A scalar product for copulas

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Abstract

We introduce a scalar product for n-dimensional copulas, based on the Sobolev scalar product for $W^{1,2}$-functions. The corresponding norm has quite remarkable properties and provides a new geometric framework for copulas. We show that, in the bivariate case, it measures invertibility properties with respect to the $*$-product for copulas defined by Darsow et al. The unique copula of minimal norm is the null element for the $*$-multiplication, whereas the copulas of maximal norm are precisely the invertible elements.

Key words: Copula, Scalar product, Sobolev space

1 Introduction

Let $I = [0, 1]$ denote the closed unit interval. An n-dimensional copula is a joint distribution function on the unit cube I^n with uniform margins. The true importance of copulas to probability theory, however, stems from the well known Sklar theorem [6, 5] which states that, when a joint distribution function has continuous marginal distribution functions, it can be decomposed into the margins and a unique copula. It follows that in this case the dependence structure is fully captured by the copula, and therefore, copulas provide a convenient framework for the study of dependence relations in probability theory and mathematical statistics.

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From an analytical perspective, copulas are Lipschitz continuous functions from I^n to I with a uniform Lipschitz constant. The set $\mathfrak{C}_n$ of copulas is a subset of any Sobolev space $W^{1,p}(I^n, \mathbb{R})$ with $p \in [1, \infty]$, so that $\mathfrak{C}_n$ can be equipped with any L^p-topology. In fact, all the L^p-topologies on $\mathfrak{C}_n$ with $1 \leq p < \infty$ coincide, and are different from the L^∞-topology; see [2] whose proof for the case $n = 2$ immediately generalizes to arbitrary dimensions. Moreover, the set $\mathfrak{C}_2$ of two-dimensional copulas becomes a monoid under the $*$-product

$$(A * B)(x, y) = \int_0^1 \partial_2 A(x, t)\, \partial_1 B(t, y)\, dt$$

introduced by Darsow et al. in [1].

In this paper, we introduce a new structure for copulas, based on the fact that the particular Sobolev space $W^{1,2}(I^n, \mathbb{R})$ is a Hilbert space. We show that

$$\langle f, g \rangle = \int_{I^n} \nabla f \cdot \nabla g \, d\lambda$$

defines a scalar product for copulas with corresponding norm

$$\|C\| = \left(\int_{I^n} |\nabla C|^2 \, d\lambda \right)^{1/2}$$

where λ denotes the Lebesgue measure. This scalar product structure yields a new geometric way of looking at copulas.

We show that, especially in the case $n = 2$, this norm for copulas possesses a variety of new unexpected features. First of all, it admits a representation via the $*$-product and, therefore, provides a link between geometric and algebraic properties of copulas. Even more importantly, the norm $\| \; \|$ detects stochastic properties of the copula and, in a very precise sense, measures stochastic dependence. Indeed, we prove that the set of copulas is contained in the shell between the spheres of radius $\sqrt{2/3}$ and 1, respectively. There is a unique copula with minimal norm which is the null element for the $*$-product; it is the product copula which corresponds to stochastically independent random variables. On the other side, the copulas of maximal norm are precisely the invertible elements; they link random variables X, Y with $Y = f(X)$ a.e. for some bijection f. Finally, using the Sobolev norm instead of the L^∞-norm resolves the well known paradox in the theory of copulas that the invertible elements are dense; see [4].

The paper is organized as follows. Section 2 sets up the notation and reviews basic properties of copulas in n dimensions. Section 3 introduces the scalar product for copulas and its corresponding norm and distance. In the bivariate case, we show that the scalar product allows a representation via the $*$-product. In Section 4, we deduce fundamental geometric properties of the set of two-dimensional copulas and relate them to the algebraic structure given by the $*$-product. Finally, we compare the topology induced by our scalar product with the topology of uniform convergence in Section 5.

2 Basic properties of copulas

In this section, we collect some basic properties of multivariate copulas. We present some proofs because the standard literature deals mostly with the bivariate situation.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the extended real line, and $\overline{\mathbb{R}}^n$ the n-fold Cartesian product $\overline{\mathbb{R}} \times \ldots \times \overline{\mathbb{R}}$. For two vectors $a, b \in \overline{\mathbb{R}}^n$ with components $a_k \leq b_k$ we write $[a, b]$ for the parallelepiped $[a_1, b_1] \times \ldots \times [a_n, b_n]$. The vertices of $[a, b]$ are the points $c = (c_1, \ldots, c_n)$ with $c_k \in \{a_k, b_k\}$ for each k.

Definition 1 *Let $S_k, 1 \leq k \leq n$, be nonempty subsets of $\overline{\mathbb{R}}$, $F : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ be a function, and $B = [a, b]$ be a parallelepiped all of whose vertices c are in S. Then the F-volume of B is given by*

$$V_F(B) = \sum_c \operatorname{sgn}(c)\, F(c)$$

where $\operatorname{sgn}(c)$ is defined to be 1 if $c_k = a_k$ for an even number of k's, and -1 otherwise.

The function $F : S \to \mathbb{R}$ is called n-increasing if $V_F(B) \geq 0$ for all parallelepipeds B whose vertices lie in S.

Suppose $\alpha_k = \min S_k$ and $\beta_k = \max S_k$ for each k. Then F is called grounded if $F(t) = 0$ for all $t \in S$ such that $t_k = \alpha_k$ for at least one k. The functions $F_k : S_k \to \mathbb{R}$ with $F_k(t) = F(\beta_1, \ldots, \beta_{k-1}, t, \beta_{k+1}, \ldots, \beta_n)$ for $1 \leq k \leq n$ are called the (one-dimensional) margins of F.

Example 2 *Consider the bivariate case where $n = 2$ and $S = [a_1, b_1] \times [a_2, b_2]$ is a rectangle. Then a function $F : S \to \mathbb{R}$ is 2-increasing if, and only if,*

$$F(y_1, y_2) - F(y_1, x_2) - F(x_1, y_2) + F(x_1, x_2) \geq 0 \tag{1}$$

for all rectangles $B = [x_1, y_1] \times [x_2, y_2] \subset S$. The expression on the left hand side of (1) can be interpreted as the second order difference of F on B.

F is grounded if, and only if, $F(a_1, y) = F(x, a_2) = 0$ for all $x \in [a_1, b_1]$ and $y \in [a_2, b_2]$. In other words, F satisfies a zero boundary condition on the two "lower" boundary parts of S.

The margins of F are the two functions $F_1(x) = F(x, b_2)$ on $[a_1, b_1]$ and $F_2(y) = F(b_1, y)$ on $[a_2, b_2]$. These represent the boundary values of F on the remaining "upper" boundary parts.

Lemma 3 *Let $F : S \to \mathbb{R}$ be grounded and n-increasing. Then F is monotonically increasing in each argument.*

PROOF. Let $(t_1, \ldots, x, \ldots, t_n)$ and $(t_1, \ldots, y, \ldots, t_n)$ be two points in S with $x \leq y$, and consider the parallelepiped

$$B = [\alpha_1, t_1] \times \ldots \times [\alpha_{k-1}, t_{k-1}] \times [x, y] \times [\alpha_{k+1}, t_{k+1}] \times \ldots \times [\alpha_n, t_n].$$

Since F is n-increasing we have $V_F(B) \geq 0$, and by the groundedness of F the F-volume of B calculates to

$$0 \leq V_F(B) = F(t_1, \ldots, y, \ldots, t_n) - F(t_1, \ldots, x, \ldots, t_n).$$

□

Lemma 4 *Let $F : S \to \mathbb{R}$ be a grounded, n-increasing function with margins F_k. Then*

$$|F(x_1, \ldots, x_n) - F(y_1, \ldots, y_n)| \leq \sum_{k=1}^{n} |F_k(x_k) - F_k(y_k)|$$

for any two points $x, y \in S$.

PROOF. The proof follows easily from the triangle inequality in combination with Lemma 3; see, e.g., [5] for the case $n = 2$. □

For the following, we denote by ∂_k the partial derivative with respect to the k-th variable.

Theorem 5 *Let $F : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ be a grounded, n-increasing function. Then, for any $k \in \{1, \ldots, n\}$, the following holds true.*

(i) The partial derivative $\partial_k F(x_1, \ldots, x_n)$ exists for all $x_j \in S_j$ with $j \neq k$ and almost all $x_k \in S_k$.
(ii) The partial derivatives of F satisfy

$$0 \leq \partial_k F \leq 1.$$

(iii) For each $j \neq k$, the functions

$$t \mapsto \partial_j F(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)$$

are monotonically increasing a.e. on S_k.

(iv) If the margin F_k of F is Lipschitz continuous, then F satisfies the Fundamental Theorem of Calculus with respect to x_k, i.e.,

$$F(x_1, \ldots, x_k, \ldots, x_n) = F(x_1, \ldots, \alpha_k, \ldots, x_n) + \int_{\alpha_k}^{x_k} \partial_k F(x_1, \ldots, t, \ldots, x_n)\, dt.$$

PROOF. Because monotone functions are differentiable a.e., Lemma 3 implies the statement about the existence of $\partial_k F$, as well as $\partial_k F \geq 0$. The inequality $\partial_k F \leq 1$ follows from Lemma 4. This proves (i) and (ii).

Concerning (iii), let $j \neq k$ and consider the parallelepiped

$$B = [\alpha_1, t_1] \times \ldots \times [x_j, y_j] \times \ldots \times [x_k, y_k] \times \ldots \times [\alpha_n, t_n].$$

Then $V_F(B) \geq 0$ is equivalent to

$$F(\ldots, y_j, \ldots, x_k, \ldots) - F(\ldots, x_j, \ldots, x_k, \ldots) \\ \leq F(\ldots, y_j, \ldots, y_k, \ldots) - F(\ldots, x_j, \ldots, y_k, \ldots).$$

If y_j tends to x_j this implies

$$\partial_j F(\ldots, x_k, \ldots) \leq \partial_j F(\ldots, y_k, \ldots).$$

Finally, if F_k is Lipschitz continuous, Lemma 4 implies that also the function $t \mapsto F(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)$ is Lipschitz continuous, and hence absolutely continuous. Therefore, it satisfies the Fundamental Theorem of Calculus formulated in (iv). □

Definition 6 *An n-dimensional copula is a grounded and n-increasing function $C : I^n \to I$ with margins $C_k(t) = t$. The set of all n-dimensional copulas is denoted by $\mathfrak{C}_n$.*

We point out that, in view of Lemma 4, each copula $C \in \mathfrak{C}_n$ is Lipschitz continuous on I^n, with a uniform Lipschitz constant depending only on n.

It is elementary to prove lower and upper bounds for copulas; see [5]. In fact, for each $C \in \mathfrak{C}_n$ one has

$$C^-(x_1, \ldots, x_n) \leq C(x_1, \ldots, x_n) \leq C^+(x_1, \ldots, x_n) \tag{2}$$

for all $(x_1, \ldots, x_n) \in I^n$ where C^+ and C^- are the so-called Fréchet-Hoeffding upper and lower bound given by

$$C^-(x_1, \ldots, x_n) = \max(x_1 + \ldots + x_n - n + 1, 0)$$
$$C^+(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$$

The upper bound C^+ is always a copula itself, whereas the lower bound C^- is a copula only for $n = 2$. For proofs and further results we refer to [5]. Finally, there is a third distinguished copula, namely the product copula

$$P(x_1, \ldots, x_n) = x_1 \ldots x_n.$$

The probabilistic role of copulas is best described by the following result due to Sklar; see [6, 5].

Theorem 7 (Sklar's theorem) *For every n-dimensional distribution function H with marginal distribution functions $F_1, \ldots, F_n$, there exists a $C \in \mathfrak{C}_n$ such that*

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Moreover, if the F_k are continuous then C is unique.

Conversely, given any $C \in \mathfrak{C}_n$ and distribution functions $F_1, \ldots, F_n$, the above equation defines an n-dimensional distribution function H with marginal distribution functions $F_1, \ldots, F_n$.

In the continuous case, the unique copula given by Sklar's theorem will often be denoted by $C_{X_1,\ldots,X_n}$. Moreover, in this setting, the random variables $X_1, \ldots, X_n$ are independent if and only if $C_{X_1,\ldots,X_n} = P$.

Of particular interest are bivariate copulas where one considers the set $\mathfrak{C}_2$ of all two-dimensional copulas. The following theorem collects once more the most important properties of copulas in $\mathfrak{C}_2$.

Theorem 8 *For each $C \in \mathfrak{C}_2$ the following holds:*

(i) *C is increasing in each argument.*
(ii) *C is Lipschitz continuous.*
(iii) *For every $x \in I$, the partial derivative $\partial_2 C(x, y)$ exists for almost all $y \in I$; similarly, for every $y \in I$, the partial derivative $\partial_1 C(x, y)$ exists for almost all $x \in I$.*
(iv) *The partial derivatives of C satisfy*

$$0 \leq \partial_i C(x, y) \leq 1$$

for $i = 1, 2$ as well as

$$C(x, y) = \int_0^x \partial_1 C(t, y)\, dt = \int_0^y \partial_2 C(x, t)\, dt.$$

(v) *The functions $t \mapsto \partial_1 C(x, t)$ and $t \mapsto \partial_2 C(t, y)$ are defined and increasing a.e. on I.*

It has been shown by Darsow et al. [1] that the set $\mathfrak{C}_2$ carries a distinguished algebraic structure, namely, it becomes a monoid when equipped with the so-called $*$-multiplication

$$(A * B)(x, y) = \int_0^1 \partial_2 A(x, t)\, \partial_1 B(t, y)\, dt. \tag{3}$$

Note that $(A * B)(x, y)$ is well defined because each partial derivative is an L^1-function with respect to t, as well as an L^∞-function; we refer to [1] for proofs and more details. Direct calculations show that for any copula $C \in \mathfrak{C}_2$ we have

$$\begin{aligned} C^+ * C = C * C^+ &= C \\ P * C = C * P &= P \\ (C^- * C)(x, y) &= y - C(1 - x, y) \\ (C * C^-)(x, y) &= x - C(x, 1 - y) \end{aligned} \tag{4}$$

In particular, the copula C^+ represents the unit element in $(\mathfrak{C}_2, *)$.

Definition 9 *Given a copula $C \in \mathfrak{C}_2$, the transposed copula $C^\top$ is defined by*

$$C^\top(x, y) = C(y, x).$$

A copula $C \in \mathfrak{C}_2$ is called symmetric if $C = C^\top$.

Note that P, C^+ and C^- are all symmetric. Moreover, it is easy to see that for any $A, B \in \mathfrak{C}_2$ we have

$$(A * B)^\top = B^\top * A^\top. \tag{5}$$

Definition 10 *A copula $C \in \mathfrak{C}_2$ is left invertible if there is a copula A, called a left inverse, such that $A * C = C^+$. It is right invertible if there is a copula A, called a right inverse, such that $C * A = C^+$. A copula is called invertible if it is both left and right invertible.*

The following result is given in [1, Theorem 7.1].

Theorem 11 *The left and right inverse of a copula $C \in \mathfrak{C}_2$ are unique, if they exist, and given by the transposed copula $C^\top$. Moreover, the following statements hold true:*

(i) C is left invertible if, and only if, for each $y \in I$, $\partial_1 C(x, y) \in \{0, 1\}$ for almost all $x \in I$.
(ii) C is right invertible if, and only if, for each $x \in I$, $\partial_2 C(x, y) \in \{0, 1\}$ for almost all $y \in I$.

3 The Sobolev scalar product for copulas

Let us denote by $\cdot$ the Euclidean scalar product, by $|\,|$ the Euclidean norm on $\mathbb{R}^n$, and by λ the n-dimensional Lebesgue measure. Recall that the set of n-dimensional copulas is denoted by $\mathfrak{C}_n$.

It follows immediately from Theorem 8, and has been noticed in [2], that

$$\mathfrak{C}_n \subset W^{1,p}(I^n, \mathbb{R})$$

for every $p \in [1, \infty]$ where $W^{1,p}(I^n, \mathbb{R})$ is the standard Sobolev space. However, it has not been exploited in this context that $W^{1,2}(I^n, \mathbb{R})$ is a Hilbert space with respect to the scalar product

$$\langle f, g \rangle_{W^{1,2}} = \int_{I^n} fg \, d\lambda + \int_{I^n} \nabla f \cdot \nabla g \, d\lambda \tag{6}$$

where ∇f denotes the vector consisting of the weak partial derivatives of f. We refer to [3] for more details.

Since copulas are continuous functions, there is an even simpler way to define a scalar product for them. Define a subspace of $W^{1,2}(I^n, \mathbb{R})$ by

$$W_0^{1,2}(I^n, \mathbb{R}) = \{ f \in W^{1,2}(I^n, \mathbb{R}) \mid f \in C^0(I^n, \mathbb{R}), f(0) = 0 \}.$$

Proposition 12 *Setting*

$$\langle f, g \rangle = \int_{I^n} \nabla f \cdot \nabla g \, d\lambda$$

defines a scalar product on $W_0^{1,2}(I^n, \mathbb{R})$.

PROOF. Indeed, $\langle \; , \; \rangle$ is a symmetric bilinear form which is nondegenerate on $W_0^{1,2}(I^n, \mathbb{R})$ since $\langle f, f \rangle = 0$ implies that f is constant a.e. so that, by continuity and $f(0) = 0$, we conclude that $f = 0$. $\square$

Note that the subspace generated by $\mathfrak{C}_n$ is contained in $W_0^{1,2}(I^n, \mathbb{R})$ and, hence, inherits the scalar product $\langle \; , \; \rangle$ from $W_0^{1,2}(I^n, \mathbb{R})$. Therefore, with a slight abuse of notation because $\mathfrak{C}_n$ is not a vector space itself, we can make the following definition.

Definition 13 *The restriction of* $\langle \; , \; \rangle$ *to* $\mathfrak{C}_n$ *is called the Sobolev scalar product on* $\mathfrak{C}_n$.

Then we define, as usual, the corresponding Sobolev norm on $\mathfrak{C}_n$ by

$$\|C\| = \Big(\int_{I^n} |\nabla C|^2 \, d\lambda \Big)^{1/2} \tag{7}$$

and the Sobolev distance function on $\mathfrak{C}_n \times \mathfrak{C}_n$ by

$$d(A, B) = \Big(\int_{I^n} |\nabla A - \nabla B|^2 \, d\lambda \Big)^{1/2}. \tag{8}$$

Remark 14 *(i) Obviously, $\| \ \|$ defines a semi-norm on the whole Sobolev space $W^{1,2}(I^n, \mathbb{R})$.*

(ii) The Sobolev norm $\| \ \|$ is reminiscent of the classical energy functional, which is well known in PDEs and differential geometry. In fact, one might call

$$E(C) = \frac{1}{2} \|C\|^2 = \frac{1}{2} \int_{I^n} |\nabla C|^2 \, d\lambda$$

the energy of a copula $C \in \mathfrak{C}_n$.

(iii) $(\mathfrak{C}_n, d)$ is a complete metric space [2, Theorem 4.5], and the $*$-product on $\mathfrak{C}_2$ is continuous with respect to d [2, Theorem 4.2].

We have seen that the Sobolev scalar product for copulas appears very naturally from an analytical point of view. However, for the case $n = 2$, it also allows a representation via the algebraic product structure on $\mathfrak{C}_2$, defined by the $*$-multiplication in (3).

Theorem 15 *For all $A, B \in \mathfrak{C}_2$ we have*

$$\begin{aligned} \langle A, B \rangle &= \int_0^1 (A^\top * B + A * B^\top)(t, t) \, dt \\ &= \int_0^1 (A^\top * B + B * A^\top)(t, t) \, dt. \end{aligned}$$

PROOF. The partial derivatives of the transposed copula are given by

$$\begin{aligned} \partial_1 A^\top(x, y) &= \partial_2 A(y, x) \\ \partial_2 A^\top(x, y) &= \partial_1 A(y, x) \end{aligned} \tag{9}$$

Using (3) and (9) we can write

$$\begin{aligned} \int_0^1 \int_0^1 \partial_1 A(x, y) \, \partial_1 B(x, y) \, dx \, dy &= \int_0^1 \Big(\int_0^1 \partial_2 A^\top(y, x) \, \partial_1 B(x, y) \, dx \Big) dy \\ &= \int_0^1 (A^\top * B)(y, y) \, dy \\ \int_0^1 \int_0^1 \partial_2 A(x, y) \, \partial_2 B(x, y) \, dx \, dy &= \int_0^1 \Big(\int_0^1 \partial_2 A(x, y) \, \partial_1 B^\top(y, x) \, dy \Big) dx \\ &= \int_0^1 (A * B^\top)(x, x) \, dx. \end{aligned}$$

Adding up both terms we obtain the first identity.

The second equation follows from the fact that, along the diagonal, we have $(A * B^\top)(t,t) = (A * B^\top)^\top(t,t) = (B * A^\top)(t,t)$ for each $t \in I$. $\square$

Corollary 16 *If $A, B \in \mathfrak{C}_2$ are symmetric, then*

$$\langle A, B \rangle = 2 \int_0^1 (A * B)(t,t)\, dt.$$

4 The Sobolev geometry of $\mathfrak{C}_2$

In this section, we continue our study of the Sobolev scalar product, respectively, the corresponding Sobolev norm on $\mathfrak{C}_2$.

Theorem 17 *Let $A, B \in \mathfrak{C}_2$. Then*

$$\frac{1}{2} \leq \langle A, B \rangle \leq 1,$$

where both bounds are sharp.

PROOF. Theorem 15, in connection with the bounds for copulas given in (2), implies that

$$2 \int_0^1 C^-(t,t)\, dt \leq \langle A, B \rangle \leq 2 \int_0^1 C^+(t,t)\, dt.$$

Simple calculations yield $\int_0^1 C^-(t,t)\, dt = 1/4$ and $\int_0^1 C^+(t,t)\, dt = 1/2$.

Finally, one easily computes that

$$\begin{aligned} \langle C^-, C^- \rangle &= \langle C^+, C^+ \rangle = 1 \\ \langle C^-, C^+ \rangle &= \frac{1}{2}\,. \end{aligned} \tag{10}$$

This shows that the bounds in the statement are sharp, and the proof is complete. $\square$

Corollary 18 *We have*

$$d(A, B) \leq 1 = d(C^-, C^+)$$

for all $A, B \in \mathfrak{C}_2$; in particular, the diameter of $(\mathfrak{C}_2, d)$ is 1.

PROOF. The inequality $d(A,B) \leq 1$ is a consequence of the identity

$$d(A,B)^2 = \|A-B\|^2 = \|A\|^2 + \|B\|^2 - 2\langle A,B\rangle \tag{11}$$

in connection with Theorem 17. The equality $d(C^-,C^+) = 1$ follows from (11) and (10). $\square$

Corollary 19 *For any $A,B \in \mathfrak{C}_2$ the following holds true.*

(i) $d(A,B) = 1$ if, and only if, $\|A\| = \|B\| = 1$ and $\langle A,B\rangle = 1/2$.
(ii) $\langle A,B\rangle = 1$ if, and only if, $\|A\| = \|B\| = 1$ and $A = B$.

PROOF. Again, both statements follow from (11) in combination with Theorem 17. $\square$

Proposition 20 *The transposition map $C \mapsto C^\top$ is an isometry on the metric space $(\mathfrak{C}_2, d)$. Moreover, for every $C \in \mathfrak{C}_2$ we have*

$$\|C\|^2 \geq 2\int_0^1 (C*C)(t,t)\,dt,$$

with equality if and only if C is symmetric.

PROOF. The fact that $\|C\| = \|C^\top\|$ follows readily from the definitions. Using Theorem 15, as well as $C^\top * C^\top = (C*C)^\top$, we can therefore write

$$\begin{aligned}
0 &\leq \|C - C^\top\|^2 \\
&= \|C\|^2 + \|C^\top\|^2 - 2\langle C, C^\top\rangle \\
&= 2\left(\|C\|^2 - \int_0^1 ((C*C)^\top + (C*C))(t,t)\,dt\right) \\
&= 2\left(\|C\|^2 - 2\int_0^1 (C*C)(t,t)\,dt\right)
\end{aligned}$$

from which the second assertion follows. $\square$

Theorem 21 *For all $C \in \mathfrak{C}_2$, the following hold:*

(i)

$$\begin{aligned}
\langle P, C\rangle &= \frac{2}{3} \\
\langle C^+, C\rangle &= 2\int_0^1 C(t,t)\,dt \\
\langle C^-, C\rangle &= 1 - 2\int_0^1 C(t,1-t)\,dt
\end{aligned}$$

(ii)

$$\|C - P\|^2 = \|C\|^2 - \frac{2}{3}$$

(iii)

$$\frac{2}{3} \leq \|C\|^2 \leq 1.$$

PROOF. Recall that P, C^+ and C^- are symmetric. Then, using Theorem 15 and (4), we can write

$$\langle P, C \rangle = \int_0^1 (P * C + C * P)(t,t)\, dt = 2 \int_0^1 P(t,t)\, dt = \frac{2}{3}.$$

The identities for $\langle C^+, C \rangle$ and $\langle C^-, C \rangle$ are shown analogously. This proves (i). Now, (ii) and (iii) are immediate consequences from (i) and Theorem 17. □

Corollary 22 *For all $C \in \mathfrak{C}_2$, we have*

$$\langle C - P, P \rangle = 0.$$

The next theorem is one of the main results of the paper. It describes fundamental features of the Sobolev norm on $\mathfrak{C}_2$, and shows that the geometric Sobolev norm detects algebraic properties of copulas. Loosely speaking, the Sobolev norm measures the "degree of invertibility" of two-dimensional copulas.

Theorem 23 *The Sobolev norm on $\mathfrak{C}_2$ satisfies*

$$\frac{2}{3} \leq \|C\|^2 \leq 1$$

for all $C \in \mathfrak{C}_2$. Moreover, the following assertions hold:

(i) $\|C\|^2 = 2/3$ *if, and only if,* $C = P$.
(ii) $\|C\|^2 \in [5/6, 1]$ *if C is left or right invertible.*
(iii) $\|C\|^2 = 1$ *if, and only if, C is invertible.*

PROOF. The foremost statement is contained in Theorem 21(iii). The first item in the list is an immediate consequence of Theorem 21(ii).

As for the second assertion, it follows from (7) that

$$\|C\|^2 = \int_0^1 \int_0^1 \partial_1 C(x,y)^2\, dx\, dy + \int_0^1 \int_0^1 \partial_2 C(x,y)^2\, dx\, dy. \tag{12}$$

If C is left invertible we know from Theorem 11 that $(\partial_1 C)^2 = \partial_1 C$ a.e., so the first summand in (12) is always equal to

$$\int_0^1 \int_0^1 \partial_1 C(x,y)\,dx\,dy = \int_0^1 y\,dy = \frac{1}{2}\,.$$

To estimate the second term in (12), we consider the inequality

$$\begin{aligned}
0 &\le \int_0^1 \int_0^1 (\partial_2 C(x,y) - x)^2\,dx\,dy \\
&= \int_0^1 \int_0^1 \partial_2 C(x,y)^2\,dx\,dy - 2\int_0^1 x \int_0^1 \partial_2 C(x,y)\,dy\,dx + \int_0^1 \int_0^1 x^2\,dx\,dy \\
&= \int_0^1 \int_0^1 \partial_2 C(x,y)^2\,dx\,dy - \frac{1}{3}.
\end{aligned}$$

Hence the second summand is at least $1/3$. Since $1/2 + 1/3 = 5/6$, we have proved the second statement.

Finally, in view of Proposition 8 (iii), we have $(\partial_i C)^2 \le \partial_i C$ for $i = 1, 2$ with equality if, and only if, $\partial_i C \in \{0, 1\}$. Consequently, (12) implies that

$$\|C\|^2 \le \int_0^1 \int_0^1 \partial_1 C(x,y)\,dx\,dy + \int_0^1 \int_0^1 \partial_2 C(x,y)\,dx\,dy = \frac{1}{2} + \frac{1}{2} = 1$$

with equality if, and only if, $\partial_1 C, \partial_2 C \in \{0, 1\}$ a.e. In view of Theorem 11, this is equivalent to C being invertible. □

Corollary 24 *For any $C \in \mathfrak{C}_2$, the following are equivalent:*

(i) $\|C\| = 1$.
(ii) $\partial_1 C, \partial_2 C \in \{0, 1\}$ *a.e.*
(iii) *C is invertible, i.e.,* $C * C^\top = C^\top * C = C^+$.
(iv) $\int_0^1 (C * C^\top + C^\top * C)(t,t)\,dt = 1$.

PROOF. This follows immediately from Theorem 11, Theorem 23 and Theorem 15. □

As a consequence, we obtain the following geometric picture for the set $\mathfrak{C}_2$ of copulas. [1]

First of all, the set $\mathfrak{C}_2$ has diameter 1 and lies between the spheres of radius $\sqrt{2/3}$ and 1, respectively; furthermore, it is contained in the affine hyperplane perpendicular to P. Due to the boundary conditions, any ray in the vector space $W_0^{1,2}(I^2, \mathbb{R})$ emanating from the origin intersects $\mathfrak{C}_2$ in at most one point.

[1] As an aside we mention that $\mathfrak{C}_2$ is convex; this is easy to check.

The unique copula of minimal norm is the product copula P, whereas copulas of maximal norm are precisely those which are invertible with respect to the $*$-multiplication. In between, the copulas which are left or right invertible are contained in the shell of radii $\sqrt{5/6}$ and 1.

Finally, we point out that the above results can also be seen from a probabilistic viewpoint. In fact, Darsow et al. prove in [1, Theorem 11.1] that a continuous random variable Y is completely dependent on X (i.e., there is a Borel measurable function f such that $Y = f(X)$ a.s.) if and only if the copula $C = C_{X,Y}$ linking X and Y is left invertible. Consequently, two continuous random variables X, Y are mutually completely dependent (i.e., there is a Borel measurable bijection f such that $Y = f(X)$ a.s.) if and only if the copula C is invertible. Therefore, the Sobolev norm measures probabilistic dependence properties, and the geometric picture stemming from Theorem 23 possesses a probabilistic counterpart.

5 Comparison with the L^∞-topology for copulas

We conclude this paper with some remarks concerning the topology induced by the Sobolev norm $\|\ \|$ on $W_0^{1,2}(I^n, \mathbb{R})$, which contains the subspace generated by $\mathfrak{C}_n$. Recall from (6) that the standard Sobolev norm is given by

$$\|f\|_{W^{1,2}}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 = \|f\|_{L^2}^2 + \|f\|^2.$$

Proposition 25 *The norms $\|\ \|$ and $\|\ \|_{W^{1,2}}$ are equivalent on the subspace generated by $\mathfrak{C}_n$.*

PROOF. Trivially, we have $\|f\| \leq \|f\|_{W^{1,2}}$. In order to prove the proposition, we will show an adapted version of the Poincaré inequality

$$\|f\|_{L^2}^2 \leq \frac{1}{2}\,\|\nabla f\|_{L^2}^2 \tag{13}$$

yielding $\|f\|_{W^{1,2}} \leq 3/2\,\|f\|$.

In order to prove (13) we may assume, in view of the theorem of Meyers and Serrin, that f is smooth. Then, due to the boundary conditions of copulas, we can write

$$|f(x)| = \Big| \int_0^{x_n} \frac{\partial f}{\partial x_n}(x', t)\, dt \Big| \leq x_n^{1/2} \Big(\int_0^{x_n} \Big| \frac{\partial f}{\partial x_n}(x', t) \Big|^2 dt \Big)^{1/2}$$

by Hölder's inequality where $x = (x', t) \in I^n = I^{n-1} \times I$. Therefore,

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int_{I^n} |f(x)|^2 \, dx \\
&\leq \int_{I^n} x_n \int_0^1 \Big| \frac{\partial f}{\partial x_n}(x', t) \Big|^2 \, dt \, dx \\
&\leq \int_{I^{n-1}} \Big(\int_0^1 x_n \, dx_n \int_0^1 |\nabla f(x', t)|^2 \, dt \Big) dx' \\
&= \frac{1}{2} \int_{I^n} |\nabla f(x)|^2 \, dx \\
&= \frac{1}{2} \|\nabla f\|_{L^2}^2 .
\end{aligned}$$

This proves (13) and, hence, the proposition. □

From a probabilistic viewpoint, the notion of convergence in distribution plays a central role. It is equivalent to the pointwise convergence of the copulas. Since copulas are Lipschitz continuous function on I^n with a uniform Lipschitz constant, this is the same as L^∞-convergence for copulas. It follows from (13) that convergence of copulas w.r.t. the Sobolev norm $\| \; \|$ implies L^2-convergence, hence also L^∞-convergence.

On the other hand, convergence in distribution does not imply convergence w.r.t. $\| \; \|$. This is most prominently illustrated by the somewhat paradoxical phenomenon that any copula can be L^∞-approximated by a sequence of invertible copulas, i.e., copulas which correspond to mutually completely dependent random variables [4]. This implies that, from a practical point of view, the product copula which describes stochastic independence cannot be distinguished in the L^∞-topology from copulas describing mutually completely dependent behavior.

The Sobolev norm, however, resolves this paradox—invertible copulas can only approximate invertible copulas, as the following result shows.

Theorem 26 *If $(C_k)_{k \in \mathbb{N}}$ is a sequence of invertible copulas $C_k \in \mathfrak{C}_2$ with $\lim_{k\to\infty} \|C_k - C\| = 0$ for some $C \in \mathfrak{C}_2$, then C is invertible.*

PROOF. In view of Corollary 24, we have $\|C_k\| = 1$ for all k which implies $\|C\| = 1$ since $\lim_{k\to\infty} \|C_k - C\| = 0$. Applying Corollary 24 again proves the theorem. □

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