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An Epic Drama: The Development of the Prime Number Theorem

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An Epic Drama: The Development of the Prime Number Theorem

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1 Introduction

From the time human beings first learned to count we have been fascinated by numbers and their almost magical properties. While the mathematical world was thrilled with the final proof of Fermat's Big Theorem this result is only one of a vast array of amazing results in number theory. We will concentrate on one of these, the prime number theorem, that describes the density of the set of prime numbers, among all the natural numbers, in terms of the natural logarithm function ln. This theorem, and the techniques surrounding its proof, has been touted as the most surprsing result in all of mathematics (see the article by Apostol [Apo]). Why should a result about the natural numbers - the counting numbers - involve the natural logarithm function? Why should the proof of a result about natural numbers involve the location of the zeros of a complex function defined in terms of a complex power series? Why should an *elementary proof* be more complicated and involved than a nonelementary proof? These are mysteries that still astound even workers in the field and mysteries that we will try to explain in this article.

What we propose to do in this paper is retell an epic drama in the history of mathematics that is still continuing. As with all goood drama it should be retold to each new generation of mathematicians. Further as with all great epics, it has great heros. Here the heros are many of history's greatest mathemtaicians; Euler, Gauss, Legendre, Riemann, Hadamard, de la Vallee Poussin, Hardy, Seleberg, Erdos and others.

The saga begins with two conjectures about the density of primes, one by Gauss and one by Legendre, after some earlier suggestions by Euler. It continues until a formal proof of these conjectures is found almost a hundred years later. Along the way, the search for a proof initiates a whole new branch of mathematics, **analytic number theory** and introduces the use of complex analysis into the study of number theory. Out of the search for a proof comes a conjecture, **the Riemann hypothesis** that is now perhaps the outstanding open problem in mathematics. Fifty years after the initial proof an *elementary proof* not using complex analysis was discovered by Selberg and Erdos. This elementary proof is in many ways much more involved than the nonelementary proof. After all of these discoveries the fascination with primes continues. Most recently, Ben Green and Terence Tao (see [GT]), proved another astounding result concerning arbitarily long sequences of primes. Terence Tao was awarded the Fields Medal in part for this result.

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2 The Prime Number Theorem:Development and Formulation

Most of modern mathematics traces back, in one way or another, to the real number system. Going further backward, the real numbers depend on the integers, and hence on number theory. The atoms or building blocks of the integers, via the Fundamental Theorem of Arithmetic, are the set of primes. Hence in quite a strong sense the study of the sequence of primes is truly fundamental in mathematics.

The first important fact one observes about the primes is that there are infinitely many of them. A proof of this fact appears in Euclid's Elements and today there are a vast number of other proofs of this basic result. In [FR] over thirty different independent proofs from distinct areas of number theory were presented. These proofs illuminated mnay aspects of the general theory of numbers. In distinction to the infinitude of primes, an inspection of the positive integers clearly indicates that the primes "thin out" . This is perhaps most strikingly quantified by the result that there are arbitrarily large gaps in the sequence of primes. More specifically, given any positive integer k, no matter how large, one can find a set of k consecutive integers all of which are composite. Hence the natural question arises as to the distribution or density of the primes. Here interest centers on the **prime number function** $\pi(x)$ defined for all real numbers x by

$$\pi(x) = \text{ number or primes } \leq x.$$

The basic question is whether there exists some easily defined function that either computes $\pi(x)$ or approximates $\pi(x)$. Clearly $\pi(x) \to \infty$ as $x \to \infty$ so the appropriate question on the distribution of primes is what is the growth rate of this function. The **Prime Number Theorem** asserts that asymptotically $\pi(x)$ is given by $\frac{x}{\ln x}$. **Asymptotically** means as x goes to ∞. It has been touted as one of the most surprising results in mathematics given that it ties together the primes and the natural logarithm function in a simple way that is most unexpected. Formally stated this result is:

Theorem 2.1. *(Prime Number Theorem) If $\pi(x)$ is the prime number function then*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

Whether or not it is the most surprising is of course open to debate. However as pointed out by Apostol it certainly is surprising that the natural log function ln appears so prominently in a result about natural numbers. Further the original and most commonly presented proofs of this result depend on complex analysis. This is again surprising in that a result concerning a subset of the natural numbers would depend upon the theory of analytic functions. What is certainly true is that the Prime Number Theorem is a result whose statement is known to most mathematicians yet very few go through its proof or know much about its development. The purpose of this article is to trace the development of the formulation and eventual proofs of this important theorem.

Some ideas on $\pi(x)$ were hinted at by Euler however the first real conjecture concerning the prime number function was given by Legendre in 1808. Shortly thereafter Gauss presented a different but asymptotically equivalent formulation.

Legendre, by looking at the list of primes up to 1,000,000, conjectured the following concerning the prime number function.

$$\pi(x) \approxeq \frac{x}{\ln x - 1.08366}.$$

Legendre gave no indication of how he arrived at the constant 1.08366. It must have arisen from some sort of experimentation with approximations of the form

$$\pi(x) \approxeq \frac{x}{\ln x - a}$$

where a is a real constant. Notice that for all such real constants a

$$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{x}{\ln x - a}$$

It follows that the prime number theorem is equivalent to

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x - a} = 1$$

for any constant a. The question arises as to whether there is an optimal value for a. Empirical evidence is that $a = 1$ is an optimal choice and generally better for very large x than Legendre's 1.08366.

Legendre did however attempt a proof of his conjecture. He did this by trying to quantify the Sieve of Eratosthenes. Before we describe Legendre's technique we recall some notation which makes the presentation of some of the statments much easier. Suppose that $f(x), g(x)$ are positive real valued functions. Then

(1) $f(x) = O(g(x))$ (read $f(x)$ is big O of $g(x)$) if there exists a constant A independent of x and an x_0 such that

$$f(x) \leq Ag(x) \text{ for all } x \geq x_0$$

(2) $f(x) = o(g(x))$ (read $f(x)$ is little o of $g(x)$) if

$$\frac{f(x)}{g(x)} \to 0 \text{ as } x \to \infty$$

In other words $g(x)$ is of a **higher order of magnitude** than $f(x)$.

(3) If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, that is there exist constants A_1, A_2 independent of x and an x_0 such that

$$A_1 g(x) \leq f(x) \leq A_2 g(x) \text{ for all } x \geq x_0,$$

then we say that $f(x)$ and $g(x)$ are of the **same order of magnitude** and write

$$f(x) \approxeq g(x)$$

(4) If

$$\frac{f(x)}{g(x)} \to 1 \text{ as } x \to \infty$$

then we say that $f(x)$ and $g(x)$ are **asymptotically equal** and we write

$$f(x) \sim g(x)$$

In general we write $O(g)$ or $o(g)$ to signify an unspecified function f such that $f = O(g)$ or $f = o(g)$. Hence for example writing $f = g + o(x)$ means that $\frac{f-g}{x} \to 0$ and saying that f is $o(1)$ means that $f(x) \to 0$ as $x \to \infty$.

It is clear that being $o(g)$ implies being $O(g)$ but not necessarily the other way around. Further it is easy to see that

$$f \sim g \text{ is equivalent to } f = g + o(g) = g(1 + o(1)).$$

In terms of this notation the prime number theorem can be expressed by

$$\pi(x) \sim \frac{x}{\ln x}$$

or equivalently

$$\pi(x) = \frac{x}{\ln x}(1 + o(1)).$$

Now we return to Legendre's attempted proof. Recall that the **Sieve of Eratosthenes** is a straightforward method to obtain all the primes less than or equal to a fixed bound x. It is ascribed (as the name suggests) to Eratosthenes (276-194 B.C.) who was the chief librarian of the great ancient library in Alexandria. Besides the sieve method he was an influential scientist and scholar in the ancient world, developing a chronology of ancient history (up to that point) and helping to obtain an accurate measure (within the measurement errors of his time) of the dimensions of the Earth.

The method of the Sieve of Eratosthenes is direct and works as follows. Given $x > 0$ list all the positive integers less than or equal to x. Starting with 2, which is prime, cross out all multiples of 2 on the list. The next number on the list, not crossed out, which is 3, is prime. Now cross out all the multiples of 3 not already eliminated. The next number left uneliminated, 5, is prime. Continue in this manner. This must only be done for numbers $\leq \sqrt{x}$. Upon completion of this process, any number not crossed out must be a prime.

Below we exhibit the Sieve of Eratosthenes for numbers $\leq$ 100. In beginning each round of elimination we must only consider numbers $\leq \sqrt{100} = 10$.

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

After completing the sieving operation we obtain the list

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 83, 89, 97\}$$

which comprises all the primes less than or equal to 100.

Given positive integers m, x, by a slight modification, the Sieve of Eratosthenes can be used to determine all the positive integers relatively prime to m and less than or equal to x.

Here suppose we are given m and x. Let $p_1, ..., p_k$ be the distinct prime factors of m arranged in ascending order, that is $p_1 < p_2 < ... < p_k$. Next list all the positive integers less than or equal to x as we did for the ordinary sieve. Start with p_1 and eliminate all multiples of p_1 on the list. Then successively do the same for p_2 through p_k. The numbers remaining on the list are precisely those relatively prime to m that are also less than or equal to x. If $p_i > x$ ignore this prime and all higher primes.

Legendre in attempting to prove his conjecture derived a computational formula for the Sieve of Eratosthenes. Given a positive integer m and a positive x let

$$N_m(x) = \text{ number of integers } \leq x \text{ and relatively prime to } m.$$

This is precisely the size of the list obtained in the modified Sieve of Eratosthenes derived above. Then:

Theorem 2.2. *(Legendre's Formula for the Sieve of Eratosthenes) Let $m \in \mathbb{N}, x \geq 0$ then*

$$N_m(x) = \sum_{d|m} \mu(d)[\frac{x}{d}]$$

where $\mu(d)$ is the Moebius function and $[\ ]$ is the greatest integer function.

Now given $x \geq 0$ let

$$m = \prod_{(p \leq \sqrt{x})} p$$

where p is prime. Then $N_m(x)$ counts the number of primes in the interval $[\sqrt{x}, x]$. It follows that

$$N_m(x) = \pi(x) - \pi(\sqrt{x}) + 1.$$

Substituting Legendre's formula into this expression we obtain as a corollary

Corollary 2.1. *For $x \geq 2$,*

$$\pi(x) = -1 + \pi(\sqrt{x}) + \sum_{\nu(d) \leq \sqrt{x}} \mu(d)[\frac{x}{d}]$$

where $\nu(d)$ is the greatest prime factor of d.

Although this gives a formula for $\pi(x)$, it is essentially useless in truly computing $\pi(x)$ for large x, or in shedding any light on the prime number theorem. First of all if we estimate $[\frac{x}{d}]$ by $\frac{x}{d} + O(1)$ and substitute in the formula we have

$$\begin{aligned} \pi(x) - \pi(\sqrt{x}) + 1 \quad &= \textstyle\sum_{\nu(d) \leq \sqrt{x}} \mu(d)(\frac{x}{d} + O(1)) \\ &= x \textstyle\prod_{p \leq \sqrt{x}} (1 - \frac{1}{p}) + O(2^{\pi(\sqrt{x})}) \end{aligned}$$

Hence the error term is exponentially larger than the main term.

Meisel in 1870 gave an improvement to Legendre's formula and was able to use this technique to compute $\pi(x)$ correctly up to $x = 10^8$.

Theorem 2.3. *(Meisel's Formula) Let $p_1 < p_2 < ... < p_n < ...$ be the listing of the primes in increasing order so that p_j is the jth prime. Let $x \geq 4$, $n = \pi(\sqrt{x})$ and $m_n = p_1 ... p_n$. Then*

$$\pi(x) = N_{m_n}(x) + m(1+s) + \frac{1}{2}s(s-1) - 1 - \sum_{j=1}^{s} \pi(\frac{x}{p_{m+j}})$$

where $m = \pi(x^{\frac{1}{3}})$ and $s = n - m$.

Gauss a bit after Legendre presented a different conjecture concerning the prime number function. By examining the list of primes less than 3,000,000 Gauss conjectured that the prime number function is given asymptotically by the logarithmic integral function $Li(x)$ defined as

$$Li(x) = \int_2^x \frac{1}{\ln t} dt.$$

Gauss' observation was then that

$$\pi(x) \approxeq Li(x).$$

If integration by parts is used on the integral defining $Li(x)$, and we take the limit as $x \to \infty$, it is clear that this integral is asymptotically $\frac{x}{\ln x}$. Hence Gauss's observation is then that

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \ln x} = 1$$

and therefore equivalent to Legendre's conjecture and the Prime Number Theorem.

Essentially Gauss' conjecture is that the function $\frac{1}{\ln x}$ is a density function for the set of prime numbers. Along these lines a very interesting interpretation of the prime number theorem is the following. The ratio $\frac{\pi(x)}{x}$ represents the probability of randomly choosing a prime less than or equal to x. The prime number theorem says that asymptotically this probability is given by $\frac{1}{\ln x}$, Gauss' density function.

What is of further interest here is that even for very large x the value $\frac{1}{\ln x}$ is not that small. Henec probabilistically it is not that hard to randomly choose a very large prime. This has applications in cryptography, esepcially in the implementation of the RSA algorithm. The book [FR] has a discussion and explanation of this.

The table below compares various approximations to $\pi(x)$.

x	$\pi(x)$	$\frac{x}{\ln x}$	$Li(x)$	$\frac{x}{\ln x - 1.08366}$	$\frac{x}{\ln x - 1}$
10^3	168	145	178	172	169
10^4	1229	1086	1246	1231	1218
10^5	9592	8686	9630	9588	9512
10^6	78498	72382	78628	78534	78030
10^7	664579	620420	664918	665138	661459
10^8	5761455	5428681	5762209	5769341	5740304

Observing the table above one notices that $Li(x) > \pi(x)$. The question arose as to whether this is always true. Littlewood in 1914 [Li] proved that $\pi(x) - Li(x)$ assumes both positive and negative values infinitely often. Te Riele in 1986 [Re] showed that there are greater than 10^{180} consecutive integers for which $\pi(x) > Li(x)$ in the range $6.62 \times 10^{370} < x < 6.69 \times 10^{370}$.

The prime number function $\pi(x)$ and the prime number theorem answer the basic questions concerning the density of primes. A related question concerns the the function

$$p(n) = p_n$$

where p_n is the nth prime. That is the question of whether there is a closed form function which estimates the nth prime. The answer to this is yes and turns out to be equivalent to the prime number theorem. We state it below.

Theorem 2.4. *The nth prime p_n is given asymptotically by*

$$p_n \approxeq n \ln n.$$

3 The Use of Analysis in Number Theory

The proof of the prime number theorem was finally accomplished in 1896 independently by Hadamard and de la Vallee-Poussin. Both proofs built on a brilliant method introduced by Riemann in 1860. Riemann's method introduced the use of complex analysis into number theory and led ultimately to the development of that branch of mathematics now called **analytic number theory**. As with most brilliant ideas, Riemann's idea had precursors and the use of analysis in studying number theoretical problems predates Riemann. The first use of analysis seems to have been done by Euler who was examining the density of primes. In particular he proved the following theorem.

Theorem 3.1. *The sum over the set or primes*

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges. In particular as a consequence the set of primes must be infinite.

To interpret this result notice that it says for example that the set of primes, although they thin out, is still more numerous that the set of perfect squares

$$\{1, 4, 9, 16, ...n^2...\}.$$

To see why this is true, recall that by the p-series test, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

To prove Theorem 2.1 Euler introduced the **zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where for Euler s was a real variable. From the p-series test this will converge if $s > 1$ and hence will define a function over the interval $(1, \infty)$

By using the fundamental theorem of arithmetic that each n can be expressed as a product of primes Euler showed that the zeta function can be written as the following product

$$\zeta(s) = \prod_{p,\ \text{prime}} (\frac{1}{1-p^{-s}}).$$

This product is known as that **Euler product expansion**. Then, by examining $\ln(\zeta(s))$ he was able to prove the divergence result.

Euler's ideas were extended by Dirichlet who used them to prove that there are infinitely many primes in any arithmetic progression $an + b$ with a and b relatively prime. This result is now known as **Dirichlet's Theorem**.

Theorem 3.2. *(Dirichlet's Theorem) Let a,b be natural numbers with* $(a, b) = 1$. *Then there are infinitely many primes of the form* $an + b$.

Dirichlet's proof is a beautiful amalgam of number theory and analysis. The proof rests on two concepts; **Dirichlet characters** and **Dirichlet series**. The basic idea is to build for each integer a, a series, which would converge if there were only finitely many primes congruent to b mod a and then show that this series actually diverges.

For each natural number k, Dirichlet introduces a complex-valued function

$$\chi_k : \mathbb{Z} \to \mathbb{C}$$

called a **Dirichlet character**. Specifically, for any integer k, a **Dirichlet character** modulo k, is a complex valued function on the integers $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

(1) $\chi(a) = 0$ if $(a, k) > 1$

(2) $\chi(1) \neq 0$

(3) $\chi(a_1 a_2) = \chi(a_1)\chi(a_2)$ for all $a_1, a_2 \in \mathbb{Z}$

(4) $\chi(a_1) = \chi(a_2)$ whenever $a_1 \equiv a_2$ mod k.

From (3) and (4) it is clear that a Dirichlet character can be considered as a multiplicative complex function on the set of residue classes modulo k. We will shorten the notation and use the word **character** to mean a Dirichlet character modulo k.

From a group theoretical point of view a Dirichlet character is just a character of a finite complex representation of the unit group $U(\mathbb{Z}_k)$ of the modular ring $\mathbb{Z}_k$.

Dirichlet then introduces what is now known as a **Dirichlet L-series**. This is defined in the following manner.

If χ is a character mod k then the **Dirichlet L-series** is defined for complex values s by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This is clearly an extension of the zeta function of Euler. From this he shows an analog of the Euler product expansion

$$L(s,\chi) = \prod_{pprime} (1 - \frac{\chi(p)}{p^s})^{-1}.$$

This is valid for $s > 1$.

Then to prove the main result Dirichlet shows that if $(a,b) = 1$ and there were only finitely many primes of the form $an + b$ then a series related to the L-series for a would converge. He then shows that this series must diverge. It follows then that there must be infinitely many primes of the form $an + b$. (A complete proof of Dirichlet's Theorem can be found in [FR].)

4 Chebyshev's Estimate

The first significant progress in developing a proof of the prime number theorem was obtained by Chebyshev in 1848. He proved that the functions $\pi(x)$ and $\frac{x}{\ln x}$ are of the same order of magnitude and that if

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x}$$

existed then the limit would have to be 1. At first glance it appeared that he was quite close to a proof of the prime number theorem. However it would take another fifty years and the development of some completely new ideas from complex analysis to actually accomplish this. A proof, along the lines of Chebyshev's methods, without recourse to complex analysis, would not be done until the work of Selberg and Erdos in the late 1940's.

Chebyshev proved the following result, now known as **Chebyshev's estimate**.

Theorem 4.1. *There exist positive constants A_1 and A_2 such that*

$$A_1 \frac{x}{\ln x} < \pi(x) < A_2 \frac{x}{\ln x}$$

for all $x \geq 2$. In the notation introduced in section 1 this says that

$$\pi(x) \approxeq \frac{x}{\ln x}$$

that is the prime number function is of the same order of magnitude as the function $\frac{x}{\ln x}$.

In Chebyshev's original proof he obtained the values $A_1 = .922$ and $A_2 = 1.105$. His proof actually involved a careful analysis of a form of Stirling's approximation. The values of these constants in Chebyshev's inequality have been improved upon many times. Sylvester in 1882 improved the values to $A_1 = .95695$ and $A_2 = 1.04423$ for sufficiently large x. It can now be shown that for all $x > 10$, $A_1 = 1$ can be used.

This following is an immediate corollary of the estimate, independent of the values of A_1 and A_2.

Corollary 4.1. $\frac{\pi(x)}{x} \to 0$ *as* $x \to \infty$.

This corollary further quantifies the fact that the primes become relatively scarcer as x gets larger. In probabilistic terms it says that the probability of randomly choosing a prime less than or equal to x goes to zero as x goes to infinity.

Chebyshev further proved that if

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \ln x}$$

existed then the limit would have to be one. Hence showing that the above limit existed would prove the prime number theorem. Chebyshev however could not prove that the limit existed.

It was mentioned at the end of section 1 that the prime number theorem is equivalent to $p_n \sim n \ln n$ where p_n denotes the nth prime. Chebyshev's estimate immediately shows that p_n and $n \ln n$ are of the same order of magnitude.

Theorem 4.2. *There exist positive constants B_1, B_2 such that*

$$B_1 n \ln n \leq p_n \leq B_2 n \ln n.$$

Equivalently

$$p_n \approxeq n \ln n.$$

Using essentially the same techniques Chebyshev proved what is called Bertrand's Theorem. This result says that given any natural number n there is always a prime between n and $2n$. The proof actually shows that given any real number $x > 1$ there exists a prime between x and $2x$. Bertrand verified this empirically for a large number of natural numbers and conjectured the result. The first proof was Chebyshev's.

Theorem 4.3. *(Bertrand's Theorem) For every natural number $n > 1$ there is a prime p such that $n < p < 2n$.*

5 Riemann's Method

From Chebyshev's estimate and its consequences it seemed that a proof of the prime number theorem was close at hand. In 1860 B.G. Riemann attempted to prove this main result. Riemann eventually wrote only one paper in number theory, and although he failed in his primary goal of proving the prime number theorem, this paper had a profound effect on both number theory in particular and mathematics in general. Riemann's basic new idea was to extend the zeta function $\zeta(s)$ of Euler by allowing s to be a complex number. This idea of Riemann initiated the use of complex analysis, specifically the theory of analytic functions and complex integration, into number theory and laid the ground work for a new discipline in mathematics called analytic number theory. Although the use of analysis begins with the Euler zeta function and continues through the work of Dirichlet it is in this paper of Riemann and the introduction of complex analytic methods that really is the beginning of analytic number theory. In modern parlance an **elementary method** in number theory is any technique that does not involve analysis.

Riemann, in allowing a complex argument s, showed that the resulting function $\zeta(s)$ is an analytic function for Re $(s) > 1$ and further can be continued analytically to a function, also denoted $\zeta(s)$, that is analytic in all of $\mathbb{C}$ except $s = 1$. Further $s = 1$ is a simple pole with residue 1, that is

$$\zeta(s) = \frac{1}{s-1} + H(s)$$

where $H(s)$ is an entire function. Riemann then showed that knowledge of the location of the complex zeroes of $\zeta(s)$ describes the density of primes. In particular, if there are no zeroes along the line Re $(s) = 1$, this would then imply the prime number theorem. This was precisely the main step in the proofs of Hadamard and de la Vallee-Poussin (given independently) of the prime number theorem given thirty-six years after Riemann's paper.

The **Riemann zeta function** is then

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ where } s = \sigma + it \text{ and } \sigma, t \in \mathbb{R}$$

By the p-series test this series converges absolutely for Re $(s) > 1$ and hence defines $\zeta(s)$ as an analytic function in this region. Further the Euler product decomposition holds on a connected arc (the part of the real line $s > 1$), and hence by analytic continuation they are still valid for complex arguments within the region of analyticity Re $s > 1$. Thus we have

$$\zeta(s) = \prod_{p,\text{prime}} (\frac{1}{1-p^{-s}}), s \in \mathbb{C}, \text{Re } s > 1;$$

It follows that $\zeta(s)$ has no zeros for Re $s > 1$.

A crucial concept in studying the zeta function is that of **analytic continuation**. The basic idea is the following: suppose a complex analytic function $f(z)$ is given by an analytic expression which holds in a region S in $\mathbb{C}$. Suppose that this is equivalent within S or within a subset of S to another analytic expression which holds in a larger region S_1. Then the second expression can be used to analytically extend or continue $f(z)$ to the larger region S_1. We make this precise.

Suppose that $f_1(z)$ is analytic on a region S_1 and $f_2(z)$ is analytic on a region S_2. Suppose that $S_1 \cap S_2 \neq \emptyset$ and $f_1(z) = f_2(z)$ on $S_1 \cap S_2$. Then $(f_2(z), S_2)$ is said to be a **direct analytic continuation** of $(f_1(z), S_1)$. The individual pairs (f_1, S_1) and (f_2, S_2) are called **function elements**. A function element (f, S) is an **analytic continuation** of (f_1, S_1) if there is a chain (f_i, S_i) of function elements connecting (f_1, S_1) to (f, S) and with each neighboring pair a direct analytic continuation. A **global analytic function** is a nonempty collection of function elements $F = \{(f_\alpha, S_\alpha)\}$ such that any two in this collection are analytic continuations of each other. A global analytic function is **complete** if it contains all analytic continuations of any of its function elements.

Finally analytic continuation is essentially unique in the sense that two analytic functions which agree on a sufficiently large domain, for example a curve, are identical.

Riemann first proves by using complex integration that $\zeta(s)$ can be continued analytically to a function analytic for Re $(s) \geq \frac{1}{2}$. He then establishes the following functional relation concerning the zeta function:

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(s+1)/2}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

or equivalently

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(s-1), s \neq 0, 1.$$

In this relation $\Gamma(s)$ is the complex gamma function which is analytic except at $z = 0, -1, -2, ...$

The functional relation then has the form

$$\zeta(s) = K(s)\zeta(s-1)$$

where

$$K(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s).$$

The transformation $s \to 1-s$ has $s = \frac{1}{2}$ as its center of symmetry. Therefore since $\zeta(s)$ is defined for Re $s \geq \frac{1}{2}$ the functional equation can be used to continue $\zeta(s)$ to a function defined for Re $s \leq \frac{1}{2}$ and hence defined over the whole complex plane. Therefore Riemann establishes the following theorem.

Theorem 5.1. *The Riemann zeta function $\zeta(s)$ can be analytically continued to a function, also denoted $\zeta(s)$, which is meromorphic in the whole plane. The only singularity of $\zeta(s)$ is a simple pole at $s = 1$ with residue 1, that is*

$$\zeta(s) = \frac{1}{s-1} + H(s)$$

where $H(s)$ is an entire function.

From the singularities of the complex Gamma function it follows that the function $K(s)$ has singularities, that is becomes infinite at the positive odd integers $2n+1, n \geq 1$. However $\zeta(2n+1)$ is finite for all $n \geq 1$. Hence from the functional relation this is possible only if $\zeta(1-s) = 0$ if $s = 2n+1$. Therefore $\zeta(s) = 0$ at all the negative even integers $-2, -4, ...$ These are called the **trivial zeros** of $\zeta(s)$. What becomes crucial in applying the zeta function to the proof of the prime number theorem is the location of its nontrivial zeros. Riemann showed that any nontrivial zeros must fall in the **critical strip** $0 \leq$ Re $s \leq 1$. Further he conjectured that all the nontrivial zeros lie along the line Re $s = \frac{1}{2}$ which is called the **critical line**. This conjecture is called the **Riemann hypothesis** and is still an open question. It has resisted solution for almost a hundred and fifty years and has had tremendous impact on both Number Theory and other branches of mathematics. We will say more later about the Riemann hypothesis. What was most important for the proof of the prime number theorem was the following.

Theorem 5.2. *If the Riemann zeta function $\zeta(s)$ has no zeros on the line* Re $s = 1$ *then the prime number theorem holds.*

In order to prove the above result Riemann introduced and analyzed several other related functions, called the **Chebyshev functions**. The first, denoted $\theta(x)$, is defined for a real variable x by

$$\theta(x) = \sum_{p \leq x} \ln p \text{ with } p \text{ prime}$$

while the second, denoted $\psi(x)$, is defined, again for a real variable x, by

$$\psi(x) = \sum_{p^k \leq x; k \geq 1} \ln p \text{ with } p \text{ prime}$$

These functions count respectively the number of primes $p \leq x$ and the number of prime powers $p^k \leq x$ weighted by $\ln p$. The **van Mangoldt function** $\Lambda(n)$ is defined for positive integers by

$$\begin{aligned} \Lambda(n) \quad &= \ln p \text{ if } n = p^c, c \geq 1 \\ &= 0 \text{ for all other } n > 0. \end{aligned}$$

Hence the Chebyshev function $\psi(x)$ is actually the summation function of $\Lambda(n)$. That is

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

The crucial result for Riemann was that the prime number theorem is equivalent to certain limit results involving the Chebyshev functions. Specifically:

Theorem 5.3. *The following are all equivalent formulations of the prime number theorem*

(a) $\pi(x) \sim \frac{x}{\ln x}$

(b) $\theta(x) \sim x$

(c) $\psi(x) \sim x$.

What Riemann actually showed is that the absence of zeros on the line Re $s = 1$ implies part (b) of the theorem above which in turn implies the prime number theorem.

6 The Proof and Some Consequences

In 1896, some 36 years after Riemann's paper, Hadamard, and independently de la Vallee-Poussin, proved the prime number theorem by finally establishing that $\zeta(s)$ has no zeros on the line Re $s = 1$.

Theorem 6.1. *The Riemann zeta function* $\zeta(s)$ *has no zeros on the line* Re $s = 1$.

The original proofs given by Hadamard and de la Valle-Poussin were quite complicated. The proof was somewhat simplified by Wiener using what are known as Tauberian theorems, but still remained quite difficult. Wiener also pointed out that the converse of Theorem 4.2 is also true - that is the prime number theorem is actually *equivalent* to the fact that there are no zeros of $\zeta(s)$ on the line Re $s = 1$.

Theorem 6.2. *The prime number theorem is equivalent to the fact that there are no zeros of $\zeta(s)$ on the line* Re $s = 1$.

In 1980 D.J. Newman found a way to give a proof using only fairly straightforward facts about complex integration and which allowed a relatively short proof to be presented.

Gauss' original approximation to $\pi(x)$ was the logarithmic integral function $Li(x)$. Riemann attempted to improve on this in the following manner. His work suggested that $\frac{\pi(x)}{x}$ would be closer to $\frac{1}{\ln x}$, that is the probability of choosing a prime randomly less than x, would be closer to $\frac{1}{\ln x}$ if one counted not only the primes but also the "weighted powers" of the primes. That is counting a p^2 as half a prime, p^3 as a third of a prime and so on. This would lead to an approximation for $Li(x)$ given by

$$Li(x) \approxeq \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + ..$$

Upon inverting this

$$\pi(x) \approxeq Li(x) - \frac{1}{2}Li(x^{\frac{1}{2}}) - \frac{1}{3}Li(x^{\frac{1}{3}})...$$

Based on this approach Riemann proposed the following **explicit formula** for $\pi(x)$,

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}})$$

where $\mu(n)$ is the **Moebius function** defined for natural numbers n by

$$\begin{aligned} \mu(n) \quad &= 1 \text{ if } n = 1 \\ &= (-1)^r \text{ if } n = p_1p_2...p_r \\ &\text{with } p_1, ..., p_r \text{ distinct primes} \\ &= 0 \text{ otherwise.} \end{aligned}$$

The series on the right side of the explicit formula can be shown to converge for $x \geq 2$ and is called the **Riemann function** $R(x)$, that is

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}}), x \geq 2.$$

Riemann's conjecture was then that $\pi(x) = R(x)$. It turns out that this is asymptotically correct. That is

Theorem 6.3. *We have $\pi(x) \sim R(x)$ where $R(x)$ is the Riemann function.*

In fact this approximation is remarkably close for large x. For $x = 400,000,000$ we have

$$\pi(400,000,000) = 21,336,326 \text{ and } R(400,000,000) = 21,355,517$$

while for $x = 1,000,000,000$

$$\pi(1,000,000,000) = 50,847,534 \text{ and } R(1,000,000,000) = 50,847,455.$$

Bertrand's Theorem showed that for any real number x there is always a prime in the interval $[x, 2x]$. Further the proof used the same methods as the proof of Chebyshev's estimate. As an immediate consequence of the prime number theorem the following result is obtained.

Theorem 6.4. *For any $\epsilon > 0$ there exists an $x_0 = x_0(\epsilon)$ such that there is always a prime in the interval $[x, (1+\epsilon)x]$ for $x > x_0$. Equivalently $\pi(x+y) > \pi(x)$ for $y = \epsilon x$.*

The above theorem and its proof has the following interesting interpretation. For large x

$$\pi(2x) - \pi(x) \sim \pi(x).$$

Hence for large x there are as many primes asymptotically between x and $2x$ as there are less than x, despite the fact that by the Prime Number Theorem the density of primes tends to thin out. However it can be shown that

$$2\pi(x) - \pi(2x) \to \infty$$

as $x \to \infty$.

The result given in Theorem 6.4 has been improved upon in various ways. Huxley in 1972 continuing a long line of research in this direction showed that there is always a prime in the interval $[x, x + x^c]$ if $c > \frac{7}{12}$ for large enough x. The value of c has subsequently been improved, the most recent being done by Baker and Harman who reduced c to .535 again for large enough x. Further Baker and Harman show that

$$\pi(x + x^{.535}) - \pi(x) > \frac{x^{.535}}{20 \ln x}$$

for large enough x.

Earlier Erdos, using a formula due to Selberg, had proved that for each $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that in the interval $[x, (1+\epsilon)x]$ there are at least $\frac{c(\epsilon)x}{\ln x}$ primes.

Finally we mention the following remarkable result which is a consequence of Bertrand's Theorem.

Theorem 6.5. *Given any positive integer n the set of integers $\{1, 2, ..., 2n\}$ can be partitioned into n disjoint pairs so that the sum of each pair is a prime.*

For example

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

can be partitioned into

$$\{1, 10\}, \{2, 9\}, \{3, 4\}, \{5, 8\}, \{6, 7\}.$$

The result is in the same spirit as the **Goldbach conjecture** which states that any even integer is the sum of two primes.

7 The Riemann Hypothesis

As we have described, the functional relation

$$\zeta(s) = K(s)\zeta(s-1)$$

where

$$K(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)$$

established that $\zeta(s) = 0$ at all the negative even integers $-2, -4, ...$ These are called the **trivial zeros** of $\zeta(s)$. Riemann in his original paper showed that any nontrivial zeros must fall in the **critical strip** $0 \leq \text{Re } s \leq 1$. He further showed that if $\zeta(s)$ has no zeros on the line Re $s = 1$ this was sufficent to prove the prime number theorem which was the method of proof for both Hadamard and de la Valle-Poussin. In the course of this investigation Riemann conjectured that all the nontrivial zeros lie along the line Re $s = \frac{1}{2}$ which is called the **critical line**. This statement is the common form of the **Riemann hypothesis**.

Riemann Hypothesis: All the nontrivial zeros of the Riemann zeta function lie along the line $Re(s) = \frac{1}{2}$.

The Riemann hypothesis has resisted solution for almost a hundred and fifty years and has had tremendous impact on both Number Theory and other branches of mathematics. Now that Fermat's Last Theorem and the Poincare Conjecture have been settled the Riemann hypothesis can be considered the outstanding open problem in mathematics. It is included among the five millenium problems.

There are various further results concerning the Riemann hypothesis and the zeros of the zeta function. Hardy in 1914 proved that $\zeta(s)$ has infinitely many zeros along the critical line Re $s = \frac{1}{2}$. As of 2006 it is known that at least the first billion and a half nontrivial zeros of $\zeta(s)$ lie along the critical line.

Selberg in 1942 showed that a positive proportion of the nontrivial zeros lie along the critical line. Levinson in 1974 improved this to show that at least $\frac{1}{3}$ of the nontrivial zeros are on the critical line. This has subsequently been improved to at least 40% of the nontrivial zeros are on the critical line.

There are several quantitative statements that are equivalent to the Riemann hypothesis. Koch in 1901 showed that the Riemann hypothesis was equivalent to

$$\pi(x) = Li(x) + O(\sqrt{x}\ln x)$$

where $Li(x)$ is the logarithmic integral function of Gauss

$$Li(x) = \int_2^x \frac{1}{\ln t} dt.$$

In a similar manner the Riemann hypothesis can be shown to be equivalent to

$$\pi(x) = Li(x) + O(x^{\frac{1}{2}+\epsilon}) \ \ \forall \epsilon > 0.$$

The equality (6.1) was also conjectured by Riemann in his original paper and is often called the prime number theorem form of the Riemann Hypothesis.

There are many other computational variations of both the prime number theorem and the Riemann hypothesis. Several of these involve the Moebius function $\mu(n)$ and **Merten's function** defined by

$$M(x) = \sum_{n \le x} \mu(x).$$

Merten's function is related to the Riemann zeta function by the following

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx.$$

Van Mangoldt proved the following.

Theorem 7.1. *The prime number theorem is equivalent to the statement*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.$$

Further the following is also known.

Theorem 7.2. *If $M(x)$ is Merten's function then:*

(1) the prime number theorem is equivalent to

$$M(x) = o(x).$$

(2) the Riemann hypothesis is equivalent to

$$M(x) = O(x^{\frac{1}{2}+\epsilon}) \text{ for any fixed } \epsilon > 0.$$

In observing computed values up to the bounds that were available to him Riemann proposed that $Li(x) > \pi(x)$ for all sufficently large x. This turned out to be incorrect. In 1914 Littlewood proved the following.

Theorem 7.3. *The difference $\pi(x) - Li(x)$ assumes both positive and negative values infinitely often.*

Littelwood's proof was interesting in that it used the following technique which has become extremely useful in analytic number theory. First he assumed that the Riemann hypothesis is true and proved that $\pi(x) - Li(x)$ changes sign infinitely often. He then showed that the same is true if the Riemann hypothesis is assumed to be false. In 1986 Te Riele showed that there are greater than 10^{180} consecutive integers for which $\pi(x) > Li(x)$ in the range $6.62 \times 10^{370} < x < 6.69 \times 10^{370}$.

The proof of Dirichlet's theorem, giving that there are infinitely many primes in any arithmetic progression $an + b$ with $(a, b) = 1$, Dirichlet L-series. Such a series is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

where χ is a character mod k, and s is a complex variable. Both the prime number theorem and the Riemann hypothesis can be extended to primes in arithmetic progressions.

For $(a, b) = 1$ let

$$\pi(x; a, b) = \text{ numbers of primes congruent to } b \text{ mod } a \text{ and } \leq x.$$

The Prime Number Theorem for Arithmetic Progressions can then be expressed as:

Theorem 7.4. *(The Prime Number Theorem for Arithmetic Progressions) For fixed $a, b > 0$ with $(a, b) = 1$ then*

$$\pi(x; a, b) \sim \frac{1}{\phi(a)} \pi(x) \sim \frac{1}{\phi(a)} \frac{x}{\ln x} \sim \frac{1}{\phi(a)} Li(x).$$

Here $\phi(n)$ is the Euler phi function.

The result can be expressed in probabilistic terms by saying that the primes are uniformly distributed in the $\phi(a)$ residue classes relatively prime to a. In fact much of the material on the prime number theorem can be rephrased in terms of probability theory. The prime mumber theorem itself can be expressed as:

Theorem 7.5. *(The Prime Number Theorem) The probability of randomly choosing a prime less than or equal to x is asymptotically given by $\frac{1}{\ln x}$.*

The extension of the Riemann hypothesis to the case of arithmetic progressions is called the **generalized Riemann hypothesis** or the **extended Riemann hypothesis**. This says that the zeros of any Dirichlet L-series also lie along the critical line Re $s = \frac{1}{2}$.

Generalized Riemann Hypothesis: For an integer k and any character χ mod k then the nontrivial zeros of the L-series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

all lie along the critical line Re $s = \frac{1}{2}$.

8 The Elementary Proof

Chebyshev's estimate (Theorem 4.1) appeared to be quite close to the prime number theorem. It provided the right bounds and further Chebyshev showed that if $\lim_{x \to \infty} \frac{\pi(x) \ln x}{x}$ existed then the value of the limit must be one. Chebyshev's methods were elementary in the sense that they involved no analysis more complicated than simple real integration and the properties of the logarithmic function (although the proofs themselves were complicated). This would seem appropriate for a proof of a theorem about primes since primes are in the realm of arithmetic and should not require deep analytic notions. However Chebyshev could not establish that the limit existed and then Riemann, ten years or so later, tried a different approach using the theory of complex analytic functions. The proof of the prime number theorem was reduced to knowing the location of the zeros of the complex analytic Riemann zeta function. Still, even with Riemann's ideas, the proof resisted solution for another thirty-six years and during this time many mathematicians began to doubt that the limit $\lim_{x\to\infty} \frac{\pi(x)\ln x}{x}$ existed. These doubts were put to rest with the proofs of Hadamard and de la Valle-Poussin. The Prime Number Theorem, a result seemingly arising in arithmetic, is equivalent to the result that there are no zeros of the Riemann zeta function $\zeta(s)$ along the line $Re(s) = 1$, a result really in complex analysis. This raised the question of the actual relationship between the distribution of primes and complex function theory. This led to the further question of whether there could exist an elementary proof of the prime number theorem along the lines of Chebyshev's methods.

The opinion that came to prevail was that it was doubtful that such a proof existed. The feeling was that complex analysis was somehow *deeper* than real analysis and in view of the equivalence mentioned above it would be unlikely to prove the prime number theorem using just the methods of real analysis. On the other hand it was felt that if such a proof existed it would open up all sorts of new avenues in number theory.

The English mathematician G.H. Hardy, who made major contributions to the study of the relationship between the prime number function $\pi(x)$ and Gauss's logarithmic integral function $Li(x)$, described the situation this way in a lecture in 1921 (see [N]).

G.H. Hardy *No elementary proof of the prime number theorem is known and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's zeta function has no roots on a certain line. A proof of such a theorem, not fundamentally dependent upon the ideas of the theory of functions, seems to me to be extraordinarily unlikely. It is rash to assert that a mathematical theorem cannot be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think that some theorems, as we say "lie deep" and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten.*

However what actually occurred was even more surprising. Selberg and then Erdos and then Erdos and Selberg together in 1948 developed elementary proofs of the prime number theorem along the lines of Chebyshev's methods. All of these proofs depended on asymptotic estimates for an extension of the von Mangoldt function. These asymptotic estimates are now called **Selberg formulae**. The discovery of this elementary proof put to rest the discussion of the relative profoundness of complex analysis versus real analysis. However, despite the brilliance of the Selberg-Erdos approach, it did not produce the startling consequences in understanding both the distribution of primes and the zeros of the Riemann zeta function that were predicted. There are now many so-called elementary proofs and the techniques involved have become standard in analytic number theory. It may be that in time these methods will lead to a deeper understanding of the basic questions.

The Selberg formula, from which the elementary proof, can be derived is the following.

Theorem 8.1. *(Selberg Formula) For $x \geq 1$,*

$$\sum_{p \leq x} (\ln p)^2 + \sum_{p,q \leq x} \ln p \ln q = 2x \ln x + O(x)$$

where p, q run over all the primes $\leq x$.

Several alternative formulations of this result are used in the elementary proof. First, the formula can be expressed in terms of the von Mangoldt function.

Theorem 8.2. *(Selberg Formula) For $x \geq 1$,*

$$\sum_{n \leq x} \Lambda(n) \ln n + \sum_{n,m \leq x} \Lambda(n)\Lambda(n) = 2x \ln x + O(x)$$

where $\Lambda(n)$ is the von Mangoldt function.

The elementary proof requires two more equivalent formulations which tie the Selberg formula to the Chebyshev functions $\theta(x)$ and $\psi(x)$.

Theorem 8.3. *(Selberg Formula) For $x \geq 1$*

$$(1)\ \ \theta(x) \ln x + \sum_{p \leq x} \ln p \theta(\frac{x}{p}) = 2x \ln x + O(x)$$

$$(2)\ \ \psi(x) \ln x + \sum_{n \leq x} \Lambda(n) \psi(\frac{x}{n}) = 2x \ln x + O(x)$$

The prime number theorem is equivalent to $\theta(x) \sim x$ and to $\psi(x) \sim x$. The Selberg proof shows $\theta(x) \sim x$. This is proved via a series of estimates whose proofs all work with, or start with, the Selberg formula (in one of its formulations), and then use tricky and difficult manipulation of series.

It is an easy consequence of the prime number theorem that if p_n is the nth prime then

$$\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = 1. \quad (8.1)$$

This fact however plays a role in the history of the elementary proof. When Selberg first gave his formula Erdos used it to give an elementary proof of (8.1). Selberg then used his formula along with the methods of Erdos' proof to develop the first elementary proof of the prime number theorem. Erdos then gave a second elementary proof. There now exist several elementary proofs of the prime number theorem that do not depend on Selberg's formula.

9 Some Extensions

Related to Riemann's explicit formula it can be shown that the distribution of the number of zeros of the Riemann zeta function along the critical line can be given asymptotically by

$$N(t) = \frac{t}{2\pi}\ln(\frac{t}{2\pi}) - \frac{t}{2\pi}$$

where $N(t)$ is the number of zeros z with $z = \frac{1}{2} + is$ along the critical line with $0 < s < t$.

There are also some surprising relationships between some physical phenomena and the location of the zeros of the Riemann zeta function. This is further related to the distribution of eigenvalues of certain operators.

An entirely elementary formulation of the Riemann hypothesis is the following (see [P]). Define a positive squarefree integer n to be **red** if it is the product of an even number of distinct primes and **blue** if it is the product of an odd number of distinct primes. Let $R(n)$ be the number of red integers not exceeding n and $B(n)$ the number of blue integers not exceeding n. The Riemann hypothesis is equivalent to the statement that for any $\epsilon > 0$ there exists an N such that for all $n > N$

$$|R(n) - B(n)| < n^{\frac{1}{2}+\epsilon}.$$

If p_n denotes the nth prime then it is a straightforward consequence of the prime number theorem that

$$p_n \sim n \ln n$$

and hence

$$\lim \frac{p_{n+1}}{p_n} = 1$$

even though there are arbitrarily large gaps in the primes. It was noted in the last section that when Selberg first gave his formula Erdos then used it to give an elementary proof of the second fact above. Subsequently Selberg then used his formula along with the methods of Erdos' proof to develop the first elementary proof of the prime number theorem.

There are two well-known conjectures concerning the difference $p_{n+1} - p_n$. The first is called **Cramer's conjecture**.

Cramer's Conjecture: $p_{n+1} - p_n \leq (1 + o(1))(\ln n)^2$

It follows from Koch's equivalence to the Riemann hypotheis that if the Riemann hypothesis is true then

$$p_{n+1} - p_n = O(p_n^{\frac{1}{2}+\epsilon}) \text{ for any fixed } \epsilon > 0.$$

The second conjecture is called **Lindelof's hypothesis**.

Lindelof's Hypothesis: $\sum_{p_n \leq x}(p_{n+1} - p_n)^2 \leq x^{1+o(1)}$

It can be shown that the Riemann hypothesis implies the Lindelof hypothesis.

10 Literature

Below we give the references for most of the material in this article. It is by no means meant to be exhaustive. The book by Narkiewicz [Na] has over a hundred pages of references and is an excellent guide to the literature. It also contains a version of Riemann's original; paper and the original proofs of Hadamard and de la Vallee-Poussin. Complete versions of Selberg's original proof can be found in the book of Nathanson [N] while a self-contained exposition of another elementary proof is in the book of Tenenbaum and Mendes-France [TMF]. A slightly different approach based on Selberg's methods can also be found in Hardy and Wright [HW]. The article by Diamond [Di] is a nice survey on the use of elementary methods in the study of primes. Nathanson's book is also an excellent source of historical comments. The book by Apostol [A] is an excllent source on analytic number theory in general. A complete proof of Dirichlet's theorem appears in the books of Fine and Rosenberger [FR] and Landau [L], while a clear discussion and outline of the proof is in the book by Tenenbaum and Mendes-France [TMF]. A wealth of material on computational aspects of number theory can be found in the excellent and comprehensive book by Crandall and Pomerance [CP]. This book also contains many suggestions for research projects. The books by Eliot [E] and [E1] present probabilistic number theory while [HR] by H. Halberstam and H.E. Richert is a source for the use sieving methods in number theory. The nice article by Goldstein [Go] covers some of the aspects of the same material as this paper while the paper by Bateman and Diamond [BT], written on the hundredth anniversary of the proof of the prime number theorem looks at the development of analytic number theory in general.

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