that the quantum physics despite the classical physics is a non-local theory, and that nonseparability is valid. The results are the following:

-

marks	0-20	20-40	40-60	60-80	80-100
Ratio (%)	3,2	16,1	12,9	19,4	48,4



Remark: The results are satisfactory .

COMPARISONS

A. Comparisons of lessons

1) From the results of the questionnaires 1,2,3 and 4we calculate the mean value (**M. V.)**

   of the total results .
   Questionnaire 1. M.V.= 57,15
   Questionnaire 2. M.V = 57,52
   Questionnaire 3. M.V = 70,03
   Questionnaire 4. M .V =56,58

   The total mean value of all of them is

   M.V.=60,3

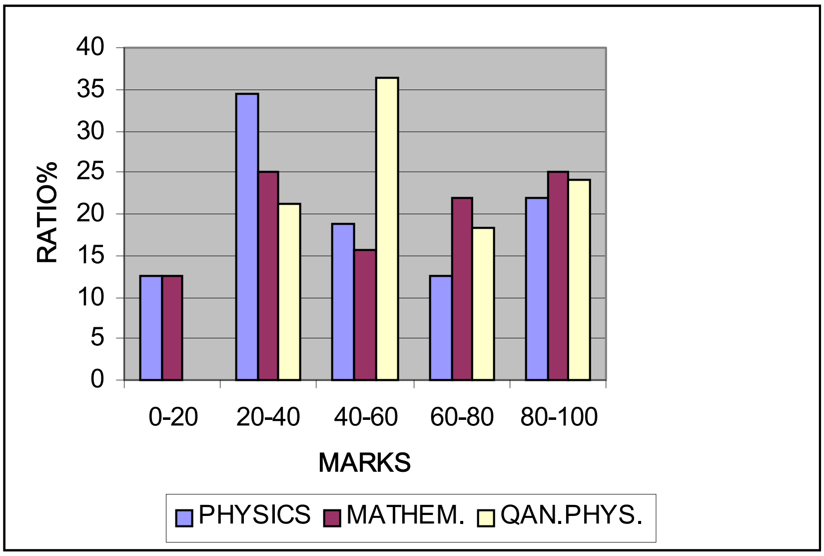
2) The pupils of the C' class of practical direction (specializing in natural sciences) have two lessons of physics , the first one is the same for both directions and the second one is more special and it is taught only to practical direction. It is named <<physics for the practical direction>> .The same is happens for the mathematics . We examined the marks of the lessons <<physics for the practical direction >> and <<mathematics for the practical direction>> of the writing tests and we compared these with the results of our questionnaires . The results and the corresponding mean values are the following

:

MARKS	0-20	20-40	40-60	60-80	80-100
PHYSICS	12,5%	34,4%	18,7%	12,5%	21,9%
MATHEMATICS	12,5%	25%	15,6%	21,9%	25%
QUANTUM PHYSICS	0%	21,2%	36,4%	18,2%	24,2%

LESSONS	PHYSICS	MATHEMATICS	QUANTUM PHYSICS
M.V.(mark)	49	52,8	60,3

The corresponding comparative diagram is the following:



Remarks:

a) We observe a stability for the mean value of all questionnaires except the questionnaire 3. where someone could see a noticeable difference from the remainder. It is due to the fact that at that stage , the pupils were able to work with the new mathematics and the new concepts were not so difficult . In spite of the fact that the mathematics in the questionnaire 4. were simple, the mean value is lower because of the fact that the new concepts were rather difficult.

b) The better performance on the quantum physics is due to the fact that the form of questionnaires are a little different from those of two other lessons. I think that under the same conditions the results would be the same.

B. Comparisons of chapters of physics

We gave to the pupils the following form:

COMPARISONS

Put degree from 1 to 5 for each object:

i. Degree of interest: (1= no interesting,……5= very interesting)

ii. Degree of difficulty: (1= no difficult,……5=very difficult)

iii. Degree of mathematical difficulties: (1= no difficult,……5=very difficult)

iv. Degree of suitability: (1= no appropriate , …5= very appropriate)

CHAPTER	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
INTEREST				
DIFFICULTIES				
MATHEMATICAL DIFFICULTIES				
SUITABILITY				

The results are the following:

i. Degree of interest

DEGREE	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
1	0%	9,7%	9,6%	0%
2	12,9%	19,5%	6,5%	12,9%
3	38,7%	22,5%	35,5%	25,7%
4	41,9%	25,8%	35,5%	41,9%
5	6,5%	22,5%	12,9%	19,5%

	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
M.V,	3,42	3,32	3,35	3,68
σ	0,80	1,3	1,1	0,9

ii. Degree of difficulty

DEGREE	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
1	0%	0%	0%	0%

2	67,7%	25,8%	29%	9,7%
3	22,6 %	25,8%	51,6%	29%
4	9,7%	35,5%	19,4%	54,8%
5	0%	12,9%	0%	6,5%

	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
M.V,	2,42	3,35	2,9	3,58
σ	0,66	1	0,7	0,8

iii. Degree of mathematical difficulties

DEGREE	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
1	16,1%	9,7%	16,1%	0%
2	54,8%	35,4%	38,7%	32,3%
3	22,6%	38,7%	29,1%	29,0%
4	6,5%	9,7%	16,1%	32,3%
5	0%	6,5%	0%	6,5%

	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
M.V,	2,19	2,68	2,45	3,13
σ	0,78	0,99	0,94	0,94

iv. Degree of suitability

DEGREE	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
1	6,5%	9,7%	6,5%	9,7%
2	6,5%	29,0%	9,7%	45,2%
3	12,9%	19,4%	38,7%	19,3%
4	54,8%	25,8%	29,0%	19,3%
5	19,3%	16,1%	16,1%	6,5%

	VIBRATIONS	WAVES	RIGID BODY	QUANTUM PHYSICS
M.V,	3,75	3,10	3,39	2,68
σ	1,04	1,28	0,96	1,09

Remarks:

a) The parameter σ is the dispersion and expresses the dissent among the pupils , as the higher the value of the σ ,the higher the division of the opinion of pupils.

b) Despite the results of the questionnaires which show better performance of the pupils on the quantum physics, we observe that the pupils think the chapter of quantum mechanics more difficult but also more interesting .

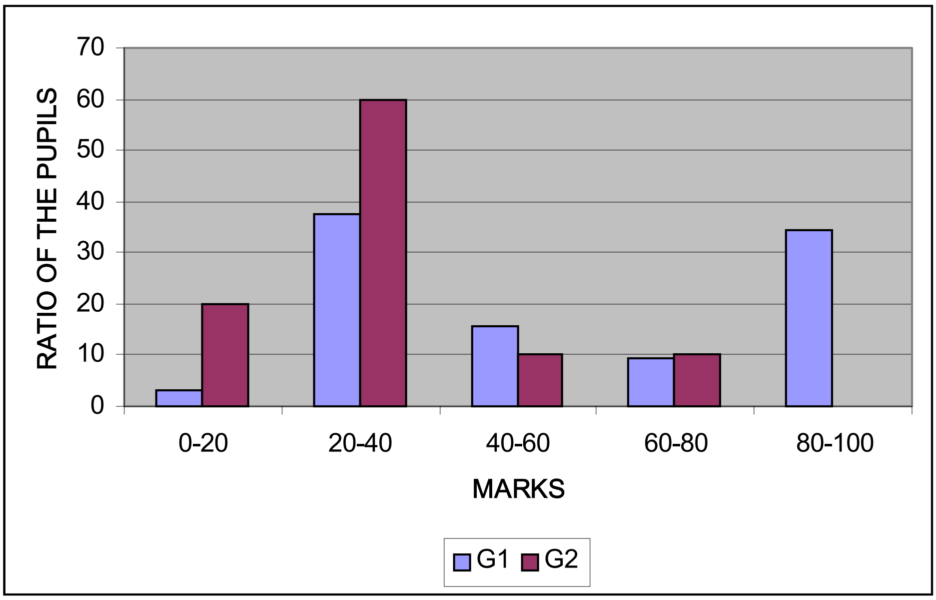
c) Very important is the fact that the pupils think quantum physics less appropriate for the high school !!! They think it very radical.

C . Comparisons of different groups

The paragraphs §1.1until § 1.9 were taught to another exact direction of C' class (G2) by another teacher . The questionnaire 1. was given to pupils .The total results of each group are the following:

MARKS	0-20	20-40	40-60	60-80	80-100
G1	3,1%	37,5%	15,6%	9,4%	34,4%
G2	10%	60%	10%	10%	0%

The corresponding comparative diagram is the following:



Remark:

We observe that the results from two groups are a little different, it is due to the fact that the number of the pupils of the G2 was very small (10) .Also the performance of the pupils of the G2 to the other lessons is lower than this of the pupils of the first class.

FINAL CONCLUSION

For establishing a concrete curriculum for the teaching program some criteria must be imposed. The main criteria for the evaluation of the teaching material are the **usefulness** and the **suitability**. In addition, it must be considered which topics can be replaced by the new ones because of the limited school program. The aim of the given questionnaires was to search and evaluate these two topics, i.e., the usefulness and the suitability.

Usefulness

Through the questionnaire A' the basic concepts positivism ,determinism, physical reality ,locality ,….concepts which are not usually defined by all pupils but are commonly accepted and are in the mind of everyone ,were researched. The results show that these conceptions exist in the mind of the pupils and are connected with the philosophical frame of classical physics . The questionnaires 1,3,4 and B' contain questions which permit us to compare the philosophical conceptions of classical physics with those of quantum physics. The results show that a satisfactory fraction of pupils understood and consolidated these differences .

The new concepts which are contrary to old concepts of classical physics were faced, surprisingly, with intensive interest from a large fraction of pupils but also with aversion, inability and conservatism from a smaller one.

In conclusion the result concerning this point is considered positive because the lesson shocked, existed and made the pupils think.

Also there are objective reasons which strengthen the factor of usefulness. This way of presentation of quantum mechanics relates to our world of digital and also the hard disc of computers is an enormous number of spins. A second application is to magnetic tomography as an application of the spins rotations. The majority of the pupils to which the lesson is addressed , are going to study later on at departments of universities which have the quantum physics as part of their program and the first contact with the subject at the high school would have beneficial results. In addition the pupils would face material which is at an advanced level of the scientific field and this makes the physics more attractive. Finally, the prospect of the production of quantum computers is a great challenge.

Suitability

Of course the teaching of quantum physics to high school has demands and requirements . The content of quantum physics is not the same with the other fields of physics and the difference is apparently much larger than the difference between classical mechanics and electrodynamics . The aim of questionnaires 1, 2, 3 and 4 is to investigate the suitability of the lesson as a school lesson. The results, taking into account the fact that the lesson was not obligatory and there was not pressure for reading and additional homework, are considered satisfied. Also the comparison of these results with the results of the lessons physics and mathematics of the existing curriculum strengthens the suitability of quantum physics.

A negative factor is that there is no possibility of experimental displays. Also the answers of the pupils to this point is a little discouraging. They show that the pupils are not ready to adopt this lesson and consider it a little radical.

In conclusion, my opinion is that if the quantum physics was enrolled into the curriculum, the performance of the pupils would be the same as that of the other lessons. Consequently the main factor of the decision must be the comparative usefulness because an other chapter must be left out of the curriculum and in addition the advantage of student being exposed into new concepts of quantum mechanics at an earlier age.

PART III

APPENDICES

APPENDIX I

Mathematical Background

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I. The Complex Numbers

1. Definition

The set ℂ of complex numbers is an expansion of the set ℝ of real numbers with following properties

α) The rules of summation and multiplication are the same as in R.
β) There is an element i where $i^2 = -1$.
γ) Every element z is written with unique manner as $z = a + \beta i$ where a, β real numbers. Also a is called the real part of z ($Re(z) = a$) and β is called the imaginary part of z
($Imz = \beta$).

2. Definition of Some Concepts and Actions

If $z_1 = a_1 + \beta_1 i$, $z_2 = a_2 + \beta_2 \cdot i$ then we have

i) Equality: $z_1 = z_2 \Leftrightarrow a_1 = a_2$, $\beta_1 = \beta_2$.

ii) Summation: $z_1 + z_2 = (a_1 + \beta_1 i) + (a_2 + \beta_2 i) = (a_1 + a_2) + (\beta_1 + \beta_2) \cdot i$

iii) Multiplication: $z_1 \cdot z_2 = (a_1 + \beta_1 i) \cdot (a_2 + \beta_2 \cdot i) = a_1 a_2 + a_1 \beta_2 i + \beta_1 \cdot a_2 i + \beta_1 \beta_2 \cdot i^2 =$
$= (a_1 a_2 - \beta_1 \beta_2) + (a_1 \beta_2 + \beta_1 a_2) i$

iv) Reverse of: If $z = a + \beta i$ $(z \neq 0)$ then the reverse of z is the number

$z^{-1} = x + y \cdot i = \frac{1}{z}$ such that

$$z \cdot z^{-1} = 1 \Rightarrow (a + \beta i)(x + yi) = 1 \Rightarrow x = \frac{a}{a^2 + \beta^2},\ y = \frac{-\beta}{a^2 + \beta^2}$$

so $$z^{-1} = \frac{a - \beta i}{a^2 + \beta^2}$$

v) Division: Division is defined as follows

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = z_1 \cdot \frac{1}{z_2} \qquad (z_2 \neq 0)$$

vi) The complex conjugate: If $z = a + \beta i$ then the complex conjugate is defined as

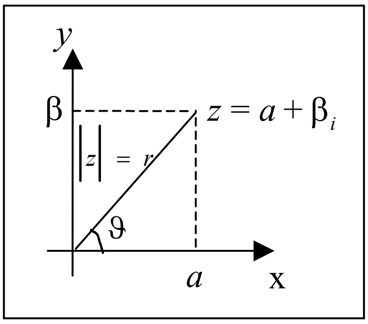
$$\overline{z} = a - \beta \cdot i$$

vii) Modules of z: It is defined as follows

$$|z| = \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + \beta^2} \quad , \ (z = a + \beta i)$$

viii) Geometric representation of z:

If $z = a + \beta i$ then we can represent it as a point of x, y-plane.
The angle ϑ is called the argument of z (arg (z)) and takes the values $0 \le \vartheta < 2n$.
One can easily show that



$$a = r \cdot \cos\vartheta, \quad \beta = r \cdot \sin\vartheta$$

where $r = |z| = \sqrt{a^2 + \beta^2}$ and $\vartheta = \tan^{-1}\left(\frac{\beta}{a}\right)$.

Also we define as

$$\boxed{e^{i\vartheta} = \cos\vartheta + i\sin\vartheta}$$

where e is the basis of Neper's logarithm $(e = 2,71....)$.

Then $z = r \cdot \cos\vartheta + r\sin\vartheta \cdot i = r(\cos\vartheta + i\sin\vartheta) \Rightarrow$

$$\boxed{z = r \cdot e^{i\vartheta}}$$

the last form is the exponential form of a complex number.

II. Matrices-Determinants

Matrix $m \times n$ is an orthogonal arrangement $n \cdot m$ numbers in m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Specially we will work with matrices of the form $n \times n$ (square matrices), $n \times 1$ (column matrices) and $1 \times n$ (row matrices).

For example $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a 2 x 2 matrix

$B = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a 2 x 1 matrix

$\Gamma = \begin{pmatrix} \alpha & \beta \end{pmatrix}$ is an 1 x 2 matrix.

We denote by abbreviation a $m \times n$ matrix A as

$$\mathrm{A} = \left[a_{ij} \right] \qquad 1 \le i \le n \ , \qquad 1 \le j \le m .$$

1. Addition of matrices

If $\mathrm{A} = \left[a_{ij} \right]$, $\mathrm{B} = \left[\beta_{\mathrm{ij}} \right]$ both of them $m \times n$ matrices we define as

α) <u>Addition</u>: $\mathrm{A} + \mathrm{B} = \left[a_{ij} + \beta_{\mathrm{ij}} \right]$

<u>Example</u>: $\begin{pmatrix} 2 & 3 \\ -1 & 7 \end{pmatrix} + \begin{pmatrix} 4 & 5 \\ 1 & -6 \end{pmatrix} = \begin{pmatrix} 2+4 & 3+5 \\ -1+1 & 7+(-6) \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 0 & 1 \end{pmatrix}$

β) Product of a number λ with a matrix A :

$$\lambda \mathrm{A} = \lambda \left[a_{ij} \right] = \left[\lambda a_{ij} \right]$$

for example $2 \cdot \begin{pmatrix} 3 & 1 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -10 & 8 \end{pmatrix}$

From the last two definitions we obtain

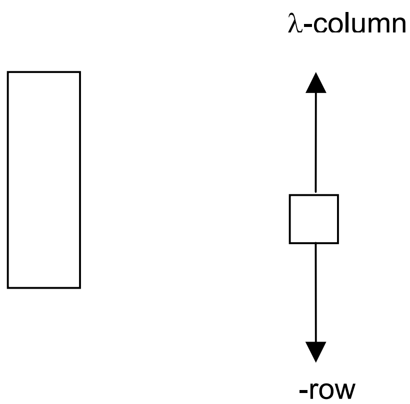
$$\lambda \mathrm{A} + \mu \mathrm{B} = \left[\lambda a_{ij} + \mu \beta_{\mathrm{ij}} \right]$$

2. Product between matrices

If $\mathrm{A} = \left[a_{ij} \right]$ $m \times n$ matrix and $\mathrm{B} = \left[\beta_{\kappa\lambda} \right]$ is n ´ r matrix then the product of two matrices is a matrix defined as follows

where

we have the form



Example:

If

Then

and

It is not necessary to be = . Generally

Properties of products: i)

ii)

iii) $(B + G)A = BA + GA$

3. Unit Matrix

Unit matrix is a square matrix with all diagonal elements equal to one and the rest equal zero.

Obviously for any matrix A it is valid

4. Reverse matrix

A matrix is said reversible if there is a matrix B also such that

Then the B is called the reverse of A and it is denoted as

i. e.

Also it is valid

Example:

Let and

then

Consequently

6. Transpose

For a matrix $A_{(m \times n)}$ we define its transpose matrix as the matrix which has as rows the columns of A and as columns the rows of A. That is

if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then

Properties:

i)

ii)

7. Conjugate matrix

For a matrix A we denote its conjugate matrix as follows:

That is we take the and replace any element with its complex conjugate.

Example: If then

Obviously .

8. Hermitian or self conjugate matrix

If for a $(n \times n)$ square matrix A it is valid $A^{+} = A$ then the matrix A is called Hermitian or self conjugate matrix.

Example:

The matrix is hermitian

because .

9. Unitary matrix

An matrix U is said to be unitary matrix if it is valid

Example:

The matrix

is unitary. Really

and

10. Determinant of square matrix

If as we know its determinant is defined as

For a 3x3 matrix we define the determinant as follows

The following properties are valid

i) $\det(AB) = \det A \cdot \det B$

ii)

iii)

11. Trace

The trace of a matrix A is the summation of all diagonal elements i.e.

Property:

<u>III. Vector Space</u>

Let us suppose the set V of all matrices column 2x1 with its elements to be complex numbers i.e. the matrices of the form with

We denote, for reason which we will see later. as

$|\ \rangle$ where $|\ \rangle$ is called ket.

We can write for any $|\ \rangle$

$|\ \rangle$

Denoting as $|\ \rangle$ and $|\ \rangle$ we obtain

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle$$

Consequently every element of v is written uniquely as linear combination of $|n_1\rangle$ and $|n_2\rangle$.

Then the pair (v , □) is a vector space over □ and the set $|\ \rangle\ |\ \rangle$ is a basis of the space. The basis is not unique. Each set $□|v_1\rangle, |v_2\rangle□$, with $|v_1\rangle$ and $|v_2\rangle$ linearly independent constitute a basis of the vector space. Two vectors $|\ \rangle\ |\ \rangle$ are linearly independent if they satisfy the property

$|\ \rangle \quad |\ \rangle$ if and only if

Example:

The vectors $|\ \rangle$ and $|\ \rangle$ constitute a basis of v . Really for every

$|\ \rangle$ we have $|\ \rangle \quad |\ \rangle \quad |\ \rangle$ where

Consequently $|\ \rangle$ —— $|\ \rangle$ —— $|\ \rangle$

Remark:

The number of the element of any basis is unique and constant and it is called the dimension of v .

IV. Inner Product

Definition

We consider the two vectors $|\ \rangle$ and $|\ \rangle$. Then the inner product of them is defined as the complex number .

But we have $\bar{\ } \quad \bar{\ } \quad \bar{\ }\ \bar{\ }$. For this reason it is useful to define for any

vector $|\ \rangle$ (ket) a corresponding $\langle\ |$ (bra) as follows

$$\langle\ | \qquad \bar{\ }\ \bar{\ }$$

Obviously if $|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad \langle\ | \quad \bar{\ }\langle\ | \quad \bar{\ }\langle\ |$.

Consequently the inner product of two vectors

$|\ \rangle$ and $|\ \rangle$ is defined as

$\langle\ |\ \rangle \quad \bar{\ }\ \bar{\ } \quad \bar{\ } \quad \bar{\ }$.

We can easily verify the properties

i) $\langle\ |\ \rangle \quad \overline{\langle\ |\ \rangle}$

ii) $\langle a | a \rangle \square 0 \qquad \langle\ |\ \rangle \qquad |\ \rangle$

iii) $\langle\ |\quad |\ \rangle \quad |\ \rangle \quad \langle\ |\ \rangle \quad \langle\ |\ \rangle$

iv) If $|\ \rangle \quad |\ \rangle \quad |\ \rangle$ then $\langle\ |\ \rangle \quad \bar{\ }\langle\ | \quad \bar{\ }\langle\ \ |\ |\ \rangle \quad \bar{\ }\langle\ \ |\ \rangle \quad \bar{\ }\langle\ \ |\ \rangle$.

A vector space which is supplied with an inner product is called Hilbert space or an inner product space.

2. Orthogonal Vectors

Two vectors $|\ \rangle\ |\ \ \rangle$ are orthogonal if $\langle b | a \rangle = \langle a | b \rangle = 0$.

3. Norm of a Vector - Unit Vector

It is defined as $\| |\ \rangle \| \quad \sqrt{\langle\ |\ \rangle}$.
If the norm of a vector is equal to unity, then it is a unit vector

$\langle a | a \rangle = 1 \quad \square \quad | a \rangle$ is a unit vector.

4. Orthonormal Vectors

Two vectors $|\ \rangle$ and $|b\rangle$ are orthonormal if

$\langle\ |\ \rangle$ and $\||a\rangle\| = \||b\rangle\| = 1$.

5. Orthonormal Basis

A basis which consists of orthonormal elements is called orthonormal basis.

<u>Example:</u>

Then basis which consists of the vector $|\ \rangle$ and $|\ \rangle$ is an orthonormal basis because $\langle n_1|n_2\rangle = 0$ and $\langle\ |\ \rangle$ $\langle\ |\ \rangle$. On the other hand the basis $\{|v_1\rangle, |v_2\rangle\}$ where $|\ \rangle$ and $|\ \rangle$ is not orthonormal because $\||\ \rangle\|$ $\sqrt{\langle\ |\ \rangle}$ $\sqrt{\ }$.

But the vectors $|v_1'\rangle = \frac{1}{\sqrt{2}}|v_1\rangle$ and $|\ \rangle$ $\frac{\ }{\sqrt{\ }}|\ \rangle$ constitute an orthonormal basis.

For the next chapters all our vectors will have norm one and all bases will be orthonormal.

<u>V. Linear Operator</u>

1. Definition

Let V be a vector space over the $\square$. Operator A is a map

i. e. $|\ \rangle$ $|\ \rangle$ where $|a\rangle, |b\rangle \in V$ …….

Linear operator is the operator with the property

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle$$

If A and B are linear operators then their sum is defined by the relation

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle$$

and their product by the relation

$|\ \rangle \qquad |\ \rangle$

Generally $\quad AB \neq BA$

2. Commutaror

The commutaror of two operators A, B is defined as follows

If $AB = BA$ then $[A, B] = 0$

and we say that the operators A and B commute.

VI. Representation of Operator by Matrix

Let $|\ \rangle|\ \rangle$ be an orthonormal basis from which we keep in mind the order of them, $|n_1\rangle$ is the first and $|n_2\rangle$ is the second. Let A be a linear operator and

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle$$
$$|\ \rangle \quad |\ \rangle \quad |\ \rangle$$

Then we correspond to A the matrix

$$\langle\ |\ |\ \rangle \quad \langle\ |\ |\ \rangle \qquad \langle\ |\ |\ \rangle$$
$$\langle\ |\ |\ \rangle \quad \langle\ |\ |\ \rangle$$

Thus the corresponding matrix of operator A is the matrix

For any vector $|\ \rangle \quad |\ \rangle \quad |\ \rangle$ we suppose that

$$|\ \rangle \ |\ \rangle \quad \text{where } |\ \rangle \quad |\ \rangle \quad |\ \rangle$$

We find that

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle$$
$$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle.$$

Consequently

The last relations can be written in the form

Remark:

1. Any relation which is satisfied by operators is also satisfied by the corresponding matrices.
2. Also the matrix representation of an operator depends on the choice of the basis.

Example:

Let $| \rangle | \rangle$ be an orthonormal basis of a vector space over the □ and three operators where it is valid

$$| \rangle \; | \rangle \qquad | \rangle \; | \rangle$$
$$| \rangle \; | \rangle \qquad | \rangle \; | \rangle$$
$$| \rangle \; | \rangle \qquad | \rangle \; | \rangle$$

α) Find the corresponding matrices

β) Prove the relations

i)

ii)

Solution

α) $$\langle \; | \quad | \; \rangle \quad \langle \; | \quad | \; \rangle$$
$$\langle \; | \quad | \; \rangle \quad \langle \; | \quad | \; \rangle$$

Similarly and

β) i) For any vector $| \rangle \quad | \rangle \quad | \rangle$ we find

$$| \rangle \qquad | \rangle \qquad | \rangle$$
$$| \rangle \qquad | \rangle$$

$|\ \rangle \qquad |\ \rangle$

$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle$

Also $|\ \rangle \qquad |\ \rangle \quad |\ \rangle \qquad |\ \rangle \qquad |\ \rangle$

$|\ \rangle \quad |\ \rangle$

Consequently

$\sigma_x \sigma_y |\alpha\rangle = i\sigma_z |\alpha\rangle$ for any vector $|\ \rangle$

Thus

.

The same relation could be proved using the corresponding matrices. Really

ii) Using the matrices we obtain

Thus

4. Unit Operator

Unit operator is the operator A where

$A|\alpha\rangle = |\alpha\rangle$ for any $|\ \rangle$.

Its representation in any basis is the unit matrix I. We use to denote A with I

.

VII. Eigenvectors and Eigenvalues

A vector $|\ \rangle$ is an eigenvector of an operator A with the corresponding eigenvalue λ if

$A|\alpha\rangle = \lambda|\alpha\rangle$.

Example:

As we have seen in the preceding example the vectors $|n_1\rangle$ and $|n_2\rangle$ are eigenvectors of the operator s_z with eigenvalues $+1$ and -1 respectively, but they are not eigenvectors of the operator s_x. However for the vectors $| \;\rangle \quad \frac{}{\sqrt{}} \; | \;\rangle \; | \;\rangle$

$$|v_2\rangle = \frac{1}{\sqrt{2}}(|n_1\rangle - |n_2\rangle)$$

we find that

$$| \;\rangle \qquad \frac{}{\sqrt{}} \; | \;\rangle \; | \;\rangle \quad \frac{}{\sqrt{}} \; | \;\rangle \; | \;\rangle \qquad | \;\rangle$$

and $$| \;\rangle \qquad \frac{}{\sqrt{}} \; | \;\rangle \; | \;\rangle \quad \frac{}{\sqrt{}} \; | \;\rangle \; | \;\rangle \qquad | \;\rangle .$$

Consequently the vectors $|v_1\rangle$ and $|v_2\rangle$ are eigenvectors of the operator s_x with eigenvalues $+1$ and -1 respectively.

General Problem:

An important problem is to find the eigenvectors and eigenvalues of an operator A, if we know its representation relative to a given basis.

Let $| \;\rangle | \;\rangle$ be a given basis and the representation of the operator A relative to this basis is

$$\begin{matrix} | \;\rangle & | \;\rangle & | \;\rangle \\ | \;\rangle & | \;\rangle & | \;\rangle \end{matrix}$$

The technique is the following (we work with vectors which their norm is equal to one):

α) We solve the equation (characteristic polynomial)

$$\det(A - l I) = 0$$

We have $\left| \quad \right|$

We find the roots l_1 and l_2 which are the eigenvalues of the operator A.

β) We suppose that the corresponding eigenvectors are

$$| \;\rangle \qquad | \;\rangle \qquad | \;\rangle$$

and

$| \rangle \quad | \rangle \quad | \rangle$

From the equations

$| \rangle \quad | \rangle$

and from the equation $\langle \; | \; \rangle \qquad | \; | \quad | \; |$ we calculate the numbers and c_2. In the same way we also calculate the coefficients c_3 and c_4.

<u>Remark:</u>

The number of the solutions is infinite because if a vector $|v_1\rangle$ is eigenvector then any other vector of the form $|v_1'\rangle = e^{ix}|v_1\rangle$ is also eigenvector with the same eigenvalue.

<u>Example:</u>

The representation of the operator s_x relative to the basis $| \rangle | \rangle$ is .

Find the eigenvalues and eigenvectors of the operator s_x.

<u>Solution:</u>

For its eigenvalues we have

$| \qquad |$

Let $| \rangle \quad | \rangle \quad | \rangle$

$| \rangle \quad | \rangle \quad | \rangle$

are the eigenvectors of s_x with corresponding eigenvalues $+1$ and -1.

Then $| \rangle \quad | \rangle$

Also $\langle \; | \; \rangle \qquad | \; | \quad | \; |$

One solution is $\frac{}{\sqrt{}}$. Consequently $| \ \rangle$ $\frac{}{\sqrt{}}$ $| \ \rangle$ $| \ \rangle$

Similarly $| \ \rangle$ $| \ \rangle$.

Also $\langle \ | \ \rangle$ $| \ | \ | \ |$.

One solution is $\frac{}{\sqrt{}}$ and $\frac{}{\sqrt{}}$. Thus $|v_2\rangle = \frac{1}{\sqrt{2}}(|n_1\rangle - |n_2\rangle)$.

Degeneracy:

If for an operator it is valid that $l_1 = l_2$ that is two different eigenvectors of an operator have the same corresponding eigenvalue then it is said to be a degeneracy.

Remark:

If we have an orthonormal basis $| \ \rangle | \ \rangle$ then any vector $| \ \rangle$ can be written as a linear combination of $|n_1\rangle$ and $|n_2\rangle$ that is

$$| \ \rangle \quad | \ \rangle \quad | \ \rangle$$

Then we can substitute the ket $| \ \rangle$ by a column vector and the bra $\langle \ |$ by a row vector $(\bar{c}_1 \ \bar{c}_2)$. Then the action of an operator A on $| \ \rangle$ is the same as the action of the corresponding matrix A on the column vector .

That is $| \ \rangle \ | \ \rangle$ $\qquad$ $| \ \rangle \quad | \ \rangle \quad | \ \rangle$

↓
matrix

Many times we denote as $| \ \rangle$ but we must keep in mind what we mean.

VIII. Hermitian Operators

Conjugate operator

We suppose a basis $| \ \rangle | \ \rangle$ and let the representation of an operator A relative to this basis be the matrix

Then if $| \ \rangle$ and $| \ \rangle$ are two arbitrary vectors

we find $\langle \ | \ | \ \rangle$

(1)

We define as conjugate operator of A the operator B such that

$$\langle \ | \quad | \ \rangle \ \langle \ | \quad | \ \rangle \tag{2}$$

We consider that the B has the form

Then

$$| \ \rangle \quad | \ \rangle$$

$$\langle \ | \quad \langle \ |$$

We obtain the last result using the property for the matrices.

Consequently $\langle \ | \quad | \ \rangle$ (3)

From (1), (2), (3) we obtain

and because of it is valid for any it must be valid

That is the matrix of a conjugate operator is the conjugate matrix.

If $A^{+} = A$ then A is called Hermitian operator or selfconjugate operator. Then we write

$$\langle \; | \quad | \; \rangle \qquad \langle \; | \qquad | \; \rangle \; \langle \; | \quad | \; \rangle$$

The Hermitian operators play an important rule to quantum mechanics.

Below we give without proof four very important theorems concerning the Hermitian operators.

<u>1st Theorem:</u> The eigenvalues of a Hermitian operator are real numbers.

<u>2nd Theorem:</u> The eigenvectors of a Hermitian operator are orthogonal.

<u>3rd Theorem:</u> If two Hermitian operators commute i. e. $[A, B] = 0$ someone can have a complete set of simultaneous eigenvectors of A and B.

<u>4th Theorem:</u> If A is a Hermitian operator acting on a space V then the space V has an orthonormal basis which consists of eigenvectors of operator A (spectral theorem)

Unitary Operators

An operator U is called a unitary operator if it preserves the inner product. That is for any two vectors $|a\rangle, |b\rangle$ we have

$$| \; \rangle \quad | \; \rangle$$

$$U|b\rangle = |b'\rangle$$

such that $\langle b'|a'\rangle = \langle b|a\rangle$

we find $\langle \; | \; \rangle \; \langle \; | \qquad | \; \rangle \; \langle \; | \; \rangle$

thus $U^{+}U = I$.

Obviously the corresponding matrix is a unitary matrix.

APPENDIX II

DENSITY MATRIX

A. Pure State

We consider a basis of Hilbert space

$$\{|a_1\rangle, |a_2\rangle, \ldots. |a_n\rangle\}$$

and a linear operator A where:

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle$$
$$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle$$

$$|\ \rangle \quad |\ \rangle \quad |\ \rangle \quad |\ \rangle$$

Through the above set of relations, the operator A is completely defined. An elegant form to write the operator A is

$$|\ \rangle\langle\ |$$

It is readily to show that the last form satisfy all the relations defining the operator A .

Operators of the form

$$|\ \rangle\langle\ |$$

are projection operators. Also any operator of the form

$$|\ \rangle\langle\ |$$

is also a projection operator. The action of them is to project any vector $|F\rangle$ to vector $|Q\rangle$. That is

$$|\ \rangle \quad |\ \rangle\langle\ |\ \rangle \quad |\ \rangle \ \langle\ |\ \rangle|\ \rangle$$

Also

$$|\ \rangle \quad |\ \rangle\langle\ |\langle\ |\ \rangle|\ \rangle \ \langle\ |\ \rangle|\ \rangle\langle\ |\ \rangle$$

Consequently

$$\boxed{\qquad}$$

As we know a complete set of

$$p_i = |a_i\rangle\langle a_i|$$

has the property

$$| \;\rangle\langle\; |$$

Specially if a system is in $|\Psi\rangle$ state, then the projection operator

$$| \;\rangle\langle\; |$$

is called **density operator**. We will see follow the uses and some properties of this operator.

Remark: For any operator of the form

$$| \;\rangle\langle\; |$$

we find that

$$TrD = \sum_n \langle a_n|F\rangle\langle Q|a_n\rangle = \sum_n \langle Q|a_n\rangle\langle a_n|F\rangle =$$

$$\langle\; | \quad | \;\rangle\langle\; || \;\rangle$$

$$\langle\; | \;\rangle$$

Properties:

α) ρ is a Hermitian operator

Proof:

$$| \;\rangle\langle\; | \qquad | \;\rangle\langle\; |$$

$$| \;\rangle\langle\; |$$

β)

Proof:

$$| \;\rangle\langle\; | \quad \langle\; | \;\rangle \; || \qquad ||$$

γ) ρ is positive

Proof:

For any vector $|F\rangle$

$$\langle\; | \; | \;\rangle \quad \langle\; | \;\rangle\langle\; | \;\rangle \quad |\langle\; | \;\rangle|$$

δ) For any observable A with corresponding operator A it is valid

Proof:

⟨ ⟩ ⟨ | | ⟩ ⟨ | ⟩⟨ | | ⟩

⟨ | | ⟩⟨ | ⟩ ⟨ | | ⟩

ε) Its eigenvalues are real nonnegative numbers and sum to one.

Proof:

Let the | ⟩ | ⟩ | ⟩ is the basis consists of the eigenvectors of r and $p_1, \ldots, p_n$ the corresponding eigenvalues.

Obviously

⟨ | | ⟩ |⟨ | ⟩|

As we know the trace is invariant, irrespectively of the basis. Thus

στ)

Proof:

Density Matrix for spin state

The general form of spin state is , thus

The density matrix ρ is a 2x2 matrix where its elements are complex numbers. As we know constitute a basis of these matrices. We write ρ in the form

$$\rho = \frac{1}{2}(1 + \vec{p}\cdot\vec{\sigma})$$

We put the factor $\frac{1}{2}$ because $Tr\sigma_i = 0$ and .

Then

Consequently $\vec{p}$ is a unit vector.

Also

But

Thus

and

$$\rho = \frac{1}{2}(1 + \hat{n}\cdot\vec{\sigma})$$

B. Mixed State

Generally, density matrix, expressed in the basis in which it is diagonal, has the form

$$| \rangle\langle |$$

where $0 \,\square\, p_i \,\square\, 1$ and

If the state is pure then the above form has only one term so that $| \rangle\langle |$ and .
That is the state is a ray. However if it has two or more terms then

$$r^2 \,\square\, r \qquad \text{because}$$

In this case we say that r is an **incoherent** superposition of states $| \rangle$ and the state of the system is **mixed state**.

Also in this case r has the properties.

α)
β) $r^+ = r$
γ) r is positive
δ) For any observable A we have
$$\langle \rangle \qquad \langle | \; | \rangle$$
ε) $\det r \,\square\, 0$

ζ) $\langle | \quad | \rangle\langle | \, | \rangle$

As we have already discussed we can construct a new space where from the mixed state we take a state which is pure state. We suppose the orthonormal vectors $| \rangle | \rangle \quad | \rangle$ which generate a space H □. The new space is

where H is the Hilbert space produced by $| \rangle | \rangle \quad | \rangle$.

We consider the state

$$| \rangle \qquad \sqrt{} \; | \rangle \; |\rangle$$

of H_t space. Then the operator

$$r \,\square\!= |\Psi\rangle\langle\Psi|$$

is of course a projection operator and the state $|\Psi\rangle$ is a pure state. This is the density operator of the new space.

We find that

$$\langle | \quad | \rangle \qquad \langle | \quad \rangle\langle \quad | \rangle$$

$$\langle | \qquad \sqrt{} | \quad \rangle \quad | \quad \rangle \sqrt{} \langle \quad | \quad \langle \; | \; | \; \rangle$$

$$| \quad \rangle\langle \quad |$$

Consequently

If we have an observable A in H space we define the an operator which acts on H_t space.

We find that

$$\langle \quad \rangle \quad \langle \; | \qquad | \; \rangle \qquad \langle \; | \quad | \; \rangle \quad \langle \quad \rangle$$

Also

$$\langle \quad \rangle \qquad\qquad \langle \quad \rangle$$

Pure and mixed state and Stern-Gerlach experiment

We will give another argument trying to persuade that pure and mixed states are two different things.

We consider two beams like these emerging from the inhomogeneous magnetic field in Stern-Gerlach experiment with the following properties

1^{st} beam is emerging from an inhomogeneous magnetic field and is in the pure state $|z\,-\rangle$. This for a system which arises after a rotation □ (right handed sense) about y-axis is also a pure state

$$| \quad \rangle \qquad -| \quad \rangle \qquad -| \quad \rangle$$

2^{nd} beam consisting of two different pure beams $|\square\,-\rangle$ and $|\square\,\square\rangle$ with statistical weight $\cos^2\frac{\square}{2}$ and $\sin^2\frac{\square}{2}$ respectively.

If each beam passes through an inhomogeneous magnetic field then we will read off the same results. But if each one pass through an inhomogeneous magnetic field then for the first one we will have ⟨ ⟩ ⟨ | | ⟩ – but for the second one ⟨ ⟩ –⟨ | | ⟩ –⟨ | | ⟩ .

As we can see the two beams are not the same thing.

Evolution of the density operator

The time evolution of a state |Y ⟩ is given through the relation

| ⟩ | ⟩

Consequently

⟨ | ⟨ |

As we have pointed out r is defined for the definite state |Y ⟩. So we can say that

| ⟩⟨ |

| ⟩⟨ |

For the mixed state we have also the same result.

APPENDIX III

ENTANGLEMENTS

Let us suppose that we have two spin particles and . The state of the system is

where and are the individual states for each particle. For the particle the Hilbert space is two dimensional space and a basis of it is the set .
For simplicity we write them as . Similarly for the particle the is the expansion of the set .

The Hilbert space of the two particles and is the expansion of the basis

(α)

that is, it is , a four dimensional space.

Let ,

be the spin operators for and particles respectively. Obviously and are operators acting on . Also they commute because they refer to different particles.

The total spin angular momentum of the system is

.

It is easy to show that

Consequently the total spin is also angular momentum. Thus we can construct a basis constituted of eigenvectors of and . We labelled these vectors as and it is valid

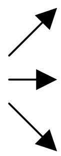
(we have put for simplicity).

Each of the states (α) is eigenvector of .

Thus

Also

The possible values of are only



and .

In any other case the total number of the states would be more than four.

i) The state is the state . We can show that

.

Also we have seen that

Similarly we can show that the state is the state .

ii) The state is the state , because

and

Eventually we have a new basis of

triplet states

single states

What is the Entanglement?

The two spins system could be in any state of the form

Some of them states are characterized as entanglement and the others as not entanglements. What is the criterion by which we can discrite a state as entanglement or not? We will give this criterion later.

Let us suppose a system of two spins and which we will call biparticle quantum system. The Hilbert space of the system is and an arbitrary pure state has the form

where

.

If we measure an observable concerning only the particle, its expectation value is given through the relation

Where

Similarly

Where is the density matrix operator in Hilbert space and is a pure state.

We can say that is the density operator for subsystem and is the density operator for subsystem .

It is easy to show that

i)
ii) is positive
iii)

We can find the eigenvectors and and its nonnegative eigenvalues and .

Then

where

If the sum contains only one term then and state of subsystem is pure state (also). Then the state is NOT ENTANGLEMENT.

On the other hand if it contains two terms then

and the state is a mixed state. Then the state is ENTANGLEMENT. The particle and are entangled!!

We can show that the states

and

are not entanglements

but

are both of them entanglement.

In mixed state case

so we can interpret as describing an ensemble of pure quantum states, in which the state occurs with probability .

Above we have seen how we can take an incoherence state. The two systems and interact each other, become entangled (correlated), the entanglement destroys the coherence states which the and were in. Also we use to say that the coherence state of collapses.

BLOCH SPHERE

As we have said the general form of density matrix is

and

If then

.

So in this case the state is pure.

If then

and the state is a mixed state.

Consequently we can correspond any possible density matrix with a point in unit 3-ball . When the point belongs to surface of the ball (is the sphere) and the state is pure. In other case is mixed. This ball is called Bloch-sphere. We will give the physical significance of the vector . Let is the corresponding state having density matrix

Then

If then and the state is pure and completely polarized. On the other hand if then the state is mixed and completely unpolarized. Thus the parameterises the polarization of the spin. We can determine if we measure the along each of three axes (-x, -y, -z).

CONVEXITY

For any operator

We can find two other operators (density matrices)

,

such that

(1)

It is valid also and the converse. For any , the linear combination

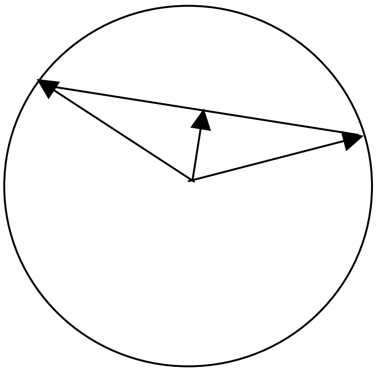
is also density matrix if , are density matrices. Set with above property are called convex set. A set is convex set if all points of the segment formed by the straight which connects any two points of it, also belongs to it.

An example is the Bloch sphere (ball). The points of the surface cannot be written through relation (1). In this case it is valid . These points are called extremes and are pure states.
Any mixed state can be written through infinite ways. We can prepare a mixed state with

as a combination of pure states, the edges of any chord passing through the point (see figure a).

This that we can write it through many ways is called <u>ambiguity</u>.

fig a



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