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Karl Friedrich Siburg, Pavel A. Stoimenov

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Karl Friedrich Siburg [a,*], Pavel A. Stoimenov [b]

[a]*Fakultät für Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, 44227 Dortmund, Germany*

[b]*Fakultät für Statistik, Technische Universität Dortmund, Vogelpothsweg 78, 44227 Dortmund, Germany*

Abstract

We introduce a scalar product for n-dimensional copulas, based on the Sobolev scalar product for $W^{1,2}$-functions. The corresponding norm has quite remarkable properties and provides a new, geometric framework for copulas. We show that, in the bivariate case, it measures invertibility properties of copulas with respect to the $*$-operation introduced by Darsow et al. (1992). The unique copula of minimal norm is the null element for the $*$-operation, whereas the copulas of maximal norm are precisely the invertible elements.

Key words: Copula, Scalar product, Sobolev space

1 Introduction

Let I be the closed unit interval [0,1]. For any integer $n \geq 2$, an n-dimensional copula (or n-copula) is a function $C : I^n \to I$ with the following properties:

(C1) $C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for all $i = 1, \ldots, n$ and for all $x_j \in I$ with $j = 1, \ldots, i-1, i+1, \ldots, n$.
(C2) $C(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$ for all $i = 1, \ldots, n$ and for all $x_i \in I$.
(C3) C is n-increasing, i.e., for all $B = \times_{i=1}^n [a_i, b_i] \subseteq I^n$ we have

$$\sum_{v \in B} sgn(v) \, C(v) \geq 0$$

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* Corresponding author.
  Email addresses: `karl.f.siburg@math.tu-dortmund.de` (Karl Friedrich Siburg), `stoimenov@gmx.de` (Pavel A. Stoimenov).

where the sum is taken over all vertices $v = (v_1, \ldots, v_n)$ of B, with $v_i \in \{a_i, b_i\}$ for each $i = 1, \ldots, n$, and $sgn(v)$ is defined to be 1 if $v_i = a_i$ for an even number of i's, and -1 otherwise.

Copulas were introduced by Sklar [14] for the investigation of how joint distribution functions are related to their univariate margins. In particular, Sklar's theorem (see [12, 13, 14]) states that for all real valued random variables $X_1, \ldots, X_n$ with joint distribution function H and univariate margins $F_1, \ldots, F_n$ there exists a copula C such that

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

If $F_1, \ldots, F_n$ are all continuous, then C is unique and is called the copula of $X_1, \ldots, X_n$; otherwise C is uniquely determined on $\mathrm{Range}\, F_1 \times \cdots \times \mathrm{Range}\, F_n$. Conversely, given a copula C and univariate distribution functions $F_1, \ldots, F_n$, then the function H defined by (1) is an n-dimensional distribution function with univariate margins $F_1, \ldots, F_n$. It follows that copulas fully capture the dependence structure of random variables, irrespective of their univariate distributions.

Further applications of copulas in probability theory arise from the fact that, after an appropriate extension to $\mathbb{R}^n$, every n-copula is an n-dimensional joint distribution function with uniform margins on I. In fact, for $n = 2$, a copula is simply the joint distribution function of a doubly stochastic probability measure defined on the Borel subsets of I^2, thus there is a one-to-one correspondence between these two concepts; see, e.g., [6, 8]. Furthermore, Brown [1] showed that there is a one-to-one correspondence between Markov operators on $L^\infty(I)$ and doubly stochastic measures on the Borel subsets of I^2, and hence between Markov operators on $L^\infty(I)$ and 2-copulas. In particular, Olsen et al. [10] established an isomorphism between 2-copulas under the $*$-product (see the definition below), and Markov operators on $L^\infty(I)$ under composition. This isomorphism has been exploited by Li et al. [7], who proposed another type of convergence for 2-copulas. Finally, we mention the importance of copulas to the theory of probabilistic metric spaces and refer to [11, 12].

Copulas exhibit several nice analytic properties. In particular, copulas are Lipschitz continuous functions from I^n to I with Lipschitz constant equal to 1, which immediately implies that they are absolutely continuous in each argument and their partial derivatives exist almost everywhere (a.e.). For arbitrary $n \geq 2$, let $\mathfrak{C}_n$ denote the set of all n-dimensional copulas. The differentiability properties of copulas imply that $\mathfrak{C}_n$ is a subset of any Sobolev space $W^{1,p}(I^n, \mathbb{R})$ with $p \in [1, \infty]$; see [4]. Moreover, Darsow et al. [3] introduce a product operation on the set $\mathfrak{C}_2$ of 2-dimensional copulas, given by

$$(A * B)(x, y) = \int_0^1 \partial_2 A(x, t)\, \partial_1 B(t, y)\, dt$$

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where $\partial_i A$, $i = 1, 2$, denotes the partial derivative of A with respect to the i-th variable.

In this paper, we introduce a new structure for n-dimensional copulas, exploiting the fact that the particular Sobolev space $W^{1,2}(I^n, \mathbb{R})$ is a Hilbert space. Namely, we show that

$$\langle f, g \rangle = \int_{I^n} \nabla f \cdot \nabla g \, d\lambda$$

defines a scalar product on the linear span of $\mathfrak{C}_n$ with corresponding norm

$$\|f\| = \Big(\int_{I^n} |\nabla f|^2 \, d\lambda \Big)^{1/2}$$

where $\cdot$ and $|\ |$ denote, respectively, the Euclidean scalar product and norm on $\mathbb{R}^n$, and λ denotes the n-dimensional Lebesgue measure.

The existence of a scalar product yields a new, geometric way of looking at copulas. In addition, for 2-copulas the scalar product admits a representation via the $*$-product and, therefore, provides a link between their geometric and algebraic properties. In particular, the set $\mathfrak{C}_2$ has diameter 1 and lies in the shell of radii $\sqrt{2/3}$ and 1. There is a unique copula of minimal norm, which is the null element in $(\mathfrak{C}_2, *)$, whereas the copulas of maximal norm are precisely those which are invertible with respect to the $*$-product. Thus, loosely speaking, the Sobolev norm measures the "degree of invertibility" of 2-copulas. We point out that, since copulas can be interpreted as dependence functions, the above results also have a probabilistic counterpart.

Briefly, the paper is organized as follows. In Section 2 we collect some basic properties of multivariate copulas. Section 3 introduces the scalar product for n-copulas and its corresponding norm and distance. In Section 4, we deduce fundamental geometric properties of the set of 2-copulas and relate them to the algebraic structure given by the $*$-product. The final Section 5 addresses issues concerned with the topology of the set of n-copulas.

2 Basic properties of copulas

We state here some key properties of copulas which we will need throughout. These follow easily from the definition; for proofs, we refer to [2, 9, 12]. Recall that, for $n \geq 2$, $\mathfrak{C}_n$ denotes the set of all n-copulas and $\partial_i C$, $i = 1, \ldots, n$, denotes the partial derivative of $C \in \mathfrak{C}_n$ with respect to the i-th variable.

Theorem 1 *For any $C \in \mathfrak{C}_n$, the following statements are true:*

(i) *C is increasing in each argument.*

(ii) C is Lipschitz continuous and for all $x_i, y_i \in I$ with $i = 1, \ldots, n$,

$$|C(x_1, \ldots, x_n) - C(y_1, \ldots, y_n)| \le \sum_{i=1}^{n} |x_i - y_i|. \qquad (2)$$

(iii) For each $i = 1, \ldots, n$ the partial derivative $\partial_i C(x_1, \ldots, x_n)$ exists for all $x_j \in I$ with $j = 1, \ldots, i-1, i+1, \ldots, n$, and for almost all $x_i \in I$, and satisfies

$$0 \le \partial_i C(x_1, \ldots, x_n) \le 1. \qquad (3)$$

Remark 2 *Note that (ii) implies that a copula is absolutely continuous in each argument, so that it can be recovered from any of its partial derivatives by integration. Moreover, by (iii) we have $\partial_i C \in L^p(I^n, \mathbb{R})$ for all $p \in [1, \infty]$.*

Furthermore, it can be shown that for any $C \in \mathfrak{C}_n$ and for all $(x_1, \ldots, x_n) \in I^n$

$$C^-(x_1, \ldots, x_n) \le C(x_1, \ldots, x_n) \le C^+(x_1, \ldots, x_n) \qquad (4)$$

where C^- and C^+ are the so-called Fréchet-Hoeffding bounds given by

$$C^-(x_1, \ldots, x_n) = \max(x_1 + \ldots + x_n - n + 1, 0),$$
$$C^+(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n).$$

The upper bound C^+ is a copula itself for all $n \ge 2$, whereas the lower bound C^- is a copula only for $n = 2$. Another distinguished copula is the product copula

$$P(x_1, \ldots, x_n) = x_1 \ldots x_n.$$

Observe that for any copula $C \in \mathfrak{C}_2$, it is possible to define a copula $C^\top$ by

$$C^\top(x, y) = C(y, x), \qquad (5)$$

which is called the transposed copula of C. C is called symmetric if $C = C^\top$.

The set $\mathfrak{C}_2$ of all 2-dimensional copulas is of particular interest since it carries an algebraic structure, the so-called $*$-product introduced by Darsow et al. [3]. For any $A, B \in \mathfrak{C}_2$ and $x, y \in I$, set

$$(A * B)(x, y) = \int_0^1 \partial_2 A(x, t)\, \partial_1 B(t, y)\, dt. \qquad (6)$$

Theorem 3 *For any $A, B \in \mathfrak{C}_2$, $A * B$ is in $\mathfrak{C}_2$.*

In particular, Darsow et al. [3] show that $(\mathfrak{C}_2, *)$ is a monoid with P and C^+ as null and unit element, respectively, since for any copula $C \in \mathfrak{C}_2$ we have

$$P * C = C * P = P, \tag{7}$$
$$C^+ * C = C * C^+ = C. \tag{8}$$

In addition, by direct calculation, the $*$-product of C with C^- is given by

$$\begin{aligned} (C^- * C)(x,y) &= y - C(1-x,y), \\ (C * C^-)(x,y) &= x - C(x,1-y). \end{aligned} \tag{9}$$

Concerning transpositions of copulas, note that $P, C^+, C^- \in \mathfrak{C}_2$ are all symmetric and that

$$(A * B)^\top = B^\top * A^\top \tag{10}$$

for any $A, B \in \mathfrak{C}_2$.

A copula $C \in \mathfrak{C}_2$ is called left invertible if there is a copula A, called a left inverse, such that $A * C = C^+$. It is right invertible if there is a copula A, called a right inverse, such that $C * A = C^+$. A copula is called invertible if it is both left and right invertible. Darsow et al. [3] show that these algebraic properties can be translated into analytical ones.

Theorem 4 *If they exist, left and right inverses of a copula $C \in \mathfrak{C}_2$ are unique and given by the transposed copula $C^\top$. Also, the following statements hold:*

  (i) *C is left invertible if and only if for each $y \in I$, $\partial_1 C(x,y) \in \{0,1\}$ for almost all $x \in I$.*
 (ii) *C is right invertible if and only if for each $x \in I$, $\partial_2 C(x,y) \in \{0,1\}$ for almost all $y \in I$.*

The next result is established in Mikusiński et al. [8].

Theorem 5 *Invertible copulas are L^∞-dense in $\mathfrak{C}_2$.*

Remark 6 *Theorem 5 implies that the $*$-product is not (jointly) continuous under the uniform norm, as has been shown in [3]. Indeed, if it were, any sequence of invertible copulas $(C_k)_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} \|C_k - P\|_{L^\infty} = 0$ would satisfy $C_k^\top * C_k = C^+$ for all k, so that, since $\lim_{k \to \infty} \|C_k^\top - P^\top\|_{L^\infty} = 0$, $P^\top$ would be a left inverse for P, contradicting (7).*

In view of Sklar's theorem, the preceding results have interesting probabilistic interpretations. Consider two continuous random variables X_1 and X_2 with (unique) copula C. It is well-known that $C = P$ if and only if X_1 and X_2 are independent. Furthermore, Darsow et al. [3] show that C is left invertible if and only if there is a Borel measurable function f such that $X_2 = f(X_1)$ almost

surely (a.s.). An analogous statement holds if C is right invertible, which implies that C is invertible if and only if there is a Borel measurable bijection f such that $X_2 = f(X_1)$ a.s. Thus, Theorem 5 expresses the counterintuitive fact that any type of stochastic dependence, in particular independence, can be approximated uniformly by the dependence corresponding to an invertible deterministic relationship.

3 The Sobolev scalar product for copulas

Recall that $\cdot$ and $|\ |$ denote the Euclidean scalar product and norm on $\mathbb{R}^n$, and λ denotes the n-dimensional Lebesgue measure. It follows immediately from Theorem 1, and has been noticed in [4], that

$$\mathfrak{C}_n \subset W^{1,p}(I^n, \mathbb{R})$$

for each $p \in [1, \infty]$, where $W^{1,p}(I^n, \mathbb{R})$ is the standard Sobolev space. However, it has not been exploited in this context that $W^{1,2}(I^n, \mathbb{R})$ is a Hilbert space with respect to the scalar product

$$\langle f, g \rangle_{W^{1,2}} = \int_{I^n} fg \, d\lambda + \int_{I^n} \nabla f \cdot \nabla g \, d\lambda \tag{11}$$

where ∇f denotes the vector consisting of the weak partial derivatives of f. We refer to [5] for more details.

In fact, there is an even simpler way to define a scalar product for copulas. Let $\mathrm{span}(\mathfrak{C}_n)$ denote the vector space generated by $\mathfrak{C}_n$. Obviously, $\mathrm{span}(\mathfrak{C}_n)$ is a subset of $W^{1,2}(I^n, \mathbb{R})$. For $f, g \in \mathrm{span}(\mathfrak{C}_n)$, set

$$\langle f, g \rangle = \int_{I^n} \nabla f \cdot \nabla g \, d\lambda, \tag{12}$$

$$\|f\| = \left(\int_{I^n} |\nabla f|^2 \, d\lambda \right)^{1/2}, \tag{13}$$

$$d(f, g) = \left(\int_{I^n} |\nabla f - \nabla g|^2 \, d\lambda \right)^{1/2}. \tag{14}$$

Proposition 7 $\langle\ ,\ \rangle$, $\|\ \|$ *and* d *define, respectively, a scalar product, a norm and a metric on* $\mathrm{span}(\mathfrak{C}_n)$.

PROOF. We need only prove the first statement. Obviously, $\langle\ ,\ \rangle$ is a symmetric bilinear form. If $\langle f, f \rangle = 0$ then $\nabla f = 0$ a.e. which, in view of Remark 2 and $f(0) = 0$, implies that $f = 0$. Therefore, $\langle\ ,\ \rangle$ is nondegenerate. $\square$

Thus, with a slight abuse of notation (because $\mathfrak{C}_n$ is not a vector space itself), we can make the following definition.

Definition 8 *The restrictions of $\langle\ ,\ \rangle$, $\|\ \|$ and d to $\mathfrak{C}_n$ are called the Sobolev scalar product, the Sobolev norm and the Sobolev distance function on $\mathfrak{C}_n$, respectively.*

Remark 9 *(i) Darsow and Olsen [4] show that $(\mathfrak{C}_2, d)$ is a complete metric space and the $*$-product on $\mathfrak{C}_2$ is (jointly) continuous with respect to d.*
(ii) The Sobolev norm $\|\ \|$ is reminiscent of the classical energy functional, which is well known in PDEs and differential geometry. In fact,

$$E(C) = \frac{1}{2}\|C\|^2 = \frac{1}{2}\int_{I^n} |\nabla C|^2\, d\lambda$$

may be called the energy of a copula $C \in \mathfrak{C}_n$.

We have seen that the Sobolev scalar product for copulas appears naturally from an analytical perspective. Moreover, it also allows a representation via the algebraic structure of $\mathfrak{C}_2$, given by the $*$-product defined in (6).

Theorem 10 *For all $A, B \in \mathfrak{C}_2$, we have*

$$\langle A, B \rangle = \int_0^1 (A^\top * B + A * B^\top)(t, t)\, dt$$
$$= \int_0^1 (A^\top * B + B * A^\top)(t, t)\, dt.$$

PROOF. The partial derivatives of the transposed copula are given by

$$\begin{aligned}
\partial_1 A^\top(x, y) &= \partial_2 A(y, x), \\
\partial_2 A^\top(x, y) &= \partial_1 A(y, x).
\end{aligned} \tag{15}$$

Using (15) and (6) we can write

$$\int_0^1\int_0^1 \partial_1 A(x, y)\,\partial_1 B(x, y)\, dx\, dy = \int_0^1 \left(\int_0^1 \partial_2 A^\top(y, x)\,\partial_1 B(x, y)\, dx \right) dy$$
$$= \int_0^1 (A^\top * B)(y, y)\, dy$$
$$\int_0^1\int_0^1 \partial_2 A(x, y)\,\partial_2 B(x, y)\, dx\, dy = \int_0^1 \left(\int_0^1 \partial_2 A(x, y)\,\partial_1 B^\top(y, x)\, dy \right) dx$$
$$= \int_0^1 (A * B^\top)(x, x)\, dx.$$

Adding up both terms we obtain the first identity. The second one follows from the fact that along the diagonal we have $(A * B^\top)(t, t) = (A * B^\top)^\top(t, t) = (B * A^\top)(t, t)$ for each $t \in I$. $\square$

Corollary 11 *If $A, B \in \mathfrak{C}_2$ are symmetric, then*

$$\langle A, B \rangle = 2 \int_0^1 (A * B)(t, t)\, dt.$$

4 The Sobolev geometry of $\mathfrak{C}_2$

This section deals with the geometric properties of the set $\mathfrak{C}_2$ of all 2-dimensional copulas. We begin with a representation of the Sobolev scalar product for the three distinguished copulas P, C^+ and C^- defined in Section 2.

Theorem 12 *For all $C \in \mathfrak{C}_2$, we have*

$$\langle P, C \rangle = \frac{2}{3}, \tag{16}$$

$$\langle C^+, C \rangle = 2 \int_0^1 C(t, t)\, dt, \tag{17}$$

$$\langle C^-, C \rangle = 1 - 2 \int_0^1 C(t, 1 - t)\, dt. \tag{18}$$

PROOF. Since P is symmetric, Theorem 10 and (7) imply that

$$\langle P, C \rangle = \int_0^1 (P * C + C * P)(t, t)\, dt = 2 \int_0^1 P(t, t)\, dt = \frac{2}{3}.$$

(17) and (18) are shown analogously using (8) and (9), respectively. $\square$

Corollary 13 *For all $C \in \mathfrak{C}_2$, we have*

$$\langle C - P, P \rangle = 0, \tag{19}$$

$$\|C - P\|^2 = \|C\|^2 - \frac{2}{3}. \tag{20}$$

PROOF. Equation (19) follows immediately from (16), while (20) is a consequence of (16) in connection with the identity

$$d(A, B)^2 = \|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2 \langle A, B \rangle. \qquad \square \tag{21}$$

Theorem 14 *For any A, B in $\mathfrak{C}_2$, we have*

$$\frac{1}{2} \leq \langle A, B \rangle \leq 1$$

where both bounds are sharp. Moreover, $\langle A, B \rangle = 1$ if and only if $A = B$ and $\|A\| = \|B\| = 1$.

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PROOF. Theorems 3 and 10, together with (4), imply

$$2 \int_0^1 C^-(t,t)\, dt \le \langle A, B \rangle \le 2 \int_0^1 C^+(t,t)\, dt.$$

Simple calculations yield $\int_0^1 C^-(t,t)\, dt = 1/4$ and $\int_0^1 C^+(t,t)\, dt = 1/2$. Furthermore, using Theorem 12 one easily computes that

$$\langle C^-, C^- \rangle = \langle C^+, C^+ \rangle = 1 \quad \text{and} \quad \langle C^-, C^+ \rangle = \frac{1}{2}. \tag{22}$$

This shows that the bounds in the statement are sharp. Finally, since we have $\|A\|^2 = \langle A, A \rangle \le 1$, the last statement is a consequence of (21). $\square$

Corollary 15 *The diameter of $(\mathfrak{C}_2, d)$ is 1; in particular, $d(C^-, C^+) = 1$. Moreover, $d(A, B) = 1$ if and only if $\langle A, B \rangle = 1/2$ and $\|A\| = \|B\| = 1$.*

PROOF. It follows immediately from equation (21) and Theorem 14 that $d(A, B) \le 1$ for all $A, B \in \mathfrak{C}_2$. The equality $d(C^-, C^+) = 1$ follows from (21) and (22). The last statement follows again from (21) and Theorem 14. $\square$

Remark 16 *We point out that C^- and C^+ are not the only copulas realizing the lower bound $1/2$ of the Sobolev scalar product in Theorem 14. For instance, by (17), any $B \in \mathfrak{C}_2$ with $B(t,t) = C^-(t,t)$ for $t \in I$ yields $\langle C^+, B \rangle = 1/2$. An example of a copula $B \ne C^-$ with this property is given by*

$$B(x,y) = \begin{cases} \min(x, y - 1/2) & \text{if } (x,y) \in [0, 1/2] \times [1/2, 1], \\ \min(x - 1/2, y) & \text{if } (x,y) \in [1/2, 1] \times [0, 1/2], \\ C^-(x,y) & \text{otherwise.} \end{cases}$$

In addition, by direct calculation, we obtain $\|B\| = 1$ and thus, in view of Corollary 15, $d(C^+, B) = 1$. Hence C^- and C^+ are also not the only copulas realizing the diameter of $\mathfrak{C}_2$.

Proposition 17 *The transposition map $C \mapsto C^\top$ is an isometry on the metric space $(\mathfrak{C}_2, d)$. Furthermore, for every $C \in \mathfrak{C}_2$ we have*

$$\|C\|^2 \ge 2 \int_0^1 (C * C)(t,t)\, dt,$$

with equality if and only if C is symmetric.

PROOF. The fact that $\|C\| = \|C^\top\|$ follows from (13) and (15). Using (21), Theorem 10 and $C^\top * C^\top = (C * C)^\top$, we can therefore write

$$
\begin{aligned}
0 &\leq \|C - C^\top\|^2 \\
&= \|C\|^2 + \|C^\top\|^2 - 2\langle C, C^\top \rangle \\
&= 2 \left(\|C\|^2 - \int_0^1 ((C * C)^\top + (C * C))(t, t)\, dt \right) \\
&= 2 \left(\|C\|^2 - 2 \int_0^1 (C * C)(t, t)\, dt \right)
\end{aligned}
$$

from which the second assertion follows. $\square$

The next theorem is one of the main results of the paper. It describes fundamental features of the Sobolev norm on $\mathfrak{C}_2$, and shows that the norm detects algebraic properties of 2-dimensional copulas. Loosely speaking, the Sobolev norm measures the "degree of invertibility" of 2-copulas.

Theorem 18 *For all $C \in \mathfrak{C}_2$, the Sobolev norm on $\mathfrak{C}_2$ satisfies*

$$
\frac{2}{3} \leq \|C\|^2 \leq 1.
$$

Moreover, the following assertions hold:

(i) $\|C\|^2 = 2/3$ if and only if $C = P$.
(ii) $\|C\|^2 \in (5/6, 1]$ if C is left or right invertible.
(iii) $\|C\|^2 = 1$ if and only if C is invertible.

PROOF. The fact that $2/3 \leq \|C\|^2 \leq 1$ follows immediately from (20) and Theorem 14. Statement (i) is also a consequence of (20). As for (ii), we have

$$
\|C\|^2 = \int_0^1 \int_0^1 (\partial_1 C(x, y))^2 \, dx\, dy + \int_0^1 \int_0^1 (\partial_2 C(x, y))^2 \, dx\, dy. \qquad (23)
$$

If C is left invertible we know from Theorem 4 that $(\partial_1 C)^2 = \partial_1 C$ a.e., so the first summand in (23) is equal to

$$
\int_0^1 \int_0^1 \partial_1 C(x, y) \, dx\, dy = \int_0^1 y \, dy = \frac{1}{2}.
$$

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To estimate the second term in (23), consider the inequality

$$0 \leq \int_0^1 \int_0^1 (\partial_2 C(x,y) - x)^2 \, dx \, dy$$
$$= \int_0^1 \int_0^1 (\partial_2 C(x,y))^2 \, dx \, dy - 2 \int_0^1 x \int_0^1 \partial_2 C(x,y) \, dy \, dx + \int_0^1 \int_0^1 x^2 \, dx \, dy$$
$$= \int_0^1 \int_0^1 (\partial_2 C(x,y))^2 \, dx \, dy - \frac{1}{3}.$$

Hence, the second term in (23) is at least $1/3$, which proves $\|C\|^2 \geq 5/6$. Equality holds if and only if $\partial_2 C(x,y) = x$ a.e. which, by Remark 2, is equivalent to $C = P$. But this contradicts the assumption that C is left invertible, so $\|C\|^2 > 5/6$. Analogous arguments hold for right invertible copulas. This proves statement (ii).

Finally, by (3) we have $(\partial_i C)^2 \leq \partial_i C$ for $i = 1, 2$, with equality if and only if $\partial_i C \in \{0, 1\}$. Consequently, (23) implies that

$$\|C\|^2 \leq \int_0^1 \int_0^1 \partial_1 C(x,y) \, dx \, dy + \int_0^1 \int_0^1 \partial_2 C(x,y) \, dx \, dy = \frac{1}{2} + \frac{1}{2} = 1$$

with equality if and only if $\partial_1 C, \partial_2 C \in \{0, 1\}$ a.e. In view of Theorem 4, this is equivalent to C being invertible. $\square$

Remark 19 *The inverse implication in Theorem 18 (ii) is not true. For example, the copula $C = \alpha C^+ + (1 - \alpha) C^-$ with $\alpha \in (0,1)$ is, by Theorem 4, neither right nor left invertible, and $\|C\|^2 = 1 - \alpha(1 - \alpha)$ converges to 1 as α tends to 0 or 1.*

Corollary 20 *For any $C \in \mathfrak{C}_2$, the following are equivalent:*

(i) C is invertible, i.e., $C * C^\top = C^\top * C = C^+$.
(ii) $\partial_1 C, \partial_2 C \in \{0, 1\}$ a.e.
(iii) $\int_0^1 (C * C^\top + C^\top * C)(t, t) \, dt = 1$.
(iv) $\|C\| = 1$.

PROOF. See Theorems 4, 10 and 18. $\square$

In summary, the Sobolev scalar product yields the following geometric picture for the set $\mathfrak{C}_2$ of 2-dimensional copulas. First of all, $\mathfrak{C}_2$ has diameter 1 and lies in the shell of radii $\sqrt{2/3}$ and 1. The unique copula of minimal norm is the product copula P, whereas the copulas of maximal norm are precisely those which are invertible with respect to the $*$-product. In between, the copulas which are left or right invertible are contained in the shell of radii $\sqrt{5/6}$ and 1; however, it is an open question whether the lower bound is sharp. Furthermore,

$\mathfrak{C}_2$ is contained in the affine hyperplane through P perpendicular to P. Since copulas satisfy certain boundary conditions (see Section 1), any ray in the vector space span($\mathfrak{C}_n$) emanating from the origin intersects $\mathfrak{C}_2$ in at most one point. Finally, we point out that, in view of the discussion at the end of Section 2, the above results also have a probabilistic counterpart.

5 The Sobolev topology of $\mathfrak{C}_n$

We conclude the paper with some results concerning the topology induced by the Sobolev norm $\|\ \|$ on span($\mathfrak{C}_n$). Recall from (11) that the standard Sobolev norm $\|\ \|_{W^{1,2}}$ is given by

$$\|f\|_{W^{1,2}}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 = \|f\|_{L^2}^2 + \|f\|^2.$$

Proposition 21 *The norms $\|\ \|$ and $\|\ \|_{W^{1,2}}$ are equivalent on span($\mathfrak{C}_n$).*

PROOF. Let f be a function in span($\mathfrak{C}_n$). Trivially, we have $\|f\| \leq \|f\|_{W^{1,2}}$. In order to prove the proposition, we will show that

$$\|f\|_{L^2}^2 \leq \frac{1}{2}\|\nabla f\|_{L^2}^2 = \frac{1}{2}\|f\|^2 \tag{24}$$

which is a version of Poincaré's inequality, yielding $\|f\|_{W^{1,2}}^2 \leq 3/2\,\|f\|^2$. Indeed, in view of Remark 2, we can write

$$|f(x)| = \left| \int_0^{x_n} \frac{\partial f}{\partial x_n}(x',t)\,dt \right| \leq x_n^{1/2} \left(\int_0^{x_n} \left| \frac{\partial f}{\partial x_n}(x',t) \right|^2 dt \right)^{1/2}$$

by Hölder's inequality where $x = (x', x_n) \in I^n = I^{n-1} \times I$. Therefore,

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int_{I^n} |f(x)|^2\,dx \\
&\leq \int_{I^n} x_n \Big(\int_0^1 \Big| \frac{\partial f}{\partial x_n}(x',t) \Big|^2 dt \Big) dx \\
&\leq \int_{I^{n-1}} \Big(\int_0^1 x_n\,dx_n \int_0^1 |\nabla f(x',t)|^2\,dt \Big) dx' \\
&= \frac{1}{2} \int_{I^n} |\nabla f(x)|^2\,dx \\
&= \frac{1}{2} \|\nabla f\|_{L^2}^2.
\end{aligned}$$

This proves (24) and, hence, the proposition. $\square$

Recall from Theorem 5 that any 2-copula can be L^∞-approximated by an invertible copula. Probabilistically, this implies that, in the L^∞-topology, in-

vertible functional dependence cannot be distinguished from any other type of stochastic dependence, including independence. The next result shows that the Sobolev norm resolves this somewhat paradoxical phenomenon.

Theorem 22 *If $(C_k)_{k \in \mathbb{N}}$ is a sequence of left invertible copulas in $\mathfrak{C}_2$ with*

$$\lim_{k \to \infty} \|C_k - C\| = 0$$

for some $C \in span(\mathfrak{C}_2)$, then C is in $\mathfrak{C}_2$ and is left invertible. Analogous statements hold for right invertible and invertible copulas.

PROOF. By Remark 9, the Sobolev limit of a sequence of copulas is a copula and the $*$-product on $\mathfrak{C}_2$ is (jointly) continuous with respect to d. Moreover, $\lim_{k \to \infty} \|C_k - C\| = 0$ implies $\lim_{k \to \infty} \|C_k^\top - C^\top\| = 0$. Thus, if each C_k is left invertible, then

$$C^\top * C = \lim_{k \to \infty} C_k^\top * \lim_{k \to \infty} C_k = \lim_{k \to \infty} (C_k^\top * C_k) = C^+,$$

which proves that C is left invertible. The case where each C_k is right invertible is shown analogously. Thus, it follows that the Sobolev limit of a sequence of invertible copulas is again an invertible copula. $\square$

Remark 23 *Note that the above proof does not really refer to the Sobolev norm. In fact, the statement of Theorem 22 holds for any metric on $\mathfrak{C}_2$ under which $\mathfrak{C}_2$ is closed and the $*$-product, as well as the transposition map, are continuous.*

Theorem 24 *On each $\mathfrak{C}_n$, Sobolev convergence implies L^∞-convergence; however, the converse is not true.*

PROOF. In view of Poincaré's inequality (24), Sobolev convergence implies L^2-convergence on each $\mathfrak{C}_n$. Since, by (2), all functions in $\mathfrak{C}_n$ are Lipschitz continuous with Lipschitz constant 1, it is readily verified that L^2-convergence implies L^∞-convergence.

For the proof that the converse implication is not true let $(A_k)_{k \in \mathbb{N}}$ be a sequence of invertible copulas in $\mathfrak{C}_2$ with $\lim_{k \to \infty} \|A_k - P\|_{L^\infty} = 0$; by Theorem 5, $(A_k)_{k \in \mathbb{N}}$ exists. (Note that here and in the sequel the dimension of the product copula P corresponds to the dimension of the respective sequence.) On the other hand, by (20) and Theorem 18 (iii), we have

$$\|A_k - P\|^2 = \frac{1}{3} \tag{25}$$

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for all k, so $\lim_{k\to\infty} \|A_k - P\| = 1/\sqrt{3} \neq 0$. In higher dimensions, define for each k a function $B_k : I^n \to I$ by $B_k(x_1, \ldots, x_n) = A_k(x_1, x_2)x_3 \ldots x_n$. Then $B_k \in \mathfrak{C}_n$ (see [9, Thm. 3.5.3]) and $\lim_{k\to\infty} \|B_k - P\|_{L^\infty} = 0$ since for each k

$$\|B_k - P\|_{L^\infty} = \sup_{x_1,\ldots,x_n \in I} |A_k(x_1, x_2) - x_1 x_2| x_3 \ldots x_n \leq \|A_k - P\|_{L^\infty}.$$

However, we show that $(B_k)_{k \in \mathbb{N}}$ does not converge to P with respect to the Sobolev norm $\| \|$. Indeed, an easy calculation yields

$$\|B_k - P\|^2 = \sum_{i=1}^{2} \int_{I^n} (\partial_i B_k - \partial_i P)^2 \, d\lambda + \sum_{j=3}^{n} \int_{I^n} (\partial_j B_k - \partial_j P)^2 \, d\lambda \qquad (26)$$

$$= \frac{1}{3^{n-2}} \|A_k - P\|^2 + \frac{n-2}{3^{n-3}} \|A_k - P\|_{L^2}^2. \qquad (27)$$

Now observe that for any L^∞-function f on I^n with $\|f\|_{L^\infty} \leq 1$ we have

$$\|f\|_{L^p}^p = \int_{I^n} |f|^p \, d\lambda \leq \int_{I^n} \|f\|_{L^\infty}^p \, d\lambda \leq \int_{I^n} \|f\|_{L^\infty} \, d\lambda = \|f\|_{L^\infty}.$$

Therefore, $\|A_k - P\|_{L^2}^2 \leq \|A_k - P\|_{L^\infty}$, which shows that the second term in (27) goes to zero. By (25), we conclude that $\lim_{k\to\infty} \|B_k - P\|^2 = 1/3^{n-1} \neq 0$, which completes the proof. $\square$

Acknowledgements

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