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LAWS FOR THE CAPILLARY PRESSURE IN A DETERMINISTIC MODEL FOR FRONTS IN POROUS MEDIA*

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Abstract. We propose and analyze a model for sharp fronts in porous media, aiming at an investigation of the capillary pressure. Using the notion of microlocal patterns we analyze the local behavior of the system. Depending on the structure of the local patterns we can derive upscaled equations that characterize the capillary pressure and include the hysteresis effect that is known from the physical system.

Key words. homogenization, two-phase flow, front dynamics, microlocal pattern

AMS subject classifications. 74Q10, 76M50, 76T10

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1. Introduction. The investigation of fluid motion in porous media has attracted much interest in the fields of engineering, physics, and mathematics. A particular interest concerns the case when two immiscible fluids are contained in the porous material, e.g., water and oil in rock. Different suggestions were made for averaged equations for this two-fluid motion; most are variants of the very successful Muskat–Leverett equations. In these equations the motion of the two fluids is coupled via an equation

$$p_a - p_b = p_c(s), \tag{1.1}$$

where p_a and p_b are the pressure functions in the two fluids, s is the saturation of, say, fluid a, and p_c is the capillary pressure. Our aim is to derive (1.1) for a model system.

We refer the reader to [2, 5, 7, 9] for other approaches towards the justification of the Muskat–Leverett equations. Concerning modifications of the system see [3, 4]; note that the result of this work and of [10] suggests another modification. For an analysis of the upscaled system see [8].

The far aim would be the homogenization of the geometry of Figure 1(a). This goal seems to be out of reach due to the topological changes of the free boundary during its propagation. A simpler geometry is the filter geometry of Figure 1(b). Here in every single tube the free boundary has essentially only one degree of freedom, its average height. By the laws for surface tension and contact angles, the geometry implies for every tube k a relation between average height and typical pressure, $p = \mathcal{P}_0(h, k)$. We will study this simplified model. For a homogenization result for the filter geometry with the same methods see [11].

We observe the creation of local structures: when the pressure in tube k reaches its maximal value and the height exceeds the critical point, an instability occurs. The pressure lowers; therefore, locally, the flow goes toward tube k, the height increases further, and the process accelerates. Such an event has the temporal and spatial scale of the tube distance ε and we call it an explosion. We will verify the appearance of

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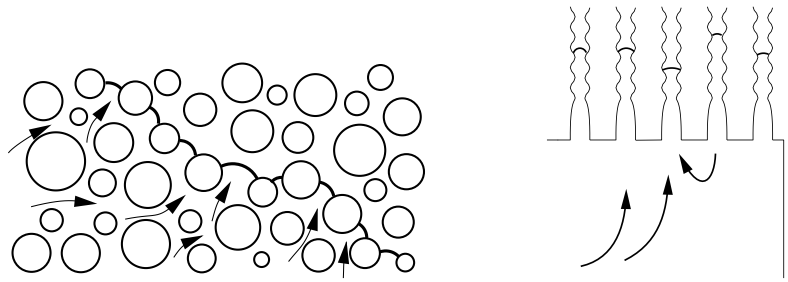

FIG. 1. *(Left) Front in porous media. (Right) Filter geometry.*

these explosions and determine their form. The distribution of the explosions can be captured using the Young measure on patterns introduced by Alberti and Müller in [1]. We will derive a conditional result for the upscaled equations, i.e., the limit $\varepsilon \to 0$: if the patterns of the limit measure all have finite length, then upscaled equations with a prescribed capillary pressure along the front are satisfied in the limit.

1.1. A model for propagating fronts.

Geometry. The fluid occupies the domain $\Omega := (-1,1)\times(-1,0)$ and is described by a Darcy law. The front is located along the upper boundary

$$\Gamma := (-1,1) \times \{0\}.$$

Γ consists of two parts. On $\Gamma_2^\varepsilon \subset \Gamma$ the fluid is in contact with the matrix; with $\gamma \in (0,1/2)$ we write

$$\Gamma_2^\varepsilon := \varepsilon \cdot (\mathbb{Z} + (\gamma, 1-\gamma)) \cap \Gamma,$$

where we used the obvious identification $\Gamma \subset \mathbb{R}$. The small parameter ε describes the pore size in the medium; we will for simplicity always assume $\varepsilon = 1/N$ with $N \in \mathbb{N}$. A fluid-gas interphase is present along

$$\Gamma_1^\varepsilon := \varepsilon \cdot (\mathbb{Z} + (-\gamma, \gamma)) \cap \Gamma.$$

The free boundary is modeled with a height function

$$h^\varepsilon : \Gamma_1^\varepsilon \to \mathbb{R}.$$

Having in mind that the free boundary in the single tube has only one degree of freedom, we reinterpret h^ε as the average height and assume that h^ε is piecewise constant. We use the space $V_\varepsilon = Q_\varepsilon L^2(\mathbb{R})$ of functions that are constant on the intervals $\varepsilon \cdot ((-\gamma,\gamma) + k)$, $k \in \mathbb{Z}$. Here Q_ε is the L^2-projection to this subspace; it is obtained by replacing a function on Γ_1^ε with its averages on the disjoint intervals of length $2\gamma\varepsilon$.

Equations. We write the Darcy law in the scaling $v^\varepsilon = -\varepsilon \nabla p^\varepsilon$. This scaling is obtained by rescaling time and is appropriate to observe microscopic processes. Note that we assumed the permeability matrix to be the identity in order to simplify notations. We study the problem

$$\partial_t h^\varepsilon(x_1,t) = -\varepsilon Q_\varepsilon \partial_2 p^\varepsilon(x_1,0,t) \qquad \forall (x_1,0)\in\Gamma_1^\varepsilon,\ \forall t, \tag{1.2}$$

$$p^\varepsilon(x_1,0,t) = \mathcal{P}_0\left(\frac{x_1}{\varepsilon}, \frac{h^\varepsilon(x_1,t)}{\varepsilon}\right) \qquad \forall (x_1,0)\in\Gamma_1^\varepsilon,\ \forall t, \tag{1.3}$$

$$\partial_2 p^\varepsilon(x_1,0,t) = 0 \qquad \forall (x_1,0)\in\Gamma_2^\varepsilon,\ \forall t, \tag{1.4}$$

$$-\Delta_x p^\varepsilon = 0 \qquad \text{in } \Omega\times(0,T). \tag{1.5}$$

Initial values for the height function are given and we always set $h^\varepsilon(x_1) = 0$ for $(x_1,0) \in \Gamma_2^\varepsilon$. The equations are complemented with the ε-independent boundary conditions. We impose periodicity on the lateral boundaries $\Sigma_\pm := \{(x_1,x_2)\in\bar\Omega \mid x_1 = \pm 1\}$. The presented methods apply also in the case of an impermeability condition. As a driving mechanism we choose a prescribed inflow on the lower boundary $\Gamma_0 := (-1,1)\times\{-1\}$,

$$-\partial_2 p^\varepsilon(x_1,-1,t) = V_0(x_1) \qquad \forall (x_1,-1)\in\Gamma_0,\ \forall t. \tag{1.6}$$

It is also possible to prescribe the pressure at the lower boundary. In either case and throughout our investigations we demand $p^\varepsilon|_{\Gamma_0}(t) > 0$ for all t. This assumption is made to simplify notations; upscaled equations in the general case follow by symmetry.

It is left to specify the material law $\mathcal{P}_0$ which prescribes the pressure-height dependence in each cell (we will always assume that $\mathcal{P}_0(\cdot,s)$ is constant on $(k-\gamma,k+\gamma)$ for every $k\in\mathbb{Z}$). A reasonable choice is the following. The material law of the cells is the same in every cell and a sawtooth function in $s = \frac{h}{\varepsilon}$,

$$\mathcal{P}_0(k,s) \equiv \mathcal{P}_0(s) = a_0\cdot s \mod a_0 s_0. \tag{1.7}$$

Here s_0 represents the volume of the single cell. The maximal pressure that is needed to advance the free boundary is

$$p_{\max} = a_0\cdot s_0.$$

A possible modification of this model is to allow the physical parameters $a_0 = a_0(k)$ and $s_0 = s_0(k)$ to depend on the position index k. If (a_0,s_0) is periodic in k, all results remain valid. We collect some first observations on the ε-problem. The proofs are straightforward and can be found in [11].

Remark 1.1. The ε-problem has a unique solution with $p^\varepsilon(t)\in H^1(\Omega)$ for all $t\in[0,T]$. The solution sequence p^ε satisfies uniform bounds in $L^\infty((0,T),L^\infty(\Omega))$ and in $L^2((0,T),H^1(\Omega))$.

The uniform bound of $p^\varepsilon\in L^\infty(\Omega)$ allows us to choose a subsequence $\varepsilon\to 0$ such that p^ε has a limit $p^0\in L^\infty(\Omega)$ in the sense of the weak-$\star$ convergence in L^∞,

$$p^\varepsilon \to p^0 \quad \text{in } L^\infty_w.$$

1.2. Main result. Our aim is to find equations for p^0 in order to describe the averaged behavior of the solutions p^ε. It turns out that, in general, p^0 is not uniquely determined. We will have to study the microscopic behavior of the family p^ε in order to find equations for p^0.

In the model equations with relation (1.7) the pressure p^ε has discontinuities. There are points $(x_1,0,t)\in\varepsilon\mathbb{Z}\times(0,T)$ in which the pressure drops from $p_{\max}$ to 0; we call them explosion points. We have already seen that we can expect a nontrivial behavior in an ε-space-time neighborhood of explosion points. Generically, we expect that the explosion points do not cluster and that the limit patterns of explosions are finite (see Definition 2.8 for a precise statement). In fact, in [10] we analyze a stochastic system and find that explosion points cluster only with probability 0.

If all realized explosion patterns are finite, then the limit pressure p^0 satisfies the following upscaled system in the distributional sense (see Theorem 4.6 for details):

$$\Delta p^0 = 0 \qquad \text{in } \Omega, \tag{1.8}$$

$$-\partial_2 p^0 = V_0 \qquad \text{on } \Gamma_0, \tag{1.9}$$

and p^0 is periodic across $\Sigma_\pm$. On the boundary Γ it satisfies

$$p^0 \le p_{\max}, \tag{1.10}$$

$$\partial_t(\Theta \circ p^0) \le -\partial_2 p^0, \tag{1.11}$$

$$\partial_t(\Theta \circ p^0) = -\partial_2 p^0 \quad \text{on } \{(x_1,0) \in \Gamma \mid p^0(x_1,0) < p_{\max}\}. \tag{1.12}$$

The function Θ is defined in the stochastic case as an expected value. Since we consider the deterministic equation (1.7), its definition reduces to

$$\Theta'(\rho) = \left\langle \frac{2\gamma}{\mathcal{P}_0'(k, s_k(\rho))} \right\rangle = \frac{2\gamma}{a_0}, \tag{1.13}$$

where $s_k(\rho)$ is defined by $\mathcal{P}_0(k, s_k(\rho)) = \rho$. The microlocal patterns of the functions p^ε can be described.

Remarks on Theorem 4.6. (1) An assumption on the distribution of explosions is indeed necessary. Consider a family of solutions to the ε-problems that is ε-periodic in x_1-direction. In this case, the limiting pressure p^0 is constant along Γ, but has jumps from $p_{\max}$ to 0 at discrete times. The function p^0 does not solve the upscaled equation (1.12).

(2) We study a sawtooth function as a material law in (1.7). This law is not satisfactory for all applications. Unfortunately, in our proofs we need a condition on positivity of $\mathcal{P}_0'$ away from discontinuities, and this restricts our choices for $\mathcal{P}_0$. We conjecture that the overall picture about solution sequences and the upscaled equations remain valid for continuous functions $\mathcal{P}_0$.

(3) One can formally relate the upscaled equation (1.12) to well-known effective conductivity formulae. Differentiating (1.3) with respect to time and inserting (1.3), we find that $\partial_t p^\varepsilon$ essentially equals $-a_0\partial_2 p^\varepsilon$. Dividing (1.12) by Θ', we find that $\partial_t p^0$ essentially equals $-\bar{a}\partial_2 p^\varepsilon$, where $\bar{a}$ is the harmonic mean of the a_0.

2. Microlocal patterns and possible patterns for fronts. We study solutions $(p^\varepsilon, h^\varepsilon)$ of (1.2)–(1.6) and are interested in the averaged behavior as it is expressed by the weak limit p^0. The goal is to derive averaged equations that characterize p^0.

We already observed that the equations have a nontrivial behavior on an ε-scale in time and in space. In this scaling we expect to see the filling procedure of the single pore: fluid enters the cell $\varepsilon(k-\gamma, k+\gamma)$ until the pressure reaches its maximal value $p_{\max}$. Now the pressure is set to zero and a pressure gradient of order $1/\varepsilon$ drives a refill procedure in which mass is transported from neighboring cells to cell k. It takes a time span of order ε and a spatial area with diameter of order ε to essentially reach again the pressure $p_{\max}$.

We want to find descriptions of this microscopic process. An adequate tool is that of microlocal patterns, introduced by Alberti and Müller in [1]. We outline aspects of this tool in this section.

Notation for measures. Let E be a locally compact Hausdorff space. We denote by $\mathcal{M}(E)$ the space of all finite real Borel measures on E. Let $C(E)$ be the space of continuous functions on E with compact support. Then $\mathcal{M}(E)$ can be identified with the dual of $C(E)$. Therefore bounded sequences in $\mathcal{M}(E)$ are precompact.

2.1. Construction of microlocal patterns. Let $u^\varepsilon : S \to \mathbb{R}$ be a sequence of functions on a compact set $S \subset \mathbb{R}^m$. We assume that for $C \in \mathbb{R}$ there holds $\|u^\varepsilon\|_{L^\infty} \le C$ for all $\varepsilon > 0$. Our aim is now to study the behavior of u^ε on the length scale ε. We therefore consider the blowup of u^ε, just as in asymptotic expansions or in the theory of two-scale convergence. Together with u^ε we consider the local pattern around $s \in S$, that is, the function

$$\mathbb{R}^m \ni t \mapsto R_s^\varepsilon u^\varepsilon(t) := u^\varepsilon(s + \varepsilon t)$$

(we assume that we extended trivially the original function u^ε outside S). The pattern $R_s^\varepsilon u^\varepsilon$ is bounded in $L^\infty(\mathbb{R}^m)$ by C. As the space of patterns we use the closed ball

$$K := \bar{B}_C(0) \subset L^\infty_w(\mathbb{R}^m),$$

where L^∞_w indicates that we use the weak-$\star$ topology on K. This makes K compact.

Since the pattern depends in an oscillatory fashion on s, one proceeds as in the construction of Young measures and considers instead of the values $R_s^\varepsilon u^\varepsilon \in K$ the measure ν_s^ε, the Dirac measure on K in the point $R_s^\varepsilon u^\varepsilon$.

Definition 2.1 (measure of microlocal patterns). *Given a sequence $u^\varepsilon \in \bar{B}_C(0) \subset L^\infty(S)$ and the corresponding Dirac measures ν_s^ε, we define the measure ν^ε on $S \times K$ by*

$$S \times K \supset \bar{S} \times \bar{K} \mapsto \nu^\varepsilon(\bar{S}, \bar{K}) := \int_{\bar{S}} \nu_s^\varepsilon(\bar{K})\ de^\varepsilon(s)$$

for all Borel sets $\bar{S} \subset S$ and $\bar{K} \subset K$. The energy density $e^\varepsilon(s)$ still has to be specified; we always assume $\int_S de^\varepsilon(s) \le C'$. Then we can choose a subsequence $\varepsilon \to 0$ such that for a finite limit measure ν there holds in the sense of weak-$\star$ convergence

$$\nu = \lim_{\varepsilon \to 0} \nu^\varepsilon \in \mathcal{M}(S \times K).$$

We call ν the measure of microlocal patterns of the (sub)sequence.

We will also use the following two projections of ν. The projection μ of ν to S, which coincides with the energy density of ν,

$$S \supset \bar{S} \mapsto \mu(\bar{S}) := \nu(\bar{S} \times K) = \lim_{\varepsilon \to 0} \int_{\bar{S}} de^\varepsilon(s),$$

and the projection ν_K of ν to K,

$$\bar{K} \mapsto \nu_K(\bar{K}) := \nu(S \times \bar{K}).$$

The elements in the support of ν_K are the realized patterns.

Possible micropatterns. Our aim is to characterize the realized patterns of solution sequences in our model problem. The method is to successively exclude possibilities. A first concept is that of *possible micropatterns*. In the subsequent definition we assume that the sequence u^ε is a sequence of solutions to a given family of equations.

DEFINITION 2.2 (possible micropattern). *For a point $s \in S$ we say that $U_s : \mathbb{R}^m \to \mathbb{R}$ is a possible micropattern in s if there exist a sequence $\varepsilon \to 0$, boundary and initial conditions with a solution sequence u^ε, and a sequence of points $s_\varepsilon \to s$, such that*

$$U^\varepsilon_{s_\varepsilon} := R^\varepsilon_{s_\varepsilon} u^\varepsilon \to U_s \quad \text{in } K.$$

U is a possible micropattern, if it is a possible micropattern in some point s.

The concept of possible micropatterns does not allow us to study the distribution of patterns, but it gives a necessary condition for a pattern to be contained in the support of ν_K. In fact, an elementary proof yields the following result.

Remark 2.3. Every measure on patterns ν_K of a solution sequence u^ε has its support contained in the set of possible micropatterns.

The fundamental property of possible patterns is that they satisfy the rescaled equations (note that we loose boundary and initial conditions). This fact will lead to a characterization of possible micropatterns.

2.2. Possible patterns for the motion of fronts. As described, we expect ε-scale phenomena in the neighborhood of points (x,t) where p^ε has a jump, i.e., in explosion points. In order to study the local behavior, we define for a given $(x,t) = (x_1, 0, t)$ the blowup of solutions

$$\begin{aligned} P^\varepsilon(y,\tau) = P^\varepsilon_{(x,t)}(y,\tau) &:= (R^\varepsilon_{(x,t)} p^\varepsilon)(y,\tau) = p^\varepsilon(x+\varepsilon y, t+\varepsilon\tau), \\ H^\varepsilon(y_1,\tau) = H^\varepsilon_{(x_1,t)}(y_1,\tau) &:= \frac{1}{\varepsilon}(R^\varepsilon_{(x_1,t)} h^\varepsilon)(y_1,\tau) \\ &= \frac{1}{\varepsilon} h^\varepsilon(x_1+\varepsilon y_1, t+\varepsilon\tau). \end{aligned}$$

For explosion points (x,t) we expect nontrivial limits of $(P^\varepsilon, H^\varepsilon)$ and our next aim is to determine the possible limits. We will call such limits explosion patterns.

The equations for $(P^\varepsilon, H^\varepsilon)$ are identical to (1.2)–(1.5), except that the factors ε are replaced by the factor 1 in (1.2) and (1.3). We have to substitute the boundary condition (1.6) by a condition on the asymptotic behavior at infinity.

The solutions P^ε are physically meaningful on space-time domains of size $\frac{1}{\varepsilon} \times \frac{1}{\varepsilon}$. After the trivial extension of the functions, their domains are

$$\begin{aligned} \bar\Omega &:= \mathbb{R} \times \mathbb{R}_-, & \bar\Gamma &:= \mathbb{R} \times \{0\} \equiv \mathbb{R}, \\ \bar\Gamma_1 &:= \mathbb{Z} + (-\gamma,\gamma), & \bar\Gamma_2 &:= \mathbb{Z} + (\gamma, 1-\gamma). \end{aligned}$$

In order to define the microlocal patterns we consider the functions $p^\varepsilon|_\Gamma : [-1,1] \times [0,T] \to \mathbb{R}$ and set $S := [-1,1] \times [0,T]$. A limit measure is found as $\nu \in \mathcal{M}(S \times K)$ with $K = \bar B_C(0) \subset L^\infty_w(\mathbb{R}^2)$. Note that $p^\varepsilon|_\Gamma$ determines uniquely its harmonic extension p^ε and the height function h^ε up to multiples of s_0. With this identification we can consider limit patterns also as functions $P : \bar\Omega \times \mathbb{R} \to \mathbb{R}$ and $H : \bar\Gamma \times \mathbb{R} \to [0, s_0)_{\text{per}}$.

We use the projection Q_1 defined as

$$(Q_1 f)(y) = \begin{cases} \dfrac{1}{2\gamma} \displaystyle\int_{k-\gamma}^{k+\gamma} f(\zeta)\, d\zeta & \text{for } y \in (k-\gamma, k+\gamma),\ k \in \mathbb{Z}, \\ f(y) & \text{for } y \in (k+\gamma, k+1-\gamma),\ k \in \mathbb{Z}. \end{cases}$$

Using Q_1 we can write the rescaled equations as (2.1)–(2.4). The next lemma states that every possible pattern is in fact a solution of these rescaled equations.

LEMMA 2.4. *Every possible micropattern (P, H) in a point $(x_1, 0, t)$ satisfies the rescaled equations*

$$\partial_\tau H(y_1, \tau) = -Q_1 \partial_2 P(y_1, 0, \tau) \qquad \forall (y_1, 0) \in \bar{\Gamma}_1, \tag{2.1}$$

$$P(y_1, 0, \tau) \in \hat{\mathcal{P}}_0(y_1, Q_1 H(y_1, \tau)) \qquad \forall (y_1, 0) \in \bar{\Gamma}_1, \tag{2.2}$$

$$\partial_2 P(y_1, 0, \tau) = 0 \qquad \forall (y_1, 0) \in \bar{\Gamma}_2, \tag{2.3}$$

$$-\Delta_y P(\cdot, \tau) = 0 \qquad \textit{in } \bar{\Omega}, \tag{2.4}$$

for almost every $\tau \in \mathbb{R}$; $\hat{\mathcal{P}}_0$ is the multivalued function

$$\hat{\mathcal{P}}_0(k, s) = \begin{cases} \mathcal{P}_0(k, s) & \textit{for } s \notin s_0\mathbb{Z}, \\ \{0, p_{\max}(k)\} & \textit{for } s \in s_0\mathbb{Z}. \end{cases}$$

Additionally there holds

$$\|P\|_{L^\infty(\bar{\Omega})} \le C \qquad \forall t \in \mathbb{R}. \tag{2.5}$$

Remarks. (a) Inequality (2.5) must be interpreted as a boundary condition. (b) The well-posedness of the system can be shown by a limiting procedure with the help of Remark 1.1. (c) Remark A.1 suggests that the decay for $t \to +\infty$ is like t^{-1} (see also the appendix of [10]).

Proof. Let (P, H) be a possible micropattern. By definition there exist a solution sequence $(p^\varepsilon, h^\varepsilon)$ and points $s_\varepsilon = (x_\varepsilon, t_\varepsilon) \in \Gamma \times [0, T]$, such that the rescaled solutions $(P^\varepsilon_{s_\varepsilon}, H^\varepsilon_{s_\varepsilon})$ converge in L^∞_w to (P, H). For $H \ne 0$ one verifies $\varepsilon^{-1}\mathrm{dist}(x_\varepsilon, \varepsilon\mathbb{Z}) \to 0$; it is therefore no loss of generality to assume $x_\varepsilon \in \varepsilon\mathbb{Z}$. The sequence of rescaled solutions satisfies on increasing domains the rescaled equations (2.1)–(2.4).

We use the following weak form of the rescaled equations:

$$\int_{\mathbb{R}} \int_{\bar{\Omega}} \Delta_y \Phi \cdot P^\varepsilon_{s_\varepsilon} = -\int_{\mathbb{R}} \int_{\bar{\Gamma}_1} H^\varepsilon_{s_\varepsilon} \cdot \partial_\tau \Phi + \int_{\mathbb{R}} \int_{\bar{\Gamma}} P^\varepsilon_{s_\varepsilon}(y_1, 0, \tau) \cdot \partial_2 \Phi, \tag{2.6}$$

$$\int_{\mathbb{R}} Q_1 P^\varepsilon_{s_\varepsilon}(k, 0, \tau) \cdot \varphi(\tau)\, d\tau = \int_{\mathbb{R}} \mathcal{P}_0(k, Q_1 H^\varepsilon_{s_\varepsilon}(k, \tau)) \cdot \varphi(\tau)\, d\tau, \tag{2.7}$$

$$H^\varepsilon_{s_\varepsilon} - Q_1 H^\varepsilon_{s_\varepsilon} = 0, \quad P^\varepsilon_{s_\varepsilon}(\cdot, 0, \cdot) - Q_1 P^\varepsilon_{s_\varepsilon}(\cdot, 0, \cdot) = 0, \tag{2.8}$$

satisfied for all $\Phi \in C^2_0(\bar{\Omega} \cup \bar{\Gamma} \times \mathbb{R})$ with $\partial_2 \Phi = 0$ on $\bar{\Gamma}_2$ and $Q_1 \Phi|_{\bar{\Gamma}} = \Phi|_{\bar{\Gamma}}$, and all $\varphi \in C^0_0(\mathbb{R})$, $k \in \mathbb{Z}$. The equations are satisfied for all $\varepsilon < \varepsilon_0$, a threshold that depends on the support of the test functions and on k.

We can take the limit $\varepsilon \to 0$ along the subsequence in the weak equations. We find that (P, H) again solves the weak equations (2.6) and (2.8). In particular, P is harmonic, satisfies a homogeneous Neumann condition on $\bar{\Gamma}_2$, and is piecewise constant on $\bar{\Gamma}_1$ for a.e. τ. Together with the L^∞-estimate this implies spatial continuity of P, and an L^∞-bound for $Q_1\partial_\tau H$. Therefore (2.1), (2.3), and (2.4) are satisfied in the strong sense.

It remains to take the limit in the nonlinear material law in (2.7). We exploit the fact that $Q_1 H^\varepsilon$ converges uniformly on compact sets to $Q_1 H$, and conclude (2.2). ∎

Remark 2.5. If the maximal pressure $p_{\max} = p_{\max}(k)$ is independent of k, then every solution of (2.1)–(2.5) satisfies

$$P(x,t) \le p_{\max} \qquad \forall x \in \bar{\Omega},\ t \in \mathbb{R}.$$

Proof. By the comparison principle for harmonic functions, $P(\cdot,t)$ is bounded on the finite domain $(-M,M)\times(-M,0)$ by the periodic harmonic functions q_M with

$$\begin{aligned} q_M &= p_{\max} \quad \text{on } \bar{\Gamma}_1, \qquad \partial_2 q_M = 0 \quad \text{on } \bar{\Gamma}_2,\\ q_M &= C \quad \text{on } (-M,M)\times\{-M\}. \end{aligned}$$

For $M \to \infty$ the sequence of functions q_M tends to $p_{\max}$ on every bounded set. This implies the result. ∎

PROPOSITION 2.6. *We consider a solution (P,H) of system (2.1)–(2.5). On the material law we assume $\partial_s \mathcal{P}_0 > 0$ on $(0,s_0)$. If $p_{\max}$ is independent of k, then (P,H) can only be*

(a) *a constant solution, $P(\cdot) \equiv p^* \in [0,p_{\max}]$ in $\bar{\Omega}\times\mathbb{R}$,*

(b) *a solution with simultaneous explosions at an explosion time T_0 and with an outlet pattern $\alpha \in \{0,1\}^{\mathbb{Z}}$,*

$$\begin{aligned} P(\cdot, T_0+\tau) &= p_{\max} \qquad \forall \tau < 0,\\ P(\cdot, T_0+\tau) &= P_\alpha(\cdot,\tau) \quad \forall \tau > 0. \end{aligned}$$

Here P_α is the explosion solution to the opening pattern α: P_α is the unique solution of (2.1)–(2.5) to the initial values

$$P_\alpha(y_1,0,t=0) = \begin{cases} p_{\max} & \text{for } y_1 \in (k-\gamma,k+\gamma),\ \alpha(k)=1,\\ 0 & \text{for } y_1 \in (k-\gamma,k+\gamma),\ \alpha(k)=0. \end{cases}$$

Proof. Let (P,H) be given. We claim that if the solution satisfies $P(\cdot,T_0) \not\equiv p_{\max}$ for some $T_0 \in \mathbb{R}$, then no explosion can happen at a later time.

We compare P in the neighborhood of a single cell, say

$$(y_1,y_2) \in R := (-1/2,1/2)\times(-1,0),$$

with the solution q of

$$\begin{aligned} \Delta_y q(t) &= 0 && \text{in } R,\ \forall t \in (T_0,\infty),\\ q(\cdot,0,t) &= q_0(t) && \text{on } (-\gamma,\gamma),\ \forall t \in (T_0,\infty),\\ \partial_2 q &= 0 && \text{on } (-1/2,-\gamma)\cup(\gamma,1/2),\\ q &= p_{\max} && \text{on } \partial R \backslash \bar{\Gamma},\\ \partial_t q_0 &= -c_q \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} \partial_2 q. && \end{aligned}$$

The initial value for q_0 and the constant c_q remain to be chosen.

We claim that for $q_0(t = 0) < p_{\max}$ the function q_0 remains below $p_{\max}$ for all times. Since the minimum of q is on Γ_1, there holds $-\partial_2 q \geq 0$ on Γ_1. For some positive $\bar{c}$ we have the estimate

$$\frac{1}{2\gamma}\int_{-\gamma}^{\gamma}(-\partial_2 q) \leq \bar{c}(p_{\max} - q_0).$$

This holds since the left-hand side is finite for fixed q_0 and a linear function of the difference $p_{\max} - q_0$. The estimate implies that q_0 grows with a speed at most proportional to $p_{\max} - q_0$. This implies that a convergence of q_0 to $p_{\max}$ has at most exponential rate.

We now use q as a comparison function for P. In a cell with $P(y_1, 0, T_0) < p_{\max}$ we choose $c_q > \sup_s \partial_s \mathcal{P}_0$ and $p_{\max} > q_0(t = T_0) > P(y_1, 0, T_0)$. Then $P \leq q$ holds for $t = T_0$ on R. If $P = q$ for the first time at some point in R, coincidence holds also on a point of the boundary Γ_1, and we have $P = q$ along Γ_1, since both functions are constant. Then $-\partial_2 q \geq -\partial_2 P$ along Γ_1 and we find

$$\begin{aligned}\partial_t P = \partial_s \mathcal{P}_0 \cdot \partial_t H &= -\partial_s \mathcal{P}_0 \cdot \frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\partial_2 P \\ &\leq -\partial_s \mathcal{P}_0 \cdot \frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\partial_2 q < -c_q \frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\partial_2 q = \partial_t q_0.\end{aligned}$$

Therefore $P \leq q$ holds for all times and also P does not reach the value $p_{\max}$ in finite time. In cells with $P(y_1, t) = p_{\max}$ the Hopf lemma implies $\partial_t P(y_1, 0, t) < 0$, and we can apply the above argument for $|t - T_0|$ small.

We claim that solutions without explosions are constants. In (A.4) of Proposition A.3, we demonstrate that a solution without explosions on the time interval $(0, \infty)$ converges to a constant function for $t \to \infty$, independent of the initial values. Since the solution (P, H) is defined for all negative times we conclude that $P(t)$ is constant for all t. The uniqueness of the solution P_α follows from the linearized stability. □

By the above proposition we know that there exist only a few possible micropatterns. Next we want to use this as information about microlocal patterns. To this end we have to choose an energy density.

DEFINITION 2.7. *The measure of microlocal patterns ν of the sequence p^ε is defined via the energy density e^ε, which we construct as a sum of Dirac measures. We define the finite set of explosion points by*

$$M^\varepsilon := \{s = (x_1, 0, t) \in \Gamma \times (0, T) \mid p^\varepsilon(s) = 0,\, x_1 \in \varepsilon\mathbb{Z}\},$$

and set

$$e^\varepsilon(\bar{S}) := \varepsilon \sum_{(x_1,0,t)\in M^\varepsilon \cap \bar{S}} s_0(x_1 \varepsilon^{-1}).$$

The set M^ε is finite, since $\partial_t p^\varepsilon > 0$ holds in points of M^ε. The sequence of measures e^ε is bounded. Since H^ε never passes a value in $s_0\mathbb{Z}$ in the negative direction, every explosion corresponds to a loss $\varepsilon 2\gamma s_0$ of fluid mass. Therefore in the space-time volume $\bar{S} = \Gamma \times (t_1, t_2)$ there holds by conservation of mass

$$|M^\varepsilon| \leq \frac{2}{\varepsilon} + \frac{\|V_0\|_{L^1}|t_2 - t_1|}{2\gamma\varepsilon \inf\{s_0(\cdot)\}},$$

which implies

$$e^{\varepsilon}(\bar{S}) \leq \left(2 + \frac{\|V_0\|_{L^1}|t_2 - t_1|}{2\gamma \inf\{s_0(\cdot)\}}\right) \sup\{s_0(\cdot)\}.$$

Remark. The disadvantage of the above energy density e^{ε} is that it can be defined only in the case that explosions can be localized to a point; for a continuous $\mathcal{P}_0$-function we cannot define an analogue of it.

In the continuous case one could introduce an energy density by setting

$$\tilde{e}^{\varepsilon}(x,t) := \frac{1}{2\gamma}(v_2^{\varepsilon}(x,0,t))_+.$$

The two energy densities have many similarities; note the disadvantage of $\tilde{e}^{\varepsilon}$ that it can become positive also because of oscillations without explosions.

We saw that the qualitative behavior of the limit p^0 is different in the periodic situation and in the (expected) physical situation of explosions that are far from each other. The limit measure ν allows us to distinguish the two situations in terms of the explosion patterns. In the periodic case, the patterns will also be periodic; i.e., ν is supported on periodic and therefore infinite patterns. With the subsequent definition we distinguish the two cases.

Definition 2.8. *We say that ν is of* finite type*, if for some constant $C_f \in \mathbb{N}$ there holds the following: every observable explosion pattern has at most C_f explosion points, i.e.,*

$$P_\alpha \in \operatorname{supp}(\nu_K) \Rightarrow |\alpha| := |\{k \in \mathbb{Z} : \alpha(k) = 1\}| \leq C_f.$$

Consider the energy density e^{ε} and assume that ν is of finite type. Then

$$\operatorname{supp}(\nu_K) \subset \bigcup_{\alpha \in A} \{P_\alpha\} \quad \text{with } A \subset \{\alpha \in \{0,1\}^{\mathbb{Z}} : |\alpha| \leq C_f\}.$$

The limit measure ν can then be written as

$$\nu(\bar{S} \times \bar{K}) = \sum_{\alpha \in A} \int_{\bar{S}} \eta_\alpha(\bar{K}) d\mu_\alpha(s), \tag{2.9}$$

where μ_α are measures on $\Gamma \times [0,T]$ and η_α is the Dirac measure on K on the explosion solution P_α.

We exploited that the constant pressure solutions do not contribute to the energy and in (2.9) that A is countable.

3. Limit measures of finite type and regions without explosions. We continue our study of a sequence of solutions $(p^{\varepsilon}, h^{\varepsilon})$ and their L^{∞}_w-limit p^0. In order to derive and even to formulate upscaled equations for p^0 we need some regularity result for p^{ε} and p^0. In the case that the limit measure ν is of finite type, a fundamental regularity statement holds: loosely speaking, spatial averages of the pressure p^0 cannot have jumps in time. This, in turn, helps us to find regions without explosions: points with $p^0 < p_{\max}$ have neighborhoods in which the ε-system is without explosions for all small ε. Note that the pointwise statement $p^0 < p_{\max}$ has to be interpreted in an appropriate way for the L^{∞}-function p^0. Regularity properties of p^{ε} in regions without explosions will be exploited in section 4.

We start with a crucial observation: if the average pressure of the ε-system is below $p_{\max}$ in a small area, then there cannot be explosions.

LEMMA 3.1 (quantitative Hopf lemma/near field effect). *Let $q : [-1,1]\times[-1,0] \to \mathbb{R}$ be a harmonic function, periodic in the first variable, and continuous up to the boundary, with $0 \le q \le p_{\max}$. Let $x \in [-1,1] \times \{0\} \equiv [-1,1]_{\rm per}$ be a point with $q(x) = p_{\max}$ and*

$$\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} q(\xi)\ d\xi = \rho < p_{\max}.$$

Then the Neumann derivative in x has a lower bound

$$\partial_2 q(x) \ge c_H(\delta,\rho)$$

with $c_H(\delta,\rho) \to +\infty$ for fixed $\rho < p_{\max}$ and $\delta \to 0$.

COROLLARY 3.2. *There exists $\delta_0 > 0$ depending only on $\|V_0\|_{L^\infty}$, $p_{\max}$, and ρ, such that for small ε every solution p^ε of (1.2)–(1.6) with*

$$p^\varepsilon_\delta(x,t) := \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} p^\varepsilon(\xi,t)\ d\xi \le \rho \qquad \forall t\in(t_1,t_2)$$

satisfies the following for $\delta < \delta_0$: p^ε has no explosion in $(x-\delta,x+\delta)\times(t_1,t_2)$.

Proof. In order to have an explosion in (x,t) we must have $\lim_{\tau\nearrow t} p^\varepsilon(x,\tau) = p_{\max}$ and $\lim_{\tau\nearrow t}(-\partial_2 p^\varepsilon(x,t)) \ge 0$. If we assume that $p^\varepsilon \le p_{\max}$ holds in the whole domain, Lemma 3.1 yields that $-\partial_2 p^\varepsilon$ is negative for small δ, and no explosion is possible. In the general case we decompose p^ε into one part p_A with the boundary values p^ε on Γ_1^ε and $\partial_2 p_A = 0$ on Γ_0, and a remainder p_B with vanishing values on Γ_1^ε and $-\partial_2 p_B = V_0$ on Γ_0, both with $\partial_2 p_{A,B} = 0$ on Γ_2^ε. Then $|\partial_2 p_B|$ is uniformly bounded and p_A is bounded by $p_{\max}$, so Lemma 3.1 applies to p_A. We conclude that $\partial_2 p_A(x,t)$ is large and that no explosion is possible. $\square$

Proof of Lemma 3.1. We can assume $(x,t) = (0,0)$. The lemma follows from an argument that is related to rearrangement. For given ρ and δ, the Neumann derivative is minimal if $q = p_{\max}$ in a neighborhood $(-s,s)$ of x and $q = 0$ in the region $s < |x| < \delta$, where s is chosen such that $2sp_{\max} = \int_{-\delta}^{\delta} q = 2\delta\rho$. This can be seen as follows. The quantity $\partial_2 q(0,0)$ is decreased if we modify the Dirichlet boundary values on $(-\delta,\delta)$ by adding a nonnegative multiple of the function $v : (-1,1) \to \mathbb{R}$,

$$v(x) := \begin{cases} +1 & \text{for } |x| \in (x_1, x_1 + r),\\ -1 & \text{for } |x| \in (x_2, x_2 + r),\\ 0 & \text{else}\end{cases}$$

for $0 < x_1 < x_1 + r < x_2$ or $0 > x_1 > x_1 - r > x_2$. The harmonic extension $\bar v$ of v satisfies $\partial_2 \bar v(0) \le 0$, since $\bar v$ is nonnegative in a neighborhood of 0 by the monotonicity of the Green's function.

The above modifications allow a redistribution of the boundary values of q. It allows us to compare $\partial_2 q(0)$ with $\partial_2 q_\delta(0)$, where q_δ is the solution of the periodic problem

$$\begin{aligned} \Delta q &= 0 && \text{in } (-1,1)\times(-1,0),\\ q &= w && \text{on } (-1,1)\times\{0\},\\ q &= p_{\max} && \text{on } (-1,1)\times\{-1\}, \end{aligned}$$

with the boundary values

$$w(x) := \begin{cases} 0 & \text{for } s < |x| < \delta, \\ p_{\max} & \text{else.} \end{cases}$$

The number $s := \sigma\delta := \frac{\rho}{p_{\max}}\delta$ is determined by the integral condition $\int_{-\delta}^{\delta} w = 2\delta\rho$. From the Hopf lemma we know that $c_H(\delta, \rho) := \partial_2 q_\delta(0) > 0$.

In order to show $c_H(\delta, \rho) \to +\infty$ for $\delta \to 0$ it remains to consider the family of harmonic functions q_δ for $s = \sigma\delta$ and σ fixed. On the domain $\mathbb{R} \times \mathbb{R}_-$ we find the general solution q_δ by rescaling q_1: $q_\delta(x) = q_1(x/\delta)$. We calculate

$$\partial_2 q_\delta(0) = \frac{1}{\delta}\partial_2 q_1(0) \to \infty$$

for $\delta \to 0$. On the bounded domain the asymptotic behavior remains unchanged and we find the result. ◻

LEMMA 3.3. *Assume that the measure ν is of finite type. Let $0 < c < 1$ be an arbitrary number that we interpret as an explosion density. Then there is no subsequence $\varepsilon_k \to 0$ such that in the ε_k-systems there happen $\frac{c}{\varepsilon_k}$ explosions simultaneously.*

Proof. We present here the proof in the case that ν_K is supported only on the pattern P_0 of the single explosion. The proof in the general case requires only additional notational effort.

Step 1. We consider a sequence $\beta_N \in \{0,1\}^{\{-N,\dots,N\}}$ (patterns) with $|\beta_N| := \big|\{x \in \{-N,\dots,N\} \mid \beta_N(x) = 1\}\big| \geq c \cdot 2N$.

Claim. There exist a number $\rho > 0$, a distance $d \in \mathbb{N}$, and a subsequence $N \to \infty$ such that all β_N realize the distance d at least with density ρ, i.e.,

$$\big|\{x \in \{-N,\dots,N\} \mid \beta_N(x) = 1,\, \beta_N(x+d) = 1\}\big| \geq \rho \cdot 2N \quad \forall N.$$

We argue by contradiction and assume that the claim is not true. With d_0 applications we find that for every $\rho > 0$ and $d_0 \in \mathbb{N}$, there exists $N_0 > 0$ such that for all $N \geq N_0$ the distances $d = 1, 2, \dots, d_0$ are realized with density less than ρ. We calculate for large N the density of β_N. On $d_0\rho \cdot 2N$ points we have no restriction, on the remaining places we have at most $2N/(d_0+1)$ values 1. We calculate for the density

$$c \leq \frac{|\beta_N|}{2N} \leq \frac{d_0\rho \cdot 2N + 2N/(d_0+1)}{2N} = d_0\rho + \frac{1}{d_0+1}.$$

Since ρ and d_0 were arbitrary we arrive at a contradiction.

Step 2. We assume that for a subsequence $\varepsilon_k \to 0$ there are $\frac{c}{\varepsilon_k}$ simultaneous explosions. We claim that this contradicts the fact that no pattern of length 2 is contained in the limit measure ν.

We set $N = 1/\varepsilon$ and denote by t^ε the time instance of the simultaneous explosions. For $x \in \mathbb{Z}$ we set $\beta_N(x) = 1$ if at position $(\varepsilon x, t^\varepsilon)$ the ε-problem has an explosion, i.e., $p^\varepsilon(\varepsilon x, t^\varepsilon) = 0$, $\beta_N(x) = 0$ otherwise. By the above claim, for some $\rho > 0$, $d \in \mathbb{N}$, and a subsequence $N \to \infty$, β_N realizes the distance d of values 1 at least ρN times, say at positions $\{x_i\}$. Let $\bar{K}$ be a neighborhood of the single explosion P_0 in K, such that all patterns $k \in K$ with an explosion in $x_1 = d$ are not contained in $\bar{K}$. Then there are ρ/ε points $s_i = (\varepsilon x_i, t^\varepsilon)$ at which $R_s^\varepsilon p^\varepsilon$ is not contained in $\bar{K}$. Therefore

$$\nu^\varepsilon(S \times \bar{K}) \leq \nu^\varepsilon(S \times K) - \rho \inf\{s_0\}.$$

On the other hand, by assumption, we have

$$\nu^\varepsilon \to \mu \otimes \delta_{P_0} \in \mathcal{M}(S \times K),$$

which implies

$$\begin{aligned}\mu(S) &= \lim_{\varepsilon\to 0} \nu^\varepsilon(S \times \bar{K}) \le \lim_{\varepsilon\to 0} \nu^\varepsilon(S \times K) - \rho \inf\{s_0\} \\ &= \mu(S) - \rho \inf\{s_0\},\end{aligned}$$

a contradiction. ∎

Note that the result of the above lemma could be improved: not only can we not have $O(N)$ explosions at the same time instance, but it is also impossible to realize $O(N)$ explosions in a time span of length $O(\varepsilon)$. Even if in Proposition 2.6 we showed that in the limit patterns explosions happen simultaneously, this need not be true for the ε-problem. Nevertheless, the statement of Lemma 3.3 will be sufficient for our purposes.

We next prove an auxiliary result on the maximal gain of fluid-mass in a test volume in a short time.

LEMMA 3.4. *In every domain $W = B_\delta(x) \times (-1, 0)$, the maximal total inward flow through the lateral boundaries $\Sigma_\pm = \{x \pm \delta\} \times (-1, 0)$ can be estimated with arbitrary $\delta_0 > \delta$ by*

$$\begin{aligned}\int_{t_1}^{t_2} \int_\Sigma \partial_n p^\varepsilon &\le 2 p_{\max} \frac{t_2 - t_1}{\delta_0 - \delta} \\ &\quad + 4\gamma(\delta_0 - \delta) \sup\{s_0\} + 2(t_2 - t_1)(\delta_0 - \delta)\|V_0\|_\infty. \end{aligned} \tag{3.1}$$

In particular, the bound can be chosen arbitrarily small for $t_2 - t_1$ small.

Proof. We do the calculations for the right boundary $\Sigma_{x+\delta} := \{x + \delta\} \times (-1, 0)$. The average total flow to the left between $x + \delta$ and $x + \delta_0$ is

$$\begin{aligned}\int_{t_1}^{t_2} \frac{1}{\delta_0 - \delta} \int_{x+\delta}^{x+\delta_0} \int_{-1}^{0} \partial_1 p^\varepsilon &\le \frac{1}{\delta_0 - \delta} \int_{t_1}^{t_2} \left(\int_{\Sigma_{x+\delta_0}} p^\varepsilon - \int_{\Sigma_{x+\delta}} p^\varepsilon \right) \\ &\le p_{\max} \frac{t_2 - t_1}{\delta_0 - \delta}.\end{aligned}$$

Then there is an intermediate value $z \in (x + \delta, x + \delta_0)$ where the average total flow is realized,

$$\int_{\Sigma_z} \int_{t_1}^{t_2} \partial_1 p^\varepsilon \le p_{\max} \frac{t_2 - t_1}{\delta_0 - \delta}.$$

By incompressibility, the maximal total flow through $\Sigma_{x+\delta}$ is the sum of two contributions: (1) the flow through Σ_z in the time interval (t_1, t_2), (2) the total inward flow through the upper boundary $(x+\delta, z) \times \{0\}$ and the lower boundary $(x+\delta, z) \times \{-1\}$. The total volume that can be released on Γ between $x + \delta$ and z is bounded by $2\gamma(\delta_0 - \delta) \cdot \sup\{s_0\}$. Formula (3.1) follows, with the factor 2 we include the left boundary. ∎

If the limit measure ν is of finite type we expect that the typical distance between explosions is large compared to ε. The following proposition verifies and sharpens this statement. Let us imagine that at temporal distances $O(\sqrt{\varepsilon})$ there happen $1/\sqrt{\varepsilon}$

explosions. Then the spatio-temporal distance between two explosions is always large compared to ε. Nevertheless, $1/\varepsilon$ explosions happen in a given spatio-temporal region. With such a construction it is also possible to have $1/\varepsilon$ explosions in an arbitrarily short time span Δt. The next proposition excludes this possibility for our evolution equations.

PROPOSITION 3.5. *If the measure ν is of finite type, then the marginal $\mu(\Gamma, \cdot) \in \mathcal{M}([0,T])$ of the measure ν, $\mu(\Gamma,(t_1,t_2)) := \nu(\Gamma \times (t_1,t_2) \times K)$, contains no atoms,*

$$\mu(\Gamma \times \{t\}) = 0 \quad \forall t \in (0,T]. \tag{3.2}$$

Proof. We assume that the limit measure $\mu(\Gamma,\cdot)$ contains an atom. Then for some $\bar{t}$,

$$e := \liminf_{\Delta t \to 0} \mu(\Gamma \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) > 0. \tag{3.3}$$

The interpretation is that along the subsequence $\varepsilon \to 0$ we find $O(e/\varepsilon)$ explosions in the time span $(\bar{t} - \Delta t, \bar{t} + \Delta t)$.

To arrive at a contradiction we first choose $\Delta\rho$ small compared to e. The smallness will be specified towards the end of our calculations. We now fix $\delta > 0$. The required smallness for δ depends only on the numbers e and $\Delta\rho$, and on the boundary data V_0. Note that choosing $\delta > 0$ small we find, by Corollary 3.2,

$$\begin{aligned} &p_\delta^\varepsilon(x,t) < p_{\max} - \Delta\rho \quad \forall t \in (t_1,t_2) \\ &\qquad \Rightarrow \text{no explosions happen in } B_\delta(x) \times (t_1,t_2). \end{aligned} \tag{3.4}$$

Given δ, we find a position $x \in [-1,1]$ such that

$$\limsup_{\Delta t \to 0} \mu(B_\delta(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) \geq e \cdot \delta.$$

Using periodicity of the domain we can assume $x \in (-1+\delta, 1-\delta)$. We now fix Δt small, such that $\mu(B_\delta(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) \geq \frac{e\delta}{2}$, and such that additionally for a given constant $c_g > 0$ (depending on δ) there holds

$$\frac{e\delta}{16} \geq \|V_0\|_\infty \cdot 4\delta|\Delta t| + c_g\sqrt{\Delta t}.$$

We next choose ε small enough to have

$$\mu^\varepsilon(B_\delta(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) \geq \frac{1}{2}\mu(B_\delta(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)),$$

and that there are no c/ε explosions at the same time instance (a small c is chosen in dependence of e and δ). The latter property is insured for small ε by Lemma 3.3. For this ε, we set $t_1 \in [\bar{t} - \Delta t, \bar{t} + \Delta t]$ to be the moment of the first explosion in $B_\delta(x)$, and set $t_2 = \bar{t} + \Delta t$. We introduce the time instance t^ε at which half of the explosions in $B_\delta(x) \times [t_1,t_2]$ have happened. At this point Lemma 3.3 guarantees that some explosions must happen after this time instance.

We can estimate the number of explosions of the ε-system in the test volume by

$$\begin{aligned} \mu^\varepsilon(B_\delta(x) \times (t_1, t^\varepsilon]) &\geq \frac{1}{2}\mu^\varepsilon(B_\delta(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) \\ &\geq \frac{1}{4}\mu(B_\delta(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) \geq \frac{e\delta}{8}. \end{aligned}$$

This means that in the ε-system at least $(e\delta)/(8\varepsilon \sup\{s_0\})$ explosions happen in $B_\delta(x) \times [t_1, t^\varepsilon]$. We will show that this implies that the average pressure is below $p_{\max}$ in t^ε and will conclude with Corollary 3.2. It remains to verify the implication

$$\text{loss of mass in explosions} \;\Rightarrow\; \text{loss of pressure.}$$

In what follows we will use the estimate for the lateral inflow

$$\int_{t_1}^{t_2} \int_\Sigma \partial_n p^\varepsilon \le c_g \sqrt{\Delta t} \tag{3.5}$$

for small Δt, which follows from Lemma 3.4 if we choose $\delta_0 = \delta + \sqrt{t_2 - t_1}$. We calculate the gain of fluid mass in the ε-system by adding the inflow into the box from below and the loss due to explosions; in the following we consider only values of h^ε in $[0, s_0]$; i.e., in an explosion we set h^ε to zero. For all $t \in (t^\varepsilon, \bar{t} + \Delta t)$,

$$\begin{aligned}
&\int_{B_\delta(x)\cap\Gamma_1^\varepsilon} \varepsilon^{-1} h^\varepsilon(\xi, \tau)\, d\xi \bigg|_{\tau=t_1}^{t} \\
&\quad \le \|V_0\|_\infty \cdot 2\delta 2|\Delta t| + c_g\sqrt{\Delta t} - \mu^\varepsilon(B_\delta(x) \times (t_1, t^\varepsilon)) \\
&\quad \le \|V_0\|_\infty \cdot 4\delta|\Delta t| + c_g\sqrt{\Delta t} - \frac{e\delta}{8} \le -\frac{e\delta}{16}.
\end{aligned}$$

We see that a decrease of fluid mass of order $O(1)$ took place in the test volume. We want to conclude from this that the average pressure also decreased by the order $O(1)$. We have to compare the effect of loss of mass with an effect that has the potential to increase the average pressure: redistribution of mass.

This effect is controlled in the following. We know that the average pressure at time t_1 satisfies $p_\delta^\varepsilon(x) \ge p_{\max} - \Delta\rho$, since an explosion happens at this time instance. Until time t the values of h^ε change, but although some of them might increase, we verify that this is not a large contribution. We sum over $x_i \in \varepsilon\mathbb{Z}$ with $x_i \in B_\delta(x)$,

$$\begin{aligned}
\frac{1}{2\delta}\sum_i (h^\varepsilon(x_i, t) - h^\varepsilon(x_i, t_1))_+ &\le \frac{1}{2\delta}\sum_i \left(\varepsilon\frac{p_{\max}}{a_0(i)} - h^\varepsilon(x_i, t_1)\right) \\
&= \frac{\varepsilon}{2\delta}\sum_i \frac{1}{a_0(i)}(p_{\max} - p^\varepsilon(x_i, t_1)) \\
&\le \frac{1}{\inf\{a_0\}}\frac{1}{2\delta}\frac{1}{2\gamma}\int_{B_\delta(x)\cap\Gamma_1^\varepsilon}(p_{\max} - p^\varepsilon(\cdot, t_1)) \\
&\le \frac{1}{\inf\{a_0\}} \cdot \Delta\rho + o(1).
\end{aligned}$$

In the last line we used that in the limit $\varepsilon \to 0$ the two averages $\frac{1}{2\delta\cdot 2\gamma}\int_{B_\delta(x)\cap\Gamma_1^\varepsilon} p$ and $\frac{1}{2\delta}\int_{B_\delta(x)} p$ coincide. Except for boundary effects, both expressions define linear and translation invariant averages of the values $p(x_i)$. The boundary effects vanish for $\varepsilon \to 0$ and the two expressions asymptotically coincide with the arithmetic mean of the values $p(x_i)$.

We can now calculate an upper bound for the average pressure at an arbitrary time instance $t \in (t^\varepsilon, t_2]$,

$$\begin{aligned}\int_{B_\delta(x)} p^\varepsilon(\xi,t)\ d\xi \le \int_{B_\delta(x)} p^\varepsilon(\xi,t_1)\ d\xi \\ + \sum_i (h^\varepsilon(x_i,t) - h^\varepsilon(x_i,t_1))_- \cdot \inf\{a_0\} \\ + \sum_i (h^\varepsilon(x_i,t) - h^\varepsilon(x_i,t_1))_+ \cdot \sup\{a_0\} + o(1) \\ \le 2\delta p_{\max} + \sum_i (h^\varepsilon(x_i,t) - h^\varepsilon(x_i,t_1)) \cdot \inf\{a_0\} \\ + \sum_i (h^\varepsilon(x_i,t) - h^\varepsilon(x_i,t_1))_+ \cdot (\sup\{a_0\} - \inf\{a_0\}) + o(1) \\ \le 2\delta p_{\max} - \frac{1}{2\gamma}\frac{e\delta}{16} \cdot \inf\{a_0\} + 2\delta \cdot C\Delta\rho + o(1)\end{aligned}$$

for $\varepsilon \to 0$. The corrector $o(1)$ takes into account that the p-average over $B_\delta(x)$ coincides only asymptotically with the p-average over $B_\delta(x) \cap \Gamma_1^\varepsilon$. Dividing by 2δ we find for small ε

$$\frac{1}{2\delta}\int_{B_\delta(x)} p^\varepsilon(\xi,t)\ d\xi \le p_{\max} - \Delta\rho \qquad \forall t \in (t^\varepsilon, t_2),$$

if $\Delta\rho$ was chosen small compared to e. We know that in the ball $B_\delta(x)$ there happen explosions in the time interval (t^ε, t_2). This is in contradiction with the fact that for our choice of δ, below the average pressure $p_{\max} - \Delta\rho$, there can be no explosions by Corollary 3.2. □

In case that ν is of finite type, the measure μ has a direct physical interpretation. For every set $\bar{S} = \Gamma \times (t_1, t_2)$ the number $\mu(\bar{S})$ is the weighted number of explosions in $\bar{S}$. If $s_0(k) = 1$ for all k, then

$$\mu(\bar{S}) = \nu(\bar{S} \times K) = \lim_{\varepsilon \to 0} \left(\varepsilon \cdot \#\{\text{explosions in } \bar{S}\}\right),$$

the limit taken along the chosen subsequence. In the general case μ measures the total mass of fluid that is lost in explosions.

Our next aim is to show the following relation between the measure of limit patterns and spatial averages of the limit pressure $p_\delta^0(x,t)$. Loosely speaking, we show

$$\nu \text{ of finite type } \Rightarrow p_\delta^0 \text{ has no jumps.}$$

In order to show this statement, by Proposition 3.5, it remains to verify that if $\mu(\Gamma, \cdot)$ contains no atoms, then p_δ^0 has no jumps. By definition, the functions p_δ^0 are Lipschitz continuous in x for every t. In order to state regularity properties in time, we choose as a representative of p_δ^0 the temporal maximal function,

$$p_\delta^0(x,t) = \limsup_{r \searrow 0} \frac{1}{2r}\int_{t-r}^{t+r} p_\delta^0(x,\tau)\ d\tau.$$

In the following we work with this representative of p_δ^0.

PROPOSITION 3.6. *Assume that the measure $\mu(\Gamma,\cdot)$ contains no atoms. If in a point $(x,\bar t\,)$ the average pressure is not maximal,*

$$p^0_\delta(x,\bar t\,) = \rho < p_{\max}, \tag{3.6}$$

then there exist $\rho_0 < p_{\max}$, $\varepsilon_0 > 0$, and $t_1 < \bar t < t_2$ with

$$p^\varepsilon_\delta(x,t) \le \rho_0 \quad \forall t \in (t_1,t_2),\ \varepsilon < \varepsilon_0, \tag{3.7}$$

$$p^0_\delta(x,t) \le \rho_0 \quad \forall t \in (t_1,t_2). \tag{3.8}$$

The number ρ_0 does not depend on δ.

Proof. Our emphasis in this proposition lies on finding $t_1 < \bar t$; in this part the assumption on μ is used. The proof has similarity with the proof of Proposition 3.5, but this time we use the converse implication

$$\text{loss of pressure} \;\Rightarrow\; \text{loss of mass in explosions.}$$

We choose $\Delta\rho$ small compared to $p_{\max} - \rho$. Let us assume that for a subsequence $\varepsilon \to 0$ and a sequence $t_1^\varepsilon \nearrow \bar t$ there holds

$$p^\varepsilon_\delta(x,t_1^\varepsilon) \ge p_{\max} - \Delta\rho.$$

From (3.6) we conclude that there exists $t^\varepsilon > t_1^\varepsilon$ arbitrarily close to $\bar t$ with

$$p^\varepsilon_\delta(x,t^\varepsilon) \le \rho + \Delta\rho.$$

This follows from the fact that spatio-temporal averages of p^ε converge to the corresponding averages of p^0. Exploiting that the pressure in t_1^ε is large, we verify that redistribution of mass between the cells is a small effect,

$$\begin{aligned}\frac{1}{2\delta}\sum_i (h^\varepsilon(x_i,t^\varepsilon) - h^\varepsilon(x_i,t_1^\varepsilon))_+ &\le \frac{1}{2\delta}\sum_i \left(\varepsilon\frac{p_{\max}}{a_0(i)} - h^\varepsilon(x_i,t_1^\varepsilon)\right)\\ &= \frac{\varepsilon}{2\delta}\sum_i \frac{1}{a_0(i)}(p_{\max} - p^\varepsilon(x_i,t_1^\varepsilon)) \le c_h.\end{aligned}$$

For small ε the constant c_h can be chosen as $c_h = C\Delta\rho$ with C independent of δ. Our next aim is to conclude that the average height is decreased. We calculate

$$\begin{aligned}\rho - p_{\max} + 2\Delta\rho &\ge \frac{1}{2\delta}\int_{B_\delta(x)} p^\varepsilon(\xi,\tau)\, d\xi\Big|_{t_1^\varepsilon}^{t^\varepsilon}\\ &\ge \frac{1}{2\delta 2\gamma}\int_{B_\delta(x)\cap\Gamma_1^\varepsilon} p^\varepsilon(\xi,\tau)\, d\xi\Big|_{t_1^\varepsilon}^{t^\varepsilon} + o(1)\\ &\ge \frac{1}{2\delta}\sum_i (h^\varepsilon(x_i,t^\varepsilon) - h^\varepsilon(x_i,t_1^\varepsilon))_+ \cdot \inf\{a_0\}\\ &\quad + \frac{1}{2\delta}\sum_i (h^\varepsilon(x_i,t^\varepsilon) - h^\varepsilon(x_i,t_1^\varepsilon))_- \cdot \sup\{a_0\} + o(1)\\ &\ge c_h\cdot(\inf\{a_0\} - \sup\{a_0\}) + o(1)\\ &\quad + \frac{1}{2\delta}\sum_i (h^\varepsilon(x_i,t^\varepsilon) - h^\varepsilon(x_i,t_1^\varepsilon))\cdot \sup\{a_0\}.\end{aligned}$$

For small ε and small $\Delta\rho$ we conclude for the change in the average height

$$\frac{1}{2\delta}\sum_i (h^\varepsilon(x_i,t^\varepsilon) - h^\varepsilon(x_i,t_1^\varepsilon)) \cdot \sup\{a_0\} \leq \rho - p_{\max} + (3+C)\Delta\rho < 0,$$

an order $O(1)$ loss of volume. As in the preceding proposition, for $t^\varepsilon - t_1^\varepsilon$ small enough, this cannot be induced by outflow on lateral or lower sides. Therefore, there are $O(1/\varepsilon)$ explosions and we have $\mu^\varepsilon(\Gamma \times (t_1,t_2)) \geq c > 0$ for all $t_2 > \bar{t}$ and for a subsequence $\varepsilon \to 0$. In the limit we find $\mu(\Gamma \times (t_1,t_2)) \geq c$; since t_1 and t_2 are arbitrary we found an atom of $\mu(\Gamma,\cdot)$ and thus a contradiction.

The construction of t_2 follows the same pattern. We calculate that an increase in pressure requires an increase in volume of order $O(1)$. This cannot be compensated by lateral inflow, inflow from below, or redistribution effects. Then there must be a gain of volume through the upper boundary Γ—a contradiction since no "negative explosions" are possible.

Property (3.8) follows from (3.7). Note that at first we find

$$p_\delta^0(x,t) \leq \rho_0 \qquad \text{a.e. } t \in (t_1,t_2).$$

By the choice of the representative p_δ^0 we conclude that the inequality holds for all t. □

COROLLARY 3.7. *Assume that ν is of finite type. Let $s = (x,\bar{t}) \in \Gamma \times (0,T)$ be a point with*

$$\limsup_{\delta\to 0} p_\delta^0(s) = \rho < p_{\max}.$$

Then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$, and $\Delta t > 0$ such that for all $\varepsilon < \varepsilon_0$ there are no explosions in $B_{\delta_0}(x) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)$, i.e.,

$$\mu^\varepsilon((x-\delta_0, x+\delta_0) \times (\bar{t} - \Delta t, \bar{t} + \Delta t)) = 0.$$

Proof. The result follows from inequality (3.7) using Corollary 3.2. Note that we have to pick a small δ in dependence of $\rho_0 < p_{\max}$; here we use that ρ_0 in Proposition 3.6 does not depend on δ. The conclusion remains valid if lim sup is replaced by lim inf in the assumption. □

4. Upscaled equations. Our aim is to derive the physical laws for the averaged pressure. On the boundary we expect a law relating the increase of pressure with the parameters pressure and normal velocity, and we write

$$\begin{aligned} p^0 < p_{\max} &\Rightarrow \partial_t p^0 = \alpha(p^0, -\partial_2 p^0), \\ p^0 = p_{\max} &\Rightarrow \partial_t p^0 = \alpha(p^0, -\partial_2 p^0)_- \end{aligned}$$

for some function α. The first implication expresses that, as long as the maximally sustained pressure is not yet achieved, there is an increase of the pressure according to the local rules of filling pores. A flow towards the boundary increases the filling height of the single pore and, due to the monotonicity of the law $\mathcal{P}_0$, the pressure will increase. The second implication describes the situation once the maximally sustained pressure is achieved. A backward flow lowers the pressure according to the averaged law. A further flow towards the boundary results in explosions and cannot increase the pressure.

For the mathematical interpretation of the above equations some care must be applied. For $p^0 \in L^\infty$ we will interpret the condition $p^0 < p_{\max}$ as

$$\limsup_{\delta\to 0} p^0_\delta(x,t) < p_{\max}.$$

On the right-hand side of the equations $\partial_2 p^0|_\Gamma$ has a meaning as a distribution. But, for the second implication, it is not clear how to take the negative part of this distribution in some parts of the boundary, the full expression in others. We should show the following relations along Γ:

$$p^0 \le p_{\max} \quad \text{a.e. on } \Gamma, \tag{4.1}$$

$$p^0(x,t) < p_{\max} \Rightarrow \partial_t p^0 = \alpha(p^0, -\partial_2 p^0), \tag{4.2}$$

$$\partial_t p^0 \le \alpha(p^0, -\partial_2 p^0) \quad \text{as distributions on } \Gamma. \tag{4.3}$$

The first inequality is immediate, since every p^ε satisfies the inequality. The evolution equation is (4.2) and it is interpreted in the sense of distributions in a neighborhood of (x,t). Inequality (4.3) is a lift-off condition. In the case of linear laws it can be shown just as (4.2). In the nonlinear case the proof is more involved, since in the situation of (4.3) the solution has less regularity properties than in the situation of (4.2).

Our aim is to homogenize the law $p^\varepsilon(\varepsilon i) = \mathcal{P}_0(i, \varepsilon^{-1} h^\varepsilon(\varepsilon i))$. The function $\mathcal{P}_0$ depends in an oscillatory fashion on x, and the function h^ε will in general also have an oscillatory behavior. Therefore a homogenization limit has to be performed. The key in the proofs is to assure regularity properties of p^ε. We want to analyze not only linear laws as in (1.7), but also more general nonlinear models.

Definition 4.1. *We speak of a nonlinear model if the laws $\mathcal{P}_0(i,\cdot)$ are nonlinear $s_0(i)$-periodic functions with* $\max \mathcal{P}_0(i,\cdot) = \mathcal{P}_0(i, s_0(i)) = p_{\max}$ *and* $0 < a_1 \le \mathcal{P}_0'(i,\cdot) \le a_2 < \infty$ *for all i, $s \in (0, s_0(i))$. We say that the nonlinear model satisfies the linear regularity properties if the statement of Lemma* 4.2 *for the linear law holds also for the laws $\mathcal{P}_0(i,\cdot)$.*

An example of a nonlinear model that satisfies the linear regularity is given by the choice $\mathcal{P}_0(i,\cdot) = \mathcal{P}_0(s)$ with some strictly monotonically increasing function $\mathcal{P}_0 : [0, s_0) \to \mathbb{R}$.

We modify the function p^0 on a set of vanishing measure such that

$$p^0(x,t) = \limsup_{\delta\to 0} p^0_\delta(x,t).$$

Lemma 4.2. *We consider linear laws $\mathcal{P}_0(i,\cdot)$ and we assume that the box $(x-\delta_0, x+\delta_0) \times (t-\Delta t, t+\Delta t)$ contains no explosions. Then there is a neighborhood $U \subset \Gamma \times \mathbb{R}$ of (x,t) in which the pressure $p^\varepsilon(\cdot,t)$ is continuous with modulus of continuity independent of ε and t.*

For every $\Delta\rho > 0$ there exist $\delta, \varepsilon_0 > 0$ such that for all $(x_1,\tau), (x_2,\tau) \in U$,

$$|p^\varepsilon(x_1,\tau) - p^\varepsilon(x_2,\tau)| \le \Delta\rho \qquad \forall |x_1 - x_2| < \delta,\ \varepsilon < \varepsilon_0, \tag{4.4}$$

$$|p^0(x_1,\tau) - p^0(x_2,\tau)| \le \Delta\rho \qquad \forall |x_1 - x_2| < \delta, \tag{4.5}$$

$$|p^\varepsilon(\xi,\tau) - p^\varepsilon_{\delta'}(x_1,\tau)| \le \Delta\rho \qquad \forall \xi \in B_{\delta'}(x_1),\ \varepsilon < \varepsilon_0,\ 0 < \delta' < \delta. \tag{4.6}$$

In the nonlinear case, (4.4) *and* (4.6) *hold for all τ except for an exceptional set of arbitrary small measure which can be prescribed together with $\Delta\rho$.*

Proof. In Proposition A.3 we show (4.4). It implies that local averages of p^0 satisfy the same inequality and we can conclude (4.5) by the theorem of Lebesgue. Inequality (4.6) is a direct consequence of (4.4).

In the nonlinear case, given $\Delta\rho$, we choose first κ and ε_0 such that the errors introduced by p_B and $p_{A,2}$ are small compared to $\Delta\rho$. By the uniform continuity of $p_{A,1}$ for most of the time we can choose δ small in order to satisfy (4.4). ☐

Remark 4.3 (partial continuity of p^0). Assume finiteness of ν and linearity of the laws $\mathcal{P}_0$. Let $s_0 = (x_0, t_0)$ be a boundary point with $p^0(s_0) < p_{\max}$. Then in a neighborhood U of s_0 the function p^0 is continuous in (x, t). Everywhere holds the equality

$$p^0(x,t) = \lim_{\delta\to 0} p^0_\delta(x,t). \tag{4.7}$$

Proof. By definition of the representative p^0 there holds

$$\limsup_{\delta\to 0} p^0_\delta(s_0) < p_{\max}.$$

Then Corollary 3.7 yields the existence of a neighborhood without explosions. Note that this holds also in points with $\liminf_{\delta\to 0} p^0_\delta(s_0) < p_{\max}$. Lemma 4.2 yields the existence of a smaller neighborhood U of (x_0, t_0) such that p^0 is uniformly continuous in x for a.e. t, with modulus of continuity independent of t. Furthermore Proposition A.3 yields uniform estimates for $\partial_t p^0_A|_\Gamma \in L^2$ in a space-time neighborhood of (x_0, t_0). They imply that

$$p^0_\delta(x,t) \quad \text{is continuous in } t.$$

We conclude that p^0 is continuous in (x, t).

We can now conclude (4.7). In points s with $\liminf_{\delta\to 0} p^0_\delta(s) < p_{\max}$ it is a consequence of the continuity of p^0. In the other case we have

$$p_{\max} = \liminf_{\delta\to 0} p^0_\delta(s) \le \limsup_{\delta\to 0} p^0_\delta(s) \le p_{\max}.$$

This implies again (4.7). ☐

With the above regularity properties of p^ε and p^0 we can now homogenize the law $\mathcal{P}_0$. As a model we have chosen a uniform law with $s_0(i) = s_0$ and $a_0(i) = a_0$ independent of the position i. In this case the expression (4.8) can be trivially calculated and equals $1/a_0$. We use the general expressions below in order to include stochastic and nonlinear models.

Assumption 4.4 (ergodicity). *Consider for* $\rho \in (0, p_{\max})$ *the expression*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{2\delta} \sum_i \frac{1}{\mathcal{P}_0'(i, s_i)}, \tag{4.8}$$

where s_i *are the unique solutions of* $\mathcal{P}_0(i, s_i) = \rho$. *The sum is taken over all indices i with* $\varepsilon i \in B_\delta(x)$.

We assume on the function $\mathcal{P}_0$ *that the above limit exists for all* x *and* δ *and that it is independent of* x *and* δ.

An example of an ergodic material is a k_0-periodic function $\mathcal{P}_0$.

Proposition 4.5 (a law for p^0 in regions without explosions). *Let the ergodicity assumption, Assumption 4.4, be satisfied and* $R = (x - \delta_0, x + \delta_0) \times (t - \Delta t, t + \Delta t)$

be a region without explosions. Then in R there holds in the sense of distributions in (x,t)

$$\partial_t \left[\Theta(p^0(x,t))\right] = -\partial_2 p^0(x,t), \tag{4.9}$$

where the function Θ *satisfies*

$$\Theta'(\rho) = \lim_{\varepsilon\to 0} 2\gamma \frac{\varepsilon}{2\delta} \sum_i \frac{1}{\mathcal{P}_0'(i,s_i)} \quad \forall \rho \in (0, p_{\max}). \tag{4.10}$$

On the right-hand side appears the expression of Assumption 4.4.

We emphasize that the above proposition holds also in the case of a nonlinear model without additional regularity assumptions.

Proof. We start the proof from the differentiated version of the microscopic pressure law $p^\varepsilon = \mathcal{P}_0(\varepsilon^{-1}h^\varepsilon)$ in a point $x_i = \varepsilon i$,

$$\partial_t p^\varepsilon(x_i,t) = \mathcal{P}_0'(i, \varepsilon^{-1}h^\varepsilon(x_i,t)) \cdot \varepsilon^{-1}\partial_t h^\varepsilon(x_i,t). \tag{4.11}$$

We have a one-to-one correspondence between pressure p^ε and height h^ε in every point x_i,

$$p^\varepsilon(x_i) = \mathcal{P}_0(i, \varepsilon^{-1}h^\varepsilon(x_i)) \quad \text{or} \quad h^\varepsilon(x_i) = \varepsilon H_0(i, p^\varepsilon(x_i)).$$

We introduce the functions Φ_i satisfying $\Phi_i(0) = 0$ and

$$\Phi_i'(\rho) = \frac{1}{\mathcal{P}_0'(i, H_0(i,\rho))}.$$

We now divide (4.11) by $\mathcal{P}_0'$ and, using (1.2), write the equation as

$$\frac{d}{dt}\Phi_i(p^\varepsilon(x_i,t)) = \varepsilon^{-1}\partial_t h^\varepsilon(x_i,t) = \frac{1}{\varepsilon 2\gamma}\int_{x_i-\gamma\varepsilon}^{x_i+\gamma\varepsilon} (-\partial_2 p^\varepsilon(\xi,t))\, d\xi.$$

Since we do not have knowledge on limits of time derivatives, we use the time integrated form. We additionally have to average over the spatial variable and use therefore the following time and space integrated equation:

$$\begin{aligned} &\frac{1}{\Delta t}\frac{\varepsilon}{2\delta}\sum_i [\Phi_i(p^\varepsilon(x_i, t+\Delta t)) - \Phi_i(p^\varepsilon(x_i,t))] \\ &\quad = \frac{1}{\Delta t}\frac{1}{2\gamma}\int_t^{t+\Delta t} (-\partial_2 p_\delta^\varepsilon(x,\tau))\, d\tau. \end{aligned} \tag{4.12}$$

We used here that taking x_1-spatial averages and the operator ∂_2 can be interchanged. The right-hand side converges for $\varepsilon \to 0$ as a distribution,

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} (-\partial_2 p_\delta^\varepsilon)\, d\tau \to \frac{1}{\Delta t}\int_t^{t+\Delta t} (-\partial_2 p_\delta^0)\, d\tau.$$

Here the convergence of the integrand is interpreted as

$$\begin{aligned} &\int_\Gamma (-\partial_2 p_\delta^\varepsilon(x,\tau)) \cdot \varphi(x)\, dx \\ &\quad := -\int_\Gamma p_\delta^\varepsilon(x,\tau)\cdot \partial_2\varphi(x)\, dx + \int_\Omega p_\delta^\varepsilon(x,y,\tau)\cdot \Delta\varphi(x,y)\, dx\, dy \\ &\quad \to -\int_\Gamma p_\delta^0(x,\tau)\cdot \partial_2\varphi(x)\, dx + \int_\Omega p_\delta^0(x,y,\tau)\cdot \Delta\varphi(x,y)\, dx\, dy \\ &\quad =: \int_\Gamma (-\partial_2 p_\delta^0(x,\tau))\cdot\varphi(x)\, dx \end{aligned}$$

for all periodic $\varphi \in C^2(\Omega)$ with compact support in $\Omega \cup \Gamma$. In the convergence we used that $p^\varepsilon|_\Gamma \to p^0|_\Gamma$ in L^∞_w, and accordingly the convergence of $p^\varepsilon_\delta|_\Gamma$.

We next consider the left-hand side of (4.12) and its limit as $\varepsilon \to 0$. We choose the function $\Theta(\rho)$ of (4.10) as the average

$$\Theta(\rho) := \lim_{\varepsilon\to 0} 2\gamma \frac{\varepsilon}{2\delta} \sum_i \Phi_i(\rho).$$

Since averages of Φ_i' exist, averages of Φ_i also exist. We now have to use the fact that p^ε has no oscillations in x for most values of t. We pick a small $\Delta\rho > 0$ and choose $\delta_0 > 0$ small to satisfy

$$|p^\varepsilon(x_i, t) - p^\varepsilon_\delta(x, t)| \le \Delta\rho \qquad \forall x_i = \varepsilon i \in B_\delta(x),\, \delta < \delta_0,\, t \in \mathcal{T}_{\delta_0}.$$

Here we use Lemma 4.2. In the linear case we can choose $\mathcal{T}_{\delta_0} = (t_1, t_2)$; in the nonlinear case $\mathcal{T}_{\delta_0}$ is an (ε-dependent) subset of (t_1, t_2). For small $\delta_0 > 0$ the measure $|\mathcal{T}_{\delta_0}|$ is arbitrarily close to $|t_2 - t_1|$.

We next exploit that the averages p^ε_δ are uniformly continuous (Proposition A.3), and that we can choose a subsequence $\varepsilon \to 0$ such that $p^\varepsilon_\delta \to p^0_\delta$ uniformly in R. We use this to write

$$|p^\varepsilon(x_i, t) - p^0_\delta(x, t)| \le \Delta\rho + o(1) \qquad \forall i,\, \forall t \in \mathcal{T}_{\delta_0}.$$

With our knowledge on oscillations of p^ε we can now use

$$\begin{aligned} |\Phi_i(p^\varepsilon(x_i,t)) - \Phi_i(p^0_\delta(x,t))| &\le \sup_i \|\Phi_i'\|_\infty \cdot |p^\varepsilon(x_i,t) - p^0_\delta(x,t)| \\ &\le \frac{1}{a_1}(\Delta\rho + o(1)) \qquad \forall i,\, \forall t \in \mathcal{T}_{\delta_0} \end{aligned}$$

to perform the replacement

$$\frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^\varepsilon(x_i,t)) = \frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^0_\delta(x,t)) + O(\Delta\rho) + o(1) \quad \forall t \in \mathcal{T}_{\delta_0}.$$

In what follows we have to consider the expressions as distributions in time and use a test function $\phi(t)$ with compact support in (t_1, t_2). We conclude

$$\begin{aligned} &\int_{t_1}^{t_2} \frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^\varepsilon(x_i,t))\phi(t)\,dt \\ &\quad = \int_{t_1}^{t_2} \frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^0_\delta(x,t))\phi(t)\,dt + O(\Delta\rho) + o(1) + o_\delta(1) \\ &\quad \to \frac{1}{2\gamma}\int_{t_1}^{t_2} \Theta(p^0_\delta(x,t))\phi(t)\,dt + O(\Delta\rho) + o_\delta(1), \end{aligned}$$

with $o_\delta(1) \to 0$ for $\delta_0 \to 0$, since averages of Φ_i are bounded and $\mathcal{T}_{\delta_0}$ has large measure. In taking the limit $\varepsilon \to 0$ we used the ergodicity assumption.

We write (4.12) now with a discrete integration by parts,

$$\begin{aligned} &-\int_{t_1}^{t_2} \Theta(p^0_\delta(x,t)) \frac{\phi(t) - \phi(t-\Delta t)}{\Delta t}\,dt \\ &\quad = -\int_{t_1}^{t_2} \partial_2 p^0_\delta(x,t)\ \frac{1}{\Delta t}\int_t^{t+\Delta t} \phi(\tau)\,d\tau\,dt + \frac{2}{\Delta t} O(\Delta\rho) + o_\delta(1). \end{aligned}$$

We take the limit $\delta \to 0$ using that along a subsequence $\delta \to 0$ the functions p^0_δ converge to p^0 pointwise a.e. Since now δ_0 can also be chosen small, we find in the sense of distributions in x

$$-\int_{t_1}^{t_2} \Theta(p^0(\cdot,t)) \frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\, dt$$
$$= -\int_{t_1}^{t_2} \partial_2 p^0(\cdot,t)\ \frac{1}{\Delta t}\int_t^{t+\Delta t} \phi(\tau)\, d\tau\, dt + \frac{2}{\Delta t} O(\Delta\rho).$$

Since $\Delta\rho$ was arbitrary, the equation holds also without the error term. Since Δt also was arbitrary, we find the result. □

In the following theorem we collect all the upscaled equations. The principal assumption of the theorem is that ν is of finite type. Thinking of periodic solutions of period $k\varepsilon$ in x_1, we know that such an assumption is necessary. The assumption can be replaced by "with probability 1" in the case of stochastic equations.

THEOREM 4.6. *We consider a subsequence p^ε of solutions to* (1.2)–(1.6) *with a limit measure ν of finite type. Let p^0 be the limit of p^ε in L^∞_w and in the weak topology of $L^2((0,T), H^1(\Omega))$. Then there exists a representative p^0 that is harmonic for all t and which satisfies*

$$0 \le p^0(x,t) \le p_{\max} \qquad \forall x \in \Gamma,\ \forall t. \tag{4.13}$$

Every point $(x,t) \in \Gamma \times (0,T)$ with

$$p^0(x,t) < p_{\max}$$

has a neighborhood in $\Gamma \times (0,T)$ on which

$$\partial_t \Theta(p^0) = -\partial_2 p^0 \tag{4.14}$$

holds in the sense of distributions. Everywhere on $\Gamma \times (0,T)$ holds the corresponding inequality

$$\partial_t \Theta(p^0) \le -\partial_2 p^0. \tag{4.15}$$

In the nonlinear case the same properties hold. To conclude (4.15) *we have to assume that the nonlinear equations satisfy the linear regularity property of Definition* 4.1*. Without this assumption we only have the weak lift-off condition* (4.22)*.*

We make some remarks on this theorem.

Boundary values on Γ. The formal definition of the limit function is as follows. We choose a subsequence $\varepsilon \to 0$ such that $p^\varepsilon|_\Gamma$ converge in L^∞_w to some limit $p^0_\Gamma \in L^\infty(\Gamma)$. The weak limit $p^0 \in L^2((0,T), H^1(\Omega))$ satisfies

$$\int_\Gamma p^0_\Gamma \cdot \partial_2 \varphi = \int_\Omega p^0 \cdot \Delta\varphi \qquad \text{a.e. } t \in (0,T)$$

for all $\varphi \in C^2_0(\Omega \cup \Gamma)$ with $\varphi = 0$ on Γ. This shows that p^0 is a harmonic function with boundary values p^0_Γ.

Initial values. If $p^\varepsilon(t=0) = P_0$ is continuous and satisfies $P_0|_\Gamma < p_{\max}$, then by the regularity results all functions p^0_δ are continuous in a neighborhood and have the initial values $(P_0)_\delta$. Therefore $p^0(t=0) = P_0$.

The lift-off condition (4.15). In all points $p^0 < p_{\max}$, (4.14) describes the evolution. Without a lift-off condition p^0 could remain on the level $p_{\max}$ even if the fluid flows backward into the domain. Therefore a condition of lift-off is necessary in order to close the system. We emphasize that weaker conditions may be sufficient; relevant is that the left-hand side in (4.15) is negative in regions where the right-hand side is negative.

In the derivation of (4.15) we face the problem that a pointwise analysis is necessary. Loosely speaking, in some points we have to argue with the help of regularity of p^ε in order to find the law, in other points we use $p^\varepsilon \geq p_{\max} - \Delta\rho$ in order to find the law. Such pointwise analysis is in conflict with the use of distributional limits as in the proof of Proposition 4.5.

The analysis proceeds in three steps. In section 2.2 we characterized the possible patterns of the system. In section 3 we showed that, if all limit patterns are finite, averages of the pressure cannot have jumps (Proposition 3.6) and that every point with nonmaximal limit pressure has a neighborhood without explosions (Corollary 3.7). Based on these observations we derive the upscaled equations.

Proof. All assertions of the theorem are already shown except for (4.15). In the case of linear laws it can easily be derived following the lines of Proposition 4.5, starting from the inequality in (4.12). The point is that for linear laws the information that p^ε is close to p^ε_δ is not needed.

In the general case the subsequent proposition establishes with (4.16) a pointwise inequality for the pressure decay. It describes on a microscopic scale the condition of lift-off and is the key for the proof.

Using (4.16) in the first inequality and the Lebesgue convergence theorem in the second (the boundedness from below of the integrand is verified in Proposition 4.7), we calculate for nonnegative smooth test functions ϕ with compact support in $(\Omega \cup \Gamma) \times (0, T)$

$$\begin{aligned}
&\int_0^T \int_\Gamma \frac{\Theta(p^0(x,0,t+\Delta t)) - \Theta(p^0(x,0,t))}{\Delta t} \phi(x,t)\, dx\, dt \\
&\quad \leq \int_0^T \int_\Gamma \liminf_{\delta\to 0} \liminf_{y \nearrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} (-\partial_2 p^0_\delta(x,y,\tau))\, d\tau \phi(x,t)\, dx\, dt \\
&\quad \leq \liminf_{\delta\to 0} \liminf_{y \nearrow 0} \int_0^T \int_\Gamma \frac{1}{\Delta t} \int_t^{t+\Delta t} (-\partial_2 p^0_\delta(x,y,\tau))\, d\tau \phi(x,t)\, dx\, dt \\
&\quad = \liminf_{\delta\to 0} \liminf_{y \nearrow 0} \int_0^T \left\{ \int_{\Omega \cap \{x_2 < y\}} \frac{1}{\Delta t} \int_t^{t+\Delta t} p^0_\delta(\tau)\, d\tau \cdot \Delta\phi\, dx_1\, dx_2 \right. \\
&\qquad\qquad \left. - \int_{\Omega \cap \{x_2 = y\}} \frac{1}{\Delta t} \int_t^{t+\Delta t} p^0_\delta(\tau)\, d\tau \cdot \partial_2 \phi\, dx_1 \right\} dt \\
&\quad = \liminf_{\delta\to 0} \int_0^T \left\{ \int_\Omega \frac{1}{\Delta t} \int_t^{t+\Delta t} p^0_\delta(\tau)\, d\tau \cdot \Delta\phi\, dx_1\, dx_2 \right. \\
&\qquad\qquad \left. - \int_\Gamma \frac{1}{\Delta t} \int_t^{t+\Delta t} p^0_\delta(\tau)\, d\tau \cdot \partial_2 \phi\, dx_1 \right\} dt
\end{aligned}$$

$$= \int_0^T \left\{ \int_\Omega \frac{1}{\Delta t} \int_t^{t+\Delta t} p^0(\tau)\ d\tau \cdot \Delta\phi \, dx_1 \, dx_2 - \int_\Gamma \frac{1}{\Delta t} \int_t^{t+\Delta t} p^0(\tau)\, d\tau \cdot \partial_2 \phi \, dx_1 \right\} dt$$

$$= \int_0^T \left\langle \frac{1}{\Delta t} \int_t^{t+\Delta t} (-\partial_2 p^0(\tau))\, d\tau, \phi \right\rangle dt.$$

Since Δt was arbitrary, this proves the claim in the case of linear regularity. The general case is treated in Corollary 4.8. ∎

PROPOSITION 4.7 (pointwise lift-off condition). *Let $\Delta t > 0$ be given. We assume that the equations satisfy the linear regularity. Then for a.e. point $(x, 0, t_0)$ there holds with*

$$V := \liminf_{\delta \to 0} \liminf_{y \nearrow 0} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \left(-\partial_2 p_\delta^0(x, y, t)\right) dt$$

the inequality

$$\frac{\Theta(p^0(x, 0, t_0 + \Delta t)) - \Theta(p^0(x, 0, t_0))}{\Delta t} \le V. \tag{4.16}$$

The expression $\liminf_{y \nearrow 0} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} (-\partial_2 p_\delta^0(x, y, t))\, dt$ is bounded from below independent of δ and (x, t_0), and $\frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} (-\partial_2 p_\delta^0(x, y, t))\, dt$ is bounded from below for every fixed δ by a constant independent of y.

Proof. An inspection of the subsequent proof and in particular inequality (4.19) shows the bounds

$$\frac{\Theta(0) - \Theta(p_{\max})}{\Delta t} \le \liminf_{y \nearrow 0} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} (-\partial_2 p_\delta^0(x, y, t))\, dt,$$

$$\frac{\Theta(0) - \Theta(p_{\max})}{\Delta t} - \sup_{-1<y<0} g_\delta(y) \le \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} (-\partial_2 p_\delta^0(x, y, t))\, dt.$$

These imply the uniform estimates that are necessary in order to apply the Lebesgue convergence theorem. In particular V is bounded from below and we can use in the following $V > -\infty$.

We will consider here the most interesting case of maximal pressure in (x, t_0), $\liminf_{\delta \to 0} p_\delta^0(x, 0, t_0) = p_{\max}$. In the other case we find a region without explosions and can base the proof on the regularity of p^0 in $(x, 0, t_0)$. The claimed inequality is immediate in the case $V \ge 0$, since Θ is monotonically increasing. We can therefore assume from now on that $V < 0$.

To show (4.16) we first fix ΔV small compared to $|V|$, and $\Delta\rho$ small compared to $|V| \cdot \Delta t$. Next we choose $\delta > 0$ small enough to have

$$\liminf_{y \nearrow 0} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} (-\partial_2 p_\delta^0(x, y, t))\, dt \le V + \Delta V,$$

$p_\delta^0(x, 0, t_0) \ge p_{\max} - \Delta\rho$, and a third smallness condition which depends on V and Δt and is made explicit later. Now we choose $y < 0$ close to 0 to have $g_\delta(y)$ small

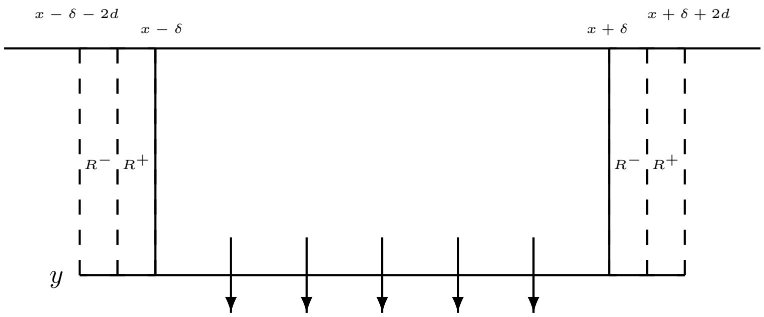

FIG. 2. *For the proof of Proposition* 4.7.

compared to $|V|$ ($g_\delta(y)$ is introduced later), and

$$\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t}(-\partial_2 p_\delta^0(x,y,t))\,dt \le V + 2\Delta V. \tag{4.17}$$

We finally choose ε_0 small in order to have $p_\delta^\varepsilon(x,0,t_0^\varepsilon) \ge p_{\max} - 2\Delta\rho$ for all $\varepsilon < \varepsilon_0$ along the subsequence and in points $t_0^\varepsilon \to t_0$. Furthermore, ε_0 is chosen small enough to have

$$\frac{1}{\Delta t}\int_{t_0^\varepsilon}^{t_0+\Delta t}(-\partial_2 p_\delta^\varepsilon(x,y,t))\,dt \le V + 3\Delta V \tag{4.18}$$

for all $\varepsilon < \varepsilon_0$. At this point we exploited to have $y < 0$ fixed; in the interior of Ω spatial derivatives of time averages of the pressure p^ε converge.

By calculating the total flow into the box $R_{\delta,y}(x) := \{(x_1,x_2) : x-\delta < x_1 < x+\delta,\ y < x_2 < 0\}$ as illustrated in Figure 2, we verify the inequality

$$2\gamma\frac{1}{\Delta t}\left[\frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^\varepsilon(x_i,\tau))\right]_{\tau=t_0^\varepsilon}^{t_0+\Delta t} \le V + 3\Delta V + g_\delta(y) + o(1) \tag{4.19}$$

for $\varepsilon \to 0$. The left-hand side measures the increase of volume on the upper boundary. On the right-hand side $V + 3\Delta V$ measures the maximal inflow into $R_{\delta,y}$ through the lower boundary as it was calculated in (4.18). $g_\delta(y)$ shall be a bound for ε-limits of the inflow through the lateral boundaries, multiplied by γ/δ. Then (4.19) follows as (4.12), since explosions only lower the left-hand side. The crucial point is now to find a bound $g_\delta(y)$ with $\lim_{y\nearrow 0} g_\delta(y) = 0$.

Construction of the bound $g_\delta(y)$. In the case of C^0 flows, that is, C^1 pressure fields, one concludes immediately that the flow through a slice of length $|y|$ is of order $|y|$. Here we only have some kind of continuity of the pressure field. Therefore, the estimate will be only of lower order, and the proof becomes more involved.

The basis for the proof is the following observation. For every (x,t) we consider the rectangles $R^- := R^-_{d,y}(x) := \{(x_1,x_2) : x < x_1 < x+d,\ y < x_2 < 0\}$ and $R^+ := R^+_{d,y}(x) := \{(x_1,x_2) : x+d < x_1 < x+2d,\ y < x_2 < 0\}$. Then for every aspect ratio $r := d/y$ the following limits coincide:

$$\lim_{y\to 0,\, d=ry}\frac{1}{|R^-|}\int_{R^-}\int_{t_0}^{t_0+\Delta t} p^0 = \lim_{y\to 0,\, d=ry}\frac{1}{|R^+|}\int_{R^+}\int_{t_0}^{t_0+\Delta t} p^0. \tag{4.20}$$

This follows from $p^0 \in L^2(0,T;H^1(\Omega))$. Time integrals of p^0 are in $H^1(\Omega)$, and for $q \in H^1(\Omega)$ holds (we use $R^- + te_1$ to denote the box R^-, translated to the right by t),

$$\begin{aligned}\frac{1}{d\cdot y}\left(\int_{R^+} q - \int_{R^-} q\right) &= \frac{1}{d\cdot y}\int_0^d dt \int_{R^- + te_1} (\partial_1 q)\\ &\le \frac{1}{y}\left(\int_{R^-\cup R^+} |\partial_1 q|^2\right)^{1/2} |2d\cdot y|^{1/2} \to 0\end{aligned}$$

for $y \to 0$ and fixed aspect ratio. We used that $|\partial_1 q|^2$ is in $L^1(\Omega)$ and therefore integrals over vanishing domains vanish.

We now conclude from (4.20) the estimate on $g_\delta(y)$. A weighted average of the horizontal flow in the box $R := \{(x_1,x_2) : x+\delta < x_1 < x+\delta+2d,\ y < x_2 < 0\}$ is (with $x_0 := x+\delta+d$)

$$\begin{aligned}&\frac{1}{d}\int_R \partial_1 p^\varepsilon(x_1,x_2)\cdot\frac{d-|x_1-x_0|}{d}\,dx_1\,dx_2 = \frac{1}{d^2}\left[\int_{R^+} p^\varepsilon - \int_{R^-} p^\varepsilon\right]\\ &\quad\to \frac{1}{r}\left[\frac{1}{|R^+|}\int_{R^+} p^0 - \frac{1}{|R^-|}\int_{R^-} p^0\right].\end{aligned}$$

By (4.20) time integrals over this term become arbitrarily small for small y. For every small $|y|$ we find a corresponding d' such that the flow through the lateral side $\{(x_1,x_2) \mid x_1 = x+\delta+d',\ y < x_2 < 0\}$ is small.

We are not allowed to change the parameter δ, but we must show that the inflow through the lateral side $x_1 = x+\delta$ is small. To this end we use the aspect ratio r that can be chosen freely. The pressure along Γ is bounded and therefore the vertical velocity satisfies in the interior an estimate $|\partial_2 p^\varepsilon(x,y)| \le C_1|y|^{-1}$ by the representation formula (see, e.g., [6, p. 22]). We find that the total vertical inflow through $\{(x_1,x_2) \mid x+\delta < x_1 < x+\delta+2d,\ x_2 = y\}$ can be bounded by $d\cdot C_1/|y| = C_1\cdot r$. Choosing a small aspect ratio r we have a bound for the vertical inflow from below.

The vertical inflow from above is bounded by $C_2\cdot d$ since no negative explosions can occur. This contribution to $g_\delta(y)$ is therefore also small for small d (independent of the aspect ratio). Putting the results together we find that choosing r and then $|y|$ small, the total inflow through $x+\delta$ is bounded by a small number $g_\delta(y)$.

Conclusions from (4.19). As a first step we write (4.19) as

$$\begin{aligned}\frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^\varepsilon(x_i,t_0+\Delta t)) &\le \frac{\varepsilon}{2\delta}\sum_i \Phi_i(p^\varepsilon(x_i,t_0^\varepsilon))\\ &\quad+\frac{\Delta t}{2\gamma}\left(V+3\Delta V+g_\delta(y)\right)+o(1).\end{aligned}$$

Since the derivatives of Φ_i are bounded, we conclude

$$p_\delta^\varepsilon(x,0,t_0+\Delta t) \le \rho < p_{\max},$$

where ρ depends only on $V<0$ and Δt, and not on δ. As in the proof of Proposition 3.6 we conclude that there is some $\rho_0 < p_{\max}$ independent of δ and $t_1 < t_0+\Delta t$ such that

$$p_\delta^\varepsilon(x,0,t) \le \rho_0 \qquad \forall t \in [t_1, t_0+\Delta t].$$

We said that a third smallness condition on δ should be satisfied. We demand that δ is small compared to $\delta_H(\rho_0)$ with δ_H from Lemma 3.1. With this choice we know that there are no explosions in the region $(x-\delta, x+\delta) \times (t_1, t_0+\Delta t)$.

We now fix a new small parameter $\sigma > 0$. Given σ we pick a new, smaller $\delta > 0$, repeat the above steps of the proof and find new $y < 0$ and $\varepsilon_0 > 0$. For the new δ we can assume by Lemma 4.2 that

$$|p^\varepsilon(\xi, 0, t_0+\Delta t) - p^\varepsilon_\delta(x, 0, t_0+\Delta t)| \leq \sigma \qquad \forall \xi \in (x-\delta, x+\delta).$$

Here we used the linear regularity property.

The functions p^ε_δ are uniformly continuous in a neighborhood of $(x, 0, t_0+\Delta t)$ by Proposition A.3 and therefore $p^\varepsilon_\delta \to p^0_\delta$ is a uniform convergence for a subsequence. Along this subsequence we now take limits in (4.19),

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} 2\gamma \frac{\varepsilon}{2\delta} \sum_i \Phi_i(p^\varepsilon(x_i, t_0+\Delta t)) \\
&\quad\leq \limsup_{\varepsilon\to 0} 2\gamma \frac{\varepsilon}{2\delta} \sum_i \Phi_i(p^\varepsilon_\delta(x, 0, t_0+\Delta t)) + C\sigma \\
&\quad= \limsup_{\varepsilon\to 0} 2\gamma \frac{\varepsilon}{2\delta} \sum_i \Phi_i(p^0_\delta(x, 0, t_0+\Delta t)) + C\sigma \\
&\quad= \Theta(p^0_\delta(x, 0, t_0+\Delta t)) + C\sigma.
\end{aligned}$$

For the second term we have, by $p^\varepsilon_\delta(x, t^\varepsilon_0) \geq p_{\max} - 2\Delta\rho$,

$$\liminf_{\varepsilon\to 0} 2\gamma \frac{\varepsilon}{2\delta} \sum_i \Phi_i(p^\varepsilon(x_i, t^\varepsilon_0)) \geq \Theta(p_{\max}) - C\Delta\rho.$$

We take $\limsup_{\varepsilon\to 0}$ in (4.19) and find

$$\begin{aligned}
&\frac{\Theta(p^0_\delta(x, 0, t_0+\Delta t)) - \Theta(p^0_\delta(x, 0, t_0))}{\Delta t} \\
&\qquad\leq V + 3\Delta V + g_\delta(y) + C\sigma + C\Delta\rho.
\end{aligned} \tag{4.21}$$

We take the limit $y \nearrow 0$ and then $\delta \to 0$ using that $p^0_\delta \to p^0$ for a.e. $(x, 0, t)$. We find inequality (4.16) up to the error terms. Since ΔV, σ, and $\Delta\rho$ were arbitrary, we find the result. □

The assumption of linear regularity was used in the above proof only towards the end in order to replace p^ε by p^ε_δ in the evaluation of the laws Φ_i. We can also restrict ourselves with the conclusion that

$$\frac{\varepsilon}{2\delta} \sum_i \Phi_i(p^\varepsilon(x_i, t_0+\Delta t)) \leq \Theta(p_{\max}) - C_1$$

for all small ε implies for some $c > 0$

$$p^0_\delta(x, t_0+\Delta t) \leq p_{\max} - c\, C_1,$$

since the derivatives of Φ_i are bounded. We find the following corollary to the above proof.

Corollary 4.8 (weak lift-off condition). *We consider the nonlinear case and do not assume linear regularity. There exists $c > 0$ such that in every point (x, t_0) and for every $\Delta t > 0$,*

$$\liminf_{\delta\to 0} p^0_\delta(x, t_0+\Delta t) \leq p_{\max} + c\; V\Delta t \tag{4.22}$$

with V as in Proposition 4.7.

With Theorem 4.6 we have derived a system of upscaled equations that is satisfied by every weak limit p^0 of the pressure functions p^ε. We have to verify that we have found all necessary information on the limit system. To this end we showed in [11] that solutions of the upscaled system of Theorem 4.6 are unique, at least for a linear function Θ. The uniqueness also implies the weak convergence of the initial sequence p^ε to the solution p^0 of the limit system.

5. Conclusions and outlook. We performed an analysis of a deterministic model for the motion of fronts in porous media. Upscaled equations were found under the hypothesis that limit patterns are finite. The limit equations include a hysteresis effect of the system: during imbibition, i.e., under inflow conditions and after a transition time, the pressure along the boundary coincides everywhere with $p_{\max}$; this value can therefore be interpreted as the capillary pressure of imbibition. When changing the boundary conditions to drainage, i.e., an outflow condition along the bottom, the system undergoes again a transitional regime before the capillary pressure of drainage (zero in our case) is reached.

An open question concerns the uniqueness of solutions of the upscaled system in the nonlinear case, that is, with the weak lift-off condition. It is desirable to extend the results to more general geometries and to more general equations for the fluid. We expect that in such systems the principle feature, the appearance of isolated explosions, remains the same.

We emphasize that the upscaled equations derived in this work form a mesoscopic model of a two-phase flow since the position of the front is still resolved. Desirable is the derivation of macroscopic laws from our mesoscopic results.

Appendix A. Regularity properties away from explosions. In this appendix we consider only regions without explosions. We expect that in this case the solution is regular. In order to get a feeling for the smoothing properties of the equations, we first consider the above equations omitting the projection Q_ε.

Remark A.1. The unique classical solution u^0 of

$$\Delta u^0(t) = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_-, \tag{A.1}$$

$$\partial_t u^0 = -\lambda \partial_2 u^0 \qquad \text{on } \mathbb{R}, \tag{A.2}$$

with initial condition $u^0(x_1, 0, 0) = \mathrm{sgn}(x_1)$ and satisfying the uniform bounds $0 \leq u^0 \leq 1$ is given by

$$u^0(t, x) = \frac{2}{\pi} \arctan\left(\frac{x_1}{\lambda t}\right). \tag{A.3}$$

Proof. We demonstrate how to find u^0 in the self-similar form $u^0(x,t) = U(\frac{x}{t})$. The function U is harmonic in the lower half-plane and on $\{(x_1, x_2) \mid x_2 = 0\}$ it satisfies

$$x_1 \cdot \partial_1 U = \partial_2 U.$$

We use a complex differentiable function $f : \mathbb{C}^- \to \mathbb{C}$ to find $U = \mathrm{Re}\, f$. The condition on the real line translates into

$$\mathrm{Re}(z \cdot f' - if') = 0.$$

We set $z \cdot f' - if' = ci$ for a real number c and find

$$f' = c\frac{i}{z - i} = c\frac{-1}{1 + iz} = c\frac{1 - i\bar{z}}{|1 + iz|^2}.$$

This implies

$$\partial_1 U = \operatorname{Re} f' = \frac{c}{1+|x_1|^2}$$

on the real line. Using $U(x_1) \to \pm 1$ for $x_1 \to \pm\infty$ determines the constant c to be $2/\pi$ and yields the result. The x_1-derivative of U inside the domain is calculated as $\partial_1 U(x_1, x_2) = \frac{1-x_2}{(1-x_2)^2+x_1^2}$; this yields the complete form of U,

$$U(x_1, x_2) = \frac{2}{\pi} \arctan\left(\frac{x_1}{1-x_2}\right). \qquad \square$$

The explicit solution above gives us an idea of how solutions to the original equations (1.2)–(1.5) behave qualitatively. Unfortunately, the picture may change in many respects once the coefficients in $\partial_t p^\varepsilon(\cdot, 0, t) = -a(\cdot) Q_\varepsilon \partial_2 p^\varepsilon(\cdot, 0, t)$ depend on x_1. But one useful property remains valid as one can see by using the exact solution as a comparison function.

Remark A.2. Consider the original equations (1.2)–(1.5) with initial values

$$p^\varepsilon(x_1, 0, 0) = \begin{cases} 0 & \text{for } x_1 < 0, \\ 1 & \text{for } x_1 > 0. \end{cases}$$

Then for every $\delta > 0$ there exists a constant C such that

$$p^\varepsilon(-\delta, 0, t) \le Ct \qquad \forall t.$$

PROPOSITION A.3. *We study solutions $p^\varepsilon : (-1,1) \times (-1,0) \times (0,t) \to \mathbb{R}$ of the original equations* (1.2)–(1.5) *for $V_0 \in C^1$ and with (piecewise) linear laws $\mathcal{P}_0$. If no explosions happen on $(-\delta_0, \delta_0) \times (0,t)$, then, for every $0 < \delta < \delta_0$, the family $p^\varepsilon(\cdot, 0, t)$ is uniformly continuous in $(-\delta, \delta)$.*

In the case of nonlinear laws, for every $\delta < \delta_0$, $0 < t_1 < t$, and arbitrary $\kappa > 0$ we can write $p^\varepsilon|_\Gamma$ as $p^\varepsilon|_\Gamma = p_A + p_B = p_{A,1} + p_{A,2} + p_B$ with

$$\begin{aligned} \partial_t p_A &\in L^2((t_1,t) \times (-\delta,\delta)), & \|p_A\| &\le C(\kappa), \\ p_{A,1} &\in L^2((t_1,t), H^1(-\delta,\delta)), & \|p_{A,1}\| &\le C(\kappa), \\ p_{A,2} &\in L^2((t_1,t), L^\infty(-\delta,\delta)), & \|p_{A,2}\| &\le C\varepsilon^\alpha, \\ p_B &\in L^\infty((t_1,t) \times (-\delta,\delta)), & \|p_B\| &\le \kappa. \end{aligned}$$

In particular, all spatial averages $p^\varepsilon_{\delta'}$ of p^ε are uniformly continuous on $(t_1,t) \times (-\delta,\delta)$. The constant $\alpha > 0$ is independent of ε, p^ε, δ, and κ.

A solution for $\varepsilon = 1$ on the extended domain $\mathbb{R} \times (-\infty, 0) \times (0,\infty)$ without explosions satisfies

$$|p^\varepsilon(x_1, 0, t) - p^\varepsilon(0,0,t)| \to 0 \quad \textit{for} \quad t \to \infty, \tag{A.4}$$

independent of the initial values.

Proof. Idea: Assume that there are no explosions at all. Then we write the linear laws as

$$\partial_t p^\varepsilon(\cdot, 0, \tau) = -a^\varepsilon(\cdot) Q_\varepsilon \partial_2 p^\varepsilon(\cdot, 0, \tau) \quad \text{on } \Gamma_1^\varepsilon,\ \forall \tau \in (0,t).$$

In this case we can show uniform estimates for $\partial_t p^\varepsilon|_\Gamma \in L^\infty((3t/4,t),L^2(\Gamma))$. These are at the same time bounds for $Q_\varepsilon\partial_2 p^\varepsilon|_\Gamma = -\frac{1}{a^\varepsilon(\cdot)}\partial_t p^\varepsilon|_\Gamma$. We can decompose p^ε into the "macroscopic" part, the harmonic, periodic function p_m satisfying $\partial_2 p_m = Q_\varepsilon\partial_2 p^\varepsilon$ on Γ, and a remainder $u^\varepsilon = p^\varepsilon - p_m$. In Lemma A.4 we show that u^ε is small in L^∞. The elliptic regularity theory yields uniform estimates for $p_m \in L^\infty((t/2,t),C^\alpha(\Gamma))$ for $\alpha < 1/2$. This yields the claim.

We now show the estimate for $\partial_t p^\varepsilon$ in the case of $\delta_0 = 1$ (no explosions along Γ). By the energy estimate for $p^\varepsilon \in L^2((0,t),H^1(\Omega))$ we find a time instance $t_1 < t/2$ such that $p^\varepsilon(t_1) \in H^1(\Omega)$ satisfies an ε-independent bound. We multiply $\Delta p^\varepsilon = 0$ by $\partial_t p^\varepsilon$ and find

$$\begin{aligned}\int_{\Gamma_0} V_0\partial_t p^\varepsilon &= \int_\Omega \nabla p^\varepsilon\cdot\partial_t\nabla p^\varepsilon - \int_{\Gamma_1^\varepsilon}\partial_2 p^\varepsilon\cdot\partial_t p^\varepsilon\\ &= \int_\Omega \partial_t\frac{1}{2}|\nabla p^\varepsilon|^2 + \int_{\Gamma_1^\varepsilon}\frac{1}{a^\varepsilon(\cdot)}|\partial_t p^\varepsilon|^2.\end{aligned}$$

This yields an estimate for $\partial_t p^\varepsilon|_{\Gamma_1^\varepsilon} \in L^2((t_1,T)\times\Gamma_1^\varepsilon)$. We find a time instance t_2, $t_1 < t_2 < 3t/4$, with bounded (by an ε-independent constant) $\partial_t p^\varepsilon(t_2)|_{\Gamma_1^\varepsilon} \in L^2$. We multiply the differentiated equation $\partial_t\Delta p^\varepsilon = 0$ by $\partial_t p^\varepsilon$ and find

$$\begin{aligned}\int_{\Gamma_0}\partial_t V_0\partial_t p^\varepsilon &= \int_\Omega|\partial_t\nabla p^\varepsilon|^2 - \int_{\Gamma_1^\varepsilon}\partial_t\partial_2 p^\varepsilon\cdot\partial_t p^\varepsilon\\ &= \int_\Omega|\partial_t\nabla p^\varepsilon|^2 + \partial_t\int_{\Gamma_1^\varepsilon}\frac{1}{2}\frac{1}{a^\varepsilon(\cdot)}|\partial_t p^\varepsilon|^2 - \int_{\Gamma_1^\varepsilon}\frac{1}{2}\frac{\partial_t(a^\varepsilon(p^\varepsilon))}{(a^\varepsilon)^2}|\partial_t p^\varepsilon|^2.\end{aligned}$$

In the case of a linear law a^ε is independent of p^ε and therefore the last term vanishes. An integration yields the claimed estimate for $\partial_t p^\varepsilon|_\Gamma \in L^\infty((t_2,t),L^2(\Gamma))$.

In the case of a nonlinear law the coefficient $a(x_1,t)$ depends on $p^\varepsilon(x_1,0,t)$, and the term containing $\partial_t a^\varepsilon|\partial_t p^\varepsilon|^2$ cannot be controlled by the other two terms. Nevertheless, the argument leading to the estimate for

$$\partial_t p^\varepsilon|_{\Gamma_1^\varepsilon} \in L^2((t_1,T)\times\Gamma_1^\varepsilon)$$

remains valid. With $p_A = p^\varepsilon$ and $p_B = 0$ we found the claimed decomposition. The estimates for the x_1-derivative of $p_{A,1} := p_m$ follow for harmonic functions from the estimates for the Neumann boundary values.

We now study the general case $\delta_0 < 1$. We choose $\Delta t < t$ small (depending on δ_0 and ρ), and consider from now on the solution only on the time interval $(t-\Delta t,t)$, the coefficients a^ε are always given by the original solution. We now decompose the solution into a part p_A with the initial values of p^ε and without explosions on Γ, and a second part p_B that captures the evolution of the explosions. Then on p_A the above arguments for $\delta_0 = 1$ can be applied. The function p_B is small on $(t-\Delta t,t)\times(-\delta_0/2,\delta_0/2)$ for Δt small by the maximum principle of Remark A.2.

Formula (A.4) follows from a scaling argument,

$$|p^\varepsilon(x_1,0,t) - p^\varepsilon(0,0,t)| = |p^{\varepsilon/t}(x_1/t,0,1) - p^{\varepsilon/t}(0,0,1)|,$$

where the scaled solution $p^{\varepsilon/t}$ has initial values $p^{\varepsilon/t}(x_1/t,0,0) = p^\varepsilon(x_1,0,0)$. The result follows from the uniform continuity of $p^{\varepsilon/t}$ at time 1, since x_1/t is arbitrarily close to 0. Note that we have to modify the solutions $p^{\varepsilon/t}|_{(-1,1)\times(-1,0)}$ on the boundary in order to guarantee their periodicity. Applying the above argument with an initial time close to 1, this change introduces only a small error term in the expression $p^{\varepsilon/t}(x_1/t,0,1) - p^{\varepsilon/t}(0,0,1)$. □

LEMMA A.4. *Let u^ε be a sequence of periodic harmonic functions on $\Omega = (-1,1)\times(-1,0)$ satisfying a harmonic Neumann condition on the lower boundary and with the following properties on the upper boundary $\Gamma = (-1,1)\times\{0\}$:*

$$Q_\varepsilon \partial_2 u^\varepsilon = 0,$$
$$g^\varepsilon := u^\varepsilon - Q_\varepsilon u^\varepsilon \quad \text{satisfies } \|\partial_{x_1} g^\varepsilon|_{\Gamma_1^\varepsilon}\|_{L^2} \le C.$$

Then $u^\varepsilon|_\Gamma \to 0$ in $L^\infty(\Gamma)$ for $\varepsilon \to 0$ independent of the sequence g^ε.

Proof. Note that we have the technical difficulty of $g^\varepsilon \notin H^1$ in general. We therefore introduce a new projection $\tilde{Q}_\varepsilon$ such that $\tilde{Q}_\varepsilon v = Q_\varepsilon v$ on Γ_1^ε with $\tilde{Q}_\varepsilon$ bounded in $\mathcal{L}(H^1(\Gamma), H^1(\Gamma))$. For $v \in L^2(\Gamma)$ we can use $\tilde{v}$, the harmonic function on Ω that satisfies $\tilde{v}|_{\Gamma_1^\varepsilon} = Q_\varepsilon v|_{\Gamma_1^\varepsilon}$ and $\partial_2 \tilde{v} = 0$ on Γ_2^ε, together with periodicity and a harmonic Neumann condition on the lower boundary. We set $\tilde{Q}_\varepsilon v := \tilde{v}$.

With this modified projection we can consider the bounded sequence $w^\varepsilon := u^\varepsilon|_\Gamma - \tilde{Q}_\varepsilon u^\varepsilon|_\Gamma \in H^1(\Gamma)$. We multiply $\Delta u^\varepsilon = 0$ by u^ε to find

$$\int_\Omega |\nabla u^\varepsilon|^2 = \int_\Gamma \partial_2 u^\varepsilon\ u^\varepsilon = \int_{\Gamma_1^\varepsilon} \partial_2 u^\varepsilon\ u^\varepsilon$$
$$= \int_{\Gamma_1^\varepsilon} \partial_2 u^\varepsilon\ (u^\varepsilon - \tilde{Q}_\varepsilon u^\varepsilon) = \int_\Gamma \partial_2 u^\varepsilon\ w^\varepsilon.$$

Since the family w^ε is bounded in $H^1(\Gamma)$ and $\partial_2 u^\varepsilon|_\Gamma \in H^{-1}(\Gamma)$ is bounded by $u^\varepsilon|_\Gamma \in L^2(\Gamma)$, we find an a priori bound for $u^\varepsilon \in H^1(\Omega)$.

In order to show the L^∞-convergence of u^ε we use again the above calculation and the fact that the family of functions $\partial_2 u^\varepsilon|_\Gamma \in H^{-1/2}(\Gamma)$ is bounded. We claim that the functions w^ε vanish in $H^{1/2}(\Gamma)$ at the rate $\varepsilon^{1/4}$. The functions w^ε are bounded in $C^{1/2}(\Gamma)$ and have vanishing averages on all intervals $\varepsilon(k-\gamma, k+\gamma)$. Therefore they satisfy an L^∞-estimate $\|w^\varepsilon\|_{L^\infty} \le C\sqrt{\varepsilon}$. Additionally $w^\varepsilon \in H^1(\Gamma)$ is bounded. By an interpolation between $L^2(\Gamma)$ and $H^1(\Gamma)$ we conclude that $\|w^\varepsilon\|_{H^{1/2}(\Gamma)} \le C\varepsilon^{1/4}$. We conclude that $u^\varepsilon \in H^1(\Omega)$ vanishes at the rate $\varepsilon^{1/8}$. We use an inverse estimate of an L^∞-norm in terms of an L^q-norm (exploiting that u^ε is constant on ε-intervals up to the error w^ε of order $O(\sqrt{\varepsilon})$), and a trace theorem with $q > 8$ to find

$$\|u^\varepsilon\|_{L^\infty(\Gamma)} \le C\left[\sqrt{\varepsilon} + \varepsilon^{-1/q}\|u^\varepsilon\|_{L^q(\Gamma)}\right]$$
$$\le C(q)\left[\sqrt{\varepsilon} + \varepsilon^{-1/q}\|u^\varepsilon\|_{H^1(\Omega)}\right] \le C\varepsilon^{1/8-1/q}.$$

This shows the assertion. □

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