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BIFURCATION ANALYSIS FOR SURFACE WAVES GENERATED BY WIND*

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Abstract. We study the generation of surface waves on water as a bifurcation phenomenon. For a critical wind-speed there appear traveling wave solutions. While linear waves do not transport mass (in the mean), nonlinear effects create a shear-flow and result in a net mass transport in the direction of the wind. We derive an asymptotic formula for the average tangential velocity along the free surface. Numerical investigations confirm the appearance of the shear-flow and yield results on the bifurcation picture.

Key words. free boundary problems, bifurcation analysis, viscous flows, finite element discretization

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1. Introduction.

We investigate the generation of surface waves by wind. The dynamics of the water are described with the incompressible Navier–Stokes equations. The effect of wind is modelled by pressure differences between backward- and forward-pointing faces of the wave. For a critical wind-speed the system undergoes a Hopf bifurcation, and there appear traveling wave solutions [12]. In the study at hand we are interested in the effect of the nonlinearity that is given by the convective term and by the changes of the domain. We show that for low viscosity the wave exhibits a tangential net flow in the direction of the wave propagation. The quantitative result is given in Theorem 4.3: An asymptotic formula relates the average tangential flow on the surface to the square of the wave-height. All other quantities are purely geometrical. We present numerical results that confirm the formula. Additionally, they yield the bifurcation picture, that is, the dependence of the height of the wave on the strength of the wind.

Before describing our method we compare this article with existing literature. There are extensive studies on the case of an inviscid fluid. Traveling wave solutions without external forcing are constructed in [1], [2], [5], and others, and waves in inviscid fluids with an external force are studied, e.g., in [6]. In contrast to these studies, we are interested in the Navier–Stokes equations; that is, we include a small viscosity. Therefore, in our case, energy is dissipated, and traveling waves can exist only if an exterior force supports the system with energy. This source of energy is the wind. Another source of energy is studied by Longuet-Higgins in [9], [10]: The capillary waves are perturbations of a larger gravity wave; the latter supports the capillary waves with energy ("parasitic waves").

The starting point for our analysis is a model for wind. Wind can support the system with energy in two ways: It can create pressure distributions along the surface, and it can exert tangential forces. If we suppose that air is inviscid, then we are bound to work only with pressure distributions. (A study of the coupled system is found in [13].) As in [12] we consider a simplified model in which wind creates a pressure

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distribution proportional to the slope of the free surface. This assumption is justified by measurements reported in [3].

Regarding the physics, our results can be interpreted as follows. In our model the wind exerts only normal forces on the water surface but nevertheless creates a tangential momentum in the water. This means that no tangential forces are needed to explain the tangential movement of water particles with the wind. The quantitative result of Theorem 4.3 makes it possible to verify this statement.

This paper is organized as follows. In section 2 we analyze the linear problem. For critical values of the wind-speed γ and the wave-velocity c we find traveling wave solutions. We derive explicit formulas for the bifurcation parameters (γ, c) and the profiles of the corresponding eigenfunctions with the classical methods of [7]. The explicit expressions are needed later to evaluate the nonlinear terms. In section 3 we transform the free boundary problem to a reference domain. We find an equivalent set of equations that is posed on a fixed domain. The deformation of the domain is expressed in the form of additional nonlinear terms.

Section 4 is devoted to the study of the bifurcating branch. In the linear case, the parameters γ and c and the velocity profiles do not depend on the height of the wave. What is the dependence in the nonlinear case? For a partial answer to this question it is sufficient to insert the linear profiles into the nonlinear terms. We can show that the velocity profiles change: A tangential velocity appears along the free boundary. This shear-flow grows quadratically with the height of the wave. We can derive an asymptotic formula for the limit of small viscosities.

The dependence of the parameters on the height of the wave (i.e., the bifurcation picture) can only be calculated numerically; sections 6 and 7 are devoted to the numerical analysis of this problem. In section 6 we describe the choice of a discretization in a finite element setting. We describe the Newton method that can be used to calculate the bifurcation parameters numerically. In section 7 we present numerical results. They confirm the appearance of the boundary shear-flow. They further indicate that the bifurcation is super-critical for physical quantities corresponding to water. That is, the wave-velocity c and the wind-speed γ both depend in a monotonically increasing way on the height of the wave.

In order to control the reliability of these results, we derive in section 5 various relations that are satisfied by the continuous solutions along the bifurcation branch. We monitor the discrete versions of these relations and find a reliable agreement with the continuous quantities.

This article studies equations that describe a traveling wave in a free boundary problem. For a wave velocity c we choose the frame of reference of the wave. We pose periodicity conditions on the left and right boundaries. The (unknown) domain Ω is written as $\Omega = \{(x,y) | x \in S^1, H < y < h(x)\}$ with a constant depth $H = -h_0 < 0$. For evaluating integrals we write $x \in (0, L) = (0, 1)$. The Navier–Stokes equations are posed on Ω. In the moving frame of coordinates they read

$$c\, \partial_x v - \nu \Delta v + (v \cdot \nabla) v + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

and the boundary conditions on $\Gamma = \text{graph}(h)$ are

$$c\, \partial_x h - v_2 + h' v_1 = 0, \tag{1.3}$$

$$\partial_n v_\tau + \partial_\tau v_n = 0, \tag{1.4}$$

$$p - 2\nu \partial_n v_n + \beta \mathcal{H}(h) = -\gamma \partial_x h. \tag{1.5}$$

Here $\mathcal{H}(h)$ is the mean curvature of the curve given by h, and β is the physical parameter for the surface tension. The last two equations are the balance of forces along the free boundary. The stress tensor for a viscous fluid reads $T(v,p) = -pI_2 + \nu(\nabla v + (\nabla v)^T)$, and the model assumptions are first, that the wind does not exert tangential forces (therefore $\tau \cdot T(v,p) \cdot n = 0$ in (1.4)), and second, that the normal force is balanced by surface tension and the outer pressure distribution as generated by the wind in (1.5). In our model the outer pressure distribution is given by $-\gamma\partial_x h$, and the strength of the wind is measured by the parameter γ. Concerning the physical derivation of the balance equations, compare, e.g., [11]. The equations are completed with a no-slip boundary condition

$$(v_1, v_2) = 0 \tag{1.6}$$

on the bottom line $\{y = H\}$. We finally have to impose a volume constraint, for which we choose $\int_0^L h = 0$.

The linearized equations define an operator $A_{(\gamma,c)}$, and we can write them as

$$A_{(\gamma,c)}(v,p,h) = 0. \tag{1.7}$$

In the time-dependent free boundary problem a Hopf bifurcation occurs. (For abstract results on Hopf bifurcation, see, e.g., [8].) We collect the results of [12] in the following theorem.

THEOREM 1.1. *There exists a critical force $\gamma = \gamma_0$ and a wave-speed $c = c_0$ such that the operator $A_{(\gamma_0,c_0)}$ has two conjugate complex imaginary eigenvalues $\lambda_{1,2}(\gamma_0) = \pm i\omega$. They transversely cross the imaginary axes from the stable into the unstable region. If a nonresonance condition is satisfied, then the nonlinear equations possess a 1-d family of periodic solutions to the time-dependent problem.*

On the basis of this theorem we will assume in the following that we are given a differentiable branch of nontrivial solutions

$$(-\varepsilon_0, +\varepsilon_0) \ni \varepsilon \mapsto (v_\varepsilon, p_\varepsilon, h_\varepsilon, \gamma_\varepsilon, c_\varepsilon)$$

of the nonlinear equations (1.1)–(1.6). By symmetry we can additionally assume

$$\partial_\varepsilon \gamma_\varepsilon|_{\varepsilon=0} = \partial_\varepsilon \lambda_\varepsilon|_{\varepsilon=0} = 0.$$

2. The linear equations. If we linearize the above equations we find equations on the fixed domain $R = S^1 \times (-h_0, 0)$. We denote the horizontal and vertical coordinates by x and y, respectively. The equations thus read

$$c\,\partial_x v - \nu\Delta v + \nabla p = 0, \tag{2.1}$$

$$\nabla \cdot v = 0; \tag{2.2}$$

in R and on $\Gamma = S^1 \times \{0\}$ they become

$$c\,\partial_x h - v_2 = 0, \tag{2.3}$$

$$\partial_x v_2 + \partial_y v_1 = 0, \tag{2.4}$$

$$p - 2\nu\partial_y v_2 + \beta\Delta_x h = -\gamma\partial_x h. \tag{2.5}$$

We observe that explicit solutions to these linear equations can be given in the periodic setting. We normalize the height function h and introduce the stream function ψ,

$$h = e^{ikx}, \qquad v = \nabla^\perp\psi = (-\partial_y, \partial_x)\psi.$$

Equation (2.2) is solved and the curl of (2.1) gives

$$c\partial_x \Delta\psi - \nu\Delta^2\psi = 0. \tag{2.6}$$

We solve this equation with the Ansatz

$$\psi(x,y) = a_1 e^{ky}e^{ikx} + a_2 e^{-ky}e^{ikx} + b_1 e^{\mu y}e^{ikx} + b_2 e^{-\mu y}e^{ikx}. \tag{2.7}$$

Equation (2.6) is satisfied if

$$ikc - \nu(\mu^2 - k^2) = 0 \tag{2.8}$$

holds. We choose the solution μ with $\mathrm{Re}(\mu) > 0$. The numbers $a_1, a_2, b_1, b_2 \in \mathbb{C}$ are determined by the boundary conditions on $\Gamma = S^1 \times \{0\}$ and $\Gamma_0 = S^1 \times \{H\}$:

$$\partial_x\psi = ikce^{ikx} \quad \text{on } \Gamma, \tag{2.9}$$

$$\partial_x\psi = 0 \quad \text{on } \Gamma_0, \tag{2.10}$$

$$(\partial_x^2 - \partial_y^2)\psi = 0 \quad \text{on } \Gamma, \tag{2.11}$$

$$\partial_y\psi = 0 \quad \text{on } \Gamma_0. \tag{2.12}$$

Inserting the Ansatz (2.7), the above is equivalent to

$$a_1 + a_2 + b_1 + b_2 = c, \tag{2.13}$$

$$a_1 e^{kH} + a_2 e^{-kH} + b_1 e^{\mu H} + b_2 e^{-\mu H} = 0, \tag{2.14}$$

$$-2k^2(a_1 + a_2) - (k^2 + \mu^2)(b_1 + b_2) = 0, \tag{2.15}$$

$$ka_1 e^{kH} - ka_2 e^{-kH} + \mu b_1 e^{\mu H} - \mu b_2 e^{-\mu H} = 0. \tag{2.16}$$

This set of equations uniquely determines a_1, a_2, b_1, b_2, and therefore ψ. We turn to the calculation of the pressure function p. Equation (2.1) reads

$$\begin{aligned} -\nabla p = c\,\partial_x v - \nu\Delta v &= ikc\nabla^\perp(a_1 e^{ky}e^{ikx} + a_2 e^{-ky}e^{ikx}) \\ &= ikc(-k, ik)a_1 e^{ky}e^{ikx} + ikc(k, ik)a_2 e^{-ky}e^{ikx}. \end{aligned}$$

We find

$$p = kca_1 e^{ky}e^{ikx} - kca_2 e^{-ky}e^{ikx}. \tag{2.17}$$

This can be inserted into the boundary condition (2.5). We then find the following complex relation between c, γ, and the material constant β:

$$kca_1 - kca_2 - 2\nu ki(ka_1 - ka_2 + \mu b_1 - \mu b_2) - \beta k^2 = -ik\gamma. \tag{2.18}$$

LEMMA 2.1. *In the limit $H \to -\infty$ there holds $a_2 \to 0$, $b_2 \to 0$, and*

$$a_1 \to c - 2\nu ki, \quad b_1 \to 2\nu ki.$$

Furthermore, γ and c are determined by

$$kc^2 - \beta k^2 = -(2\nu k)^2(\mathrm{Re}(\mu) - k), \tag{2.19}$$

$$\gamma k - 2 \cdot 2\nu k^2 c = -(2\nu k)^2 \mathrm{Im}(\mu). \tag{2.20}$$

Proof. From (2.14) and (2.16) we conclude that a_2 and b_2 are vanishing at an exponential rate for $H \to -\infty$. Then (2.13) and (2.15) imply

$$a_1 + b_1 \to c, \qquad a_1 + b_1 \cdot \left(1 + \frac{ic}{2k\nu}\right) \to 0,$$

and therefore the result. Equation (2.18) reduces to

$$a_1 kc - 2\nu k i a_1 k - 2\nu k i b_1 \mu - \beta k^2 + i\gamma k \to 0. \tag{2.21}$$

Taking separately the real part and the imaginary parts of the above equation yields the result. □

We are in particular interested in the behavior of the coefficients $a_1, \dots, b_2$ for $\nu \to 0$. This is the focus of the following lemma.

LEMMA 2.2. *Let $h_0 = |H|$ and $\kappa := e^{kH} < 1$ be fixed. We assume that c remains finite. We are interested in the solution (a_1, a_2, b_1, b_2) of* (2.13)–(2.16) *in the limit $\nu \to 0$. For some $C_0 > 0$ there holds $\mathrm{Re}(\mu) \geq \frac{C_0}{\sqrt{\nu}}$ and*

$$b_2 = |e^{2\mu H}|\, O(|b_1|), \tag{2.22}$$

$$b_1 = 2\nu i k + O(\nu^2), \tag{2.23}$$

$$a_1 + a_2 = c - 2\nu i k + O(\nu^2), \tag{2.24}$$

$$\kappa a_1 \left(1 + \frac{k}{\mu}\right) + \kappa^{-1} a_2 \left(1 - \frac{k}{\mu}\right) = o(|e^{\mu H}|). \tag{2.25}$$

Proof. The asymptotic behavior of μ follows from (2.8).

Relation (2.15) implies that $|a_1 + a_2| = O(\nu^{-1} \cdot |b_1 + b_2|)$, and (2.13) then yields $a_1 + a_2 = c + O(\nu)$ and $b_1 + b_2 = O(\nu)$.

We claim that both a_1 and a_2 remain finite. Assume the contrary. Since $a_1 + a_2$ remains finite, a_1 and a_2 tend to infinity at the same rate. Then also $|b_1 e^{\mu H} + b_2 e^{-\mu H}|$ and $|\mu| \cdot |b_1 e^{\mu H} - b_2 e^{-\mu H}|$ tend to infinity at equal rates. This is possible only if there are cancellations in $|b_1 e^{\mu H} - b_2 e^{-\mu H}|$; then b_1/b_2 tends to infinity. Recalling $b_1 + b_2 = O(\nu)$, we conclude that b_1 is of order ν. Then $b_2 = O(\nu e^{2\mu H})$, and by (2.13) and (2.14) both a_1 and a_2 remain finite. This is a contradiction.

Since a_1 and a_2 remain finite, (2.14) and (2.16) together imply (2.22). From (2.13) and (2.15) we see that for any $K \in \mathbb{N}$

$$-2k^2(c - b_1) = (k^2 + \mu^2) b_1 + O(\nu^K) = \left(2k^2 + \frac{ikc}{\nu}\right) b_1 + O(\nu^K),$$

and we conclude (2.23). Now (2.24) follows immediately from (2.13). We combine (2.14) and (2.16) to find

$$\kappa a_1 \left(1 + \frac{k}{\mu}\right) + \kappa^{-1} a_2 \left(1 - \frac{k}{\mu}\right) = o(|e^{\mu H}|).$$

This is (2.25). □

We draw a conclusion concerning the asymptotics of μ and γ in the following.

LEMMA 2.3. *There hold*

$$\frac{1}{\mu} = \frac{1 - i}{\sqrt{2kc}} \sqrt{\nu} + O(\nu^{3/2}), \tag{2.26}$$

$$\gamma = -2c\, \mathrm{Im}(a_1) + 2\nu k c \left(\frac{2\kappa^2}{1 - \kappa^2}\right) + o(\nu). \tag{2.27}$$

Proof. We start from the formula for μ:

$$\nu\mu^2 = ikc + k^2\nu.$$

We take the square root on both sides and do a Taylor expansion for the right-hand side. We find

$$\sqrt{\nu}\mu = \frac{1+i}{\sqrt{2}}\sqrt{kc} + O(\nu).$$

Taking the inverse we find the result for μ.

To find an asymptotic expression for γ we take the imaginary part of (2.18):

$$k\gamma + ck\ \mathrm{Im}(a_1 - a_2) - 2\nu k^2 \mathrm{Re}(a_1 - a_2) = o(\nu).$$

Using $\mathrm{Im}(a_1 - a_2) = \mathrm{Im}(2a_1 + b_1) + o(\nu)$ and, following from (2.24), (2.25), $\mathrm{Re}(a_1 - a_2) = c\frac{1+\kappa^2}{1-\kappa^2} + O(\sqrt{\nu})$, we find

$$\gamma + 2c\ \mathrm{Im}(a_1) + c\ \mathrm{Im}(b_1) - 2\nu kc\left(\frac{1+\kappa^2}{1-\kappa^2}\right) = o(\nu).$$

Inserting expression (2.23) for b_1, we have proved the claim. □

3. The domain transformation. In what follows we consider real functions (v, p, h) describing the velocity, pressure, and height of the wave. Our aim is to study the effect of the nonlinear contributions in the bifurcation problem. In this way we can get a better understanding of the bifurcation picture and find properties of the solutions that are not present in the linearized model. Several nonlinearities play a role; one of them is the dependence of the domain on the solution. We deal with this nonlinearity by transforming the equations onto a reference domain, here the rectangle R. Finding nonlinear equations for the new quantities is the aim of this section.

We start the analysis with the weak form of the nonlinear equations. We write $\tilde{v}$ and $\tilde{p}$ for the physical quantities and use the symmetrized first derivatives $Du := \frac{1}{2}(\nabla u + (\nabla u)^T)$. Thus we obtain

$$\begin{aligned} &\int_\Omega c\ \partial_x \tilde{v}\tilde{\Phi} + 2\nu \int_\Omega D\tilde{v} : D\tilde{\Phi} - \int_\Omega \tilde{v}\otimes\tilde{v} : \nabla\tilde{\Phi} \\ &\quad + \int_\Gamma (\tilde{v}\cdot n)\tilde{v}\cdot\tilde{\Phi} - \int_\Omega \tilde{p}\ \nabla\cdot\tilde{\Phi} + \int_\Gamma t(h) n\cdot\tilde{\Phi} = 0 \qquad \forall\tilde{\Phi}, \end{aligned} \tag{3.1}$$

$$\int_\Omega \nabla\cdot\tilde{v}\tilde{\Psi} = 0 \qquad \forall\tilde{\Psi}, \tag{3.2}$$

$$t(h) = -\beta\mathcal{H}(h) - \gamma h'. \tag{3.3}$$

Next we want the unknown functions to be defined on the reference domain R. We introduce the domain transformation

$$\Theta : \Omega \to R, \qquad (x, y) \mapsto \left(x, \left(\frac{h_0 + y}{h_0 + h(x)}\right) h_0 - h_0\right).$$

We can now set

$$\tilde{v} = v\circ\Theta, \quad \tilde{p} = p\circ\Theta,$$

with $v : R \to \mathbb{R}^2$ and $p : R \to \mathbb{R}$. We define Φ and Ψ analogously. Our next aim is to find nonlinear equations for the functions (v, p), to which end we introduce

$$J := |\det(D\Theta)|^{-1} = \frac{h_0 + h(x)}{h_0}, \qquad \tilde{\vartheta} := \partial_x \Theta_2 \cdot J = -\left(\frac{y + h_0}{h_0 + h(x)}\right) \partial_x h.$$

We can now transform the above equation term by term. The first yields

$$\begin{aligned}\int_\Omega c\ \partial_x \tilde{v}\tilde{\Phi} &= c \int_R (\partial_x v \cdot 1 + \partial_y v \cdot \partial_x \Theta_2)\, \Phi |\det(D\Theta)|^{-1} \\ &= c \int_R \partial_x v \Phi J + c \int_R \partial_y v \Phi \tilde{\vartheta}.\end{aligned}$$

We proceed in this way with all the terms. We simplify the system by collecting only quadratic terms in (v, p, h), and we use $\vartheta = -(y + h_0)h_0^{-1}\ \partial_x h(x)$ instead of $\tilde{\vartheta}$. Furthermore, we neglect quadratic contributions that involve a factor ν. This is in accordance with the analysis of the next section, where we consider the limit $\nu \to 0$. The new equations read

$$\begin{aligned}&\int_R c\ \partial_x v\ \Phi J + \int_R c\ \partial_y v\ \Phi \vartheta + 2\nu \int_R Dv : D\Phi - \int_R v \otimes v : \nabla \Phi \\ &+ \int_0^L (v \cdot e_2) v \cdot \Phi - \int_R p\ (J\partial_x \Phi_1 + \vartheta \partial_y \Phi_1 + \partial_y \Phi_2) \\ &+ \int_0^L t(h)(-h', 1) \cdot \Phi = 0 \qquad \forall \Phi,\end{aligned} \tag{3.4}$$

$$\int_R (J\partial_x v_1 + \vartheta \partial_y v_1 + J\partial_y v_2)\Psi = 0 \qquad \forall \Psi. \tag{3.5}$$

Since the mean-curvature operator coincides with Δ_x up to cubic terms in h, the above equations are complemented by

$$t(h) = -\beta \Delta_x h - \gamma h', \tag{3.6}$$

$$c\partial_x h = v_2 - h'\ v_1. \tag{3.7}$$

We consider (3.4)–(3.7) in what follows.

4. Bifurcation analysis. We introduce $\lambda_\varepsilon = (c_\varepsilon, \gamma_\varepsilon)$ and $u_\varepsilon = (v_\varepsilon, p_\varepsilon, h_\varepsilon)$. The equations can be written in the form

$$F(\lambda_\varepsilon, u_\varepsilon) = 0, \tag{4.1}$$

with

$$u_0 = 0, \qquad \partial_\varepsilon u_\varepsilon|_{\varepsilon=0} = u, \qquad \partial_\varepsilon^2 u_\varepsilon|_{\varepsilon=0} = \bar{u}.$$

Differentiating (4.1) twice with respect to ε yields, with the notation

$$\mathcal{L}\tilde{u} = \frac{\partial F}{\partial u} \cdot \tilde{u}, \tag{4.2}$$

$$\mathcal{N}(\tilde{u}) = \frac{1}{2}\left(\frac{\partial^2 F}{\partial u^2}\right) \langle \tilde{u}, \tilde{u} \rangle, \tag{4.3}$$

the equation

$$\frac{\partial \mathcal{L}}{\partial \lambda} \cdot u \; \frac{\partial \lambda_\varepsilon}{\partial \varepsilon} + \mathcal{L} \cdot \bar{u} + \mathcal{N}(u) = 0. \tag{4.4}$$

We assume that we are in the generic situation $\frac{\partial \lambda_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = 0$; this simplifies the above equation. We now calculate (4.4) in our case by inserting $u_\varepsilon = \varepsilon u + \varepsilon^2 \bar{u}$ and $\lambda_\varepsilon = \lambda + \varepsilon^2 \bar{\lambda}$ into (3.4)–(3.7). Here (u, λ) is the solution to the linear problem. We calculate the set of equations for $\bar{u}$ by comparing terms of order ε^2. For all smooth $\Phi : R \to \mathbb{R}^2$ and $\Psi : R \to \mathbb{R}$ there hold

$$\int_R c \; \partial_x \bar{v} \; \Phi + 2\nu \int_R D\bar{v} : D\Phi - \int_R \bar{p} \text{ div } (\Phi) - \beta \int_0^L \Delta_x \bar{h} \Phi_2 - \gamma \int_0^L \bar{h}' \cdot \Phi_2$$
$$+ \int_R c \; \partial_x v \; \Phi \; h h_0^{-1} - \int_R c \; \partial_y v \; \Phi h' \; \left(\frac{y + h_0}{h_0} \right) - \int_R v \otimes v : \nabla \Phi + \int_0^L (v \cdot e_2) v \cdot \Phi$$
$$- \int_R p \left(h \partial_x \Phi_1 - h' \; \partial_y \Phi_1 \frac{y + h_0}{h_0} \right) + \int_0^L (\beta \Delta_x h + \gamma h') \; h' \; \Phi_1 = 0, \tag{4.5}$$

$$\int_R (\partial_x \bar{v}_1 + \partial_y \bar{v}_2) \Psi - \int_R h' \; y \; \partial_y v_1 \; \Psi = 0. \tag{4.6}$$

These equations are complemented by

$$c\bar{h}' - \bar{v}_2 = -h' \; v_1. \tag{4.7}$$

So far the system has no normalizing condition. We use the orthogonal projection π_1 onto a subspace Ψ_1 and impose $\pi_1 h_\varepsilon = \pi_1(\varepsilon h)$ on the bifurcating branch. Then we find the normalizing condition

$$\bar{h} \perp \Psi_1 := \{ a \sin(kx) + b \cos(kx) | a, b \in \mathbb{R} \} . \tag{4.8}$$

Equations (4.5)–(4.8) form a complete set of equations and determine $(\bar{v}, \bar{p}, \bar{h})$. The velocity field $\bar{v}$ that is induced by the nonlinearity satisfies the following identity.

PROPOSITION 4.1. *There holds*

$$\nu \int_0^L \bar{v}_1 + h_0 \gamma \|h'\|^2 = \int_R v_1 \left(v_2 - c h' \left(\frac{y + h_0}{h_0} \right) \right) - \int_R p h' \left(\frac{y + h_0}{h_0} \right). \tag{4.9}$$

The right-hand side is bounded for $h_0 \to \infty$. Therefore for deep canals (h_0 large) there appears to be an average flow to the left.

Proof. We multiply (4.5) with the test-function $\Phi(x, y) = (h_0 + y) \; e_1$ that represents parallel shear-flows to the right. The only terms remaining are

$$2\nu \int_R D\bar{v} : D(\Phi) + \int_R c \; \partial_x v_1 \; \Phi_1 h h_0^{-1} - \int_R c \; \partial_y v_1 \; \Phi_1 h' \; \left(\frac{y + h_0}{h_0} \right) - \int_R v_1 v_2 \partial_y \Phi_1$$
$$+ \int_0^L v_2 v_1 \; \Phi_1(.,0) + \int_R p \; h' \; \left(\frac{y + h_0}{h_0} \right) \partial_y \Phi_1 + \int_0^L \gamma h' \; h' \; h_0 = 0.$$

We now evaluate

$$2\nu \int_R D\bar{v} : D(\Phi) = \nu \int_R D\bar{v} : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \nu \int_R \partial_y \bar{v}_1.$$

With one integration by parts and using $ch' = v_2$ on the upper boundary we calculate

$$\int_R c\ \partial_x v_1\ (y+h_0)hh_0^{-1} - \int_R c\ \partial_y v_1\ (y+h_0)^2 h'\ h_0^{-1} + \int_0^L v_2 v_1\ h_0 = \int_R ch'\ v_1\ \left(\frac{y+h_0}{h_0}\right).$$

This proves the claim (4.9). ∎

The next step in our analysis is to evaluate the integrals on the right-hand side of (4.9). We evaluate terms up to order $O(\nu)$.

LEMMA 4.2. *The three terms on the right-hand side of* (4.9) *satisfy*

$$2\int_R v_1 v_2 = 2k^2 h_0(\mathrm{Re}(a_1)\mathrm{Im}(a_2) - \mathrm{Im}(a_1)\mathrm{Re}(a_2)) - k\mathrm{Im}(b_1)\mathrm{Re}(a_1+a_2) + o(\nu), \tag{4.10}$$

$$-2\int_R v_1 ch'\left(\frac{y+h_0}{h_0}\right) = -\frac{c}{h_0}\mathrm{Im}(a_1)(1-e^{kH}) + \frac{c}{h_0}\mathrm{Im}(a_2)(1-e^{-kH}) + o(\nu), \tag{4.11}$$

$$-2\int_R ph'\left(\frac{y+h_0}{h_0}\right) = -ck\ \mathrm{Im}(a_1+a_2) + \frac{c}{h_0}\mathrm{Im}(a_1)(1-e^{kH}) - \frac{c}{h_0}\mathrm{Im}(a_2)(1-e^{-kH}) + o(\nu). \tag{4.12}$$

Proof. The proof is a direct calculation. We use the linear solution $v = \mathrm{Re}((-\partial_y, \partial_x)\psi)$, $h = \mathrm{Re}(e^{ikx})$, and obtain

$$\begin{aligned}
2\int_R v_1\cdot v_2 &= \mathrm{Re}\int_R (-\partial_2\bar\psi)\cdot(\partial_1\psi) = k\mathrm{Im}\int_R \partial_2\bar\psi\ \psi \\
&= k\mathrm{Im}\int_H^0 (k\bar a_1 e^{ky} - k\bar a_2 e^{-ky} + \bar\mu\bar b_1 e^{\bar\mu y})\cdot(a_1 e^{ky} + a_2 e^{-ky} + b_1 e^{\mu y})\ dy + o(\nu) \\
&= k\mathrm{Im}\int_H^0 k|a_1|^2 e^{2ky} + \bar a_1 a_2\ k - \bar a_2 a_1\ k - k|a_2|^2 e^{-2ky}\ dy \\
&\quad + k\mathrm{Im}\int_H^0 (k\bar a_1 e^{ky} - k\bar a_2 e^{-ky} + \bar\mu\bar b_1 e^{\bar\mu y})b_1 e^{\mu y}\ dy \\
&\quad + k\mathrm{Im}\int_H^0 \bar\mu\bar b_1 e^{\bar\mu y}(a_1 e^{ky} + a_2 e^{-ky})\ dy + o(\nu) \\
&= 2h_0 k^2 \mathrm{Im}\bar a_1 a_2 + o(\nu) - k\mathrm{Im}(\bar b_1(a_1+a_2)).
\end{aligned}$$

We used the fact that

$$\int_H^0 (k\bar a_1 e^{ky} - k\bar a_2 e^{-ky})b_1 e^{\mu y}\ dy = o(\nu)$$

and

$$\int_H^0 \bar\mu\bar b_1 e^{\bar\mu y}(a_1 e^{ky} + a_2 e^{-ky})\ dy = \bar b_1(a_1+a_2) - \int_H^0 \bar b_1 e^{\bar\mu y} k(a_1 e^{ky} - a_2 e^{-ky})\ dy = \bar b_1(a_1+a_2) + o(\nu).$$

Thus (4.10) is shown.

We turn to the proof of (4.11).

$$
\begin{aligned}
2\int_R v_1 h' \cdot (y+h_0) &= \mathrm{Re}\int_R (-\partial_2\bar\psi)\cdot h'\cdot(y+h_0)\\
&= -\mathrm{Re}\int_R (k\bar a_1 e^{ky} - k\bar a_2 e^{-ky} + \bar\mu \bar b_1 e^{\bar\mu y})\cdot e^{-ikx} ike^{ikx}(y+h_0) + o(\nu)\\
&= k\ \mathrm{Im}\int_H^0 (k\bar a_1 e^{ky} - k\bar a_2 e^{-ky} + \bar\mu \bar b_1 e^{\bar\mu y})\ (y+h_0)\ dy + o(\nu)\\
&= h_0 k\ \mathrm{Im}(\bar a_1 + \bar a_2 + \bar b_1) + k\ \mathrm{Im}\int_H^0 (-\bar a_1 e^{ky} - \bar a_2 e^{-ky} - \bar b_1 e^{\bar\mu y}) + o(\nu)\\
&= \mathrm{Im}(a_1)(1-e^{kH}) - \mathrm{Im}(a_2)(1-e^{-kH}) + o(\nu).
\end{aligned}
$$

We turn next to the proof of (4.12). From the expression for p we read off

$$
-2\int_R ph'\left(\frac{y+h_0}{h_0}\right) = -\frac{c}{h_0}\mathrm{Re}\int_R k\ (\bar a_1 e^{ky} - \bar a_2 e^{-ky})e^{-ikx} ike^{ikx}(y+h_0).
$$

Up to the factor $\frac{c}{h_0}$ and the term containing $\mathrm{Im}(b_1)$ this coincides with the second line of the previous calculation. We thus conclude (4.12). ∎

THEOREM 4.3. *In the limit $\nu \to 0$ there holds*

$$
\int_0^L \bar v_1 = -2k^3 h_0 c\left(\frac{1+\kappa^2}{1-\kappa^2}\right) + o(1). \tag{4.13}
$$

We recall that $\kappa = e^{-kh_0}$. Note that for $h_0 \to 0$ the above expression tends to $-4k^2c \neq 0$. The tangential flow on a wave of height ε has the average velocity $-2\varepsilon^2 k^3 h_0 c\frac{1+\kappa^2}{1-\kappa^2}$.

Proof. From Proposition 4.1 and Lemma 4.2 we conclude that

$$
\begin{aligned}
\nu\int_0^L \bar v_1 + h_0\gamma\|h'\|^2 = {}& k^2 h_0(\mathrm{Re}(a_1)\mathrm{Im}(a_2) - \mathrm{Im}(a_1)\mathrm{Re}(a_2))\\
&-\frac{1}{2}k\mathrm{Im}(b_1)\mathrm{Re}(a_1+a_2) + \frac{1}{2}ck\mathrm{Im}(b_1) + o(\nu).
\end{aligned}
$$

From the asymptotic expressions for a_1, a_2, and b_1 we now insert the relations

$$
\mathrm{Im}(a_2) = -\mathrm{Im}(a_1) - \mathrm{Im}(b_1) + o(\nu), \quad \mathrm{Re}(a_1+a_2) = c + o(\nu).
$$

The last two terms in the above expression cancel and we find

$$
\begin{aligned}
\nu\int_0^L \bar v_1 + h_0\gamma\|h'\|^2 &= k^2 h_0(\mathrm{Re}(a_1)\mathrm{Im}(a_2) - \mathrm{Im}(a_1)\mathrm{Re}(a_2)) + o(\nu)\\
&= k^2 h_0(\mathrm{Re}(a_1)(-\mathrm{Im}(a_1) - \mathrm{Im}(b_1)) - \mathrm{Im}(a_1)\mathrm{Re}(a_2)) + o(\nu)\\
&= -k^2 h_0(\mathrm{Im}(a_1)\mathrm{Re}(a_1+a_2) - \mathrm{Re}(a_1)\mathrm{Im}(b_1)) + o(\nu)\\
&= -k^2 h_0 c\ \mathrm{Im}(a_1) - \mathrm{Im}(b_1)\ k^2 h_0 \mathrm{Re}(a_1) + o(\nu)\\
&= -k^2 h_0 c\ \mathrm{Im}(a_1) - 2\nu k^3 h_0\left(\frac{c}{1-\kappa^2}\right) + o(\nu).
\end{aligned}
$$

We used (2.23) in order to evaluate $\mathrm{Im}(b_1)$ and (2.25) to find the expression $\mathrm{Re}(a_1) = \frac{c}{1-\kappa^2} + o(\nu)$. We finally insert the asymptotic expression for γ:

$$h_0\gamma\|h'\|^2 = -h_0\,\frac{k^2}{2}2c\mathrm{Im}(a_1) + h_0\,\frac{k^2}{2}\cdot 2\nu kc\left(\frac{2\kappa^2}{1-\kappa^2}\right) + o(\nu).$$

We find

$$\begin{aligned}\nu\int_0^L \bar{v}_1 &= -\nu k^3 h_0 c\,\left(\frac{2\kappa^2}{1-\kappa^2}\right) - 2\nu k^3 h_0\left(\frac{c}{1-\kappa^2}\right) + o(\nu)\\ &= -2\nu k^3 h_0 c\left(\frac{1+\kappa^2}{1-\kappa^2}\right) + o(\nu).\end{aligned}$$

This is the result. ◻

5. Further equations for nontrivial solutions. In this section we derive some relations that are satisfied by the traveling wave solutions. Later on we will use these relations to check the quality of numerical results. We again use the scalar fields $J = \frac{h_0+h(x)}{h_0}$ and $\vartheta = -\frac{y+h_0}{h_0}\partial_x h$ to derive relations for the following quantities:

$$\begin{aligned}\langle Av, v\rangle &:= \int_R Dv : Dv, \qquad \langle Uv, v\rangle := \int_\Omega (v\cdot\nabla)v\, v,\\ \langle Cv, v\rangle &:= \int_R \partial_x v\, v\cdot J + \int_R \partial_y v\, v\cdot\vartheta,\\ \langle J_1 h, v_1\rangle &:= \int_0^L (\partial_x h)^2 v_1\, dx, \qquad \langle J_2 h, v_1\rangle := \int_0^L \partial_x h\, v_2\, dx,\\ \langle T_1 h, v_1\rangle &:= \int_0^L \Delta_x h\,(-\partial_x h)v_1\, dx,\\ \langle T_2 h, v_2\rangle &:= \int_0^L \Delta_x h\, v_2\, dx.\end{aligned}$$

LEMMA 5.1. *For a solution* $(v^\varepsilon, p^\varepsilon, h^\varepsilon)$ *of the bifurcation equations* (3.4)–(3.7) *the following relations are satisfied:*

$$\langle Uv^\varepsilon, v^\varepsilon\rangle + c\langle Cv^\varepsilon, v^\varepsilon\rangle = 0, \tag{5.1}$$

$$\langle T_1 h^\varepsilon, v_1^\varepsilon\rangle + \langle T_2 h^\varepsilon, v_2^\varepsilon\rangle = 0, \tag{5.2}$$

$$2\nu\langle Av^\varepsilon, v^\varepsilon\rangle + \gamma\langle J_1 h^\varepsilon, v_1^\varepsilon\rangle + \gamma\langle J_1 h^\varepsilon, v_1^\varepsilon\rangle = 0. \tag{5.3}$$

Proof. We omit the superscript ε in the calculation. The convective term is

$$\langle Uv, v\rangle = \int_R v_i\nabla_i v_j\; v_j = \int_0^L v_2\frac{1}{2}|v|^2,$$

and the frame-speed term is

$$\begin{aligned}\langle Cv, v\rangle &= \int_R \left(\frac{h_0+h}{h_0}\right)\partial_x v\cdot v - \int_R h'\left(\frac{y+h_0}{h_0}\right)\partial_y v\cdot v\\ &= -\int_R h'/h_0\,\frac{1}{2}|v|^2 + \int_R h'/h_0\,\frac{1}{2}|v|^2 - \int_0^L h'\frac{1}{2}|v|^2\\ &= -\int_0^L h'\frac{1}{2}|v|^2.\end{aligned}$$

This implies (5.1). The surface-tension term (5.2) is

$$\begin{aligned}\langle T_1 h, v_1\rangle + \langle T_2 h, v_2\rangle &= \int_0^L \Delta_x h\ (v_2 - h' v_1)\\ &= c\int_0^L \Delta_x h\ h' = 0.\end{aligned}$$

Equality (5.3) follows from the fact that the sum vanishes over all terms :

$$\begin{aligned}&2\nu\langle Av, v\rangle + \gamma\langle Jh, v\rangle + \langle Uv, v\rangle\\ &\qquad + c\langle Cv, v\rangle + \beta\langle Th, v\rangle = 0.\end{aligned}$$

This is a consequence of (3.4). $\square$

6. Discretization. We follow the standard approach of finite-element discretization as outlined, e.g., in [4]. We solve for the primary physical quantities velocity $v \in V$, pressure $p \in W$, and the height function of the free boundary $h \in X$. The function spaces are

$$V = H^1(\Omega), \qquad W = L^2(\Omega), \qquad X = H^1(\Gamma),$$

with $\Omega = [0,1] \times [-h_0, 0]$ and $\Gamma = [0,1] \equiv [0,1] \times \{0\} \subset \Omega$, both equipped with periodicity conditions in the horizontal end-points. We approximate the above spaces with finite element basis functions

$$\Phi_i \in V, \qquad \Psi_k \in W, \qquad \varphi_l \in X.$$

We have chosen a rectangular grid on Ω. The finite element (FE)-functions Φ_i are piecewise biquadratic polynomials, Ψ_k are piecewise bilinear polynomials, and φ_l are piecewise quadratic polynomials. Note that, in our calculation, velocities and pressure are defined on a fixed domain. To take into account the domain transformation we introduce nonlinear quadratic forms in what follows.

The FE-discretization yields divergence matrices $\tilde{B}_1$ and $\tilde{B}_2$. The stiffness matrix $\tilde{A}$ discretizes the product of the symmetrized gradients

$$\langle D\Phi_i, D\Phi_j\rangle_{L^2(\Omega)} \longrightarrow \tilde{A} := \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}.$$

We need matrices that describe coupling between volume forces and boundary forces. They are defined as follows:

$$\begin{aligned}(R^0)_{il} &:= ((\Phi_i \cdot e_2)|_\Gamma, \varphi_l)_{L^2(\Gamma)},\\ (R^1)_{il} &:= (\partial_x(\Phi_i \cdot e_2)|_\Gamma, \varphi_l)_{L^2(\Gamma)},\\ (R^2)_{il} &:= (\partial_x(\Phi_i \cdot e_2)|_\Gamma, \partial_x\varphi_l)_{L^2(\Gamma)}.\end{aligned}$$

The eigenfunctions of imaginary eigenvalues correspond to traveling waves. In the numerical scheme we do not calculate eigenvalues but rather follow the lines of the continuous considerations and do all calculations in a moving coordinate system. In order to do so, we define matrices that correspond to x-derivatives:

$$\begin{aligned}\tilde{H}_{lm} &:= (\partial_x\varphi_l, \varphi_m)_{L^2(\Gamma)},\\ \tilde{C}_{ij} &:= (\partial_x\phi_i, \phi_j)_{L^2(\Omega)}.\end{aligned}$$

Our aim is to calculate real factors γ (wind-speed) and c (wave-speed) such that the system has a nontrivial solution. The numerical scheme implements a Newton method to find this solution. In every step the matrix T below is calculated from the approximate solution. T depends on the approximate solution in two ways: 1) the factors γ and c are taken from the last approximation, and 2) the matrices C, B, T, etc. discretize the nonlinear operators of (3.4), corresponding to $\tilde{C}$, $\tilde{B}$, $\tilde{T}$, etc. This captures the effects of the domain transformation. The operators have to be constructed using the approximate height function of the last step. The transport operator U discretizes the operator $(v \cdot \nabla)$ and has to be constructed using the approximate velocity profiles:

$$T := \begin{bmatrix} 2\nu A_{11} + U + c\, C & 2\nu A_{12} & B_1^t & \beta T_1 + \gamma J_1 \\ 2\nu A_{21} & 2\nu A_{22} + U + c\, C & B_2^t & \beta T_2 + \gamma J_2 \\ B_1 & B_2 & 0 & 0 \\ K_1 & K_2 & 0 & c\, H \end{bmatrix}.$$

Here $K_2 = R^0$, $J_2 = R^1$, and $T_2 = R^2$. Note that for every (γ, c) the matrix T is singular since the vector $(v, p, h) = (0, 0, \text{const})$ is mapped to zero. This corresponds to the fact that we have to pose a volume condition such as $\int h = 0$. We do so by adding the volume equation to one row of the matrix H. After this change, still, for the solution parameters $\gamma = \gamma_0$ and $c = c_0$, the matrix $T_{(c,\gamma)}$ has a two-dimensional kernel, corresponding to two eigenvectors with $h \sim \sin(x)$ and $h \sim \cos(x)$. We normalize the solutions with the two additional rows

$$r_{sin} \cdot h := \langle \sin(.), h(.) \rangle_{L^2(\Gamma)} = \frac{1}{2} h_{max},$$
$$r_{cos} \cdot h := \langle \cos(.), h(.) \rangle_{L^2(\Gamma)} = 0.$$

In every step of the Newton method we have to solve a system of the form

$$\begin{bmatrix} & & & & J_1 \cdot h & C \cdot v_1 \\ & & T & & J_2 \cdot h & C \cdot v_2 \\ & & & & 0 & 0 \\ & & & & 0 & H \cdot h \\ 0 & 0 & 0 & r_{sin} & 0 & 0 \\ 0 & 0 & 0 & r_{cos} & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{p} \\ \tilde{h} \\ \tilde{\gamma} \\ \tilde{c} \end{pmatrix} = f.$$

We perform an LU-decomposition of this matrix for the linear equation. To solve the nonlinear system we implement a fixed point iteration, using the linear LU-decomposition as a preconditioner. For every small amplitude $h_{max} \in \mathbb{R}$ we find solutions (v, p, h, γ, c) of the nonlinear system.

7. Numerical results. All calculations are done in the cgs-system of units. We have chosen the viscosity and surface tension for a water-air interface, that is, $\nu = 0.01, \beta = 74$. The domain is a box of length 1 and height $h_0 = 0.33$; the according wave number is $k = 2\pi$. The grid is shown in Figure 1.

The maximal height of the wave is set to be $h_{max} = 0.01$. The Newton iteration yields a solution corresponding to every wave-height $\|h\|$, together with the values for c and γ. Figure 2 shows the profile of the wave: it is close to the sinusoidal shape of the linear solution. The parameter γ quantifies the wind force—it is therefore appropriate to call Figure 3 the bifurcation diagram. It shows the dependence of the wave-height on the force. Figure 4 shows the dependence of the height on the wave-speed c. All units are in the *cgs*-system.

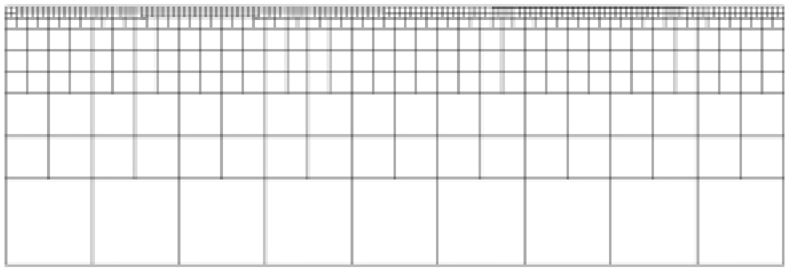

FIG. 1. *The grid.*

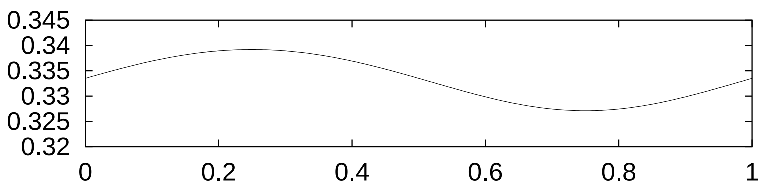

FIG. 2. *The wave profile.*

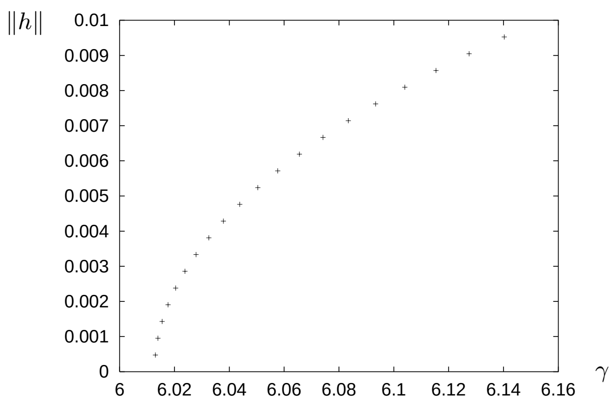

FIG. 3. *The bifurcation diagram.*

TABLE 1
Numerical values of the control parameters.

Bilinear form	Value	$\sum$	Rel. error
$\langle Uv, v\rangle$	0.229034		
$c\langle Cv, v\rangle$	-0.225985	0.003	1.3 %
$\langle T_1 h, v_1\rangle$	-0.014579		
$\langle T_2 h, v_2\rangle$	0.014578	0.000	$<$ 0.1 %
$2\nu\langle Av, v\rangle$	0.972864		
$\gamma\langle J_1 h, v_1\rangle$	-0.055459		
$\gamma\langle J_2 h, v_2\rangle$	-0.920452	-0.003	0.3 %

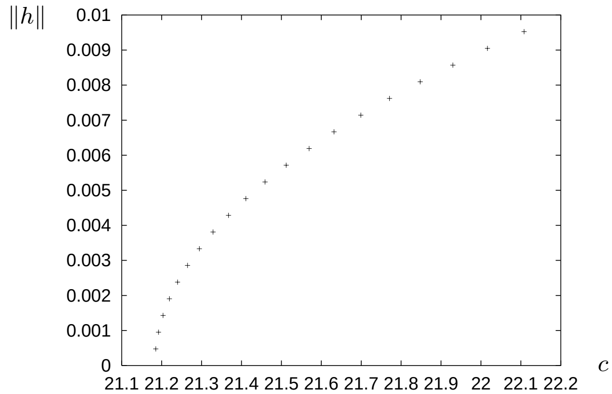

FIG. 4. *The wave-speed on the bifurcating branch.*

TABLE 2
Tangential speed for $\nu = 0.01$ *and* $h_{max} = 0.01$.

$\|h\|$	S_{theory}	$S_{discrete}$	Rel. error
0.0019	0.0525	0.0495	5.7 %
0.0038	0.2111	0.1986	5.9 %
0.0057	0.4789	0.4483	6.4 %
0.0076	0.8617	0.8008	7.1 %
0.0095	1.3673	1.2592	7.9 %

To have an indication of the precision of the calculation we evaluate the control parameters that were derived in section 5; they are listed in Table 1. We know from Lemma 5.1 that two or three consecutive lines, respectively, should add up to zero.

We next check the accuracy of formula (4.13) describing the appearance of the shear-flow. Table 2 shows, for different values of the height, the limiting value

$$S_{theory} = 2k^3 h_0 c \left(\frac{1+\kappa^2}{1-\kappa^2}\right) \|h\|^2$$

of (4.13) (calculated with the numerical value for c) and the discrete evaluation of the quantity $S_{discrete} = -\int_0^L \bar{v}_1$. Figure 5 illustrates the table: We plot $S_{discrete}$ versus S_{theory}. The diagonal corresponds to exact coincidence of theoretical (asymptotic) value and discrete results. We find that the measured points lie below the diagonal.

Figure 6 gives an idea of the velocity field of the nonlinear traveling wave solution propagating to the left; the corresponding height-function is shown in Figure 2. As a result of the tangential surface-flow to the left, the velocity at the boundary points $x = 0$ and $x = 0.5$ has a component to the left. This is in contrast with the linear situation, in which the velocity is oriented in the vertical direction at these two points. The pressure distribution is indicated with gray-scale isolines in Figure 6. While in the linear situation the pressure is proportional to $\sin(x)$, in our case it is no longer symmetric, but ranges from -60.0 to 47.4.

8. Conclusions and outlook. We have presented a study on the generation of water waves by wind. Our approach was to consider the phenomenon as a bifurcation problem—this is only possible if viscosity is included. We were able to derive integral identities for the solution. In order to evaluate these identities we had to consider the

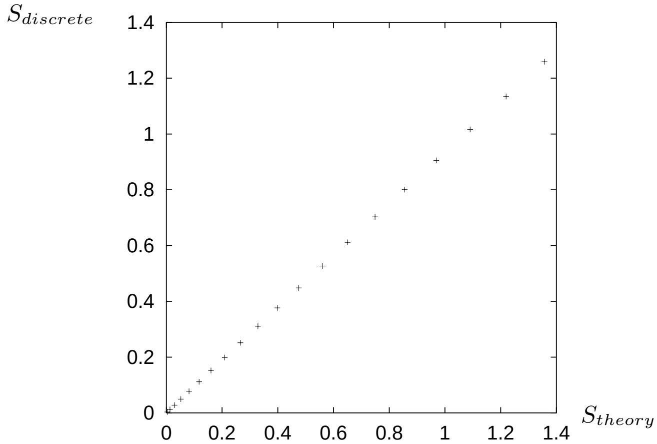

FIG. 5. *The average tangential speed: Prediction by theory vs. numerical observation.*

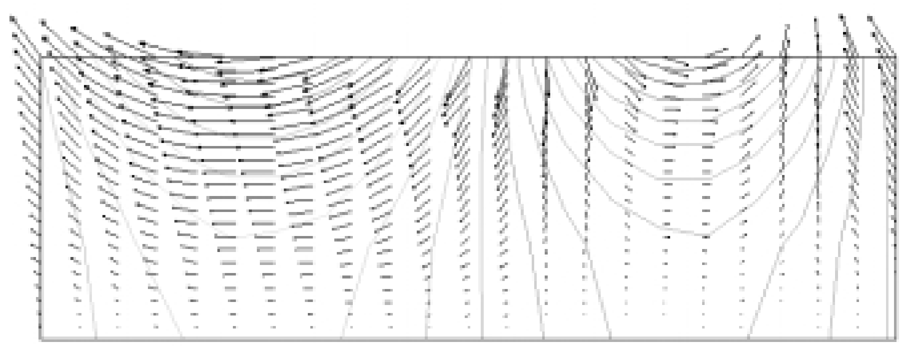

FIG. 6. *The numerical solution.*

limit of vanishing viscosity. In this limit we could show that wind generates a drift, and we could give the quantitative result with (4.13). The numerical findings are in good agreement.

The observed phenomenon has similarities to the Stokes drift for inviscid fluids. A clear distinction is the behavior in deep canals. While the Stokes drift remains finite, the effect studied here can generate an arbitrarily large drift: For large depth the fluid behaves as if it were exposed to a tangential force on the boundary.

We found that the numerical values for the drift are always below the theoretical values. We have to keep in mind that the numerical data are bound to discretization errors. On the other hand, the theoretical formula was derived in the limit of vanishing viscosity. Our interpretation (based on the fact that small discretization errors appear in Table 1) is that a finite viscosity diminishes the drift.

In its present form the numerical scheme allows only for relatively small values of the wave-elevation ($\|h\| = 0.01$). It is desirable to calculate a continuation of the bifurcating branch. Nevertheless we are already far from the linearized situation: In the upper left corner of Figure 6 we see that the vertical speed (which is as in a linear solution) has the same magnitude as the horizontal speed (which is absent in the linear solution). Linear and nonlinear effects are already of the same order.

Another open question concerns the observed wave length. Given a wave number we calculated properties of solution branches, but we did not touch the question of which wave number bifurcates first. This question requires a further analysis of the effect of wind: the dependence of the wind force γ on the wave number must be investigated.

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