

Non-equilibrium Dynamics of Symmetry Breaking and Gauge Fields in Quantum Field Theory

Dissertation

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Chapter 1

Introduction

1.1 Non-equilibrium Dynamics in QFT

Non-equilibrium dynamics have become a very active field of research in the last few years in nearly all parts of physics. In condensed matter physics for example the description of the dynamics of non-equilibrium phase transitions plays an important role [1]. Such phase transitions occur in ferromagnets, superfluids, and liquid crystals to name only a few. They are subjects of intensive studies, both theoretical and experimental.

Also in cosmology some phenomena require the use of non-equilibrium technics. One example is the electroweak phase transition which took place 10^{-12} seconds after the Big Bang. If the electroweak phase transition is a phase transition of first order then it leads to a possibility to explain the observed asymmetry between matter and anti-matter. The mechanism which is responsible for the asymmetry is called baryogenesis. It can only occur if the so-called Sakharov conditions [2] are fulfilled. They are non-conservation of baryon number, C and CP violation and a system out of thermal equilibrium. This problem is investigated by several groups, e.g., [3, 4], using non-equilibrium methods.

Another phenomenon in cosmology where non-equilibrium dynamics are important is the inflationary phase of the early universe which is studied intensively by different groups [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Inflation refers to an epoch during the evolution of the universe in which it underwent an accelerated expansion phase. This would resolve some of the short comings of the Standard hot Big Bang model, e.g., the flatness problem, concerning the energy density of the universe and the horizon problem, related by the large scale smoothness of the universe, indicated by the Cosmic Microwave Background Radiation (CMBR). For a general introduction to inflation, see e.g. [15, 16].

At lower energies heavy ion collisions are under consideration as non-equilibrium processes [17, 18, 19, 20, 21, 22]. In such heavy ion collisions a new state of matter could be reached if the short range impulsive forces between nucleons could be overcome and if squeezed nucleons would merge into each other. This new state should be a Quark Gluon Plasma (QGP), in which quarks and gluons, the fundamental constituents of matter, are no longer confined, but free to move around over a volume in which a high enough temperature and/or density reveals. Heavy ion collisions are studied experimentally at current and

forthcoming accelerators, the Relativistic Heavy Ion Collider RHIC at Brookhaven and the Large Hadron Collider LHC at CERN. The occurring Quantum Chromodynamic (QCD) phase transition in these processes could be out of equilibrium and lead to formations of coherent condensates of low energy pions, so called Disoriented Chiral Condensates (DCC). Recent results reported from CERN-SPS [23] seem to indicate a strong evidence for the existence of a QGP in Pb-Pb collisions.

The described systems cannot be analyzed in equilibrium because of their extreme environments. For example a rapid decrease of temperature as it occurs in the early universe does not allow an equilibrium description. In the last years many efforts have been made in developing a formalism which allows a description of systems out of equilibrium. It is based on the CTP formalism developed by Schwinger [24] and independently by Keldysh [25]. This is only a perturbative approach and in the last decades different nonperturbative approximation schemes were developed and improved. Classical approximations were used for example by Khlebnikov and Tkachev [26] to study the non-equilibrium evolution during inflation and reheating. In this approach the classical equations of motions are directly solved on a computer. Since the theories contain an infinite number of degrees of freedom the regularization is performed on a lattice in space. Questions about the dependence of the results on the lattice space and if a classical description of the physics in the early universe leads to the right predictions were studied by Smit and Aarts [27]. Beyond the classical approximation there are several different methods, semiclassical, large N , and mean field methods, under investigation. The Hartree approximation as a mean field method was extensively analyzed for example by Boyanovsky et al. [28]. They have also worked on the large N approximation [28, 29] beside other groups as for example Cooper et al. [21]. We have concentrated ourselves on the semiclassical one loop approximation and the large N expansion and study them in this work in detail.

We are especially interested in the quantum field theoretical aspects of non-equilibrium dynamics. In our previous works we have developed a method which allows a clean separation between divergent and finite parts of the equations and thus makes a numerical implementation rather simple. The method is based on [30, 31, 32] and we have extended it to different models and approximations. The simplest model we have analyzed is the ϕ^4 theory in the one loop approximation [33]. This paper constitute the basic for all our further studies. Here, we extend the approximation to the large N model. The ϕ^4 theory offers, as a simple toy model, the possibility to study different approximation schemes and analyze the behavior of the system in detail without leading to further problems induced by the complexity of the model, as is, for example, the case in gauge theories. We are especially interested in the effects caused by spontaneous symmetry breaking. For the unbroken case we have published our results in [34] and here we show only some special examples of the numerical results. Furthermore, we investigate finite temperature effects. Due to the spontaneous symmetry breaking the possibility for a phase transition occurs. Such a phase transition is important for many physical phenomena and the non-equilibrium description is necessary for systems as diverse as formation and evolution of defects in ${}^4\text{He}$ after a rapid quench, or the efficiency of baryogenesis in the electroweak phase transition as mentioned above.

As for models discussed in the context of the electroweak phase transition gauge fields and fermions are of importance, we discuss, beside the ϕ^4 theory, gauge theories in detail. The inclusion of fermion fields is studied in [35]. It is not part of this thesis. Gauge theories form the basis for all modern elementary particle theories. An unalterable requirement of such theories is that they are renormalizable. The renormalization of the parameters of a theory is necessary to find a relation between the computed and the measured observables. Important for the success of gauge theories is the fact, that they can be always renormalized if the gauge bosons are massless. This was shown in the fundamental works of 't Hooft and Veltman [36] and Lee and Zinn-Justin. Herewith, their important role as models describing interactions was founded. Due to the various degrees of freedom they offer a wide range for numerical experiments. We are interested in two different aspects: the first one concerns gauge invariance. We investigate the dependence of the effective action of the gauge parameter ξ for general R_ξ -gauges. We start with the analysis of the gauge invariance of the effective action for the static case of bubble nucleation. Motivated by the result that the exact one loop correction to the nucleation rate is gauge independent we then study the fluctuation operator for a system out of equilibrium for general R_ξ -gauges. We find after a suitable transformation a new fluctuation operator equivalent to that of the Coulomb gauge. Therefore, we investigate the non-equilibrium system in the Coulomb gauge and in a gauge invariant formalism also leading to the same operator. Our second purpose concerns the influence of the different degrees of freedom on the behavior of a system out of equilibrium. Hence, we carry out some numerics for different parameter sets for the Coulomb gauge and as a special case of R_ξ -gauges the 't Hooft-Feynman background gauge.

1.2 Content of the Thesis

This work is organized as follows. In chapter 2 we analyze the ϕ^4 theory in the large N approximation. We describe the model and the approximation in section 2.2 and carry out the renormalization in section 2.3 and 2.4 in order to implement the model numerically. We are especially interested in the influence of spontaneous symmetry breaking on the development of the system. Hence, we study in section 2.5 numerically the broken and unbroken symmetry case and analyze the different effects which occur. We discuss the phase structure of the system as a function of temperature and initial conditions. Furthermore, we generalize some predictions which were made in [29] for the long time behavior of the system to finite temperature.

In chapter 3 we investigate gauge theories under different aspects. Our interest concerns the dynamical evolution of gauge theories as well as the problem of gauge invariance, which is discussed in section 3.1. We describe the model under consideration in section 3.2. We calculate the gauge and the gauge fixing mode and explain their role for the search of a gauge invariant description of the effective action of the Higgs field in the SU(2) Higgs model. Then in section 3.3, an explicit example for the computation of the effective action independently of the gauge fixing parameter ξ is given. We discuss the bubble nucleation

as a time independent, metastable, and radially symmetric configuration. As a second example we study in section 3.4 the SU(2)-Higgs model in the non-equilibrium context. We derive the mode functions for arbitrary ξ and create a new fluctuation operator whose diagonal elements are independent of the gauge fixing parameter.

Another approach we investigate in the next chapter is based on the gauge invariant description of the effective potential developed by Boyanovsky et al. [37]. We extend their method to the non-equilibrium case and show which problems arise in this context. Then we study the Coulomb gauge, which is very similar to the gauge invariant description in the one loop approximation. After a renormalization procedure, we carry out some numerical simulations and analyze the behavior of the system for different masses and couplings. We compare the results with those we have found in our previous work [38] where we have used a 't Hooft-Feynman-background gauge in order to investigate the influence of different gauges in non-equilibrium systems.

Finally, we present in the last chapter our conclusions and give an outlook.

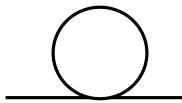
Chapter 2

ϕ^4 Theory in the Large N Approximation

2.1 Introduction

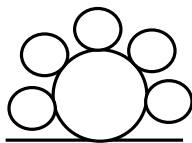
The time evolution of a system out of equilibrium is described in the Heisenberg picture by the Heisenberg equation of motion. After specifying an initial density matrix the dynamics of the system can be in principle described. The problem which arises is that in general the equations of motion cannot be solved exactly. Therefore, approximation schemes have to be developed. Perturbative expansions are only useful if the identification of a small parameter is possible. For a non-equilibrium system this identification is very difficult because, even for a small coupling constant, the field amplitudes can become nonperturbatively large and the expansion breaks down. Different nonperturbative approximation schemes have been developed recently in a non-equilibrium context. They include the quantum fluctuations explicitly in the effective equations of motion. The simplest nonperturbative approximation is the one loop approximation where the effective mass of the mode functions includes the second derivative of the classical potential and not just m^2 . Extensions of this approximation are, e.g., the large N approximation and the Hartree approximation since in these approximations also the fluctuation terms are included in the effective mass. Here, we study the one loop approximation and the large N approximation in order to investigate the behavior of a non-equilibrium system under the different approximations schemes. We discuss the main properties of these two schemes and their main differences. We analyze numerically both of them and show for which setting the one loop approximation is not reliable anymore. For this purpose we choose the simplest model for the description of a phase transition, the ϕ^4 theory with spontaneous symmetry breaking. The results can be used to make predictions for the development of extended models, like the SU(2) Higgs model which is studied in the next chapter.

In the one loop approximation we deal with a single field theory. We expand the field Φ about its mean value and include only the one loop self energy diagram of the form



without further resummation. This leads to an effective mass in the mode functions which contains only the second derivative of the classical potential and no fluctuation terms. This approximation is also often referred to as one loop resummed approximation. As we will see this kind of resummation leads to problems in a spontaneously broken theory for the complex part of the effective potential. During the evolution of the zero mode in the complex part of the effective potential instabilities arise. They lead to an exponential growth of the modes and therefore, the approximation breaks down.

Better results can be obtained with mean field approximations. Two of them are discussed extensively in the literature for non-equilibrium processes, the Hartree and the large N approximation. They are closely related to each other. Both are based on graphs of the form



whereby the propagators corresponding to the lines differ for the two approximations. For both approximations the second derivative of the classical potential for the mass term of the mode functions in the one loop approximation is replaced by its Gaussian mean value. This leads to self consistent equations. The explicit structure of the effective mass is of course different in the two approximations, see e.g. [21]. Since the Hartree approximation is only a variational ansatz and not a consistent expansion in any small parameter the renormalization procedure is not clear.

In the large N approximation the scalar field Φ is replicated in i components $\Phi \rightarrow \Phi_i$, $i = 1, \dots, N$ which leads to the possibility for a systematic power series in the parameter $1/N$. The leading order in the large N corresponds to a self-consistent mean field approximation. We are especially interested in the effects of the spontaneous breakdown of the global $O(N)$ symmetry which leads to the existence of $N - 1$ massless Goldstone bosons. They dominate the dynamics in the large N limit and lead to an efficient mechanism for the mean field to continuously transfer its kinetic energy to the massless modes over time. It is important to notice that mean field approximations contain no mode-mode collision terms which means that the particles interact with each other only through the mean field. Since the large N approximation is a well defined expansion in powers of N it has two advantages over the Hartree approximation: the renormalization can be done in a consistent way as we will show in the following section, and furthermore, it leads to

the possibility to improve the approximation in a systematic way by retaining higher order terms in the series. A detailed description and comparison of the different approximation schemes can be found e.g. in [21].

In the following section we consider the ϕ^4 theory in different approximation schemes. We investigate in detail the model with spontaneous symmetry breaking at finite temperature in the large N approximation with special emphasis on the renormalization procedure. We carry out numerical studies for the one loop approximation at $T = 0$ and show in which regime it fails. In order to study the effects of the symmetry breaking we compare for $T = 0$ numerical results for the unbroken and the broken symmetry case in the large N limit. We discuss the effects induced by finite temperature and calculate the critical temperature for a phase transition.

2.2 The Model

We consider the $O(N)$ model with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i - \frac{\lambda}{4N} \left[(\phi^i \phi^i) - N v^2 \right]^2, \quad (2.1)$$

where $\phi^i, i = 1, \dots, N$ are N real scalar fields. The non-equilibrium state of the system is characterized by a classical expectation value which we take in the direction of ϕ^N . We split the field into its expectation value ϕ and the quantum fluctuations ψ via

$$\phi^i(\vec{x}, t) = \delta_N^i \sqrt{N} \phi(t) + \psi^i(\vec{x}, t). \quad (2.2)$$

In the large N limit all terms which are not for the order N are neglected in the Lagrangian. In particular terms containing the fluctuation ψ^N of the component ϕ^N are at most of order \sqrt{N} and are therefore dropped. This is in contrast to the Hartree approximation where the fluctuations of ϕ_N are included. The fluctuations of the other components are identical, their summation produces factors $N - 1 = N(1 + O(1/N))$. Identifying all the fields $\psi^1, \dots, \psi^{N-1}$ as ψ the leading order term in the Lagrangian then takes the form

$$\mathcal{L} = N (\mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_I), \quad (2.3)$$

with

$$\mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2, \quad (2.4)$$

$$\mathcal{L}_\psi = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi + \frac{\lambda}{2} v^2 \psi^2 - \frac{\lambda}{4} (\psi^2)^2, \quad (2.5)$$

$$\mathcal{L}_I = -\frac{\lambda}{2} \psi^2 \phi^2, \quad (2.6)$$

where ψ^2 is to be identified with $\sum \psi^i \psi^i / N$.

We decompose the fluctuating field into momentum eigenfunctions via

$$\psi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \left[a_k U_k(t) e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger U_k^*(t) e^{-i\vec{k}\cdot\vec{x}} \right], \quad (2.7)$$

with $\omega_{k0}^2 = m_0^2 + k^2$. The mass m_0 will be specified below. This field decomposition defines a vacuum state as being annihilated by the operators a_k .

The equations of motion for the field $\phi(t)$ and of the fluctuations $U_k(t)$ have been derived in this formalism by various authors [21, 22, 39]. In addition to the large N Lagrangian (2.3, 2.4) we use, on averaging over the quantum fluctuations, rules like

$$(\psi^2)^2 \Rightarrow \langle \psi^2 \rangle^2, \quad (2.8)$$

$$\frac{\partial(\psi^2)^2}{\partial\psi} \Rightarrow 4\psi\langle\psi^2\rangle, \quad \text{or} \quad (2.9)$$

$$\frac{\partial^2(\psi^2)^2}{\partial\psi^2} \Rightarrow 4\langle\psi^2\rangle, \quad (2.10)$$

which follow at large N from the identification $\psi^2 \simeq \sum \psi^i \psi^i / N$.

We include in the following the counter terms that we will need later in order to write the renormalized equations. Then the equation of motion for the field ϕ becomes

$$\ddot{\phi}(t) + \lambda\phi \left[\phi^2(t) - v^2 \right] + \delta\lambda\phi^3(t) + \delta m^2\phi(t) + (\lambda + \delta\lambda)\phi(t)\mathcal{F}(t, T) = 0. \quad (2.11)$$

Here $\mathcal{F}(t, T)$ is the divergent fluctuation integral, it is given by the average of the fluctuation fields defined by the initial density matrix. For a thermal initial state of quanta with energy ω_{k0} it reads

$$\mathcal{F}(t, T) = \langle \psi^2(\vec{x}, t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} |U_k(t)|^2. \quad (2.12)$$

The mode functions satisfy the equation

$$\left[\frac{d^2}{dt^2} + \omega_k^2(t) \right] U_k(t) = 0, \quad (2.13)$$

and the initial conditions

$$U_k(0) = 1, \quad \dot{U}_k(0) = -i\omega_{k0}. \quad (2.14)$$

The time dependent frequency $\omega_k(t)$ is given by

$$\omega_k^2(t) = k^2 + \mathcal{M}^2(t), \quad (2.15)$$

with the time dependent mass

$$\mathcal{M}^2(t) = \lambda(\phi^2 - v^2) + \delta\lambda\phi^2 + \delta m^2 + (\lambda + \delta\lambda)\mathcal{F}(t). \quad (2.16)$$

We rewrite the mode equation in the form

$$\left[\frac{d^2}{dt^2} + \omega_{k0}^2 \right] U_k(t) = -\mathcal{V}(t)U_k(t), \quad (2.17)$$

whereby we have defined the time dependent potential $\mathcal{V}(t) = \mathcal{M}^2(t) - \mathcal{M}^2(0)$. We further identify $m_0 = \mathcal{M}(0)$ as the initial mass. The equation of motion for the expectation value can also be rewritten as

$$\ddot{\phi}(t) + \mathcal{M}^2(t)\phi(t) = 0, \quad (2.18)$$

which is of the same form as (2.13) with $k = 0$. The expectation value is also referred to as zero mode, average field or mean field. The average of energy with respect to the initial density matrix is given by

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\dot{\phi}^2(t) + \frac{1}{4}\lambda(\phi^2 - v^2)^2 + \delta\Lambda + \frac{1}{2}\delta m^2\phi^2 + \frac{1}{4}\delta\lambda\phi^4 \\ &+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} \left\{ \frac{1}{2}|\dot{U}_k(t)|^2 + \frac{1}{2}\omega_k^2(t)|U_k(t)|^2 \right\} \\ &- \frac{\lambda + \delta\lambda}{4} \mathcal{F}^2(t, T). \end{aligned} \quad (2.19)$$

Note that twice the last term, with positive sign, is included in the fluctuation energy, since $\omega_k^2(t)$ contains $\mathcal{F}(t, T)$. It is easy to check, using the equations of motion (2.18) and (2.13), that the energy is conserved. The energy density is the 00 component of the energy-momentum tensor. The average of the energy momentum tensor for our system is diagonal, its space-space components define the pressure which is given by

$$\begin{aligned} p &= \dot{\phi}^2(t) - \mathcal{E} + A \frac{d^2}{dt^2} [\phi^2(t) + \mathcal{F}(t, T)] \\ &+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} \left(\omega_{k0}^2 + \frac{k^2}{3} \right) |U_k(t)|^2. \end{aligned} \quad (2.20)$$

The term proportional to A as introduced by Callan et al. [40] is the space-space component of the improvement term $A(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\phi^2$ for the energy momentum tensor. Here it serves as a renormalization counter term. The determination of the pressure is essentially for the consideration of a system in expanding space time, since there the covariant conservation of the energy includes the energy itself as well as the pressure.

2.3 Perturbative Expansion

In order to prepare the renormalized version of the equations we introduce a suitable expansion of the mode functions. We have used this method exhaustively in our previous publications for the inflaton field coupled to itself [33] and to gauge bosons [38, 41] in Minkowski-space and for the inflaton field coupled to itself in a spatially flat FRW-universe [42]. All these calculations have been done for $T = 0$. The renormalization procedure does not change for $T \neq 0$. Therefore, we give here only a brief review of the perturbative expansion. For details the reader is referred to our previous work.

The mode functions can be written as

$$\left[\frac{d^2}{dt^2} + \omega_{k0}^2 \right] U_k(t) = -\mathcal{V}(t)U_k(t), \quad (2.21)$$

with

$$\mathcal{V}(t) = \mathcal{M}^2(t) - \mathcal{M}^2(0) , \quad (2.22)$$

$$\omega_{k0}^2 = \vec{k}^2 + \mathcal{M}^2(0) . \quad (2.23)$$

The mode functions satisfy the equivalent integral equation

$$U_k(t) = e^{-i\omega_{k0}t} + \int_0^\infty dt' \Delta_{k,\text{ret}}(t-t') \mathcal{V}(t') U_k(t') , \quad (2.24)$$

with

$$\Delta_{k,\text{ret}}(t-t') = -\frac{1}{\omega_{k0}} \theta(t-t') \sin[\omega_{k0}(t-t')] . \quad (2.25)$$

For $U_k(t)$ we choose the following ansatz

$$U_k(t) = e^{-i\omega_{k0}t} [1 + f_k(t)] , \quad (2.26)$$

to separate $U_k(t)$ into the trivial part corresponding to the case $\mathcal{V}(t) = 0$ and a function $f_k(t)$ which represents the reaction to the potential. $f_k(t)$ satisfies the differential equation

$$\ddot{f}_k(t) - 2i\omega_{k0}\dot{f}_k(t) = -\mathcal{V}(t)[1 + f_k(t)] , \quad (2.27)$$

with the initial conditions $f_k(0) = \dot{f}_k(0) = 0$ or the equivalent integral equation

$$f_k(t) = \int_0^\infty dt' \Delta_{k,\text{ret}}(t-t') \mathcal{V}(t') [1 + f_k(t')] e^{i\omega_{k0}(t-t')} . \quad (2.28)$$

We now expand $f_k(t)$ with respect to orders in $\mathcal{V}(t)$ by writing

$$f_k(t) = f_k^{(1)}(t) + f_k^{(2)}(t) + f_k^{(3)}(t) + \dots \quad (2.29)$$

$$= f_k^{(1)}(t) + \overline{f_k^{(2)}}(t) , \quad (2.30)$$

where $f_k^{(n)}(t)$ is of n th order in $\mathcal{V}(t)$ and $\overline{f_k^{(n)}}(t)$ is the sum over all orders beginning with the n th one:

$$\overline{f_k^{(n)}}(t) = \sum_{l=n}^\infty f_k^{(l)}(t) . \quad (2.31)$$

The function $\overline{f_k^{(1)}}(t)$ is identical to the function $f_k(t)$ itself which is obtained by solving (2.27). The function $\overline{f_k^{(2)}}(t)$ can be computed by using the differential equation, via

$$\ddot{\overline{f_k^{(2)}}}(t) - 2i\omega_{k0}\dot{\overline{f_k^{(2)}}}(t) = -\mathcal{V}(t)\overline{f_k^{(1)}}(t) , \quad (2.32)$$

or by iteration via

$$\overline{f_k^{(2)}}(t) = \int_0^\infty dt' \Delta_{k,\text{ret}}(t-t') \mathcal{V}(t') \overline{f_k^{(1)}}(t') e^{i\omega_{k0}(t-t')} . \quad (2.33)$$

This iteration has the advantage for the numerical computation that it avoids computing $f_k^{(2)}$ via the small difference $f_k^{(1)} - f_k^{(1)}$. However, the integral equations are used as well in order to derive the asymptotic behavior as $\omega_{k0} \rightarrow \infty$ and to separate divergent and finite contributions. The leading orders of $f_k(t)$ are discussed in detail in [33, 42, 38] and we do not want to repeat them here. Some more details can be found in section 3.5.4. In this chapter we are more interested in the effects of the finite contributions at finite temperature and in the self consistent solving of the large N limit.

2.4 Renormalization

2.4.1 Equation of Motion

We use the expansion and the definition introduced in the previous section in order to single out the divergent terms from the fluctuation integral. We have

$$\begin{aligned} \mathcal{M}^2(t) = & \lambda(\phi^2 - v^2) + \delta m^2 + \delta\lambda\phi^2 + (\lambda + \delta\lambda) \{I_{-1}(m_0, T) \\ & - I_{-3}(m_0, T) [\mathcal{M}^2(t) - \mathcal{M}^2(0)] + \mathcal{F}_{\text{fin}}(t, T)\} , \end{aligned} \quad (2.34)$$

where the finite part of $\mathcal{F}(t, T)$ can be written as

$$\begin{aligned} \mathcal{F}_{\text{fin}}(t, T) = & \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{k0}^3} \int_0^t dt' \cos[2\omega_{k0}(t-t')] \dot{\mathcal{V}}(t') \coth \frac{\beta\omega_{k0}}{2} \\ & + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \left[2\text{Re} f_k^{(2)}(t) + |f_k^{(1)}(t)|^2 \right] \coth \frac{\beta\omega_{k0}}{2} , \end{aligned} \quad (2.35)$$

and where the divergent integrals are defined as

$$I_{-1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \left(1 + \frac{2}{e^{\beta\omega_{k0}} - 1} \right) = I_{-1}(m_0) + \Sigma_{-1}(m_0, T) , \quad (2.36)$$

$$I_{-3}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{k0}^3} \left(1 + \frac{2}{e^{\beta\omega_{k0}} - 1} \right) = I_{-3}(m_0) + \Sigma_{-3}(m_0, T) . \quad (2.37)$$

The integrals $I_{-k}(m_0)$ are those which occur in the renormalization at $T = 0$. Their dimensionally regularized form will be given below. The additional temperature dependent terms $\Sigma_{-k}(m_0, T)$ are finite. They are defined as

$$\Sigma_{-1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_{k0} (e^{\beta\omega_{k0}} - 1)} , \quad (2.38)$$

$$\Sigma_{-3}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}^3 (e^{\beta\omega_{k0}} - 1)} . \quad (2.39)$$

We derive some useful explicit expressions for these integrals in Appendix A.

It is convenient to include these finite terms into the definition of $\mathcal{F}_{\text{fin}}(t, T)$. Then the time dependent mass takes the form

$$\mathcal{M}^2(t) = \lambda(\phi^2 - v^2) + \delta\lambda\phi^2 + \delta m^2 + (\lambda + \delta\lambda) \left[I_{-1}(m_0) - I_{-3}(m_0)\mathcal{V}(t) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) \right], \quad (2.40)$$

with

$$\tilde{\mathcal{F}}_{\text{fin}}(t, T) = \Sigma_{-1}(m_0, T) - \mathcal{V}(t)\Sigma_{-3}(m_0, T) + \mathcal{F}_{\text{fin}}(t, T). \quad (2.41)$$

The time dependent mass (2.40) contains both renormalization constants δm and $\delta\lambda$. Furthermore, its definition by this equation is implicit, $\mathcal{M}^2(t)$ appears also on the right hand side of (2.40) in $\mathcal{V}(t)$.

We now have to fix the renormalization counter terms in such a way that the relation between the time dependent mass and $\phi(t)$ becomes finite. An additional constraint derives from the requirement that the renormalization counter terms should not depend on the initial condition but only on the parameters appearing in the Lagrangian, i.e., λ and m . For the simpler case of the one loop equations this has been achieved [33].

We first determine $\delta\lambda$ by considering the difference

$$\begin{aligned} \mathcal{V}(t) &= \mathcal{M}^2(t) - \mathcal{M}^2(0) \\ &= (\lambda + \delta\lambda) \left[\phi^2(t) - \phi^2(0) - I_{-3}(m_0)\mathcal{V}(t) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) - \tilde{\mathcal{F}}_{\text{fin}}(0, T) \right]. \end{aligned} \quad (2.42)$$

The divergent parts depend on the initial mass m_0 . We have to replace this by a renormalization scale independent of the initial conditions. In [34] we chose the scale m , where m was the mass parameter appearing in the Lagrangian. Here the analogous mass squared would be $m^2 = -\lambda v^2$ and therefore, m would be imaginary. In order to circumvent this problem we introduce another scale m_1 which we do not specify here. In the numerical computations we have used the physical mass $m_1^2 = m_h^2 = 2\lambda v^2$. We rewrite the implicit equation for $\mathcal{V}(t)$ as

$$\begin{aligned} \mathcal{V}(t) [1 + (\lambda + \delta\lambda)I_{-3}(m_1)] &= (\lambda + \delta\lambda) \left\{ \phi^2(t) - \phi^2(0) - [I_{-3}(m_0) - I_{-3}(m_1)]\mathcal{V}(t) \right. \\ &\quad \left. + \tilde{\mathcal{F}}_{\text{fin}}(t, T) - \tilde{\mathcal{F}}_{\text{fin}}(0, T) \right\}. \end{aligned} \quad (2.43)$$

We now require

$$\frac{\lambda + \delta\lambda}{1 + (\lambda + \delta\lambda)I_{-3}(m_1)} = \lambda. \quad (2.44)$$

Solving with respect to $\delta\lambda$ we find

$$\delta\lambda = \frac{\lambda^2 I_{-3}(m_1)}{1 - \lambda I_{-3}(m_1)}. \quad (2.45)$$

Inserting this relation into (2.43) we have

$$\mathcal{V}(t) = \lambda \left\{ \phi^2(t) - \phi^2(0) - [I_{-3}(m_0) - I_{-3}(m_1)]\mathcal{V}(t) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) - \tilde{\mathcal{F}}_{\text{fin}}(0, T) \right\} \quad (2.46)$$

or

$$\mathcal{V}(t) = \frac{\lambda}{1 + \lambda [I_{-3}(m_0) - I_{-3}(m_1)]} \left[\phi^2(t) - \phi^2(0) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) - \tilde{\mathcal{F}}_{\text{fin}}(0, T) \right]. \quad (2.47)$$

This is a finite relation for the potential $\mathcal{V}(t)$ since the difference $[I_{-3}(m_0) - I_{-3}(m_1)]$ is finite. Using dimensional regularization (for some more details see Appendix C)

$$I_{-3}(m_0) = \left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{k_0}^3} \right\}_{\text{reg}} = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma \right\}, \quad (2.48)$$

we find

$$I_{-3}(m_0) - I_{-3}(m_1) = \frac{1}{16\pi^2} \ln \frac{m_1^2}{m_0^2}. \quad (2.49)$$

We now go back to equation (2.34) which we take at the initial time $t = 0$:

$$m_0^2 \equiv \mathcal{M}^2(0) = \lambda[\phi^2(0) - v^2] + \delta\lambda\phi^2(0) + \delta m^2 + (\lambda + \delta\lambda) [I_{-1}(m_0) + \tilde{\mathcal{F}}_{\text{fin}}(0, T)]. \quad (2.50)$$

This is an implicit relation between m_0 and $\phi(0)$ which, however, contains still the infinite quantities $\delta\lambda$, δm and $I_{-1}(m_0)$. In order to proceed we note the following explicit relation between I_{-1} and I_{-3} which follows from the dimensionally regularized expressions for these quantities

$$\begin{aligned} I_{-1}(m_0) &= \left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{k_0}} \right\}_{\text{reg}} = -\frac{m_0^2}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma + 1 \right\} \\ &= -m_0^2 I_{-3}(m_0) - \frac{m_0^2}{16\pi^2}. \end{aligned} \quad (2.51)$$

Therefore, we can rewrite (2.50) as

$$m_0^2 = (-\lambda v^2 + \delta m^2) + (\lambda + \delta\lambda) \left[\phi^2(0) - m_0^2 I_{-3}(m_0) - \frac{m_0^2}{16\pi^2} + \tilde{\mathcal{F}}_{\text{fin}}(0, T) \right]. \quad (2.52)$$

As renormalization condition we require $m_0 = 0$ for $T = 0$ at the minimum of the potential $\phi = v$ so that the effective potential and the tree level potential have the same value at their minimum. For $T = 0$ we have $\tilde{\mathcal{F}}_{\text{fin}}(t = 0, T = 0) = \Sigma_{-1}(m_0, T = 0) = 0$. Setting $m_0 = 0$ and $\phi(0) = 0$ in (2.52) we find

$$\delta m^2 = -\delta\lambda v^2 = -\frac{\lambda^2 v^2 I_{-3}(m_1)}{1 - \lambda I_{-3}(m_1)}. \quad (2.53)$$

Inserted into (2.52) we obtain the renormalized gap equation

$$m_0^2 = \lambda \mathcal{C} \left[\phi^2(0) - v^2 - \frac{m_0^2}{16\pi^2} + \Sigma_{-1}(m_0, T) \right], \quad (2.54)$$

with

$$\mathcal{C} = \left(1 + \frac{\lambda}{16\pi^2} \ln \frac{m_1^2}{m_0^2} \right)^{-1}. \quad (2.55)$$

The gap equation (2.54) and the renormalized definition of the potential (2.47) constitute, along with the equations of motion the basic renormalized equations for the self consistent large N dynamics.

The gap equation has to be solved at $t = 0$ and determines the relation between m_0 and $\phi(0)$. For later times we have

$$\begin{aligned} \mathcal{M}^2(t) &= m_0^2 + \mathcal{V}(t) \\ &= m_0^2 + \mathcal{C}\lambda \left[\phi^2(t) - \phi^2(0) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) - \tilde{\mathcal{F}}_{\text{fin}}(0, T) \right]. \end{aligned} \quad (2.56)$$

Since the gap equation can be cast into different forms we can obtain several equivalent forms of this equation. Solving the gap equation for $\phi^2(0)$ we find

$$\lambda\mathcal{C}\phi^2(0) = m_0^2 + \lambda\mathcal{C} \left[v^2 + \frac{m_0^2}{16\pi^2} - \Sigma_{-1}(m_0, T) \right], \quad (2.57)$$

or equivalent

$$\phi^2(0) = \frac{m_0^2}{\lambda} + v^2 + \frac{m_0^2}{16\pi^2} \left(1 + \ln \frac{m_1^2}{m_0^2} \right) - \Sigma_{-1}(m_0, T), \quad (2.58)$$

so that

$$\mathcal{M}^2(t) = \lambda\mathcal{C} \left[\phi^2(t) - v^2 - \frac{m_0^2}{16\pi^2} + \tilde{\mathcal{F}}_{\text{fin}}(t, T) \right]. \quad (2.59)$$

Having obtained a finite relation between $\phi(t)$ and $\mathcal{M}(t)$ the equations of motion for the zero mode $\phi(t)$ and for the modes $U_k(t)$ are well defined and finite.

Here we have included the corrections of leading order, proportional to $\Sigma_{-1}(m_0, T)$, into the finite part of the fluctuation integral. These terms are important at high temperature. They appear in the gap equation via $\tilde{\mathcal{F}}_{\text{fin}}(0, T) = \Sigma_{-1}(m_0, T) \simeq T^2/12$. Omitting terms of order $\lambda/16\pi^2$ the gap equation (2.54) becomes

$$m_0^2 \simeq -\lambda v^2 + \lambda\phi^2(0) + \frac{\lambda}{12}T^2. \quad (2.60)$$

Therefore, at high temperature the mass circulating in the loop is dominated by the hard λT^2 term. In the following we need the fluctuation integral $\mathcal{F}(t, T)$ which is and will remain divergent. We need an expression in which these divergences appear explicitly. Using

$$\mathcal{F}(t, T) = I_{-1}(m_0) - I_{-3}(m_0) \left[\mathcal{M}^2(t) - \mathcal{M}^2(0) \right] + \tilde{\mathcal{F}}_{\text{fin}}(t, T) \quad (2.61)$$

and inserting the expression for $\mathcal{M}^2(t)$ we have just derived, we obtain

$$\begin{aligned} \mathcal{F}(t, T) &= -\frac{m_0^2}{16\pi^2} - \mathcal{C}\lambda I_{-3}(m_0)\phi^2(t) + \lambda v^2 \mathcal{C} I_{-3}(m_0) + \lambda\mathcal{C} \frac{m_0^2}{16\pi^2} I_{-3}(m_0) \\ &\quad + \mathcal{C} \frac{\lambda}{16\pi^2} I_{-3}(m_0)(m_0 + \lambda v^2) + \mathcal{C} [1 - \lambda I_{-3}(m_1)] \tilde{\mathcal{F}}_{\text{fin}}(t, T). \end{aligned} \quad (2.62)$$

While we have found here the gap equation as a self-consistency condition, it can also be derived [12, 43] from a potential (free energy) which here takes the form

$$\begin{aligned}
 V(m_0^2, \phi^2, T) &= \frac{m_0^2}{2} \left\{ \phi^2 - v^2 - \frac{m_0^2}{2\lambda} + \frac{m_0^2}{32\pi^2} \left[\ln \left(\frac{m_0^2}{m_1^2} \right) - \frac{3}{2} \right] \right\} \\
 &+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\beta} \ln [1 - \exp(-\beta\omega_0)] .
 \end{aligned} \tag{2.63}$$

Then the gap equation follows from the condition

$$\frac{\partial V(m_0^2, \phi^2, T)}{\partial m_0^2} = 0 . \tag{2.64}$$

2.4.2 Energy and Pressure

The unrenormalized expressions for the energy density and for the pressure have been given in section 2.2. Apart from the counter terms which we have already fixed in renormalizing the equation of motion, two new counter terms appear, the cosmological constant term $\delta\Lambda$ in the energy density and the improvement term $Ad^2(\phi^2 + \langle\psi^2\rangle)/dt^2$ in the pressure. These terms must suffice for rendering the expressions for energy density and pressure finite.

We start with the expression (2.19) for the energy which we rewrite as

$$\begin{aligned}
 \mathcal{E} &= \frac{1}{2}\dot{\phi}^2(t) + \frac{1}{4}(\lambda + \delta\lambda)(\phi^2 - v^2)^2 \\
 &+ \mathcal{E}_{\text{fl}}(t, T) - \frac{\lambda + \delta\lambda}{4}\mathcal{F}^2(t, T) + \delta\Lambda ,
 \end{aligned} \tag{2.65}$$

with

$$\mathcal{E}_{\text{fl}}(t, T) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} \left\{ \frac{1}{2}|\dot{U}_k(t)|^2 + \frac{1}{2}\omega_k^2(t)|U_k(t)|^2 \right\} . \tag{2.66}$$

Here we have used already the relation $\delta m^2 = -\delta\lambda v^2$, and part of the cosmological constant counter term $\delta\Lambda$ is included in $\delta\lambda v^4/4$. We split the Bose factor as before

$$\coth \frac{\beta\omega_{k0}}{2} = 1 + \frac{2}{e^{\beta\omega_{k0}} - 1} . \tag{2.67}$$

The integrations involving the second term are finite. We define

$$\Delta\mathcal{E}_{\text{fl}}(t, T) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \frac{2}{e^{\beta\omega_{k0}} - 1} \left\{ \frac{1}{2}|\dot{U}_k(t)|^2 + \frac{1}{2}\omega_k^2(t)|U_k(t)|^2 \right\} . \tag{2.68}$$

The integrations involving the first term have been discussed in [33]. Following this discussion we can decompose the integral via

$$\mathcal{E}_{\text{fl}}(t, 0) = I_1(m_0) + \frac{1}{2}\mathcal{V}(t)I_{-1}(m_0) - \frac{1}{4}\mathcal{V}^2(t)I_{-3}(m_0) + \mathcal{E}_{\text{fl,fin}}(t, 0) , \tag{2.69}$$

with

$$\begin{aligned}\mathcal{E}_{\text{fl,fin}}(t, 0) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k_0}} \left\{ \frac{1}{2} |\dot{f}_k^{(1)}|^2 + \frac{\mathcal{V}(t)}{2} \left[2\text{Re} f_k^{(1)} + |f_k^{(1)}|^2 \right] + \frac{\mathcal{V}^2(t)}{8\omega_{k_0}^2} \right\}, \quad (2.70) \\ I_1(m_0) &= \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{k_0}}{2} \right\}_{\text{reg}} = -\frac{m_0^4}{64\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma + \frac{3}{2} \right\} \\ &= -\frac{m_0^4}{4} I_{-3}(m_0) - \frac{3m_0^4}{128\pi^2}.\end{aligned}\quad (2.71)$$

We denote the sum of both finite contributions as $\mathcal{E}_{\text{fl,fin}}(t, T)$. The expression for the energy now takes the form

$$\begin{aligned}\mathcal{E} &= \frac{1}{2} \dot{\phi}^2 + \frac{\lambda + \delta\lambda}{4} (\phi^2 - v^2)^2 + \mathcal{E}_{\text{fl,fin}}(t, T) + I_1(m_0) + \frac{1}{2} \mathcal{V}(t) I_{-1}(m_0) - \frac{1}{4} \mathcal{V}^2(t) I_{-3}(m_0) \\ &\quad - \frac{\lambda + \delta\lambda}{4} \mathcal{F}^2(t, T) + \delta\Lambda.\end{aligned}\quad (2.72)$$

In addition to the divergences arising from $\mathcal{E}_{\text{fl}}(t, T)$ we have to take into consideration those of $\mathcal{F}^2(t, T)$ which we have analyzed above. If all divergences and the renormalization constant $\delta\lambda$ are inserted, the expression turns out to be finite, i.e., the remaining counter term $\delta\Lambda$ is needed only for a finite renormalization. We require the energy to vanish at $T = 0$ for $\phi(t) \equiv v$, which implies $m_0 = 0$. Then $\delta\Lambda = 0$. There remains a finite constant depending on the initial condition

$$\Delta\Lambda = \frac{m_0^4}{128\pi^2} \left(1 + \frac{2\lambda\mathcal{C}}{16\pi^2} \right). \quad (2.73)$$

Then the energy is given by

$$\begin{aligned}\mathcal{E} &= \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{4} \mathcal{C} (\phi^2 - v^2)^2 + \frac{1}{2} \Delta m^2 (\phi^2 - v^2) \\ &\quad + \mathcal{E}_{\text{fl,fin}}(t, T) - \frac{\lambda}{4} \mathcal{C} \tilde{\mathcal{F}}_{\text{fin}}^2(t, T) + \Delta\Lambda,\end{aligned}\quad (2.74)$$

with

$$\Delta m^2 = -\lambda\mathcal{C} \frac{m_0^2}{16\pi^2}. \quad (2.75)$$

Finally, we have to give a finite expression for the pressure, using our last free counter term. We write the pressure in the form

$$p = \dot{\phi}^2(t) - \mathcal{E} + p_{\text{fl}}(t, T) + A \frac{d^2}{dt^2} \left[\phi^2(t) + \mathcal{F}(t, T) \right]. \quad (2.76)$$

Here we have anticipated a special form of the counter term, indeed for the expression in brackets it is possible choose a priori an arbitrary Lorentz scalar, the additional piece of the energy momentum tensor being trivially conserved on account of its tensor structure $\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2$. Of course it has to be suited for the renormalization procedure. The

fluctuation part of the pressure consists again of three parts, a divergent one, a finite one independent of the temperature and a finite integral involving the thermal distribution function $1/[\exp(\omega_{k0}/T) - 1]$. The analysis for $T = 0$ has been performed in [33]. Following the discussion there we can write p_{fl} as

$$p_{\text{fl}}(t, T) = p_{\text{fl,fin}}(t, 0) + \Delta p_{\text{fl}}(t, T) - \frac{m_0^4}{96\pi^2} - \frac{m_0^2}{48\pi^2} \mathcal{V}(t) - \frac{1}{6} \left[I_{-3}(m_0) + \frac{1}{48\pi^2} \right] \ddot{\mathcal{V}}(t). \quad (2.77)$$

$\Delta p_{\text{fl}}(t, T)$ is given by

$$\Delta p_{\text{fl}}(t, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \frac{2}{e^{\beta\omega_{k0}} - 1} \left(\omega_{k0}^2 + \frac{k^2}{3} \right) |U_k(t)|^2, \quad (2.78)$$

the finite part for $T = 0$ by

$$\begin{aligned} p_{\text{fl,fin}}(t, 0) = & \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \left\{ \left(\omega_{k0}^2 + \frac{k^2}{3} \right) \left[2\text{Re} f_k^{(2)}(t) + |f_k^{(1)}(t)|^2 \right] \right. \\ & + \left(\frac{1}{6\omega_{k0}^2} - \frac{m_0^2}{24\omega_{k0}^4} \right) \int_0^t dt' \cos 2\omega_{k0}(t-t') \ddot{\mathcal{V}}(t') \\ & + \left(\frac{1}{12\omega_{k0}^2} + \frac{m_0^2}{24\omega_{k0}^4} \right) \cos(2\omega_{k0}t) \ddot{\mathcal{V}}(0) \\ & \left. + |f_k^{(1)}(t)|^2 - 2\text{Re} \left[i\omega_{k0} \dot{f}_k^{(1)}(t) + i\omega_{k0} f_k^{(1)}(t) f_k^{(1)*}(t) \right] \right\}. \quad (2.79) \end{aligned}$$

We call the sum of both finite fluctuation integrals $p_{\text{fl,fin}}(t, T)$. Now we have to consider the divergent terms. We observe that $\ddot{\mathcal{V}}(t)$ is given by

$$\ddot{\mathcal{V}}(t) = \lambda \mathcal{C} \frac{d^2}{dt^2} \left[\phi^2(t) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) \right]. \quad (2.80)$$

On the other hand, using (2.62) we have

$$\frac{d^2}{dt^2} \mathcal{F}(t, T) = \frac{d^2}{dt^2} \left[-\lambda \mathcal{C} I_{-3}(m_0) \phi^2(t) + \mathcal{C} (1 - \lambda I_{-3}(m_1)) \tilde{\mathcal{F}}_{\text{fin}}(t, T) \right] \quad (2.81)$$

and therefore

$$A \frac{d^2}{dt^2} \left[\phi^2(t) + \mathcal{F}(t, T) \right] = A \frac{d^2}{dt^2} \mathcal{C} [1 - \lambda I_{-3}(m_1)] \left[\phi^2(t) + \tilde{\mathcal{F}}_{\text{fin}}(t, T) \right]. \quad (2.82)$$

As apparent from (2.80), this matches in form with the divergent term $I_{-3}(m_0) \ddot{\mathcal{V}}(t)/6$. Insisting again in choosing the counter term independent of the initial condition we fix

$$A = \frac{\lambda I_{-3}(m_1)}{6[1 - \lambda I_{-3}(m_1)]} \quad (2.83)$$

and retain a finite term

$$-\frac{1}{96\pi^2} \left[\ln \left(\frac{m_1^2}{m_0^2} \right) + 2 \right] \ddot{\mathcal{V}}(t). \quad (2.84)$$

The final result for the pressure reads

$$p = \dot{\phi}^2(t) - \mathcal{E} + p_{\text{fl,fin}}(t, T) - \frac{m_0^4}{96\pi^2} - \frac{m_0^2}{48\pi^2} \mathcal{V}(t) - \frac{1}{96\pi^2} \left[\ln \left(\frac{m_1^2}{m_0^2} \right) + 2 \right] \ddot{\mathcal{V}}(t). \quad (2.85)$$

A further important quantity is the particle number density. The production of particles is of interest primarily in connection with inflationary cosmology. By the end of inflation only a scalar field remains in the universe, and the density of all other forms of matter is exponentially small. Matter is produced again as a result of coherent oscillations of the field $\phi(t)$. The amplitude of the field decreases as a result of the transfer of the energy to the produced particles. This epoch is called preheating in the literature, introduced in [44]. The particles created during this stage are far from equilibrium, thermalization and equilibration will be achieved via collision relaxation. This stage is called reheating. In the approximation, we study, collisions are absent and therefore, we can not describe the reheating process. Nevertheless, e.g. in [12] an estimate for the reheating temperature is made under some reasonable assumptions based on the analysis of the particle spectrum. Therefore, it is important that our approach leads to a possibility to determine the particle number in an easy way. Since we are mainly interested in the technical details for a suitable implementation of the large N approximation we do not give an interpretation of the particle number in the cosmological context. The expression for the particle number takes a rather simple form expressed by the truncated mode functions:

$$\begin{aligned} N(t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \coth \frac{\beta\omega_{k0}}{2} \left\{ \frac{1}{4} \left[|U_k(t)|^2 + \frac{1}{\omega_{k0}^2} |\dot{U}_k(t)|^2 \right] - \frac{1}{2} \right\} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{8\omega_{k0}^3} \coth \frac{\beta\omega_{k0}}{2} |\dot{f}_k^{(1)}(t)|^2. \end{aligned} \quad (2.86)$$

Since the particle number is finite, we do not have to renormalize it.

2.5 Numerics

We have carried out different numerical calculations. First, we discuss the ϕ^4 theory with spontaneous symmetry breaking in the large N limit at $T = 0$ and $T \neq 0$ [45]. Then we investigate the one loop approximation and compare it with the large N approximation. We choose for these considerations the coupling $\lambda = 1$ and also the squared of the vacuum expectation value (vev) $v^2 = 1$. At the end we show some results for the unbroken theory [34].

2.5.1 The Broken Symmetry Case

The dynamical evolution of the non-equilibrium system depends on two parameters, the temperature T and the initial value of the zero mode $\phi(0) = \phi_0$ which in analogy with

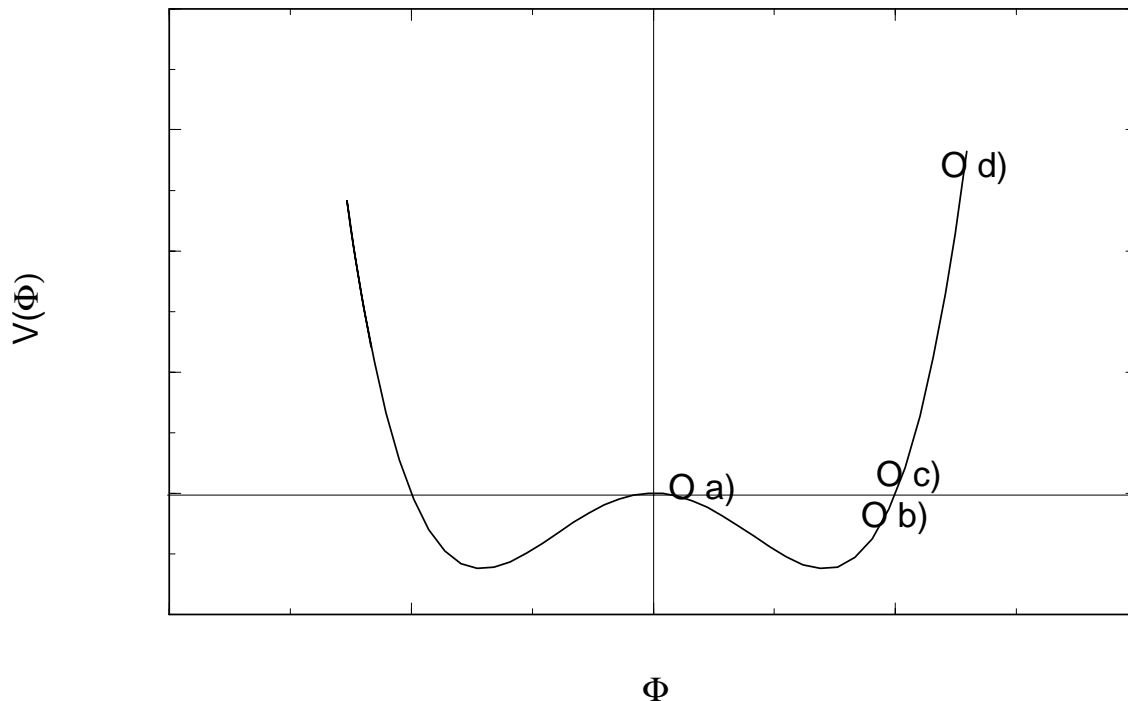
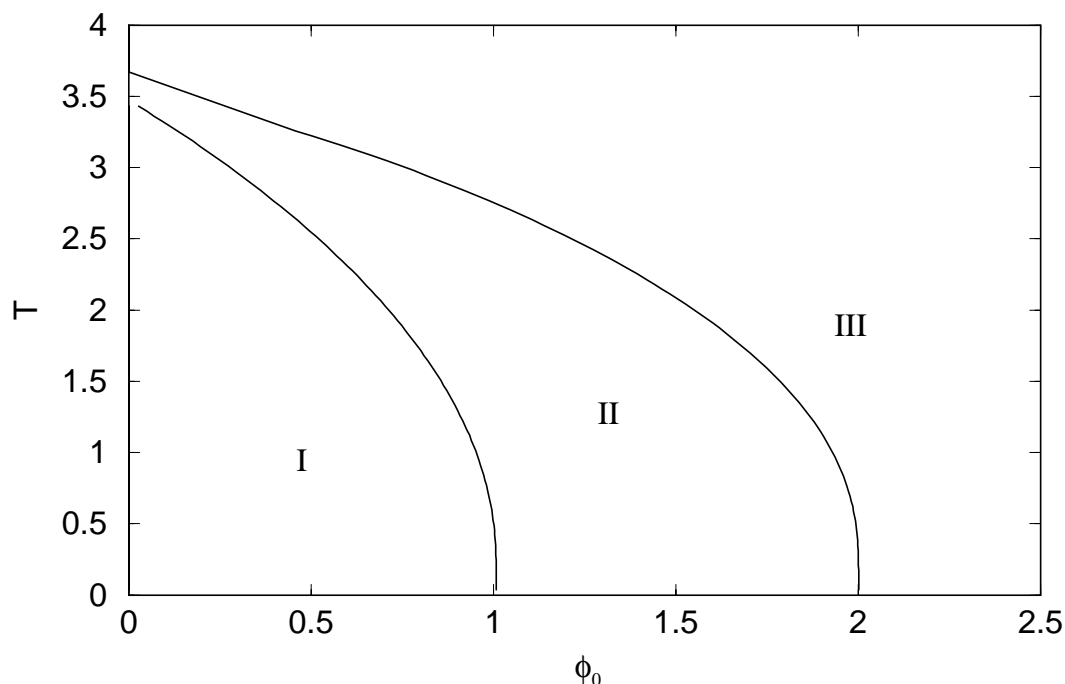


Figure 2.1: Classical potential

thermal equilibrium systems can be considered as an external parameter. We have plotted in Fig. 2.1 a typical classical potential with four different initial values for the zero mode. Clearly, this picture simplifies the situation too much. Since we consider quantum fluctuations in our analysis the effective potential would describe the system in a more realistic way. In 1987 Weinberg and Wu [46] discussed the effective potential for a spontaneously broken theory. They investigated the complex region of the effective potential which leads to unstable states. In the non-equilibrium context this region of instabilities for homogenous configurations is known as the spinodal region and is discussed widely in recent papers in different contexts, e.g. [47, 48, 49, 50]. But also the effective potential leads to wrong predictions for a system out of equilibrium. The effective potential gives a static description of the physical setting and cannot describe, e.g., the dynamics of a phase transition. It can be used to determine the nature of a phase transition and the description of static quantities like critical temperatures. It corresponds to the equilibrium free energy as a function of order parameters. The use of the effective potential for the description of the dynamics of a phase transition has been criticized by many authors [39, 51, 52]. Fig. 2.1 has to give only a basic impression about the physical setting.

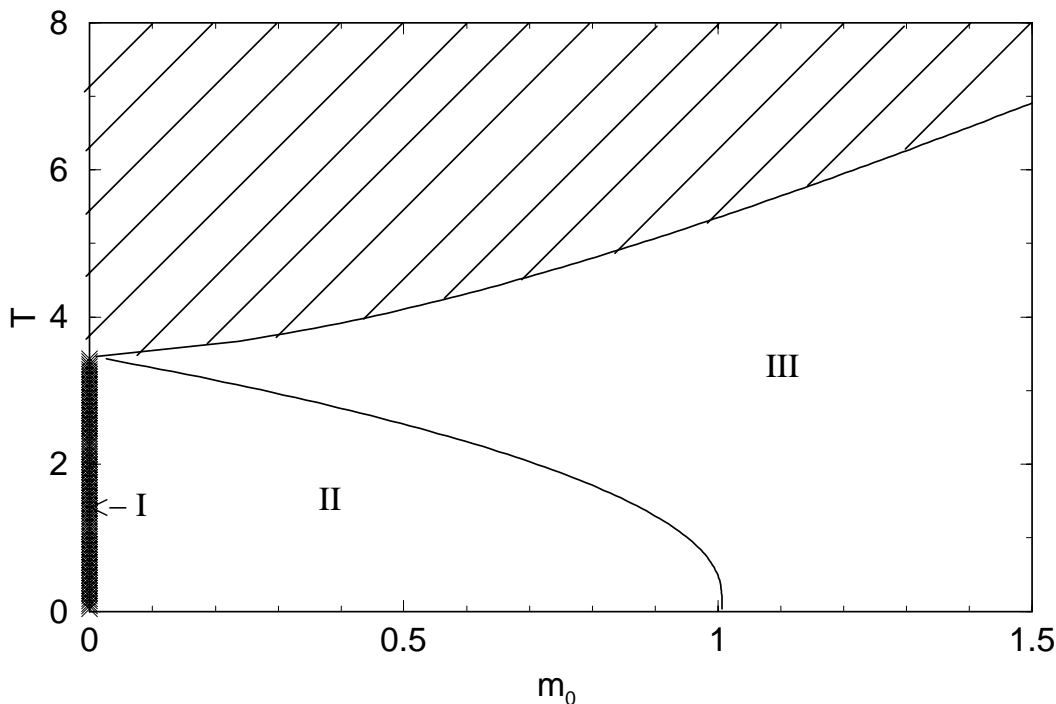
We begin our analysis by discussing the special features of the different starting points and two related phase diagrams displayed in Fig. 2.2 and Fig. 2.3. These phase diagrams, plotted in the $\phi_0 - T$ plane and in the $m_0 - T$ plane, show different regions in which the system can evolve. We explain the boundaries for these regions below. The first region,


 Figure 2.2: Phase diagram in the $\phi_0 - T$ plane

labeled by I, corresponds to a negative initial mass squared. In this case the gap equation has no real solution and the initial value becomes complex. In Fig. 2.1 an example for such an initial value is labeled by a). In the second region, called II, m_0^2 is positive. The initial mass for the zero mode is chosen smaller than $V[\phi(0)]$ in the real part of the effective potential. A possible choice is displayed in Fig. 2.1 at starting point b). In this case the zero mode has not enough energy to reach the maximum of the potential and we find a final state for which the symmetry is spontaneously broken. The third region in the phase diagrams corresponds to the initial values c) and d). Here we find a final state with restored symmetry. In the following we will characterize the regions in detail, evaluate the boundaries explicitly and describe the dynamical evolution of the system. In [29, 53] some analysis of the expected long time behavior were carried out, which we generalize here to finite temperature.

Region I: $m_0^2 < 0$

The initial value ϕ_0 and the initial mass m_0 are connected via the gap equation (2.54). It requires m_0^2 to be positive for $T = 0$ in order to find a real initial value ϕ_0 . The point where m_0 vanishes marks an initial value that leads to a solution $\phi = \text{const}$. For $T = 0$ this stationary amplitude is given by $\phi = v$. For $T > 0$ we can find this amplitude as well. For


 Figure 2.3: Phase diagram in the $m_0^2 - T$ plane

$m_0 = 0$ the thermal integral $\Sigma_{-1}(m_0, T)$ is given by its value for massless quanta, i.e.,

$$\Sigma_{-1}(0, T) = \frac{T^2}{12}, \quad (2.87)$$

and therefore

$$\phi_1^2(T) = \phi_0^2|_{m_0=0} = v^2 - \frac{T^2}{12}. \quad (2.88)$$

The region below this boundary (2.88) we call region I.

In our considerations it is not possible to start the field in the unstable region I. For $\phi(0) = 0$ the initial frequency for the fluctuation field is complex for low momenta because $\mathcal{F}(0) = 0$. In order to circumvent this problem it is possible to prepare the initial state in a different way

$$\omega_{k0}^2 = k^2 - m_0^2 \quad \text{for } k^2 > |m_0^2|, \quad (2.89)$$

$$\omega_{k0}^2 = k^2 + m_0^2 \quad \text{for } k^2 \leq |m_0^2|. \quad (2.90)$$

This choice corresponds to the ground state of an upright harmonic oscillator and to a quench type of situation in which the initial state is evolved in an inverted parabolic potential (for early time $t > 0$) as explained e.g. in [12]. We do not consider this possibility here.

Region II: $m_0^2 > 0$, $\mathcal{M}_\infty^2 = 0$

Now we choose the initial value above the boundary condition (2.88) but lower than $V[\phi(0)]$ as displayed in b) in Fig. 2.1. The initial value leads to the possibility of entering the unstable region. The quantum fluctuations begin to grow exponentially and lead to an effective damping of the field which starts to oscillate in the minimum as it is shown in Fig. 2.4. Since the fluctuations $\mathcal{F}(t)$ are growing, the effective mass

$$\mathcal{M}^2(t) = \lambda[\phi^2(t) - v^2] + \mathcal{F}(t) \quad (2.91)$$

becomes positive (we do not care about the renormalization here, because we are at the moment only interested in qualitative features) and the exponential growth of the fluctuations stops. Therefore, the development of the field stabilizes and ϕ_∞ reaches a finite constant value. If the initial amplitude $\phi(0)$ is sufficiently small, the back reaction leads $\mathcal{M}^2(t)$, which is plotted in Fig. 2.5, to settle down at zero. In contrast to the dissipative effects via parametric amplification this effect is induced by spinodal instabilities. Here the asymptotic mass vanishes and the zero mode reaches a finite value. Another effect missing in the unbroken theory which strengthens the damping was already explained in the introduction. The existence of massless Goldstone bosons leads to the possibility of the mean field transferring its kinetic energy. Another physical quantity of interest is the particle number density shown in Fig. 2.6. It increases at the beginning until it also ends up at a constant value. The numerical results for $T = 2.5$ are shown in Figs. 2.7, 2.8. The effects are stronger than for $T = 0$ but the qualitative features are the same.

In the one loop approximation, where we do not take the fluctuations in the mass term for the mode functions into account, $\mathcal{M}^2(t)$ can become negative, instabilities arise and the approximation breaks down. We will describe this phenomenon in section 2.5.2.

Now we consider some empirical relations describing the long time behavior of the system which were developed in [29]. We extend their results to finite temperature [45]. At $T = 0$ the final value of ϕ_∞ was found to be related to the initial value ϕ_0 by an empirical relation

$$\phi_\infty^4 = \phi_0^2(2v^2 - \phi_0^2), \quad T = 0. \quad (2.92)$$

The generalization of this relation to finite temperature is not obvious. In [29] it was remarked that the relation depends only on the initial zero mode part of the energy, which is given by $E = \lambda(\phi^2 - v^2)^2/4$. It satisfies the constraints that $\phi_\infty^2 = v^2$ if $\phi_0^2 = v^2$, and that $\phi_\infty = 0$ if classically the system can reach the maximum of the potential, which happens if $\phi_0^2 = \phi_2^2(T = 0) = 2v^2$. We further observe that the classical turning point is at $\bar{\phi}^2 = 2v^2 - \phi_0^2$ so that we may write (2.92) as the geometric mean

$$\phi_\infty^2 = \sqrt{\phi_0^2 \bar{\phi}_0^2}. \quad (2.93)$$

This form turns out to lead to the correct generalization for finite temperature. Obviously, the relation is characterized by the motion at early times when the quantum fluctuations have not yet evolved. When discussing renormalization we have made an expansion with respect to the potential $\mathcal{V}(t)$ which vanishes at $t = 0$. The same expansion can be used

to study the early time behavior. In the expression for the energy the coefficients of the terms of first and second order in \mathcal{V} have been absorbed into renormalization constants. However, the thermal fluctuations are not absorbed in this way and will add to the zero mode terms in an early time expansion. These appear in the energy, see (2.72), via

$$\Delta\mathcal{E}_{\text{fl}}(t, T) = \Sigma_1(m_0, T) + \frac{1}{2}\mathcal{V}(t)\Sigma_{-1}(m_0, T) - \frac{1}{4}\mathcal{V}^2(t)\Sigma_{-3}(m_0, T) + \mathcal{O}(\mathcal{V}^3), \quad (2.94)$$

as a part of $\mathcal{E}_{\text{fl,fin}}(t, T)$ and via (2.41) in $\tilde{\mathcal{F}}_{\text{fin}}(t, T)$. Taking these expansions into account the energy can be written in the form

$$E \simeq \frac{\lambda}{4}\mathcal{C} \left[a\phi^4 + \tilde{a}\phi_0^4 + b\phi^2 + \tilde{b}\phi_0^2 + c\phi^2\phi_0^2 \right] + \text{const.}, \quad (2.95)$$

up to terms of order \mathcal{V}^3 . We need the coefficients

$$a = 1 - \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T), \quad (2.96)$$

$$b = -2 \left[v^2 - \Sigma_{-1}(m_0, T) \right], \quad (2.97)$$

$$c = \lambda\mathcal{C}\mathcal{C}_T\Sigma_{-3}(m_0, T), \quad (2.98)$$

where we have introduced

$$\mathcal{C}_T = \left[1 + \frac{\lambda}{16\pi^2} \ln \frac{m_1^2}{m_0^2} + \lambda\Sigma_{-3}(m_0, T) \right]^{-1}. \quad (2.99)$$

This enables us to shorten the expression for the potential in the following way

$$\mathcal{V}(t) = \lambda\mathcal{C}_T \left[\phi^2(t) - \phi^2(0) + \mathcal{F}_{\text{fin}}(t, T) \right]. \quad (2.100)$$

The classical turning point is given by

$$\bar{\phi}_0^2 = -\frac{b + (a + c)\phi_0^2}{a} = \frac{2v^2 - \Sigma_{-1}(m_0, T) - [1 + \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)]\phi_0^2}{1 - \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)}, \quad (2.101)$$

so that we are led to suppose

$$\phi_\infty^2(T) = \sqrt{\frac{1}{1 - \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)}} \sqrt{\phi_0^2 \{2v^2 - 2\Sigma_{-1}(m_0, T) - [1 + \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)]\phi_0^2\}}. \quad (2.102)$$

We find that this relation is very well fulfilled numerically. We show in Figs. 2.9-2.11 the results for different temperatures. According to this formula the region II is limited by the requirement that the expression in the square root has to be positive. This leads to a boundary between region II and new region III of the form

$$\phi_2^2 = 2 \frac{v^2 - \Sigma_{-1}(m_0, T)}{1 + \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)}. \quad (2.103)$$

We note that the relation is implicit, the value of m_0 that appears on the right hand side is related to ϕ_2^2 on the left hand side by the gap equation.

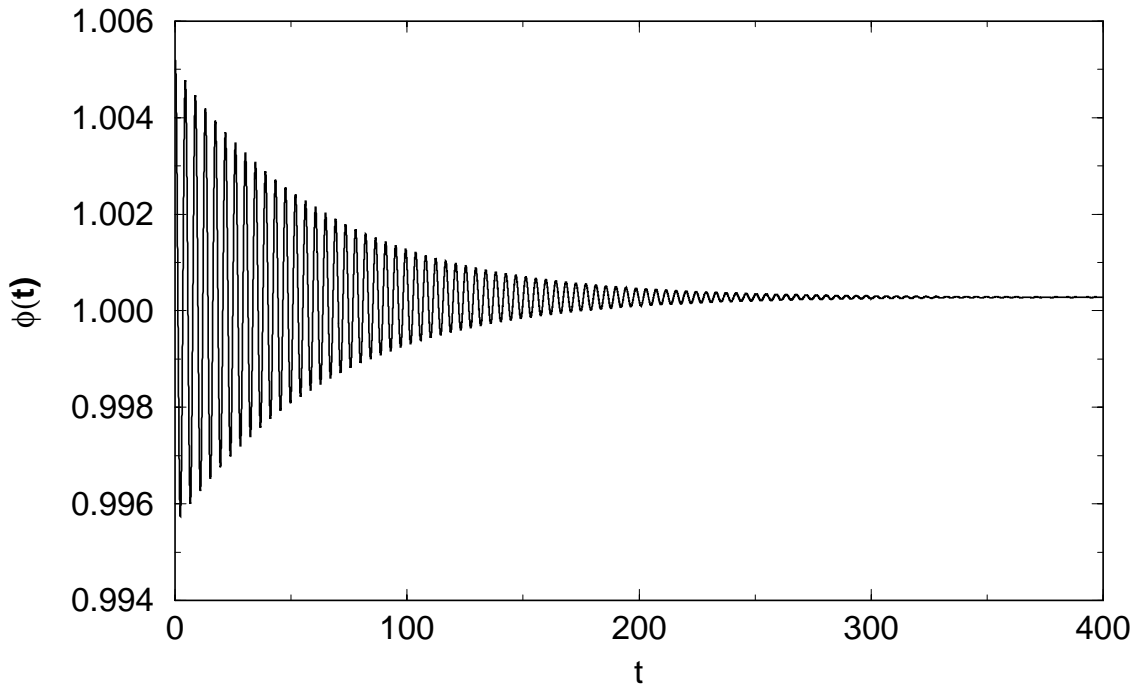


Figure 2.4: Zero mode versus t for $T = 0$, $m_0 = 0.1$, and $\phi(0) = 1.005$

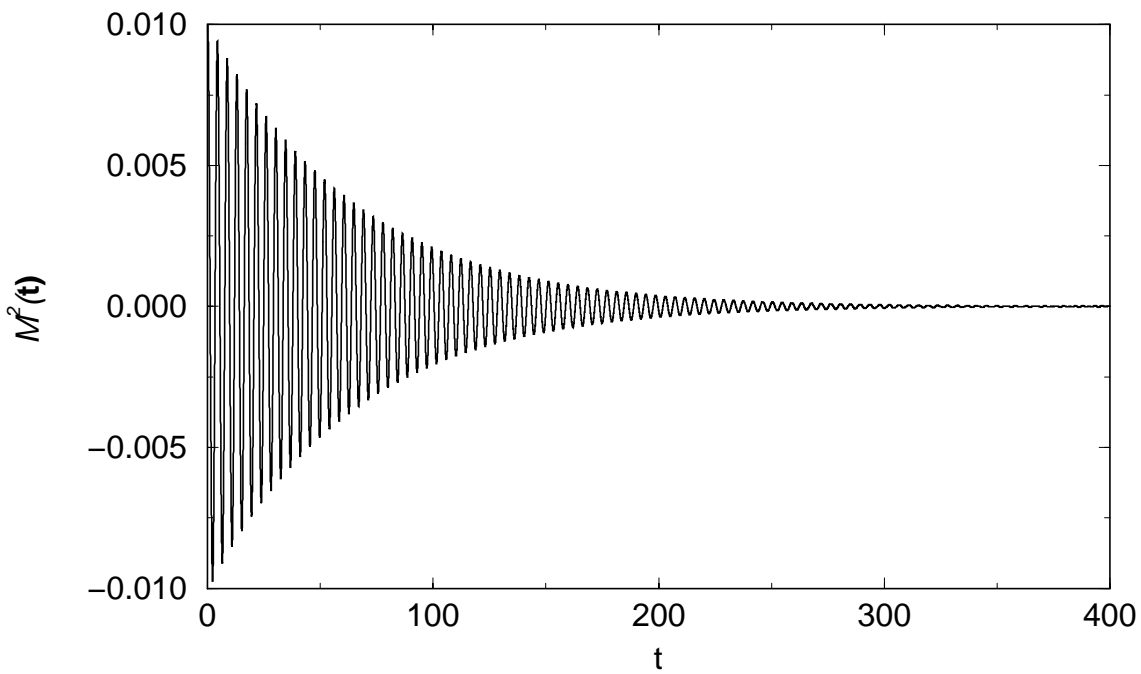


Figure 2.5: Mass squared versus t for $T = 0$, $m_0 = 0.1$, and $\phi(0) = 1.005$

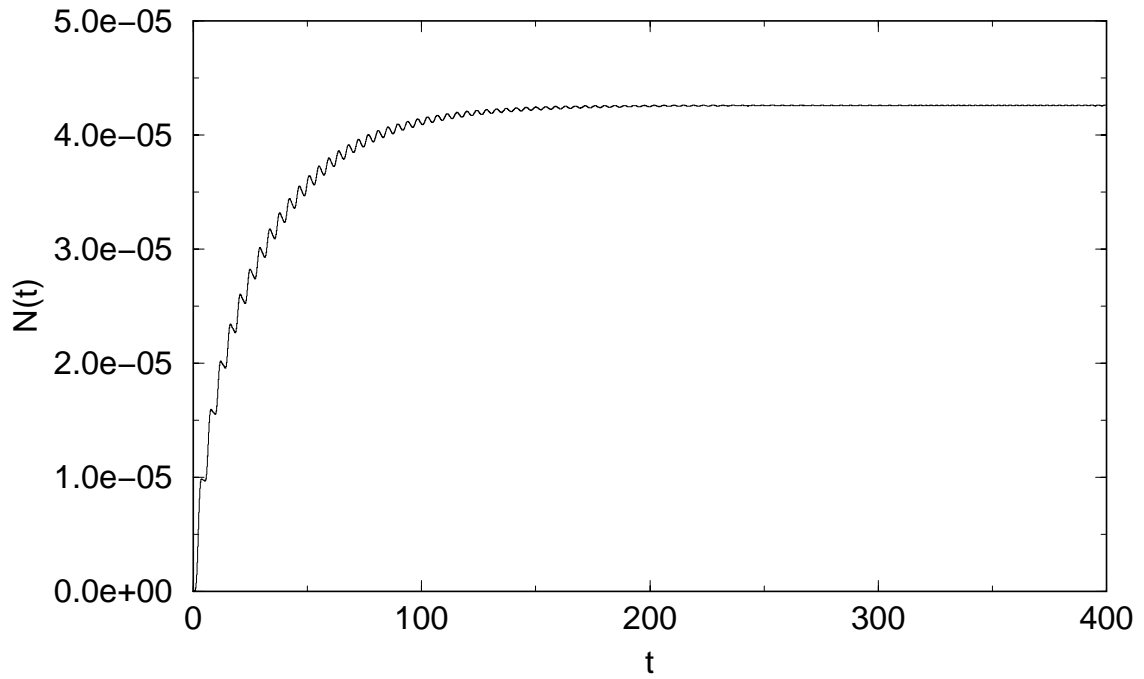


Figure 2.6: Particle number versus t for $T = 0$, $m_0 = 0.1$, and $\phi(0) = 1.005$

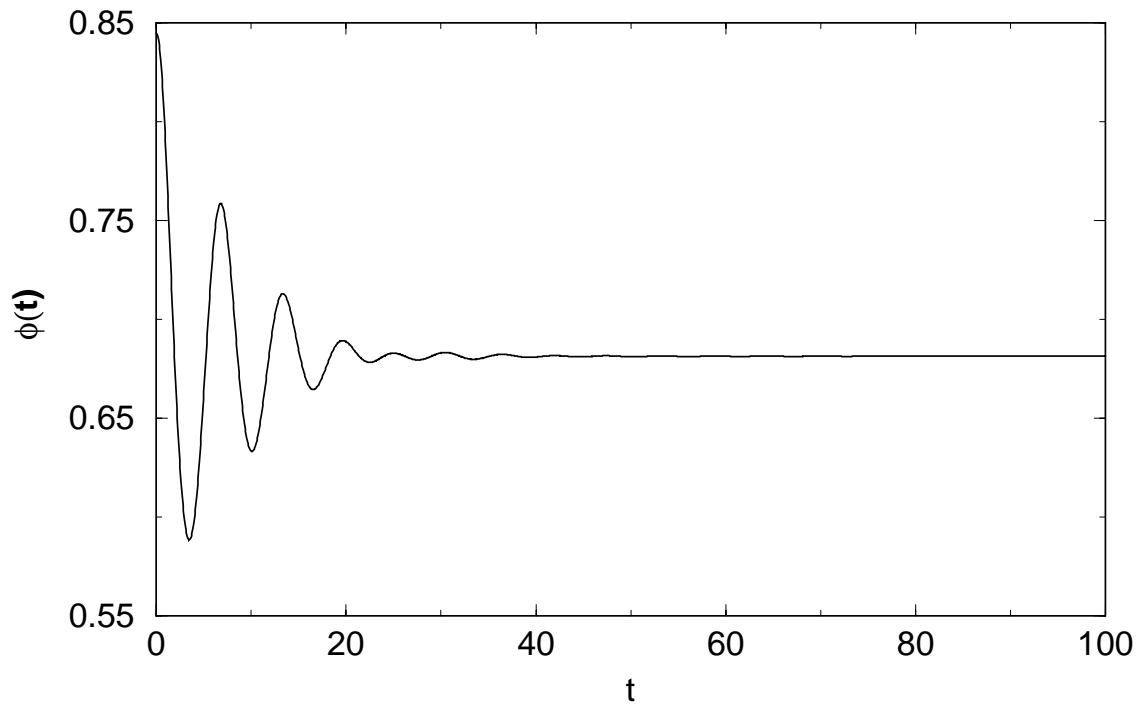


Figure 2.7: Zero mode versus t for $T = 2.5$, $m_0 = 0.4$, and $\phi(0) = 1.845$

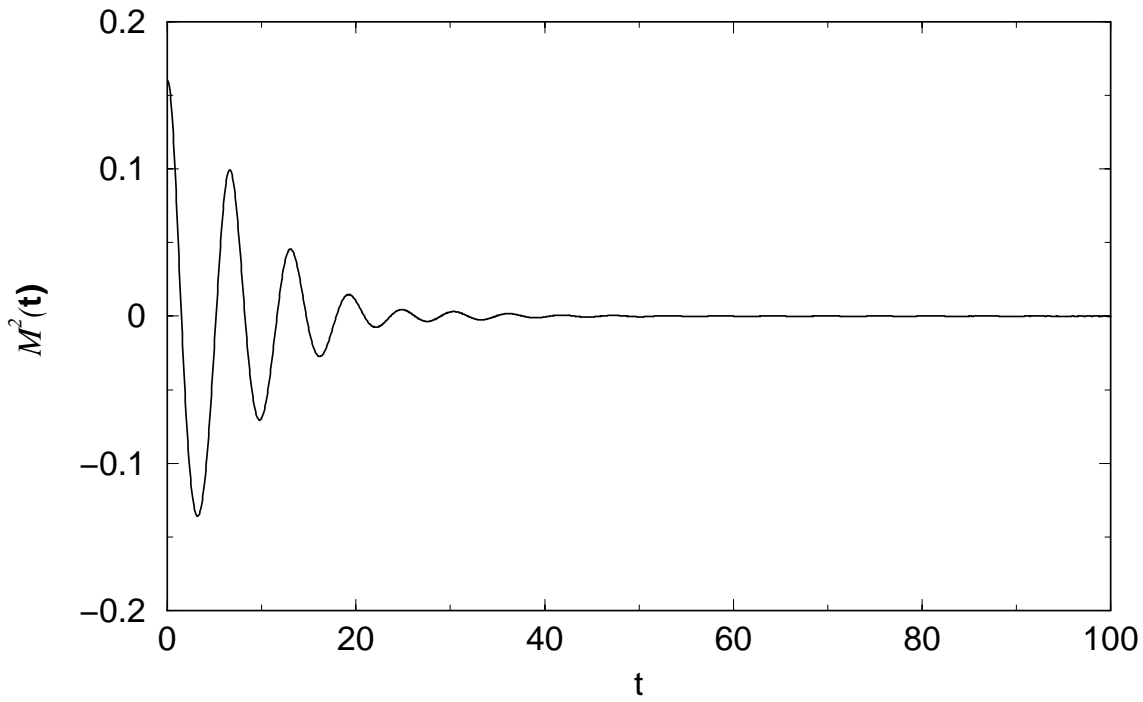


Figure 2.8: Mass squared versus t for $T = 2.5$, $m_0 = 0.4$, and $\phi(0) = 1.845$

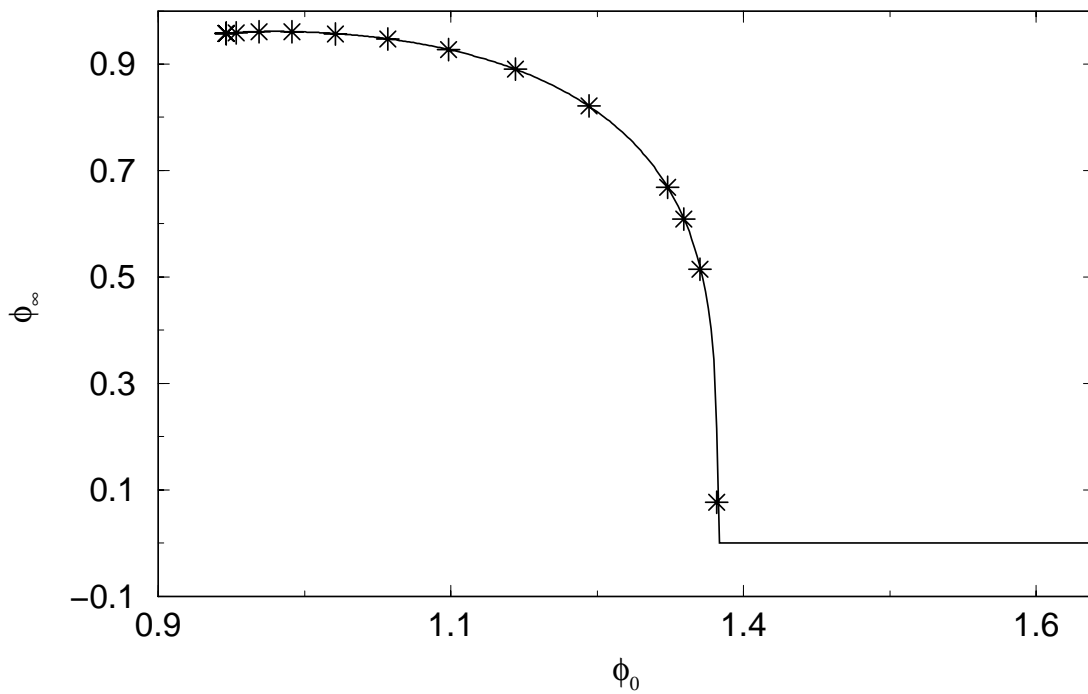


Figure 2.9: Late time amplitude ϕ_∞ versus ϕ_0 for $T = 1$

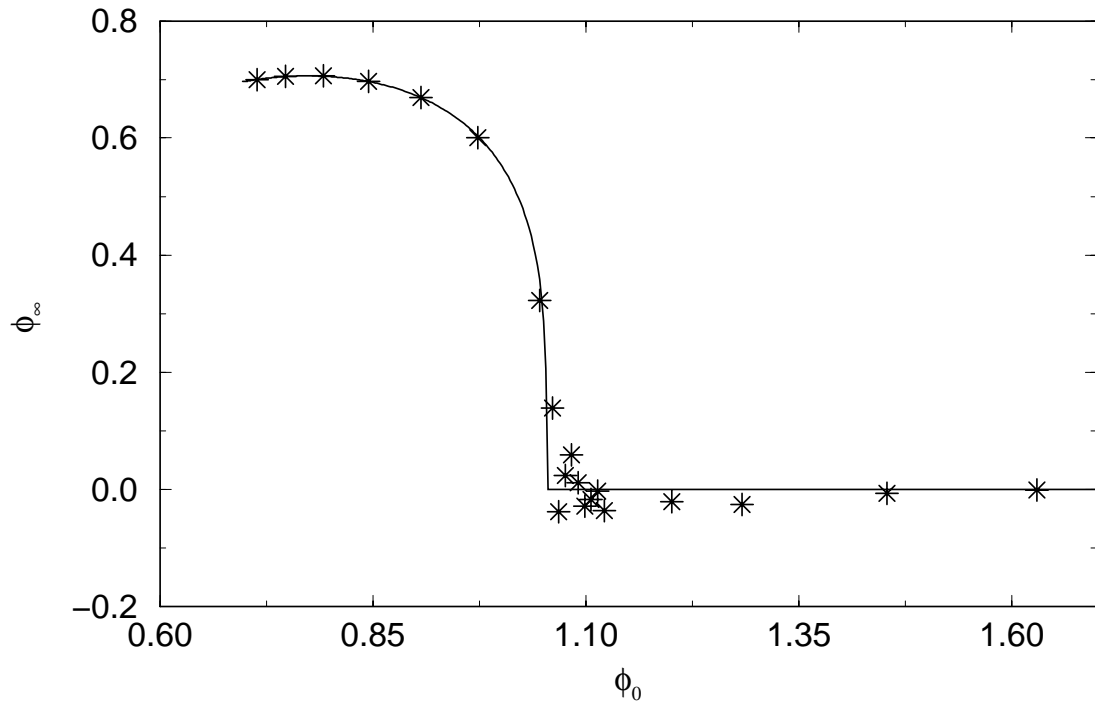


Figure 2.10: Late time amplitude ϕ_∞ versus ϕ_0 for $T = 2.5$

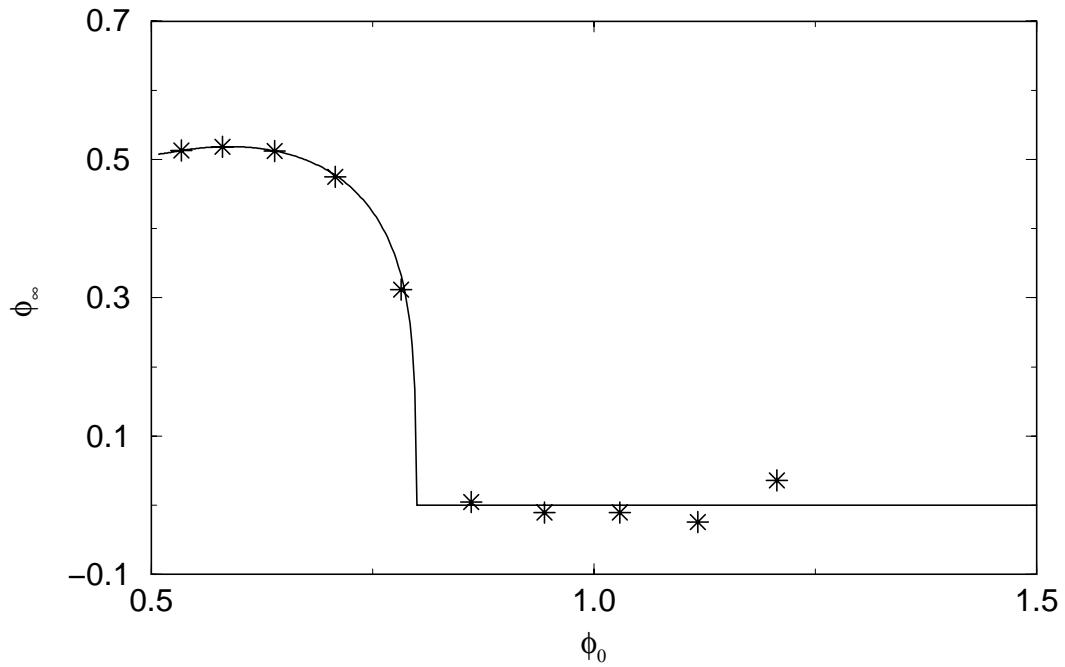


Figure 2.11: Late time amplitude ϕ_∞ versus ϕ_0 for $T = 3$

Region III: $\phi_\infty = 0$, $\mathcal{M}_\infty^2 > 0$

If we choose now an initial value for the zero mode high above the maximum of the potential, this point is considered in Fig. 2.1 in c), the zero mode has enough energy to cross the maximum, begins to oscillate around zero and a stable situation occurs. The region of initial values ϕ_0 leading to this behavior is called region III. We have simulated this case by choosing $T = 0$ and $m_0 = 3$ which leads to the initial value $\phi(0) = 3.185$ for the zero mode. The behavior of the zero mode is shown in Fig. 2.12. The field reaches a stationary oscillation around zero. In Fig. 2.13 we have displayed the corresponding mass squared which also reaches a stable finite value. This behavior coincides with the behavior in the unbroken theory that we will show in subsection 2.5.3. The zero mode is damped via parametric amplification. This phenomenon occurs if the mass reaches a finite stationary state and the zero mode oscillates around zero. It is discussed e.g. in [28]. In order to check our numerical calculations we have also investigated energy conservation (Fig. 2.14). The zero mode part of the energy decreases whereas the fluctuation energy increases. The solid line shows the total energy which is conserved. Since we have computed some finite corrections Δm^2 , $\Delta \lambda$ and $\Delta \Lambda$ which we have handled as a contribution to the zero mode part of the energy, this part of the energy can become negative. As a second cross check we have computed the pressure. The total pressure shown in Fig. 2.15 reaches an asymptotic value of nearly $\mathcal{E}/3$ typical for an ultra relativistic gas.

The fourth initial value, in Fig. 2.1 labeled with b), is taken only a little bit higher than the maximum. The behavior of the zero mode is shown in Fig. 2.16. At the beginning it is efficiently damped but then it oscillates around zero. The mass squared plotted in Fig. 2.17 very quickly reaches nearly zero. This behavior can be understood as follows. The mass term in the mode function is given by (2.91). We have prepared the initial conditions in such a way that m_0^2 is positive. Evolving in time $\phi(t)$ decreases and vanishes at the maximum. The first term in (2.91) becomes negative. On the other hand the fluctuations $\mathcal{F}(t)$ increase and lead therefore to a positive value for $\mathcal{M}^2(t)$. Since the asymptotic value of \mathcal{M}^2 is only slightly larger than zero as one can see in Fig. 2.18 the oscillation period of the field is very large.

There are two phenomena that characterize the transition to region III. On the one hand, the stabilization of the system is taken over by the phenomenon of parametric resonance. On the other hand the system has enough energy so that $\phi(t)$ can move over the maximum of the potential at $\phi = 0$, and indeed will oscillate around $\phi = 0$. Accordingly, the threshold value of ϕ_0 at which these two phenomena set in can be characterized by two - a priori unrelated - criteria. Both rely on plausible assumptions, which at $T = 0$ lead to the same prediction for the critical value of ϕ_0 .

The criterion based on the energy consideration has been presented in the previous subsection, we now describe the criterion supplied by the phenomenon of parametric resonance. For the case of unbroken symmetry it was found at zero [53] and finite temperature [34], that the late time behavior is described by an empirical sum rule which relates \mathcal{M}_∞^2 to the initial amplitude. We show some results for the unbroken case in subsection 2.5.3. For $T = 0$ an analogous sum rule was found to hold for the case of spontaneously broken

symmetry as well [29]. It is given by

$$\mu_\infty^2 = -1 + \frac{\eta_0^2}{2} . \quad (2.104)$$

Here μ and η are normalized in such a way that the classical equation of motion at early times, i.e. in the parametric resonance regime without back reaction, reads

$$\eta'' - \eta + \eta^3 = 0 , \quad (2.105)$$

where the prime denotes a derivative with respect to $\tau = \alpha t$ and where $\eta = \beta\phi$, also $\mu = \mathcal{M}/\alpha$. With $\eta(\tau)$ a solution of (2.105) the mode equation becomes a Lamé equation. The sum rule implies [53], that the frequencies $\omega_k^2(t) = \mathcal{M}^2(t) + k^2$ are shifted outside the parametric resonance band of the Lamé equation. Although there is no rigorous derivation for the sum rule, it accordingly seems related to the parametric resonance phenomenon.

As the shift of the frequencies outside the parametric resonance region must have happened at the end of the phase where the evolution of the system is described by parametric resonance, we will again consider the initial classical evolution. Again, in addition to the zero mode terms we have to take into account the terms due to the thermal fluctuations. In terms of the parameters introduced in the previous section the equation of motion is given by

$$\ddot{\phi} + \lambda\mathcal{C}a\phi^3 + \frac{\lambda}{2}\mathcal{C}(b + c\phi_0^2)\phi = 0 . \quad (2.106)$$

Comparing to the normalized equation (2.105) we determine the factors α and β to be

$$\alpha = \sqrt{\frac{\lambda\mathcal{C}}{2}}\sqrt{b + c\phi_0^2} , \quad (2.107)$$

$$\beta = \sqrt{-\frac{2a}{b + c\phi_0^2}} , \quad (2.108)$$

so that the asymptotic mass is given by

$$\begin{aligned} \mathcal{M}_\infty^2 &= \alpha^2 \left(-1 + \frac{1}{2}\beta^2\phi_0^2 \right) \\ &= \lambda\mathcal{C} \left\{ -v^2 + \Sigma_{-1}(m_0, T) + \frac{1}{2} [1 + \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)] \phi_0^2 \right\} . \end{aligned} \quad (2.109)$$

Again ϕ_0 and m_0 are related by the gap equation. At the transition from region II to region III the asymptotic mass vanishes. It is easily seen that this criterion leads to an identical equation for the boundary, i.e., (2.103).

The field amplitude decreases to zero at late times, in this regime. So the symmetry is restored dynamically at high excitation characterized by a high value of ϕ_0 .

At the critical temperature $T_C = \sqrt{12}v$ both boundaries $\phi_1(T)$ and $\phi_2(T)$ become zero. Above T_C the behavior of the system is the same as for region III, for all initial values

of ϕ_0 . While at the border between region I and II there was a lowest value for ϕ_0 for obtaining real solutions of the gap equation, now there is a lowest value of m_0 , the one for which $\phi_0 = 0$. It is obtained by solving the gap equation for $\phi_0 = 0$ and agrees with the thermodynamical equilibrium value m_β at that temperature, as defined, e.g., in (3.38) in a paper by Dolan and Jackiw [54]. Of course with $\phi_0 = 0$ the system remains static. The sum rule for the asymptotic value (2.109) is compared to the data in Fig. 2.19. The agreement is excellent.

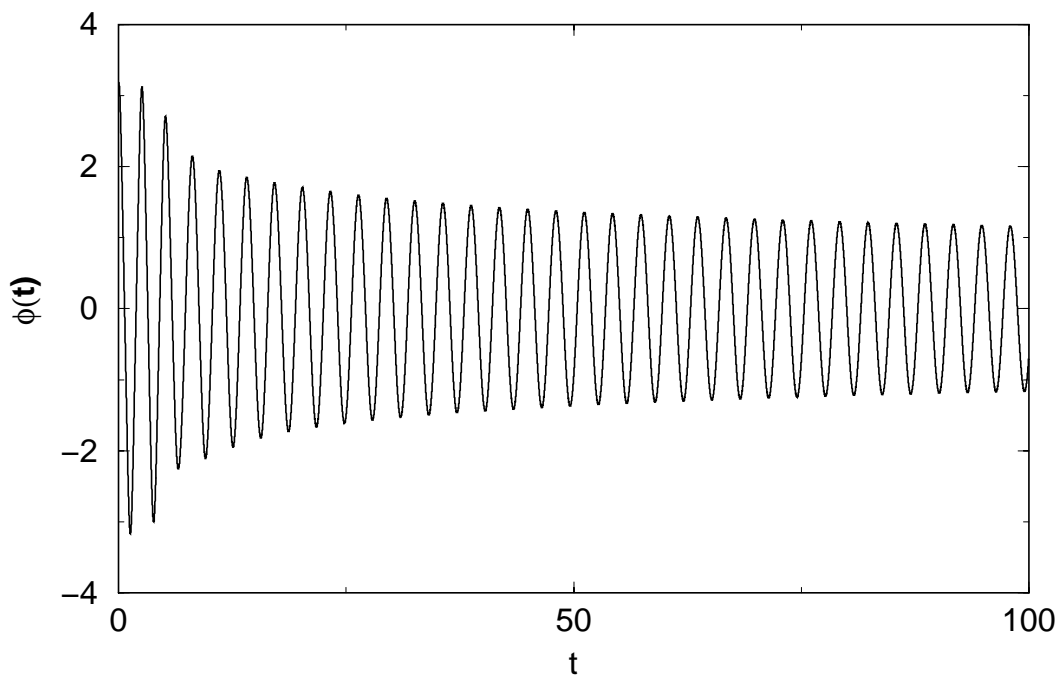


Figure 2.12: Zero mode versus t for $T = 0$, $m_0 = 3$, and $\phi(0) = 3.185$

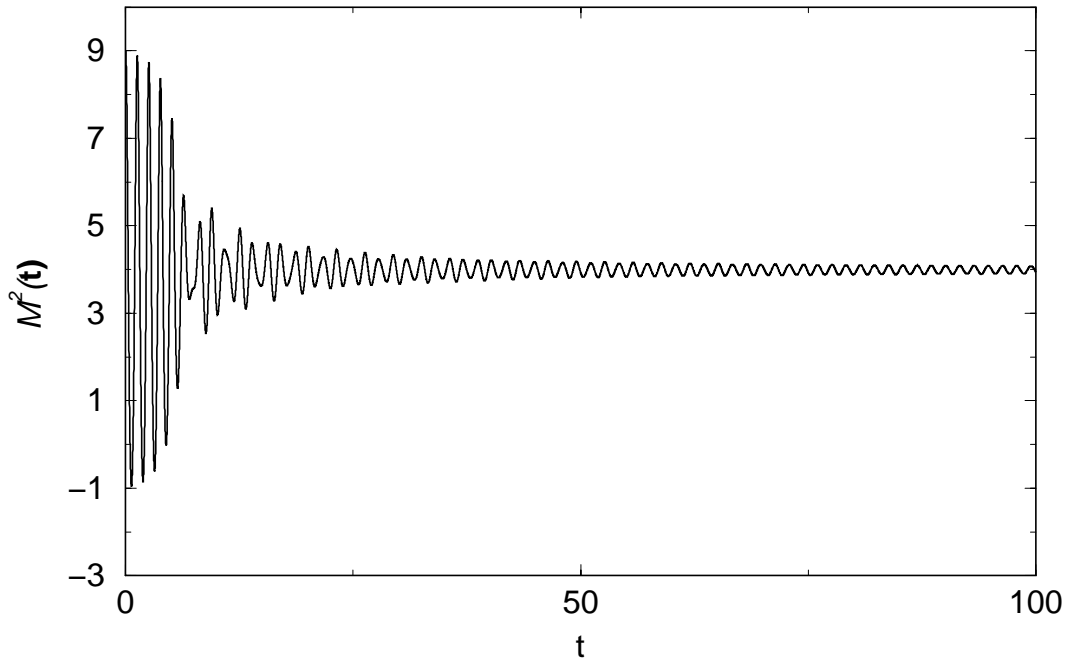


Figure 2.13: Mass squared versus t for $T = 0$, $m_0 = 3$, and $\phi(0) = 3.185$

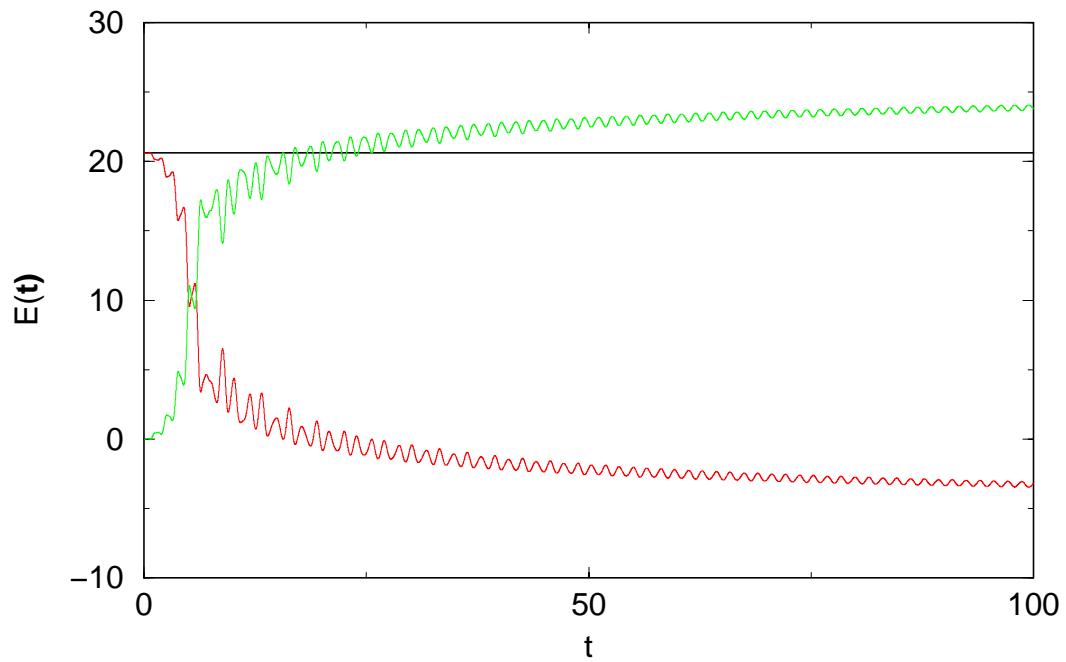


Figure 2.14: Mode energies and their sum versus t for $T = 0$, $m_0 = 3$, and $\phi(0) = 3.185$

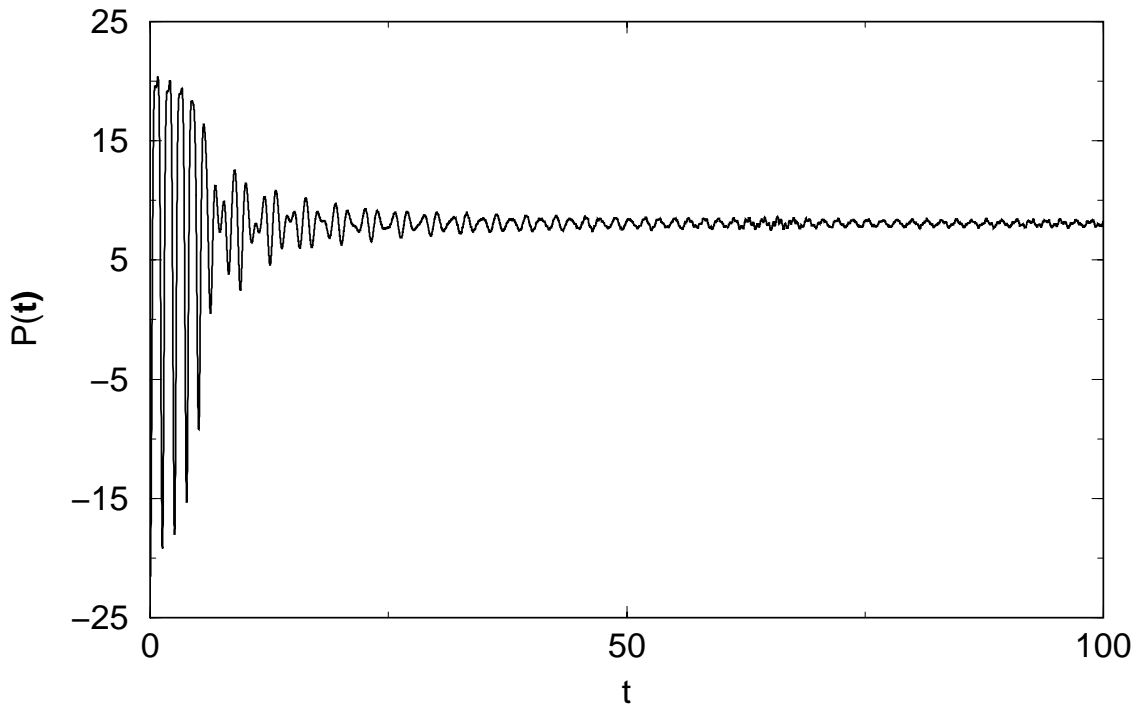


Figure 2.15: Pressure versus t for $T = 0$, $m_0 = 3$, and $\phi(0) = 3.185$

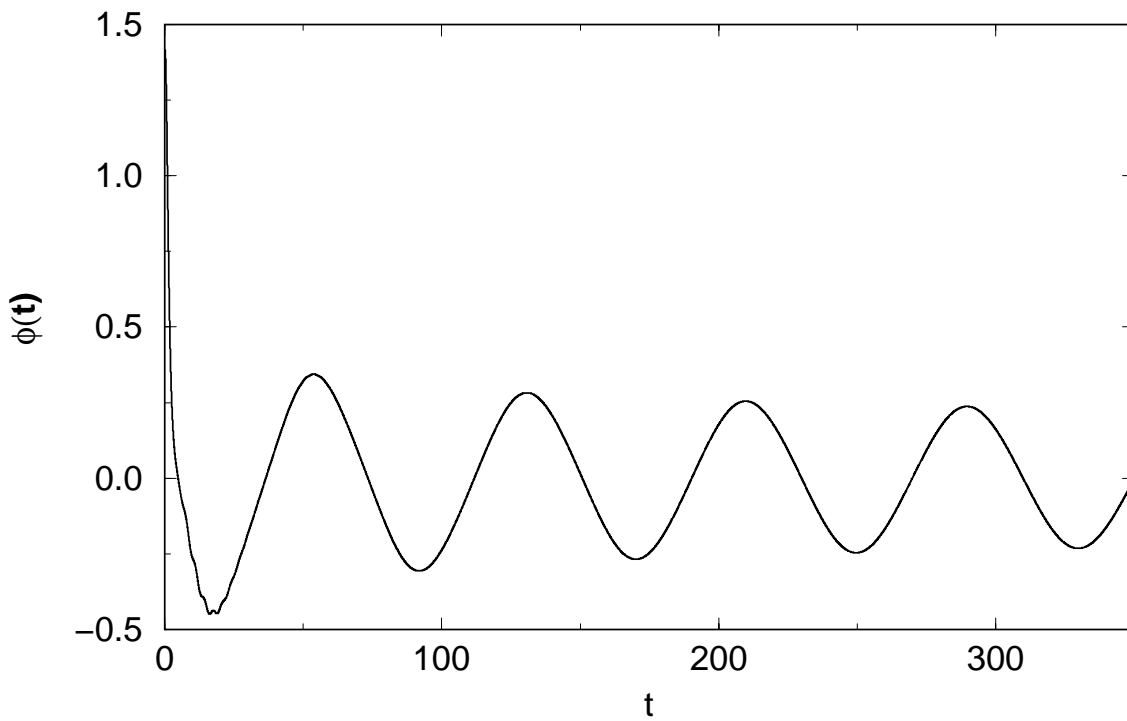


Figure 2.16: Zero mode versus t for $T = 0$, $m_0 = 1$, and $\phi(0) = 1.415$

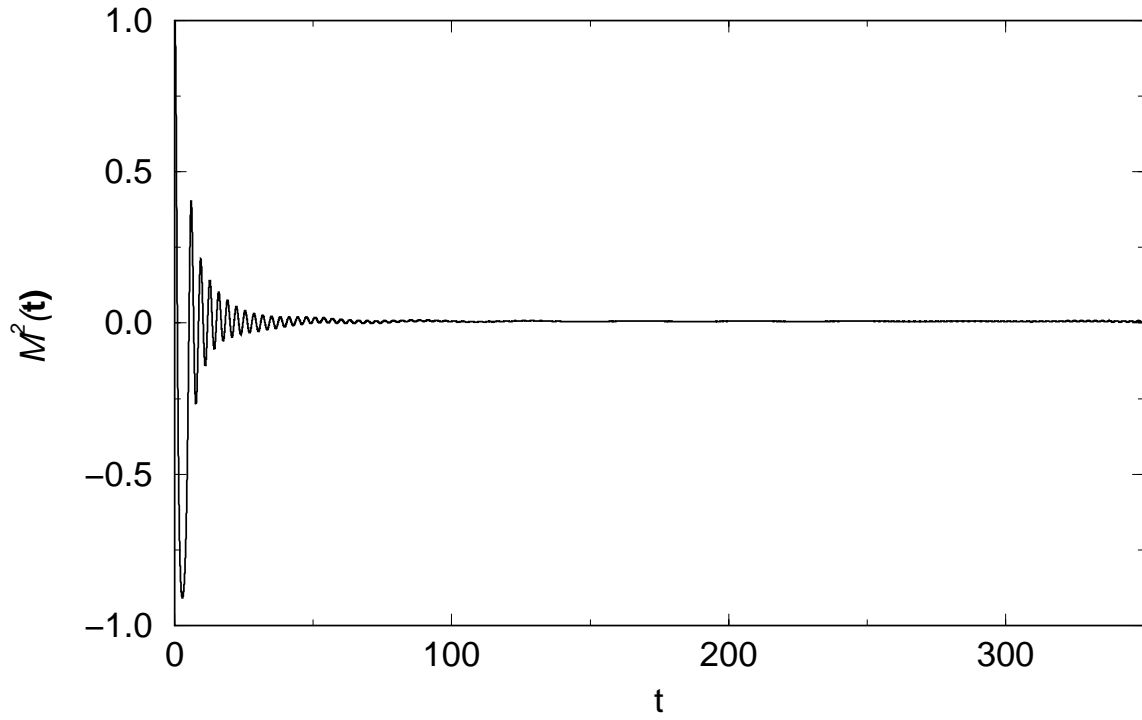


Figure 2.17: Mass squared versus t for $T = 0$, $m_0 = 1$, and $\phi(0) = 1.415$

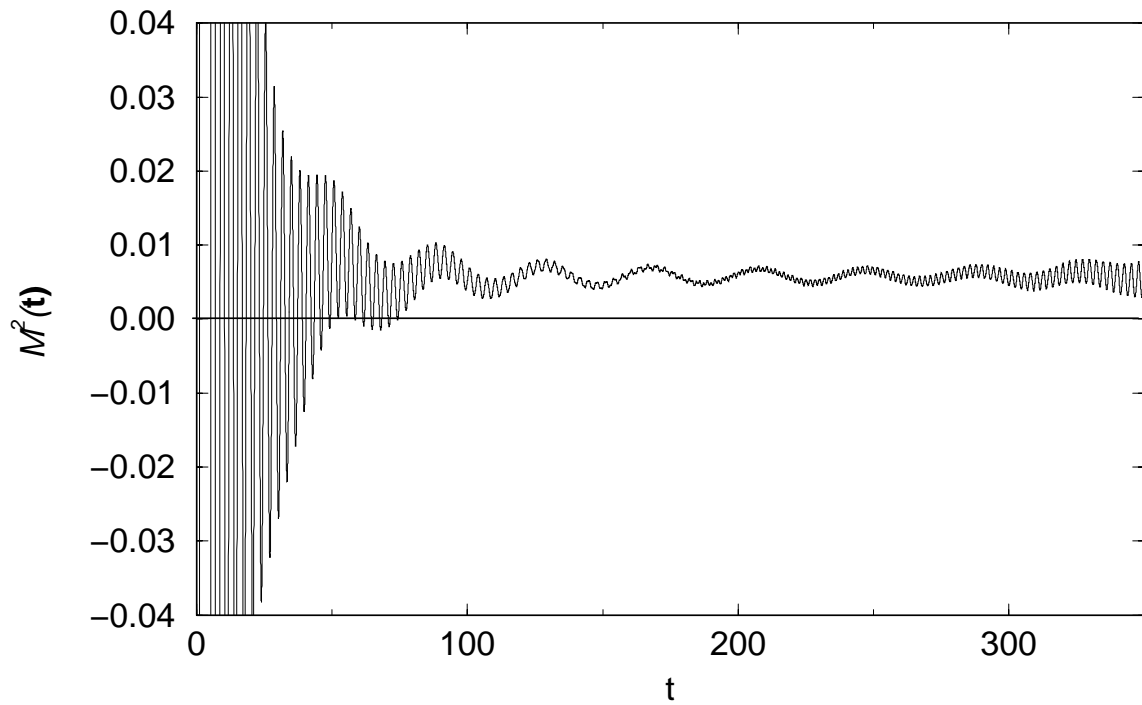


Figure 2.18: The same as in Fig. 2.17 with another y -range

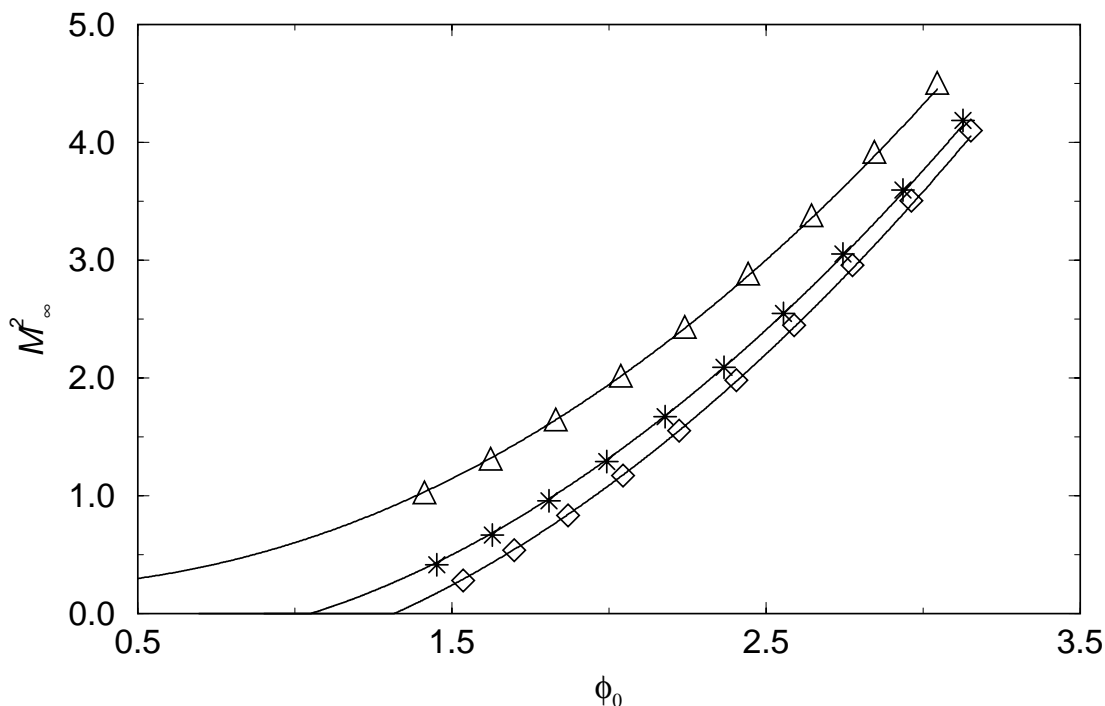


Figure 2.19: Asymptotic sum rule for \mathcal{M}^2 . The data for $T = 1.5$ (diamonds), $T = 2.5$ (asterisks) and $T = 4$ (triangles) are compared to (2.109) (solid line)

2.5.2 Comparison with the One Loop Approximation

In this section we compare the large N approximation with one loop calculations by setting $N = 1$. For the unbroken case we have discussed the renormalization and numerics in [33]. Since we are here only interested in a qualitative comparison we only review the unrenormalized equations. The renormalization is straightforward along the line in [33]. The equation of motion for the zero mode reads in the one loop approximation for $T = 0$

$$\ddot{\phi}(t) + \lambda\phi(t) [\phi^2(t) - v^2] + 3\lambda\phi(t)\mathcal{F}(t) = 0, \quad (2.110)$$

with

$$\mathcal{F}(t) = \int \frac{d^3k}{(2\pi)^3} \frac{|U_k(t)|^2}{2\omega_{k0}}. \quad (2.111)$$

Ignoring the counter terms this equation is similar to (2.11) up to a factor 3 in the fluctuation integral. The mode function fulfills the second order differential equation

$$\left\{ \frac{d^2}{dt^2} + k^2 + m_h^2 + 3\lambda [\phi^2(t) - v^2] \right\} U_k(t) = 0. \quad (2.112)$$

The difference to the large N approximation now becomes obvious, we do not include the fluctuation integral in the time dependent mass for the modes and therefore, we neglect the back reaction. Another point is that we deal only with a theory of a single field.

We now consider numerically the two cases b) and c) in Fig. 2.1. Starting point d) is analogous to the unbroken symmetry case and not very interesting. The initial value a) leads also in this case to a complex initial frequency and is therefore not considered. For the evolution of the field in the minimum (Fig. 2.20), where we have chosen the initial value for the zero mode as low that the field does not reach the unstable part of the effective potential we find that the damping is much less efficient than in the large N approximation. Since the zero mode does not enter the spinodal region this effect has to be expected. Only dissipation via parametric amplification occurs. Also the possibility of the decay of the mean field is absent because we have no Goldstone bosons in the theory. We will see in the case of the gauge fields, where we have various degrees of freedom, that the effect induced by the decay of the zero mode leads even in the one loop approximation to efficient damping. We have also plotted the fluctuation integral in Fig. 2.21. It increases in time, reaches a stationary state and oscillates forever.

In the second case we let the field evolve in the unstable region. As already mentioned in the discussion for the large N approximation the frequency for the modes becomes complex for low momenta and near the origin. Therefore, the modes blow up exponentially as it is shown in Fig. 2.23. In the large N approximation the contribution of the fluctuations in the effective mass leads during the development of the system to a positive effective mass and stops the exponential growth of the modes. The zero mode settles down to a constant value. In the one loop approximation the effective mass is given by

$$\mathcal{M}^2(t) = -\frac{m_h^2}{2} + 3\lambda\phi^2(t). \quad (2.113)$$

Since the mean field decreases the effective mass becomes complex if $\phi^2(t) < m_h^2/6\lambda$ and stays complex. The exponential growth of the modes is not stopped, the oscillations of the zero mode get a very high frequency and after a while the approximation breaks down (Fig. 2.22). For our choice of parameters ($\lambda = 1$, $m_h^2 = 2\lambda v^2 = 2$) this happens for $\phi(t) < \pm\sqrt{\frac{1}{3}}$ displayed by the dashed line in Fig. 2.22. The zero mode reaches this value at $t \approx 21$, the effective mass squared, which is shown in Fig. 2.23 by the dashed line, becomes negative and the fluctuation integral growth exponentially as explained above. This indicates that the one loop approximation is not reliable in the unstable region. For our considerations concerning the gauge fields in the next chapter we consider only the one loop approximation. For gauge theories already this approximation becomes rather complicated and it is not clear yet how to improve it in a systematic way. Therefore, we only study numerically stable configurations and initial conditions.

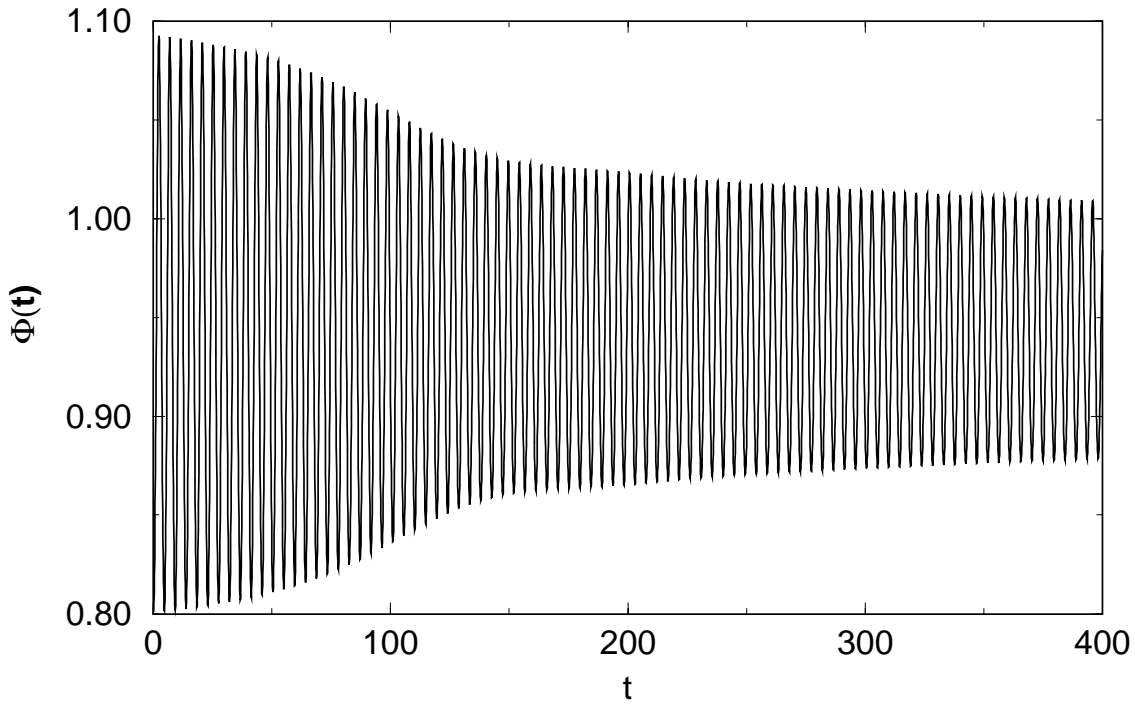


Figure 2.20: Zero mode versus t for $m_0 = 0.96$ and $\phi(0) = 0.8$

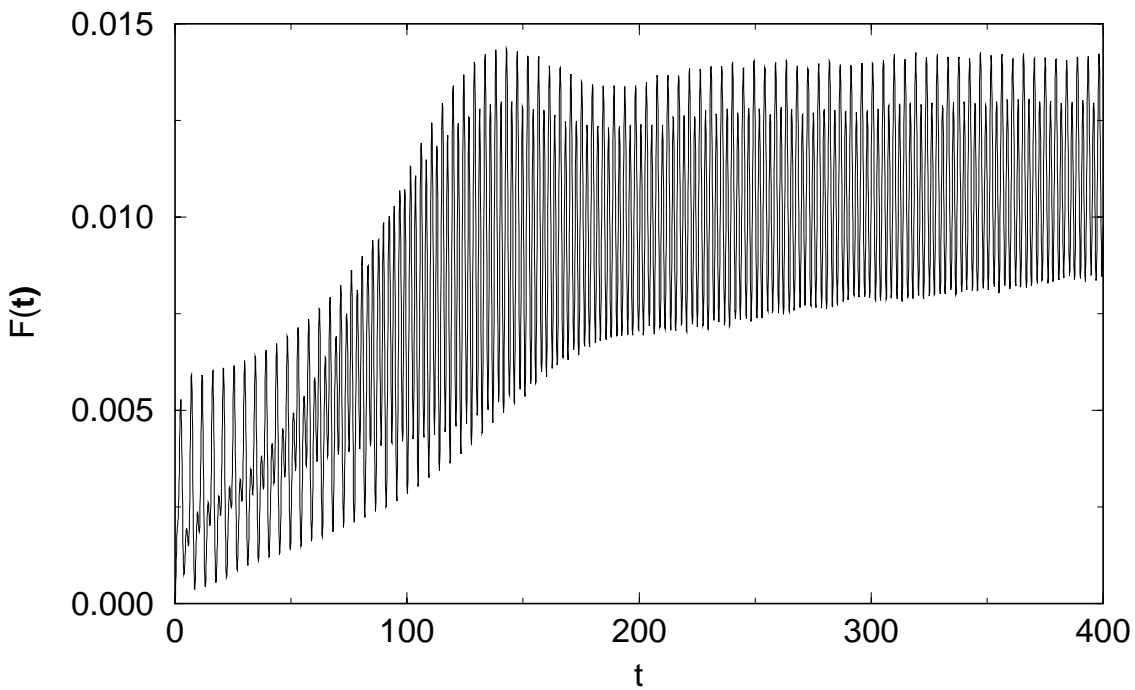


Figure 2.21: Fluctuation integral versus t for $m_0 = 0.96$ and $\phi(0) = 0.8$

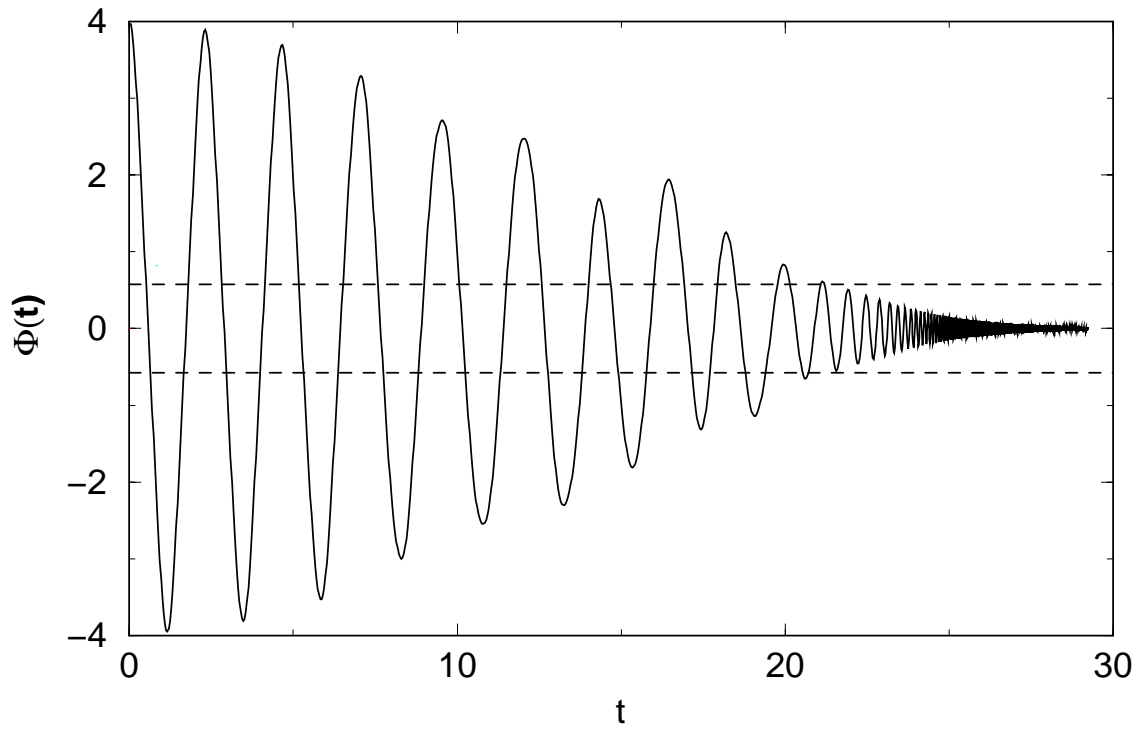


Figure 2.22: Zero mode versus t for $m_0 = 6.86$ and $\phi(0) = 4$

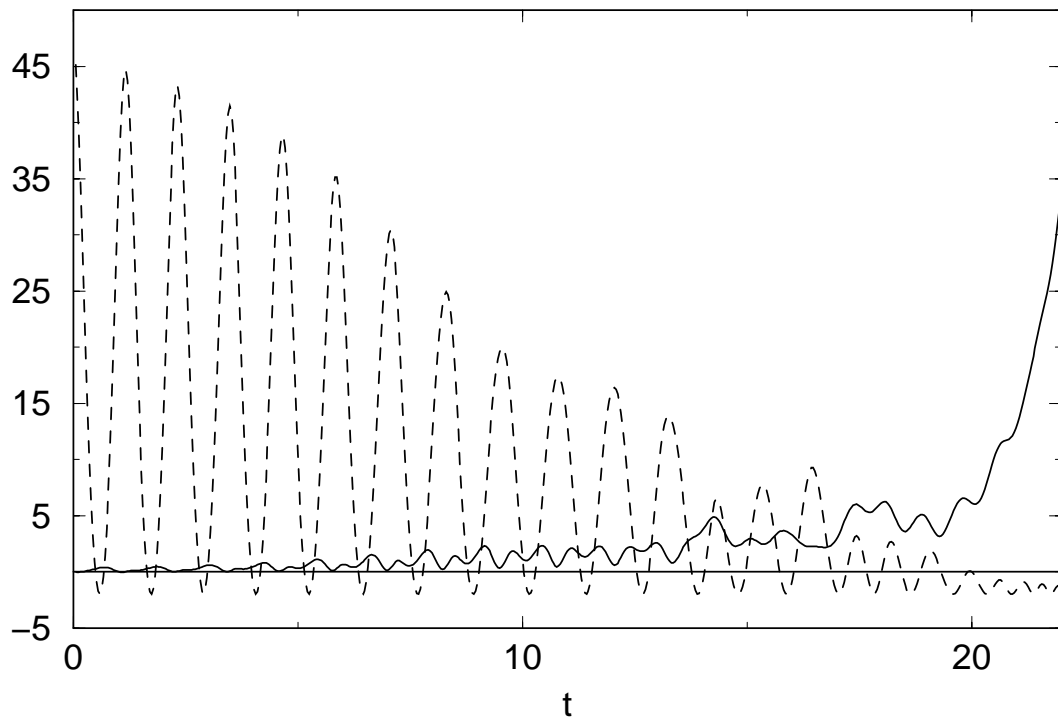


Figure 2.23: Fluctuation integral (solid line) and mass squared (dashed line) versus t for $m_0 = 6.86$ and $\phi(0) = 4$

2.5.3 Comparison with the Unbroken Theory

In this section we consider numerical studies of the unbroken theory in the large N expansion at $T = 0$ and $T \neq 0$. We have published a detailed analysis of the corresponding equations and the renormalization for finite temperature in [34]. We present here only the results. We have analyzed the behavior of the system for two parameter sets at $T = 0$, where we have varied the coupling constant λ and fixed m_h^2 and m_0 which leads to different initial values because m_0 and ϕ_0 are connected via the gap equation. We have also chosen one parameter set for $T \neq 0$.

	λ	T	m^2	m_0	$\phi(0)$	$\mathcal{M}^2(\infty)$	right hand side
Parameter set 1	1	0	1	3	2.82	5.0	4.98
Parameter set 2	5	0	1	3	1.24	4.95	4.90
Parameter set 3	1	10	1	3	1.25	8.24	8.25

Table 2.1: Parameter sets for the unbroken theory

In Fig. 2.24 we have plotted the zero mode for parameter set 1. The damping of the field is not very efficient and the field reaches a stable state after a short time. The effective mass squared is displayed in Fig. 2.25. It ends up in an asymptotic average of 5 with small oscillations. The sum rule for the unbroken theory adapted to our notation and definitions in [34] is given at finite temperature by ¹

$$\mathcal{M}_\infty^2 = \mathcal{C} \left\{ m^2 - \frac{\lambda}{16\pi^2}(m_0^2 - m^2) + \lambda\Sigma_{-1}(m_0, T) + \frac{1}{2} [1 + \lambda\mathcal{C}_T\Sigma_{-3}(m_0, T)] \phi_0^2 \right\}. \quad (2.114)$$

The additional term $\lambda(m_0^2 - m^2)/(16\pi^2)$ in comparison to (2.109) is due to another choice of the renormalization point. We have verified this sum rule for our parameter sets, the left and right hand sides of the sum rule are compared in Table 2.1, the agreement is excellent.

As cross check for our numerics we have also plotted energy conservation in Fig. 2.26 and the pressure in Fig. 2.27. The zero mode part of the energy decreases and the fluctuation energy increases like in the case of spontaneous symmetry breaking. The production of fluctuation energy leads to an increase of the particle number shown in Fig. 2.28.

We have considered a second parameter set with a larger coupling constant λ . This leads to a smaller initial value for the mean field. The behavior of the zero mode shown in Fig. 2.29 is qualitative the same as for the first parameter set. In order to check the sum rule we have also plotted the mass squared $\mathcal{M}^2(t)$ in Fig. 2.30. The agreement is very good for this parameter choice, too.

The behavior of the zero mode in the unbroken theory is the same as for the broken theory in region III as to be expected. In Figs. 2.31-2.33 we have displayed the behavior of the system for finite temperature.

¹In [34] we have not considered the contribution from $\Sigma_{-3}(m_0, T)$.

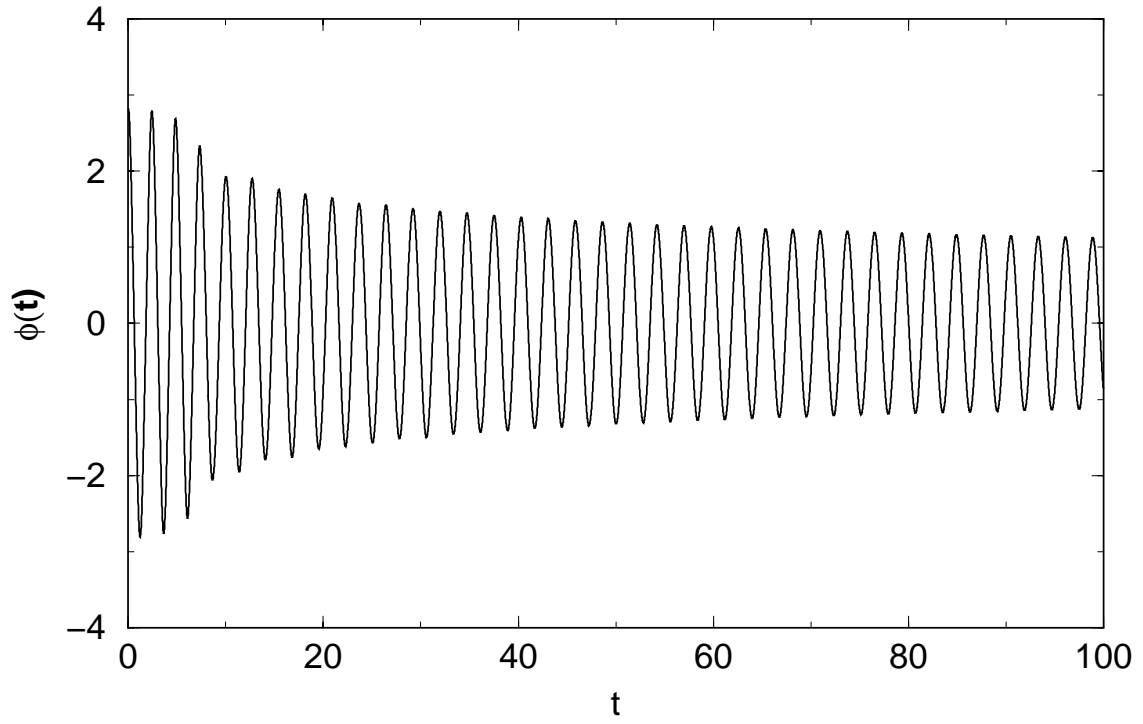


Figure 2.24: Zero mode versus t for parameter set 1

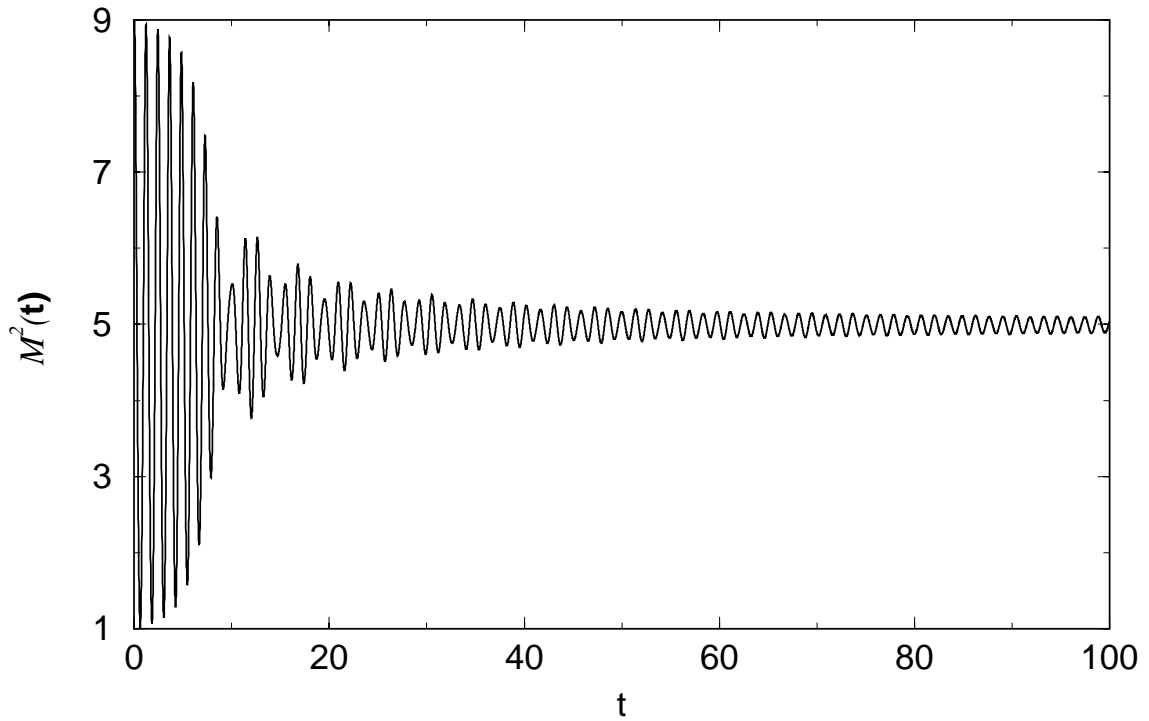


Figure 2.25: Mass squared versus t for parameter set 1

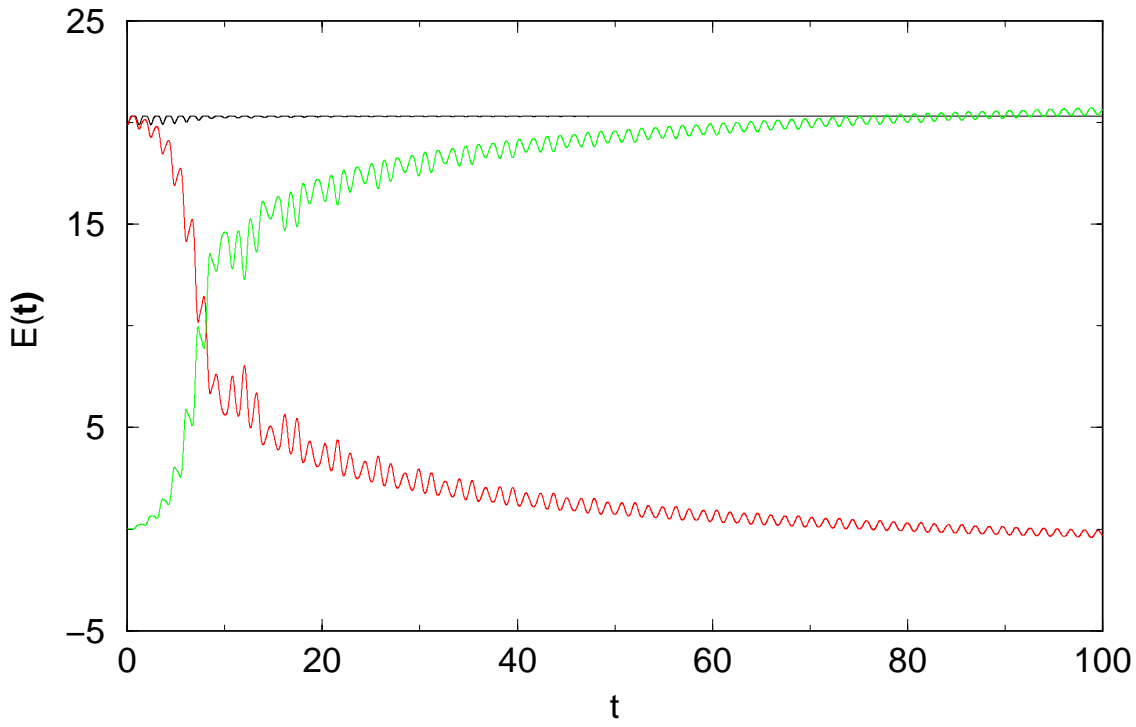


Figure 2.26: Mode energies and their sum versus t for parameter set 1

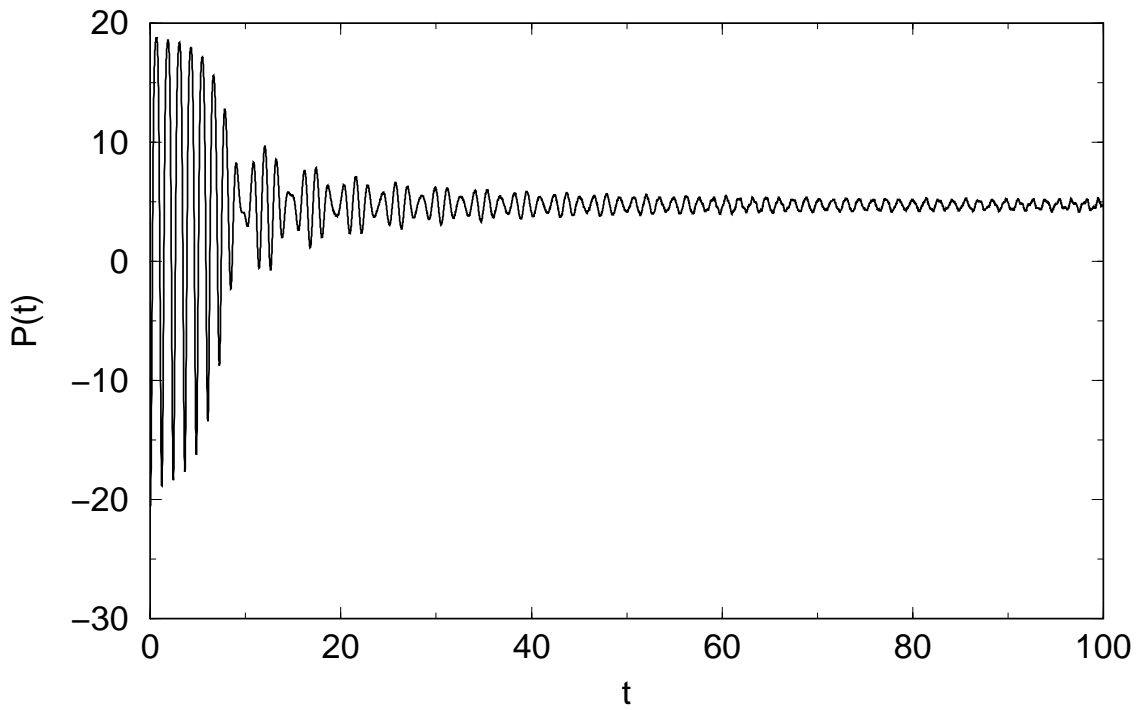


Figure 2.27: Pressure versus t for parameter set 1

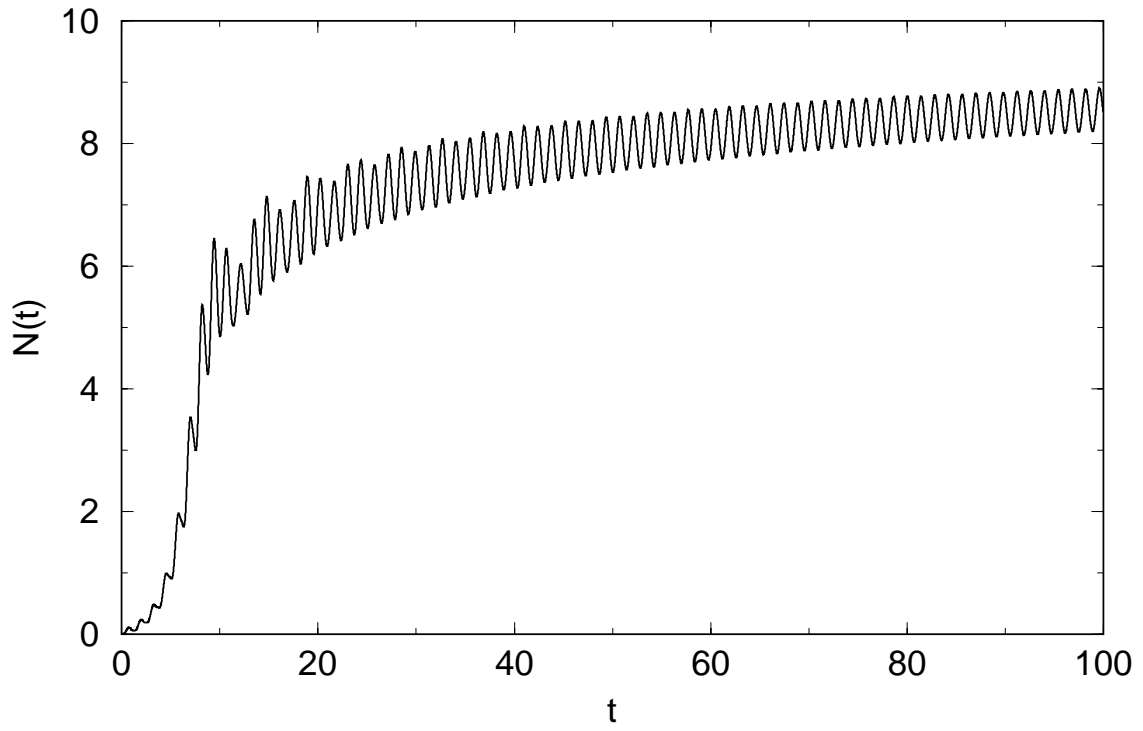


Figure 2.28: Particle number versus t for parameter set 1

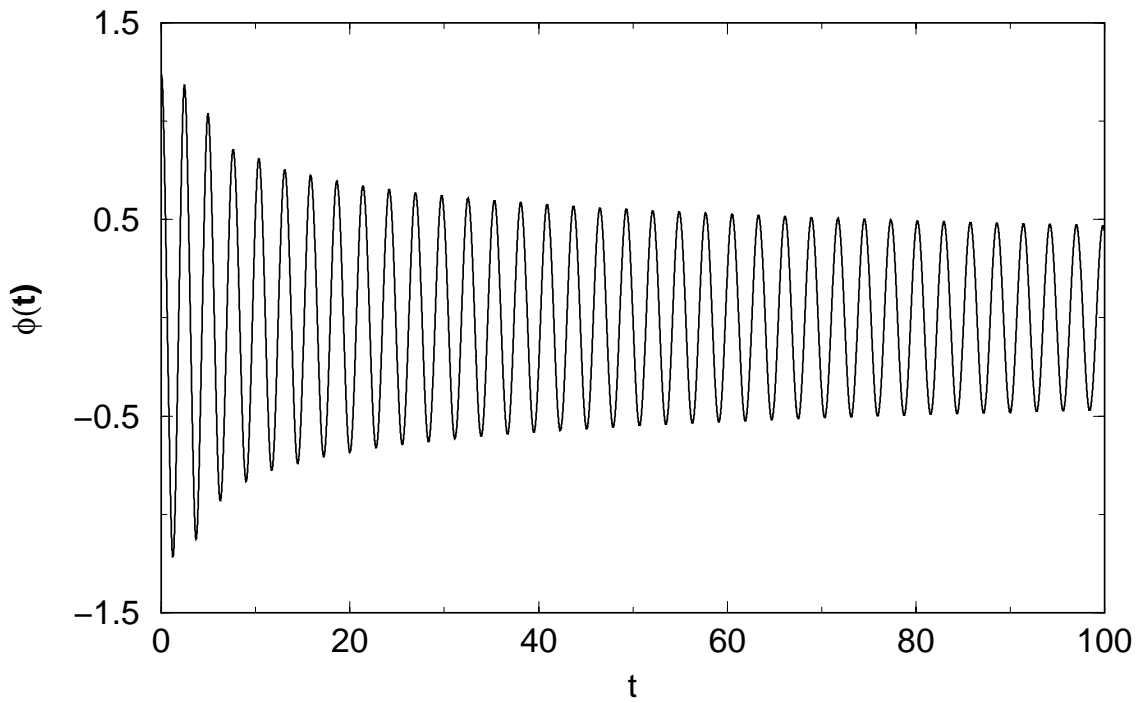


Figure 2.29: Zero mode versus t for parameter set 2

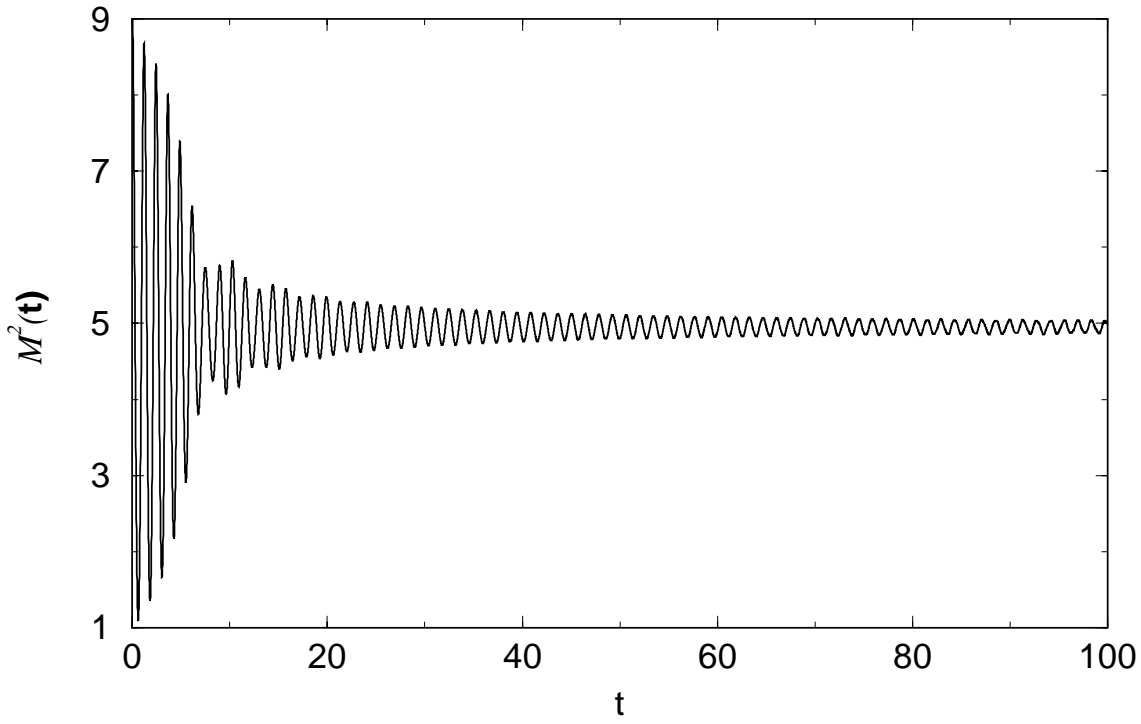


Figure 2.30: Mass squared versus t for parameter set 2

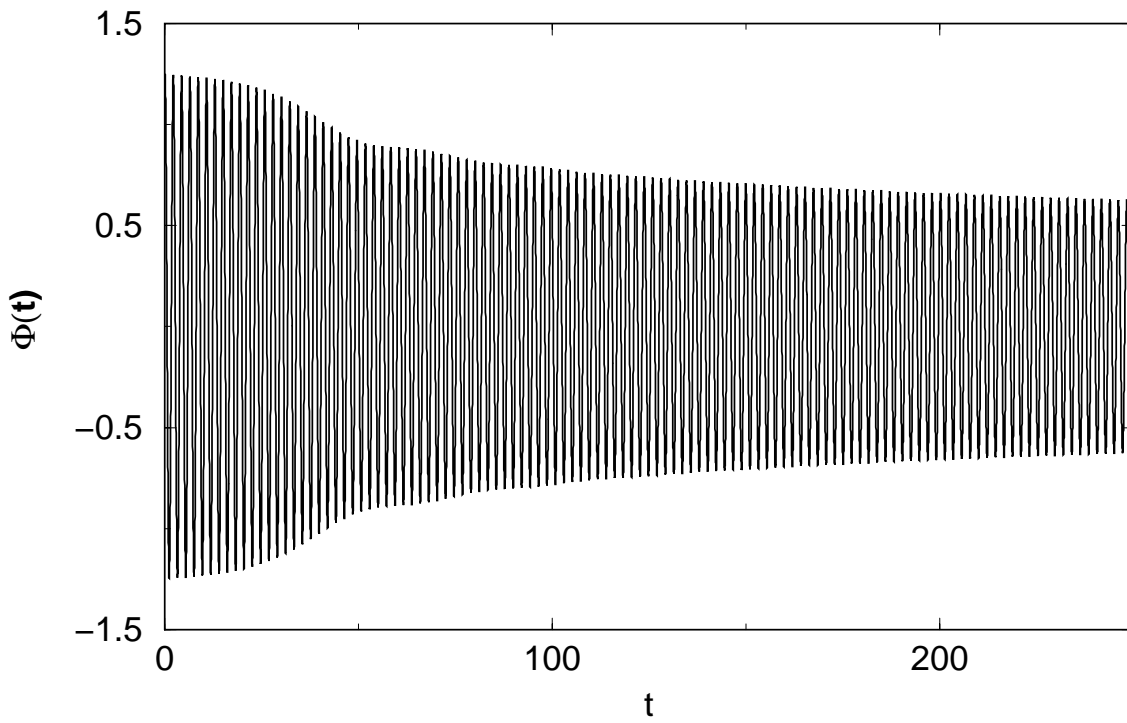


Figure 2.31: Zero mode versus t for parameter set 3

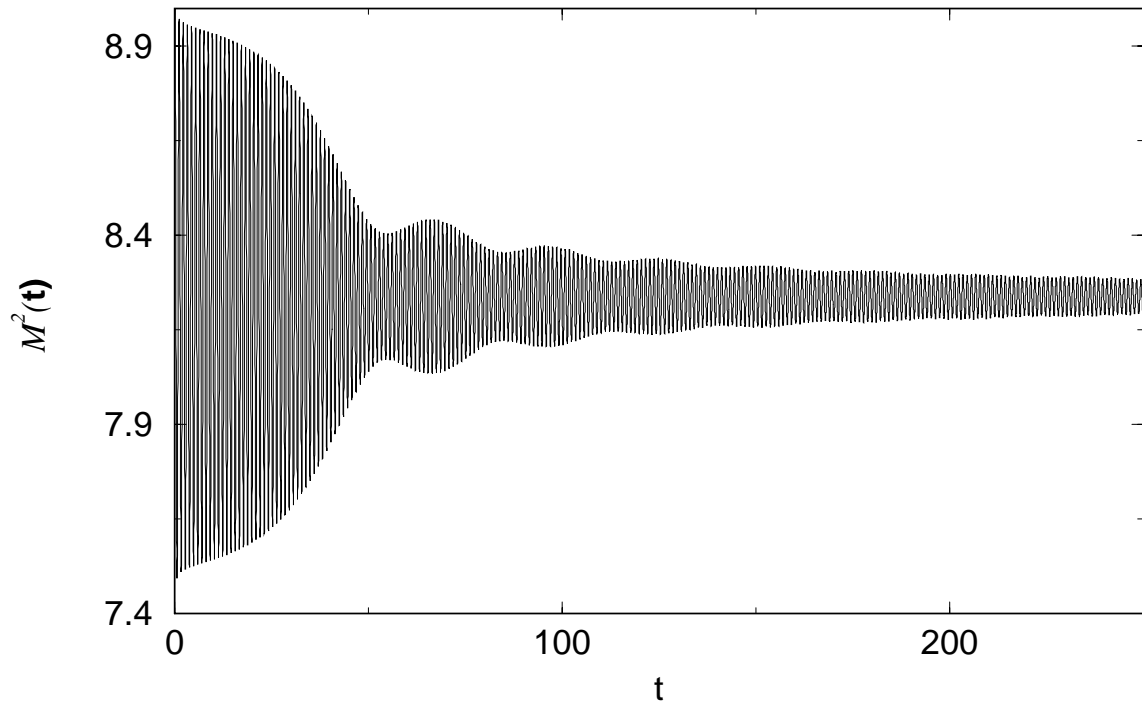


Figure 2.32: Mass squared versus t for parameter set 3

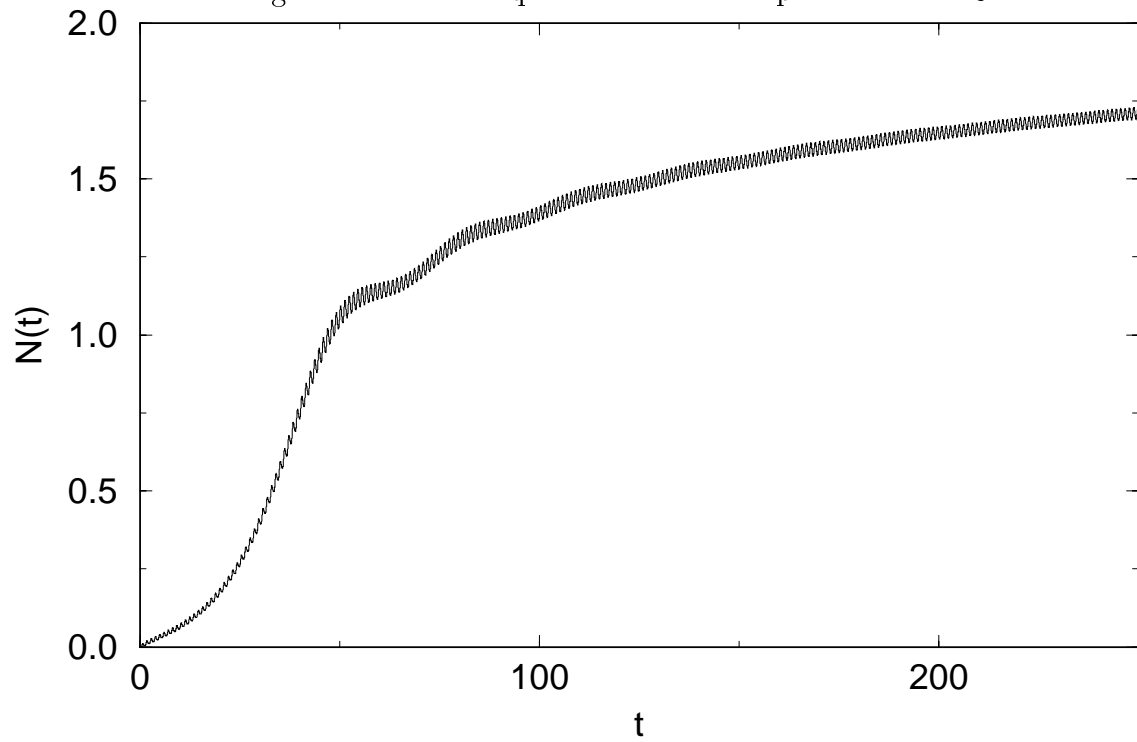


Figure 2.33: Particle number versus t for parameter set 3

Chapter 3

Gauge Theories

3.1 Introduction

In the last few decades gauge field theories have become an important part of elementary particle physics and cosmology. They successfully describe the strong interactions of quarks and the weak forces of quarks and leptons. The development and progress in this field since the fundamental work of Yang and Mills [55] in 1954 have been manifold. We provide some comments on gauge theories with emphasis on the aspect of gauge invariance.

First, we recall some of the key works describing the evolution of the field. In 1954, Yang and Mills generalized the principle of local gauge invariance to a non-Abelian gauge group. This generalization made it possible to build a model in quantum field theory of interacting elementary particles.

The quantization and the perturbative renormalization of such a theory was unclear for a long time until 1967 when Faddeev and Popov [56] and de Witt [57, 58, 59] constructed a scheme for the quantization of massless Yang-Mills theories. In the same year Weinberg [60] and Salam [61] independently proposed a unified model of weak and electromagnetic interactions. It was based on the Higgs-mechanism [62] which generates masses for vector bosons by spontaneous symmetry breaking.

In 1971, G. 't Hooft showed [63] that it is possible to construct a self consistent quantum theory of massive vector fields by generalization of the quantization of massless Yang-Mills fields including spontaneous symmetry breaking.

In 1972, by analyzing the perturbative behavior of the theory, the frame work for a quantum field theory of gauge fields was completed. In papers by 't Hooft and Veltmann [36], Lee and Zinn-Justin [64], Slavnov [65] and Taylor [66], different methods of regularization were developed, renormalization in a perturbative approach was discussed and the Ward identities were derived.

Since then, the quantum theory of gauge fields has developed rather fast. In 1973 [67] QCD was constructed as a gauge field theory and in 1974 [68], the first attempts for the unification of the strong, the weak and the electromagnetic interaction in the so-called Grand Unified Theories were successful. The development of supergravity theories began in 1976 [69], and for the unification of all interactions superstring-theories [70] are very

important.

The fast development of elementary particle physics also influenced the forthcoming in cosmology. We are interested in this work on gauge field theories with special regard to the early stage of the universe. Many processes in cosmology can be described by using gauge field theories. They offer the possibility of the description of phase transitions which have occurred in the early universe. We consider the SU(2)-Higgs model as it is the simplest non-Abelian gauge theory. This model is suitable, for example, to describe the electroweak phase transition which plays an important role in the explanation of baryogenesis. It can also be used as a starting point for more complex models to investigate an even earlier epoch of the universe the so-called inflationary phase. For comparison we have considered the Abelian Higgs model, too. Our considerations have the aim of examining gauge invariance. Furthermore, we investigate the influence of different quantization schemes by considering the Abelian Higgs model in a gauge invariant formulation and compare it with gauge fixed results for the SU(2) Higgs model. This comparison is possible because we choose the Higgs field as a background field and no classical gauge field. The difference between these two models then results only in degeneracy factors.

3.1.1 Gauge Invariance

In general, there exist two kinds of symmetry: symmetry under translation and rotation and internal symmetries. Noether's theorem states that for every symmetry of the Lagrangian there exists a conserved quantity. In gauge theories the internal symmetries are of special interest. First, we consider the global gauge invariance or invariance of the first kind. As a simple example we investigate the conservation of the electric charge. We define a finite gauge transformation

$$\phi(x) \rightarrow \phi'(x) = e^{-iq}\phi(x) , \quad (3.1)$$

where q is the electronic charge belonging to the field ϕ . In a Lagrangian with m fields $\phi_i(x)$, the sum over the charges vanishes $\sum_i q_i = 0$ and therefore, \mathcal{L} is invariant under the gauge transformation. This implies that the Lagrangian is electrically neutral and all interactions conserve the charge. The symmetry group of these unitary transformations is the U(1) group in one dimension. In Quantum Electrodynamics (QED), the uncharged photon has $q = 0$ while the electric field and its conjugate transform respectively according to

$$\psi \rightarrow \psi' = e^{-iq\theta}\psi , \quad \bar{\psi} \rightarrow \bar{\psi}' = e^{iq\theta}\bar{\psi} . \quad (3.2)$$

θ is constant, and therefore the gauge transformation has to be equal at all points in space-time. Thus, it is a global invariance. If we consider θ as an arbitrary function depending on x , we find that \mathcal{L} is invariant under a much larger group of transformations, the local gauge transformations

$$\psi \rightarrow \psi' = e^{-i\theta(x)}\psi , \quad (3.3)$$

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta(x) . \quad (3.4)$$

If we apply these transformation to a simple Lagrangian like

$$\mathcal{L} = \sum_i^n \left(\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - m \bar{\psi}_i \psi_i \right) , \quad (3.5)$$

we find that \mathcal{L} due to the derivative ∂_μ which now also acts on $\theta(x)$, is not locally gauge invariant. One has to introduce a new derivative

$$D_\mu = \partial_\mu + ieq_i A_\mu(x) , \quad (3.6)$$

which is called covariant derivative. One can also show that the field strength tensor $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ is locally gauge invariant. Therefore, the gauge invariant Lagrangian of QED takes the form

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}^2 . \quad (3.7)$$

The local gauge group is the group $U(1)$. The photon mass term $m^2 A_\mu A^\mu$ is not locally gauge invariant. Therefore, local gauge invariance physically corresponds to the fact that the photon is massless. These considerations can be extended to non-Abelian gauge theories. The simplest one is the $SU(2)$ gauge group. Clearly, the form of the covariant derivative and of the field strength tensor depends on the underlying gauge group. Some problems arise in the quantization of gauge theories. For example the electromagnetic field has, as usual for a massless field, only two independent components, but is described by a 4-vector A_μ . Therefore, we have to introduce a gauge fixing. If we choose two components as the physical ones, manifest covariance is lost. In order to preserve covariance, one is left with two redundant degrees of freedom. These choices correspond to two types of gauge fixings: the physical gauges like Coulomb or Lorentz gauge, and the covariant gauges like R_ξ -gauge, with the special cases for $\xi = 1$ which is called Feynman-gauge and $\xi = 0$ known as Landau-gauge. We investigate in this work as a physical gauge the Coulomb gauge and a closely related gauge invariant formalism. In order to compare them with a covariant gauge we also study R_ξ -gauges. We analyze the different degrees of freedom in order to show how the different approaches are connected.

Another problem arises by the derivation of the gauge field propagator. In the generating functional

$$Z = \int \mathcal{D}A_\mu e^{i \int \mathcal{L} dx} , \quad (3.8)$$

where \mathcal{L} is invariant under local gauge transformations, one integrates over all A_μ including those only related by simple gauge transformations. This leads to an infinite contribution to Z and thus to the Green functions. We need a gauge fixing so that the integral does not extent over values simply related by gauge transformations. The quantization of gauge field theories has been unclear for a long time because of the freedom to make gauge transformations. In 1962, Feynman [71] showed that the naive quantization of the theory was not unitary. In order to cancel the nonunitary terms from the theory, Feynman postulated the existence of a term that did not emerge from the standard quantization procedure.

Faddeev and Popov [56] gave a derivation of this term which is today known as Faddeev-Popov term. Since then, different methods for the quantization of gauge field theories have been developed. We have summarized some of them with a closer look to their special features concerning gauge theories.

- Canonical quantization

This is the most direct method for quantizing a field theory. It is closely related to the development of quantum mechanics. Time is singled out as a special coordinate and Lorentz invariance is lost. The advantage of the method is that only physical states are quantized and thus, unitarity is guaranteed. For simple field theories it is a convenient way to quantize the system but for more complex systems like non-Abelian gauge theories it becomes rather complicated.

- Gupta-Bleuler quantization

The Gupta-Bleuler method is closely related to canonical quantization but it has the advantage of maintaining full Lorentz invariance. The disadvantage of this quantization is the appearance of propagating ghosts or unphysical states. They have to be eliminated by constraints.

- Path integral method

A very powerful method has been developed by Feynman and Hibbs [72] and is known as the path integral formalism. The great advantage of quantizing gauge field theories using this method is the freedom to choose or change the gauge fixing in a simple way. In the canonical formalism the gauge has to be fixed from the beginning. In the path integral the gauge fixing is performed by introducing certain delta functions and the gauge can be changed simply by replacing these factors. The path integral formalism for gauge fields was developed by Faddeev and Popov [56]. A disadvantage of the path integral formulation is that the functional integration is mathematically not well defined.

- Dirac quantization with constraints

Dirac quantization [73, 74] is based on a Hamiltonian description of the system. The advantage of this method is that it exhibits gauge invariant states and operators. It begins by recognizing the first class constraints. Then, these constraints become operators in the quantum theory and are imposed onto the physical states, thus defining the physical subspace of the Hilbert space and gauge invariant operators. In Dirac's formulation, the projection onto the gauge invariant subspace of the full Hilbert space is achieved by imposing the first class constraints onto the states. Physical operators are those which commute with the first class constraints. The first class constraints are the generators of gauge transformations. In [37], this approach is used to formulate a gauge invariant description of the Abelian Higgs model. It is not clear yet whether one can extend their formalism to non-Abelian gauge theories. We will discuss some details of this procedure in section 3.5.1.

- BRST quantization

This approach developed by Becchi, Rouet, and Stora [75], and independently by Tyutin [76], is a convenient method for quantizing gauge field theories. As in the Gupta-Bleuler quantization, ghosts and unphysical states are allowed to propagate. They are eliminated by the BRST condition. This condition can be derived as follows. After fixing the gauge, the theory has lost its local gauge invariance. But one can prove that the gauge fixed Lagrangian of the general form

$$\mathcal{L} = -\frac{1}{2}(F_{\mu\nu}^a)^2 - \frac{1}{2\xi}(\partial \cdot A)^2 - \bar{\eta}^a \partial^\mu D_\mu \eta^a \quad (3.9)$$

is gauge invariant under a new global symmetry

$$\delta A_\mu^a = -\frac{1}{g}(D_\mu \eta^a)\lambda, \quad (3.10)$$

$$\delta \eta^a = -\frac{1}{2}f^{abc}\eta^b\eta^c\lambda, \quad (3.11)$$

$$\delta \bar{\eta}^a = -\frac{1}{\xi g}(\partial^\mu A_\mu^a)\lambda, \quad (3.12)$$

where λ and η^a are both Grassmann variables and λ is constant. Using Noether's theorem it is possible to construct a current J_μ belonging to the BRST variation and also a BRST charge Q_{BRST} . One can show [77] that the condition for physical states is given by

$$Q_{\text{BRST}}|\Psi\rangle = 0. \quad (3.13)$$

Thus, this method leads to a compact and elegant statement of the physical state condition.

It is clear that one cannot distinguish cleanly between the different methods. For example the BRST quantization is compatible with the path integral formulation as well as with the canonical quantization; and into the Dirac quantization one can implement a gauge fixing which leads to the usual gauge-fixed path integral representation in terms of Faddeev-Popov determinants and ghosts. Here we have given only a short overview of the common methods and their advantages and disadvantages. In the first part of our work, we use the path integral method and a general R_ξ -gauge which leads to a description with ghost fields. We investigate the ξ -dependence of the one loop effective action in the SU(2) Higgs model. In the second part, we use the gauge invariant description obtained by Dirac's quantization in the Abelian Higgs model and compare the results with those found in our previous work [38], where we have used a 't Hooft-Feynman background gauge in order to give some statements about the influence of the gauge in the non-equilibrium case. We also discuss the Coulomb gauge which is strongly related to the gauge invariant approach. It has the advantage that the approximation to one loop order is more transparent than in the gauge invariant approach.

3.1.2 The Effective Potential

The problem of the gauge invariance of the effective potential is widely discussed in literature. Since the effective potential is closely related to the effective action we give some remarks on this problem.

The effective potential is the sum of one particle irreducible (1-PI) Green functions at zero external momentum. In a gauge theory with massive scalars, these are off-shell quantities, and thus, in general, gauge dependent. The question arises how to extract physical quantities from a gauge dependent effective potential. Shortly after the paper by Coleman and Weinberg [78] showing that radiative corrections could induce spontaneous symmetry breaking, the gauge dependence of the effective potential $V[\bar{\phi}]$ was pointed out by Jackiw [79]. This gave rise to a long controversy. Dolan and Jackiw [54] found that the critical temperature extracted from the effective potential is gauge independent. They analyzed different gauges, e.g. Lorentz gauge, R_ξ -gauges and unitary gauge, and claimed that the unitary gauge is the relevant one because it is a physical gauge. Fukuda and Kugo [80] showed that the value of the effective potential is gauge invariant at any stationary point. A solution of the problem how to find gauge independent quantities from a gauge dependent potential was found by Nielsen [81]. He derived a set of identities (the so-called Nielsen identities) which give a functional connection between the generating functional and the gauge parameter ξ . For the effective potential this connection is of the form

$$\frac{\partial V[\bar{\phi}]}{\partial \xi} + \frac{\partial \bar{\phi}}{\partial \xi} \frac{\partial V[\bar{\phi}]}{\partial \bar{\phi}} = 0 . \quad (3.14)$$

This means that the total derivative of the effective potential with respect to the gauge parameter vanishes when the corresponding shift in the expectation value of the quantum field is taken into account. In other words the effective potential is gauge invariant at the classical extrema. This implies that spontaneous symmetry breaking through radiative corrections is a gauge independent phenomenon derived from a gauge fixed effective potential. With these identities it is also possible to derive physical quantities like masses, couplings, etc. in the correct form. There was a lot of progress made in this field; e.g., Aitchison and Fraser [82] explicitly computed the Nielsen identities for Scalar QED in R_ξ -gauge. Metaxas and Weinberg [83] verified, using the gradient expansion for the leading orders in the coupling, the gauge independence of the quantum corrections to the bubble nucleation rate.

There are many attempts to find an alternative way to derive of a gauge invariant effective potential. Buchmüller, Fodor, and Hebecker [84] developed a method for calculating the effective potential by introducing so-called composite operators $\phi^\dagger \phi$. They are, in contrast to ϕ , gauge invariant. A group in Pittsburgh worked on the problem of the Higgs mass bound in the Abelian Higgs model. In a first paper, Loinaz and Willey [85] considered the effective potential in the R_ξ -gauge and found a ξ -dependent Higgs mass bound. They also verified that the effective potential is gauge invariant in the minimum of the classical potential. In a second paper by Duncan, Loinaz, and Willey [86], the problem of the gauge dependence was solved by using the Coulomb gauge and a description in terms of gauge

invariant composite operators. The last publication on this field by Boyanovsky, Loinaz, and Willey [87] was based on [37]. In [37], a concept for a gauge invariant effective potential was developed. They used the canonical formulation to find gauge invariant states and operators. For quantizing the theory they applied Dirac's method of quantization (see e.g. [74]). After selecting the gauge invariant states, namely, those that are annihilated by the first class constraints they were able to fix gauge invariant order parameters. This enabled them to construct the effective potential for the order parameter without involving any gauge fixing. They discussed some non-equilibrium features of their effective potential and compared it with gauge fixed results. They also made some statements about higher-loop resummation and gauge invariance and found that also in these higher orders no gauge invariance can be found. Since we extend their approach in this work for the non-equilibrium case in order to compare it with our results [38] for the 't Hooft-Feynman-gauge we give a more detailed description of the method in section 3.5.1. In [87], the Higgs-mass bound in this approach is discussed. They also compared the effective potential with gauge-fixed results. They found by numerical computations that the differences between the various descriptions are very small. But they stated that in the one loop approximation only small coupling constants were chosen and, therefore, the effect is negligible.

There are many other papers concerning the problem of gauge invariance of the effective potential and the effective action. We have only stressed a few of them. For our purpose the Hamiltonian approach seems to be very interesting for comparison with our results and therefore, we analyze and extend it in detail.

Our investigations concerning the effective action [88] are based on Nielsen's theorem. We substantiate the general statement that gauge invariance is expected if the classical background field is an extremum of the classical action, as computed by mode functions. We make a general analysis of the modes in the gauge-Higgs sector in order to find modes that cancel the unphysical ghosts. We apply this method to the computation of the fluctuation determinant for bubble nucleation in the SU(2)-Higgs model in the 't Hooft-background gauge with general gauge parameter ξ , and for the same model in the non-equilibrium context.

3.1.3 Systems under Consideration

As a first example, we investigate the SU(2)-Higgs model with an isoscalar Higgs background field stationary in time and inhomogeneous in space. Such a configuration plays an important role in the discussion of the electroweak phase transition. If the Higgs mass is not too large a phase transition of first order occurs via bubble nucleation [89, 90, 91, 92, 93]. This scenario yields the possibility of explaining baryogenesis within the minimal standard model [94]. Since we are interested in a gauge invariant formulation of the fluctuation operator and do not want to give physical results, we do not go into the details of the physics of bubble nucleation. We give a short introduction about the assumptions we make for the model in section 3.3. For a general overview about bubble nucleation, see e.g. [95].

The second system we consider is the SU(2)-Higgs model in the non-equilibrium context. The purpose of the investigation is twofold: In the first part we are only interested in a

gauge invariant description of the fluctuation operator. At this point we are not considering the problem of renormalization. In the second part we use a gauge invariant approach developed by Boyanovsky et al. [37] and apply it to non-equilibrium dynamics in the Abelian Higgs model. Here, we use our formalism for renormalization. We find some problems in the infrared region. They are due to higher loop effects. In order to find a well defined approximation, we investigate the Coulomb gauge. We will show that a linearized form of the gauge invariant approach, where we neglect all contributions higher than one loop, is equivalent to the Coulomb gauge. After renormalization, we implement the equations numerically. We compare the results with the 't Hooft-Feynman gauge fixed theory.

3.2 The Model

First of all, we give a general description of the model we consider in the R_ξ -gauge. We choose the simplest non-Abelian gauge theory the SU(2) Higgs model. We analyze the model in a general 't Hooft- R_ξ gauge where we use the Higgs field as the background field. We do not consider the gauge field as a background field.

3.2.1 The Lagrangian

The Lagrangian of the SU(2) Higgs model reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi^\dagger\Phi), \quad (3.15)$$

with the field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c, \quad (3.16)$$

and the covariant derivative

$$D_\mu \equiv \partial_\mu - i\frac{g}{2}A_\mu^a \tau^a. \quad (3.17)$$

The potential has the form

$$V(\Phi^\dagger\Phi) = \frac{\lambda}{4}(\Phi^\dagger\Phi - v^2)^2. \quad (3.18)$$

In the following we will assume a classical field (condensate)

$$\Phi(x) = \phi(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.19)$$

Its space-time dependence is not further specified here. A time independent, metastable, radially symmetric configuration will be relevant for bubble nucleation; a spatially homogeneous time dependent field describes a non-equilibrium situation, as considered in [38, 41]. The fluctuations around this space-time dependent condensate are parameterized as

$$\Phi(x) = [\phi(x) + h(x) + i\tau_a\varphi_a(x)] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.20)$$

with the isoscalar Higgs mode $h(x)$ and the would-be Goldstone fields $\varphi_a(x)$, $a = 1 \dots 3$. This generates the would-be Goldstone which has the mass of an isoscalar Higgs field in the case of $v = 0$, i.e., in the symmetric case, and in the case of broken symmetry it has the mass of a gauge field in the minimum $v = \phi$. Since there is no classical gauge field, we have

$$A_a^\mu(x) = a_a^\mu(x) . \quad (3.21)$$

Inserting the field expansions leads to

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} \left(\partial_\mu a_\nu^a - \partial_\nu a_\mu^a + g\epsilon^{abc} a_\mu^b a_\nu^c \right) \left(\partial^\mu a^{\nu a} - \partial^\nu a^{\mu a} + g\epsilon^{abc} a^{\mu b} a^{\nu c} \right) \\ &+ \frac{1}{2} \left[\left(\partial_\mu - i\frac{g}{2}\tau^a a_\mu^a \right) \Phi \right]^\dagger \left[\left(\partial^\mu - i\frac{g}{2}\tau^a a^{\mu a} \right) \Phi \right] \\ &- \frac{\lambda}{4} \left(\Phi^\dagger \Phi - v^2 \right)^2 \\ &= -\frac{1}{2} \left(\partial_\mu a_\nu^a \partial^\mu a^{\nu a} - \partial_\mu a_\nu^a \partial^\nu a^{\mu a} \right. \\ &\quad \left. + 2g\epsilon^{abc} a_\mu^b a_\nu^c \partial^\mu a^{\nu a} + \frac{g^2}{2} \epsilon^{abc} a_\mu^b a_\nu^c \epsilon^{ade} a^{\mu d} a^{\nu e} \right) \\ &+ \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + 2\partial_\mu \phi \partial^\mu h + \partial_\mu h \partial^\mu h + \partial_\mu \varphi^a \partial^\mu \varphi^a \right) \\ &+ \frac{g}{2} \left[\left(\partial_\mu \phi \right) a^{\mu a} \varphi^a + \left(\partial_\mu h \right) a^{\mu a} \varphi^a \right. \\ &\quad \left. - \phi \left(\partial_\mu \varphi^a \right) a^{\mu a} - h \left(\partial_\mu \varphi^a \right) a^{\mu a} + \epsilon^{abc} \left(\partial_\mu \varphi^a \right) a^{\mu b} \varphi^c \right] \\ &+ \frac{g^2}{8} \left[\phi^2 a^{\mu a} a_\mu^a + 2\phi h a^{\mu a} a_\mu^a + h^2 a^{\mu a} a_\mu^a + a^{\mu a} a_\mu^a \varphi^b \varphi^b \right] \\ &- \frac{\lambda}{4} \phi^4 - \lambda h \phi^3 - \frac{3}{2} \lambda h^2 \phi^2 - \frac{\lambda}{2} \phi^2 \varphi^a \varphi^a - \lambda h^3 \phi \\ &- \lambda h \phi \varphi^a \varphi^a - \frac{\lambda}{4} h^4 - \frac{\lambda}{2} h^2 \varphi^a \varphi^a - \frac{\lambda}{4} \varphi^a \varphi^a \varphi^b \varphi^b \\ &+ \frac{\lambda}{2} v^2 \phi^2 + \lambda v^2 h \phi + \frac{\lambda}{2} v^2 h^2 + \frac{\lambda}{2} \varphi^a \varphi^a v^2 - \frac{\lambda}{4} v^4 . \end{aligned} \quad (3.22)$$

The Lagrangian can be split into a classical part

$$\mathcal{L}_{\text{cl}}(x) = \frac{1}{2} \left[\partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right] , \quad (3.23)$$

and a fluctuation Lagrangian. The part of first order in the fluctuating field vanishes, if the classical equation of motion

$$\square \phi + \lambda(\phi^2 - v^2)\phi = 0 , \quad (3.24)$$

is fulfilled. The part of second order in the fluctuations reads

$$\mathcal{L}^{(2)} = \frac{1}{2} \left\{ -\partial_\mu a_\nu^a \partial^\mu a_a^\nu + \partial_\mu a_\nu^a \partial^\nu a_a^\mu \right.$$

$$\begin{aligned}
 & + \frac{g^2}{4} \phi^2 a_\mu^a a_a^\mu + \partial_\mu \varphi_a \partial^\mu \varphi_a + g \partial_\mu \phi a_a^\mu \varphi_a - g \phi a_\mu \partial^\mu \varphi_a \\
 & - \lambda (\phi^2 - v^2) \varphi_a \varphi_a + \partial_\mu h \partial^\mu h - \lambda (3\phi^2 - v^2) h^2 \} .
 \end{aligned} \tag{3.25}$$

In the one loop approximation we do not have to consider higher order terms, but for the leading Feynman graphs up to second order, also fluctuations of the third order are of interest. They are discussed in the non-equilibrium context.

The gauge-fixing term, in the 't Hooft background gauge is given by

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} \mathcal{F}_a \mathcal{F}_a , \tag{3.26}$$

with the gauge conditions

$$\mathcal{F}_a = \partial_\mu a_a^\mu + \frac{1}{2} \xi g (\phi + h) \varphi_a , \tag{3.27}$$

which leads to

$$\begin{aligned}
 \mathcal{L}_{\text{gf}} = & -\frac{1}{2\xi} \partial_\mu a^{\mu a} \partial_\nu a^{\nu a} + \frac{g}{2} \phi a^{\mu a} \partial_\mu \varphi^a + \frac{g}{2} h a^{\mu a} \partial_\mu \varphi^a \\
 & + \frac{g}{2} a^{\mu a} \varphi^a \partial_\mu \phi + \frac{g}{2} a^{\mu a} \varphi^a \partial_\mu h - \frac{g^2}{8} \xi \phi^2 \varphi^a \varphi^a \\
 & - \frac{g^2}{8} \xi h^2 \varphi^a \varphi^a - \frac{g^2}{4} \xi \phi h \varphi^a \varphi^a .
 \end{aligned} \tag{3.28}$$

The background field method has the advantage that it is gauge invariant under infinitesimal gauge transformation which was shown by Abbott [96]. Normally, explicit gauge invariance is lost when quantum corrections are included. The background field method allows the fixing of a gauge without losing explicit gauge invariance. This gauge invariance does not lead to an independence of the gauge fixing parameter ξ . In the following we use the terminology *gauge invariance* for the independence of ξ and not for invariance under infinitesimal gauge transformation. If one is only interested in the fluctuations up to second order, the gauge fixing term reduces to

$$\mathcal{F}_a = \partial_\mu a_a^\mu + \frac{1}{2} \xi g \phi \varphi_a . \tag{3.29}$$

The corresponding Faddeev-Popov Lagrangian which is relevant for our calculations is

$$\mathcal{L}_{\text{FP}} = \left\{ \partial_\mu \eta_a^\dagger \partial^\mu \eta_a - \xi \frac{g^2}{4} \phi^2 \eta_a^\dagger \eta_a \right\} . \tag{3.30}$$

The whole Lagrangian then reads

$$\begin{aligned}
 \mathcal{L}_0 = & -\frac{1}{2} \partial_\mu a_\nu^a \partial^\mu a^{\nu a} + \frac{1}{2} \left(1 - \frac{1}{\xi} \right) \partial_\mu a^{\mu a} \partial_\nu a^{\nu a} + \frac{g^2}{8} v^2 a_\mu^a a^{\mu a} \\
 & + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2
 \end{aligned}$$

$$\begin{aligned}
 & +\frac{1}{2}\partial_\mu h\partial^\mu h - \lambda v^2 h^2 \\
 & +\frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a - \frac{g^2\xi}{8}v^2\varphi^a\varphi^a \\
 & +\partial_\mu\eta^{\dagger a}\partial^\mu\eta^a - \frac{g^2\xi}{4}v^2\eta^{\dagger a}\eta^a, \tag{3.31}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{L}_1 = & -g\epsilon^{abc}a_\mu^b a_\nu^c\partial^\mu a^{\nu a} - \frac{g^2}{4}\epsilon^{abc}a_\mu^b a_\nu^c\epsilon^{ade}a^{\mu d}a^{\nu e} \\
 & +\partial_\mu\phi\partial^\mu h + g(\partial_\mu\phi)a^{\mu a}\varphi^a + g(\partial_\mu h)a^{\mu a}\varphi^a + \frac{g}{2}\epsilon^{abc}(\partial_\mu\varphi^a)a^{\mu b}\varphi^c \\
 & +\frac{g^2}{8}(\phi^2 - v^2)a^{\mu a}a_\mu^a + \frac{g^2}{4}\phi h a^{\mu a}a_\mu^a + \frac{g^2}{8}h^2 a^{\mu a}a_\mu^a + \frac{g^2}{8}a^{\mu a}a_\mu^a\varphi^b\varphi^b \\
 & -\lambda h\phi^3 - \frac{3}{2}\lambda h^2(\phi^2 - v^2) - \frac{\lambda}{2}(\phi^2 - v^2)\varphi^a\varphi^a - \lambda h^3\phi \\
 & -\lambda h\phi\varphi^a\varphi^a - \frac{\lambda}{4}h^4 - \frac{\lambda}{2}h^2\varphi^a\varphi^a - \frac{\lambda}{4}\varphi^a\varphi^a\varphi^b\varphi^b + \lambda v^2 h\phi \\
 & -\frac{g^2\xi}{8}(\phi^2 - v^2)\varphi^a\varphi^a - \frac{g^2\xi}{8}h^2\varphi^a\varphi^a - \frac{g^2\xi}{4}\phi h\varphi^a\varphi^a \\
 & -\frac{g^2\xi}{4}\eta^{\dagger a}(\phi^2 - v^2)\eta^a. \tag{3.32}
 \end{aligned}$$

The separation into a free and an interacting part makes it more convenient to read off the Feynman rules. As already mentioned for the one loop calculations, we are only interested in the part of second order in the Lagrangian, but for the Feynman-graphs we also need some vertices up to third order. The fourth order terms are irrelevant and they are only given for completeness.

3.2.2 Gauge Mode and Gauge-fixing Mode

Before we discuss the fluctuation operator for a specific physical setting we specify here the unphysical degrees of freedom in the gauge field and would-be Goldstone sector whose cancellation against the Faddeev-Popov modes will lead to a gauge invariant fluctuation determinant. The fluctuation operator of the isoscalar Higgs mode $h(x)$ is gauge invariant from the outset.

We arrange the gauge field fluctuations a_a^μ and the would-be Goldstone fields φ_a in a $(4+1)$ column vector

$$\psi_a = \left\{ \begin{array}{c} a_a^\mu \\ \varphi_a \end{array} \right\}. \tag{3.33}$$

We start with the equations of motion obtained *without* the gauge-fixing term. The differential operator (fluctuation operator) governing the mode evolution then takes the form

$$\mathcal{M} = \left\{ \begin{array}{cc} -(\square + \frac{g^2}{4}\phi^2)\delta_\mu^\nu + \partial^\nu\partial_\mu & -\frac{g}{2}\partial^\nu\phi + \frac{g}{2}\phi\partial^\nu \\ -g\partial_\mu\phi - \frac{g}{2}\phi\partial_\mu & \square + \lambda(\phi^2 - v^2) \end{array} \right\}. \tag{3.34}$$

The mode equations are the same for all $a = 1, 2, 3$:

$$\mathcal{M}\psi_a = 0. \quad (3.35)$$

An infinitesimal gauge transformation is given by

$$\psi_a^g(x) = \left\{ \begin{array}{c} \partial^\mu \\ \frac{g}{2}\phi(x) \end{array} \right\} f_a(x). \quad (3.36)$$

These modes satisfy the mode equation (3.35) if $\phi(x)$ satisfies the classical field equation (3.24). The latter condition is crucial. It arises from the mode equation for φ_a ; the one for the vector potentials is fulfilled trivially.

If the gauge mode is substituted into the gauge condition we find

$$(\mathcal{F}_a)_g = \left[\square + \xi \frac{g^2}{4} \phi^2(x) \right] f_a, \quad (3.37)$$

where the differential operator on the right hand side is just the Faddeev-Popov operator. So, if the gauge mode is inserted into the Lagrangian, the gauge-fixing term contains the Faddeev-Popov operator *squared*. It is very suggestive that the contribution of this squared operator to the effective action, i.e., to the log det of the fluctuation operator, is cancelled by twice the log det of the Faddeev-Popov operator.

If the gauge-fixing term is included, the fluctuation operator takes the form

$$\mathcal{M}_f = \left\{ \begin{array}{cc} -(\square + \frac{g^2}{4}\phi^2)\delta_\mu^\nu + \left(1 - \frac{1}{\xi}\right)\partial^\nu\partial_\mu & -g\partial^\nu\phi \\ -g\partial_\mu\phi & \square + \lambda(\phi^2 - v^2) + \frac{g^2}{4}\xi\phi^2 \end{array} \right\}. \quad (3.38)$$

If we apply the fluctuation operator to the gauge mode and use the classical equation of motion, we obtain

$$\mathcal{M}_f\psi_a^g(x) = \left\{ \begin{array}{c} -\frac{1}{\xi}\partial_\mu \\ \frac{g}{2}\phi(x) \end{array} \right\} \left[\square + \xi \frac{g^2}{4} \phi^2(x) \right] f_a(x) = \left\{ \begin{array}{c} -\frac{1}{\xi}\partial_\mu \\ \frac{g}{2}\phi(x) \end{array} \right\} \mathcal{M}_{\text{FP}} f_a(x). \quad (3.39)$$

The differential operator appearing on the right hand side is just the Faddeev-Popov operator

$$\mathcal{M}_{\text{FP}} = \square + \xi \frac{g^2}{4} \phi^2(x). \quad (3.40)$$

If f_a is an eigenfunction of the Faddeev-Popov operator, $\mathcal{M}_{\text{FP}} f_a = \omega_{\text{FP}}^2 f_a$, then the associated gauge mode satisfies

$$\left\{ \begin{array}{cc} -\xi & 0 \\ 0 & 1 \end{array} \right\} \mathcal{M}_f\psi_a^g = \omega_{\text{FP}}^2 \psi_a^g. \quad (3.41)$$

The factor ξ in the matrix multiplies the four gauge field components. So the fluctuation operator modified by multiplication with a constant matrix, has a class of eigenfunctions with the same eigenvalues as the Faddeev-Popov operator. In the effective action, the modification by the constant matrix is irrelevant, as one computes the ratio between the

fluctuation determinants in the background field and in a standard vacuum configuration, to which the same arguments apply.

Now consider the gauge condition \mathcal{F}_a . We introduce the covector

$$\mathbf{u}_\xi = \left[\partial_\mu, \xi \frac{g}{2} \phi(x) \right], \quad (3.42)$$

so that

$$\mathcal{F}_a = \mathbf{u}_\xi \psi_a. \quad (3.43)$$

Consider an arbitrary mode ψ_a . We then find, using again the classical equation of motion,

$$\mathbf{u}_\xi \left\{ \begin{array}{cc} -\xi & 0 \\ 0 & 1 \end{array} \right\} \mathcal{M}_f \psi_a = \left[\square + \xi \frac{g^2}{2} \phi^2(x) \right] \mathbf{u}_\xi \psi_a = \mathcal{M}_{\text{FP}} \mathcal{F}_a. \quad (3.44)$$

Let ψ_a^α now be an eigenmode of the modified fluctuation operator with eigenvalue ω_a^2 . Then this equation entails

$$\mathbf{u}_\xi \omega_a^2 \psi_a^\alpha = (\omega_a^\alpha)^2 \mathcal{F}_a^\alpha = \mathcal{M}_{\text{FP}} \mathcal{F}_a^\alpha. \quad (3.45)$$

So if the projection on the vector \mathbf{u}_ξ is different from zero, the eigenvalue is simultaneously an eigenvalue of \mathcal{M}_{FP} . We thereby have a second class of modes on which the fluctuation operator of the gauge-Higgs system has the same spectrum as the Faddeev-Popov operator. We call them gauge-fixing modes. We have to make sure that this class of modes, obtained by a projection, is not empty, and not identical with the gauge modes.

Obviously, the modes on which the projector \mathbf{u}_ξ yields zero are those which satisfy the gauge condition; these are the physical modes. We know that out of the five components of the gauge-Higgs modes ψ only three are physical; they represent the spatial components of the massive gauge field.

We next consider the action of the projector on the gauge eigenmodes. It is convenient to introduce a vector \mathbf{v} that generates the gauge modes via

$$\psi_a^g = \mathbf{v} f_a = \left\{ \begin{array}{c} \partial^\mu \\ \frac{g}{2} \phi(x) \end{array} \right\} f_a. \quad (3.46)$$

We note that

$$\mathbf{u}_\xi \mathbf{v} = \square + \xi \frac{g^2}{4} \phi^2. \quad (3.47)$$

This implies that the gauge-fixing mode obtained by projection of a gauge mode satisfies

$$\mathcal{F}_a = \mathbf{u}_\xi \psi_a^g = \mathbf{u}_\xi \mathbf{v} f_a = \left[\square + \xi \frac{g^2}{4} \phi^2(x) \right] f_a. \quad (3.48)$$

So if f_a is an eigenfunction of the Faddeev-Popov operator, then the gauge-fixing mode generated from it does not represent a new, independent mode. However, the gauge modes and the physical modes do not exhaust the Hilbert space that is based on five field degrees of freedom, and we are sure that the projector does not give zero on the remaining subspace.

We have shown up to now, that for a background field satisfying the classical equation of motion there are two classes of modes whose contribution to the effective action will be cancelled by the one from the Faddeev-Popov sector. We have not shown, thereby, that the remaining physical part of the gauge-Higgs sector becomes independent of ξ . Furthermore, the way in which the modes are eliminated is a technical matter; it depends on the structure of the background field, and on the problem under consideration. So if we want to illustrate the application of these general results we have to consider specific models.

We will here analyze the modes introduced above, and the cancellation of their contribution to the fluctuation determinant, for the case of bubble nucleation in the SU(2) Higgs model and as a second example for a system out of equilibrium.

3.3 Bubble Nucleation

Bubble nucleation occurs in the SU(2) Higgs model if the phase transition from the symmetric high temperature phase to the broken symmetry phase at low temperature is first order. It has been considered as providing a possible mechanism for baryogenesis, a possibility ruled out by the present lower limit for the Higgs mass. Still the model is of interest; in particular, it can be studied in lattice simulations for sufficiently low Higgs masses. The phase transition is described (see, e.g., [97]), by the 3-dimensional high-temperature action

$$S_{\text{ht}} = \frac{1}{g_3(T)^2} \int d^3x \left[\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (D_i \Phi)^\dagger (D_i \Phi) + V_{\text{ht}}(\phi^\dagger \phi) + \frac{1}{2} A_0 \left(-D_i D_i + \frac{1}{4} \phi^\dagger \phi \right) A_0 \right]. \quad (3.49)$$

Here, the coordinates and fields have been rescaled as [98]

$$\vec{x} \rightarrow \frac{\vec{x}}{gv(T)}, \quad \phi \rightarrow v(T)\phi, \quad A \rightarrow v(T)A. \quad (3.50)$$

The vacuum expectation value $v(T)$ is defined as

$$v^2(T) = \frac{2D}{\lambda_T} (T_0^2 - T^2). \quad (3.51)$$

T_0 is the temperature at which the extremum at $\phi = 0$ of the high-temperature potential V_{ht} changes from a minimum for $T > T_0$ to a maximum for $T < T_0$. The temperature dependent coupling of the three-dimensional theory is defined as

$$g_3^2(T) = \frac{gT}{v(T)}. \quad (3.52)$$

We use the standard parameters

$$D = (3m_W^2 + 2m_t^2)/8v_0^2, \quad (3.53)$$

$$E = 3g^3/32\pi , \quad (3.54)$$

$$B = 3(3m_W^4 - 4m_t^4)/64\pi^2 v_0^4 , \quad (3.55)$$

$$T_0^2 = (m_H^4 - 8v_0^2 B)/4D , \quad (3.56)$$

$$\lambda_T = \lambda - 3 \left(3m_W^4 \ln \frac{m_W^2}{a_B T^2} - 4m_t^4 \ln \frac{m_t^2}{a_F T^2} \right) / 16\pi^2 v_0^4 . \quad (3.57)$$

We use in the following a different rescaling, introduced in [99, 100], based on the secondary minimum of the high-temperature potential which occurs at

$$\tilde{v}(T) = \frac{3ET}{2\lambda} + \sqrt{\left(\frac{3ET}{2\lambda}\right)^2 + v^2(T)} . \quad (3.58)$$

The high-temperature potential then takes the form

$$V_{\text{ht}}(\phi^\dagger \phi) = \frac{\lambda_T}{4g^2} \left\{ (\phi^\dagger \phi)^2 - \epsilon(T)(\phi^\dagger \phi)^{3/2} + \left[\frac{3}{2}\epsilon(T) - 2\right] \phi^\dagger \phi \right\} , \quad (3.59)$$

with

$$\epsilon(T) = \frac{4}{3} \left(1 - \frac{v(T)^2}{\tilde{v}(T)^2} \right) . \quad (3.60)$$

The standard formula [101, 102, 103, 104, 105, 106] for the bubble nucleation rate is given by

$$\Gamma/V = \frac{\omega_-}{2\pi} \left(\frac{\tilde{S}}{2\pi} \right)^{3/2} \exp(-\tilde{S}) \mathcal{J}^{-1/2} . \quad (3.61)$$

Here \tilde{S} is the high-temperature action, (3.49), minimized by a classical minimal bubble configuration (see below), while \mathcal{J} is the fluctuation determinant which describes the next-to-leading part of the semiclassical approach and which will be defined below; its logarithm is related to the one loop effective action by

$$S_{\text{eff}}^{1-l} = \frac{1}{2} \ln \mathcal{J} . \quad (3.62)$$

Finally, ω_- is the absolute value of the unstable mode frequency.

The classical bubble configuration is described by a vanishing gauge field and a real spherically symmetric Higgs field $\phi(r) = |\phi|(r)$, which is a solution of the Euler-Lagrange equation

$$-\phi''(r) - \frac{2}{r}\phi'(r) + \frac{dV_{\text{ht}}}{d\phi(r)} = 0 , \quad (3.63)$$

with the boundary conditions

$$\lim_{r \rightarrow \infty} \phi(r) = 0 \quad \text{and} \quad \phi'(0) = 0 . \quad (3.64)$$

We expand the gauge and Higgs fields around this classical configuration via

$$\begin{aligned} A_\mu^a(\vec{x}) &= a_\mu^a(\vec{x}) , \\ \phi(\vec{x}) &= [\phi(r) + h(\vec{x}) + \tau^a \varphi_a(\vec{x})] \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \end{aligned} \quad (3.65)$$

where a_μ^a , h and φ_a are the fluctuating fields, denoted collectively by φ_i .

If the action is expanded with respect to the fluctuating fields, the first order term vanishes if $\phi(r)$ satisfies the classical equation of motion (3.63). The second order part defines the fluctuation operator via

$$S^{(2)} = \frac{1}{\tilde{g}_3^2(T)} \int d^3x \frac{1}{2} \varphi_m \mathcal{M}_{mn} \varphi_n . \quad (3.66)$$

The fluctuation determinant \mathcal{J} appearing in the rate formula is defined by ¹

$$\mathcal{J} = \frac{\det \mathcal{M}}{\det \mathcal{M}_0} , \quad (3.67)$$

where \mathcal{M}_0 is the fluctuation operator obtained by expanding around a spatially homogeneous classical field that is a minimum of effective potential. The gauge conditions for the 3-dimensional theory read

$$\mathcal{F}_a = \partial_\mu a_\mu^a + \frac{\xi}{2} \phi \varphi_a = 0 . \quad (3.68)$$

The total gauge-fixed action S_t is obtained from the high-temperature action by adding to it the gauge-fixing action

$$S_{\text{gf}} = \frac{1}{\tilde{g}_3^2(T)} \int d^3x \frac{1}{2\xi} \mathcal{F}_a \mathcal{F}_a . \quad (3.69)$$

The corresponding Faddeev-Popov action reads

$$S_{\text{FP}} = \frac{1}{\tilde{g}_3^2(T)} \int d^3x \eta^\dagger \left[-\Delta + \xi \frac{\phi^2(r)}{4} \right] \eta . \quad (3.70)$$

The fluctuation operator is obtained from the total action $S_t = S_{\text{ht}} + S_{\text{gf}} + S_{\text{FP}}$. The fluctuation operator, and along with it the fluctuation determinant, decomposes under partial wave expansion into fluctuation operators for fixed angular momentum. We will consider this in the following.

The background field is isoscalar, so the isospin index a just results in multiplicity factors, we will omit it in the following. The scalar fields $h(\vec{x})$, $\varphi_a(\vec{x})$, $\eta(\vec{x})$, and $a_0(\vec{x})$ are expanded with respect to spherical harmonics $Y_\ell^m(\hat{x})$, the partial wave mode functions are denoted by $f_h^\ell(r)$, $f_\varphi^\ell(r)$, $f_\eta^\ell(r)$, and $f_0^\ell(r)$. The vector spherical harmonics $\hat{x} Y_\ell^m$, $r \nabla Y_\ell^m$, and $\vec{L} Y_\ell^m$ are used for expanding the space components of the gauge fields via

$$\mathbf{a}(\vec{x}) = \sum_{\ell m} \left[\frac{f_a^\ell(r)}{\sqrt{\ell(\ell+1)}} r \nabla Y_\ell^m + f_b^\ell(r) \hat{x} Y_\ell^m + \frac{f_c^\ell(r)}{\sqrt{\ell(\ell+1)}} \vec{x} \times \nabla Y_\ell^m \right] . \quad (3.71)$$

The fluctuation operator is block-diagonal. In the following we consider just one partial wave and omit the superscript ℓ . We denote the partial wave reduction of the fluctuation

¹We omit some sophistications related to zero and unstable modes.

operator \mathcal{M} by \mathbf{M}^ℓ ; we omit the superscript, however. The components $f_h(r)$, $f_\eta(r)$, $f_c(r)$, and $f_0(r)$ are decoupled, the operator has the form

$$\mathbf{M}_{nn} = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + m_n^2 + V_n(r). \quad (3.72)$$

The masses are $m_\eta = m_0 = m_c = 0$ and $m_h = m_H$ with the Higgs mass

$$m_H^2 = \frac{\lambda_T}{4g^2}(3\epsilon - 4). \quad (3.73)$$

The potentials are $V_0(r) = V_c(r) = \phi^2(r)/4$, $V_\eta(r) = \xi\phi^2(r)/4$ and

$$V_h(r) = \frac{\lambda_T}{4g^2} [12\phi^2(r) - 6\epsilon\phi(r)]. \quad (3.74)$$

The Faddeev-Popov fluctuations are fermionic and two-fold degenerate, as usual.

The modes f_a , f_b and f_φ are coupled. The non-vanishing components are

$$\mathbf{M}_{aa}(r) = -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{\xi r^2} + \frac{\phi^2(r)}{4}, \quad (3.75)$$

$$\mathbf{M}_{bb}(r) = -\frac{1}{\xi} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + \frac{\ell(\ell+1) + 2/\xi}{r^2} + \frac{\phi^2(r)}{4}, \quad (3.76)$$

$$\begin{aligned} \mathbf{M}_{\varphi\varphi}(r) &= -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + \xi \frac{\phi^2(r)}{4} \\ &\quad + m_\phi^2 + \frac{\lambda}{g^2} \left[\phi^2(r) - \frac{3}{4}\epsilon\phi(r) \right], \end{aligned} \quad (3.77)$$

$$\mathbf{M}_{ab}(r) = -\frac{\sqrt{\ell(\ell+1)}}{\xi r^2} \left[2 + (1-\xi)r \frac{d}{dr} \right], \quad (3.78)$$

$$\mathbf{M}_{ba}(r) = -\frac{\sqrt{\ell(\ell+1)}}{\xi r^2} \left[1 + \xi - (1-\xi)r \frac{d}{dr} \right], \quad (3.79)$$

$$\mathbf{M}_{b\varphi}(r) = \mathbf{M}_{\varphi b}(r) = -\phi'(r). \quad (3.80)$$

The fluctuation operator of this coupled system is hermitean, as it should be, because it arises from the variation of a Lagrangian. The asymmetry suggested by the explicit form arises from integrations by parts.

The gauge parameter ξ only occurs in the coupled system and for the Faddeev-Popov modes. The cancellation of the ξ dependence will have to occur between these two sectors. They will be analyzed in the next section.

3.3.1 Analysis of the Fluctuation Operator

In analyzing the gauge dependence we have to consider the coupled system of the modes f_a , f_b , and f_φ , i.e., the radial mode functions for angular momentum ℓ . In analogy to

section 3.2.2 we consider the fluctuation operator multiplied from the left by a constant matrix $\text{diag}(\xi, \xi, 1)$. The eigenvalue problem for the fluctuation operator then takes the form of the three differential equations for the radial mode functions for angular momentum ℓ :

$$-f_a'' - \frac{2}{r}f_a' + \frac{\ell(\ell+1)}{\xi r^2}f_a + \frac{\phi^2(r)}{4}f_a - \frac{\sqrt{\ell(\ell+1)}}{\xi r^2}[2f_b + (1-\xi)r f_b'] = \frac{\omega^2}{\xi}f_a, \quad (3.81)$$

$$-\frac{1}{\xi}\left(f_b'' + \frac{2}{r}f_b'\right) + \frac{\ell(\ell+1) + 2/\xi}{r^2}f_b + \frac{\phi^2(r)}{4}f_b - \frac{\sqrt{\ell(\ell+1)}}{\xi r^2}[(1+\xi)f_a - (1-\xi)r f_a'] - \phi'(r)f_\varphi = \frac{\omega^2}{\xi}f_b, \quad (3.82)$$

$$-f_\varphi'' - \frac{2}{r}f_\varphi' + \frac{\ell(\ell+1)}{r^2}f_\varphi + m_H^2 f_\varphi + \xi \frac{\phi^2(r)}{4}f_\varphi + \frac{\lambda}{g^2}\left[\phi^2(r) - \frac{3}{4}\epsilon\phi(r)\right]f_\varphi - \phi'(r)f_b = \omega^2 f_\varphi. \quad (3.83)$$

In view of the general arguments of section 3.2.2, we now should identify the gauge and the gauge-fixing modes. A general gauge transformation is parameterized by a function $\chi(\vec{x})$. It can be expanded into partial waves with respect to spherical harmonics; the radial mode function is denoted by $f_\chi(r)$. The gauge mode then takes the form

$$\begin{aligned} f_a^g(r) &= \sqrt{\ell(\ell+1)}f_\chi(r), \\ f_b^g(r) &= f_\chi'(r), \\ f_\varphi^g(r) &= -\frac{\phi(r)}{2}f_\chi(r). \end{aligned} \quad (3.84)$$

The partial wave amplitude of the gauge-fixing mode \mathcal{F} is obtained from the general definition

$$\mathcal{F}(\vec{x}) = \nabla \mathbf{a}(\vec{x}) + \xi \frac{\phi(r)}{2}\varphi(\vec{x}). \quad (3.85)$$

This equation is expanded into partial waves. The radial mode function of the mode \mathcal{F} then reads

$$f_{\mathcal{F}}(r) = f_b'(r) + \frac{2}{r}f_b(r) - \frac{\sqrt{\ell(\ell+1)}}{r}f_a(r) + \xi \frac{\phi(r)}{2}f_\varphi(r). \quad (3.86)$$

It can be checked, using the basic differential equations (3.81)-(3.83) and the differential equation for the background field (3.63), that the mode $f_{\mathcal{F}}$ satisfies the differential equation for the Faddeev-Popov modes

$$-f_{\text{FP}}'' - \frac{2}{r}f_{\text{FP}}' + \frac{\ell(\ell+1)}{r^2}f_{\text{FP}} + \xi \frac{\phi^2(r)}{4}f_{\text{FP}} = \omega^2 f_{\text{FP}}. \quad (3.87)$$

Likewise, if the gauge function $f_\chi(r)$ satisfies this differential equation then the mode functions f_n^g generated from it via (3.84) satisfy the basic differential equations (3.81)-(3.83). This is as to be expected from the general arguments.

We now try to separate the system of differential equations by introducing a suitable set of new mode functions. We first eliminate the mode $f_a(r)$ in favor of $f_{\mathcal{F}}(r)$,

$$f_a(r) = -r \frac{f_{\mathcal{F}}(r) + f'_b(r) + 2f_b(r) - (\xi/2)\phi(r)f_\varphi(r)}{\sqrt{\ell(\ell+1)}}. \quad (3.88)$$

As mentioned above, $f_{\mathcal{F}}(r)$ satisfies

$$-f_{\mathcal{F}}'' - \frac{2}{r}f_{\mathcal{F}}' + \frac{\ell(\ell+1)}{r^2}f_{\mathcal{F}} + \xi \frac{\phi^2(r)}{4}f_{\mathcal{F}} = \omega^2 f_{\mathcal{F}}. \quad (3.89)$$

Having eliminated $f_a(r)$ in this way it cannot be used anymore as a gauge mode, for which now $f_b(r)$ is a possible candidate. However, it is not possible to use a simple algebraic substitution. We introduce the new mode function $f_g(r)$, analogous to $\chi(r)$, and eliminate $f_b(r)$ with the substitution

$$f_b(r) = \frac{d}{dr}f_g(r). \quad (3.90)$$

We make the two other amplitudes gauge invariant by defining

$$\tilde{f}_\varphi(r) = f_\varphi(r) + \frac{\phi(r)}{2}f_g(r), \quad (3.91)$$

$$\tilde{f}_{\mathcal{F}}(r) = f_{\mathcal{F}}(r) + \omega^2 f_g(r). \quad (3.92)$$

The latter equation follows from the general relation (3.48). We now have to find the equation of motion for the amplitude $f_g(r)$. In view of its close relation to the gauge function $\chi(r)$ we make the ansatz

$$-f_g'' - \frac{2}{r}f_g' + \frac{\ell(\ell+1)}{r^2}f_g + \xi \frac{\phi^2(r)}{4}f_g = \omega^2 f_g + \mathcal{R}(r). \quad (3.93)$$

We insert the substitutions into the differential equations (3.82), (3.83), and (3.89) for the amplitudes $f_b(r)$, $f_\varphi(r)$, and $f_{\mathcal{F}}(r)$, respectively. We find, after some algebra, the equation

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} r^2 \mathcal{R}(r) &= \frac{1}{2r} \frac{d}{dr} r^2 \left[\xi \phi(r) \tilde{f}_\varphi(r) - \tilde{f}_{\mathcal{F}}(r) \right] \\ &\quad + \frac{1}{2} \left[\frac{d}{dr} \phi(r) \tilde{f}_\varphi(r) - \phi(r) \frac{d}{dr} \tilde{f}_\varphi(r) \right] + \frac{1}{\xi} \frac{d}{dr} \tilde{f}_{\mathcal{F}}(r) \end{aligned} \quad (3.94)$$

as a consistency condition for \mathcal{R} . It can be solved readily

$$\begin{aligned} \mathcal{R}(r) &= \frac{\xi}{2} \phi(r) \tilde{f}_\varphi(r) - \tilde{f}_{\mathcal{F}}(r) \\ &\quad + \frac{1}{2r^2} \int_0^r dr' r'^2 \left[\phi'(r') \tilde{f}_\varphi(r') - \phi(r') \tilde{f}'_\varphi(r') + \frac{2}{\xi} \tilde{f}'_{\mathcal{F}}(r') \right]. \end{aligned} \quad (3.95)$$

This fixes the right hand side of equation (3.93) for $f_g(r)$, which is one of the basic ones for the new amplitudes. The equations for the other amplitudes become

$$-\tilde{f}_\varphi'' - \frac{2}{r}\tilde{f}_\varphi' + \frac{\ell(\ell+1)}{r^2}\tilde{f}_\varphi + \left\{ m_H^2 + \frac{\lambda}{g^2} \left[\phi^2(r) - \frac{3}{4}\epsilon\phi(r) \right] \right\} \tilde{f}_\varphi \quad (3.96)$$

$$= \omega^2 \tilde{f}_\varphi - \frac{1}{2}\phi(r)\tilde{f}_\mathcal{F} + \frac{\phi(r)}{4r^2} \int_0^r dr' r'^2 \left[\phi'(r')\tilde{f}_\varphi(r') - \phi(r')\tilde{f}_\varphi'(r') + \frac{2}{\xi}\tilde{f}_\mathcal{F}'(r') \right],$$

$$-\tilde{f}_\mathcal{F}'' - \frac{2}{r}\tilde{f}_\mathcal{F}' + \frac{\ell(\ell+1)}{r^2}\tilde{f}_\mathcal{F} + \xi \frac{\phi^2(r)}{4}\tilde{f}_\mathcal{F} \quad (3.97)$$

$$= \omega^2 \left\{ -\xi \frac{\phi(r)}{2}\tilde{f}_\varphi - \frac{1}{2r^2} \int_0^r dr' r'^2 \left[\phi'(r')\tilde{f}_\varphi(r') - \phi(r')\tilde{f}_\varphi'(r') + \frac{2}{\xi}\tilde{f}_\mathcal{F}'(r') \right] \right\}.$$

Obviously, we have not succeeded in separating the system. However, in this form the gauge and gauge-fixing modes are easy to identify. We see that with the choice $\tilde{f}_\varphi = 0$ and $\tilde{f}_\mathcal{F} = 0$ the function $\mathcal{R}(r)$ vanishes and the differential equation for f_g becomes the Faddeev-Popov equation again, with a corresponding energy spectrum. Likewise, the combination $f_\mathcal{F} = \tilde{f}_\mathcal{F} + \omega^2 f_g$ still satisfies (3.89) and has a Faddeev-Popov eigenvalue spectrum as well. However, we do not find another linearly independent combination of amplitudes involving the amplitude \tilde{f}_φ that would satisfy a differential equation independent of ξ . So that part of the energy spectrum that is not compensated by the Faddeev-Popov contributions apparently still depends on the choice of ξ .

Matters are different, however, if we evaluate the effective action. This can be done using the fluctuation modes at $\omega = 0$, using a general theorem on fluctuation determinants [107], generalized to coupled systems, that has been used, e.g., for computing the fluctuation corrections to bubble nucleation [91]. It is based on the equation ²

$$\mathcal{J}(\nu) \equiv \frac{\det(\mathbf{M} + \nu^2)}{\det(\mathbf{M}_0 + \nu^2)} = \lim_{r \rightarrow \infty} \frac{\det \mathbf{f}(\nu, r)}{\det \mathbf{f}_0(\nu, r)}. \quad (3.98)$$

Here, \mathbf{M} is the partial wave fluctuation operator as defined previously, and the matrix $\mathbf{f}(\nu, r)$ is an $(n \times n)$ matrix formed by a fundamental system of n linearly independent n -tuples of solutions for a given ν , regular at $r = 0$. The operator \mathbf{M}_0 and the solutions \mathbf{f}_0 refer to a trivial background field configuration, in the present case to the symmetric vacuum state characterized by $\phi(r) \equiv 0$. It is understood, that both systems \mathbf{f} and \mathbf{f}_0 are started at $r = 0$ with identical initial conditions. Finally, the desired fluctuation determinant is given by $\mathcal{J} \equiv \mathcal{J}(0)$.

If we apply the theorem we only need the coupled system of differential equations for $\omega = i\nu = 0$, and then it decouples in a triangular way. The right hand side of the equation for $\tilde{f}_\mathcal{F}$ vanishes entirely, the right hand side of the differential equation for \tilde{f}_φ only depends on $\tilde{f}_\mathcal{F}$, while both \tilde{f}_φ and $f_\mathcal{F}$ appear on the right hand side of the equation for f_g . Furthermore, for $\tilde{f}_\mathcal{F} = 0$, the differential equation for \tilde{f}_φ becomes independent of ξ . We can choose the following set of linearly independent solutions:

²For a short proof along the lines of [107] see [108].

(i) a gauge mode solution f_n^g with $\tilde{f}_{\mathcal{F}}^g \equiv 0$ and $\tilde{f}_{\varphi}^g \equiv 0$, for which f_n^g evolves in the same way as a pure Faddeev-Popov mode;

(ii) a physical solution f_n^{φ} with $\tilde{f}_{\mathcal{F}}^{\varphi} \equiv 0$; then $\tilde{f}_{\varphi}^{\varphi}$ evolves independently; it appears on the right hand side of the differential equation for f_n^{φ} , which can be obtained by using the Green function of the homogenous equation, and finally

(iii) a gauge-fixing mode solution $f_n^{\mathcal{F}}$, where $\tilde{f}_{\mathcal{F}}^{\mathcal{F}}$ is different from zero. For $\nu = 0$ the right hand side of (3.97) vanishes and $f_n^{\mathcal{F}}$ evolves like a Faddeev-Popov mode. Both other amplitudes are different from zero in this case. Note that the second type of solution is determined only modulo an arbitrary multiple of the first one, and the third one only modulo arbitrary multiples of both other ones. This does not affect the determinant $\det \mathbf{f}(0, r)$, however.

The structure of the matrix $\mathbf{f}(0, r)$ now is triangular and its determinant is obtained from the diagonal elements as

$$\det \mathbf{f}(0, r) = f_n^g(0, r) \tilde{f}_{\mathcal{F}}^{\mathcal{F}}(0, r) \tilde{f}_{\varphi}^{\varphi}(0, r) = f_{\text{FP}}^2(0, r) \tilde{f}_{\varphi}^{\varphi}(0, r). \quad (3.99)$$

The same structure holds for the free solutions which have to be started at $r = 0$ with identical initial conditions, i.e. with the same coefficients of the lowest powers of r , as determined by the centrifugal barriers. We have considered the behavior at $r = 0$ in detail and have verified that an appropriate choice is possible.

The effective action is obtained by adding the logarithms of the various fluctuation determinants for all independent systems, and for all partial waves. The only ξ dependence occurs in the gauge and gauge-fixing modes of the coupled system, and for the two Faddeev-Popov modes. Since these compensate each other the total effective action becomes independent of ξ .

For the practical computation this means that for the coupled system we just have to solve the integro-differential equation for \tilde{f}_{φ} with $\tilde{f}_{\mathcal{F}} = 0$, i.e.,

$$\begin{aligned} & -\tilde{f}_{\varphi}'' - \frac{2}{r} \tilde{f}_{\varphi}' + \frac{\ell(\ell+1)}{r^2} \tilde{f}_{\varphi} + \left\{ m_H^2 + \frac{\lambda}{g^2} \left[\phi^2(r) - \frac{3}{4} \epsilon \phi(r) \right] \right\} \tilde{f}_{\varphi} \\ & = \frac{\phi(r)}{4r^2} \int_0^r dr' r'^2 \left[\phi'(r') \tilde{f}_{\varphi}(r') - \phi(r') \tilde{f}_{\varphi}'(r') \right]. \end{aligned} \quad (3.100)$$

From this derivation and discussion it is clear that the gauge independence only holds for the effective action, and not for other physical quantities. The non-diagonal parts of the mode solutions still depend on ξ , so other expectation values are affected by the gauge parameter ξ .

3.4 Non-equilibrium Dynamics: R_{ξ} -gauges

3.4.1 Perturbative Expansion: The Leading Feynman-Diagrams

In this section we consider the R_{ξ} -gauges in the context of non-equilibrium dynamics. First of all, we want to get a deeper insight into the relevant degrees of freedom for our model.

Therefore, it is useful to analyze the leading Feynman-graphs. It allows us to extract the UV divergences and to investigate the ξ -dependence of the counter terms. Since the counter terms are unphysical quantities they are not gauge invariant and contain the gauge parameter ξ . We derive the relevant propagators and vertices and calculate the leading diagrams with the CTP-formalism which was developed for systems out of equilibrium by Schwinger [24] and Keldysh [25]. It allows the description of the time development of quantum fields with given initial conditions. In recent works it is implemented to the use of functional technics. Usually the in-out-formalism is used to calculate the generating functional W and the effective action Γ . The CTP-formalism is a generalization of the in-out-formalism. It is based on the sum over paths first going forward in time in the presence of one external source from an in vacuum to a state defined on a hypersurface of constant time in the future, and then backwards in time in the presence of a different source to the same in vacuum. Therefore, it is also called in-in-formalism. It yields a real and causal effective action, field equations, and expectation values. The number of Green function and propagators is doubled, they are labeled with G^{++} , G^{--} , G^{+-} , and G^{-+} . They are also causal. A detailed discussion of the formalism can be found e.g. in [109, 110].

We also want to make some comparison with [37]. We use the method which they have developed in the next section in order to compare different gauges. Here, we identify the degrees of freedom in the R_ξ -gauge with their degrees of freedom. From the kinetic part of the Lagrangian (3.31), we get the following free propagators. ³

1. The gauge boson propagator is :

$$\begin{array}{c} \text{~~~~~} \end{array} \quad i\Delta_{\mu\nu}^{ab} = -\frac{i\delta^{ab}}{k^2 - m_W^2 + i\epsilon} \left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m_W^2} \right) + \frac{i\delta^{ab}}{k^2 - \xi m_W^2 + i\epsilon} \frac{\partial_\mu \partial_\nu}{m_W^2},$$

with $m_W^2 = e^2 v^2$.

(3.101)

2. The propagator for the isoscalar Higgs field is given by:

$$\begin{array}{c} \text{-----} \end{array} \quad i\Delta_h = i\frac{1}{k^2 - m_h^2 + i\epsilon},$$

with $m_h^2 = 2\lambda v^2$.

(3.102)

3. The propagator for the isovector Higgs field is given by:

$$\begin{array}{c} \text{~~~~~} \end{array} \quad i\Delta_\varphi^{ab} = i\frac{\delta^{ab}}{k^2 - \xi m_W^2 + i\epsilon}.$$

(3.103)

4. The propagator for the ghost field is given by:

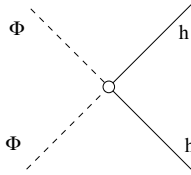
$$\begin{array}{c} \text{-----} \end{array} \quad i\Delta_\eta^{ab} = i\frac{\delta^{ab}}{k^2 - \xi m_W^2 + i\epsilon}.$$

(3.104)

³For a simpler comparison with [37], we have replaced the gauge coupling g by $2e$.

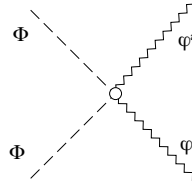
The connection between the free propagators and the propagators developed from the CTP-formalism is discussed in [111] great detail. There, the discussion is limited to the Feynman gauge, i.e. to the choice $\xi = 1$. Since the proceeding is straightforward and can simply be expanded to general ξ by introducing the new propagators, we do not repeat the whole formalism here. For the perturbative expansion, we also need the vertices read off from (3.32). We have listed only the relevant ones, which are:

V1)



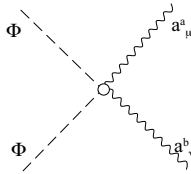
$$i\Gamma = -i\frac{3}{2}\lambda [\phi^2(t) - v^2]$$

V2)



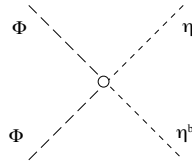
$$i\Gamma = -i\left(\frac{\lambda}{2} + \xi\frac{e^2}{2}\right) \delta^{ab} [\phi^2(t) - v^2]$$

V3)



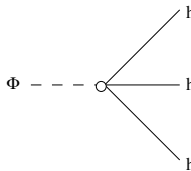
$$i\Gamma = i\frac{e^2}{2} \delta^{ab} g_{\mu\nu} [\phi^2(t) - v^2]$$

V4)



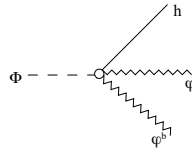
$$i\Gamma = -i\xi e^2 \delta^{ab} [\phi^2(t) - v^2]$$

V5)



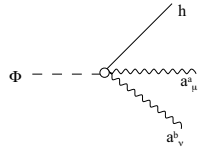
$$i\Gamma = -i\lambda\phi(t)$$

V6)



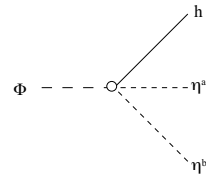
$$i\Gamma = -i\delta^{ab}\phi(t) (\lambda + \xi e^2)$$

V7)



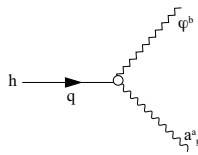
$$i\Gamma = ie^2 \delta^{ab} g_{\mu\nu} \phi(t)$$

V8)



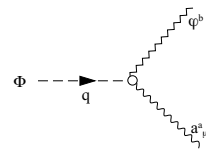
$$i\Gamma = -2ie\delta^{ab}\phi(t)$$

V9)



$$i\Gamma = ie^2 \epsilon^{abc} q_\mu \phi(t)$$

V10)



$$i\Gamma = 2ie\delta^{ab} q_\mu \phi(t)$$

In zeroth order in the fluctuation, we obtain the classical equation of motion. The corresponding Feynman graph has the following form:

$$+ \text{---} + \boxed{} \tag{3.105}$$

The plus signs at the propagator indicate the use of propagators in the CTP-formalism. The related equation of motion then reads

$$\ddot{\phi} + \lambda\phi(\phi^2 - v^2) = 0 . \tag{3.106}$$

In the first order we get tadpole-graphs which lead to quadratic divergences. They have to be removed in a mass counter term. Graphically, the following Feynman diagrams contribute:



The equation of motion in this order is given by

$$\begin{aligned}
 \ddot{\phi} + \lambda(\phi^2 - v^2) &+ 3\lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_h} \\
 &+ 3(\lambda + \xi e^2)\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_\varphi} \\
 &+ 9e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_a} + 3\xi e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_\xi} \\
 &- 6\xi e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_\xi} = 0, \tag{3.107}
 \end{aligned}$$

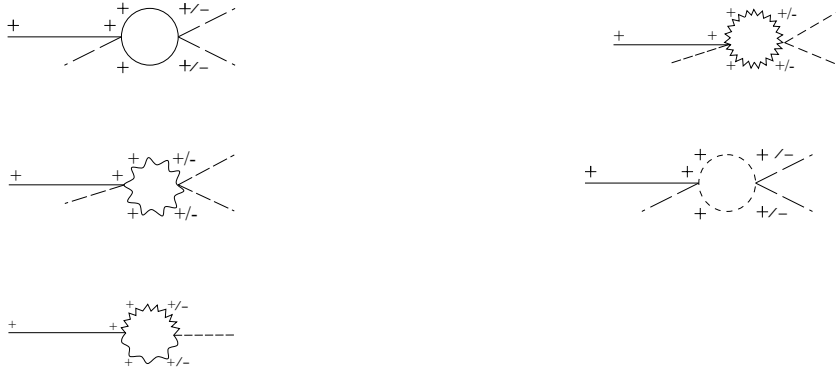
with

$$\omega_h^2 = k^2 + m_h^2, \tag{3.108}$$

$$\omega_\varphi^2 = \omega_\xi^2 = k^2 + \xi m_W^2, \tag{3.109}$$

$$\omega_a^2 = k^2 + m_W^2. \tag{3.110}$$

The first momentum integral is due to the isoscalar Higgs field. It is gauge invariant from the outset and clearly independent of ξ . In [37], this field is denoted with ϕ . The second fluctuation field is the Goldstone boson. It depends on ξ and is therefore a gauge dependent field. In Landau gauge, i.e., $\xi = 0$ this field belongs to the Goldstone mode in [37]. But they have compared their results with the Landau gauge and found that the effective potential in Landau gauge is not equivalent to their gauge invariant approach. The third and the fourth integral represent the gauge sector. They contain two transversal degrees of freedom, a longitudinal and a temporal one. The two transversal fields are the physical ones and they are gauge invariant. They are the same as in [37]. The other two degrees of freedom do not exist in [37] because they only have taken physical fields into account. In the next order, five additional graphs contribute:



$$= \phi(t) \int_{-\infty}^t dt' [\phi^2(t') - v^2] \int \frac{d^3k}{(2\pi)^3} \left\{ -\frac{9\lambda^2}{2\omega_h} \sin(2\omega_h \Delta t) - \frac{3(\lambda + \xi e^2)^2}{2\omega_\varphi} \sin(2\omega_\varphi \Delta t) \right.$$

$$\begin{aligned}
 & + \frac{6\xi^2 g^4}{2\omega_\xi} \sin(2\omega_\xi \Delta t) - 6e^4 \left[\left(\frac{k^2}{m_W^4} + \frac{3}{4\omega_a^2} \right) \sin(2\omega_a \Delta t) \right. \\
 & \left. - \frac{k^2}{2m_W^4 \omega_a \omega_\xi} (\omega_a + \omega_\xi)^2 \sin[(\omega_a + \omega_\xi) \Delta t] + \left(\frac{k^2}{m_W^4} + \frac{\xi^2}{4\omega_a^2} \right) \sin(2\omega_\xi \Delta t) \right] \Big\} \\
 & - 12e^2 \int_{-\infty}^t dt' \dot{\phi}(t') \int \frac{d^3 k}{(2\pi)^3} \left\{ \frac{\omega_a + \omega_\varphi}{2\omega_a \omega_\varphi} \cos[(\omega_a + \omega_\varphi) \Delta t] \right. \\
 & \left. + \frac{\omega_a(\omega_a + \omega_\varphi)}{2\omega_\varphi m_W^2} \cos[(\omega_a + \omega_\varphi) \Delta t] - \frac{\omega_\xi(\omega_a + \omega_\varphi)}{2\omega_\varphi m_W^2} \cos[(\omega_\xi + \omega_\varphi) \Delta t] \right\} \quad (3.111) \\
 = & \phi(t) [\phi^2(t) - v^2] \int \frac{d^3 k}{(2\pi)^3} \left\{ -\frac{9\lambda^2}{4\omega_h^3} - \frac{3(\lambda + \xi e^2)^2}{4\omega_\varphi^3} - \frac{9e^4}{4\omega_a^3} - \frac{3\xi^2 e^4}{4\omega_a^2 \omega_\xi} + \frac{6\xi^2 e^4}{4\omega_\xi^3} \right\} \\
 & - 12e^2 \ddot{\phi}(t) \int \frac{d^3 k}{(2\pi)^3} \left\{ \frac{1}{2\omega_a \omega_\xi (\omega_a + \omega_\varphi)} + \frac{\omega_a}{2\omega_\varphi m_W^2 (\omega_a + \omega_\varphi)} - \frac{\omega_\xi}{2\omega_\varphi m_W^2 (\omega_\xi + \omega_\varphi)} \right\} \\
 & + \mathcal{O}(\omega^{-4}), \quad \Delta t = t - t'.
 \end{aligned}$$

The first four graphs lead to coupling constant renormalization and the fifth one to wave function renormalization. This graph is due to the coupled channel of the Goldstone-gauge-sector. The counter terms are given by

$$\delta m_h^2 = 3\lambda m_h^2 I_{-3}(m_h) + 3(\lambda + \xi e^2) m_h^2 I_{-3}(m_h), \quad (3.112)$$

$$\delta \lambda = 9\lambda^2 I_{-3}(m_h) + 3(\lambda + \xi e^2)^2 I_{-3}(m_h) + 9e^4 I_{-3}(m_W) - 3\xi^2 e^4 I_{-3}(m_W), \quad (3.113)$$

$$\delta Z = -12e^2 \left(\frac{5}{4} - \frac{1}{4}\xi \right) I_{-3}, \quad (3.114)$$

where we have used

$$\left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega} \right\}_{\text{reg}} = -\frac{m^2}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma + 1 \right\} = -m^2 I_{-3}(m) - \frac{m^2}{16\pi^2} \quad (3.115)$$

$$\left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega^3} \right\}_{\text{reg}} = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma \right\} = I_{-3}(m), \quad (3.116)$$

$$I_{-3}(m) = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma \right\}. \quad (3.117)$$

For non-equilibrium systems, the perturbative expansion is not useful to get information about the development of the zero mode under influence of quantum fluctuations. In order to go beyond the early time regime, we have to include the full back reaction and, therefore, we have to use nonperturbative methods. Nevertheless, the analysis of the leading terms and divergences in the equation of motion leads to some interesting results. First of all, we have been able to make some comments on the arising degrees of freedom and their gauge dependence. The graphs are also instructive to get an insight into the renormalization. The isoscalar and the isovector fields contribute to both the mass counter term and to the coupling constant. The gauge field and ghost field only lead to a coupling constant

renormalization. This corresponds to the gauge invariant results. The explicit form of the counter terms is of course different, because they are unphysical. The coupled channel between the gauge field a and the isovector field φ leads to the wave function renormalization. The counter terms we have obtained are the same as we would get if we were to consider the full nonperturbative equations in the R_ξ -gauge. The ξ -dependent counter term must be proportional to the classical equation of motion because the effective action is gauge independent if the classical equation of motion is fulfilled. From the functional derivative of the counter terms in the effective action

$$\delta S = \int d^4x \frac{1}{4} \phi \left(-2\delta Z \ddot{\phi} + \delta m_h^2 \phi^2 - \delta \lambda \phi^3 \right) , \quad (3.118)$$

we expect for the ξ -dependent parts of the counter terms the following relation

$$\delta_\xi \lambda = \delta_\xi m_h^2 / v^2 = 2\lambda \delta_\xi Z = 6\lambda e^2 \xi v^2 , \quad (3.119)$$

which is obviously fulfilled. Therefore, the gauge dependence of the counter terms vanishes if the classical equation of motion is fulfilled.

3.4.2 The Fluctuation Operator for Non-equilibrium Dynamics

In this section we extend our considerations from section 3.2.2 to non-equilibrium dynamics. We derive the fluctuation operator in the one loop approximation and investigate the different modes in order to construct an analogous effective action as in section 3.3 for the bubble nucleation. In our discussion of non-equilibrium dynamics, the field $\phi(x)$ is assumed to depend on time only. Then, in the presence of spatial translation invariance, it is appropriate to perform a spatial Fourier transform. We introduce a vector of mode functions U_n , where n denotes the various components $h, \perp, 0, L, \varphi, \eta$ for the isoscalar Higgs mode, the transverse, time and longitudinal components of the gauge fields, the would-be Goldstone fields, and the two Faddeev-Popov fluctuations, respectively. We omit the isospin indices entirely, as they will lead to combinatorial factors only. The field fluctuations then takes the form

$$\psi_n(x) = U_n(k, t) \exp(i\vec{k} \cdot \vec{x}) . \quad (3.120)$$

The functions U_n only depend on $k = |\vec{k}|$.

The isoscalar Higgs mode, the transverse gauge field modes and the Faddeev-Popov modes are decoupled; their fluctuation operators are diagonal elements given by

$$\mathcal{M}_{nn} = \frac{d^2}{dt^2} + \omega_n^2(t) , \quad (3.121)$$

with

$$\omega_h^2(t) = k^2 + m_h^2 + 3\lambda \left[\phi^2(t) - v^2 \right] , \quad (3.122)$$

$$\omega_\perp^2(t) = k^2 + e^2 \phi^2(t) , \quad (3.123)$$

$$\omega_\eta^2(t) = k^2 + \xi e^2 \phi^2(t) . \quad (3.124)$$

The isoscalar Higgs and the transverse gauge field modes are clearly gauge independent.

The fluctuation operator of the remaining components $0, \varphi, L$ is given by the 3×3 matrix

$$\mathcal{M}_\times = \begin{pmatrix} -\frac{d^2}{dt^2} - \xi\omega_a^2(t) & -2\xi e\dot{\phi} & (\xi - 1)ik\frac{d}{dt} \\ -2e\dot{\phi} & \frac{d^2}{dt^2} + \omega_{\xi e\lambda}^2(t) & 0 \\ ik(1 - \xi^{-1})\frac{d}{dt} & 0 & \frac{d^2}{dt^2} + \xi^{-1}\omega_\xi^2(t) \end{pmatrix}, \quad (3.125)$$

where $\omega_a^2(t) = \omega_\perp^2(t)$, $\omega_\xi^2(t) = \omega_\eta^2(t)$, and

$$\omega_{\xi e\lambda}^2(t) = k^2 + \xi e^2 \phi^2(t) + \lambda [\phi^2(t) - v^2]. \quad (3.126)$$

The gauge mode introduced previously in section 3.2.2 is

$$U_0^g(k, t) = \dot{f}(k, t), \quad (3.127)$$

$$U_L^g(k, t) = -ikf(k, t), \quad (3.128)$$

$$U_\varphi^g(k, t) = e\phi(t)f(k, t), \quad (3.129)$$

and the gauge fixing mode becomes

$$U_{\mathcal{F}}(k, t) = \dot{U}_0(k, t) + ikU_L(k, t) + \xi e\phi(t)U_\varphi(k, t). \quad (3.130)$$

Using the differential equation for the modes and for the zero mode $\phi(t)$, we find

$$\ddot{U}_{\mathcal{F}}(k, t) + [k^2 + \xi e^2 \phi^2(t)] U_{\mathcal{F}}(k, t) = 0, \quad (3.131)$$

$$\ddot{f}(k, t) + [k^2 + \xi e^2 \phi^2(t)] f(k, t) = 0. \quad (3.132)$$

We now define two mode functions that are independent of gauge transformations via

$$\tilde{U}_0(k, t) = U_0(k, t) - \frac{i}{k}\dot{U}_L(k, t), \quad (3.133)$$

$$\tilde{U}_\varphi(k, t) = U_\varphi(k, t) - \frac{i}{k}e\phi(t)U_L(k, t). \quad (3.134)$$

In terms of these functions we find

$$\ddot{U}_L(k, t) + \omega_\xi^2(t)U_L(k, t) = (1 - \xi)ik\dot{\tilde{U}}_0(k, t). \quad (3.135)$$

Furthermore, the new functions satisfy

$$\ddot{\tilde{U}}_0(k, t) + \omega_a^2(t)\tilde{U}_0(k, t) = -2e\dot{\phi}(t)\tilde{U}_\varphi(k, t), \quad (3.136)$$

$$\ddot{\tilde{U}}_\varphi(k, t) + \omega_{\xi e\lambda}^2(t)\tilde{U}_\varphi(k, t) = 2e\dot{\phi}(t)\tilde{U}_0(k, t) + e\phi(t)(1 - \xi)\dot{\tilde{U}}_0(k, t). \quad (3.137)$$

Note that now U_L does not appear anymore on the right hand sides of the equation for \tilde{U}_0 and \tilde{U}_φ .

We find furthermore, that the gauge fixing mode $U_{\mathcal{F}}$ can be expressed entirely in terms of the functions \tilde{U}_0 and \tilde{U}_{φ} :

$$U_{\mathcal{F}}(k, t) = \xi[e\phi\tilde{U}_{\varphi}(k, t) + \dot{\tilde{U}}_0(k, t)] = \xi\tilde{U}_{\mathcal{F}}(k, t) . \quad (3.138)$$

Inserting the differential equation for the new modes we find

$$\ddot{\tilde{U}}_{\mathcal{F}}(k, t) + [k^2 + \xi e^2 \phi^2(t)] \tilde{U}_{\mathcal{F}}(k, t) = \ddot{\tilde{U}}_0(k, t) + \omega_{\xi}^2(t)\tilde{U}_{\mathcal{F}}(k, t) = 0 . \quad (3.139)$$

We now use (3.139) to eliminate \tilde{U}_0 in favor of $\tilde{U}_{\mathcal{F}}$. We can rewrite the differential equation for \tilde{U}_0 in the form

$$\omega_a^2(t)\tilde{U}_0(k, t) = -\dot{\tilde{U}}_{\mathcal{F}}(k, t) + e\dot{\phi}(t)\dot{\tilde{U}}_{\varphi}(k, t) - e\dot{\phi}(t)\tilde{U}_{\varphi}(k, t) , \quad (3.140)$$

so that \tilde{U}_0 becomes a dependent variable, expressed in terms of the modes $\tilde{U}_{\mathcal{F}}$ and \tilde{U}_{φ} .

The fluctuation operator for the modes $\tilde{\mathcal{F}}$, $\tilde{\varphi}$ and L now takes the triangular form

$$\tilde{\mathcal{M}}_{\times} = \begin{pmatrix} \frac{d^2}{dt^2} + \omega_{\xi}^2(t) & 0 & 0 \\ (\xi - 1)e\phi(t) + \frac{2e\dot{\phi}(t)}{\omega_a^2(t)} \frac{d}{dt} & \mathcal{M}_{\varphi\varphi} & 0 \\ (\xi - 1)ik & (1 - \xi)ike\phi(t) & \frac{d^2}{dt^2} + \omega_{\xi}^2(t) \end{pmatrix} , \quad (3.141)$$

with

$$\mathcal{M}_{\varphi\varphi} = \frac{d^2}{dt^2} + k^2 + e^2\phi^2(t) + \lambda [\phi^2(t) - v^2] - \frac{2e^2\dot{\phi}(t)}{\omega_a^2(t)} \left[\phi(t) \frac{d}{dt} - \dot{\phi}(t) \right] . \quad (3.142)$$

Since $\omega_{\xi}^2(t) = \omega_{\eta}^2(t)$, the diagonal elements \mathcal{M}_{LL} and $\mathcal{M}_{\mathcal{F}\mathcal{F}}$ are equal to the Faddeev-Popov fluctuation operator. The diagonal element $\mathcal{M}_{\varphi\varphi}$ is independent of ξ .

We thereby have reduced the fluctuation operator to a triangular form. We can choose the following set of linearly independent solutions in the same way as for the case of bubble nucleation:

(i) a gauge mode solution U_n^L , with $\tilde{U}_{\mathcal{F}}^L \equiv 0$ and $\tilde{U}_{\varphi}^L \equiv 0$, for which U_n^L evolves in the same way as a pure Faddeev-Popov mode;

(ii) a physical solution \tilde{U}_n^{φ} with $\tilde{U}_{\mathcal{F}}^{\varphi} \equiv 0$; then \tilde{U}_n^{φ} evolves independently, and is independent of the gauge parameter; it appears on the right hand side of the differential equation for U_n^{φ} , which can be obtained by using the Green function of the homogenous equation; and finally

(iii) a gauge fixing mode solution $U_n^{\mathcal{F}}$, where $\tilde{U}_{\mathcal{F}}^{\mathcal{F}}$ is different from zero and $\tilde{U}_{\mathcal{F}}$ evolves like a Faddeev-Popov mode. Both other amplitudes are different from zero in this case.

This structure indicates that the effective action, i.e., the fluctuation determinant, will be independent of ξ , after cancellation of the FP modes. This will now be demonstrated explicitly.

3.4.3 Gauge Independence of the Effective Action

The one loop effective action is given by

$$\begin{aligned} S_{\text{eff}} &= \frac{i}{2} \ln \det \mathcal{M} \\ &= \frac{i}{2} (\ln \det \mathcal{M}_h + 2 \ln \det \mathcal{M}_\perp + \ln \det \mathcal{M}_\times - 2 \ln \det \mathcal{M}_\eta) . \end{aligned} \quad (3.143)$$

We will now relate the fluctuation operator \mathcal{M}_\times to the triangular operator $\tilde{\mathcal{M}}_\times$ introduced in the previous section.

The various substitutions leading to the new mode functions $\tilde{U}_\mathcal{F}(k, t)$, $\tilde{U}_\varphi(k, t)$ and $\tilde{U}_L(k, t)$ can be represented by the relation

$$\mathcal{N} \begin{Bmatrix} \tilde{U}_\mathcal{F} \\ \tilde{U}_\varphi \\ U_L \end{Bmatrix} = \begin{Bmatrix} U_0 \\ U_\varphi \\ U_L \end{Bmatrix} , \quad (3.144)$$

where the matrix \mathcal{N} is given by

$$\mathcal{N} = \begin{pmatrix} -\frac{1}{\omega_a^2(t)} \partial_t & -\frac{e}{\omega_a^2(t)} [\dot{\phi}(t) - \phi(t) \partial_t] & \frac{i}{k} \partial_t \\ 0 & 1 & \frac{i}{k} e \phi(t) \\ 0 & 0 & 1 \end{pmatrix} . \quad (3.145)$$

At the same time, in order to obtain the triangular form of the operator $\tilde{\mathcal{M}}$ we have formed various linear combinations of the basic differential equations for $U_0(t)$, $U_\varphi(t)$ and $U_L(t)$. The relation between \mathcal{M} and $\tilde{\mathcal{M}}$ can therefore be written as

$$\mathcal{M} \mathcal{N} = \mathcal{U} \tilde{\mathcal{M}} . \quad (3.146)$$

The matrix \mathcal{U} is found to be given by

$$\mathcal{U} = \begin{pmatrix} \partial_t \frac{1}{\omega_a^2(t)} & -e \partial_t \frac{\phi(t)}{\omega_a^2(t)} & -\frac{i}{k} \partial_t \\ 0 & 1 & \frac{i}{k} e \phi(t) \\ \frac{ik}{\omega_a^2(t)} (\xi^{-1} - 1) & -\frac{ik}{\omega_a^2(t)} e \phi(t) (\xi^{-1} - 1) & \xi^{-1} \end{pmatrix} . \quad (3.147)$$

Therefore, in (3.143), we have to substitute \mathcal{M} by $\mathcal{U} \tilde{\mathcal{M}} \mathcal{N}^{-1}$. This leads to

$$\ln \det \mathcal{M}_\times = \ln \det \mathcal{U} + \ln \det \tilde{\mathcal{M}}_\times - \ln \det \mathcal{N} . \quad (3.148)$$

We can factorize \mathcal{N} and \mathcal{U} in the following way

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_1 \mathcal{N}_2 \\ &= \begin{pmatrix} \frac{1}{\omega_a^2(t)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\partial_t & -e [\dot{\phi}(t) - \phi(t) \partial_t] & \frac{i}{k} \omega_a^2(t) \partial_t \\ 0 & 1 & \frac{i}{k} e \phi(t) \\ 0 & 0 & 1 \end{pmatrix} , \end{aligned} \quad (3.149)$$

and

$$\begin{aligned} \mathcal{U} &= \mathcal{U}_1 \mathcal{U}_2 \mathcal{U}_3 \tag{3.150} \\ &= \begin{pmatrix} \partial_t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -e\phi(t) & -i\frac{\omega_a^2(t)}{k} \\ 0 & 1 & \frac{i}{k}e\phi(t) \\ \frac{ik(\xi^{-1}-1)}{\omega_a^2(t)} & -\frac{ik(\xi^{-1}-1)e\phi(t)}{\omega_a^2(t)} & \xi^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_a^2(t)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

In the same way as in (3.148), the $\ln \det$ of the products can be decomposed into a sum of terms for the single factors. The contributions of \mathcal{N}_1 and \mathcal{U}_3 are identical, so they cancel in the difference $\ln \det \mathcal{U} - \ln \det \mathcal{N}$. The same is true for the contributions of \mathcal{N}_2 and \mathcal{U}_1 , the former one being triangular. Finally, \mathcal{U}_2 is a c -number matrix; its determinant turns out to be one, so $\ln \det \mathcal{U}_2 = 0$. Therefore, we obtain

$$\begin{aligned} \ln \det \mathcal{M}_\times &= \ln \det \tilde{\mathcal{M}}_\times \tag{3.151} \\ &= \ln \det \mathcal{M}_{\varphi\varphi} + 2 \ln \det \mathcal{M}_\eta, \end{aligned}$$

using the triangular form of $\tilde{\mathcal{M}}_\times$. The contributions of \mathcal{M}_η cancel against the Faddeev-Popov contributions as expected, and $\ln \det \tilde{\mathcal{M}}_\times$ is independent of the gauge parameter ξ . The remaining contributions of the isoscalar Higgs and the transverse modes are independent of ξ as well.

We have shown so far that the one loop effective action for the $SU(2)$ Higgs model, and for its Abelian counterpart, are independent of the gauge parameter ξ , if evaluated for a classical background field that satisfies the classical equation of motion. At the minimum of the effective potential $\phi = v$ we find $\mathcal{M}_{\varphi\varphi} = m_\perp$ as expected for a massive gauge field. In general it can be shown that the fluctuation operator $\mathcal{M}_{\varphi\varphi}$ is identical to the fluctuation operator of the Goldstone mode in the Coulomb gauge. This will be done in section 3.7.

For the calculation of the equation of motion we have to compute the first derivative of the effective action. Unfortunately, if we evaluate this derivative for the classical minimum $\phi(t) = v$ we find a ξ -dependent result.

Considering a general variation of the background field, $H(t) + \delta H(t)$, will not, in general, satisfy the equation of motion. In order to allow such general variation we should not use the equation of motion when computing the effective action. (3.146) then gets replaced by

$$\mathcal{M}_\times \mathcal{N} = \mathcal{U} \tilde{\mathcal{M}}_\times + \Delta. \tag{3.152}$$

Δ vanishes if $H(t)$ is a solution of the classical equation of motion. Otherwise it can be found to have the form

$$\Delta = \begin{pmatrix} 0 & \frac{e}{\omega_a^2(t)} \partial_t & 0 \\ 0 & 0 & -\frac{i}{k} e \\ 0 & i\frac{k}{\omega_a^2(t)} (\xi^{-1} - 1) & 0 \end{pmatrix} [\ddot{\phi} + \lambda(\phi^2 - v^2)\phi], \tag{3.153}$$

displaying explicitly the classical equation of motion.

We have in general the relation

$$\delta S_{\text{eff}} = \frac{i}{2} \delta \ln \det \mathcal{M} = \frac{i}{2} \text{Tr} \mathcal{M}^{-1} \delta \mathcal{M}. \quad (3.154)$$

Using

$$\mathcal{M}_\times = \mathcal{U} \tilde{\mathcal{M}}_\times \mathcal{N}^{-1} + \Delta \mathcal{N}^{-1}, \quad (3.155)$$

and the vanishing of Δ along the classical path, we find

$$\begin{aligned} \text{Tr} \mathcal{M}_\times^{-1} \delta \mathcal{M}_\times &= \text{Tr} \mathcal{N} \tilde{\mathcal{M}}_\times^{-1} \mathcal{U}^{-1} \left[\delta (\mathcal{U} \tilde{\mathcal{M}}_\times \mathcal{N}^{-1}) + \delta \Delta \mathcal{N}^{-1} \right] \\ &= \text{Tr} \tilde{\mathcal{M}}_\times^{-1} \delta \tilde{\mathcal{M}}_\times + \text{Tr} \mathcal{U}^{-1} \delta \mathcal{U} - \text{Tr} \mathcal{N}^{-1} \delta \mathcal{N} + \text{Tr} \delta \Delta \tilde{\mathcal{M}}_\times^{-1} \mathcal{U}^{-1}. \end{aligned} \quad (3.156)$$

We now use the factorization of the matrices \mathcal{N} and \mathcal{U} , as in (3.149), (3.150). In the same way as in (3.156) the trace over the products can be decomposed into a sum of traces. The functional derivative of \mathcal{N}_2 and \mathcal{U}_1 vanishes and, therefore, also the contributions $\text{Tr} \mathcal{N}_1^{-1} \delta \mathcal{N}_1$ and $\text{Tr} \mathcal{U}_1^{-1} \delta \mathcal{U}_1$. The contributions from \mathcal{N}_1 and \mathcal{U}_3 are the same. As they are subtracted in (3.156), they cancel. Next consider the contribution of \mathcal{U}_2 . The inverse is given by

$$\mathcal{U}_2^{-1} = \begin{pmatrix} \xi^{-1} \frac{\omega_\xi^2(t)}{\omega_a^2(t)} & eH(t) & ik \\ \frac{eH(t)(1-\xi^{-1})}{\omega_a^2(t)} & 1 & -\frac{i}{k} eH(t) \\ \frac{ik(1-\xi^{-1})}{\omega_a^2(t)} & 0 & 1 \end{pmatrix}, \quad (3.157)$$

and the functional derivative reads

$$\delta \mathcal{U}_2 = \begin{pmatrix} 0 & -e & -2\frac{i}{k} e^2 H(t) \\ 0 & 0 & e\frac{i}{k} \\ \frac{2ike^2 H(t)(1-\xi^{-1})}{\omega_a^4(t)} & \frac{ike(1-\xi^{-1})}{\omega_a^2(t)} \left[1 - \frac{2e^2 H(t)}{\omega_a^2(t)} \right] & 0 \end{pmatrix}. \quad (3.158)$$

It is easy to see that the trace of the product of the two matrices vanishes. For the calculation of the equation of motion we are left with

$$\text{Tr} \mathcal{M}_\times^{-1} \delta \mathcal{M}_\times = \text{Tr} \tilde{\mathcal{M}}_\times^{-1} \delta \tilde{\mathcal{M}}_\times + \text{Tr} \delta \Delta \tilde{\mathcal{M}}_\times^{-1} \mathcal{U}^{-1}. \quad (3.159)$$

The last term is explicit ξ -dependent. It seems not possible in the non-equilibrium case to formulate a gauge parameter independent equation of motion. Nevertheless, we have formulated a fluctuation operator whose diagonal element $\mathcal{M}_{\varphi\varphi}$ coincides with the Goldstone mode in the Coulomb gauge. This result indicates the importance of this mode and a detailed analysis of it is worthwhile. We will investigate $\mathcal{M}_{\varphi\varphi}$ in subsection 3.6.2 in detail.

3.5 Non-equilibrium Dynamics: Gauge Invariant Approach

In this section we consider the Abelian Higgs model in a gauge invariant formulation in order to compare it with our previous results found for the Feynman-gauge [38]. There, we

investigated the SU(2)-Higgs model. Since we have chosen the isoscalar field as background field, the only difference between the models arises in some multiplicity factors. Another choice for the background field would not allow a comparison of the two approaches. As it is common for the Abelian Higgs model, we choose the gauge coupling e in contrast to our paper where we used the coupling constant $g = 2e$.

3.5.1 The Formalism

We study the Abelian Higgs model in a gauge invariant formulation. The basic ideas for this description are developed in [37, 87]. We give here a short overview about the derivation of the Hamiltonian. For details the reader is referred to the two papers mentioned above. The Lagrangian density for the Abelian Higgs model reads

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + D_\mu\Phi^\dagger D^\mu\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi - v^2)^2, \quad (3.160)$$

$$D_\mu = \partial_\mu + ieA_\mu\Phi. \quad (3.161)$$

We want to formulate a gauge invariant Hamiltonian. Therefore, we use as in [37, 87] the canonical formulation. We have to identify the canonical field variables and constrains. The canonical momenta conjugate to the scalar and vector fields are given by

$$\Pi^0 = 0, \quad (3.162)$$

$$\Pi^i = \dot{A}^i + \nabla^i A^0 = -E^i, \quad (3.163)$$

$$\pi^\dagger = \dot{\phi} + ieA^0\phi, \quad (3.164)$$

$$\pi = \dot{\phi}^\dagger - ieA^0\phi^\dagger. \quad (3.165)$$

Therefore, the Hamiltonian reads

$$H = \int d^3x \left\{ \frac{1}{2}\vec{\Pi} \cdot \vec{\Pi}^\dagger + \pi^\dagger\pi + (\vec{\nabla}\phi - ie\vec{A}\phi) \cdot (\vec{\nabla}\phi^\dagger) + \frac{1}{2}(\vec{\nabla} \times \vec{A})^2 + \frac{\lambda}{4}(\phi^\dagger\phi - v^2)^2 + A_0 [\vec{\nabla} \cdot \vec{\Pi} - ie(\pi\phi - \pi^\dagger\phi^\dagger)] \right\}. \quad (3.166)$$

We will quantize this system with Dirac's method [74]. Therefore, we have to recognize the first class constraints (mutually vanishing Poisson brackets). Then the constraints become operators in the quantum theory and are imposed onto the physical states, thus defining the physical subspace of the Hilbert space and gauge invariant operators. We have two first class constrains

$$\Pi^0 = \frac{\delta\mathcal{L}}{\delta A^0} = 0, \quad (3.167)$$

and Gauss' law

$$\mathcal{G}(\vec{x}, t) = \nabla^i \pi^i - \rho = 0, \quad (3.168)$$

$$\rho = ie(\phi\pi - \phi^\dagger\pi^\dagger), \quad (3.169)$$

with ρ being the matter field charge density. We can now quantize the system by imposing the canonical equal time commutation relations

$$[\Pi^0(\vec{x}, t), A^0(\vec{y}, t)] = -i\delta(\vec{x} - \vec{y}) , \quad (3.170)$$

$$[\Pi^i(\vec{x}, t), A^j(\vec{y}, t)] = -i\delta^{ij}\delta(\vec{x} - \vec{y}) , \quad (3.171)$$

$$[\pi^\dagger(\vec{x}, t), \phi^\dagger(\vec{y}, t)] = -i\delta(\vec{x} - \vec{y}) , \quad (3.172)$$

$$[\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta(\vec{x} - \vec{y}) . \quad (3.173)$$

In Dirac's formulation, physical operators are those that commute with the first class constraints. Since $\Pi^0(\vec{x}, t)$ and $\mathcal{G}(\vec{x}, t)$ are generators of local gauge transformations, operators that commute with the first class constraints are gauge invariant [37, 87]. As shown in [37] the fields and the conjugate momenta can be written in the following form

$$\Phi(\vec{x}) = \phi(\vec{x}) \exp \left[ie \int d^3y \vec{A}(\vec{y}) \cdot \vec{\nabla}_y G(\vec{y} - \vec{x}) \right] , \quad (3.174)$$

$$\Pi(\vec{x}) = \pi(\vec{x}) \exp \left[-ie^2 \int d^3y \vec{A}(\vec{y}) \cdot \vec{\nabla}_y G(\vec{y} - \vec{x}) \right] , \quad (3.175)$$

with $G(\vec{y} - \vec{x})$ the Coulomb Green's function that satisfies

$$\Delta G(\vec{y} - \vec{x}) = \delta^3(\vec{y} - \vec{x}) . \quad (3.176)$$

They are invariant under gauge transformations [37]. The gauge field can be separated into transverse and longitudinal components

$$\vec{A}(\vec{x}) = \vec{A}_L(\vec{x}) + \vec{A}_T(\vec{x}) , \quad (3.177)$$

$$\vec{\nabla} \times \vec{A}_L(\vec{x}) = 0 , \quad (3.178)$$

$$\vec{\nabla} \cdot \vec{A}_T(\vec{x}) = 0 . \quad (3.179)$$

$$(3.180)$$

Since the fields \vec{A}_T and Φ and their canonical momenta commute with the constraints, they are gauge invariant. It is also possible to write the momentum canonical to the gauge field in longitudinal and transverse components

$$\vec{\Pi}(\vec{x}) = \vec{\Pi}_L(\vec{x}) + \vec{\Pi}_T(\vec{x}) , \quad (3.181)$$

where both components are gauge invariant. In [37], it is mentioned that in all matrix elements between gauge invariant states the longitudinal component can be replaced by the charge density

$$\vec{\Pi}_L(\vec{x}) \rightarrow ie \left[\Phi(\vec{y})\Pi(\vec{y}) - \Phi^\dagger(\vec{y})\Pi^\dagger(\vec{y}) \right] = \rho . \quad (3.182)$$

Finally, the Hamiltonian becomes

$$\begin{aligned} H = & \int d^3x \left\{ \frac{1}{2} \vec{\Pi}_T \cdot \vec{\Pi}_T + \Pi^\dagger \Pi + (\vec{\nabla} \Phi - ie \vec{A}_T \Phi) \cdot (\vec{\nabla} \Phi^\dagger + ie \vec{A}_T \Phi^\dagger) + \frac{1}{2} (\vec{\nabla} \times \vec{A}_T)^2 \right. \\ & \left. + \frac{\lambda}{4} (\Phi^\dagger \Phi - v^2)^2 + \frac{1}{2} \int d^3x \int d^3y \rho(\vec{x}) G(\vec{x} - \vec{y}) \rho(\vec{y}) \right\} . \end{aligned} \quad (3.183)$$

The features of this Hamiltonian are discussed at length in [37, 87]. One of the striking points is the equivalence with the Hamiltonian in the Coulomb gauge. This similarity is not uncommon because, in the Coulomb gauge, only the physical degrees of freedom are taken into account. We do not want to go deeper into the discussion of the gauge invariance of the Hamiltonian. We focus our interest on the non-equilibrium aspects of the theory. In [37, 87], the effective potential was derived and Some aspects of non-equilibrium dynamics were discussed. But as they explain and we have explained in our discussion of the ϕ^4 theory, the effective potential is not suitable for non-equilibrium dynamics. We derive here the full non-equilibrium equations. We include not only the terms which are quadratic in the zero mode, but all terms up to second order which are relevant for the one loop approximation. We consider the derivate terms of the zero mode and its conjugate momentum. This yields the wave function renormalization which is not considered in [37, 87] or has to be introduced by hand. We will also see that the formalism does not give a clear statement about the loop order which is included. A linearization of the equations leads to the Coulomb gauge, which we consider in detail in chapter 3.6. There, we show the correspondence of the Hamiltonian approach and the Coulomb gauge. As we will show, the inclusion of higher loop terms in the Hamiltonian approach leads to problems in the IR-region. First of all, we derive the equation of motion for the fields. Therefore, we separate the expectation value of Φ and of its canonical momentum into a mean value and a fluctuation part

$$\Phi(\vec{x}, t) = \phi(t) + \varphi(\vec{x}, t) , \quad (3.184)$$

$$\Pi(\vec{x}, t) = \Pi(t) + \pi(\vec{x}, t) . \quad (3.185)$$

We also introduce real fields and canonical momenta as follows

$$\varphi = \frac{1}{\sqrt{2}}(h + i\varphi) , \quad (3.186)$$

$$\pi = \frac{1}{\sqrt{2}}(\pi_h - i\pi_\varphi) . \quad (3.187)$$

Therefore, we find as a gauge invariant Hamiltonian

$$\begin{aligned} H = & \Omega \left[\frac{1}{2}\Pi^2 + U(\phi^2) \right] \\ & + \int d^3x \left[\frac{1}{2}\vec{\pi}_\perp^2 - \frac{1}{2}(\vec{\nabla} \times \vec{a}_\perp)^2 + \frac{1}{2}e^2 a_\perp^2 \phi^2 + \frac{1}{2}\pi_h^2 + \frac{1}{2}\pi_\varphi^2 \right. \\ & \left. + \frac{1}{2}(\vec{\nabla}h)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{\lambda}{2}(\phi^2 - v^2)(h^2 + \varphi^2) + \lambda\phi^2 h^2 \right] \\ & + \int d^3x \int d^3y \frac{e^2}{2} [\phi\pi_\varphi(\vec{x}) - \Pi\varphi(\vec{x})] G(\vec{x} - \vec{y}) [\phi\pi_\varphi(\vec{y}) - \Pi\varphi(\vec{y})] , \end{aligned} \quad (3.188)$$

with the following commutation relations:

$$[\phi, \Pi] = \frac{i}{\Omega} , \quad (3.189)$$

$$[a_{\perp i}(\vec{x}), \pi_{\perp j}(\vec{y})] = i \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\Delta} \right) \delta^3(\vec{x} - \vec{y}) , \quad (3.190)$$

$$[\varphi(\vec{x}), \pi(\vec{y})] = i \delta^3(\vec{x} - \vec{y}) . \quad (3.191)$$

3.5.2 Equation of Motion

We are now able to find the equations of motion for the zero mode and the fluctuation fields. In the Hamiltonian formalism we find the equation of motion by calculating the commutator between the Hamiltonian and the field

$$\dot{f} = i[H, f] . \quad (3.192)$$

We get for the zero mode ϕ

$$\begin{aligned} \dot{\phi} &= i[H, \phi] \\ &= \Pi \left[1 + \frac{e^2}{\Omega} \int d^3x \int d^3y \varphi(\vec{x}) G(\vec{x} - \vec{y}) \varphi(\vec{y}) \right] \\ &\quad - \frac{e^2}{2\Omega} \phi \int d^3x \int d^3y [\varphi(\vec{x}) G(\vec{x} - \vec{y}) \pi_\varphi(\vec{y}) + \pi_\varphi(\vec{x}) G(\vec{x} - \vec{y}) \varphi(\vec{y})] , \end{aligned} \quad (3.193)$$

and for the canonical momentum

$$\begin{aligned} \dot{\Pi} &= i[H, \Pi] \\ &= -U'(\phi) - \frac{\phi}{\Omega} \int d^3x [e^2 \vec{a}_\perp^2 + \lambda(h^2 + \varphi^2) + 2\lambda h^2] \\ &\quad - \frac{e^2}{\Omega} \phi \int d^3x \int d^3y \pi_\varphi(\vec{x}) G(\vec{x} - \vec{y}) \pi_\varphi(\vec{y}) \\ &\quad + \frac{e^2}{2\Omega} \Pi \int d^3x \int d^3y [\pi_\varphi(\vec{x}) G(\vec{x} - \vec{y}) \varphi(\vec{y}) + \varphi(\vec{x}) G(\vec{x} - \vec{y}) \pi_\varphi(\vec{y})] . \end{aligned} \quad (3.194)$$

We also need the equations of motion for the three different quantum fluctuations. The first one is the transverse gauge field. We find an equation for the field $a_{\perp i}$ itself and for its canonical momentum. We can combine these two expressions to a second order differential equation for the gauge field:

$$\begin{aligned} \dot{a}_{\perp i} &= \pi_{\perp i} , \quad \dot{\pi}_{\perp i} = (\Delta - e^2 \phi^2) a_{\perp i} \\ \Rightarrow \ddot{a}_{\perp i} &= (\Delta - e^2 \phi^2) a_{\perp i} . \end{aligned} \quad (3.195)$$

In the same way, we get the equation of motion for the real component of the Higgs fluctuation

$$\ddot{h} = [\Delta - \lambda(3\phi^2 - v^2)] h . \quad (3.196)$$

More difficulties arise for the Goldstone sector φ because the field and its canonical momentum are coupled via Green functions. We find for the field itself

$$\dot{\varphi} = \pi_\varphi + \frac{e^2}{2} \int d^3x [\phi G_{xy} (\phi \pi_\varphi - \Pi \varphi) + (\phi \pi_\varphi - \Pi \varphi) G_{xy} \phi] , \quad (3.197)$$

and for the momentum

$$\dot{\pi}_\varphi = \Delta\varphi - \lambda\varphi(\phi^2 - v^2) - \frac{e^2}{2} \int d^3x \left[\Pi^2 G_{xy}\varphi - (\Pi\phi + \phi\Pi)G_{xy}\pi_\varphi \right] . \quad (3.198)$$

Now we transform the equations of motion into Fourier space. Therefore, we expand the fluctuation fields in the following way (see also (2.7))

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_0} \left\{ a_k U(k, t) e^{i\vec{k}\cdot\vec{x} - i\omega t} + a_k^\dagger U^*(k, t) e^{-i\vec{k}\cdot\vec{x} + i\omega t} \right\} , \quad (3.199)$$

with the usual commutator relations for the annihilation and creation operators a_k, a_k^\dagger :

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 2\omega_0 \delta^3(\vec{k} - \vec{k}') . \quad (3.200)$$

The $U(k, t)$ are the mode functions for the fluctuations depending on k . For convenience we omit this dependence in the following. The Fourier transform for the Green function leads to a factor $1/k^2$. With these expansions, the equation of motion for the zero mode reads

$$\begin{aligned} \dot{\phi}(t) &= \Pi(t) \left[1 + e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{p0} k^2} |U_\varphi(t)|^2 \right] \\ &\quad - \frac{e^2}{2} \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{p0} k^2} \left[U_\varphi(t) U_{\pi_\varphi}^*(t) + U_{\pi_\varphi}(t) U_\varphi^*(t) \right] , \end{aligned} \quad (3.201)$$

and for the canonical momentum

$$\begin{aligned} \dot{\Pi}(t) &= -U'[\phi(t)] \\ &\quad - \phi(t) \int \frac{d^3k}{(2\pi)^3} \left[\frac{e^2}{\omega_{a0}} |U_\perp(t)|^2 + \frac{3\lambda}{2\omega_{h0}} |U_h(t)|^2 + \frac{\lambda}{2\omega_{p0}} |U_\varphi(t)|^2 + \frac{e^2}{2\omega_{p0} k^2} |U_{\pi_\varphi}(t)|^2 \right] \\ &\quad + \Pi(t) \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{4\omega_{p0} k^2} \left[U_{\pi_\varphi}(t) U_\varphi^*(t) + U_{\pi_\varphi}^*(t) U_\varphi(t) \right] , \end{aligned} \quad (3.202)$$

where we have introduced the following frequencies

$$\omega_{p0}^2 = \left[\vec{k}^2 + \lambda(\phi_0^2 - v^2) \right] (\vec{k}^2 + e^2\phi_0^2)/k^2 = \omega_{\varphi_0}^2 \omega_{a0}^2 / k^2 , \quad (3.203)$$

$$\omega_{\varphi_0}^2 = \vec{k}^2 + \lambda(\phi_0^2 - v^2) , \quad (3.204)$$

$$\omega_{a0}^2 = \vec{k}^2 + m_W^2 + e^2(\phi_0^2 - v^2) , \quad (3.205)$$

$$\omega_{h0}^2 = \vec{k}^2 + m_h^2 + 3\lambda(\phi_0^2 - v^2) , \quad (3.206)$$

$$m_h^2 = 2\lambda v^2 , \quad m_W^2 = e^2 v^2 . \quad (3.207)$$

The index 0 indicates the choice of $t = 0$. Notice, that we have included a factor two for the two transverse gauge freedoms in the equation of motion for $\dot{\Pi}$. The choice of ω_{p0} will become more transparent in the next section. By comparing these results with the zero

mode equation in the gauge fixed theory we find some analogies in the fluctuation integrals. The transverse gauge field and the Higgs field component h lead to the same contribution in both theories. The Goldstone channel φ fulfills a coupled differential equation. In the gauge invariant description we have a coupling between the field itself and its canonical momentum. In the R_ξ -gauge, it couples to the time and longitudinal component of the gauge field. These components do not appear in the new description because they are unphysical.

In the same way we have found the zero mode equation, we can derive the mode functions for the fluctuation fields. We find

$$\left[\frac{d^2}{dt^2} + \omega_a^2(t) \right] U_a(t) = 0, \quad (3.208)$$

$$\left[\frac{d^2}{dt^2} + \omega_h^2(t) \right] U_h(t) = 0, \quad (3.209)$$

$$\left[\frac{d}{dt} + \frac{e^2}{k^2} \Pi(t) \phi(t) \right] U_\varphi(t) - \frac{\omega_a^2(t)}{k^2} U_{\pi_\varphi}(t) = 0, \quad (3.210)$$

$$\left[\frac{d}{dt} - \frac{e^2}{k^2} \Pi(t) \phi(t) \right] U_{\pi_\varphi}(t) + \left[\omega_\varphi^2(t) + \frac{e^2}{k^2} \Pi^2(t) \right] U_\varphi(t) = 0. \quad (3.211)$$

In the mode equations for U_φ and U_{π_φ} , a problem in the IR region arises for the first time. The denominator with k^2 will lead to problems as we will see later on. During the discussion of the comparison of the different approaches in chapter 3.7, it will become clear that this IR-instability is caused by higher loop effects. It is possible by combining the differential equation for the field (3.210) and its conjugate momentum (3.211) to find an IR-stable mode equation by neglecting all terms of higher order than one loop.

3.5.3 Energy Density

The energy density can easily be calculated from the Hamiltonian. We have to insert the expansion of the fields in dependence of the mode functions with the corresponding creators and annihilators. Then we have to take the expectation value in the ground state. The field expansion is analogous to the one for the equation of motion (3.199). We find for the energy density in the Fourier space

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \Pi^2(t) + U[\phi(t)] \\ &+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left[|\dot{U}_\perp(t)|^2 + \omega_a^2(t) |U_\perp(t)|^2 \right] \\ &+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{h0}} \left[|\dot{U}_h(t)|^2 + \omega_h^2(t) |U_h(t)|^2 \right] \\ &+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left[|U_{\pi_\varphi}(t)|^2 + \omega_\varphi^2(t) |U_\varphi(t)|^2 \right] \end{aligned}$$

$$\begin{aligned}
 & + \int \frac{d^3 k}{(2\pi)^3} \frac{e^2}{4\omega_{p0} k^2} \left\{ \phi^2(t) |U_{\pi_\varphi}(t)|^2 + \Pi^2(t) |U_\varphi(t)|^2 \right. \\
 & \quad \left. - \Pi(t) \phi(t) [U_{\pi_\varphi}^*(t) U_\varphi(t) + U_{\pi_\varphi}(t) U_\varphi^*(t)] \right\} . \quad (3.212)
 \end{aligned}$$

It is easy to check the conservation of the energy by determining the time derivative. With the equations of motion we can show that it vanishes.

3.5.4 Perturbative Expansion

We are interested in the behavior of the zero mode under the influence of the quantum fluctuation. Therefore, we need well defined, finite relations, and we have to renormalize the equation of motion and the energy density. In the differential equation for $\phi(t)$ and $\Pi(t)$ we have some momentum integrals over the quantum fluctuations. For the analysis of the behavior of these integrals we make a perturbative expansion of the mode functions. Then we are able to fix the leading terms of the integrals in powers of ω and to regularize and renormalize them. For the single channels a_\perp and h , the mode functions are the same as in the R_ξ -gauge and we have already analyzed them in [38]. Nevertheless, for completeness we give a short review for these two fields.

The Single Channels

The perturbative expansion of a single mode is already explained in section 2.3. For completeness we review the general results with special emphasis on the leading behavior of the mode functions for a_\perp and h . For the transverse gauge field a_\perp and the real component of the Higgs field h , the mode functions read

$$\left[\frac{d^2}{dt^2} + \omega_{j0}^2 \right] U_j(t) = -V_j(t) U_j(t) , \quad (3.213)$$

with $j = a, h$. Here we have introduced the potentials

$$V_a(t) = e^2 [\phi^2(t) - \phi_0^2] , \quad (3.214)$$

$$V_h(t) = 3\lambda [\phi^2(t) - \phi_0^2] . \quad (3.215)$$

We separate $U_j(t)$ into a trivial part corresponding to the case $V(t) = 0$ and a function $f_j(t)$ which represents the reaction to the potential by making the ansatz

$$U_j(t) = e^{-i\omega_{j0}t} [1 + f_j(t)] . \quad (3.216)$$

Then $f_j(t)$ satisfies the differential equation

$$\ddot{f}_j(t) - 2i\omega_{j0}\dot{f}_j(t) = -V_j(t)[1 + f_j(t)] , \quad (3.217)$$

with the initial conditions $f_j(0) = \dot{f}_j(0) = 0$. Expanding now $f_j(t)$ with respect to orders in the potential by writing

$$f_j(t) = f_j^{(1)} + f_j^{(2)} + f_j^{(3)} + \dots , \quad (3.218)$$

we can extract the leading behavior for $f_j(t)$:

$$f_j^{(1)}(t) = -\frac{i}{2\omega_{j0}} \int_0^t dt' V_j(t') - \frac{V_j(t)}{4\omega_{j0}^2} + \frac{1}{4\omega_{j0}^2} \int_0^t dt' e^{2i\omega_{j0}\Delta t} \dot{V}_j(t') + \mathcal{O}(\omega_{j0}^{-3}), \quad (3.219)$$

$$f_j^{(2)}(t) = -\frac{1}{4\omega_{j0}^2} \int_0^t dt' \int_0^{t'} dt'' V_j(t') V_j(t'') + \mathcal{O}(\omega_{j0}^{-3}), \quad (3.220)$$

with $\Delta t = t - t'$. A more detailed analysis can be found e.g. in [33, 38].

The Goldstone Channel

For the Goldstone mode, the equation of motion is a little bit more complicated due to the system of the coupled differential equations. We can write the system in matrix form

$$\begin{pmatrix} \dot{U}_\varphi(t) \\ \dot{U}_{\pi_\varphi}(t) \end{pmatrix} = -\mathbf{A}(\mathbf{t}) \begin{pmatrix} U_\varphi(t) \\ U_{\pi_\varphi}(t) \end{pmatrix}, \quad (3.221)$$

with

$$\mathbf{A}(\mathbf{t}) = \begin{pmatrix} -\frac{e^2}{k^2} \Pi(t) \phi(t) & \frac{\omega_a^2(t)}{k^2} \\ -\omega_\varphi^2(t) - \frac{e^2}{k^2} \Pi^2(t) & \frac{e^2}{k^2} \Pi(t) \phi(t) \end{pmatrix}. \quad (3.222)$$

In order to get an equivalent differential equation as for the single channels, we add $A(0)$ on both sides and take the time derivative of the equation. Thus, we get a second order differential equation system of the following form

$$\begin{pmatrix} \ddot{U}_\varphi(t) \\ \ddot{U}_{\pi_\varphi}(t) \end{pmatrix} - \mathbf{A}^2(\mathbf{0}) \begin{pmatrix} U_\varphi(t) \\ U_{\pi_\varphi}(t) \end{pmatrix} = [\mathbf{A}^2(\mathbf{t}) - \mathbf{A}^2(\mathbf{0}) - \dot{\mathbf{A}}(\mathbf{t})] \begin{pmatrix} U_\varphi(t) \\ U_{\pi_\varphi}(t) \end{pmatrix}, \quad (3.223)$$

where $\mathbf{A}^2(\mathbf{0})$ is a diagonal matrix with the elements

$$A_{11}^2(0) = A_{22}^2(0) = -\frac{\omega_{a0}^2 \omega_{\varphi 0}^2}{k^2} = -\omega_{p0}^2, \quad (3.224)$$

$$A_{12}^2(0) = A_{21}^2(0) = 0. \quad (3.225)$$

Now it is obvious why we have chosen ω_{p0} in the field expansion for the Goldstone field. It is a kind of plasma frequency. It is useful in the fluctuation integrals for π_φ and φ because our ansatz for the mode functions in the Goldstone sector will contain the plasma frequency. Cancellations are possible then. Now we have to compute the pre-factor on the right hand side of the differential equation system (3.223). We find

$$\mathbf{A}^2(\mathbf{t}) - \mathbf{A}^2(\mathbf{0}) - \dot{\mathbf{A}}(\mathbf{t}) = \mathbf{M}(\mathbf{t}), \quad (3.226)$$

with

$$M_{11} = -(\lambda + e^2)(\phi^2 - \phi_0^2) - \frac{e^2}{k^2} \left[\lambda \phi^2 (\phi^2 - v^2) - \lambda \phi_0^2 (\phi_0^2 - v^2) + \Pi^2 - \Pi \dot{\phi} - \dot{\Pi} \phi \right], \quad (3.227)$$

$$M_{12} = -2 \frac{e^2}{k^2} \phi \dot{\phi}, \quad (3.228)$$

$$M_{21} = 2 \left(\lambda \phi \dot{\phi} + \frac{e^2}{k^2} \Pi \dot{\Pi} \right), \quad (3.229)$$

$$M_{22} = -(\lambda + e^2)(\phi^2 - \phi_0^2) - \frac{e^2}{k^2} \left[\lambda \phi^2 (\phi^2 - v^2) - \lambda \phi_0^2 (\phi_0^2 - v^2) + \Pi^2 + \Pi \dot{\phi} + \dot{\Pi} \phi \right]. \quad (3.230)$$

From now on we omit the time dependence where it is obvious. A general solution for the differential equation system is given by

$$U_\varphi = U_\varphi(0) + \int_0^t dt' \frac{1}{\omega_{p0}} \sin(\omega_{p0} \Delta t) (M_{11} U_\varphi + M_{12} U_{\pi_\varphi}), \quad (3.231)$$

$$U_{\pi_\varphi} = U_{\pi_\varphi}(0) + \int_0^t dt' \frac{1}{\omega_{p0}} \sin(\omega_{p0} \Delta t) (M_{21} U_\varphi + M_{22} U_{\pi_\varphi}). \quad (3.232)$$

Now we have to fix the initial values for the mode functions U_φ and U_{π_φ} . It is easy to verify that $U_\varphi U_{\pi_\varphi}^* - U_\varphi^* U_{\pi_\varphi}$ is constant for all times by showing that the time derivative vanishes. With the differential equation for mode functions this is obvious. We require the constant to be $-2i\omega_{p0}$. A second claim is a well defined behavior of the mode functions for the initial time. We choose the ansatz

$$U_\varphi(0) = A e^{-i\omega_{p0} t}, \quad (3.233)$$

$$U_{\pi_\varphi}(0) = B e^{-i\omega_{p0} t}. \quad (3.234)$$

$$(3.235)$$

By inserting this in the differential equations and by utilizing the first requirement we can determine the pre-factors A and B :

$$A = \frac{\omega_{a0}}{k}, \quad B = i \frac{\omega_{p0} k}{\omega_{a0}} = i\omega_{\varphi 0}. \quad (3.236)$$

Therefore, an efficient ansatz for the mode functions seems to be

$$U_\varphi(t) = \frac{\omega_{a0}}{k} [1 + f_\varphi(t)] e^{-i\omega_{p0} t}, \quad (3.237)$$

$$U_{\pi_\varphi}(t) = i\omega_{\varphi 0} [1 + f_{\pi_\varphi}(t)] e^{-i\omega_{p0} t}. \quad (3.238)$$

We get new truncated mode functions

$$f_\varphi = - \int_0^t dt' \frac{i}{2\omega_{p0}} \left(e^{2i\omega_{p0}\Delta t} - 1 \right) \left[M_{11}(1 + f_\varphi) + \frac{i\omega_{p0}k^2}{\omega_{a0}^2} M_{12}(1 + f_{\pi_\varphi}) \right], \quad (3.239)$$

$$f_{\pi_\varphi} = - \int_0^t dt' \frac{i}{2\omega_{p0}} \left(e^{2i\omega_{p0}\Delta t} - 1 \right) \left[\frac{\omega_{a0}^2}{i\omega_{p0}k^2} M_{21}(1 + f_\varphi) + M_{22}(1 + f_{\pi_\varphi}) \right]. \quad (3.240)$$

The analysis of the leading behavior of these two functions is a straightforward but rather long calculation. We have given the details in appendix B. There, the reader can also find the differential equations which have to be solved in order to find $f_j^{(1)}$, $f_j^{(2)}$, and $f_j^{(1)}$ with $j = \varphi, \pi_\varphi$. The main results are

$$f_\varphi^{(1)} = -\frac{i}{\omega_{p0}} \int_0^t dt' (\lambda + e^2)(\phi^2 + \phi_0^2) + \frac{V_a}{2\omega_{a0}^2} - \frac{\lambda + e^2}{4\omega_{p0}^2}(\phi^2 - \phi_0^2) + \mathcal{O}(\omega^{-3}), \quad (3.241)$$

$$f_{\pi_\varphi}^{(1)} = -\frac{i}{\omega_{p0}} \int_0^t dt' (\lambda + e^2)(\phi^2 + \phi_0^2) + \frac{V_\varphi}{2\omega_{p0}^2} - \frac{\lambda + e^2}{4\omega_{p0}^2}(\phi^2 - \phi_0^2) + \mathcal{O}(\omega^{-3}), \quad (3.242)$$

with $V_\varphi = \lambda(\phi^2 - \phi_0^2)$. We have also calculated the truncated mode functions in the second order of the potential. The results are rather long and therefore, we show them only in the appendix.

3.5.5 Renormalization

With the knowledge of the leading behavior for the mode functions, we are now able to renormalize the equation of motion for the zero mode, the canonical momentum, and the energy.

Equation of Motion

In the first order differential equations for $\dot{\phi}$ (3.201) and $\dot{\Pi}$ (3.202), we have to deal with several momentum integrals. The integrals with mixed mode functions are finite. This can be seen by inserting the ansatz (3.237) for the mode functions

$$\begin{aligned} U_\varphi U_{\pi_\varphi}^* + U_\varphi^* U_{\pi_\varphi} &= -i\omega_{p0}(1 + f_\varphi)(1 + f_{\pi_\varphi}^*) + i\omega_{p0}(1 + f_\varphi^*)(1 + f_{\pi_\varphi}) \\ &= -2\omega_{p0}(\text{Im}f_{\pi_\varphi} - \text{Im}f_\varphi + \text{Re}f_\varphi \text{Im}f_{\pi_\varphi} - \text{Im}f_\varphi \text{Re}f_{\pi_\varphi}). \end{aligned} \quad (3.243)$$

As shown in appendix B the leading order of $\text{Im}f_{\pi_\varphi}$ (B.18) and $\text{Im}f_\varphi$ (B.9) cancels each other. We are left with

$$\begin{aligned} -\frac{1}{2\omega_{p0}} \left(U_\varphi U_{\pi_\varphi}^* + U_\varphi^* U_{\pi_\varphi} \right) &= \mathcal{K}_\pi^I(\omega) - \mathcal{K}_\phi^I(\omega) + \text{Im}f_{\pi_\varphi}^{(2)} - \text{Im}f_\varphi^{(2)} \\ &\quad + \text{Re}f_\varphi \text{Im}f_{\pi_\varphi} - \text{Im}f_\varphi \text{Re}f_{\pi_\varphi} \propto \mathcal{O}(\omega^{-3}), \end{aligned} \quad (3.244)$$

and therefore, the k -integrals are finite. The other integral in (3.201) leads to the wave function renormalization, the integrals in (3.202) to the mass and coupling constant renormalization. First, we fix the wave function renormalization δZ . Therefore, we have to deal with the integral

$$e^2 \Pi \int \frac{d^3 k}{(2\pi)^3} \frac{|U_\varphi|^2}{2\omega_{p_0} k^2} = e^2 \Pi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{p_0} k^2} \left(1 + \frac{m_{W_0}^2}{k^2} \right) (1 + 2\text{Re}f_\varphi + |f_\varphi|^2) , \quad (3.245)$$

because this integral is proportional to $\Pi(t)$. On the right hand side we already have inserted the ansatz for the mode function. The only divergent part of this integral is given by the one. Therefore, we have to regularize the integral

$$\left\{ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{p_0} k^2} \right\}_{\text{reg}} = 2I_{-3}(m_h) + 2C , \quad (3.246)$$

with

$$C = \frac{1}{16\pi^2} \ln \frac{m_h^2}{m_{\varphi_0}^2} . \quad (3.247)$$

A detailed analysis of the regularization of this integral is given in appendix C. The wave function renormalization takes the form

$$\delta Z = 2e^2 I_{-3}(m_h) . \quad (3.248)$$

We also get a finite contribution

$$\Delta Z = 2e^2 C . \quad (3.249)$$

The next step is to find the mass and coupling constant renormalization. Therefore, we have to analyze the first momentum integral in (3.202). The transverse gauge field mode a_\perp and the Higgs field component h give the same contribution as in the Feynman-gauge-fixed theory. We give only the results; for details the reader is referred to [38]. For the gauge field we get

$$\begin{aligned} -2e^2 \phi \int \frac{d^3 k}{(2\pi)^3} \frac{|U_\perp|^2}{2\omega_{a_0}} &= -2e^2 \phi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{a_0}} + 2e^2 \phi \int \frac{d^3 k}{(2\pi)^3} \frac{V_\perp}{4\omega_{a_0}^3} + \mathcal{F}_\perp^{\text{fin}} \\ &= 2e^4 \phi^3 I_{-3}(m_W) - \Delta\lambda_\perp \phi^3 + \frac{1}{2} \Delta m_\perp \phi + \mathcal{F}_\perp^{\text{fin}} , \end{aligned} \quad (3.250)$$

with

$$\Delta\lambda_\perp = -\frac{e^4}{8\pi^2} \ln \frac{m_W^2}{m_{W_0}^2} , \quad \Delta m_\perp = \frac{e^2}{4\pi^2} m_{W_0}^2 , \quad (3.251)$$

$$\begin{aligned} \mathcal{F}_\perp^{\text{fin}} &= -2e^2 \phi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{8\omega_{a_0}^3} \int_0^t dt' \cos(2\omega_{a_0} \Delta t) \dot{V}_\perp(t') \\ &\quad - 2e^2 \phi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{a_0}} \left(2\text{Re}f_\perp^{(2)} + |f_\perp^{(1)}|^2 \right) . \end{aligned} \quad (3.252)$$

The divergence $I_{-3}(m_W)$ contributes to the coupling constant renormalization. For the Higgs field component h we find

$$\begin{aligned}
 -3\lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{|U_h|^2}{2\omega_{h0}} &= -3\lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{h0}} + 3\lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{V_h}{4\omega_{h0}^3} + \mathcal{F}_h^{\text{fin}} \\
 &= -\frac{3}{2}m_h^2\lambda\phi I_{-3}(m_h) + 9\lambda^2\phi^3 I_{-3}(m_h) \\
 &\quad -\Delta\lambda_h\phi^3 + \frac{1}{2}\Delta m_h\phi + \mathcal{F}_h^{\text{fin}}, \tag{3.253}
 \end{aligned}$$

with

$$\Delta\lambda_h = -\frac{9\lambda^2}{16\pi^2} \ln \frac{m_h^2}{m_{h0}^2}, \quad \Delta m_h = -\frac{3\lambda}{16\pi^2} m_h^2 \ln \frac{m_h^2}{m_{h0}^2} + \frac{3}{8\pi^2} \lambda m_{h0}^2, \tag{3.254}$$

$$\begin{aligned}
 \mathcal{F}_h^{\text{fin}} &= -3\lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{8\omega_{h0}^3} \int_0^t dt' \cos(2\omega_{h0}\Delta t) \dot{V}_h(t') \\
 &\quad -3\lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{h0}} \left(2\text{Re}f_h^{(2)} + |f_h^{(1)}|^2 \right). \tag{3.255}
 \end{aligned}$$

The first integral $I_{-3}(m_h)$ contributes to the mass renormalization, the second to the coupling constant renormalization. The last integral we have to deal with is due to the Goldstone mode. This integral differs from the Feynman-gauge-fixed theory. In the Feynman gauge we have taken unphysical degrees of freedom like the time component of the gauge field into account which was coupled to the Goldstone boson. The physical degrees of freedom are the same in both descriptions. Here we have only physical degrees of freedom and thus the difference between the two calculations is to be expected. We have to analyze

$$\begin{aligned}
 &-\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{p0}} \left[\lambda|U_\varphi|^2 + \frac{e^2}{k^2}|U_{\pi_\varphi}|^2 \right] \\
 = &-\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{p0}} \left\{ \lambda \frac{\omega_{a0}^2}{k^2} \left[1 - \frac{1}{2\omega_{p0}^2}(\lambda + e^2)(\phi - \phi_0^2) + \frac{V_a}{\omega_{a0}^2} \right] \right. \\
 &\quad \left. + e^2 \frac{\omega_{\varphi 0}^2}{k^2} \left[1 - \frac{1}{2\omega_{p0}^2}(\lambda + e^2)(\phi^2 - \phi_0^2) + \frac{V_\varphi}{\omega_{p0}^2} \right] \right\} + \mathcal{F}_\varphi^{\text{fin1}}, \tag{3.256}
 \end{aligned}$$

with

$$\begin{aligned}
 \mathcal{F}_\varphi^{\text{fin1}} &= -\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{p0}} \left[\lambda \frac{\omega_{a0}^2}{k^2} \left(2\mathcal{K}_\varphi^R(\omega) + 2\text{Re}f_\varphi^{(2)} + |f_\varphi^{(1)}|^2 \right) \right. \\
 &\quad \left. + e^2 \frac{\omega_{\varphi 0}^2}{k^2} \left(2\mathcal{K}_\pi^R(\omega) + 2\text{Re}f_{\pi_\varphi}^{(2)} + |f_{\pi_\varphi}^{(1)}|^2 \right) \right]. \tag{3.257}
 \end{aligned}$$

Here we have used the ansatz (3.237) and the expansion of the truncated mode functions (B.7), (B.16) from appendix B. We can replace ω_{p0} in order to get integrals which we can

regularize dimensionally in the following way:

$$\omega_{p_0}^2 = \omega_{a_0}^2 + m_{\varphi_0}^2 \left(1 + \frac{m_{W_0}^2}{k^2} \right), \quad (3.258)$$

and we find for the divergent part of (3.256)

$$\begin{aligned} & -\phi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{p_0}} \left[(\lambda + e^2) - \frac{1}{2\omega_{p_0}^2} (\lambda + e^2)^2 (\phi^2 - \phi_0^2) + \frac{e^2 \lambda}{k^2} (2\phi_0^2 - v^2) \right] \\ & -\phi \int \frac{d^3 k}{(2\pi)^3} \frac{e^2}{2\omega_{p_0}^3 k^2} V_\varphi m_{\varphi_0}^2 \\ = & -\frac{m_h^2}{2} (\lambda - e^2) \phi I_{-3}(m_h) + (\lambda - e^2)^2 \phi^3 I_{-3}(m_h) \\ & + \frac{1}{2} \Delta m_\varphi \phi - \Delta \lambda_\varphi \phi^3 + \mathcal{F}_\varphi^{\text{fin}2}, \end{aligned} \quad (3.259)$$

with

$$\Delta m_\varphi = -m_h^2 (\lambda - e^2) C + \frac{\lambda + e^2}{8\pi^2} (m_{\varphi_0}^2 + m_{W_0}^2), \quad (3.260)$$

$$\Delta \lambda_\varphi = -(\lambda - e^2)^2 C, \quad (3.261)$$

$$\mathcal{F}_\varphi^{\text{fin}2} = -\phi \int \frac{d^3 k}{(2\pi)^3} \frac{e^2}{2\omega_{p_0}^3 k^2} V_\varphi m_{\varphi_0}^2. \quad (3.262)$$

Therefore, we have found divergences independent of the initial condition. After analyzing all infinities in the momentum integrals we can fix the counter terms and the finite contributions to the fluctuation integral. The counter terms are

$$\delta m_h = m_h^2 (4\lambda - e^2) I_{-3}(m_h), \quad (3.263)$$

$$\delta \lambda = [(\lambda - e^2)^2 + 9\lambda^2] I_{-3}(m_h) + 2e^4 I_{-3}(m_W), \quad (3.264)$$

$$\delta Z = 2e^2 I_{-3}(m_h), \quad (3.265)$$

where we have chosen as the classical potential $V_{\text{cl}} = \frac{\lambda}{4} (\phi^2 - v^2)^2$. The finite equations of motion read now

$$\dot{\phi} = \Pi (1 + \Delta Z) + \Delta \mathcal{F}_\phi^{\text{fin}}, \quad (3.266)$$

$$\dot{\Pi} = \frac{1}{2} (m_h^2 + \Delta m_h^2) \phi - (\lambda + \Delta \lambda) \phi^3 + \Delta \mathcal{F}_\Pi^{\text{fin}}, \quad (3.267)$$

with

$$\Delta Z = 2e^2 C, \quad (3.268)$$

$$\Delta m_h^2 = \Delta m_\perp + \Delta m_h + \Delta m_\varphi, \quad (3.269)$$

$$\Delta \lambda = \Delta \lambda_\perp + \Delta \lambda_h + \Delta \lambda_\varphi, \quad (3.270)$$

$$\begin{aligned} \Delta \mathcal{F}_\phi^{\text{fin}} &= e^2 \Pi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{p0} k^2} \left[2\text{Re} f_\phi^{(1)} + |f_\phi^{(1)}|^2 + \frac{m_{W0}^2}{k^2} \left(1 + 2\text{Re} f_\phi^{(1)} + |f_\phi^{(1)}|^2 \right) \right] \\ &\quad + \frac{e^2}{2} \phi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} \left[\mathcal{K}_\pi^I(\omega) - K_\phi^I(\omega) + \text{Re} f_\phi \text{Im} f_{\pi_\phi} - \text{Im} f_\phi \text{Re} f_{\pi_\phi} \right] , \end{aligned} \quad (3.271)$$

$$\begin{aligned} \Delta \mathcal{F}_\Pi^{\text{fin}} &= \mathcal{F}_\perp^{\text{fin}} + \mathcal{F}_h^{\text{fin}} + \mathcal{F}_\phi^{\text{fin1}} + \mathcal{F}_\phi^{\text{fin2}} \\ &\quad - \frac{e^2}{2} \Pi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} \left[\text{Im} f_{\pi_\phi} - \text{Im} f_\phi + \text{Re} f_\phi \text{Im} f_{\pi_\phi} - \text{Im} f_\phi \text{Re} f_{\pi_\phi} \right] . \end{aligned} \quad (3.272)$$

In the first integral in $\Delta \mathcal{F}_\phi^{\text{fin}}$ (3.271) the IR-problem appears explicitly. We find a factor k^4 in the denominator which can be reduced to a $1/k$ by writing the integral measure and ω_{p0} in powers of k . In the first order also the integrand $\text{Re} f_\phi^{(1)}$ does not fix the problem but at this stage it is not clear if the resummation of all one loop graphs leads to a IR-stable fluctuation integral. We will see later on that this problem also occurs in the Coulomb gauge if we want to take higher loops into account. In the pure one loop approximation the integrals and mode functions are well defined. At this point the disadvantage of the formalism used for calculations which do not only concern the effective potential becomes obvious. We are not able to make a clear statement of the order of the approximation. By using the fields and their momenta we loose the opportunity of symbolizing the equations by Feynman graphs even in the lowest order. We will see later on that by replacing the relation $\dot{\phi} = \Pi$ with $\dot{\phi} = \Pi + \Delta \mathcal{F}$ we consider back reactions in a not well defined nor controllable manner. These higher loop terms lead to the IR-inconsistency. In the Coulomb gauge we are able to show this more explicitly. Nevertheless, we carry out also the renormalization of the energy density in order to show that the formalism leads to consistent equations and that the UV-regularization can be carried out in the usual way.

Energy Density

For the renormalization of the energy, we have to show that it contains the same divergences and, therefore, leads to the same counter terms as the equation of motion. In addition, we get a cosmological constant from a quartic divergence. The procedure for the single channels is the same as in the Feynman gauge fixed theory [38]. We give here only a short review. The Goldstone channel is a little bit more complicated but straightforward. We have given some details in appendix B. We will discuss the renormalization channel by channel and at the end we will collect the counter terms and formulate a finite energy density.

The Gauge Field

For the gauge field we have to analyze

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left[|\dot{U}_\perp|^2 + \omega_a^2(t) |U_\perp|^2 \right] . \quad (3.273)$$

With the ansatz (3.216), the expansion of the truncated mode functions (3.219), and the relations

$$2\omega_{a0} \text{Re} f_\perp^{(1)} = \text{Re} \left(i\omega_{a0} \dot{f}_\perp^{(1)*} \right) , \quad (3.274)$$

$$\omega_{a0}^2 \left(2\text{Re}f_{\perp}^{(1)} + |f_{\perp}^{(1)}|^2 \right) = \text{Re} \left[i\omega_{a0} \left(\dot{f}_{\perp}^{(1)*} + f_{\perp}^{(1)} \dot{f}_{\perp}^{(1)*} \right) \right], \quad (3.275)$$

which we have already used in [42] we find

$$\begin{aligned} (3.273) &= \int \frac{d^3k}{(2\pi)^3} \omega_{a0} + \int \frac{d^3k}{(2\pi)^3} \frac{V_{\perp}}{2\omega_{a0}} - \int \frac{d^3k}{(2\pi)^3} \frac{V_{\perp}^2}{8\omega_{a0}^3} + \mathcal{E}_{\perp}^{\text{fin}} \\ &= -\frac{1}{2}e^4\phi^4 I_{-3}(m_W) + \frac{1}{4}\Delta\lambda_{\perp}\phi^4 - \frac{1}{4}\Delta m_{\perp}\phi^2 + \Delta\Lambda'_{\perp} + \mathcal{E}_{\perp}^{\text{fin}}, \end{aligned} \quad (3.276)$$

with

$$\begin{aligned} \mathcal{E}_{\perp}^{\text{fin}} &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left\{ |\dot{f}_{\perp}^{(2)}|^2 + 2\text{Re}(f_{\perp}^{(1)} \dot{f}_{\perp}^{(2)*}) + V_a \left(2\text{Re}f_{\perp}^{(2)} + |f_{\perp}^{(1)}|^2 \right) \right. \\ &\quad \left. + \frac{1}{4\omega_{a0}^2} \int_0^t dt' \int_0^t dt'' e^{-2i\omega_{a0}(t'-t'')} \dot{V}_{\perp}(t') \dot{V}_{\perp}(t'') \right\}, \end{aligned} \quad (3.277)$$

$$\Delta\Lambda'_{\perp} = \frac{e^4}{64\pi^2} \phi_0^4. \quad (3.278)$$

As in the equation of motion, the gauge field contributes only to the coupling constant renormalization. The divergence is still independent of the initial conditions.

The Higgs Field

Since the mode function for the gauge field and the Higgs field are the same we only have to change the notation for the potential, frequencies, and the truncated mode function. Therefore, we get the following contribution for the energy from the Higgs mode:

$$\begin{aligned} &\int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{h0}} \left[|\dot{U}_h|^2 + \omega_h^2(t) |U_h|^2 \right] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{h0}}{2} + \int \frac{d^3k}{(2\pi)^3} \frac{V_h}{4\omega_{h0}} - \int \frac{d^3k}{(2\pi)^3} \frac{V_{h0}^2}{16\omega_{h0}^3} + \mathcal{E}_h^{\text{fin}} \\ &= -\frac{1}{4} \left(9\lambda^2\phi^4 - 3\lambda m_h^2\phi^2 + \frac{m_h^4}{4} \right) I_{-3}(m_h) + \frac{1}{4}\Delta\lambda_h - \frac{1}{4}\Delta m_h + \Delta\Lambda_h + \Delta\Lambda'_h, \end{aligned} \quad (3.279)$$

with

$$\begin{aligned} \mathcal{E}_h^{\text{fin}} &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{h0}} \left\{ |\dot{f}_h^{(2)}|^2 + 2\text{Re}(f_h^{(1)} \dot{f}_h^{(2)*}) + V_h \left(2\text{Re}f_h^{(2)} + |f_h^{(1)}|^2 \right) \right. \\ &\quad \left. + \frac{1}{4\omega_{h0}^2} \int_0^t dt' \int_0^t dt'' e^{-2i\omega_{h0}(t'-t'')} \dot{V}_h(t') \dot{V}_h(t'') \right\}, \end{aligned} \quad (3.280)$$

$$\Delta\Lambda_h = -\frac{m_h^4}{256\pi^2} \ln \frac{m_h^2}{m_{h0}^2}, \quad (3.281)$$

$$\Delta\Lambda'_h = -\frac{3m_{h0}^4}{128\pi^2} + \frac{3}{32\pi^2} \lambda m_{h0}^2 \phi_0^2. \quad (3.282)$$

The term $\propto \phi^4 I_{-3}$ contributes to the coupling constant renormalization $\delta\lambda$, the one $\propto \phi^2 I_{-3}$ to mass renormalization δm , and the term $\propto m_h^4 I_{-3}$ leads to the cosmological constant $\delta\Lambda$.

The Goldstone Field

Finally, we have to investigate the Goldstone mode. It contains two integrals

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left[|U_{\pi_\varphi}|^2 + \omega_\varphi^2(t) |U_\varphi|^2 \right] \\ & + \int \frac{d^3 k}{(2\pi)^3} \frac{e^2}{4\omega_{p0} k^2} \left[\phi^2 |U_{\pi_\varphi}|^2 + \Pi^2 |U_\varphi|^2 - \phi \Pi (U_{\pi_\varphi} U_\varphi^* + U_{\pi_\varphi}^* U_\varphi) \right]. \end{aligned} \quad (3.283)$$

With the identification $f_{\pi_\varphi} \approx \dot{f}_\varphi$, the structure of the first integral corresponds to the single channels. Therefore, it should lead to the mass counter term, coupling constant renormalization, and the cosmological constant. The second integral contains a part $\propto \Pi^2$. This looks like the wave function renormalization. In addition, the first term is logarithmically divergent and could provide a part of the coupling constant counter term. As in the equation of motion the term with the mixed mode functions is already finite. The next step is the insertion of (3.237) in (3.283) and the analysis of the divergence structure. Since we have a factor $\omega^2(t)$ in the nominator we find many more infinite contributions than in the equation of motion. The terms of the form $\mathcal{K}_\varphi^R + \mathcal{K}_\pi^R$ as well as $2\text{Re}f_j^{(2)} + f_j^{(1)} f_j^{(1)*}$ which were finite in the equation of motion are now taken into account for renormalization. In particular, we find for (3.283), depending on the truncated mode functions:

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left[\omega_{\varphi 0}^2 (1 + 2\text{Re}f_{\pi_\varphi} + |f_{\pi_\varphi}|^2) + (\omega_{\varphi 0}^2 + V_\varphi) \frac{\omega_{a0}^2}{k^2} (1 + 2\text{Re}f_\varphi + |f_\varphi|^2) \right] \\ & + \int \frac{d^3 k}{(2\pi)^3} \frac{e^2}{4\omega_{p0} k^2} \left[\phi^2 \omega_{p0}^2 (1 + 2\text{Re}f_{\pi_\varphi} + |f_{\pi_\varphi}|^2) + \Pi^2 \frac{\omega_{a0}^2}{k^2} (1 + 2\text{Re}f_\varphi + |f_\varphi|^2) \right] \\ & + \frac{e^2}{2} \phi \Pi \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} (\text{Im}f_{\pi_\varphi} - \text{Im}f_\varphi + \text{Re}f_\varphi \text{Im}f_{\pi_\varphi} - \text{Im}f_\varphi \text{Re}f_{\pi_\varphi}) \\ = & \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_{\varphi 0}^2}{2\omega_{p0}} \end{aligned} \quad (3.284)$$

$$+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left[V_\varphi + e^2 \frac{\omega_{p0}^2}{k^2} (\phi^2 + \phi_0^2) + e^2 \frac{\omega_{\varphi 0}^2}{\omega_{p0}^2} \left(\frac{\omega_{p0}^2}{\omega_{a0}^2} - 1 \right) (\phi^2 - \phi_0^2) \right] \quad (3.285)$$

$$+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left\{ \frac{e^2}{k^2} \Pi^2 + V_\varphi \left(\frac{e^2}{\omega_{a0}^2} - \frac{\lambda + e^2}{2\omega_{p0}^2} \right) (\phi^2 - \phi_0^2) + \frac{e^2}{k^2} \phi_0^2 V_\varphi \right. \quad (3.286)$$

$$\left. + 2\omega_{\varphi 0}^2 (\mathcal{K}_\varphi^R + \mathcal{K}_\pi^R) + \frac{e^2}{2k^2} (\lambda - e^2) \phi^2 \frac{\omega_{\varphi 0}^2}{\omega_{p0}^2} (\phi^2 - \phi_0^2) \right.$$

$$\left. + \frac{e^2}{k^2} \phi_0^2 \frac{\omega_{\varphi 0}^2}{\omega_{p0}^2} \left[e^2 \frac{\omega_{p0}^2}{\omega_{a0}^2} - \frac{1}{2} (\lambda + e^2) \right] (\phi^2 - \phi_0^2) \right.$$

$$\left. + \omega_{\varphi 0}^2 \left[2\text{Re}f_{\pi_\varphi}^{(2)} + f_{\pi_\varphi}^{(1)} f_{\pi_\varphi}^{(1)*} + 2\text{Re}f_\varphi^{(2)} + f_\varphi^{(1)} f_\varphi^{(1)*} \right] \right\}$$

$$+\mathcal{E}_{\pi\varphi}^{\text{fin}}, \quad (3.287)$$

with

$$\begin{aligned} \mathcal{E}_{\pi\varphi}^{\text{fin}} &= \frac{e^2}{2}\phi\Pi \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \left(\text{Im}f_{\pi\varphi}^{(1)} - \text{Im}f_{\varphi}^{(1)} + \text{Re}f_{\varphi}^{(1)}\text{Im}f_{\pi\varphi}^{(1)} - \text{Im}f_{\varphi}^{(1)}\text{Re}f_{\pi\varphi}^{(1)} \right) \\ &+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left\{ e^2 \frac{\omega_{\varphi 0}^2}{k^2} \phi^2 \left(2\text{Re}f_{\pi\varphi}^{(2)} + |f_{\pi\varphi}^{(1)}|^2 \right) + 2V_{\varphi}\mathcal{K}_{\varphi}^R \right. \\ &\quad + \left(e^2 \frac{\omega_{\varphi 0}^2}{k^2} \phi_0^2 + \frac{\omega_{a0}^2}{k^2} V_{\varphi} \right) \left(2\text{Re}f_{\varphi}^{(2)} + |f_{\varphi}^{(1)}|^2 \right) + 2\frac{e^2}{k^2} \phi_0^2 V_{\varphi} \text{Re}f_{\varphi}^{(1)} \\ &\quad + \omega_{\varphi 0}^2 \left[2\text{Re}f_{\pi\varphi}^{(3)} + 2\text{Re}f_{\pi\varphi}^{(1)}\text{Im}f_{\pi\varphi}^{(2)} + 2\text{Im}_{\pi\varphi}^{(1)}\text{Re}f_{\pi\varphi}^{(2)} + |f_{\pi\varphi}^{(2)}|^2 \right. \\ &\quad \left. + 2\text{Re}f_{\varphi}^{(3)} + 2\text{Re}f_{\varphi}^{(1)}\text{Im}f_{\varphi}^{(2)} + 2\text{Im}_{\varphi}^{(1)}\text{Re}f_{\varphi}^{(2)} + |f_{\varphi}^{(2)}|^2 \right] \left. \right\} \\ &+ \Pi^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{4\omega_{p0}k^4} \left[\omega_{a0}^2 \left(2\text{Re}f_{\varphi}^{(1)} + |f_{\varphi}^{(1)}|^2 \right) \right]. \quad (3.288) \end{aligned}$$

At this point, we are able to analyze the divergences. (3.284) is quartically divergent, (3.285) quadratically, and (3.286) contains logarithmic divergences. In the last integral, an IR-divergence appears analogous to the fluctuation integral. Also, here it is not obvious whether there is a cancellation of these divergences in higher order.

The Quartic Divergence in the Goldstone Channel

The first integral we have to consider is (3.284)

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{\varphi 0}^2}{2\omega_{p0}} &= \frac{1}{4} \left[4m_{W0}^4 - (m_{W0}^2 + m_{\varphi}^2)^2 \right] I_{-3}(m_h) \\ &+ \frac{1}{4} \left(\frac{3}{8\pi^2} + C \right) \left[4m_{W0}^2 - (m_{W0}^2 + m_{\varphi 0}^2)^2 \right]. \quad (3.289) \end{aligned}$$

Details for the regularization of such an integral are given in appendix C. The result still contains infinities depending on the initial conditions. But as we will see later on, they are cancelled with divergences from the other integrals.

The Quadratic Divergences in the Goldstone Channel

The next integral which has to be analyzed is the quadratically divergent one (3.285)

$$\begin{aligned} &\int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{p0}} \left[V_{\varphi} + e^2(\phi^2 - \phi_0^2) + e^2 \frac{m_{\varphi 0}^2}{k^2} (\phi^2 + \phi_0^2) + e^2 \frac{m_{\varphi 0}^2}{\omega_{a0}^2} (\phi^2 - \phi_0^2) \right] \\ &= \left[\frac{m_h^2}{4} (\lambda - 3e^2) \phi^2 - \frac{1}{2} (e^2 - \lambda) \phi^2 \phi_0^2 + \frac{m_h^2}{4} (e^2 - \lambda) \phi_0^2 - \frac{1}{2} (e^4 - \lambda^2) \phi_0^4 \right] I_{-3}(m_h) \\ &\quad - \frac{1}{2} (m_{\varphi 0}^2 + m_{W0}^2) \left(\frac{1}{16\pi^2} + C \right) \left[V_{\varphi} + e^2 (\phi^2 - \phi_0^2) \right] + 2e^2 m_{\varphi 0}^2 C \phi^2, \quad (3.290) \end{aligned}$$

where we have used the relation (3.258) between ω_{a0} and ω_{p0} . Again we have infinite parts which depend on the initial conditions. We can also see that the divergence $\propto \phi^2$ in this

form does not lead to the mass counter term. Furthermore, we are not able to identify Δm_φ until now. At this point it is obvious that the evaluation which was sufficient for the equation of motion is insufficient here. Here we have to go deeper into the expansion of the truncated mode functions. As we will see in the next paragraph we find the necessary infinities and can show that the dependence of the initial conditions dropped out.

The Logarithmic Divergences in the Goldstone Channel

The first integral we compute leads to the wave function renormalization

$$e^2 \Pi^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p_0} k^2} = e^2 \Pi^2 I_{-3}(m_h) + e^2 \Pi^2 C = \frac{1}{2} \Pi^2 \delta Z + \frac{1}{2} \Pi^2 \Delta Z. \quad (3.291)$$

It is useful to split the rest of the integral (3.286) into three parts. The first one is the one which we have usually considered in our previous works:

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p_0}} \left\{ V_\varphi \left(\frac{e^2}{\omega_{a_0}^2} - \frac{\lambda + e^2}{2\omega_{p_0}^2} \right) (\phi^2 - \phi_0^2) + \frac{e^2}{k^2} \phi_0^2 V_\varphi \right. \\ & \quad \left. + \frac{e^2}{2k^2} (\lambda - e^2) \phi^2 \frac{\omega_{\varphi_0}^2}{\omega_{p_0}^2} (\phi^2 - \phi_0^2) \right. \\ & \quad \left. + \frac{e^2}{k^2} \phi_0^2 \frac{\omega_{\varphi_0}^2}{\omega_{p_0}^2} \left[e^2 \frac{\omega_{p_0}^2}{\omega_{a_0}^2} - \frac{1}{2} (\lambda + e^2) \right] (\phi^2 - \phi_0^2) \right\} \\ & = \left[-\frac{1}{2} (\lambda - e^2)^2 \phi^4 + (\lambda^2 - e^2 \lambda + e^4) \phi^2 \phi_0^2 \right. \\ & \quad \left. - \frac{1}{2} (e^4 + \lambda^2) \phi_0^4 \right] [I_{-3}(m_h) + C] + \mathcal{O}(\omega^{-5}). \end{aligned} \quad (3.292)$$

The second part leads in the equation of motion to a finite contribution. Here we find

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_{\varphi_0}^2}{2\omega_{p_0}} [\mathcal{K}_\varphi^R(\omega) + \mathcal{K}_\pi^R(\omega)] \\ & = e^2 \left(\frac{m_h^2}{2} - \lambda \phi^2 \right) (\phi^2 - \phi_0^2) [I_{-3}(m_h) + C] + \mathcal{O}(\omega^{-5}). \end{aligned} \quad (3.293)$$

The integrand of the last part is analyzed in appendix B. With the result (B.28) we find

$$\begin{aligned} & \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_{\varphi_0}^2}{4\omega_{p_0}} [2\text{Re} f_{\pi_\varphi}^{(2)} + f_{\pi_\varphi}^{(1)} f_{\pi_\varphi}^{(1)*} + 2\text{Re} f_\varphi^{(2)} + f_\varphi^{(1)} f_\varphi^{(1)*}] \\ & = \frac{1}{4} (\lambda + e^2)^2 (\phi^2 - \phi_0^2)^2 [I_{-3}(m_h) + C] + \mathcal{O}(\omega^{-5}). \end{aligned} \quad (3.294)$$

The Counter Terms

We now collect all divergences and show that they lead to the same counter terms as in the equation of motion. We also demonstrate that all divergences depending on the initial

conditions are cancelled. Finally, we calculate the finite contribution to the cosmological constant. From (3.289), (3.290), (3.292), (3.293), and (3.294) we find

$$\begin{aligned} \mathcal{E}_{\pi\varphi}^{\text{div}} &= -\frac{m_h^4}{16}I_{-3}(m_h) + \frac{m_h^2}{4}(\lambda - e^2)\phi^2 I_{-3}(m_h) - \frac{1}{4}(\lambda - e^2)^2\phi^4 I_{-3}(m_h) \\ &\quad -\frac{1}{4}\Delta m_\varphi + \frac{1}{4}\Delta\lambda_\varphi + \Delta\Lambda_\varphi + \Delta\Lambda'_\varphi, \end{aligned} \quad (3.295)$$

with

$$\Delta\Lambda_\varphi = \frac{m_h^4}{16}C, \quad (3.296)$$

$$\begin{aligned} \Delta\Lambda'_\varphi &= \frac{C}{2}e^2 m_h^2 \phi_0^2 + \frac{3}{128\pi^2} \left[4m_{W_0}^4 - (m_{W_0}^2 + m_{\varphi_0}^2)^2 \right] \\ &\quad + \frac{1}{128\pi^2}(\lambda - e^2)(m_{W_0}^2 + m_{\varphi_0}^2)\phi_0^2. \end{aligned} \quad (3.297)$$

The divergences are independent of the initial conditions and, with the contributions of the gauge and the Higgs field, we get the mass, coupling constant, and wave function renormalization δm_h^2 (3.263), $\delta\lambda$ (3.264), and δZ (3.265). Additionally, we find

$$\delta\Lambda = \frac{m_h^4}{8}I_{-3}(m_h), \quad (3.298)$$

and as finite contributions

$$\Delta\Lambda = \Delta\Lambda_h + \Delta\Lambda_\varphi, \quad (3.299)$$

$$\Delta\Lambda' = \Delta\Lambda'_\perp + \Delta\Lambda'_h + \Delta\Lambda'_\varphi. \quad (3.300)$$

The counter term $\delta\Lambda$ is up to a factor two the same as in the Feynman gauge fixed theory. The difference is due to the fact that we have in the SU(2) Higgs model a vector meson with isospin degeneracy three and here only a scalar particle. As a finite expression for the energy we have now

$$\mathcal{E} = \frac{1}{2}(1 + \Delta Z)\Pi^2 - \frac{1}{4}(m_h^2 + \Delta m_h^2)\phi^2 + \frac{1}{4}(\lambda + \Delta\lambda)\phi^4 + \Delta\Lambda + \Delta\Lambda' + \mathcal{E}^{\text{fin}}, \quad (3.301)$$

with

$$\mathcal{E}^{\text{fin}} = \mathcal{E}_\perp^{\text{fin}} + \mathcal{E}_h^{\text{fin}} + \mathcal{E}_{\pi\varphi}^{\text{fin}} + \mathcal{O}(\omega^{-5}). \quad (3.302)$$

The part $\propto \mathcal{O}(\omega^{-5})$ comes from the subleading terms in the logarithmically divergent contribution of the energy and can be handled numerically via subtractions.

In this section we have shown, that it is possible to extend the formalism developed in [37] for the effective potential to the full non-equilibrium equations of motion. We derived the equation of motion for the zero mode and the energy density and showed that a consistent UV-renormalization for both quantities is possible. We found counter terms independent of the initial condition. By taking into account all terms up to second order

and not only the quadratic ones like in [37], we found in addition to the mass and coupling constant counter term also the wave function counter term. Since we have worked in the Hamiltonian formalism and, therefore, with the field and its canonical momentum and not with the field and its derivative as it is usual in the Euler-Lagrange formalism we gave up the possibility of a well defined one loop approximation. By reducing the system of classical equations in (3.210) and (3.211), one can easily see that after taking the time derivative of (3.210) and inserting (3.211), the equation of motion for ϕ contains a fluctuation integral higher than one loop order. These higher loop terms lead to IR-instabilities. A better analysis of the appearance is possible in the Coulomb gauge. There, we will localize the $1/k$ poles as contributions from the two loop order.

3.6 Non-equilibrium Dynamics: Coulomb Gauge

3.6.1 The Fluctuation Operator in the Coulomb Gauge

Our starting point again is the Lagrangian (3.15). As discussed, e.g. in [37], and in the last section it is possible to find a gauge invariant description of the Abelian Higgs model by quantizing the theory with Dirac's method. It is developed from the corresponding Hamiltonian to \mathcal{L} . First, one has to find gauge invariant observables that commute with first class constraints which are Gauss' law and vanishing canonical momentum for A_0 . Then the Hamiltonian has to be written in these gauge invariant quantities. An equivalent way is to choose the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$. One gets a Hamiltonian written in terms of transverse components and including the instantaneous Coulomb interaction. This Coulomb interaction can be traded with a Lagrange multiplier field linearly coupled to the charge density. This leads to the Lagrangian in the Coulomb gauge

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \Phi^\dagger \partial^\mu \Phi + \frac{1}{2} \partial_\mu \vec{A}_T \partial^\mu \vec{A}_T - e \vec{A}_T \cdot \vec{j}_T - e^2 \vec{A}_T \cdot \vec{A}_T \Phi^\dagger \Phi \\ &\quad + \frac{1}{2} (\nabla A_0)^2 + e^2 A_0^2 \Phi^\dagger \Phi - ie A_0 \rho - V(\Phi^\dagger \Phi), \\ \vec{j}_T &= i(\Phi^\dagger \vec{\nabla}_T \Phi - \vec{\nabla}_T \Phi^\dagger \Phi), \quad \rho = -i(\Phi \dot{\Phi}^\dagger - \dot{\Phi}^\dagger \Phi), \end{aligned} \quad (3.303)$$

where A_T is the transverse component of the gauge field. The field A_0 is a gauge invariant Lagrange multiplier whose equation of motion is algebraic:

$$\nabla^2 A_0(\vec{x}, t) = \rho(\vec{x}, t). \quad (3.304)$$

Using the usual decomposition of Φ into an expectation value and a fluctuation part, and splitting the fluctuations into a real part h and an imaginary part φ , one finds for the Lagrangian in the Fourier space

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\dot{\phi} + \dot{h})^2 + \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} k^2 (\phi + h)^2 - \frac{1}{2} k^2 \varphi^2 \\ &\quad + \frac{1}{2} \dot{a}_\perp^2 - \frac{1}{2} [k^2 + e^2 (\phi + h)^2] a_\perp^2 - \frac{1}{2} \mu^2 \varphi^2 \\ &\quad - \frac{\lambda}{4} [(\phi + h)^2 + \varphi^2]^2 + \frac{1}{2} (k^2 + e^2 \phi^2) a_0^2 + e a_0 (\varphi \dot{\phi} - \dot{\phi} \varphi), \end{aligned} \quad (3.305)$$

with $\mu^2 = -\lambda v^2$. The physical degrees of freedom - the transversal gauge mode and the Higgs mode h - are the same as in the R_ξ gauge up to a factor three. This factor is the degeneracy factor of the non-Abelian model. Since we are analyzing the model with an isoscalar Higgs background, this is the only difference between the two models. We now investigate the remaining degrees of freedom, the Higgs field component φ and the Lagrange multiplier a_0 . As already mentioned, the equation that fixes a_0 is purely algebraic. The field equation reads

$$\omega_a^2(t)a_0(t) = e \left[\dot{\varphi}(t)\phi(t) - \varphi(t)\dot{\phi}(t) \right] . \quad (3.306)$$

This corresponds to our choice of \tilde{U}_0 in (3.140) as a dependent variable. For the Goldstone field we find

$$\ddot{\varphi}(t) + \left[k^2 + \mu^2 + \lambda\phi^2(t) \right] \varphi(t) = e\dot{a}_0(t)\phi(t) + 2ea_0(t)\dot{\phi}(t) . \quad (3.307)$$

It is possible as in the R_ξ -gauge to eliminate the mode function for the Lagrange multiplier a_0 in (3.307). By using the differential equation for φ and the classical equation of motion (3.24) it is easy to show that the time derivative of a_0 is given by

$$\dot{a}_0(t) = \frac{e}{\omega_a^2(t)} \left[\phi(t)\ddot{\varphi}(t) - \varphi(t)\ddot{\phi}(t) - 2e\phi(t)\dot{\phi}(t)a_0(t) \right] = -e\phi(t)\varphi(t) . \quad (3.308)$$

Inserting (3.306) and (3.308) in (3.307) leads to

$$\mathcal{M}_{\varphi\varphi}(t)\varphi(t) = 0 , \quad (3.309)$$

where $\mathcal{M}_{\varphi\varphi}$ is given by (3.142). Thus we have found the same fluctuation operator for the Goldstone mode as in the transformed R_ξ -equations. a_0 is a dependent mode and the physical modes are the two transversal gauge fields a_\perp and the scalar Higgs mode h . Therefore, the equation of motion for the mode functions in the Coulomb gauge is the same as we found in the previous section for the R_ξ -gauge. From the Lagrangian (3.305), we can also derive the equation of motion for the zero mode ϕ . We find

$$\begin{aligned} \ddot{\phi} + \lambda\phi(\phi^2 - v^2) + 3\lambda\phi\langle h^2 \rangle + 2e^2\phi\langle a_\perp^2 \rangle + \lambda\phi\langle \varphi^2 \rangle \\ - e^2\phi\langle a_0^2 \rangle + 2e\langle a_0\dot{\varphi} \rangle + e\langle \dot{a}_0\varphi \rangle = 0 , \end{aligned} \quad (3.310)$$

where the expectation values for the fields and their normalization are not specified yet. Now we eliminate the field a_0 *without* using the classical equation of motion. We set $\ddot{\phi}$ equal to

$$\ddot{\phi} = -\lambda\phi(\phi^2 - v^2) + \mathcal{R}(t) , \quad (3.311)$$

where $\mathcal{R}(t)$ contains the fluctuations. The field equations are then given by

$$a_0 = \frac{e}{\omega_a^2}(\phi\dot{\varphi} - \dot{\phi}\varphi) , \quad (3.312)$$

$$\dot{a}_0 = -e\phi\varphi + \frac{e}{k^2}\varphi\mathcal{R} , \quad (3.313)$$

$$\ddot{\varphi} = -\omega_{e\lambda}^2\varphi + \frac{2e^2\dot{\phi}}{\omega_a^2}(\phi\dot{\varphi} - \dot{\phi}\varphi) + \frac{e^2}{k^2}\phi\mathcal{R}\varphi , \quad (3.314)$$

with

$$\omega_{e\lambda}^2 = k^2 + e^2\phi^2 + \lambda(\phi^2 - v^2). \quad (3.315)$$

By taking the fluctuation integral $\mathcal{R}(t)$ into account, we receive a mode function which is of order higher than one loop. In this case we are dealing with the back reaction of the quantum fluctuations a_0 and φ . Obviously, these higher loop terms contain a factor e^2/k^2 as in the Hamiltonian approach which lead to the IR-instabilities. By choosing $\mathcal{R}(t) = 0$ we consider only one loop effects and (3.314) is identical with (3.309) We will discuss this case in detail at the end of this section. Using (3.312) and (3.313), we find for the zero mode the equation of motion in terms of the quantum fluctuations

$$\begin{aligned} \ddot{\phi} + \lambda\phi(\phi^2 - v^2) &+ 3\lambda\phi\langle h^2 \rangle + 2e^2\phi\langle a_\perp^2 \rangle \\ &+ (\lambda - e^2)\phi\langle \varphi^2 \rangle - e^4\phi\dot{\phi}^2\langle \frac{\varphi^2}{\omega_a^4} \rangle \\ &+ e^2\phi\langle \frac{2\omega_a^2 - e^2\phi^2}{\omega_a^4}\dot{\varphi}^2 \rangle \\ &- 2e^2\dot{\phi}^2\langle \frac{k^2}{\omega_a^4}\varphi\dot{\varphi} \rangle + e^2\mathcal{R}\langle \frac{\varphi^2}{k^2} \rangle = 0. \end{aligned} \quad (3.316)$$

Again we see here the appearance of a higher loop contribution which is IR-divergent. We can also compute the energy density in the Coulomb gauge. It is given by

$$\begin{aligned} \mathcal{E} = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 &+ \frac{1}{2}[\langle \dot{h}^2 \rangle + \langle \omega_h^2 h^2 \rangle] \\ &+ [\langle \dot{a}_\perp^2 \rangle + \langle \omega_a^2 a_\perp^2 \rangle] \\ &+ \frac{1}{2}[\langle \dot{\varphi}^2 \rangle + \langle \omega_\varphi^2 \varphi^2 \rangle] \\ &- \frac{1}{2}\langle \omega_a^2 a_0^2 \rangle. \end{aligned} \quad (3.317)$$

The component $\langle a_0^2 \rangle$ has a negative sign because of the indefinite metric of the time component of the gauge field. By inserting (3.312) into the last term of the energy, \mathcal{E} can be written similar to the equation of motion in the form

$$\begin{aligned} \mathcal{E} = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 &+ \frac{1}{2}[\langle \dot{h}^2 \rangle + \langle \omega_h^2 h^2 \rangle] \\ &+ [\langle \dot{a}_\perp^2 \rangle + \langle \omega_a^2 a_\perp^2 \rangle] \\ &+ \langle \frac{k^2}{2\omega_a^2}\dot{\varphi}^2 \rangle + \frac{1}{2}\langle \left(\omega_\varphi^2 - \frac{e^2\dot{\phi}^2}{\omega_a^2} \right) \varphi^2 \rangle \\ &+ e^2\phi\dot{\phi}\langle \frac{\varphi}{\omega_a^2}\dot{\varphi} \rangle. \end{aligned} \quad (3.318)$$

With (3.311), (3.314) and the equations of motion for h and a_\perp , it is straightforward to show that the time derivative of \mathcal{E} vanishes. Now we have to decide whether we neglect

$\mathcal{R}(t)$ or not. For a well defined one loop approximation which we have considered in the 't Hooft-Feynman gauge in [38] we have to take it to be zero. Numerically, this leads to some problems for the energy conservation. In order to show the energy conservation, we have to compute the time derivative of \mathcal{E} . In the energy density terms proportional $\dot{\phi}$ appear, and, after performing the time derivative, it is necessary to insert the equation of motion for the zero mode. Since we have taken $\mathcal{R}(t) = 0$, we have to insert $\ddot{\phi} = -\lambda\phi(\phi^2 - v^2)$. Analytically, this is no problem, but numerically this equation is only solved at the initial time. By solving the equation of motion for the zero mode numerically we automatically take the fluctuation integral into account. Therefore, we cannot expect exact energy conservation. On the other hand this is a good cross check for the quality of the one loop approximation. For small coupling constant e^2 the numerical energy conservation has to be acceptable.

3.6.2 Perturbative Expansion

In order to renormalize the equation of motion and the energy density, we have to investigate the Coulomb mode function in more detail. The isoscalar Higgs field h and the transversal gauge field component a_\perp are gauge invariant from the outset. Therefore, their mode functions are independent of the chosen gauge and of the taken formalism. They are the same in the R_ξ -gauge, in the Hamiltonian approach, and of course also in the Coulomb gauge, and so we can use the results found in section 3.5.4 for the perturbative expansion of these fields. The equation for the Goldstone field has a different structure. Since we are only interested in the one loop approximation, we have to choose $\mathcal{R} = 0$ in (3.314). Therefore, we have to investigate the following equation for the Goldstone field φ

$$\ddot{\varphi} + \omega_{e\lambda}^2(t)\varphi + \frac{2e^2\dot{\phi}}{\omega_a^2} [\phi\dot{\varphi} - \dot{\phi}\varphi] = 0 . \quad (3.319)$$

First of all, we have to quantize the field φ and we have to introduce the mode representation. In order to quantize the field we have to find the conjugate momentum. We can read it off from the Lagrangian (3.305):

$$\Pi_\varphi = \dot{\varphi} - ea_0\phi = \dot{\varphi} \frac{k^2}{\omega_a^2} - \frac{e^2\phi\dot{\phi}}{\omega_a^2} \varphi . \quad (3.320)$$

The commutation relation for the field and its momentum is given by

$$[\varphi, \Pi_\varphi] = i\delta(\vec{x} - \vec{x}') . \quad (3.321)$$

By computation of the commutator for the field and its time derivative, we find that it is multiplied by a factor

$$[\varphi, \dot{\varphi}] \frac{k^2}{\omega_a^2} = i\delta(\vec{x} - \vec{x}') . \quad (3.322)$$

Now we can expand the field in terms of the mode functions and the corresponding annihilation and creation operators

$$\varphi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left(a_k U_\varphi(t) e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger U_\varphi^*(t) e^{-i\vec{k}\cdot\vec{x}} \right) . \quad (3.323)$$

In order to satisfy the commutator for φ and $\dot{\varphi}$, the commutator for a_k and a_k^\dagger differs from its common form

$$[a_k, a_k^\dagger] = 2i(2\pi)^3 \delta^3(\vec{k} - \vec{k}') \frac{1}{k^2} . \quad (3.324)$$

The expectation value for the field is then given by

$$\langle \varphi \varphi \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|U_\varphi|^2}{2\omega k^2} . \quad (3.325)$$

The mode function satisfies the differential equation

$$\ddot{U}_\varphi + \omega_{e\lambda}^2 U_\varphi + \frac{2e^2 \dot{\phi}}{\omega_a^2} (\phi \dot{U}_\varphi - \dot{\phi} U_\varphi) = 0 . \quad (3.326)$$

The frequency in the expectation value has to be fixed by the determination of the Wronskian which belongs to the mode equation. Since the form of the differential equation is not standard due to the first derivative of U_φ , the following transformation is efficient

$$U_\varphi = \omega_a \tilde{U}_\varphi . \quad (3.327)$$

The \dot{U}_φ -terms are cancelled and we find a new mode equation of the form

$$\ddot{\tilde{U}}_\varphi + \left(\omega_{e\lambda}^2 + \frac{3e^2 \dot{\phi}^2 k^2}{\omega_a^4} + \frac{e^2 \phi \ddot{\phi}}{\omega_a^2} \right) \tilde{U}_\varphi = 0 . \quad (3.328)$$

Now we redefine the frequency $\omega_{e\lambda}^2$ in order to find a suitable Wronskian. We choose

$$\tilde{\omega}_{e\lambda}^2 = \omega_{e\lambda}^2 + \frac{3e^2 \dot{\phi}^2 k^2}{\omega_a^4} + \frac{e^2 \phi \ddot{\phi}}{\omega_a^2} , \quad (3.329)$$

and get the mode function

$$\ddot{\tilde{U}}_\varphi + \tilde{\omega}_{e\lambda}^2 \tilde{U}_\varphi = 0 . \quad (3.330)$$

The Wronskian then has the well known form

$$\dot{\tilde{U}}_\varphi \tilde{U}_\varphi^* - \tilde{U}_\varphi \dot{\tilde{U}}_\varphi^* = C . \quad (3.331)$$

Since \tilde{U}_φ behaves like $e^{-i\tilde{\omega}_{e\lambda} t}$ we can fix the constant to be $C = -2i\tilde{\omega}_{e\lambda}$. For the initial time $\dot{\phi}$ and the fluctuation integral vanish so that $\tilde{\omega}_{e\lambda}^2$ simplifies to

$$\tilde{\omega}_{e\lambda}^2 = k^2 + e^2 \phi_0^2 + \lambda(\phi_0^2 - v^2) - \frac{e^2 \lambda}{\omega_{a0}^2} \phi_0^2 (\phi_0^2 - v^2) . \quad (3.332)$$

For high momenta, which are important for the UV-divergences, the last term is negligible. We discuss this point in more detail within the renormalization. By introducing $\tilde{\omega}_{\lambda e0}^2$ on the left hand side of the mode equation, we find

$$\ddot{\tilde{U}}_\varphi + \tilde{\omega}_{\lambda e0}^2 \tilde{U}_\varphi = -V_{e\lambda} \tilde{U}_\varphi - \frac{3e^2 \dot{\phi}^2 k^2}{\omega_a^4} \tilde{U}_\varphi - \frac{e^2 \phi \ddot{\phi}}{\omega_a^2} \tilde{U}_\varphi - \frac{e^2 \lambda}{\omega_{a0}^2} \phi_0^2 (\phi_0^2 - v^2) , \quad (3.333)$$

with

$$V_{e\lambda} = (e^2 + \lambda) (\phi^2 - \phi_0^2) . \quad (3.334)$$

As for the h - and a_\perp -channel in (3.216) we choose the following ansatz

$$\tilde{U}_\varphi(t) = e^{-i\tilde{\omega}_{e\lambda_0}t} [1 + f_\varphi(t)] , \quad (3.335)$$

and we find for the new truncated mode function the differential equation

$$\ddot{f}_\varphi - 2i\tilde{\omega}_{e\lambda_0}\dot{f}_\varphi = - \left[V_{e\lambda} + \frac{3e^2\dot{\phi}^2 k^2}{\omega_a^4} + \frac{e^2\phi\ddot{\phi}}{\omega_a^2} + \frac{e^2\lambda\phi_0^2}{\omega_{a0}^2} (\phi_0^2 - v^2) \right] (1 + f_\varphi) . \quad (3.336)$$

In the leading order we find

$$\ddot{f}_\varphi^{(1)} - 2i\tilde{\omega}_{e\lambda_0}\dot{f}_\varphi^{(1)} = -\mathcal{V}(\phi_t) , \quad (3.337)$$

or the equivalent integral representation for the function itself and its derivative

$$f_\varphi^{(1)} = - \int_0^t dt' \frac{1}{2i\tilde{\omega}_{e\lambda_0}} (e^{2i\tilde{\omega}_{e\lambda_0}\Delta t} - 1) \mathcal{V}(\phi_{t'}) , \quad (3.338)$$

$$\dot{f}_\varphi^{(1)} = - \int_0^t dt' e^{2i\tilde{\omega}_{e\lambda_0}\Delta t} \mathcal{V}(\phi_{t'}) , \quad (3.339)$$

with

$$\mathcal{V}(\phi_t) = V_{e\lambda}(t) + \frac{3e^2\dot{\phi}(t)^2 k^2}{\omega_a^4(t)} + \frac{e^2\phi(t)\ddot{\phi}(t)}{\omega_a^2(t)} + \frac{e^2\lambda\phi_0^2(\phi_0^2 - v^2)}{\omega_{a0}^2} . \quad (3.340)$$

Integration by parts allows us to extract the leading behavior of the truncated mode functions. We find

$$\text{Re}f_\varphi^{(1)} = -\frac{\mathcal{V}(\phi_t)}{4\tilde{\omega}_{e\lambda_0}^2} + \int_0^t dt' \frac{1}{4\tilde{\omega}_{e\lambda_0}^2} \cos(2\tilde{\omega}_{e\lambda_0}\Delta t) \dot{\mathcal{V}}(\phi_{t'}) , \quad (3.341)$$

$$\text{Im}f_\varphi^{(1)} = - \int_0^t dt' \frac{\mathcal{V}(\phi_{t'})}{2\tilde{\omega}_{e\lambda_0}} + \int_0^t dt' \frac{1}{2\tilde{\omega}_{e\lambda_0}} \cos(2\tilde{\omega}_{e\lambda_0}\Delta t) \mathcal{V}(\phi_{t'}) , \quad (3.342)$$

$$\text{Re}\dot{f}_\varphi^{(1)} = -\frac{\dot{\mathcal{V}}(\phi_t)}{4\tilde{\omega}_{e\lambda_0}^2} + \int_0^t dt' \frac{1}{4\tilde{\omega}_{e\lambda_0}} \cos(2\tilde{\omega}_{e\lambda_0}\Delta t) \ddot{\mathcal{V}}(\phi_{t'}) , \quad (3.343)$$

$$\text{Im}\dot{f}_\varphi^{(1)} = -\frac{\mathcal{V}(\phi_t)}{2\tilde{\omega}_{e\lambda_0}} + \int_0^t dt' \frac{1}{2\tilde{\omega}_{e\lambda_0}} \cos(2\tilde{\omega}_{e\lambda_0}\Delta t) \dot{\mathcal{V}}(\phi_{t'}) . \quad (3.344)$$

Now we can start to analyze the divergences in the fluctuation integrals and perform the renormalization.

3.6.3 Renormalization

In order to get a well defined equation of motion and energy density, we have to extract the divergences and introduce counter terms. As already mentioned, the isoscalar Higgs field h and the transversal gauge field component a_\perp fulfill the same mode equations as in the gauge invariant approach. Also, the structure of the fluctuation integrals are the same. Therefore, we can take the results from section 3.5.5.

Equation of Motion

The fluctuation integral for the transversal gauge mode is analyzed in detail in (3.250)-(3.252). The results for the isoscalar Higgs field are given in (3.253)-(3.255). The Goldstone channel looks slightly different from the gauge invariant approach, where we have worked with the field itself and its canonical momentum. Since we started in this section from the Lagrangian and not from the Hamiltonian we have only introduced the field. We have to consider the following part of the fluctuation integral

$$\begin{aligned} \mathcal{F}_\varphi = & (\lambda - e^2)\phi\langle\varphi^2\rangle - e^4\phi\dot{\phi}^2\left\langle\frac{\varphi^2}{\omega_a^4}\right\rangle + e^2\phi\left\langle\frac{2k^2 + e^2\phi^2}{\omega_a^4}\dot{\varphi}^2\right\rangle \\ & - 2e^2\dot{\phi}\left\langle\frac{k^2}{\omega_a^4}\varphi\dot{\varphi}\right\rangle. \end{aligned} \quad (3.345)$$

By introducing the mode functions \tilde{U}_φ , and rewriting the expectation values as momentum integrals we find

$$\begin{aligned} \mathcal{F}_\varphi = & (\lambda - e^2)\phi \int \frac{d^3k}{(2\pi)^3} \frac{\omega_a^2}{2k^2\tilde{\omega}_{e\lambda 0}} |\tilde{U}_\varphi|^2 - e^4\phi\dot{\phi}^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{U}_\varphi|^2}{2k^2\tilde{\omega}_{e\lambda 0}\omega_a^2} \\ & + e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{2k^2 + e^2\phi^2}{2k^2\tilde{\omega}_{e\lambda 0}\omega_a^4} \left[\frac{e^4\phi^2\dot{\phi}^2}{\omega_a^2} |\tilde{U}_\varphi|^2 + \omega_a^2 |\dot{\tilde{U}}_\varphi|^2 + e^2\phi\dot{\phi} \left(\tilde{U}_\varphi\dot{\tilde{U}}_\varphi^* + \tilde{U}_\varphi^*\dot{\tilde{U}}_\varphi \right) \right] \\ & - e^2\dot{\phi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda 0}\omega_a^4} \left[2e^2\phi\dot{\phi} |\tilde{U}_\varphi|^2 + \omega_a^2 \left(\tilde{U}_\varphi\dot{\tilde{U}}_\varphi^* + \tilde{U}_\varphi^*\dot{\tilde{U}}_\varphi \right) \right] \\ = & (\lambda - e^2)\phi \int \frac{d^3k}{(2\pi)^3} \frac{\omega_a^2}{2k^2\tilde{\omega}_{e\lambda 0}} |\tilde{U}_\varphi|^2 + e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{2k^2 + e^2\phi^2}{2k^2\tilde{\omega}_{e\lambda 0}\omega_a^2} |\dot{\tilde{U}}_\varphi|^2 \\ & - e^4\phi\dot{\phi}^2 \int \frac{d^3k}{(2\pi)^3} \frac{2\omega_a^2 + k^2}{2\omega_a^6\tilde{\omega}_{e\lambda 0}} |\tilde{U}_\varphi|^2 + e^2\dot{\phi} \int \frac{d^3k}{(2\pi)^3} \frac{e^2\phi^2\omega_a^2 - k^4}{2k^2\tilde{\omega}_{e\lambda 0}\omega_a^4} \left(\tilde{U}_\varphi\dot{\tilde{U}}_\varphi^* + \tilde{U}_\varphi^*\dot{\tilde{U}}_\varphi \right). \end{aligned} \quad (3.346)$$

The first and the second integral contain divergences while the rest of the expression is convergent and can be treated numerically. We can rewrite the integral in the following way

$$\begin{aligned} \mathcal{F}_\varphi = & (\lambda - e^2)\phi \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{2\tilde{\omega}_{e\lambda 0}} + \frac{e^2\phi^2}{2k^2\tilde{\omega}_{e\lambda 0}} - \frac{V_{e\lambda}}{4\tilde{\omega}_{e\lambda 0}^3} \right) \\ & + e^2\phi \int \frac{d^3k}{(2\pi)^3} \left(\frac{\tilde{\omega}_{e\lambda 0}}{k^2} - \frac{e^2\phi^2}{2k^2\tilde{\omega}_{e\lambda 0}} + \frac{V_{e\lambda}}{2k^2\tilde{\omega}_{e\lambda 0}} \right) \end{aligned}$$

$$\begin{aligned}
 & + \text{finite terms} \\
 = & \lambda\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda 0}} \left(1 + \frac{e^2\phi^2}{k^2} - \frac{V_{e\lambda}}{2\tilde{\omega}_{e\lambda 0}^2} \right) \\
 & + e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda 0}} \left(1 + \frac{2m_\varphi^2}{k^2} - \frac{V_{e\lambda}}{2\tilde{\omega}_{e\lambda 0}^2} \right) \\
 & + \mathcal{F}_\varphi^{\text{fin}} , \tag{3.347}
 \end{aligned}$$

with

$$\begin{aligned}
 \mathcal{F}_\varphi^{\text{fin}} = & e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{m_W^2(m_W^2 - \tilde{m}_{e\lambda 0}^2)}{2k^2\omega_a^2\tilde{\omega}_{e\lambda 0}} - e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{V_{e\lambda}\tilde{m}_{e\lambda 0}^2}{2k^2\tilde{\omega}_{e\lambda 0}^3} - e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{m_{W0}^2 m_{\varphi 0}^2}{k^2\omega_{a0}^2\tilde{\omega}_{e\lambda 0}} \\
 & + (\lambda - e^2)\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda 0}} \left[\frac{\omega_a^2}{k^2} (2\text{Re}f_\varphi^{(1)} + |f_\varphi^{(1)}|^2) + \frac{V_{e\lambda}}{2\tilde{\omega}_{e\lambda 0}^2} \right] \\
 & + e^2\phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^2\omega_a^2\tilde{\omega}_{e\lambda 0}} \left\{ (2k^2 + e^2\phi^2) [|f_\varphi^{(1)}|^2 - \tilde{\omega}_{e\lambda 0}^2 (2\text{Re}f_\varphi^{(1)} + |f_\varphi^{(1)}|^2)] - \omega_a^2 V_{e\lambda} \right\} \\
 & - e^4\phi\dot{\phi}^2 \int \frac{d^3k}{(2\pi)^3} \frac{2\omega_a^2 + k^2}{2\omega_a^6\tilde{\omega}_{e\lambda 0}} |\tilde{U}_\varphi|^2 \\
 & + e^2\dot{\phi}^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^2\phi^2\omega_a^2 - k^4}{2k^2\tilde{\omega}_{e\lambda 0}\omega_a^4} \left(\tilde{U}_\varphi \dot{U}_\varphi^* + \tilde{U}_\varphi^* \dot{U}_\varphi \right) . \tag{3.348}
 \end{aligned}$$

In order to regularize the divergent integrals we have to consider the frequency $\tilde{\omega}_{e\lambda 0}$ again. First, we find a quadratic divergence of the form

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda 0}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{e\lambda 0}} \frac{1}{\sqrt{1 - \frac{e^2\lambda\phi_0^2}{\omega_{e\lambda 0}^2\omega_{a0}^2}(\phi_0^2 - v^2)}} . \tag{3.349}$$

For vanishing momentum, the second term in the root simplifies to

$$\frac{e^2\lambda\phi_0^2}{\omega_{e\lambda 0}^2\omega_{a0}^2} (\phi_0^2 - v^2) \stackrel{k=0}{=} \frac{\lambda(\phi_0^2 - v^2)}{e^2\phi_0^2 + \lambda(\phi_0^2 - v^2)} < 1 , \tag{3.350}$$

and for $k > 0$ it is obvious that the term is less than one. Therefore, we can expand the root

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda 0}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{e\lambda 0}} \left[1 + \frac{e^2\lambda\phi_0^2}{2\omega_{e\lambda 0}^2\omega_{a0}^2} (\phi_0^2 - v^2) + \mathcal{O}(\omega^{-8}) \right] . \tag{3.351}$$

The first term can be regularized in the usual way. The second term is already finite. For the numerical calculations of the finite contributions it is simplest just to subtract the divergence from the integral in the form

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left(\frac{1}{\tilde{\omega}_{e\lambda 0}} - \frac{1}{\omega_{e\lambda 0}} \right) = \text{finite} . \tag{3.352}$$

In the same way we can treat the logarithmic divergences. Here we find

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2k^2 \tilde{\omega}_{e\lambda 0}} = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2k^2 \omega_{e\lambda 0}} \left[1 + \frac{e^2 \lambda \phi_0^2}{2\omega_{e\lambda 0}^2 \omega_{a0}^2} (\phi_0^2 - v^2) + \mathcal{O}(\omega^{-8}) \right], \quad (3.353)$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\tilde{\omega}_{e\lambda 0}^3} = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{e\lambda 0}^3} \left[1 + \frac{3e^2 \lambda \phi_0^2}{2\omega_{e\lambda 0}^2 \omega_{a0}^2} (\phi_0^2 - v^2) + \mathcal{O}(\omega^{-8}) \right]. \quad (3.354)$$

After regularizing the divergent integrals, the fluctuation integral reads

$$\begin{aligned} \mathcal{F}_\varphi &= -(e^2 - \lambda)^2 \phi^3 I_{-3}(m_{e\lambda}) + \frac{m_h^2}{2} (\lambda - e^2) \phi I_{-3}(m_{e\lambda}) - 2e^2 \ddot{\phi} I_{-3}(m_{e\lambda}) \\ &+ \Delta\lambda_\varphi \phi^3 - \frac{1}{2} \Delta m_\varphi \phi + \Delta Z + \mathcal{F}_\varphi^{\text{fin}} + \tilde{\mathcal{F}}_\varphi^{\text{fin}}, \end{aligned} \quad (3.355)$$

with

$$\Delta Z = 2e^2 C, \quad (3.356)$$

$$\Delta m_\varphi = -m_h^2 (\lambda - e^2) C + (e^2 + \lambda) \frac{m_{e\lambda 0}^2}{8\pi^2}, \quad (3.357)$$

$$\Delta\lambda_\varphi = -(e^2 - \lambda)^2 C, \quad (3.358)$$

$$\begin{aligned} \tilde{\mathcal{F}}_\varphi^{\text{fin}}(t) &= \lambda \phi(t) \int \frac{d^3 k}{(2\pi)^3} \left[\left(\frac{1}{2\tilde{\omega}_{e\lambda 0}} - \frac{1}{2\omega_{e\lambda 0}} \right) \frac{\omega_a^2(t)}{k^2} - \frac{V_{e\lambda}(t)}{4} \left(\frac{1}{\tilde{\omega}_{e\lambda 0}^3} - \frac{1}{\omega_{e\lambda 0}^3} \right) \right] \\ &+ e^2 \phi(t) \int \frac{d^3 k}{(2\pi)^3} \left[\left(\frac{1}{2\tilde{\omega}_{e\lambda 0}} - \frac{1}{2\omega_{e\lambda 0}} \right) \left(1 + \frac{2m_\varphi^2(t)}{k^2} \right) \right. \\ &\quad \left. - \frac{V_{e\lambda}(t)}{4} \left(\frac{1}{\tilde{\omega}_{e\lambda 0}^3} - \frac{1}{\omega_{e\lambda 0}^3} \right) \right]. \end{aligned} \quad (3.359)$$

The mass parameter in the divergent part $I_{-3}(m_{e\lambda})$ is in this case not a time dependent mass but independent of time and initial condition, i.e. $m_{e\lambda}^2 = m_h^2 + m_W^2 = 2\lambda v^2 + e^2 v^2$. The finite corrections to the wave function, the mass and the coupling constant are the same as in the gauge invariant approach as it should be. Also, the counter terms are exactly the same. This is clear since the UV divergences are due to the first two orders in the expansion of the one loop graphs. In the Hamiltonian approach, the problems in the IR region are caused by higher loop terms but in the one loop order we have taken the same parts in the Hamiltonian approach and in the Coulomb gauge. Therefore, the UV divergences must be equivalent in both approaches. For completeness, we give the full renormalized equation of motion with all fluctuation integrals

$$(1 + \Delta Z) \ddot{\phi} - \frac{1}{2} (m_h^2 + \Delta m_h^2) \phi + (\lambda + \Delta\lambda) \phi^3 + \mathcal{F}_\perp^{\text{fin}} + \mathcal{F}_h^{\text{fin}} + \mathcal{F}_\varphi^{\text{fin}} + \tilde{\mathcal{F}}_\varphi^{\text{fin}} = 0. \quad (3.360)$$

Energy Density

As in the equation of motion, the channel for the isoscalar Higgs field and for the transversal gauge field component have completely the same structure in the energy density as in

the gauge invariant approach since they are independent of the gauge fixing. Therefore, also here we need only to investigate the Goldstone channel. We have to show that the divergences of the energy are the same as those we have found in the equation of motion. We have to consider the following part of the fluctuation energy

$$\mathcal{E}_\varphi = \langle \frac{k^2}{2\omega_a^2} \dot{\varphi}^2 \rangle + \frac{1}{2} \langle \omega_\varphi \varphi^2 \rangle - e^2 \dot{\varphi}^2 \langle \frac{\varphi^2}{2\omega_a^2} \rangle + e^2 \phi \dot{\phi} \langle \frac{\varphi \dot{\varphi}}{\omega_a^2} \rangle. \quad (3.361)$$

Now we have again to introduce the modified mode function \tilde{U}_φ . The energy density then becomes

$$\begin{aligned} \mathcal{E}_\varphi &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\tilde{\omega}_{e\lambda 0} \omega_a^2} \left[\frac{e^4 \phi^2 \dot{\varphi}^2}{\omega_a^2} |\tilde{U}_\varphi|^2 + \omega_a^2 |\dot{\tilde{U}}_\varphi|^2 + e^2 \phi \dot{\phi} (\tilde{U}_\varphi \dot{\tilde{U}}_\varphi^* + \tilde{U}_\varphi^* \dot{\tilde{U}}_\varphi) \right] \\ &+ \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_a^2}{4k^2 \tilde{\omega}_{e\lambda 0}} \left(\omega_\varphi^2 - \frac{e^2 \dot{\varphi}^2}{\omega_a^2} \right) |\tilde{U}_\varphi|^2 \\ &+ e^2 \phi \dot{\phi} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4k^2 \tilde{\omega}_{e\lambda 0} \omega_a^2} \left[2e^2 \phi \dot{\phi} |\tilde{U}_\varphi|^2 + \omega_a^2 \left(\tilde{U}_\varphi \dot{\tilde{U}}_\varphi^* + \tilde{U}_\varphi^* \dot{\tilde{U}}_\varphi \right) \right] \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_a^2}{4k^2 \tilde{\omega}_{e\lambda 0}} \left(\omega_\varphi^2 - \frac{e^2 \dot{\varphi}^2}{\omega_a^2} \right) |\tilde{U}_\varphi|^2 + \int \frac{d^3 k}{(2\pi)^3} \frac{|\dot{\tilde{U}}_\varphi|^2}{4\tilde{\omega}_{e\lambda 0}} \\ &+ e^4 \phi^2 \dot{\varphi}^2 \int \frac{d^3 k}{(2\pi)^3} \frac{2\omega_a^2 + k^2}{4\tilde{\omega}_{e\lambda 0} k^2 \omega_a^4} |\tilde{U}_\varphi|^2 + e^2 \phi \dot{\phi} \int \frac{d^3 k}{(2\pi)^3} \frac{2k^2 + e^2 \phi^2}{4k^2 \tilde{\omega}_{e\lambda 0} \omega_a^2} \left(\tilde{U}_\varphi \dot{\tilde{U}}_\varphi^* + \tilde{U}_\varphi^* \dot{\tilde{U}}_\varphi \right). \end{aligned} \quad (3.362)$$

As in the fluctuation integral in the equation of motion also here the two first integrals are divergent. By inserting the expansion for the mode function we can extract the divergences explicitly. We find in detail

$$\begin{aligned} \mathcal{E}_\varphi &= -e^2 \dot{\varphi}^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4k^2 \tilde{\omega}_{e\lambda 0}} \\ &+ \int \frac{d^3 k}{(2\pi)^3} \left[\frac{\tilde{\omega}_{e\lambda 0}}{2} + \frac{V_{e\lambda}}{4\tilde{\omega}_{e\lambda 0}} + \frac{m_a^2 m_\varphi^2}{4k^2 \tilde{\omega}_{e\lambda 0}} - \frac{V_{e\lambda}^2}{16\tilde{\omega}_{e\lambda 0}^3} + \frac{m_{\varphi 0}^2 m_{W 0}^2}{4\omega_{a 0}^2 \tilde{\omega}_{e\lambda 0}} \right] + \mathcal{E}_\varphi^{\text{fin}}, \end{aligned} \quad (3.363)$$

with

$$\begin{aligned} \mathcal{E}_\varphi^{\text{fin}} &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\tilde{\omega}_{e\lambda 0}} \left(\frac{m_W^2(t) m_\varphi^2(t)}{k^2} + \frac{m_{W 0}^2 m_{\varphi 0}^2}{\omega_{a 0}^2} \right) \left(2\text{Re} f_\varphi^{(1)} + |f_\varphi^{(1)}|^2 \right) \\ &+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\tilde{\omega}_{e\lambda 0}} \left(|f_\varphi|^2 - \frac{V_{e\lambda}^2}{4\tilde{\omega}_{e\lambda}^2} \right) \\ &+ \int \frac{d^3 k}{(2\pi)^3} \frac{V_{e\lambda}}{4\tilde{\omega}_{e\lambda 0}} \left(2\text{Re} f_\varphi + |f_\varphi|^2 + \frac{V_{e\lambda}}{2\tilde{\omega}_{e\lambda 0}^2} \right) \\ &- e^2 \dot{\varphi}^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\tilde{\omega}_{e\lambda 0} k^2} \left(2\text{Re} f_\varphi^{(1)} + |f_\varphi^{(1)}|^2 \right) \\ &+ e^4 \phi^2 \dot{\varphi}^2 \int \frac{d^3 k}{(2\pi)^3} \frac{2\omega_a^2 + k^2}{4\tilde{\omega}_{e\lambda 0} k^2 \omega_a^4} |\tilde{U}_\varphi|^2 \end{aligned}$$

$$+e^2 \phi \dot{\phi} \int \frac{d^3 k}{(2\pi)^3} \frac{2k^2 + e^2 \phi^2}{4k^2 \tilde{\omega}_{e\lambda 0} \omega_a^2} \left(\tilde{U}_\varphi \dot{U}_\varphi^* + \tilde{U}_\varphi^* \dot{U}_\varphi \right). \quad (3.364)$$

Now we have in addition to the quadratic and logarithmic divergences a quartic one. As for the divergent integrals in the equation of motion we can expand the frequency $\tilde{\omega}_{e\lambda 0}$ in order to get the standard form of the integral for the regularization. We find

$$\int \frac{d^3 k}{(2\pi)^3} \frac{\tilde{\omega}_{e\lambda 0}}{2} = \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_{e\lambda 0}}{2} \left[1 - \frac{e^2 \lambda \phi_0^2}{\omega_{a0}^2 \omega_{e\lambda 0}^2} (\phi_0^2 - v^2) + \mathcal{O}(\omega^{-8}) \right]. \quad (3.365)$$

The first term which is quartically divergent has to be regularized and to be absorbed in the cosmological constant. The second term is logarithmically divergent and, therefore, it must cancel with another contribution in the energy fluctuation integral. The last term in the second momentum integral in (3.363) has the same form with opposite sign. This term arose by introducing $\tilde{\omega}_{e\lambda 0}$ for $\omega_{e\lambda 0}$. After expanding $\tilde{\omega}_{e\lambda 0}$ in the denominator, the divergent part has the right form to cancel the upper divergences and we only get a finite contribution from the higher order expansion terms. Finally, the energy density for the Goldstone part becomes

$$\begin{aligned} \mathcal{E}_\varphi = & -e^2 \dot{\phi}^2 I_{-3}(m_{e\lambda}) - \frac{m_h^4}{4} I_{-3}(m_{e\lambda}) \\ & - \frac{1}{4} (\lambda - e^2)^2 \phi^4 I_{-3}(m_{e\lambda}) + \frac{m_h^2}{4} (\lambda - e^2) \phi^2 I_{-3}(m_{e\lambda}) \\ & + \frac{1}{2} \Delta Z + \frac{1}{4} \Delta \lambda_\varphi \phi^4 - \frac{1}{4} \Delta m_h \phi^2 + \Delta \Lambda_\varphi + \Delta \Lambda'_\varphi + \mathcal{E}_\varphi^{\text{fin}} + \tilde{\mathcal{E}}_\varphi^{\text{fin}}, \end{aligned} \quad (3.366)$$

with

$$\Delta \Lambda_\varphi = \frac{m_h^4}{16} C, \quad (3.367)$$

$$\Delta \Lambda'_\varphi = \frac{1}{128 \pi^2} (m_{W0}^2 + m_{\varphi 0}^2)^2, \quad (3.368)$$

$$\begin{aligned} \tilde{\mathcal{E}}_\varphi^{\text{fin}} = & \int \frac{d^3 k}{(2\pi)^3} \left[\frac{\tilde{\omega}_{e\lambda 0}}{2} - \frac{\omega_{e\lambda 0}}{2} + \frac{m_{\varphi 0}^2 m_{W0}^2}{4 \tilde{\omega}_{e\lambda 0} \omega_{a0}^2} \right. \\ & + \frac{1}{4} \left(V_{e\lambda} + \frac{m_a^2 m_\varphi^2}{k^2} - \frac{e^2 \dot{\phi}^2}{k^2} \right) \left(\frac{1}{\tilde{\omega}_{e\lambda 0}} - \frac{1}{\omega_{e\lambda 0}} \right) \\ & \left. - \frac{V_{e\lambda}^2}{16} \left(\frac{1}{\tilde{\omega}_{e\lambda 0}^3} - \frac{1}{\omega_{e\lambda 0}^3} \right) \right]. \end{aligned} \quad (3.369)$$

The second divergence in (3.363) leads together with a contribution from the isoscalar Higgs field to the cosmological constant

$$\delta \Lambda = \frac{m_h^4}{8} I_{-3}(m_h). \quad (3.370)$$

The results for the divergences of the energy density are in agreement with the counter terms we have found for the equation of motion as well as the finite terms. The finite

contribution to the cosmological constant is the same as in the gauge invariant approach, only the part which depends on the initial condition Λ'_φ looks slightly different. This is due to the changing in the perturbative expansion and the different structure of the frequencies in the quartic divergence. Since the mode functions are not the same this is no surprise.

3.7 Comparison of the Different Approaches

At this point we can summarize the comparison of the different gauges and approaches. In the R_ξ -gauges we have found, by a suitable transformation of the mode function, the Coulomb mode (3.142). It is exactly the same as we have found in the Coulomb gauge after eliminating the field a_0 and by neglecting all terms higher than one loop order (3.319). In the gauge invariant approach the Goldstone channel was described by a combination of two first order differential equations for the field itself (3.210) and its canonical momentum (3.211). Now we show that this approach also leads in the one loop order to the same equations as in the Coulomb gauge. If we differentiate (3.210) with respect to t and use the classical equation of motion $\dot{\phi} = \Pi$, we find a second order differential equation of the form

$$\ddot{U}_\varphi + \frac{e^2}{k^2} (\dot{\phi}^2 + \phi\ddot{\phi}) U_\varphi + \frac{e^2}{k^2} \phi\dot{\phi}\dot{U}_\varphi - \frac{2e^2\phi\dot{\phi}}{k^2} U_{\pi_\varphi} - \frac{\omega_a^2}{k^2} \dot{U}_{\pi_\varphi} = 0 . \quad (3.371)$$

With the relations for U_{π_φ} and \dot{U}_{π_φ}

$$U_{\pi_\varphi} = \frac{k^2}{\omega_a^2} \dot{U}_\varphi + \frac{e^2\phi\dot{\phi}}{\omega_a^2} U_\varphi , \quad (3.372)$$

$$\dot{U}_{\pi_\varphi} = \frac{e^2\phi\dot{\phi}}{\omega_a^2} \dot{U}_\varphi + \frac{e^4\phi^2\dot{\phi}^2}{k^2\omega_a^2} U_\varphi - \omega_\varphi^2 U_\varphi - \frac{e^2}{k^2} \dot{\phi}^2 U_\varphi , \quad (3.373)$$

and again with the classical equation of motion now in the form $\ddot{\phi} = -\lambda\phi(\phi^2 - v^2)$, it is straightforward to show

$$\mathcal{M}_{\varphi\varphi} U_\varphi = 0 , \quad (3.374)$$

where $\mathcal{M}_{\varphi\varphi}$ is given by (3.142). By inserting the classical equation of motion without the fluctuation part, we suppress the higher loop terms and therefore get rid of the IR problem. Therefore, we have shown that if the classical equation of motion is fulfilled, the Goldstone mode in the R_ξ -gauge, in the Coulomb gauge, and in the Hamiltonian approach has the same fluctuation operator. We also want to compare the equation of motion for the zero mode ϕ in the Coulomb gauge and in the Hamiltonian approach. In order to shorten the notation and make a comparison easier we also write here the fluctuation integrals as expectation values and do not worry about the normalization. By taking the time derivative of the zero mode equation (3.201), we find:

$$\ddot{\phi} = \dot{\Pi} + \dot{\Pi} e^2 \left\langle \frac{\varphi^2}{k^2} \right\rangle + 2e^2 \Pi \left\langle \frac{\varphi\dot{\phi}}{k^2} \right\rangle - e^2 \dot{\phi} \left\langle \frac{\varphi\pi_\varphi}{k^2} \right\rangle - e^2 \phi \left[\left\langle \frac{\dot{\varphi}\pi_\varphi}{k^2} \right\rangle + \left\langle \frac{\varphi\dot{\pi}_\varphi}{k^2} \right\rangle \right] . \quad (3.375)$$

Now we insert the differential equation for the conjugate momentum $\dot{\Pi}$ (3.202). Since we are only interested in the one loop order we can use the classical equation of motion $\dot{\phi} = \Pi$ without fluctuations and neglect terms of higher orders arising from the product of fluctuation integrals. Then the linearized field equation reads as a second order differential equation

$$\begin{aligned} \ddot{\phi} = -U'(\phi) & - \phi \left[2e^2 \langle a_{\perp}^2 \rangle + 3\lambda \langle h^2 \rangle + \lambda \langle \varphi^2 \rangle + e^2 \langle \frac{\pi_{\varphi}^2}{k^2} \rangle \right] \\ & - e^2 U'(\phi) \langle \varphi^2 k^2 \rangle + 2e^2 \dot{\phi} \langle \frac{\varphi \dot{\phi}}{k^2} \rangle - e^2 \phi \left[\langle \frac{\dot{\phi} \pi_{\varphi}}{k^2} \rangle + \langle \varphi \dot{\pi}_{\varphi} k^2 \rangle \right]. \end{aligned} \quad (3.376)$$

By using (3.372) and (3.373) for the conjugate momentum of the fluctuation field φ , we get the same result as in (3.316) if we choose $\mathcal{R}(t) = 0$.

3.8 Numerics

In order to investigate the influence of the gauge field sector on the zero mode in a system out of equilibrium, we have carried out some numerical examples. We are interested in two different aspects: first we investigate the influence of the different gauges. Therefore, we choose $\xi = 1$ in the R_{ξ} -gauge which leads to the 't Hooft-Feynman background gauge and we compare the results with those we find in the Coulomb gauge. Secondly, we investigate the effect of the φ -mode in the Coulomb gauge and the effect of the $a_0\varphi$ -channel in the 't Hooft-Feynman gauge. For this purpose we consider also the case where only the transversal gauge components and the isoscalar Higgs field act on the zero mode. For the 't Hooft-Feynman gauge we summarize the results for the equation of motion, the mode functions, and the energy density which we have published in [38]. For more details, especially in view of the renormalization, the reader is referred to our paper.

3.8.1 The 't Hooft-Feynman Gauge

Starting point for our considerations is the Lagrangian (3.31), (3.32) with the choice $\xi = 1$. We can derive the equation of motion for the zero mode as

$$\begin{aligned} \ddot{\phi} + \lambda \phi (\phi^2 - v^2) & + 3\lambda \phi \langle h^2 \rangle + \frac{3}{4} g^2 \phi \langle a_{\perp}^2 \rangle \\ & + 3 \left(\lambda + \frac{g^2}{4} \right) \phi \langle \varphi^2 \rangle - \frac{3}{4} g^2 \phi \langle a_0^2 \rangle - \frac{3}{2} g \partial_t \langle a_0 \varphi \rangle = 0. \end{aligned} \quad (3.377)$$

In this notation we have not taken care of the normalization and the renormalization or the different solutions for the mode functions of the coupled channel. We only want to give an overview of the equations and the connections of the fields and do not go into technical details. As already mentioned, the mode functions for the isoscalar Higgs field and the transversal gauge field are the same as in the Coulomb gauge (3.208), (3.209). For

the coupled sector $a_0\varphi$, they are given by the fluctuation operator (3.125). Obviously, the structure of the operator simplifies for $\xi = 1$. The longitudinal gauge component decouples from the system, and we have to consider only a 2×2 system of the form

$$\begin{Bmatrix} -\partial_t^2 - \omega_a^2(t) & g\dot{\phi}(t) \\ g\dot{\phi}(t) & \partial_t^2 + \omega_{e\lambda}^2(t) \end{Bmatrix} \begin{Bmatrix} a_0(t) \\ \varphi(t) \end{Bmatrix} = 0. \quad (3.378)$$

The operator has two interesting features which distinguishes it from the single modes. The indefinite metric of the time component of the gauge field, and the time derivative of the zero mode in the off diagonal elements which connect the two fields. We find analogous properties in the Coulomb gauge; the fluctuation part for a_0 contributes with a negative sign to the fluctuation integral (3.310), and we find time derivatives of the zero mode in the equation of motion for the zero mode (3.316) as well as in the mode function for φ (3.319).

In the Feynman gauge, the mode functions for the transversal gauge field, for the longitudinal gauge field, and for the ghost fields are the same. Two of the three gauge components are cancelled by the ghost fields and only one degree of freedom is left in contrast to the Coulomb gauge where we have two transverse gauge components. The factor three in front of the fluctuation integrals for a_\perp , a_0 and φ reflects the degeneracy factor which is due to the non-Abelian structure of the model. The energy density can be derived by integration of the equation of motion for the zero mode or from the corresponding Hamiltonian of the system. It reads (see also [38])

$$\begin{aligned} \mathcal{E} = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 &+ \frac{1}{2}[\langle \dot{h}^2 \rangle + \langle \omega_h^2 h^2 \rangle] \\ &+ \frac{3}{2}[\langle \dot{a}_\perp^2 \rangle + \langle \omega_a^2 a_\perp^2 \rangle] \\ &+ \frac{3}{2}[\langle \dot{\varphi}^2 \rangle + \langle \omega_{e\lambda}^2 \varphi^2 \rangle] \\ &- \frac{3}{2}[\langle \dot{a}_0^2 \rangle + \langle \omega_a^2 a_0^2 \rangle]. \end{aligned} \quad (3.379)$$

This expression looks very similar to the Coulomb energy (3.317) despite the fact that a_0 is dynamical and contributes with a derivate part. Again the degeneracy factors appear.

3.8.2 Results

For our numerical calculations, we have chosen four different sets of parameters listed in Table 3.1. The initial value for the zero mode ϕ and the Higgs mass m_h are the same for all sets. They are chosen in such a way that the zero mode evolves in the right minimum of the potential. With this choice of initial conditions, the zero mode can not evolve into the complex part of the effective potential. In this region, the instabilities increase dramatically and the one loop approximation breaks down as explained for the ϕ^4 theory in subsection 2.5.2.

In order to give an impression of the potential we have plotted in Fig. 3.1 for the Coulomb gauge and Fig. 3.2 for the Feynman gauge the potential energy versus the zero mode as the shape of the figures (solid lines) and also the zero mode part of the energy versus ϕ (dashed lines). The field begins to roll down the potential but the energy is not high enough for the field to reach the maximum at zero. Therefore, it starts to oscillate in the minimum. The prediction of these two plots in the context of non-equilibrium dynamics is not clear. The shape of the potential is more or less equivalent to the effective potential because we have taken the finite terms concerning from the renormalization into account, but the effective potential is an equilibrium quantity, because the expectation value of the scalar field, that serves as order parameter, is space time independent. Nevertheless, they are instructive to get an idea of the potential by which the zero mode is influenced.

For our numerical considerations we have only varied the coupling constants λ and e (or g) and therefore the masses of the different fields. We have also given the initial masses of the three different fields in Table 3.1.

For the first parameter set, we have chosen the same coupling constant for the Higgs field and the gauge field. The initial masses for the fields are all small but not zero. Since we have taken the initial value for the zero mode to be small, the effect of the quantum fluctuations is negligible. The behavior of the zero mode is the same in the Coulomb gauge, in the 't Hooft-Feynman gauge, and for the $a_{\perp}h$ -system. We have displayed it in Fig. 3.3–Fig.3.5.

The situation changes drastically, if we choose a smaller gauge coupling and therefore a nearly vanishing Goldstone mass. We have plotted the zero mode in Fig. 3.6 for the Coulomb gauge and in Fig. 3.7 for the 't Hooft-Feynman gauge. The field is strongly damped and settles down to the minimum. Since this effect does not occur for the pure $a_{\perp}h$ -system, it is obvious that it is induced by the Goldstone sector. The effect is stronger in the non-Abelian model which is caused by the additional factors three in front of the fluctuation integrals. In Figs. 3.9 and 3.10 we have displayed the fluctuation integrals for the different components for both gauges. The Goldstone fluctuation is obviously the dominant one.

In the third parameter set we have chosen a nearly vanishing mass for the isoscalar Higgs-field component. The field is damped also in this case for all three systems Fig. 3.11–Fig. 3.13, but not as strong as for the second parameter set. In the last case we have considered vanishing initial masses for all fields. The results are very similar to those found for the third parameter set, the field is damped but it oscillates forever as shown in Fig. 3.14–Fig. 3.16.

In order to check our numerics we have plotted the energy density for the Coulomb gauge in Fig. 3.17, 3.19 and the 't Hooft Feynman gauge in Fig. 3.18, 3.20 for the first and the second parameter set. The upper line displays the fluctuation energy which increases and the lower line the zero mode part of the energy which decreases. The solid line shows the total energy. For convenience we have added in all cases a constant to the zero mode part of the energy. Otherwise the curves are only straight lines due to their distance. For the first parameter set, the energy conservation is excellent in both cases. Since the energy transfer is negligible this result is to be expected. We have also chosen the second

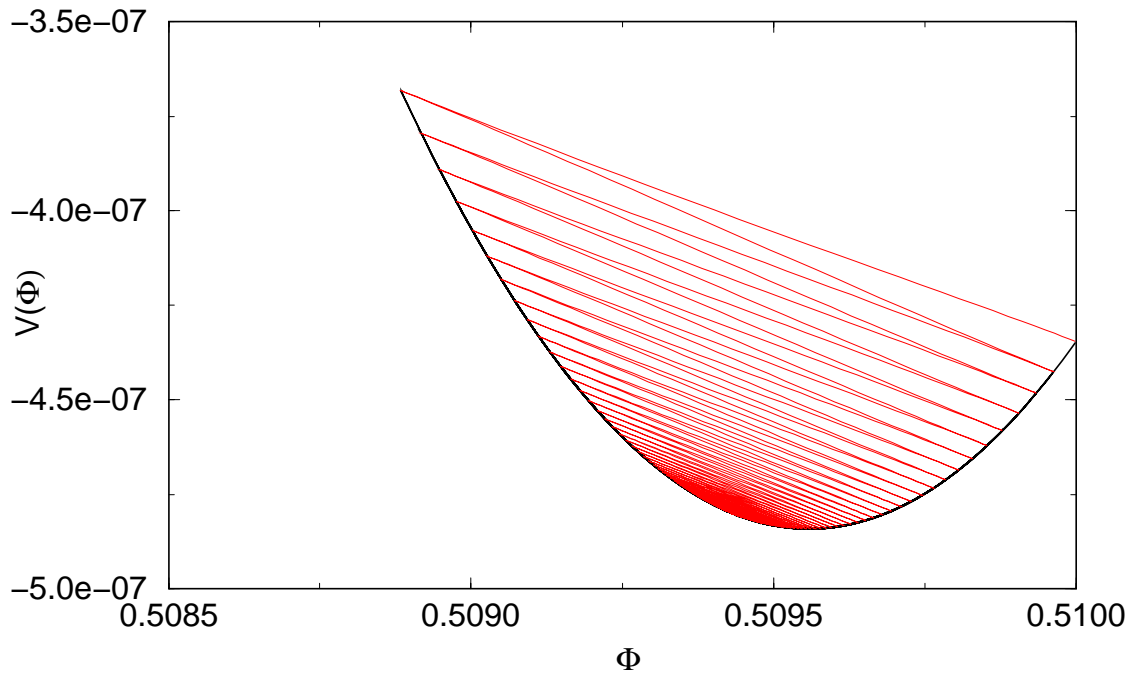
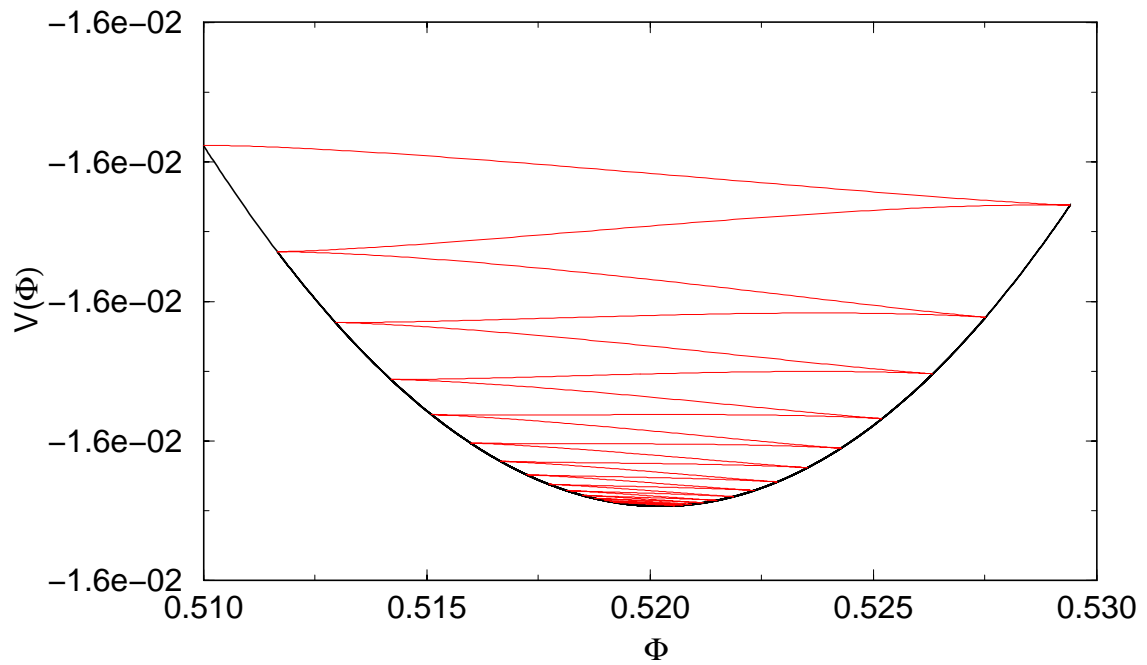
	λ	$e = \frac{g}{2}$	m_h^2	ϕ_0	m_{a0}^2	m_{h0}^2	$m_{e\lambda 0}^2$
Parameter set 1	1	1	0.5	0.51	0.26	0.53	0.27
Parameter set 2	1	0.1	0.5	0.51	$2.6 \cdot 10^{-3}$	0.78	0.01
Parameter set 3	0.33	1.3	0.5	0.51	0.44	$7.5 \cdot 10^{-3}$	0.28
Parameter set 4	0.33	0.8	0.5	0.51	0.17	$7.5 \cdot 10^{-3}$	$2.3 \cdot 10^{-3}$

Table 3.1: Parameter sets for the gauge theories

parameter set because the behavior of the field is in this case more extraordinary than in the others. For the Coulomb gauge the total energy oscillates at the beginning a little bit. We have already explained in section 3.6.1 that we do not have to expect a perfect result because we cannot insert the classical equation of motion numerically, which we have done when we have shown energy conservation by taking the time derivative of the energy density.

Summarizing the results, we have found that the damping effect is strongest for a nearly massless Goldstone field and a massive isoscalar Higgs field. In this case the isoscalar Higgs field has the possibility to decay into the other fields. We have found an analogous behavior in the ϕ^4 theory in the large N limit. There, the damping of zero mode was due to the massless Goldstone bosons. In gauge theories a special feature of the Goldstone mode is the occurrence of the time derivative of the zero mode. We find such a time derivative as well in the vertex which couples the Goldstone mode to the other fields as in the mode function itself in both gauges. We found a similar phenomenon in the ϕ^4 theory with fermionic fluctuations [35]. There, we noticed a catalyzing effect of the fermions to a scalar field. The fermionic field alone has not influenced the zero mode very much and also the scalar field alone has not shown a remarkable effect. But the two fields together have damped the zero mode enormously. In this model, we have chosen a massless fermion and the mode function of the fermion contains a derivative term of the zero mode.

We have also found that the behavior for the different gauges is qualitatively the same and that the Goldstone channel plays an important role.

Figure 3.1: Potential versus ϕ in the Coulomb gauge for parameter set 2Figure 3.2: Potential versus ϕ in the 't Hooft-Feynman gauge for parameter set 2

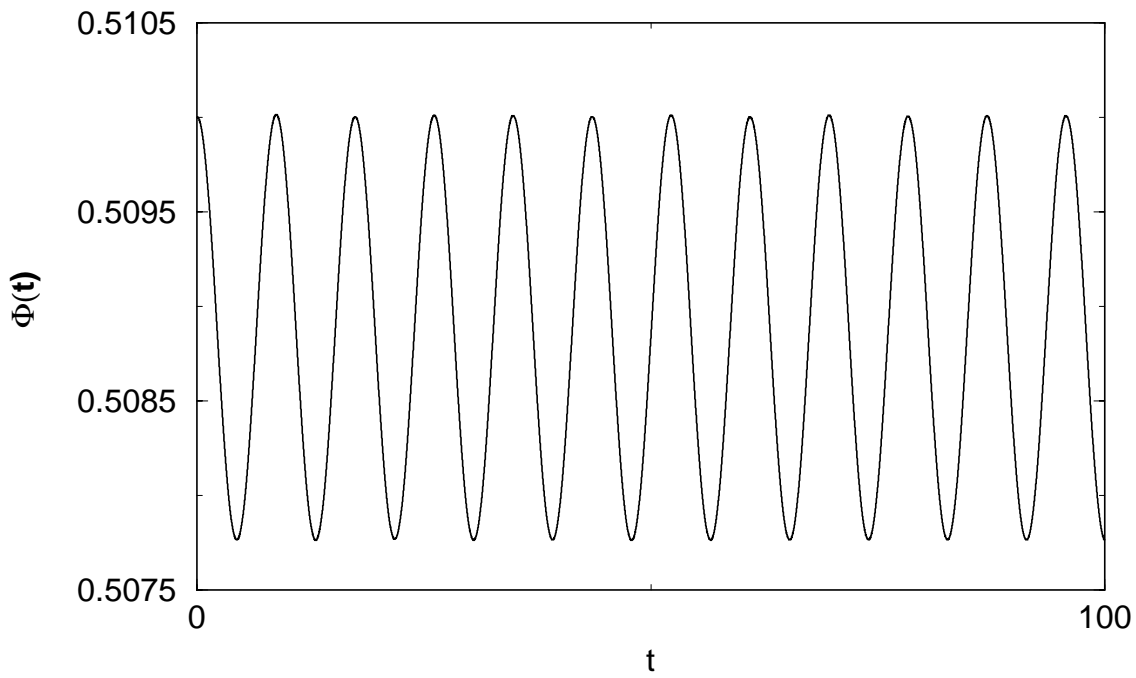


Figure 3.3: Zero mode versus t in the Coulomb gauge for parameter set 1

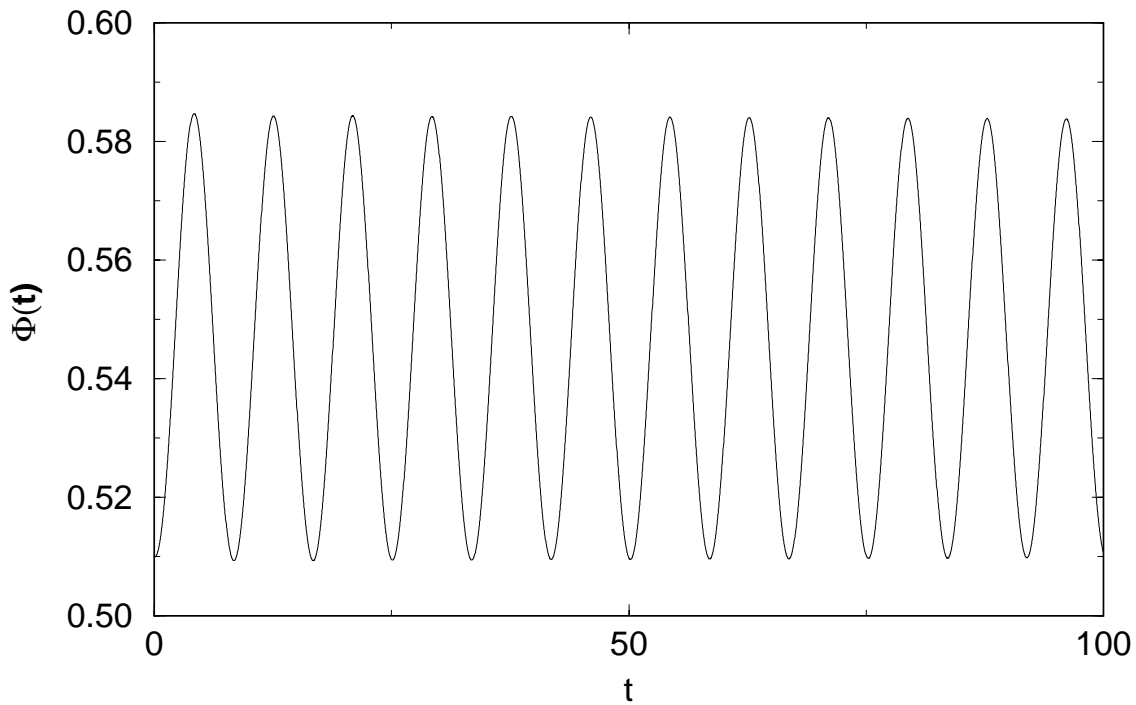


Figure 3.4: Zero mode versus t in the 't Hooft-Feynman gauge for parameter set 1

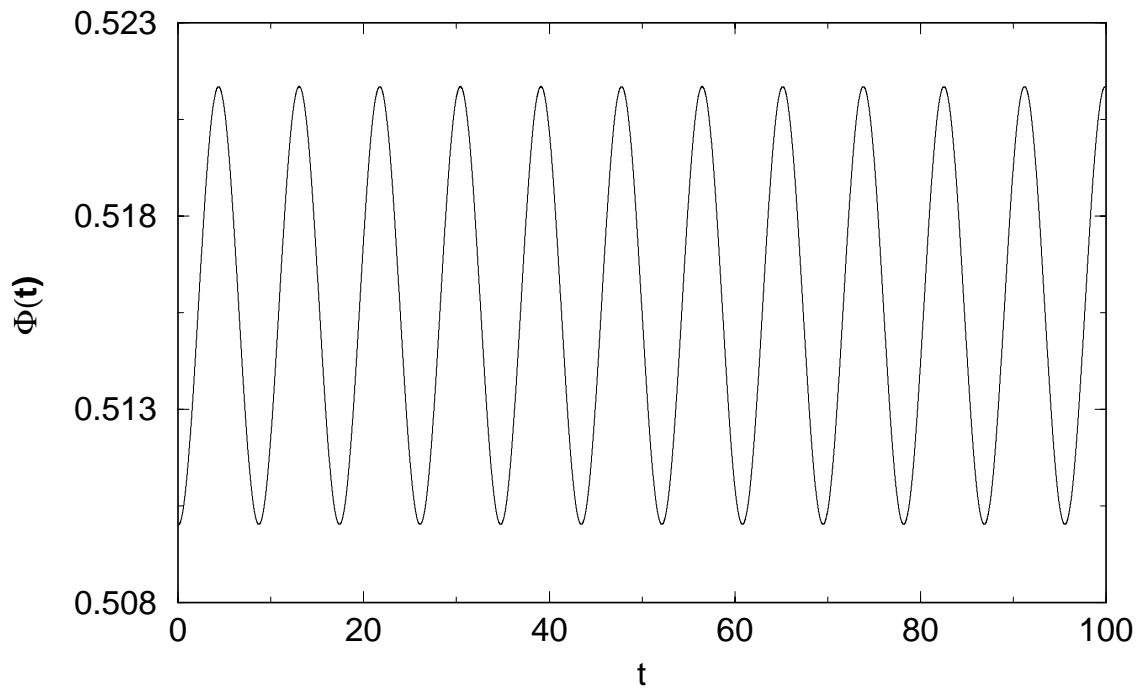


Figure 3.5: Zero mode versus t under influence of the h and a_{\perp} for parameter set 1

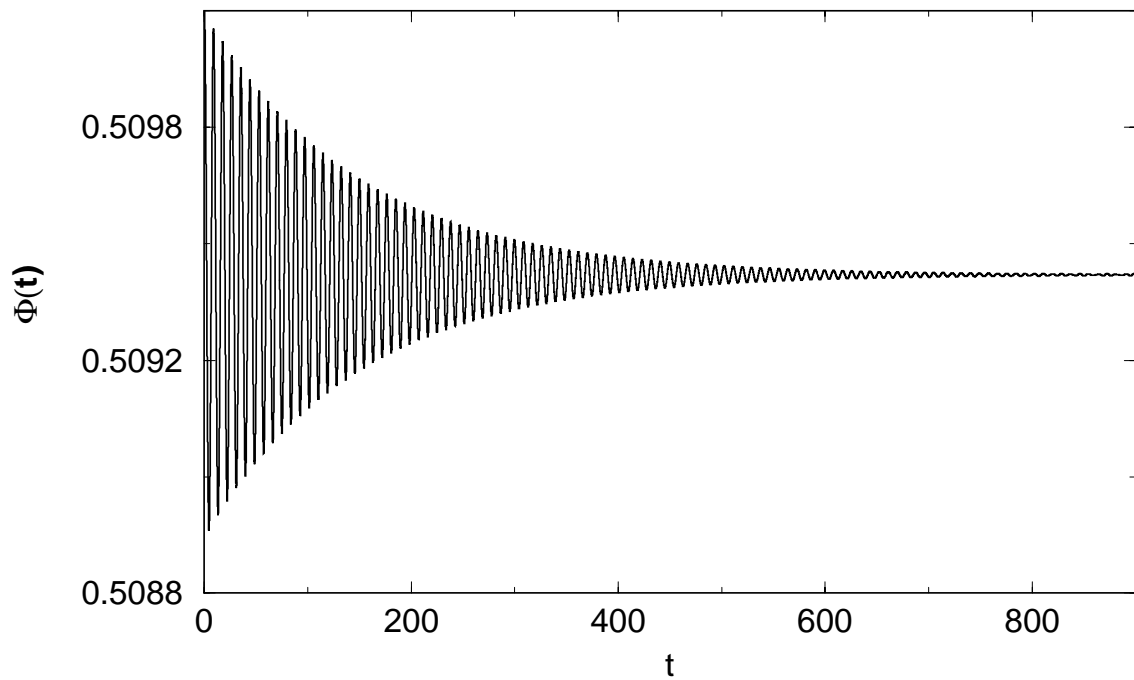


Figure 3.6: Zero mode versus t in the Coulomb gauge for parameter set 2

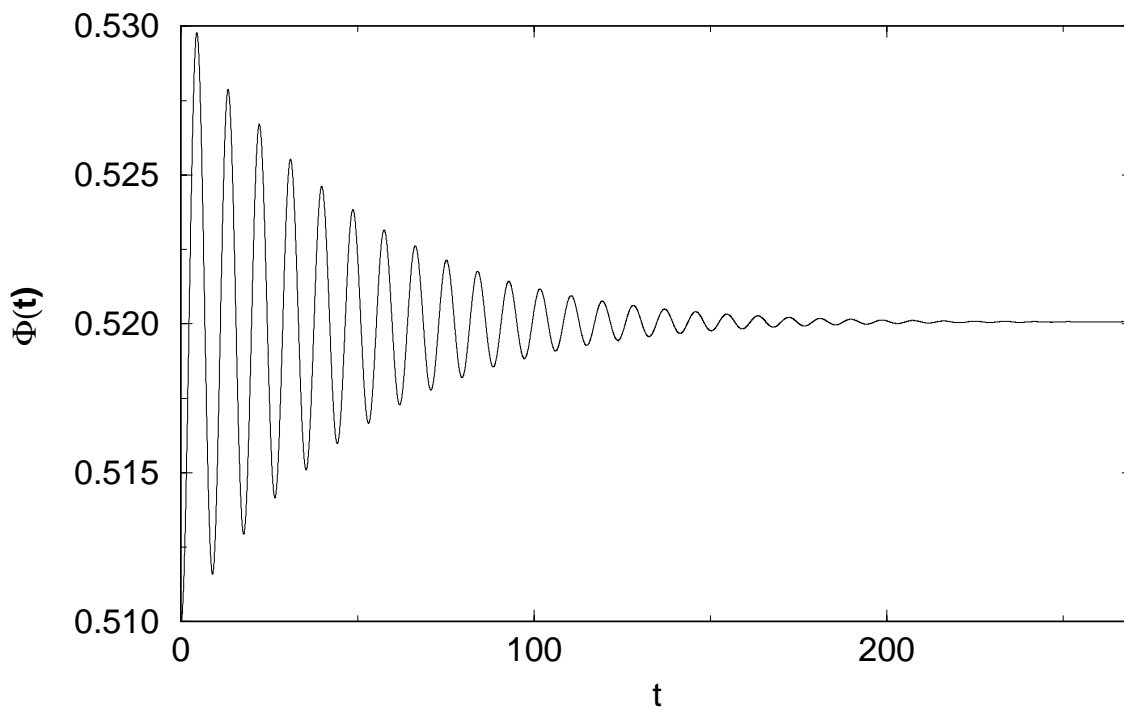


Figure 3.7: Zero mode versus t in the 't Hooft-Feynman gauge for parameter set 2

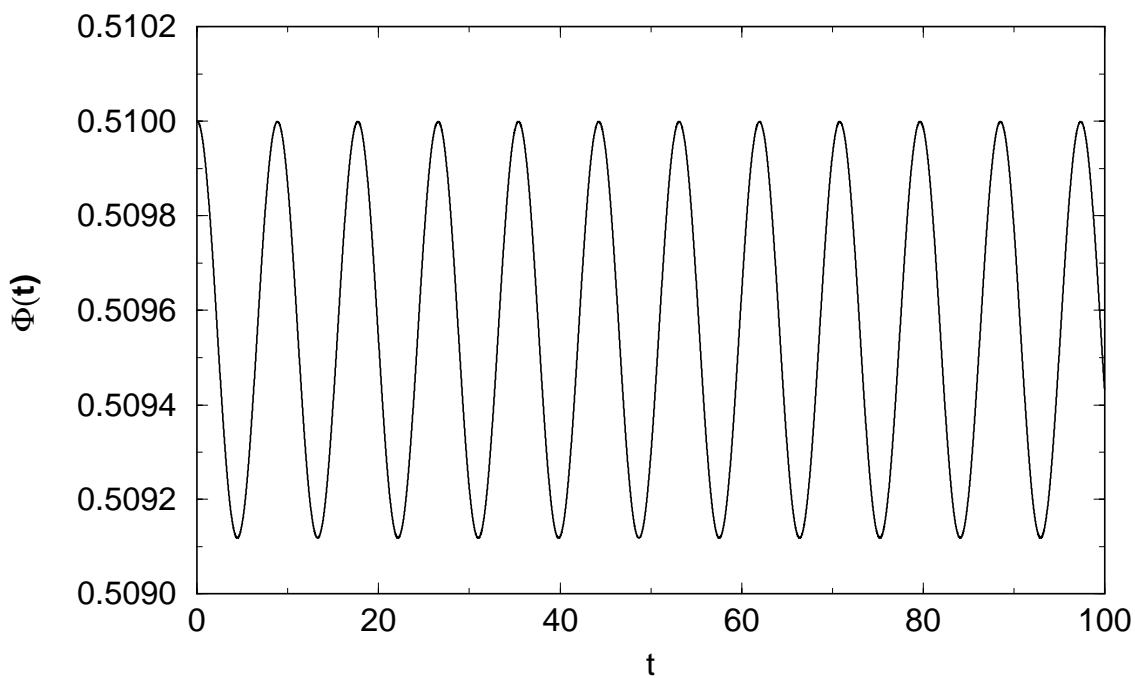


Figure 3.8: Zero mode versus t under influence of the h and a_{\perp} for parameter set 2

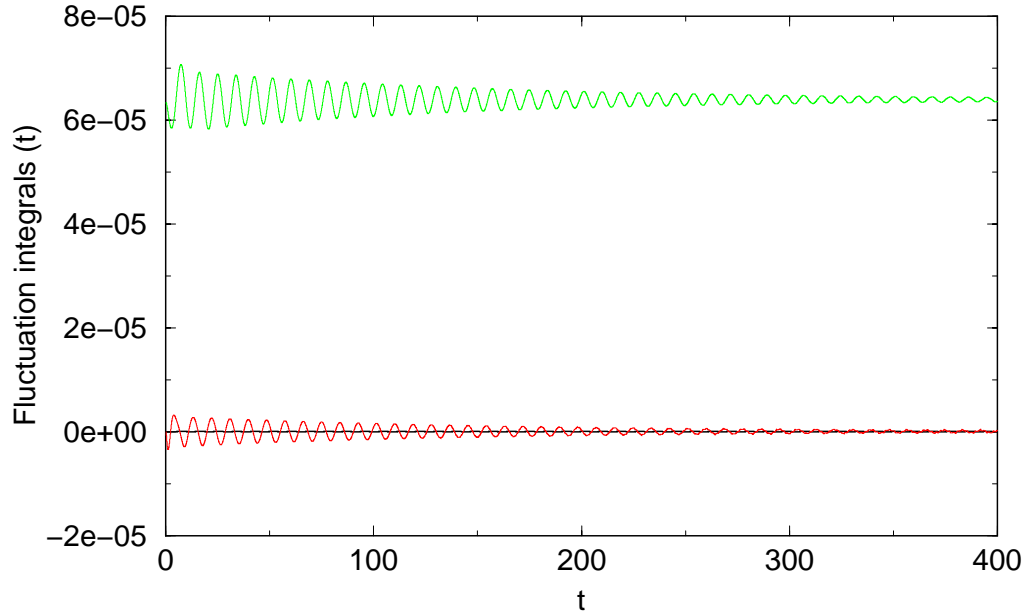


Figure 3.9: Fluctuation integrals versus t in the Coulomb gauge for parameter set 2, solid line: a_{\perp} fluctuations, dashed line: h fluctuations, dotted line: φ fluctuations

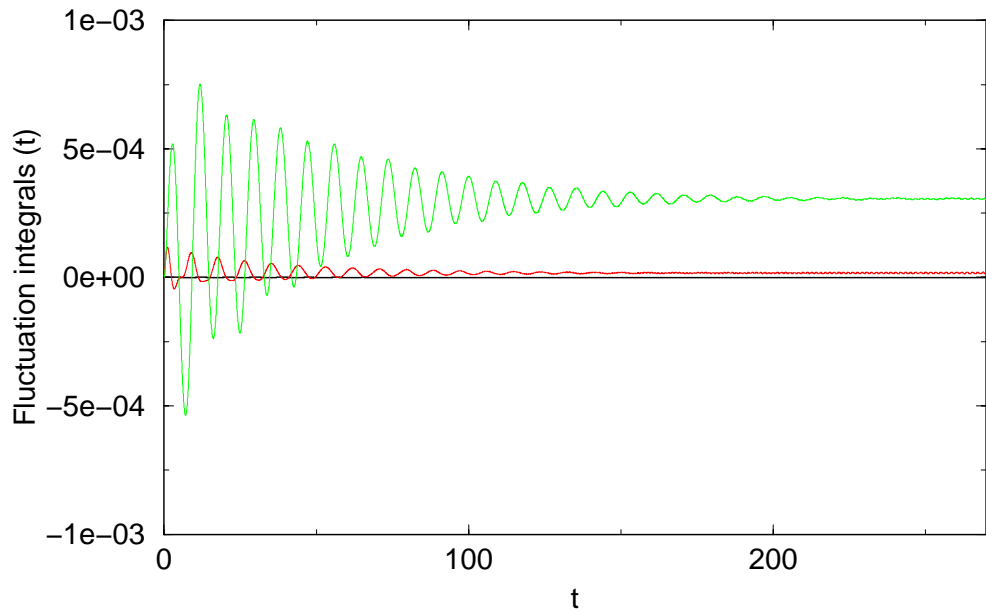


Figure 3.10: Same as in Fig. 3.9 in the 't Hooft-Feynman gauge

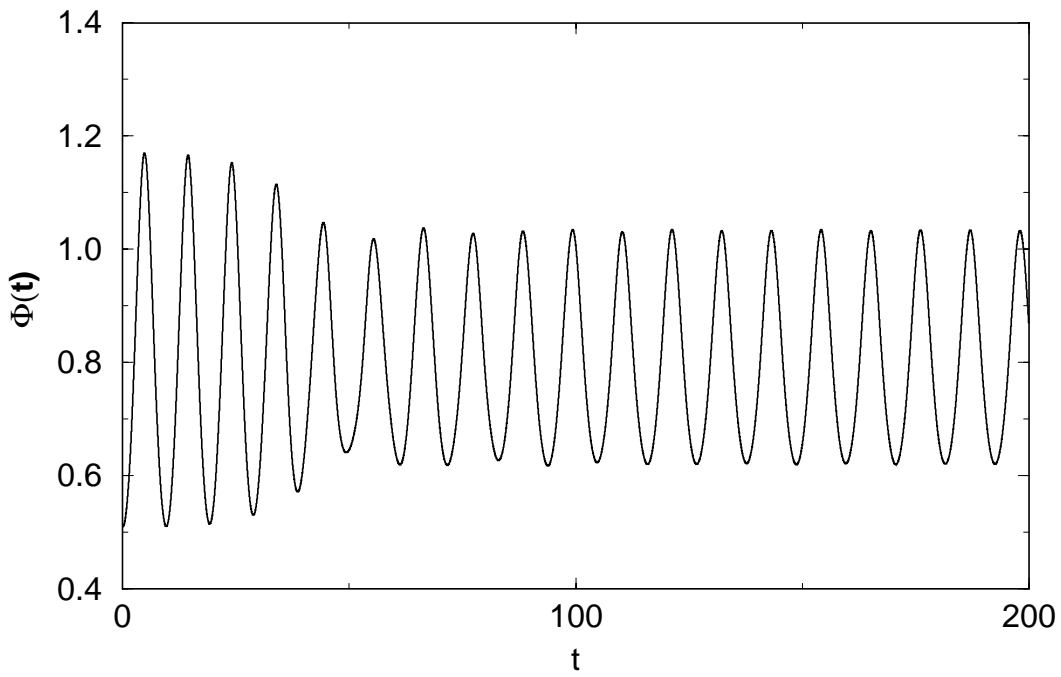


Figure 3.11: Zero mode versus t in the Coulomb gauge for parameter set 3

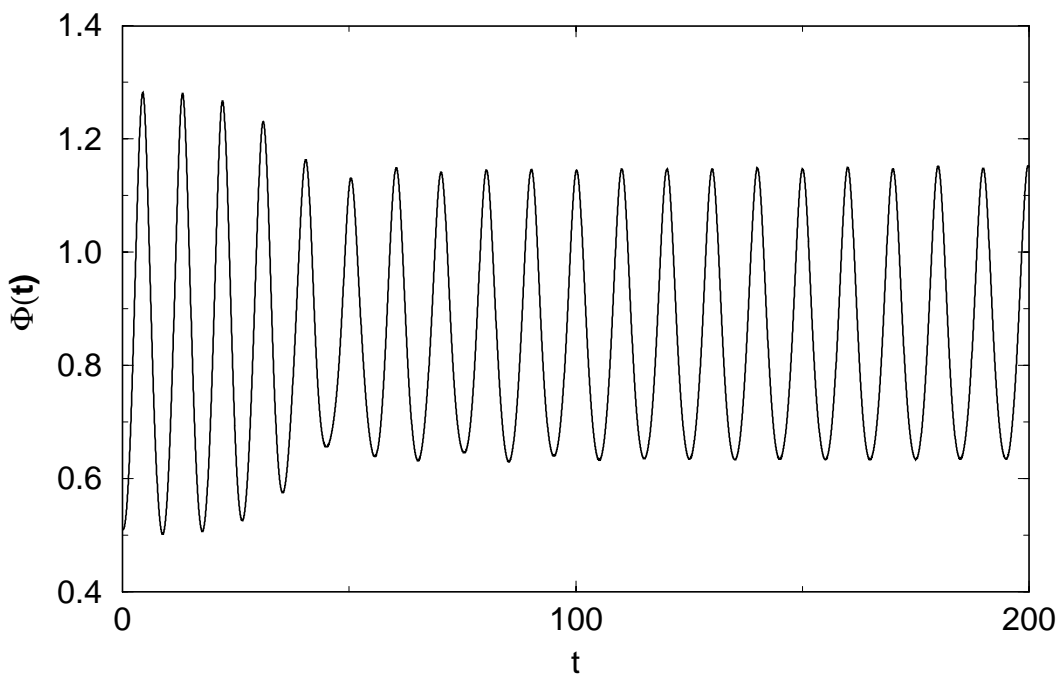


Figure 3.12: Zero mode versus t in the 't Hooft-Feynman gauge for parameter set 3

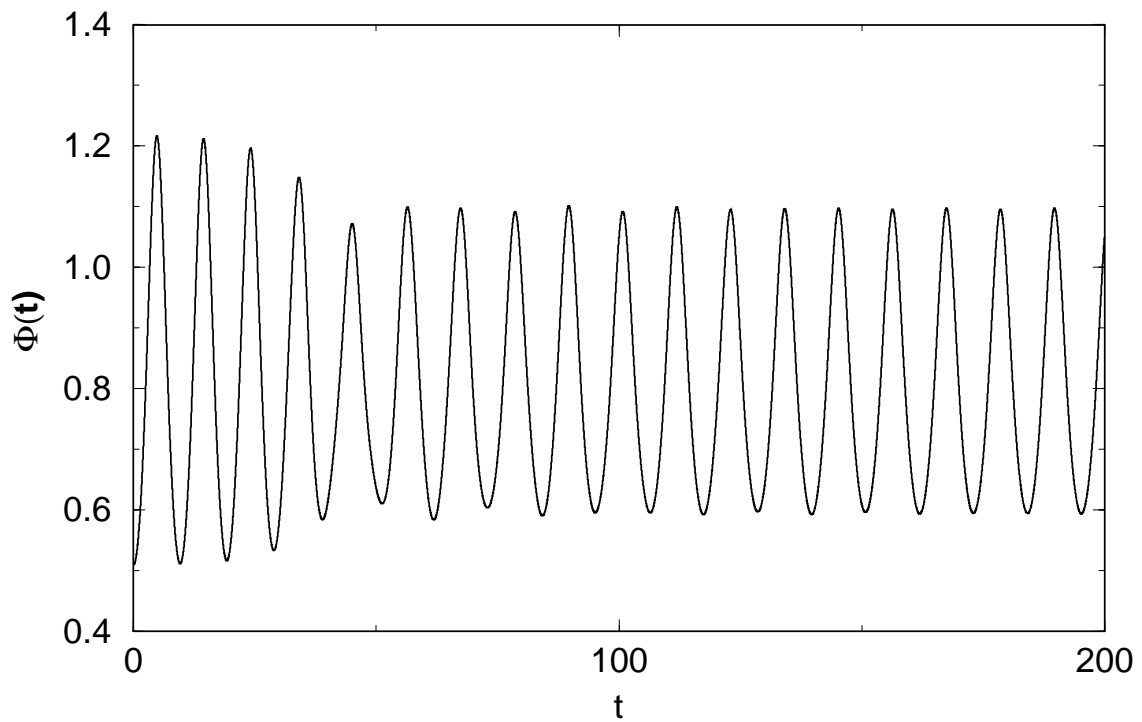


Figure 3.13: Zero mode versus t under influence of the h and a_{\perp} for parameter set 3

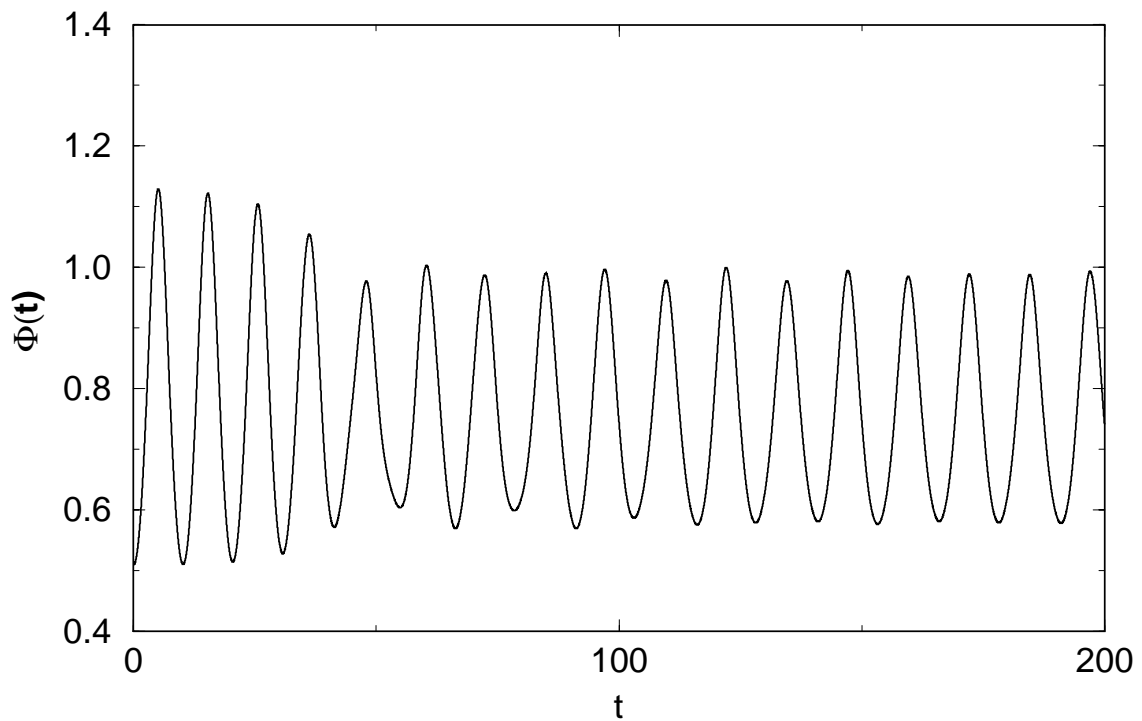


Figure 3.14: Zero mode versus t in the Coulomb gauge for parameter set 4

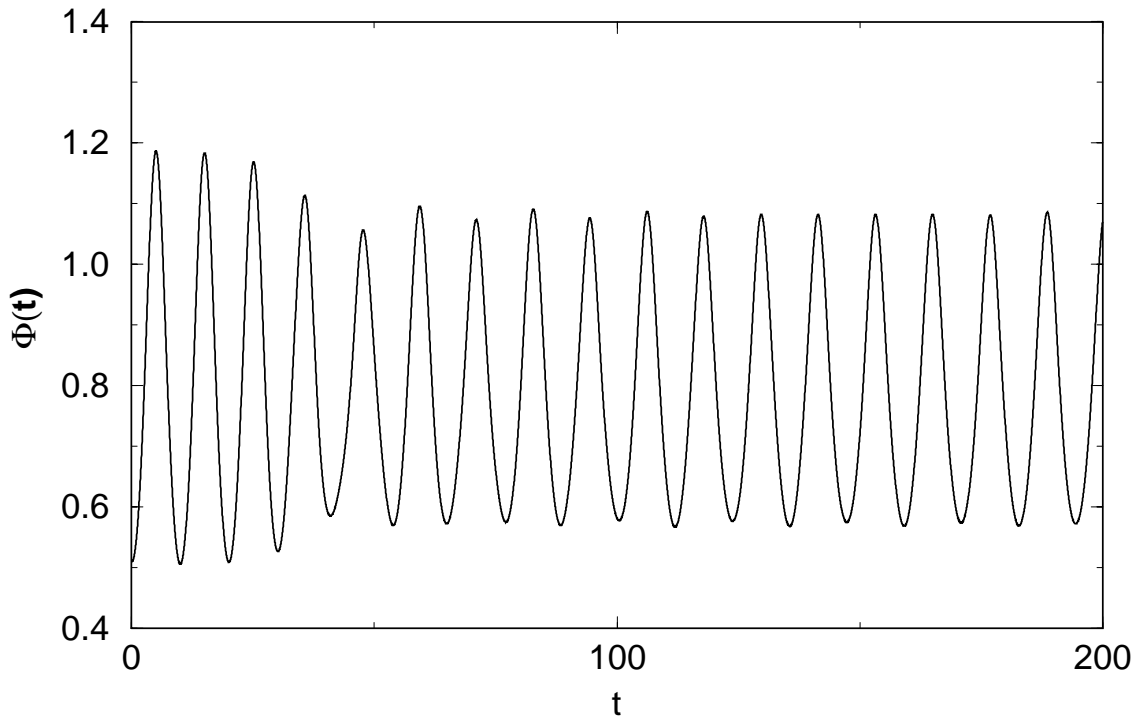


Figure 3.15: Zero mode versus t in the 't Hooft-Feynman gauge for parameter set 4

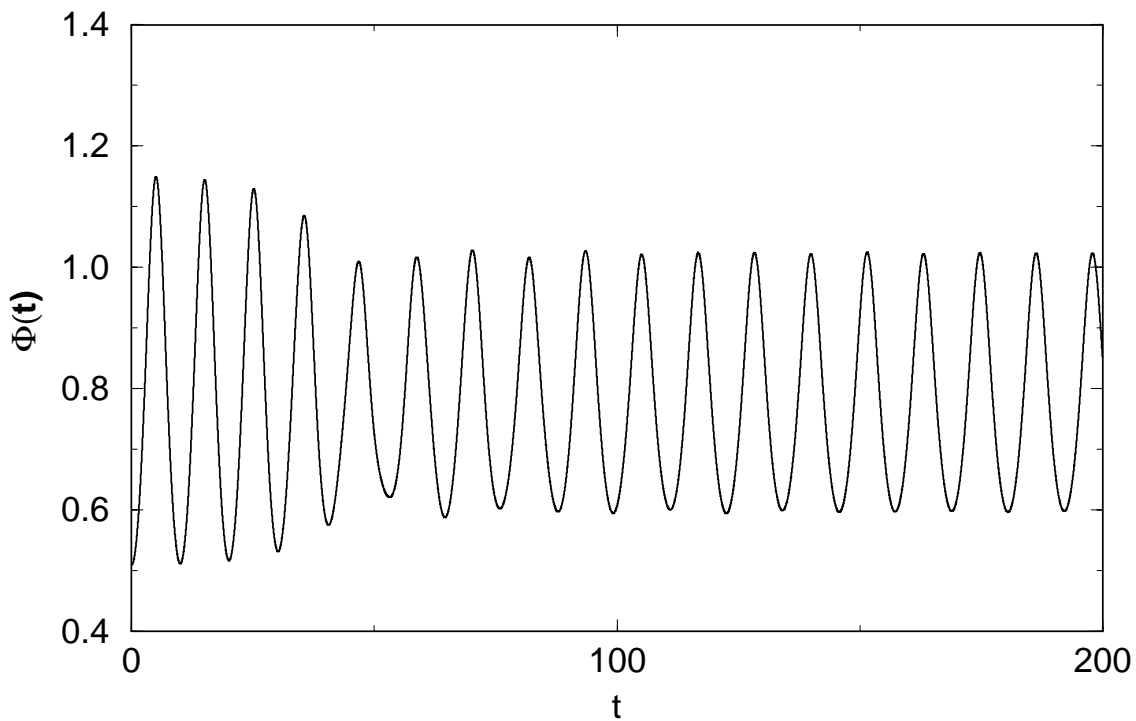


Figure 3.16: Zero mode versus t under influence of the h and a_{\perp} for parameter set 4

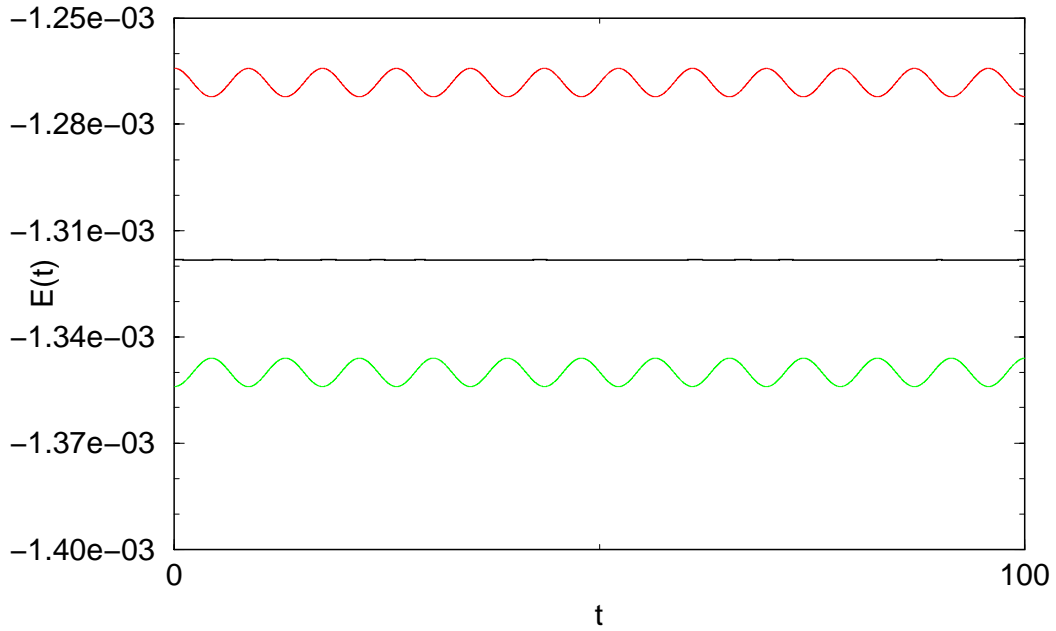


Figure 3.17: Mode energies and their sum versus t in the Coulomb gauge for parameter set 1

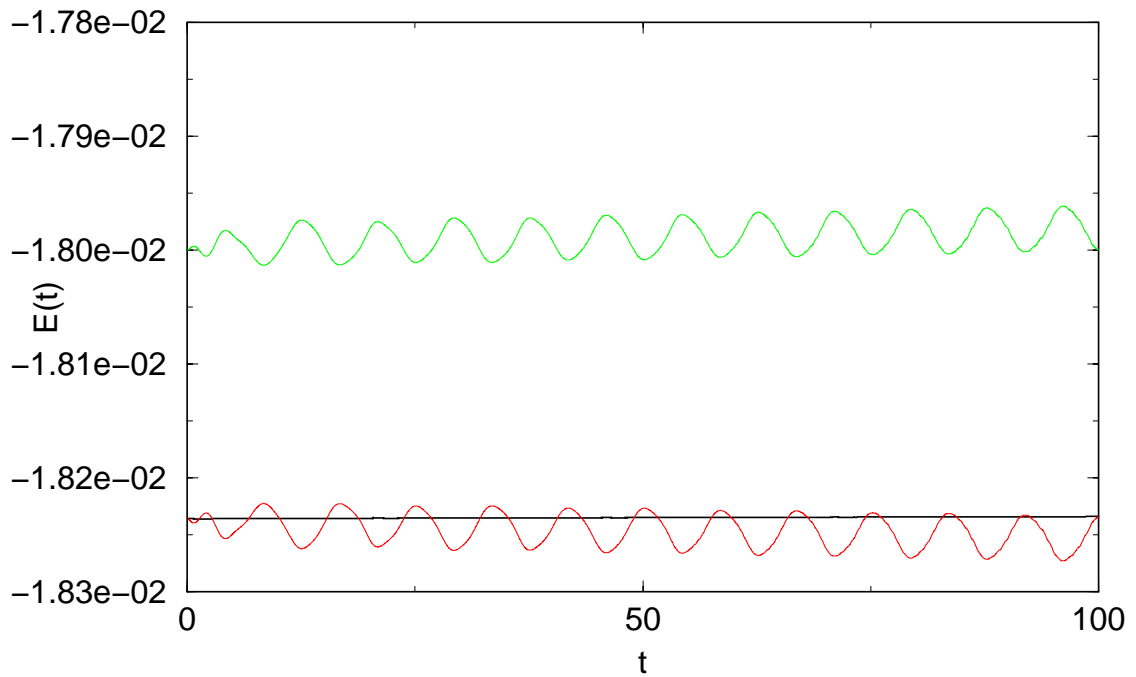


Figure 3.18: Mode energies and their sum versus t in the 't Hooft-Feynman gauge for parameter set 1

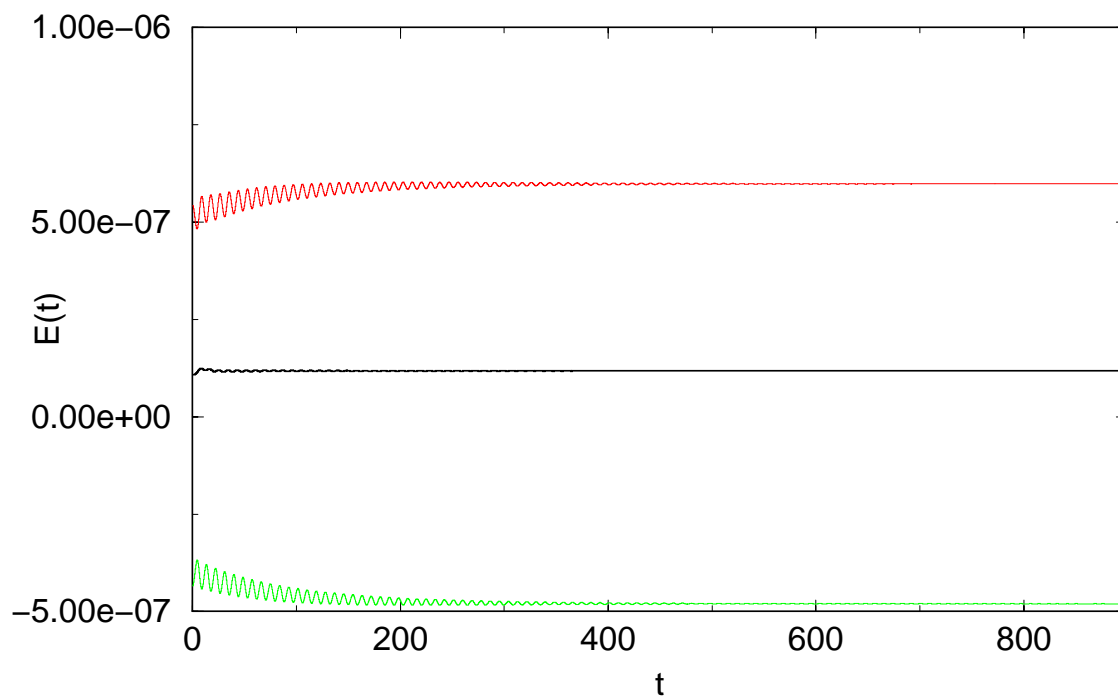


Figure 3.19: Mode energies and their sum versus t in the Coulomb gauge for parameter set 2

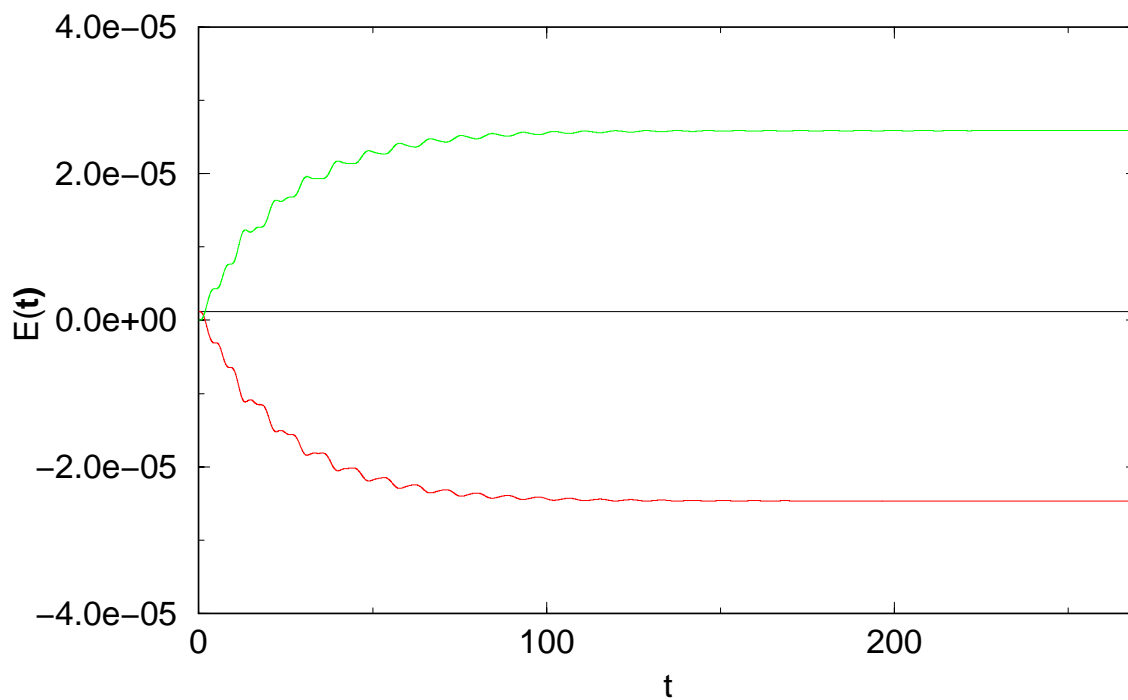


Figure 3.20: Mode energies and their sum versus t in the 't Hooft-Feynman gauge for parameter set 2

Chapter 4

Conclusion and Outlook

In this work we have studied the non-equilibrium dynamics of different models with different approximation schemes under various aspects. Thereto, we have used a scheme for the renormalization which we have developed in [33] for a system out of equilibrium. It is based on a perturbative expansion of the mode functions and allows a clean separation between divergent and finite parts. The advantages of this approach are manifold. Since we have extract explicit expressions for the divergent parts we were able to regularize them with different regularization schemes, such as dimensional or Pauli-Villars regularization. The identification of the divergences with the common Feynman graphs was possible and therefore, we have found the standard counter terms for the renormalization. This was cross check for our analytical considerations. The finite parts were treated numerically. As we have shown this scheme works for different approximations and models.

Our investigations have started with the analysis of the ϕ^4 theory with spontaneous symmetry breaking. We have studied a large N model at finite temperature and carried out the renormalization. After formulating finite and well defined equations we have implemented them numerically. By investigation of the evolution of the system we find, depending on the initial conditions, final states with restored $O(N)$ symmetry and final states for which the symmetry is spontaneously broken. The resulting phase diagrams resemble typical phase diagrams of thermodynamical systems with the temperature and an external parameter, the initial value ϕ_0 , as parameters. We have generalized two empirical formulae found by Boyanovsky et al. [29] to finite temperature, which relate the initial and asymptotic value of the field and the time dependent mass squared. We have also shown some numerical examples for the unbroken theory and investigated the behavior of various physical quantities like the particle number and the pressure which are important in connection with inflationary cosmology. The particle number leads for example to predictions concerning the reheating temperature and the computation of the pressure is necessary if we want to analyse a model in expanding space time. Furthermore, we studied the one loop approximation in order to compare the results with the large N approximation. For very low initial values of the zero mode which lead to a development in the minimum of the effective potential we have found an analogous behavior of the field as in the case of unbroken symmetry but for initial values lower than the maximum of the potential with the

possibility to reach the unstable region of the potential the approximation breaks down, because the mode functions grow exponentially. This is due to the lack of quantum back reaction onto the fluctuations. The fact that we have found such a behavior may, however, indicate the correct physics and is not necessarily a consequence of an inadequate approximation. It is known that the system is indeed unstable for spatially constant static fields, it is an instability with respect to the formation of domains [46]. For space dependent fields like minimal bubble configurations the one loop approximation for the effective action does not show any unphysical features [91, 93, 100], though the effective potential is complex in the unstable region. Hence, it is not clear if the large N approximation necessarily improves the understanding of the physics.

In the second part we have extensively studied gauge theories out of equilibrium. Here, the aim of the work was twofold. We have investigated different aspects of gauge invariance and furthermore, studied the behavior of gauge systems out of equilibrium numerically. We started our considerations with an analysis of the R_ξ -gauges. As a first example we have examined a time independent problem, the bubble nucleation. We have found a way to transform the fluctuation operator in a triangular form whereas two diagonal elements were the same as the Faddeev Popov fluctuation operator and one was ξ -independent, which led to a gauge parameter independent effective action. The final conclusion of our consideration was that the exact one loop correction to the nucleation rate is gauge independent. This goes beyond the results of Weinberg and Metaxas [83], where a similar statement was derived for the leading orders in the gauge coupling, using the gradient expansion. In the non-equilibrium case we have also shown that with an equivalent analysis for the mode functions as for the bubble nucleation it is possible to construct a fluctuation operator in a triangular form. Two of the diagonal elements are cancelled by the Faddeev-Popov ghosts and the third one is independent of ξ . We found that this mode is equivalent to the Goldstone mode in the Coulomb gauge. Furthermore, we have examined the structure of the divergences in the R_ξ -gauge by computing the leading Feynman diagrams with the CTP-formalism. The gauge dependence of the counter terms we have found vanishes if the classical equation of motion is fulfilled.

Another approach we have studied is a gauge invariant formulation of the Abelian Higgs model developed by Boyanovsky et al. [37]. We have extended their calculation to a complete set of equations, which describes the evolution of the zero mode under the influence of gauge and Higgs fluctuations. We have performed the renormalization and found some problems induced by the inclusion of terms higher than one loop order. Since these terms were included in an uncontrollable manner, we have computed a linearized form of the equations. They are equivalent to the Coulomb gauge fixed theory in the one loop approximation. In order to implement these equations numerically, we have also renormalized the Coulomb gauge fixed theory.

The numerical simulations have shown that the behavior of the zero mode in the Coulomb gauge is similar to the 't Hooft-Feynman gauge as a special case of the R_ξ -gauges. We have found that the Goldstone channel for small Goldstone masses led to an efficient damping of the zero mode. This effect is due to the possibility that the Higgs field can decay into the Goldstone field. As in the one loop approximation in the ϕ^4 theory, the

evolution of the zero mode in the unstable region was impossible.

We have analyzed different technical issues of quantum field theories out of equilibrium. It is certainly very important to develop new approaches to the evolution of quantum systems for theories with spontaneously broken symmetry. There are indications in a large N quantum mechanical system [112] that the large N limit may be misleading, as the next to leading corrections become large especially at late times. It is not clear, what the impact of these results on quantum field theory will be. Therefore, it is necessary to develop new methods and approximation schemes going beyond mean field methods. This would also lead to a possibility to include the effects of the rescattering of the produced particles.

Beside these technical considerations, the implementation of our results for the gauge fields in a cosmological context would be very interesting. Together with our studies on fermionic systems out of equilibrium we now have built the foundation to examine more realistic models to describe the physics of the early universe. The recent success in detecting neutrino masses has revived the idea of grand unification. An implementation of our method in Grand Unified Theories could lead to new and interesting results. Also the implementation of Friedman-Robertson-Walker cosmology in the model we have considered is interesting in order to get a more suitable model for describing the inflationary scenario.

Furthermore, supersymmetric models are expected to play a fundamental role at the early stages of the evolution of the universe. The special form of the potential which is important for inflation can be described very well by supersymmetric models. Further studies on these models are important for a better understanding of cosmology.

Appendix A

Some Thermal Integrals

In this Appendix we give some explicit expressions for the thermal integrals as we have used them in the numerical computations. In deriving these relations we have relied on the integral tables of Prudnikov, Brychkov and Marichev [113].

The finite temperature part of the tadpole graph, which constitutes a correction to the mass, is given by the integral

$$\begin{aligned} \Sigma_{-1}(m_0, T) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega_{k0}(e^{\beta\omega_{k0}} - 1)} \\ &= \frac{m_0^2}{2\pi^2} \sum_{n=1}^{\infty} \left\{ \mathcal{T}_{nm}^{-2} + \sum_{j=0}^{\infty} \frac{1}{4j!(j+1)!} \left[2 \ln \frac{\mathcal{T}_{nm}}{2} - \psi(j+1) - \psi(j+2) \right] \left(\frac{\mathcal{T}_{nm}^2}{4} \right)^j \right\}, \end{aligned} \quad (\text{A.1})$$

where \mathcal{T}_{nm} stands for nm_0/T . For large values of \mathcal{T}_{nm} (this means for small T) the integrand is dominated by momenta of order $k \simeq T$. Therefore, we can expand ω_{k0} with respect to powers of m/k and the integral is then well approximated by

$$\Sigma_{-1}(m_0, T) \simeq \frac{m_0^2}{2\pi^2} \sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}} e^{-\mathcal{T}_{nm}} \mathcal{T}_{nm}^{-\frac{3}{2}} \left\{ 1 + \frac{3}{8} \mathcal{T}_{nm}^{-1} - \frac{15}{128} \mathcal{T}_{nm}^{-2} + \frac{105}{1024} \mathcal{T}_{nm}^{-3} + \mathcal{O}(\mathcal{T}_{nm}^{-4}) \right\}. \quad (\text{A.2})$$

For $T \gg m_0$ we find directly from (A.1) the well-known approximation

$$\Sigma_{-1}(m_0, T) \simeq \frac{1}{2\pi^2} \zeta(2) T^2 = \frac{T^2}{12}. \quad (\text{A.3})$$

It yields the hard thermal loop corrections to the mass.

The finite temperature part of the fish graph, which can be considered as a finite correction to the coupling constant, is given by

$$\begin{aligned} \Sigma_{-3}(m_0, T) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{k0}^3} \frac{1}{e^{\beta\omega_{k0}} - 1} \\ &= \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \left\{ \frac{1}{(2j-1)j!j!} \left[\ln \frac{\mathcal{T}_{nm}}{2} - \psi(j+1) - \frac{1}{2j-1} \right] \left(\frac{\mathcal{T}_{nm}^2}{4} \right)^j \right. \\ &\quad \left. + \frac{\pi}{2} \mathcal{T}_{nm} \right\}. \end{aligned} \quad (\text{A.4})$$

For small T or large \mathcal{T}_{nm} we find the approximation

$$\Sigma_{-3}(m_0, T) \simeq -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{\sqrt{\pi}}{2} e^{-\mathcal{T}_{nm}} \mathcal{T}_{nm}^{-\frac{3}{2}} \left\{ 1 - \frac{21}{8} \mathcal{T}_{nm}^{-1} - \frac{1185}{128} \mathcal{T}_{nm}^{-2} - \frac{42735}{1024} \mathcal{T}_{nm}^{-3} + \mathcal{O}(\mathcal{T}_{nm}^{-4}) \right\}. \quad (\text{A.5})$$

For large temperatures this integral behaves linear in T , more precisely

$$\Sigma_{-3}(m_0, T) \simeq \frac{1}{8\pi} \frac{T}{m}. \quad (\text{A.6})$$

The finite temperature part associated with the quartic divergence in the energy is given by the Planck formula

$$\begin{aligned} \Sigma_1(m_0, T) &= \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_{k0}}{e^{\beta\omega_{k0}} - 1} \\ &= \frac{m_0^4}{2\pi^2} \sum_{n=1}^{\infty} \left\{ 6\mathcal{T}_{nm}^{-4} \left(1 - \frac{\mathcal{T}_{nm}}{12} \right) \right. \\ &\quad \left. + \frac{1}{16} \sum_{j=0}^{\infty} \frac{2j+1}{j!(3)_j} \left(2 \ln \frac{\mathcal{T}_{nm}}{2} - \psi(3+j) - \psi(1+j) + \frac{2}{2j+1} \right) \right\}. \end{aligned} \quad (\text{A.7})$$

As an approximation for large \mathcal{T}_{nm} or small T we find

$$\Sigma_1(m_0, T) \simeq \frac{m_0^4}{2\pi} \sum_{n=1}^{\infty} e^{-\mathcal{T}_{nm}} \sqrt{\frac{\pi}{2}} \mathcal{T}_{nm}^{-\frac{3}{2}} \left\{ 1 + \frac{27}{8} \mathcal{T}_{nm}^{-1} + \frac{705}{128} \mathcal{T}_{nm}^{-2} + \frac{2625}{1024} \mathcal{T}_{nm}^{-3} + \mathcal{O}(\mathcal{T}_{nm}^{-4}) \right\}. \quad (\text{A.8})$$

For large temperatures one obtains

$$\Sigma_1(m_0, T) \simeq \frac{\pi^2}{30} T^4. \quad (\text{A.9})$$

Appendix B

Mode Functions in the Gauge Invariant Approach

In this appendix we summarize the leading behavior of the mode functions in the gauge invariant approach. The truncated mode function for the field φ is given by (3.239):

$$f_\varphi = - \int_0^t dt' \frac{i}{2\omega_{p0}} \left(e^{2i\omega_{p0}\Delta t} - 1 \right) \left[M_{11}(1 + f_\varphi) + \frac{i\omega_{p0}k^2}{\omega_{a0}^2} M_{12}(1 + f_{\pi_\varphi}) \right]. \quad (\text{B.1})$$

As for the single channels we expand this function in orders of the potential

$$f_\varphi = f_\varphi^{(1)} + f_\varphi^{(2)} + \dots. \quad (\text{B.2})$$

B.1 The First Order

We find in the first order of the potential

$$\ddot{f}_\varphi^{(1)} - 2i\omega_{p0}\dot{f}_\varphi^{(1)} = M_{11} + \frac{ik^2\omega_{p0}}{\omega_{a0}^2} M_{12}, \quad (\text{B.3})$$

or the equivalent integral equation

$$f_\varphi^{(1)} = - \int_0^t dt' \frac{i}{2\omega_{p0}} \left(e^{2i\omega_{p0}\Delta t} - 1 \right) \left(M_{11} + \frac{i\omega_{p0}k^2}{\omega_{a0}^2} M_{12} \right). \quad (\text{B.4})$$

In order to find the leading behavior of $f_\varphi^{(1)}$ we have to insert the matrix elements M_{11} and M_{12} . After integration by parts we get

$$\begin{aligned} f_\varphi^{(1)} &= - \int_0^t dt' \frac{i}{2\omega_{p0}} \left\{ (\lambda + e^2)(\phi^2 - \phi_0^2) + 2ie^2 \frac{\omega_{p0}}{\omega_{a0}^2} \phi \dot{\phi} + \frac{e^2}{k^2} \mathcal{I}_1 \right\} \\ &\quad - \frac{1}{4\omega_{p0}^2} \left\{ (\lambda + e^2)(\phi^2 - \phi_0^2) + 2ie^2 \frac{\omega_{p0}}{\omega_{a0}^2} \phi \dot{\phi} + \frac{e^2}{k^2} \mathcal{I}_1 \right\} \end{aligned}$$

$$+ \int_0^t dt' \frac{1}{4\omega_{p0}^2} e^{2i\omega_{p0}\Delta t} \left\{ 2\phi\dot{\phi}(\lambda + e^2) + 2ie^2 \frac{\omega_{p0}}{\omega_{a0}^2} (\dot{\phi}^2 + \phi\ddot{\phi}) + \frac{e^2}{k^2} \dot{\mathcal{I}}_1 \right\}, \quad (\text{B.5})$$

with

$$\mathcal{I}_1 = \lambda\phi^2(\phi^2 - v^2) - \lambda\phi_0^2(\phi_0^2 - v^2) + \Pi^2 - \Pi\dot{\phi} - \dot{\Pi}\phi. \quad (\text{B.6})$$

For the analysis of the equation of motion we need the real part of $f_\varphi^{(1)}$. We can write it in the following way

$$\text{Re}f_\varphi^{(1)} = \frac{V_a}{2\omega_{a0}^2} - \frac{1}{4\omega_{p0}^2}(\lambda + e^2)(\phi^2 - \phi_0^2) + \mathcal{K}_\varphi^R(\omega), \quad (\text{B.7})$$

with

$$\begin{aligned} \mathcal{K}_\varphi^R(\omega) &= -\frac{e^2}{4\omega_{p0}^2 k^2} \mathcal{I}_1 + \int_0^t dt' \frac{1}{4\omega_{p0}^2} \cos(2\omega_{p0}\Delta t) \left\{ 2(\lambda + e^2)\phi\dot{\phi} + \frac{e^2}{k^2} \dot{\mathcal{I}}_1 \right\} \\ &\quad - \int_0^t dt' \frac{e^2}{2\omega_{p0}\omega_{a0}^2} \sin(2\omega_{p0}\Delta t) (\dot{\phi}^2 + \phi\ddot{\phi}). \end{aligned} \quad (\text{B.8})$$

For the imaginary part we find

$$\text{Im}f_\varphi^{(1)} = -\int_0^t dt' \frac{1}{2\omega_{p0}} (\lambda + e^2)(\phi^2 - \phi_0^2) + \mathcal{K}_\varphi^I(\omega), \quad (\text{B.9})$$

with

$$\begin{aligned} \mathcal{K}_\varphi^I(\omega) &= -\int_0^t dt' \frac{e^2}{2\omega_{p0} k^2} \mathcal{I}_1 - \frac{e^2}{2\omega_{p0}\omega_{a0}^2} \phi\dot{\phi} \\ &\quad + \int_0^t dt' \frac{1}{4\omega_{p0}^2} \sin(2\omega_{p0}\Delta t) \left\{ 2(\lambda + e^2)\phi\dot{\phi} + \frac{e^2}{k^2} \dot{\mathcal{I}}_1 \right\} \\ &\quad + \int_0^t dt' \frac{e^2}{2\omega_{p0}\omega_{a0}^2} \cos(2\omega_{p0}\Delta t) (\dot{\phi}^2 + \phi\ddot{\phi}). \end{aligned} \quad (\text{B.10})$$

In the same way we can handle the function f_{π_φ} . The function is given by (3.240)

$$f_{\pi_\varphi} = -\int_0^t dt' \frac{i}{2\omega_{p0}} \left(e^{2i\omega_{p0}\Delta t} - 1 \right) \left[\frac{\omega_{a0}^2}{i\omega_{p0} k^2} M_{21}(1 + f_\varphi) + M_{22}(1 + f_{\pi_\varphi}) \right]. \quad (\text{B.11})$$

The differential equation for f_{π_φ} in the first order of the potential is then given by

$$\ddot{f}_{\pi_\varphi}^{(1)} - 2i\omega_{p0}\dot{f}_{\pi_\varphi}^{(1)} = \frac{\omega_{a0}^2}{ik^2\omega_{p0}} M_{21} + M_{22}, \quad (\text{B.12})$$

or the equivalent integral equation

$$f_{\pi\varphi}^{(1)} = - \int_0^t dt' \frac{i}{2\omega_{p0}} \left(e^{2i\omega_{p0}\Delta t} - 1 \right) \left(\frac{\omega_{a0}^2}{i\omega_{p0}k^2} M_{11} + M_{22} \right). \quad (\text{B.13})$$

After integration by parts we get

$$\begin{aligned} f_{\pi\varphi}^{(1)} = & - \int_0^t dt' \frac{i}{2\omega_{p0}} \left\{ (\lambda + e^2)(\phi^2 - \phi_0^2) + 2i \frac{\omega_{a0}^2}{\omega_{p0}k^2} (\lambda\phi\dot{\phi} + \frac{e^2}{k^2}\Pi\dot{\Pi}) + \frac{e^2}{k^2}\mathcal{I}_2 \right\} \\ & - \frac{1}{4\omega_{p0}^2} \left\{ (\lambda + e^2)(\phi^2 - \phi_0^2) + 2i \frac{\omega_{a0}^2}{\omega_{p0}k^2} \left(\lambda\phi\dot{\phi} + \frac{e^2}{k^2}\Pi\dot{\Pi} \right) + \frac{e^2}{k^2}\mathcal{I}_2 \right\} \\ & + \int_0^t dt' \frac{1}{4\omega_{p0}^2} e^{2i\omega_{p0}\Delta t} \left\{ 2i \frac{\omega_{a0}^2}{\omega_{p0}k^2} \left[\lambda(\dot{\phi}^2 + \phi\ddot{\phi}) + \frac{e^2}{k^2}(\dot{\Pi}^2 + \Pi\dot{\Pi}) \right] \right. \\ & \left. + 2\phi\dot{\phi}(\lambda + e^2) + \frac{e^2}{k^2}\dot{\mathcal{I}}_2 \right\}, \end{aligned} \quad (\text{B.14})$$

with

$$\mathcal{I}_2 = \lambda\phi^2(\phi^2 - v^2) - \lambda\phi_0^2(\phi_0^2 - v^2) + \Pi^2 + \Pi\dot{\phi} + \dot{\Pi}\phi. \quad (\text{B.15})$$

The separation into real and imaginary part leads to

$$\text{Re} f_{\pi\varphi}^{(1)} = \frac{V_\varphi}{2\omega_{p0}^2} - \frac{1}{4\omega_{p0}^2} (\lambda + e^2)(\phi^2 - \phi_0^2) + \mathcal{K}_\pi^R(\omega), \quad (\text{B.16})$$

with

$$\begin{aligned} \mathcal{K}_\pi^R(\omega) = & - \frac{e^2}{4\omega_{p0}^2k^2}\mathcal{I}_2 + \int_0^t dt' \frac{e^2}{\omega_{p0}^2k^2} \left(\lambda\phi\dot{\phi}\phi_0^2 + \Pi\dot{\Pi} + \frac{e^2}{k^2}\Pi\dot{\Pi}\phi_0^2 \right) \\ & + \int_0^t dt' \frac{1}{4\omega_{p0}^2} \cos(2\omega_{p0}\Delta t) \left\{ 2(\lambda + e^2)\phi\dot{\phi} + \frac{e^2}{k^2}\dot{\mathcal{I}}_2 \right\} \\ & - \int_0^t dt' \frac{\omega_{a0}^2}{2\omega_{p0}^3k^2} \sin(2\omega_{p0}\Delta t) \left[\lambda(\dot{\phi}^2 + \phi\ddot{\phi}) + \frac{e^2}{k^2}(\dot{\Pi}^2 + \Pi\dot{\Pi}) \right]. \end{aligned} \quad (\text{B.17})$$

For the imaginary part we find

$$\text{Im} f_{\pi\varphi}^{(1)} = - \int_0^t dt' \frac{1}{2\omega_{p0}} (\lambda + e^2)(\phi^2 - \phi_0^2) + \mathcal{K}_\pi^I(\omega), \quad (\text{B.18})$$

with

$$\mathcal{K}_\pi^I(\omega) = - \frac{\omega_{a0}^2}{2\omega_{p0}^3k^2} \left(\lambda\phi\dot{\phi} + \frac{e^2}{k^2}\Pi\dot{\Pi} \right) - \int_0^t dt' \frac{e^2}{2\omega_{p0}k^2}\mathcal{I}_2$$

$$\begin{aligned}
 & + \int_0^t dt' \frac{1}{4\omega_{p0}^2} \sin(2\omega_{p0}\Delta t) \left\{ 2(\lambda + e^2)\phi\dot{\phi} + \frac{e^2}{k^2}\dot{\mathcal{I}}_2 \right\} \\
 & + \int_0^t dt' \frac{\omega_{a0}^2}{2\omega_{p0}^3 k^2} \cos(2\omega_{p0}\Delta t) \left[\lambda(\dot{\phi}^2 + \phi\ddot{\phi}) + \frac{e^2}{k^2}(\dot{\Pi}^2 + \Pi\dot{\Pi}) \right]. \quad (\text{B.19})
 \end{aligned}$$

B.2 The Second Order

In the equation of motion as well as in the energy we have contributions of the form $\text{Re}f_j^{(2)} + f_j^{(1)}f_j^{(1)*}$. In the single channels the leading order of the two terms is cancelled as we have seen in the Feynman gauge theory [38]. For the Goldstone mode we have to check this cancellation. As we will see we are still left with contributions of $\mathcal{O}(\omega^{-4})$. In the equation of motion they are finite but in the energy they lead to logarithmic divergences. In order to find the real part of $f_\varphi^{(2)}$ we have to investigate the following integral

$$f_\varphi^{(2)} = \int_0^t dt' \frac{i}{2\omega_{p0}} [e^{2i\omega_{p0}\Delta t} - 1] \left\{ \left[(\lambda + e^2)(\phi^2 - \phi_0^2) + \frac{e^2}{k^2}\mathcal{I}_1 \right] f_\varphi^{(1)} + 2ie^2 \frac{\omega_{p0}}{\omega_{a0}^2} \phi\dot{\phi} f_{\pi_\varphi}^{(1)} \right\}. \quad (\text{B.20})$$

We are only interested in $f_\varphi^{(2)}$ up to $\mathcal{O}(\omega^{-4})$ because these terms contributes to the divergences. Therefore, we need $f_\varphi^{(1)}$ only up to $\mathcal{O}(\omega^{-4})$ and $f_{\pi_\varphi}^{(1)}$ up to $\mathcal{O}(\omega^{-3})$:

$$\begin{aligned}
 f_\varphi^{(1)} & = -\frac{\lambda - e^2}{4\omega_{p0}^2}(\phi^2 - \phi_0^2) + i\frac{\lambda - e^2}{4\omega_{p0}^3}\phi\dot{\phi} \\
 & - \int_0^t dt' \frac{i}{2\omega_{p0}} \left[(\lambda + e^2)(\phi^2 - \phi_0^2) + \frac{e^2}{k^2}\mathcal{I}_1 \right] + \mathcal{O}(\omega^{-4}), \quad (\text{B.21})
 \end{aligned}$$

$$\begin{aligned}
 f_{\pi_\varphi}^{(1)} & = -\frac{e^2 - \lambda}{4\omega_{p0}^2}(\phi^2 - \phi_0^2) + \frac{i}{4\omega_{p0}^3}(e^2 - \lambda)\phi\dot{\phi} \\
 & - \int_0^t dt' \frac{i}{2\omega_{p0}} \left[(\lambda + e^2)(\phi^2 - \phi_0^2) + \frac{e^2}{k^2}\mathcal{I}_2 \right] + \mathcal{O}(\omega^{-4}). \quad (\text{B.22})
 \end{aligned}$$

We have given $f_{\pi_\varphi}^{(2)}$ also up to $\mathcal{O}(\omega^{-4})$ because we need it for the computation of $f_{\pi_\varphi}^{(2)}$. We have used the connection between the frequencies $\omega_{p0}^2 = \omega_{a0}^2 + \mathcal{O}(\omega^0)$ to simplify the expression for $f_\varphi^{(1)}$. Since we are only interested in the leading behavior we can identify the two frequencies. By inserting (B.21) and (B.22) into (B.20) we find

$$\begin{aligned}
 \text{Re}f_\varphi^{(2)} & = - \int_0^t dt' \int_0^{t'} dt'' \frac{(\lambda + e^2)^2}{4\omega_{p0}^2} [\phi^2(t') - \phi_0^2] [\phi^2(t'') - \phi_0^2] \\
 & - \int_0^t dt' \int_0^{t'} dt'' \frac{e^2(\lambda + e^2)}{4\omega_{p0}^2 k^2} \left\{ [\phi^2(t'') - \phi_0^2] \mathcal{I}_1(t') + [\phi^2(t') - \phi_0^2] \mathcal{I}_1(t'') \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t dt' \frac{\lambda^2 - e^4}{8\omega_{p0}^4} \phi(t') \dot{\phi}(t') [\phi^2(t') - \phi_0^2] \\
 & - \int_0^t dt' \frac{e^2(e^2 - \lambda)}{4\omega_{p0}^4} \phi(t') \dot{\phi}(t') [\phi^2(t') - \phi_0^2] \\
 & + \int_0^t dt' \frac{\lambda^2 - e^4}{8\omega_{p0}^3} [\phi^2(t') - \phi_0^2]^2 \sin(2\omega_{p0}\Delta t) \\
 & + \int_0^t dt' \int_0^{t'} dt'' \frac{(\lambda + e^2)^2}{4\omega_{p0}^2} [\phi^2(t') - \phi_0^2] [\phi^2(t'') - \phi_0^2] \cos(2\omega_{p0}\Delta t) \\
 & - \int_0^t dt' \int_0^{t'} dt'' \frac{e^2(\lambda + e^2)}{2\omega_{p0}^3} [\phi^2(t'') - \phi_0^2] \phi(t') \dot{\phi}(t') \sin(2\omega_{p0}\Delta t) + \mathcal{O}(\omega^{-5}) .
 \end{aligned} \tag{B.23}$$

For $\text{Re}f_{\pi_\varphi}^{(2)}$ we have to investigate

$$f_{\pi_\varphi}^{(2)} = \int_0^t dt' \frac{i}{2\omega_{p0}} [e^{2i\omega_{p0}\Delta t} - 1] \left\{ 2i \frac{\lambda}{\omega_{p0}} \phi \dot{\phi} f_{\pi_\varphi}^{(1)} + \left[(\lambda + e^2)(\phi^2 - \phi_0^2) + \frac{e^2}{k^2} \mathcal{I}_2 \right] f_{\pi_\varphi}^{(1)} \right\} . \tag{B.24}$$

With (B.21) and (B.22) we find

$$\begin{aligned}
 \text{Re}f_{\pi_\varphi}^{(2)} & = - \int_0^t dt' \int_0^{t'} dt'' \frac{(\lambda + e^2)^2}{4\omega_{p0}^2} [\phi^2(t') - \phi_0^2] [\phi^2(t'') - \phi_0^2] \\
 & - \int_0^t dt' \int_0^{t'} dt'' \frac{e^2(\lambda + e^2)}{4\omega_{p0}^2 k^2} \left\{ [\phi^2(t'') - \phi_0^2] \mathcal{I}_2(t') + [\phi^2(t') - \phi_0^2] \mathcal{I}_2(t'') \right\} \\
 & + \int_0^t dt' \frac{\lambda^2 - e^4}{8\omega_{p0}^4} \phi(t') \dot{\phi}(t') [\phi^2(t') - \phi_0^2] \\
 & - \int_0^t dt' \frac{\lambda(e^2 - \lambda)}{4\omega_{p0}^4} \phi(t') \dot{\phi}(t') [\phi^2(t') - \phi_0^2] \\
 & - \int_0^t dt' \frac{\lambda^2 - e^4}{8\omega_{p0}^3} [\phi^2(t') - \phi_0^2]^2 \sin(2\omega_{p0}\Delta t) \\
 & + \int_0^t dt' \int_0^{t'} dt'' \frac{(\lambda + e^2)^2}{4\omega_{p0}^2} [\phi^2(t') - \phi_0^2] [\phi^2(t'') - \phi_0^2] \cos(2\omega_{p0}\Delta t) \\
 & - \int_0^t dt' \int_0^{t'} dt'' \frac{\lambda(\lambda + e^2)}{2\omega_{p0}^3} [\phi^2(t'') - \phi_0^2] \phi(t') \dot{\phi}(t') \sin(2\omega_{p0}\Delta t) + \mathcal{O}(\omega^{-5}) .
 \end{aligned} \tag{B.25}$$

There are many cancellations in the between (B.25) and (B.23). But taking $f_j^{(1)} f_j^{(1)*}$ into account the expression becomes even more simpler. In particular we find

$$\begin{aligned}
 f_\varphi^{(1)} f_\varphi^{(1)*} &= \int_0^t dt' \int_0^t dt'' \frac{(\lambda + e^2)^2}{4\omega_{p_0}^2} [\phi^2(t') - \phi_0^2] [\phi^2(t'') - \phi_0^2] \\
 &+ \int_0^t dt' \int_0^t dt'' \frac{e^2(\lambda + e^2)}{2\omega_{p_0}^2 k^2} [\phi^2(t') - \phi_0^2] \mathcal{I}_1(t'') \\
 &+ \frac{(\lambda - e^2)^2}{16\omega_{p_0}^4} [\phi^2(t) - \phi_0^2]^2 \\
 &+ \frac{e^4 - \lambda^2}{4\omega_{p_0}^4} \phi(t) \dot{\phi}(t) \int_0^t dt' [\phi^2(t') - \phi_0^2] + \mathcal{O}(\omega^{-5}), \tag{B.26}
 \end{aligned}$$

and

$$\begin{aligned}
 f_{\pi_\varphi}^{(1)} f_{\pi_\varphi}^{(1)*} &= \int_0^t dt' \int_0^t dt'' \frac{(\lambda + e^2)^2}{4\omega_{p_0}^2} [\phi^2(t') - \phi_0^2] [\phi^2(t'') - \phi_0^2] \\
 &+ \int_0^t dt' \int_0^t dt'' \frac{e^2(\lambda + e^2)}{2\omega_{p_0}^2 k^2} [\phi^2(t') - \phi_0^2] \mathcal{I}_2(t'') \\
 &+ \frac{(\lambda - e^2)^2}{16\omega_{p_0}^4} [\phi^2(t) - \phi_0^2]^2 \\
 &- \frac{e^4 - \lambda^2}{4\omega_{p_0}^4} \phi(t) \dot{\phi}(t) \int_0^t dt' [\phi^2(t') - \phi_0^2] + \mathcal{O}(\omega^{-5}). \tag{B.27}
 \end{aligned}$$

Therefore, the relevant term in the energy becomes rather simple

$$2\text{Re} f_\varphi^{(2)} + f_\varphi^{(1)} f_\varphi^{(1)*} + 2\text{Re} f_{\pi_\varphi}^{(2)} + f_{\pi_\varphi}^{(1)} f_{\pi_\varphi}^{(1)*} = \frac{(\lambda + e^2)^2}{4\omega_{p_0}^4} (\phi^2 - \phi_0^2)^2 + \mathcal{O}(\omega^{-5}). \tag{B.28}$$

The leading behavior in ω cancels, therefore we have no problems in the equation of motion. But for the energy the term is relevant. It leads to a logarithmic divergence. This is a new feature in comparison to the Feynman gauge fixed theory. There, we haven't take into account terms from this sum, they were already finite.

Appendix C

Dimensional Regularization

In this appendix we give a short overview of the dimensional regularization of the divergent integrals. All these integrals are of the following form [113]

$$\int_0^\infty dx x^{\alpha-1} (x+y)^{-\rho} (x+z)^{-\lambda} = z^{-\lambda} y^{\alpha-\rho} B(\alpha, \rho + \lambda - \alpha) {}_2F_1\left(\alpha, \lambda; \rho + \lambda, 1 - \frac{y}{z}\right), \quad (\text{C.1})$$

with

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad {}_2F_1(a, b, c, m) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{m^n}{n!}. \quad (\text{C.2})$$

We also need the following expansion formulas for the Γ function

$$\begin{aligned} \Gamma(-n + \epsilon) &= \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon) \right], \\ \frac{1}{\Gamma(n + \epsilon)} &= \frac{1}{\Gamma(n)} - \epsilon \frac{\psi(n)}{\Gamma(n)} + \mathcal{O}(\epsilon^2). \end{aligned} \quad (\text{C.3})$$

We show the proceedings for the regularization explicitly for one integral. The calculations for the others are analogous and we will only give the results for them. A typical integral we have found is

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p_0} k^2} = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{a_0} \omega_{\varphi_0} k}, \quad (\text{C.4})$$

where we have used the relation $\omega_{p_0} k = \omega_{\varphi_0} \omega_{a_0}$ between the frequencies in order to get the same structure for the integral as in (C.1). By shifting the dimension of the integral from $3 \rightarrow 3 - \epsilon$ and substituting $k^2 = x$, we get

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{a_0} \omega_{\varphi_0} k} \rightarrow \frac{1}{4(4\pi)^{(3-\epsilon)/2}} \frac{1}{\Gamma(\frac{3}{2} - \frac{\epsilon}{2})} \int_0^\infty dx x^{-\frac{\epsilon}{2}} (x + m_{\varphi_0}^2)^{-\frac{1}{2}} (x + m_{W_0}^2)^{-\frac{1}{2}}. \quad (\text{C.5})$$

Now we can use the integral formula (C.1), expand the Γ -functions, and we are left with a finite sum over n and the typical $1/\epsilon$ -pole for dimensional regularization:

$$\begin{aligned}
 & \frac{1}{4(4\pi)^{3/2}} \frac{m_{\varphi 0}}{m_{W 0}} \left(\frac{4\pi}{m_{\varphi 0}^2} \right)^{\frac{\epsilon}{2}} \frac{1}{\Gamma(\frac{3}{2} - \frac{\epsilon}{2})} B\left(1 - \frac{\epsilon}{2}\right) {}_2F_1\left(1 - \frac{\epsilon}{2}, \frac{1}{2}; 1; 1 - \frac{m_{\varphi 0}^2}{m_{W 0}^2}\right) \\
 = & \frac{1}{4\sqrt{\pi}(4\pi)^{3/2}} \frac{m_{\varphi 0}}{m_{W 0}} \left(\frac{4\pi}{m_{\varphi 0}^2} \right)^{\frac{\epsilon}{2}} \frac{1}{\Gamma(\frac{3}{2} - \frac{\epsilon}{2})} \Gamma\left(\frac{\epsilon}{2}\right) \sum_{n=0}^{\infty} \frac{\Gamma(1 - \frac{\epsilon}{2} + n) \Gamma(\frac{1}{2} + n)}{\Gamma(1 + n) n!} \left(1 - \frac{m_{\varphi 0}^2}{m_{W 0}^2}\right)^n \\
 = & \frac{1}{2\pi(4\pi)^{3/2}} \frac{m_{\varphi 0}}{m_{W 0}} \left(\frac{4\pi}{m_{\varphi 0}^2} \right)^{\frac{\epsilon}{2}} \left[\frac{2}{\epsilon} - \gamma + \frac{\pi}{4} (1 - \gamma - 2 \ln 2) \right] \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{n!} \left(1 - \frac{m_{\varphi 0}^2}{m_{W 0}^2}\right)^n \\
 = & \frac{1}{16\pi^2} \left(\frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_h^2} - \gamma \right) + \frac{1}{16\pi^2} \left[\ln \frac{m_h^2}{m_{\varphi 0}^2} + \frac{\pi}{4} (1 - \gamma - 2 \ln 2) \right] \\
 = & I_{-3}(m_h) + C , \tag{C.6}
 \end{aligned}$$

with

$$C = \frac{1}{16\pi^2} \ln \frac{m_h^2}{m_{\varphi 0}^2} , \tag{C.7}$$

$$\gamma = 0.57721 \dots . \tag{C.8}$$

We neglect the finite contribution $\frac{1}{64\pi}(1 - \gamma - 2 \ln 2)$. In the same way we find for the other divergent integrals

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{p 0}^3} = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{a 0}^2 \omega_{p 0}} = I_{-3}(m_h) + C , \tag{C.9}$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{p 0}} = -(m_{\varphi 0}^2 + m_{W 0}^2) I_{-3}(m_h) - (m_{\varphi 0}^2 + m_{W 0}^2) \left(C + \frac{1}{16\pi^2} \right) , \tag{C.10}$$

$$\begin{aligned}
 \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_{\varphi 0}^2}{2\omega_{p 0}} &= \frac{1}{4} \left[4m_{W 0}^4 - (m_{W 0}^2 + m_{\varphi 0}^2)^2 \right] I_{-3}(m_h) \\
 &+ \frac{1}{4} \left(\frac{3}{8\pi^2} + C \right) \left[4m_{W 0}^4 - (m_{W 0}^2 + m_{\varphi 0}^2)^2 \right] . \tag{C.11}
 \end{aligned}$$

Bibliography

- [1] W. H. Zurek, Phys. Rept. **276**, 177 (1996).
- [2] A. D. Sakharov, JETP Lett. **5**, 24 (1967).
- [3] J. Garcia-Bellido, D. Y. Grigoriev, A. Kusenko and M. Shaposhnikov, Phys. Rev. **D60**, 123504 (1999).
- [4] L. M. Krauss and M. Trodden, Phys. Rev. Lett. **83**, 1502 (1999).
- [5] S. Khlebnikov, L. Kofman, A. Linde and I. Tkachev, Phys. Rev. Lett. **81**, 2012 (1998).
- [6] A. Linde, Phys. Rev. **D58**, 083514 (1998).
- [7] P. B. Greene, L. Kofman, A. Linde and A. A. Starobinskii, Phys. Rev. **D56**, 6175 (1997).
- [8] L. Kofman, A. Linde and A. A. Starobinskii, Phys. Rev. **D56**, 3258 (1997).
- [9] S. A. Ramsey and B. L. Hu, Phys. Rev. **D56**, 678 (1997).
- [10] S. A. Ramsey, B. L. Hu and A. M. Stylianopoulos, Phys. Rev. **D57**, 6003 (1998).
- [11] P. B. Greene and L. Kofman, Phys. Lett. **B448**, 6 (1999).
- [12] D. Boyanovsky, H. J. de Vega, R. Holman and J. F. J. Salgado, Phys. Rev. **D54**, 7570 (1996).
- [13] D. Boyanovsky et al., Phys. Rev. **D56**, 1939 (1997).
- [14] J. Baacke and C. Pätzold, Phys. Rev. **D61**, 024016 (2000).
- [15] A. D. Linde, *Inflation and quantum cosmology*, Academic Press, Boston, 1990.
- [16] M. S. T. E. W. Kolb, *The early universe*, Addison-Wesley, Redwood City, Calif., 1990.
- [17] D. Boyanovsky and H. J. de Vega, Phys. Rev. **D47**, 2343 (1993).

- [18] D. Boyanovsky, H. J. de Vega and R. Holman, Phys. Rev. **D51**, 734 (1995).
- [19] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper and E. Mottola, Phys. Rev. Lett. **67**, 2427 (1991).
- [20] F. Cooper, J. M. Eisenberg, Y. Kluger, E. Mottola and B. Svetitsky, Phys. Rev. **D48**, 190 (1993).
- [21] F. Cooper et al., Phys. Rev. **D50**, 2848 (1994).
- [22] F. Cooper, Y. Kluger, E. Mottola and J. P. Paz, Phys. Rev. **D51**, 2377 (1995).
- [23] U. Heinz and M. Jacob, nucl-th/0002042 (2000).
- [24] J. Schwinger, J. Math. Phys. **2**, 407 (1961).
- [25] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964).
- [26] S. Y. Khlebnikov and I. I. Tkachev, Phys. Rev. Lett. **77**, 219 (1996).
- [27] G. Aarts and J. Smit, Nucl. Phys. **B511**, 451 (1998).
- [28] D. Boyanovsky, H. J. de Vega, R. Holman, D. S. Lee and A. Singh, Phys. Rev. **D51**, 4419 (1995).
- [29] D. Boyanovsky, H. J. de Vega, R. Holman and J. Salgado, Phys. Rev. **D59**, 125009 (1999).
- [30] J. Baacke, Z. Phys. **C47**, 263 (1990).
- [31] J. Baacke, Z. Phys. **C47**, 619 (1990).
- [32] J. Baacke, Z. Phys. **C53**, 402 (1992).
- [33] J. Baacke, K. Heitmann and C. Pätzold, Phys. Rev. **D55**, 2320 (1997).
- [34] J. Baacke, K. Heitmann and C. Pätzold, Phys. Rev. **D57**, 6406 (1998).
- [35] J. Baacke, K. Heitmann and C. Pätzold, Phys. Rev. **D58**, 125013 (1998).
- [36] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).
- [37] D. Boyanovsky, D. Brahm, R. Holman and D. S. Lee, Phys. Rev. **D54**, 1763 (1996).
- [38] J. Baacke, K. Heitmann and C. Pätzold, Phys. Rev. **D55**, 7815 (1997).
- [39] D. Boyanovsky, H. J. de Vega and R. Holman, Phys. Rev. **D49**, 2769 (1994).
- [40] J. C. G. Callan, S. Coleman and R. Jackiw, Ann. Phys. **59**, 42 (1970).

- [41] J. Baacke, K. Heitmann and C. Pätzold, In *Copenhagen 1998, Strong and electroweak matter* 163- 167.
- [42] J. Baacke, K. Heitmann and C. Pätzold, Phys. Rev. **D56**, 6556 (1997).
- [43] F. Cooper, S. Habib, Y. Kluger and E. Mottola, Phys. Rev. **D55**, 6471 (1997).
- [44] L. Kofman, A. Linde and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).
- [45] J. Baacke and K. Heitmann, hep-ph/0003317 (2000).
- [46] E. J. Weinberg and A. Wu, Phys. Rev. **D36**, 2474 (1987).
- [47] D. Cormier and R. Holman, Phys. Rev. **D60**, 041301 (1999).
- [48] D. Cormier and R. Holman, hep-ph/9912483 (1999).
- [49] D. Cormier and R. Holman, hep-ph/0001168 (2000).
- [50] D. Boyanovsky, H. J. de Vega and R. Holman, hep-ph/9903534 (1999).
- [51] G. F. Mazenko, Phys. Rev. Lett. **54**, 2163 (1985).
- [52] G. F. Mazenko, R. M. Wald and W. G. Unruh, Phys. Rev. **D31**, 273 (1985).
- [53] D. Boyanovsky, C. Destri, H. J. de Vega, R. Holman and J. Salgado, Phys. Rev. **D57**, 7388 (1998).
- [54] L. Dolan and R. Jackiw, Phys. Rev. **D9**, 3320 (1974).
- [55] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
- [56] L. D. Faddeev and V. N. Popov, Phys. Lett. **25B**, 29 (1967).
- [57] B. S. Dewitt, Phys. Rev. **160**, 1113 (1967).
- [58] B. S. Dewitt, Phys. Rev. **162**, 1195 (1967).
- [59] B. S. Dewitt, Phys. Rev. **162**, 1239 (1967).
- [60] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).
- [61] A. Salam, Originally printed in *Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden*, Stockholm 1968, 367-377.
- [62] P. W. Higgs, Phys. Lett. **12**, 132 (1964).
- [63] G. 't Hooft, Nucl. Phys. **B35**, 167 (1971).
- [64] B. W. Lee and J. Zinn-Justin, Phys. Rev. **D5**, 3137 (1972).

- [65] A. A. Slavnov, *Theor. Math. Phys.* **10**, 99 (1972).
- [66] J. C. Taylor, *Nucl. Phys.* **B33**, 436 (1971).
- [67] D. J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973).
- [68] H. Georgi and S. L. Glashow, *Phys. Rev. Lett.* **32**, 438 (1974).
- [69] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Phys. Rev.* **D13**, 3214 (1976).
- [70] M. B. Green and J. H. Schwarz, *Phys. Lett.* **149B**, 117 (1984).
- [71] R. P. Feynman, *Acta Phys. Polon.* **24**, 697 (1963).
- [72] A. R. Feynman, Richard P. and Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill, New York, 1965.
- [73] P. A. M. Dirac, *Lectures in quantum mechanics*, Yeshiva University, New York, 1964.
- [74] M. Henneaux and C. Teitelboim, *Quantization of gauge systems.*, Princeton Univ. Press, Princeton, NJ, 1992.
- [75] C. Becchi, A. Rouet and R. Stora, *Annals Phys.* **98**, 287 (1976).
- [76] I. V. Tyutin, LEBEDEV-75-39.
- [77] T. Kugo and I. Ojima, *Phys. Lett.* **73B**, 459 (1978).
- [78] S. Coleman and E. Weinberg, *Phys. Rev.* **D7**, 1888 (1973).
- [79] R. Jackiw, *Phys. Rev.* **D9**, 1686 (1974).
- [80] R. Fukuda and T. Kugo, *Phys. Rev.* **D13**, 3469 (1976).
- [81] N. K. Nielsen, *Nucl. Phys.* **B101**, 173 (1975).
- [82] I. J. R. Aitchison and C. M. Fraser, *Ann. Phys.* **156**, 1 (1984).
- [83] D. Metaxas and E. J. Weinberg, *Phys. Rev.* **D53**, 836 (1996).
- [84] W. Buchmüller, Z. Fodor and A. Hebecker, *Phys. Lett.* **B331**, 131 (1994).
- [85] W. Loinaz and R. S. Willey, *Phys. Rev.* **D56**, 7416 (1997).
- [86] A. Duncan, W. Loinaz and R. S. Willey, *Phys. Rev.* **D56**, 3874 (1997).
- [87] D. Boyanovsky, W. Loinaz and R. S. Willey, *Phys. Rev.* **D57**, 100 (1998).
- [88] J. Baacke and K. Heitmann, *Phys. Rev.* **D60**, 105037 (1999).
- [89] K. K. K. Enqvist, J. Ignatius and K. Rummukainen, *Phys. Rev.* **D45**, 3415 (1992).

- [90] J. Baacke and V. G. Kiselev, Phys. Rev. **D48**, 5648 (1993).
- [91] J. Baacke, Phys. Rev. **D52**, 6760 (1995).
- [92] J. Baacke and A. Sürig, Phys. Rev. **D53**, 4499 (1996).
- [93] A. Sürig, Phys. Rev. **D57**, 5049 (1998).
- [94] V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **155B**, 36 (1985).
- [95] A. G. Cohen, D. B. Kaplan and A. E. Nelson, Ann. Rev. Nucl. Part. Sci. **43**, 27 (1993).
- [96] L. F. Abbott, Nucl. Phys. **B185**, 189 (1981).
- [97] M. Dine, R. G. Leigh, P. Huet, A. Linde and D. Linde, Phys. Rev. **D46**, 550 (1992).
- [98] L. Carson and L. McLerran, Phys. Rev. **D41**, 647 (1990).
- [99] M. Hellmund, J. Kripfganz and M. G. Schmidt, Phys. Rev. **D50**, 7650 (1994).
- [100] A. L. Jochen Kripfganz and M. G. Schmidt, Nucl. Phys. **B433**, 467 (1995).
- [101] J. S. Langer, Ann. Phys. **41**, 108 (1967).
- [102] J. S. Langer, Ann. Phys. **54**, 258 (1969).
- [103] S. Coleman, Phys. Rev. **D15**, 2929 (1977).
- [104] C. Callan and S. Coleman, Phys. Rev. **D16**, 1762 (1977).
- [105] I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).
- [106] A. D. Linde, Nucl. Phys. **B216**, 421 (1983).
- [107] S. Coleman, *Aspects of symmetry*, Cambridge Univ. Pr., Cambridge, 1985.
- [108] J. Baacke, *Fluctuation corrections to bubble nucleation, in Hot Hadronic Matter*, J. Letessier, H. H. Gutbord, and J. Rafelski, New York, 1995.
- [109] E. Calzetta and B. L. Hu, Phys. Rev. **D35**, 495 (1987).
- [110] J. P. Paz, Phys. Rev. **D42**, 529 (1990).
- [111] K. Heitmann, Diploma thesis, University of Dortmund (1996).
- [112] B. Mihaila, T. Athan, F. Cooper, J. Dawson and S. Habib, hep-ph/0003105 (2000).
- [113] A. P. Prudnikov, J. A. Bryckov and O. I. Maricev, *Integrals and series. Elementary functions.*, Gordon & Breach, New York [u.a.], 1986.

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