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Ben Schweizer

Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

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Ben Schweizer[1]

Abstract: We propose a new method for the homogenization of hysteresis models of plasticity. For the one-dimensional wave equation with an elasto-plastic stress-strain relation we derive averaged equations and perform the homogenization limit for stochastic material parameters. This generalizes results of the seminal paper by Franců and Krejčí. Our approach rests on energy methods for partial differential equations and provides short proofs without recurrence to hysteresis operator theory. It has the potential to be extended to the higher dimensional case.

Keywords: effective model, hysteresis, plasticity, Prager model, differential inclusion, nonlinear wave equation

1 Introduction

Linear models of continuum mechanics seem to be well understood in most aspects of analysis and numerics. Much different is the situation in nonlinear rheological models. In the plastic deformation of a body we encounter memory effects, the relation between deformation and force includes an hysteresis term. This complicates considerably the analysis of plastic materials.

The fundamental variables to describe a mechanical body are the displacement vector u, the strain tensor ε, and the stress tensor σ. The strain describes the infinitesimal local deformation of the body, the stress the inner forces. With density ρ and volume forces f, assuming small strains and exploiting conservation of momentum, the body is described by a set of equations on the reference volume $\Omega \subset \mathbb{R}^N$.

$$\rho\,\partial_t^2 u = \nabla\cdot\sigma + f, \quad \varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^\perp), \tag{1.1}$$

$$\text{a constitutive relation between } \varepsilon \text{ and } \sigma. \tag{1.2}$$

The simplest choice for (1.2) is the linear relation $\sigma = A\cdot\varepsilon$ for some tensor A, which is the model of linear elasticity. We are interested here in plasticity models for (1.2). Those models are non-linear and involve an hysteresis term.

[1]Fakultät für Mathematik, TU Dortmund, Vogelpothsweg 87, D-44227 Dortmund. ben.schweizer@tu-dortmund.de

Many models have been suggested, for an overview we refer to the text-books [11, 1, 6, 20]. A prominent position has the Prager model of elasto-plasticity with linear kinematic hardening. It has a wide range of applications and can be motivated by an arrangement of two elastic and one perfectly plastic element, see e.g. [21, 23, 24]. Following these contributions, we use the differential notation and write the law as

$$\partial I_K(\sigma - \alpha\varepsilon) \ni \partial_t\varepsilon - \beta\,\partial_t\sigma, \tag{1.3}$$

with parameters $\alpha, \beta \geq 0$, K a closed compact subset of $\mathbb{R}^{N\times N}$, I the characteristic function which is infinite in the complement of K, and ∂ the subdifferential. The model has the advantage to include elasto-plasticity, rigid plasticity, and elasticity as special cases. By replacing I_K with a convex function, one obtains the Kelvin-Voigt model.

In the one-dimensional case $N = 1$ we may write the hysteresis relation (1.3) in a simpler form. Assuming symmetry, the set K coincides with a symmetric interval, $K = [-\gamma, \gamma]$. The inverse $\Psi_\gamma = (\partial I_K)^{-1}$ of the subdifferential ∂I_K is then the multivalued function $\Psi_\gamma = \gamma\,\mathrm{sign}$ with $\mathrm{sign}(0) = [-1, 1]$. The plasticity model takes the form

$$\alpha\varepsilon \in \sigma - \gamma\,\mathrm{sign}(\partial_t\varepsilon - \beta\partial_t\sigma). \tag{1.4}$$

It is illustrated for $\beta = 0$ in Figure 1. We rewrite the system with the new variable $\tilde\varepsilon := \varepsilon - \beta\sigma$ and the abbreviation $\kappa = 1 - \alpha\beta$ as

$$\alpha\tilde\varepsilon \in \kappa\sigma - \gamma\,\mathrm{sign}(\partial_t\tilde\varepsilon).$$

Omitting the tilde symbol and restricting (1.1)–(1.2) to the one-dimensional case, we therefore deal with the problem

$$\rho\partial_t^2 u = \partial_x\sigma + f \tag{1.5}$$

$$\partial_x u = \varepsilon + \beta\sigma \tag{1.6}$$

$$\alpha\varepsilon \in \kappa\sigma - \gamma\,\mathrm{sign}(\partial_t\varepsilon). \tag{1.7}$$

Our main result concerns the homogenization of this wave equation with memory term. We introduce a small parameter $\eta > 0$ which refers to a small length scale and a study a stochastic material distribution to define problem (P_η) in (2.3)–(2.5) below. We furthermore introduce a homogenized problem (P_*) in (2.6)–(2.8). Our main result is the following theorem.

Theorem 1.1 (Homogenization). *Let the ergodicity of Property 2.1 be satisfied and let $\alpha, \beta, \gamma, \kappa$ be positive. Let $(u^\eta, \sigma^\eta, \varepsilon^\eta)$ be a sequence of strong solutions of problems (P_η) and let (u^*, σ^*, w) be a strong solution of (P_*) with compatible boundary and initial data as in (3.1)–(3.3). Then, for $\eta \to 0$, there holds*

$$\partial_t u^\eta \to \partial_t u^* \text{ and } \sigma^\eta \to \sigma^* \text{ in } L^2(\Omega_T) \text{ almost surely.} \tag{1.8}$$

To relate our result to the literature, we mention another way to express the hysteresis relation. To simplify, we consider $\beta = 0$, the case of rigid plasticity. In this case, two equivalent ways to express the constitutive relation are

$$\partial I_K(\sigma - \alpha\varepsilon) \ni \partial_t \varepsilon \quad \text{and} \quad \alpha\varepsilon = \mathcal{P}_K[\sigma],$$

where $\mathcal{P}_K$ is a play-type hysteresis operator corresponding to the set K.

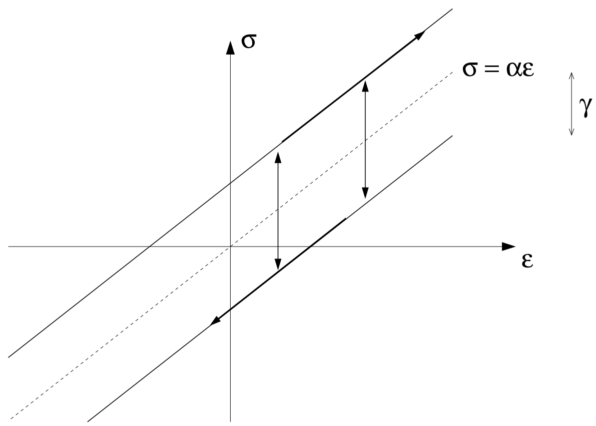

Figure 1: The hysteresis relation between ε and σ in the one-dimensional case $N = 1$, for the convex set $K = [-\gamma, \gamma]$, without kinematic hardening, i.e. $\beta = 0$. The result is a play-type hysteresis relation between ε and σ.

Existence results for (1.5)–(1.7) and related equations can be found e.g. in [6, 10, 20, 22]. The homogenized system (P_*) can be written again as a wave equation with an hysteresis relation, now with an hysteresis-operator of Prandtl-Ishlinskii type. This fact allows to apply abstract existence results in order to find also solutions of (P_*). Concerning weak solutions to the problem we refer to [20, 21]. Since our formulation of the homogenized system (P_*) and also our solution concept differs from previous approaches, we include here the construction of strong solutions.

First *homogenization results for hysteresis problems* are due to Visintin, we refer once more to [20]. The homogenization process for the one-dimensional play-type model (1.5)–(1.7) is made rigorous by Franců and Krejčí in [10]. We mention that the authors treated the deterministic case with $\alpha = 1$ and $\beta = 0$. Our Theorem 1.1 generalizes their result to oscillating parameters $\alpha > 0$ and $\beta > 0$ and to the stochastic case. Our emphasis is to introduce a new method which is flexible and which provides short proofs. The method is based on testing procedures and uses in an essential way the differential inclusion formulation in the effective system (2.6)–(2.8).

Our approach extends the *method of oscillating test-functions* introduced by Tartar, see [16]. This method is very flexible (cp. e.g. [15]), and was adapted by Kozlov for stochastic homogenization problems [12, 14]. For other stochastic approaches we refer to [5, 8, 17]. Concerning stochastic homogenization for hysteresis problems we are only aware of [18], where we introduced a method to treat play-type hysteresis in porous media equations. The common feature with the application in plasticity is the appearance of the new variable $w(y)$ which keeps track of microscopic properties of solutions. The approach in [18] was technically more challanging since we worked with weak solutions and with a structure property for $\partial_t w$.

Another powerful tool to treat homogenization problems is *two-scale convergence* of Allaire [2]. This method was used in hysteresis problems of viscoelasticity [24] and in Preisach models in porous media equations [13]. But, at least in its standard form, it cannot be applied to stochastic problems (cp. [5]). We emphasize that the viewpoint regarding the role of y is different in two-scale convergence and in our approach. In two-scale convergence, y is the microscopic independent variable. Here, just as in [3] or [18], y is an indicator variable for the material and does not necessarily stand for a relative position.

Homogenization in *higher space dimension* is studied in a series of papers by Visintin. In [21] and [23] a simple form of a limit system is given; it coincides with our formulation in the one-dimensional case. Rigorous justifications are not yet available. For the case of viscoelasticity, [24] provides a complete description and proofs for higher dimensional homogenization results. The equations are first written in a variational form with the help of the Fenchel inequality and other tools from convex analysis. The method of two-scale convergence allows to obtain a system of effective equations. It is worth noting that in the setting of viscoelasticity the effective equations are much less accessible, cp. problem 7.1 and formula (137) of [24].

Our method has the potential to be applied in higher space dimension. As pointed out by Visintin, the fundamental difference between one and higher space dimension is as follows: While the control of $\nabla \cdot \sigma$ excludes oscillations in the one-dimensional case, it still allows oscillations in the higher-dimensional case. This may indicate that, in the averaged system for $N > 1$, also the stress σ (and not only the strain ε) depends on the material indicator y.

The ultimate goal is to develop an understanding of effective theories for the Prager model in higher dimension. Concerning this problem, Visintin writes in [24], page 239: *... the choice of an appropriate functional framework for the two-scale formulation of the Prager model [...] seems less obvious.* We hope that the new method introduced here is a step towards that goal.

We emphasize that our approach provides averaged models for hysteresis effects in plastic materials. It is therefore an analysis of the time-dependent problem. Regarding the interesting field of variational problems in plasticity we mention [19] as a general reference, [4] for a homogenization result, and [7] for an analysis of microstructures that appear as a consequence of non-convexity.

2 Setting of the homogenization problem

Our aim is to homogenize the nonlinear wave equation (1.5)–(1.7) in order to justify an effective model for heterogeneous plastic materials. To this end we introduce a small parameter $\eta > 0$ which stands for the typical length scale of the microscopic variations. We study a situation in which the material parameters are oscillating,

$$\alpha = \alpha^\eta, \quad \beta = \beta^\eta, \quad \gamma = \gamma^\eta.$$

To keep the notation as simple as possible, we omit here variations of the density which can be treated along the same lines. Concerning the variations of the parameters, one may think of the standard setting of homogenization, namely of given $[0,1)^N$-periodic functions $\alpha_{per}, \beta_{per}, \gamma_{per}$ and the rapidly oscillating functions

$$\alpha^\eta(x) = \alpha_{per}\left(\frac{x}{\eta}\right), \quad \beta^\eta(x) = \beta_{per}\left(\frac{x}{\eta}\right), \quad \gamma^\eta(x) = \gamma_{per}\left(\frac{x}{\eta}\right).$$

We want to take another point of view. We imagine that the body is composed of different materials and that this fact creates the variations in the parameters. We introduce a variable $y \in [0,1] =: I$ of a material indicator which labels the different types of material. A function $\chi^\eta : \Omega \to I$ describes the distribution of material: material $\chi^\eta(x) \in I$ is present in the point $x \in \Omega$. According to this description of the physics, we can then assume that the material parameters depend only on y, with the characterizations $\alpha : [0,1] \ni y \mapsto \alpha(y) \in \mathbb{R}$ etc. We therefore assume

$$\alpha^\eta(x) = \alpha(\chi^\eta(x)), \quad \beta^\eta(x) = \beta(\chi^\eta(x)), \quad \gamma^\eta(x) = \gamma(\chi^\eta(x)), \tag{2.1}$$

with

$$\chi^\eta(x) = \chi_1\left(\frac{x}{\eta}\right), \tag{2.2}$$

$\chi_1 : \mathbb{R}^N \to I = [0,1]$. The case of one-dimensional periodic media is recovered by setting $\chi_1(x) = x \bmod 1$.

Stochastic Homogenization The more general case of a stochastic material can easily be treated in the above setting. In the simplest case one chooses a probability measure space $(\Omega_{\mathcal{P}}, \mathcal{A}, \mathcal{P})$ such that the elements $\chi_1 \equiv \omega \in \Omega_{\mathcal{P}}$ are functions, $\chi_1 : \mathbb{R}^N \to I$, and keeps (2.1), (2.2). To compare this construction with the notation of [12], we note that for a random variable of the form $A : \Omega_{\mathcal{P}} \to \mathbb{R}$, $\omega \mapsto \alpha(\omega(0))$, and the group of translations $T(x) : \omega(.) \mapsto \omega(. + x)$, there holds

$$A\left(T\left(\frac{x}{\eta}\right)\omega\right) = \alpha\left(\left(T\left(\frac{x}{\eta}\right)\omega\right)(0)\right) = \alpha(\chi_1(x/\eta)) = \alpha(\chi^\eta(x)).$$

Regarding the probability space we always assume that the ergodicity Property 2.1 below holds. The Birkhoff theorem [9, 12] guarantees that the property is satisfied if the probability space is ergodic with respect to the group of translations T. Our convergence results hold *almost surely,* i.e. for $\mathcal{P}$-almost all $\chi_1 = \omega \in \Omega_{\mathcal{P}}$. We denote the distribution of the values of $\chi_1(0) \in I$ by the probability measure $\mu \in \mathcal{M}(I)$. Expected values are averages with respect to this measure; for $g : I \to \mathbb{R}$ we set

$$\langle g \rangle := \int_I g(y)\, d\mu(y) \in \mathbb{R}.$$

Property 2.1 (Ergodicity property). *Let $g \in L^q(I, d\mu)$ for $q \geq 1$ and let $g^\eta : \Omega \to \mathbb{R}$ be defined as*

$$g^\eta(x) = g(\chi^\eta(x)).$$

Then g^η converges weakly to a constant function,

$$g^\eta(x) \rightharpoonup \langle g \rangle \;\; \textit{in } L^q(\Omega) \textit{ almost surely.}$$

Our results on stochastic homogenization exploit only this ergodicity property of the probability measure. We recall that the periodic setting is a special case. Another typical example is to choose χ_1 constant in each cell $k + [0, 1)$, $k \in \mathbb{Z}$, with the constant values chosen stochastically as independent, identically distributed random variables with distribution μ. Our choice of the coefficients in (2.1) implies that we include stochastic dependencies of the parameters, which reflects the physical situation.

Effective equations For convenience, the results are stated for a constant density $\rho = 1$. The general case and even an oscillatory density ρ^η can be treated following the proofs below. We use the variable $\tilde{\varepsilon}$ that was introduced before, but omit the tilde symbol. We always study the domain $\Omega = (0, L) \subset \mathbb{R}^1$ and equations on $\Omega_T = (0, L) \times (0, T)$.

The hysteresis problem with oscillatory parameter functions is the following problem (P_η) for $u^\eta, \sigma^\eta, \varepsilon^\eta : \Omega_T \to \mathbb{R}$ with $\kappa^\eta = 1 - \alpha^\eta \beta^\eta$.

$$\partial_t^2 u^\eta = \partial_x \sigma^\eta + f \tag{2.3}$$

$$\partial_x u^\eta = \varepsilon^\eta + \beta^\eta \sigma^\eta \tag{2.4}$$

$$\alpha^\eta \varepsilon^\eta \in \kappa^\eta \sigma^\eta - \gamma^\eta \operatorname{sign}(\partial_t \varepsilon^\eta) \tag{2.5}$$

We claim that, with the additional independent variable $y \in I := [0, 1]$, the homogenized problem is given by the following problem (P_*) for $u^*, \sigma^* : \Omega_T \to \mathbb{R}$ and $w : \Omega_T \times I \to \mathbb{R}$.

$$\partial_t^2 u^* = \partial_x \sigma^* + f \tag{2.6}$$

$$\partial_x u^* = \int_I w(y)\, d\mu(y) + \beta^* \sigma^* \tag{2.7}$$

$$\alpha(y) w(y) \in \kappa(y) \sigma^* - \gamma(y) \operatorname{sign}(\partial_t w(y)) \qquad \mu - a.e. \quad y \in I \tag{2.8}$$

where $\beta^* = \langle\beta\rangle$ is the expected value of β. In the case of periodic homogenization, β^* is the Y-average of $\beta \circ \chi_1$. In all results, we assume that the material parameters α, β, and γ are positive continuous functions on I, with $\kappa = 1 - \alpha\beta > 0$.

The loose description of the homogenized equations is as follows. The stresses σ^η have small variations and converge strongly to σ^*, equation (2.7) is the averaged version of (2.4). The strain, instead, has large variations. It depends on the material in each point, hence it is approximated with a function $w(x,t,y)$. Equation (2.8) demands that the constitutive relation (2.5) holds for every single material.

3 Estimates and existence results

We consider $N = 1$, $\Omega = (0, L) \subset \mathbb{R}^1$, and $\Omega_T = \Omega \times (0,T)$. We always impose the boundary conditions $\sigma^\eta(0) = \sigma^\eta(L) = 0$ and $\sigma^*(0) = \sigma^*(L) = 0$. In the initial conditions we must exercise care due to compatibility requirements. With $u_0^\eta, v_0 \in H^1(\Omega)$ and $s_0 \in H_0^1(\Omega)$ given, we set $e_0^\eta := \partial_x u_0^\eta - \beta^\eta s_0$ and pose the initial condition

$$u^\eta|_{t=0} = u_0^\eta, \quad (\partial_t u^\eta)|_{t=0} = v_0, \quad \varepsilon^\eta|_{t=0} = e_0^\eta. \tag{3.1}$$

In order to satisfy (2.5) we must always assume the compatibility condition

$$\alpha^\eta e_0^\eta \in \kappa^\eta s_0 + [-\gamma^\eta, \gamma^\eta] \qquad \text{on } \Omega. \tag{3.2}$$

To avoid technical difficulties in the choice of the initial values, we restrict to a situation of increasing strains until time 0. We are then given $v_0 \in H^1(\Omega)$ and $s_0 \in H_0^1(\Omega)$, and set $e_0^\eta = \frac{1}{\alpha^\eta}(\kappa^\eta s_0 - \gamma^\eta) \in L^2(\Omega)$ according to (2.5) and the history. The initial displacement $u_0^\eta \in H^1(\Omega)$ is determined by $\partial_x u_0^\eta = e_0^\eta + \beta^\eta s_0$. For the averaged equations we determine an averaged initial condition through

$$\partial_x u_0^*(x) = s_0(x)\left\langle \frac{\kappa}{\alpha} + \beta \right\rangle - \left\langle \frac{\gamma}{\alpha} \right\rangle.$$

The initial condition for problem (P_*) is set to

$$u^*|_{t=0} = u_0^*, \quad (\partial_t u^*)|_{t=0} = v_0, \quad w(y)|_{t=0} = \frac{1}{\alpha(y)}(\kappa(y) s_0 - \gamma(y)). \tag{3.3}$$

We note that the initial conditions are consistent with (2.7) because of $\langle w\rangle = \partial_x u_0^* - \beta^* s_0$.

Energy estimates and approximate solutions

Let us start with the calculation of an energy decay estimate for solutions of the above problems (P_η) and (P_*). The derivation is rigorous for the strong solutions of Definition 3.4. Our aim here is to present, in its simplest form, a calculation which is the basis for several results of this work.

Lemma 3.1 (Energy estimate)**.** *Strong solutions to problem (P_η) satisfy, in the weak sense on $(0,T)$, the energy equality*

$$\partial_t \frac{1}{2}\int_\Omega \left\{ |\partial_t u^\eta|^2 + \frac{\alpha^\eta}{\kappa^\eta}|\varepsilon^\eta|^2 + \beta^\eta |\sigma^\eta|^2 \right\} + \int_\Omega \frac{\gamma^\eta}{\kappa^\eta}|\partial_t \varepsilon^\eta| = \int_\Omega \partial_t u^\eta\, f. \tag{3.4}$$

Strong solutions to problem (P_*) satisfy

$$\begin{aligned} &\partial_t \frac{1}{2}\int_\Omega \left\{ |\partial_t u^*|^2 + \int_I \frac{\alpha(y)}{\kappa(y)}\, |w(y)|^2 \, d\mu(y) + \beta^* |\sigma^*|^2 \right\} \\ &\qquad + \int_\Omega \int_I \frac{\gamma(y)}{\kappa(y)} |\partial_t w(y)|\, d\mu(y) = \int_\Omega \partial_t u^*\, f. \end{aligned} \tag{3.5}$$

Proof. The calculations are straightforward. We present them for the averaged problem (P_*).

$$\begin{aligned} &\partial_t \left(\frac{1}{2}\int_\Omega |\partial_t u^*|^2 \right) - \int_\Omega \partial_t u^*\, f \\ &\overset{(2.6)}{=} \int_\Omega \partial_t u^* \partial_x \sigma^* = -\int_\Omega \partial_t \partial_x u^* \sigma^* \\ &\overset{(2.7)}{=} -\int_\Omega (\partial_t \int_I w(y)\, d\mu(y) + \beta^* \partial_t \sigma^*) \sigma^* \\ &\overset{(2.8)}{\in} -\partial_t \frac{1}{2}\int_\Omega \beta^* |\sigma^*|^2 - \int_\Omega \int_I \partial_t w(y) \left[\frac{\alpha}{\kappa} w + \frac{\gamma}{\kappa}\, \mathrm{sign}(\partial_t w) \right](y)\, d\mu(y) \\ &= -\partial_t \frac{1}{2}\int_\Omega \beta^* |\sigma^*|^2 - \int_\Omega \int_I \left\{ \partial_t \frac{1}{2}\frac{\alpha}{\kappa}|w(y)|^2 + \frac{\gamma}{\kappa}|\partial_t w| \right\} d\mu(y). \end{aligned}$$

This provides the energy estimate. □

The energy estimates imply that the natural function spaces for weak solutions are

$$\partial_t u^\eta, \varepsilon^\eta, \sigma^\eta \in L^\infty(0,T;L^2(\Omega)), \quad \partial_t \varepsilon^\eta \in L^1(\Omega_T)$$

for problem (P_η), and

$$\begin{aligned} &\partial_t u^*, \sigma^* \in L^\infty(0,T;L^2(\Omega)), \\ &w \in L^\infty(0,T;L^2(\Omega \times I, dx \otimes d\mu)), \quad \partial_t w \in L^1(\Omega \times (0,T) \times I, dx \otimes dt \otimes d\mu) \end{aligned}$$

for problem (P_*).

The δ-regularization of the equations

A possibility to find approximate solutions is to regularize the sign-function and its inverse. For $\delta > 0$, let $\psi_\delta : \mathbb{R} \to \mathbb{R}$ be the following approximation of sign^{-1}.

$$\psi_\delta(r) := \begin{cases} \delta r & \text{for } r \in [-1, 1], \\ \delta + \frac{1}{\delta}(r-1) & \text{for } r > 1, \\ -\delta + \frac{1}{\delta}(r+1) & \text{for } r < -1. \end{cases} \tag{3.6}$$

We will also use the inverse of ψ_δ, the function $\phi_\delta := (\psi_\delta)^{-1}$. We note that $\xi\phi_\delta(\xi) \geq |\xi| - \delta$ for all $\xi \in \mathbb{R}$.

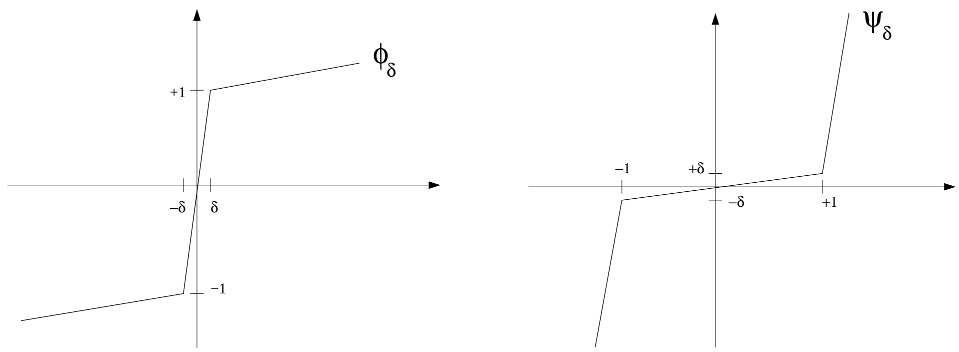

Figure 2: The functions ϕ_δ and ψ_δ

We now define the following regularized system (P_η^δ).

$$\partial_t^2 u_\delta^\eta = \partial_x \sigma_\delta^\eta + f \tag{3.7}$$

$$\sigma_\delta^\eta = \frac{1}{\beta^\eta}(\partial_x u_\delta^\eta - \varepsilon_\delta^\eta) \tag{3.8}$$

$$\partial_t \varepsilon_\delta^\eta = \psi_\delta \left(\frac{1}{\gamma^\eta} \left[\kappa^\eta \sigma_\delta^\eta - \alpha^\eta \varepsilon_\delta^\eta \right] \right) \tag{3.9}$$

complemented with the boundary conditions for σ and with the initial condition (3.1). We note that the compatibility condition (3.2) guarantees that $\partial_t \varepsilon_\delta^\eta|_{t=0}$ is bounded in $L^\infty(\Omega)$, independent of δ. Replacing σ_δ^η with the help of (3.8) in the other two equation, we see that the regularized system is a wave equation with a force-term that is determined by a family of ordinary differential equations.

The same procedure defines the regularized effective system (P_*^δ).

$$\partial_t^2 u_\delta^*(x,t) = \partial_x \sigma_\delta^*(x,t) + f(x,t) \tag{3.10}$$

$$\beta^* \sigma_\delta^*(x,t) = \partial_x u_\delta^*(x,t) - \int_I w_\delta(x,t,y)\, d\mu(y) \tag{3.11}$$

$$\partial_t w_\delta(x,t,y) = \psi_\delta \left(\frac{\kappa(y)\sigma_\delta^*(x,t) - \alpha(y) w_\delta(x,t,y)}{\gamma(y)} \right) \tag{3.12}$$

which we complement with the initial condition (3.3). The choice of the averaged initial condition guarantees $\sigma_\delta^*|_{t=0} = s_0$ and therefore that $\partial_t w_\delta|_{t=0}$ remains bounded in $L^\infty(\Omega \times I)$.

Problem (P_*^δ) is a coupled system of one wave equation in Ω (namely (3.10) with σ_δ^* replaced with the help of (3.11)), and a family of nonlinear ordinary differential equations, parametrized by y (namely (3.12) with σ_δ^* from (3.11)). The solvability of the regularized system is not demonstrated here, it can be shown e.g. with a Galerkin approximation as in [18]. In our next step we derive a priori estimates that are independent of δ.

Lemma 3.2 (Energy estimates for the regularized problem)**.** *There exists $C = C(\Omega, \alpha, \kappa) > 0$, independent of δ and η, such that the following holds. Solutions u_δ^η to the regularized problem (P_η^δ) satisfy*

$$\partial_t \frac{1}{2} \int_\Omega \left\{ |\partial_t u_\delta^\eta|^2 + \frac{\alpha^\eta}{\kappa^\eta} |\varepsilon_\delta^\eta|^2 + \beta^\eta |\sigma_\delta^\eta|^2 \right\} + \int_\Omega \frac{\gamma^\eta}{\kappa^\eta} |\partial_t \varepsilon_\delta^\eta| \le \int_\Omega \partial_t u_\delta^\eta\, f + C\delta. \quad (3.13)$$

Solutions u_δ^* to the regularized problem (P_*^δ) satisfy

$$\partial_t \frac{1}{2} \int_\Omega \left\{ |\partial_t u_\delta^*|^2 + \int_I \frac{\alpha}{\kappa} |w_\delta|^2 \, d\mu + \beta^* |\sigma_\delta^*|^2 \right\} + \int_\Omega \int_I \frac{\gamma}{\kappa} |\partial_t w_\delta| \, d\mu \le \int_\Omega \partial_t u_\delta^*\, f + C\delta. \quad (3.14)$$

Proof. The calculation is the same as in Lemma 3.2. For the averaged equations one calculates

$$\begin{aligned}
&\partial_t \frac{1}{2} \left\{ \int_\Omega |\partial_t u_\delta^*|^2 + \int_\Omega \beta^* |\sigma_\delta^*|^2 \right\} - \int_\Omega \partial_t u_\delta^*\, f \\
&\quad = - \int_\Omega \int_I \partial_t w_\delta(y)\, \sigma_\delta^* \, d\mu(y) \\
&\quad = - \int_\Omega \int_I \partial_t w_\delta(y) \left[\frac{\alpha}{\kappa} w_\delta + \frac{\gamma}{\kappa} \phi_\delta(\partial_t w) \right](y) \, d\mu(y) \\
&\quad = -\partial_t \frac{1}{2} \int_\Omega \int_I \frac{\alpha(y)}{\kappa(y)} |w_\delta(y)|^2 \, d\mu(y) - \int_\Omega \int_I \frac{\gamma}{\kappa} \partial_t w_\delta\, \phi_\delta(\partial_t w_\delta) \, d\mu \\
&\quad \le -\partial_t \frac{1}{2} \int_\Omega \int_I \frac{\alpha}{\kappa} |w_\delta|^2 \, d\mu - \int_\Omega \int_I \frac{\gamma}{\kappa} |\partial_t w_\delta| + \delta |\Omega| \int_I \frac{\gamma}{\kappa} \, d\mu,
\end{aligned}$$

where, in the last line, we exploited $\xi \phi_\delta(\xi) \ge |\xi| - \delta$.

This shows the energy estimate for the regularized system. The calculation for the regularized system (P_η^δ) is analogous. □

Strong solutions

The next Lemma provides higher order estimates for solutions of the δ-problems. It is valid for strong solutions that have derivatives $\partial_t^3 u_\delta^*, \partial_t^2 \partial_x u_\delta^*$ and $\partial_t^2 w_\delta$ in an L^2 space.

Lemma 3.3 (Higher order estimates). *Let $f \in H^1(\Omega_T)$ be given, let $v_0 \in H^1(\Omega)$ and $s_0 \in H^1_0(\Omega)$ define the initial data $u_0^\eta, u_0^* \in H^1(\Omega)$ and $w|_{t=0} \in L^2(\Omega \times I)$ as in (3.3). Then, independent of δ and η, strong solutions of the approximate equations satisfy uniform bounds.*
Solutions of problem (P_η^δ) are bounded in

$$\partial_t^2 u_\delta^\eta, \partial_t\partial_x u_\delta^\eta, \partial_x \sigma_\delta^\eta, \partial_t \sigma_\delta^\eta, \partial_t \varepsilon_\delta^\eta \in L^\infty(0,T; L^2(\Omega)).$$

Solutions of problem (P_*^δ) are bounded in

$$\begin{aligned}&\partial_t u_\delta^*, \partial_t^2 u_\delta^*, \partial_t\partial_x u_\delta^*, \partial_x \sigma_\delta^*, \partial_t \sigma_\delta^* \in L^\infty(0,T; L^2(\Omega)),\\ &w_\delta, \partial_t w_\delta \in L^\infty(0,T; L^2(\Omega \times I, dx \otimes d\mu)).\end{aligned}$$

Proof. We again restrict to the averaged equations (P_*^δ). We start with the time differentiated version of the wave equation,

$$\partial_t^3 u_\delta^*(x,t) = \partial_t \partial_x \sigma_\delta^*(x,t) + \partial_t f(x,t).$$

This equation is multiplied with $\partial_t^2 u_\delta^*$ and integrated over Ω. We calculate along the lines of the energy estimate.

$$\begin{aligned}
&\partial_t \left(\frac{1}{2} \int_\Omega |\partial_t^2 u_\delta^*|^2 \right) - \int_\Omega \partial_t^2 u_\delta^* \, \partial_t f\\
&\quad = \int_\Omega \partial_t^2 u_\delta^* \, \partial_t \partial_x \sigma_\delta^* = - \int_\Omega \partial_t^2 \partial_x u_\delta^* \, \partial_t \sigma_\delta^*\\
&\quad = - \int_\Omega \partial_t^2 \left[\beta^* \sigma_\delta^* + \int_I w_\delta(x,t,y)\, d\mu(y) \right] \partial_t \sigma_\delta^*\\
&\quad = -\partial_t \frac{1}{2} \int_\Omega \beta^* |\partial_t \sigma_\delta^*|^2 - \int_\Omega \int_I \partial_t^2 w_\delta(x,t,y)\, \partial_t \left[\frac{\alpha}{\kappa} w_\delta + \frac{\gamma}{\kappa} \phi_\delta(\partial_t w_\delta) \right](x,t,y)\, d\mu(y).
\end{aligned}$$

We exploit $\phi_\delta' \geq 0$ and obtain

$$\partial_t \frac{1}{2} \left\{ \int_\Omega |\partial_t^2 u_\delta^*|^2 + \int_\Omega \beta^* |\partial_t \sigma_\delta^*|^2 + \int_\Omega \int_I \frac{\alpha}{\kappa} |\partial_t w_\delta|^2 \, d\mu \right\} \leq \int_\Omega \partial_t^2 u_\delta^* \, \partial_t f.$$

This higher order estimate is completely analogous to the energy estimate, but we gained one time derivative. It is worth noticing that an estimate for $\partial_t^2 w_\delta$ cannot be concluded.

Our choice of initial values guarantees, as already noted, the boundedness of $\partial_t w_\delta|_{t=0}$ in L^∞, and hence also the boundedness of

$$\begin{aligned}
\partial_t^2 u_\delta^*|_{t=0} &= \partial_x s_0 + f|_{t=0} \in L^2(\Omega),\\
\partial_t \sigma_\delta^*|_{t=0} &= \frac{1}{\beta^*} \left[\partial_x v_0 - \int_I \partial_t w_\delta(x,0,y)\, d\mu(y) \right] \in L^2(\Omega),
\end{aligned}$$

A standard Gronwall-type argument provides the uniform bounds for $\partial_t^2 u_\delta^*$, $\partial_t \sigma_\delta^*$, and $\partial_t w_\delta$. The uniform estimates for the remaining derivatives can be concluded from the equations, that of $\partial_x \sigma_\delta^*$ from (3.10), that of $\partial_t \partial_x u_\delta^*$ from (3.11). ☐

We will exploit the high regularity of Lemma 3.3 to define a strong solution concept where $\partial_t \varepsilon$ and $\partial_t w$ are functions.

Definition 3.4 (Strong solutions). *Let $f \in H^1(\Omega_T)$ be given. A vector $(u^\eta, \sigma^\eta, \varepsilon^\eta) \in L^2(\Omega_T)^3$ is called a* strong solution *to problem (P_η), if the distributional derivatives satisfy*

$$\partial_t^2 u^\eta, \partial_t \sigma^\eta, \partial_t \varepsilon^\eta \in L^\infty(0, T; L^2(\Omega)),$$

and equations (2.3)–(2.5) are satisfied in the sense of distributions.

A vector $(u^*, \sigma^*, w) \in L^2(\Omega_T)^2 \times L^2(\Omega_T \times I, dx \otimes dt \otimes d\mu)$ is called a strong solution *to problem (P_*), if the distributional derivatives satisfy*

$$\begin{aligned} &\partial_t^2 u^*, \partial_t \sigma^* \in L^\infty(0, T; L^2(\Omega)),\\ &\partial_t w \in L^\infty(0, T; L^2(\Omega \times I, dx \otimes d\mu)), \end{aligned}$$

and equations (2.6)–(2.8) are satisfied in the sense of distributions.

For both systems we additionally demand that the initial conditions and boundary conditions are satisfied in the sense of traces.

We note that, for strong solutions, the equations are relations for functions and that the equations hold almost everywhere. We next analyze a slightly weaker solution concept. It will turn out to be equivalent with the previous construction, but the definition is useful in proofs.

Definition 3.5 (Strong variational solutions). *A vector $(u^\eta, \sigma^\eta, \varepsilon^\eta) \in L^2(\Omega_T)^3$ is a* strong variational solution *to problem (P_η), if it satisfies the regularity properties of a strong solution, equations (2.3)–(2.4) almost everywhere,*

$$-\alpha^\eta \varepsilon^\eta + \kappa^\eta \sigma^\eta \in [-\gamma^\eta, \gamma^\eta] \qquad \text{a.e. in } \Omega_T, \tag{3.15}$$

and the energy inequality

$$\frac{1}{2} \int_\Omega \left\{ |\partial_t u^\eta|^2 + \frac{\alpha^\eta}{\kappa^\eta} |\varepsilon^\eta|^2 + \beta^\eta |\sigma^\eta|^2 \right\} \Big|_0^t + \int_0^t \int_\Omega \frac{\gamma^\eta}{\kappa^\eta} |\partial_t \varepsilon^\eta| \leq \int_0^t \int_\Omega \partial_t u^\eta f \tag{3.16}$$

for almost every $t \in (0, T)$. The definition of a strong variational solution *to problem (P_*) is analogous.*

Lemma 3.6. *Every strong variational solution is a strong solution. This holds for the original system (P_η) and for the averaged system (P_*).*

Proof. It suffices to read the calculation of the energy estimate in Lemma 3.1 in the opposite direction. The regularity of the solution allows to perform all the necessary manipulations. Introducing $S^\eta := [\kappa^\eta\sigma^\eta - \alpha^\eta\varepsilon^\eta]/\gamma^\eta$, the well-known calculations transform the energy inequality of (3.16) into

$$\int_0^t\int_\Omega \frac{\gamma^\eta}{\kappa^\eta}|\partial_t\varepsilon^\eta| \le -\int_0^t\int_\Omega \partial_t\frac{1}{2}\left\{|\partial_t u^\eta|^2 + \frac{\alpha^\eta}{\kappa^\eta}|\varepsilon^\eta|^2 + \beta^\eta|\sigma^\eta|^2\right\} + \int_0^t\int_\Omega \partial_t u^\eta\, f$$
$$= \int_0^t\int_\Omega \partial_t\varepsilon^\eta\left[\sigma^\eta - \frac{\alpha^\eta}{\kappa^\eta}\varepsilon^\eta\right] = \int_0^t\int_\Omega \frac{\gamma^\eta}{\kappa^\eta}\partial_t\varepsilon^\eta\, S^\eta.$$

Relation (3.15) implies $S^\eta \in [-1,1]$ almost everywhere. The integral condition estimates absolute values $|\partial_t\varepsilon^\eta|$ in terms of $\partial_t\varepsilon^\eta \cdot S^\eta$. This is only posible if $S^\eta \in \operatorname{sign}(\partial_t\varepsilon^\eta)$ almost everywhere. This provides the missing relation (2.5). The same proof works for the averaged system (P_*). □

Theorem 3.7. *Let $f \in H^1(\Omega_T)$ be a right hand side, let $v_0 \in H^1(\Omega)$ and $s_0 \in H_0^1(\Omega)$ define the initial data, and let $\alpha(y)$ and $\kappa(y) = 1 - \alpha(y)\beta(y)$ be strictly positive on $I = [0,1]$. Then, for every $\eta > 0$, problems (P_η) and (P_*) have a unique strong solution.*

Proof. We use strong solutions $(u_\delta^\eta, \sigma_\delta^\eta, \varepsilon_\delta^\eta)$ of problem (P_η^δ) for $\delta > 0$. The uniform a priori estimates of Lemma 3.3 guarantee that, taking the limit $\delta \to 0$, we find a weak-* convergent subsequence and limit functions $(u^\eta, \sigma^\eta, \varepsilon^\eta)$. In the first two equations of the approximate system, (3.7)–(3.8), the weak limit $\delta \to 0$ can be performed in all terms. This provides relations (2.3)–(2.4) for the limit functions.

Instead of deriving (2.5) directly, we better show that the weak limit is a strong variational solution. Taking the limit in the energy inequality (3.13) for the approximate solutions yields the energy inequality (3.16). The L^2-boundedness of $\partial_t\varepsilon_\delta^\eta$ together with the shape of ψ_δ in (3.9) implies that the weak limit of the squared bracket, $[\kappa^\eta\sigma^\eta - \alpha^\eta\varepsilon^\eta]$, takes values in $[-\gamma^\eta, \gamma^\eta]$, for almost all $(x,t) \in \Omega_T$. This provides (3.15). We conclude that the limit $(u^\eta, \sigma^\eta, \varepsilon^\eta)$ is a strong variational solution and hence, by Lemma 3.6, also a strong solution.

The proof for problem (P_*) is completely analogous.

It remains to verify the uniqueness property for the averaged system. Let therefore (u^1, σ^1, w^1) and (u^2, σ^2, w^2) be two strong solutions of system (2.6)–(2.8) with the same right hand side and with the same initial values. We can calculate for the difference $(u^\#, \sigma^\#, w^\#) = (u^1, \sigma^1, w^1) - (u^2, \sigma^2, w^2)$

$$\partial_t\left(\frac{1}{2}\int_\Omega |\partial_t u^\#|^2\right) = \int_\Omega \partial_t u^\# \partial_x\sigma^\# = -\int_\Omega \partial_t\partial_x u^\# \sigma^\#$$
$$= -\int_\Omega (\partial_t \int_I w^\#(y)\, d\mu(y) + \beta^*\partial_t\sigma^\#)\sigma^\#$$
$$\in -\partial_t\frac{1}{2}\int_\Omega \beta^*|\sigma^\#|^2$$
$$-\int_\Omega\int_I \partial_t w^\#(y)\left[\frac{\alpha}{\kappa}w^\# + \frac{\gamma}{\kappa}\left(\operatorname{sign}(\partial_t w^1) - \operatorname{sign}(\partial_t w^2)\right)\right](y)\, d\mu(y)$$

$$\leq -\partial_t \frac{1}{2}\int_\Omega \beta^* |\sigma^\#|^2 - \int_\Omega \int_I \partial_t \frac{1}{2}\frac{\alpha}{\kappa}|w^\#(y)|^2\, d\mu(y),$$

where, in the last line, we exploited the monotonicity of the sign-function, $(\xi - \zeta)(\text{sign}(\xi) - \text{sign}(\zeta)) \geq 0$, for $\xi = \partial_t w^1$ and $\zeta = \partial_t w^2$. □

4 Homogenization

4.1 Two-scale ergodicity

To prepare the averaging procedure we first generalize the ergodicity property 2.1 to functions with oscillations on two scales. The counterexample recalled in (4.4) of Section 4.3 shows that a careful analysis is appropriate.

Definition 4.1 (Two-scale ergodicity property)**.** *We say that the stochastic process and a function $g : \Omega \times I \to \mathbb{R}$ satisfy the* two-scale ergodicity property with $q \in [1, \infty)$ *if the following holds.*

Let $\mu \in \mathcal{M}(I)$ be the distribution of the values $\chi_1(0)$, set $\chi^\eta(x) = \chi_1(x/\eta)$. Let the oscillating functions $g^\eta : \Omega \to \mathbb{R}$ and the expected values $\langle g \rangle : \Omega \to \mathbb{R}$ be defined as

$$g^\eta(x) = g(x, \chi^\eta(x)), \quad \langle g \rangle (x) = \int_I g(x, y)\, d\mu(y).$$

Then

$$g^\eta \rightharpoonup \langle g \rangle \text{ for } \eta \to 0 \text{ in } L^q(\Omega) \text{ almost surely.} \tag{4.1}$$

The next lemma shows that a certain smoothness of the function g guarantees the two-scale ergodicity property.

Lemma 4.2. *Let the process satisfy the single-scale ergodicity of Property 2.1 and let g be continuous, $g \in C^0(\Omega \times I)$. Then the two-scale ergodicity property is satisfied and (4.1) holds for every $q < \infty$.*

Proof. We choose a countable dense set of points $x_i \in \Omega$, $i \in \mathbb{N}$. An application of Property 2.1 to the countable number of functions $g(x_i, .)$ shows that

$$\int_\Omega g(x_i, \chi^\eta(x))\, \varphi(x)\, dx \to \int_\Omega \langle g \rangle (x_i)\, \varphi(x)\, dx\ \forall i \in \mathbb{N}, \varphi \in L^\infty(\Omega), \text{ almost surely.} \tag{4.2}$$

The family $g^\eta \in L^q(\Omega)$ is bounded. To verify (4.1), it suffices to show that for every weakly convergent subsequence $g^\eta \rightharpoonup \bar{g}$ in $L^q(\Omega)$ we have $\bar{g} = \langle g \rangle$.

With this aim we fix a test function $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. The function g is uniformly continuous on $\text{supp}(\varphi) \times I$. We therefore find $n \in \mathbb{N}$ and disjoint sets $\Omega_i \subset \Omega$ such that $\text{supp}(\varphi) \subset \bigcup_{i=1}^n \Omega_i$, $x_i \in \Omega_i$, with $|g(x,y) - g(x_i,y)| \le \varepsilon$ for all $x \in \Omega_i$, $y \in I$. This allows to calculate

$$\int_\Omega g^\eta(x)\,\varphi(x)\,dx = \int_\Omega g(x,\chi^\eta(x))\,\varphi(x)\,dx$$
$$= \sum_{i=1}^n \int_{\Omega_i} g(x_i,\chi^\eta(x))\,\varphi(x)\,dx + \sum_{i=1}^n \int_{\Omega_i} \left[g(x,\chi^\eta(x)) - g(x_i,\chi^\eta(x))\right]\,\varphi(x)\,dx.$$

The last sum is, in absolute value, bounded by $C\varepsilon$, with C depending only on Ω and the test-function φ. Taking the limit $\eta \to 0$ and exploiting (4.2) we find

$$\left|\int_\Omega \bar{g}(x)\,\varphi(x)\,dx - \sum_{i=1}^n \int_{\Omega_i} \langle g\rangle\,(x_i)\,\varphi(x)\,dx\right|$$
$$= \lim_{\eta\to 0}\left|\int_\Omega g^\eta(x)\,\varphi(x)\,dx - \sum_{i=1}^n \int_{\Omega_i} g(x_i,\chi^\eta(x))\,\varphi(x)\,dx\right| \le C\varepsilon.$$

The function $\langle g\rangle$ is obtained as a y-average of g and has therefore the same modulus of continuity, $|\langle g\rangle(x) - \langle g\rangle(x_i)| \le \varepsilon$ for all $x \in \Omega_i$. Increasing the constant C, we find have

$$\left|\int_\Omega \bar{g}(x)\,\varphi(x)\,dx - \int_\Omega \langle g\rangle\,(x)\,\varphi(x)\,dx\right| \le C\varepsilon.$$

Since ε was arbitrary, this demonstrates $\bar{g} = \langle g\rangle$ and hence the result. ☐

In the next lemma we study stochastic processes that take only finitely many values. In this special case, general functions g satisfy the two-scale ergodicity.

Lemma 4.3. *Let the process satisfy the single-scale ergodicity of Property 2.1 and let the support of μ be finite, $\mu = \sum_{k=1}^m \rho_k \delta_{y_k}$ with $y_k \in I$ and $\rho_k \in [0,1]$. Then, for every $q < \infty$ and every $g \in L^q(\Omega \times I, dx \otimes \mu)$, the two-scale ergodicity property (4.1) of Definition 4.1 is satisfied.*

Proof. We exploit that we can identify

$$L^q(\Omega \times I, dx \otimes \mu) \equiv L^q(\Omega)^m \text{ via } g(x,y_k) = g_k(x)\ \forall k \le m.$$

We are given a function $g \in L^q(\Omega \times I, dx \otimes \mu)$ which we identify with the m scalar functions $g_k \in L^q(\Omega)$. The function $g^\eta(x) = g(x,\chi^\eta(x))$ satisfies

$$\|g^\eta\|_{L^q(\Omega)}^q = \int_\Omega |g(x,\chi^\eta(x))|^q\,dx \le \int_\Omega \max_{k\le m} |g_k(x)|^q\,dx \le C.$$

We can therefore assume that g^η converges weakly in L^q. It remains to verify that the limit is almost surely $\langle g\rangle(x) = \sum_k \rho_k g_k(x)$.

For $i \in \mathbb{N}$ we consider $\varepsilon_i = \frac{1}{i} \searrow 0$, and approximations $g_k^i \in C^0(\Omega)$ of the functions $g_k \in L^q(\Omega)$ with

$$\sum_{k=1}^{m} \|g_k^i - g_k\|_{L^q(\Omega)}^q \leq \varepsilon_i.$$

With our identification of the function spaces, the functions g_k^i define a function $g^i \in L^q(\Omega \times I, dx \otimes \mu)$. We calculate

$$\begin{aligned} \|(g^i)^\eta - g^\eta\|_{L^q(\Omega)}^q &= \int_\Omega |g^i(x, \chi^\eta(x)) - g(x, \chi^\eta(x))|^q \, dx \\ &\leq \int_\Omega \max_{k \leq m} |g_k^i(x) - g_k(x)|^q \, dx \leq \varepsilon_i. \end{aligned}$$

Furthermore, also the averaged functions are comparable,

$$\begin{aligned} \| \left\langle g^i \right\rangle - \langle g \rangle \|_{L^q(\Omega)}^q &= \int_\Omega \left| \sum_{k=1}^{m} \rho_k (g_k^i(x) - g_k(x)) \right|^q \, dx \\ &\leq \int_\Omega \max_{k \leq m} |g_k^i(x) - g_k(x)|^q \, dx \leq \varepsilon_i. \end{aligned}$$

Lemma 4.2 provides the convergence (4.1) for the smooth approximation,

$$(g^i)^\eta \rightharpoonup \left\langle g^i \right\rangle \text{ in } L^q(\Omega) \text{ for all } i \in \mathbb{N}, \text{ almost surely.}$$

To combine our findings we write

$$g^\eta - \langle g \rangle = [g^\eta - (g^i)^\eta] + [(g^i)^\eta - \left\langle g^i \right\rangle] + [\left\langle g^i \right\rangle - \langle g \rangle]$$

and conclude that, almost surely, the left hand side converges weakly to 0 in $L^q(\Omega)$. This shows (4.1) and concludes the proof. □

In our application of the two-scale ergodicity lemmas we use functions with an additional time dependence, $g : \Omega \times (0, T) \times I \to \mathbb{R}$. We remark here that the above convergence results remain valid if Ω is replaced by Ω_T.

4.2 Homogenization for a finite number of materials

We are now in the position to present a very short proof of the homogenization result. It is particularly simple since strong solutions are available. We treat here the special case of measures μ with a finite support, corresponding to only a finite number of material properties.

Theorem 4.4 (Homogenization)**.** *Let the single-scale ergodicity of Property 2.1 be satisfied, let supp(μ) be finite, $\mu = \sum_{k=1}^{m} \rho_k \delta_{y_k}$. Let $(u^\eta, \sigma^\eta, \varepsilon^\eta)$ be a sequence of strong solutions of problems (P_η), (2.3)–(2.5). Furthermore, let (u^*, σ^*, w) be a strong solution of (P_*), (2.6)–(2.8), with compatible initial and boundary data as in (3.1)–(3.3). Then*

$$\begin{aligned} \partial_t u^\eta - \partial_t u^* \to 0, \quad \sigma^\eta - \sigma^* \to 0 \\ \varepsilon^\eta - w \circ \chi^\eta \to 0 \end{aligned} \quad \text{in } L^2(\Omega_T), \text{ almost surely.} \tag{4.3}$$

Proof. We define an appropriate quadratic distance between η-solution and a special function, which we reconstruct from the solution of the homogenized problem. We use χ^η and set $w^\eta(t,x) := w(t,x,\chi^\eta(x))$.

$$E(t) = \frac{1}{2}\int_\Omega |\partial_t u^\eta - \partial_t u^*|^2 + \frac{1}{2}\int_\Omega \frac{\alpha^\eta}{\kappa^\eta}|\varepsilon^\eta - w^\eta|^2$$

We can calculate for this quadratic distance the weak derivative

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_\Omega (\partial_t u^\eta - \partial_t u^*)\partial_x(\sigma^\eta - \sigma^*) + \int_\Omega \frac{\alpha^\eta}{\kappa^\eta}\left(\varepsilon^\eta - w^\eta\right)\left(\partial_t \varepsilon^\eta - \partial_t w^\eta\right) \\ &= -\int_\Omega \partial_t\partial_x u^\eta(\sigma^\eta - \sigma^*) + \int_\Omega \partial_t\partial_x u^*(\sigma^\eta - \sigma^*) \\ &\quad + \int_\Omega \frac{\alpha^\eta}{\kappa^\eta}\left(\varepsilon^\eta - w^\eta\right)\left(\partial_t \varepsilon^\eta - \partial_t w^\eta\right) \\ &= -\int_\Omega \partial_t \varepsilon^\eta \sigma^\eta + \int_\Omega \partial_t \varepsilon^\eta \sigma^* - \int_\Omega \beta^\eta \partial_t \sigma^\eta(\sigma^\eta - \sigma^*) \\ &\quad + \int_\Omega \left(\partial_t \int_I w(y)\, d\mu(y)\right)(\sigma^\eta - \sigma^*) + \int_\Omega \beta^* \partial_t \sigma^*(\sigma^\eta - \sigma^*) \\ &\quad + \int_\Omega (\varepsilon^\eta - w^\eta)\frac{\alpha^\eta}{\kappa^\eta}\partial_t \varepsilon^\eta - \int_\Omega (\varepsilon^\eta - w^\eta)\frac{\alpha^\eta}{\kappa^\eta}\partial_t w^\eta. \end{aligned}$$

With the help of the constitutive law we now evaluate σ^η and σ^* in the first integral. Relation (2.5) can be used directly, in (2.8) we insert $y = \chi^\eta(x)$ to obtain

$$\sigma^\eta \in \frac{\alpha^\eta}{\kappa^\eta}\varepsilon^\eta + \frac{\gamma^\eta}{\kappa^\eta}\,\mathrm{sign}(\partial_t \varepsilon^\eta), \qquad \sigma^* \in \frac{\alpha^\eta}{\kappa^\eta}w^\eta + \frac{\gamma^\eta}{\kappa^\eta}\,\mathrm{sign}(\partial_t w^\eta).$$

We evaluate ε^η and w^η with the same relation in the last integral to obtain

$$\begin{aligned} \frac{d}{dt}E(t) \in &-\int_\Omega \partial_t \varepsilon^\eta \left[\frac{\alpha^\eta}{\kappa^\eta}\varepsilon^\eta + \frac{\gamma^\eta}{\kappa^\eta}\,\mathrm{sign}(\partial_t \varepsilon^\eta)\right] \\ &+ \int_\Omega \partial_t \varepsilon^\eta \left[\frac{\alpha^\eta}{\kappa^\eta}w^\eta + \frac{\gamma^\eta}{\kappa^\eta}\,\mathrm{sign}(\partial_t w^\eta)\right] \\ &+ \int_\Omega \left(\int_I \partial_t w(y)\, d\mu(y)\right)(\sigma^\eta - \sigma^*) \end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega (\beta^\eta \partial_t \sigma^\eta - \beta^* \partial_t \sigma^*)(\sigma^\eta - \sigma^*) + \int_\Omega (\varepsilon^\eta - w^\eta) \frac{\alpha^\eta}{\kappa^\eta} \partial_t \varepsilon^\eta \\
&- \int_\Omega \left[\sigma^\eta - \frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t \varepsilon^\eta) - \sigma^* + \frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t w^\eta) \right] \partial_t w^\eta.
\end{aligned}
$$

In this expression, many terms cancel or can be neglected. The fifth integral over Ω cancels with identical terms that appear in the first two integrals. What remains from the first two integrals has a sign,

$$
- \int_\Omega \partial_t \varepsilon^\eta \frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t \varepsilon^\eta) + \int_\Omega \partial_t \varepsilon^\eta \frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t w^\eta) \le 0.
$$

The same sign appears in the last integral,

$$
- \int_\Omega \left[-\frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t \varepsilon^\eta) + \frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t w^\eta) \right] \partial_t w^\eta \le 0.
$$

We arrive at the expression

$$
\begin{aligned}
\frac{d}{dt} E(t) \le \int_\Omega \left(\int_I \partial_t w(y) \, d\mu(y) - \partial_t w^\eta \right) (\sigma^\eta - \sigma^*) \\
- \int_\Omega (\beta^\eta \partial_t \sigma^\eta - \beta^* \partial_t \sigma^*)(\sigma^\eta - \sigma^*).
\end{aligned}
$$

After a time integration, the last integral can be written as

$$
\begin{aligned}
- \int_0^t & \int_\Omega (\beta^\eta \partial_t \sigma^\eta - \beta^* \partial_t \sigma^*)(\sigma^\eta - \sigma^*) \\
&= - \int_0^t \int_\Omega \beta^\eta (\partial_t \sigma^\eta - \partial_t \sigma^*)(\sigma^\eta - \sigma^*) - \int_0^t \int_\Omega (\beta^\eta - \beta^*) \partial_t \sigma^* (\sigma^\eta - \sigma^*) \\
&= - \int_\Omega \frac{\beta^\eta}{2} |\sigma^\eta - \sigma^*|^2 \Big|_0^t - \int_0^t \int_\Omega (\beta^\eta - \beta^*) \partial_t \sigma^* (\sigma^\eta - \sigma^*).
\end{aligned}
$$

We can now exploit our choice of initial values (3.1) and (3.3). The velocities satisfy $\partial_t u^\eta|_{t=0} = v_0 = \partial_t u^*|_{t=0}$, the stresses $\sigma^\eta|_{t=0} = s_0 = \sigma^*|_{t=0}$, and the strains $\varepsilon^\eta|_{t=0} = e_0^\eta = (\kappa^\eta s_0 - \gamma^\eta)/\alpha^\eta = w^\eta|_{t=0}$.

We therefore find, for almost every $t \in (0, T)$, the time-integral

$$
\begin{aligned}
E(t) + \int_\Omega \frac{\beta^\eta}{2} |\sigma^\eta - \sigma^*|^2(t) &= \int_\Omega \int_0^t \frac{d}{dt} E + \int_\Omega \frac{\beta^\eta}{2} |\sigma^\eta - \sigma^*|^2 \Big|_0^t \\
\le \int_{\Omega_t} \left(\int_I \partial_t w(y) \, d\mu(y) - \partial_t w^\eta \right) &(\sigma^\eta - \sigma^*) - \int_{\Omega_t} (\beta^\eta - \beta^*) \partial_t \sigma^* (\sigma^\eta - \sigma^*).
\end{aligned}
$$

To the function $g(x, y) = \partial_t w(x, y)$, $\partial_t w \in L^2(\Omega_T \times I, dx \otimes dt \otimes d\mu)$, we apply Lemma 4.3, which yields the weak convergence

$$
\partial_t w^\eta \rightharpoonup \int_I \partial_t w(y) \, d\mu(y) \quad \text{in } L^2(\Omega_T) \text{ almost surely.}
$$

Since σ^η is bounded in $H^1(\Omega_T)$, a subsequence of σ^η converges strongly in $L^2(\Omega_T)$. This provides that, almost surely, for every $t < T$ holds

$$\int_{\Omega_t} \left(\int_I \partial_t w(y)\, d\mu(y) - \partial_t w^\eta \right) (\sigma^\eta - \sigma^*) \to 0 \text{ for } \eta \to 0.$$

For the second integral in the $E(t)$-estimate it suffices to apply the single-scale ergodicity, Property 2.1. It provides $\beta^\eta \rightharpoonup \beta^*$ in $L^q(\Omega_T)$ almost surely, for every $q < \infty$. Since $\partial_t \sigma^* \in L^2(\Omega_T)$ is a fixed function and σ^η converges strongly in $L^s(\Omega_T)$ for some $s > 2$, this integral vanishes in the limit. □

4.3 Infinite number of materials

The above proof of the homogenization result works whenever we have the two-scale ergodicity of Definition 4.1 for the function $g = \partial_t w$ at our disposal. But, in general, the function $\partial_t w$ is not continuous and we can therefore not conclude (4.1) with Lemma 4.2. For arbitrary L^q-functions g, relation (4.1) turns out to be intricate. This is already known from periodic homogenization problems.

Let us recall an example developed by Gérard, Murat, and Allaire (see [2], Proposition 5.8). It provides a characteristic function $g \in L^\infty([0,1] \times [0,1]) \cap C^0([0,1], L^1([0,1]))$, periodically extended in the second variable, with

$$\lim_{\eta \to 0} \int_0^1 g(x, x/\eta) \neq \int_0^1 \int_0^1 g(x,y)\, dy\, dx. \tag{4.4}$$

In our notation, this means: even in the case of periodic homogenization, i.e. with the deterministic choice $\chi_1(x) = x \bmod 1$ in one space dimension, property (4.1) fails for a general $g \in L^\infty(\Omega \times Y)$.

We solve this problem by working with the following semi-discrete approximation $(P_{*,h})$ of the limiting system. We choose $h = \frac{1}{m} > 0$, $m \in \mathbb{N}$, set $y_j = jh$ for $j = 0, ..., m$, and consider the finite set of points $I_h = \{y_0, ..., y_m\}$. With the projection $P_h : y \mapsto \sup\{y_j \in I_h | y_j \le y\}$ we impose for $u_h, \sigma_h : \Omega_T \to \mathbb{R}$ and $w_h : \Omega_T \times I_h \to \mathbb{R}$ the following equations on Ω_T.

$$\partial_t^2 u_h = \partial_x \sigma_h + f \tag{4.5}$$

$$\partial_x u_h = \int_I w_h(P_h(y))\, d\mu(y) + \beta^* \sigma_h \tag{4.6}$$

$$\alpha(y) w_h(y) \in \kappa(y) \sigma_h - \gamma(y) \operatorname{sign}(\partial_t w_h(y)) \qquad \forall y \in I_h \tag{4.7}$$

This system can be regarded as an approximation of the averaged system (P_*) with the distribution μ.

We can also take another point of view. We can regard system (4.5)–(4.7) as problem (P_*) with a new measure, namely the finitely valued measure $\mu_h = P_{h,\#}\mu$ on I_h. It is the push-forward of μ, which is defined by $\mu_h(F) = \mu(P_h^{-1}(F))$.

In particular, Theorem 3.7 can be applied and yields the existence of a strong solution to problem (4.5)–(4.7).

Furthermore, the uniform estimates of Lemma 3.3 are valid and we can conclude, along a subsequence,

$$\partial_t u_h \to \partial_t u^*, \quad \sigma_h \to \sigma^* \text{ in } L^2(\Omega_T) \tag{4.8}$$

for $h \to 0$. We need a variant of Lemma 4.3 to conclude the ergodicity limit.

Lemma 4.5. *Let the single-scale ergodicity of Property 2.1 be satisfied and let $g \in L^q(\Omega \times I)$ be constant on intervals $[y_j, y_{j+1}) \subset I$ for almost every $x \in \Omega$. Then the two-scale ergodicity of Definition 4.1 holds for g.*

Proof. It suffices to consider the new stochastic variables $\Psi_1 = P_h\chi_1$ and $\Psi^\eta = P_h\chi^\eta$ instead of χ_1 and χ^η. The distribution of the values $\Psi_1(0) \in I$ is given by $\nu = P_{h,\#}\mu$. This measure has its finite support contained in I_h, which essentially brings us back to the situation of Lemma 4.3.

We claim that the new process satisfies again the single-scale ergodicity property 2.1. To see this, let $\bar{g} : I_h \to \mathbb{R}$ be given. We identify $\bar{g}$ with a function that is constant on the intervals $[y_j, y_{j+1})$. Then, by the single-scale ergodicity property of χ_1, almost surely holds

$$\bar{g}(\Psi^\eta) = \bar{g}(\chi^\eta) \rightharpoonup \langle \bar{g} \rangle = \int_I \bar{g}(y)\, d\mu(y) = \sum_j g(y_j)\, \mu([y_j, y_{j+1})) = \int_I \bar{g}(y)\, d\nu(y).$$

We can thus apply Lemma 4.3 to Ψ_1 and conclude that a piecewise constant L^q-function $g : \Omega \times I \to \mathbb{R}$, restricted to $\Omega \times I_h$, satisfies the two-scale ergodicity for Ψ_1. We find that, almost surely in $L^q(\Omega)$,

$$g^\eta = g(.,\chi^\eta(.)) = g(.,\Psi^\eta(.)) \rightharpoonup \int_I g(.,y)\, d\nu(y) = \int_I g(.,y)\, d\mu(y).$$

This shows (4.1) for the original process. □

We are now in the position to prove our main result.

Proof of Theorem 1.1. We follow the calculations of the proof of Theorem 4.4, but as comparison functions we now use the semi-discrete solutions of the averaged system, u_h, σ_h, and w_h. As a comparison function we now use $w^\eta(t,x) := w_h(t,x,P_h\chi^\eta(x))$. We identify w_h with the function that is constant on intervals $[y_j, y_{j+1})$. We may then also write $w^\eta(t,x) := w_h(t,x,\chi^\eta(x))$. With

$$E_h(t) = \frac{1}{2}\int_\Omega |\partial_t u^\eta - \partial_t u_h|^2 + \frac{1}{2}\int_\Omega \frac{\alpha^\eta}{\kappa^\eta}|\varepsilon^\eta - w^\eta|^2$$

we can calculate

$$\begin{aligned}\frac{d}{dt}E_h(t) = &-\int_\Omega \partial_t\varepsilon^\eta\sigma^\eta + \int_\Omega \partial_t\varepsilon^\eta\sigma_h - \int_\Omega \beta^\eta\partial_t\sigma^\eta(\sigma^\eta-\sigma_h)\\ &+\int_\Omega\left(\partial_t\int_I w_h(P_h y)\,d\mu(y)\right)(\sigma^\eta-\sigma_h) + \int_\Omega \beta^*\partial_t\sigma_h(\sigma^\eta-\sigma_h)\\ &+\int_\Omega(\varepsilon^\eta-w^\eta)\frac{\alpha^\eta}{\kappa^\eta}\partial_t\varepsilon^\eta - \int_\Omega(\varepsilon^\eta-w^\eta)\frac{\alpha^\eta}{\kappa^\eta}\partial_t w^\eta.\end{aligned}$$

With the help of the constitutive law we now evaluate σ^η and σ_h in the first integral. Relation (2.5) can be used directly, in (4.7) we insert $y = P_h\chi^\eta(x)$ to obtain

$$\sigma^\eta \in \frac{\alpha^\eta}{\kappa^\eta}\varepsilon^\eta + \frac{\gamma^\eta}{\kappa^\eta}\operatorname{sign}(\partial_t\varepsilon^\eta), \qquad \sigma_h \in \frac{\bar\alpha^\eta}{\bar\kappa^\eta}w^\eta + \frac{\bar\gamma^\eta}{\bar\kappa^\eta}\operatorname{sign}(\partial_t w^\eta)$$

with

$$\bar\alpha(y) = \alpha(P_h y), \quad \bar\kappa(y) = \kappa(P_h y), \quad \bar\gamma(y) = \gamma(P_h y).$$

At the same time, we evaluate ε^η and w^η with these relations in the last integral. Note that we omit the overbar in the second integral and compensate this fact in the last integral.

$$\begin{aligned}\frac{d}{dt}E_h(t) \in &-\int_\Omega \partial_t\varepsilon^\eta\left[\frac{\alpha^\eta}{\kappa^\eta}\varepsilon^\eta + \frac{\gamma^\eta}{\kappa^\eta}\operatorname{sign}(\partial_t\varepsilon^\eta)\right]\\ &+\int_\Omega \partial_t\varepsilon^\eta\left[\frac{\alpha^\eta}{\kappa^\eta}w^\eta + \frac{\bar\gamma^\eta}{\bar\kappa^\eta}\operatorname{sign}(\partial_t w^\eta)\right]\\ &+\int_\Omega\left(\int_I \partial_t w_h(P_h y)\,d\mu(y)\right)(\sigma^\eta-\sigma_h)\\ &-\int_\Omega(\beta^\eta\partial_t\sigma^\eta - \beta^*\partial_t\sigma_h)(\sigma^\eta-\sigma_h) + \int_\Omega(\varepsilon^\eta-w^\eta)\frac{\alpha^\eta}{\kappa^\eta}\partial_t\varepsilon^\eta\\ &-\int_\Omega\left[\sigma^\eta - \frac{\gamma^\eta}{\kappa^\eta}\operatorname{sign}(\partial_t\varepsilon^\eta) - \sigma_h + \frac{\bar\gamma^\eta}{\bar\kappa^\eta}\operatorname{sign}(\partial_t w^\eta)\right]\partial_t w^\eta\\ &+\int_\Omega w^\eta\left[\frac{\alpha^\eta}{\kappa^\eta} - \frac{\bar\alpha^\eta}{\bar\kappa^\eta}\right](\partial_t w^\eta - \partial_t\varepsilon^\eta)\,.\end{aligned}$$

Again, many terms cancel or can be neglected. The fifth integral over Ω cancels with identical terms that appear in the first two integrals. What remains from the first two integrals has a sign,

$$-\int_{\Omega_t}\partial_t\varepsilon^\eta\frac{\gamma^\eta}{\kappa^\eta}\operatorname{sign}(\partial_t\varepsilon^\eta) + \int_{\Omega_t}\partial_t\varepsilon^\eta\frac{\bar\gamma^\eta}{\bar\kappa^\eta}\operatorname{sign}(\partial_t w^\eta) \le q_1(h),$$

where we introduce an error estimate $q_1(h) > 0$. The error only depends on the L^1-norm of $\partial_t \varepsilon^\eta$ and the maximal distance $\max_y(\bar\gamma/\bar\kappa - \gamma/\kappa)$. This error has, like all error terms $q_l(h)$, $l = 1, ..., 5$ of the subsequent calculation, the property

$$q_l(h) \to 0 \text{ for } h \to 0 \quad \text{ independent of } \eta.$$

The sign property appears also in

$$-\int_{\Omega_t} \left[-\frac{\gamma^\eta}{\kappa^\eta} \operatorname{sign}(\partial_t \varepsilon^\eta) + \frac{\bar\gamma^\eta}{\bar\kappa^\eta} \operatorname{sign}(\partial_t w^\eta) \right] \partial_t w^\eta \le q_2(h).$$

Since the last integral satisfies the same estimate, we arrive at the time integrated expression

$$\begin{aligned} E_h(.)|_0^t \le \int_{\Omega_t} \left(\int_I \partial_t w_h(P_h y)\, d\mu(y) - \partial_t w^\eta \right) (\sigma^\eta - \sigma_h) \\ - \int_{\Omega_t} (\beta^\eta \partial_t \sigma^\eta - \beta^* \partial_t \sigma_h)(\sigma^\eta - \sigma_h) + q_3(h) \end{aligned}$$

for almost every $t \in (0, T)$. We apply the manipulations on the σ-integral as known from the proof of Theorem 1.1. For arbitrary $t > 0$ we arrive at

$$\begin{aligned} \left\{ E_h + \int_\Omega \frac{\beta^\eta}{2} |\sigma^\eta - \sigma_h|^2 \right\}(t) \le \int_{\Omega_t} \left(\int_I \partial_t w_h(y)\, d\mu(y) - \partial_t w^\eta \right) (\sigma^\eta - \sigma_h) \\ - \int_{\Omega_t} (\beta^\eta - \beta^*) \partial_t \sigma_h (\sigma^\eta - \sigma_h) + q_4(h) \end{aligned}$$

for almost every $t \in (0, T)$. For fixed $h > 0$ we let η tend to 0. We exploit that, almost surely, along a subsequence holds: σ^η converges strongly in $L^s(\Omega_T)$ for some $s > 2$, $\partial_t \sigma_h$ is a fixed function in $L^2(\Omega_T)$, and $\beta^\eta \rightharpoonup \beta^*$ in $L^q(\Omega_T)$ for all $q < \infty$. With the two-scale ergodicity property of Lemma 4.5 for $\partial_t w_h$ we find

$$\liminf_{\eta \to 0} \int_0^T \left\{ E_h + \int_\Omega \frac{\beta^\eta}{2} |\sigma^\eta - \sigma_h|^2 \right\}(t)\ dt \le q_5(h).$$

Because of (4.8), we find that, almost surely,

$$\partial_t u^\eta \to \partial_t u^* \text{ and } \sigma^\eta \to \sigma^* \text{ in } L^2(\Omega_T),$$

and thus the claim of Theorem 1.1. □

Conclusions

We studied plasticity equations with an hysteretic constitutive law. Homogenized equations are presented in the form of differential inclusions; in this form, they are a natural extension of the original equations for a family of materials. The homogenization limit is performed rigorously, the key idea is to use the solution of the averaged equations to construct oscillating test-functions. The notion of two-scale ergodicity helps to perform the limit process. In the case of a finite number of materials the process is two-scale ergodic and limits can be taken directly. The general case is reduced to this situation with a discretization of the homogenized equations.

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