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A measure of mutual complete dependence

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Abstract

Two random variables X and Y are mutually completely dependent (m.c.d.) if there is a measurable bijection f with $P(Y = f(X)) = 1$. For continuous X and Y, a natural approach to constructing a measure of dependence is via the distance between the copula of X and Y and the independence copula. We show that this approach depends crucially on the choice of the distance function. For example, the L^p-distances, suggested by Schweizer and Wolff, cannot generate a measure of (mutual complete) dependence, since every copula is the uniform limit of copulas linking m.c.d. variables.

Instead, we propose to use a modified Sobolev norm, with respect to which, mutual complete dependence cannot approximate any other kind of dependence. This Sobolev norm yields the first nonparametric measure of dependence capturing precisely the two extremes of dependence, i.e., it equals 0 if and only if X and Y are independent, and 1 if and only if X and Y are m.c.d.

Key words: Measure of dependence, Mutual complete dependence, Copula, Sobolev norm

1991 MSC: primary 62E10;, secondary 62H20.

1 Introduction

Let X and Y be two random variables on a common probability space. Y is defined [8] to be completely dependent on X if there exists a Borel measurable

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function f such that

$$P(Y = f(X)) = 1. \tag{1}$$

X and Y are called *mutually completely dependent* (m.c.d.) if Y is completely dependent on X, and X is completely dependent on Y. In other words, X and Y are m.c.d. if and only if there is a Borel measurable bijection f satisfying (1).

Stochastic independence and mutual complete dependence are exactly opposite in character. The former case entails complete unpredictability of either random variable from the other, whereas the latter corresponds to complete predictability. Therefore, we claim that a measure of dependence for X and Y should measure the strength of mutual complete dependence with extreme values satisfying the following: (i) the measure equals 0 if and only if X and Y are independent, and (ii) it equals 1 if and only if X and Y are m.c.d.

On analyzing the implications of these requirements, several things become apparent. In particular, choosing $[0, 1]$, as the range of values, suggests that such indices aim at measuring the degree of functional relationship between two random variables rather than their concordance. The latter property can be assessed by a measure of concordance, e.g., Spearman's ρ and Kendall's τ. This concept was developed by Scarsini [13] and discussed, e.g., in [6, 11].

Condition (i) hints at the construction of a measure of dependence, in terms of a metrical distance or, in a broad sense, dissimilarity between the joint distribution and the distribution representing independence. Indeed, there is an extensive literature on measures of dependence based on this idea. We refer to [4, 5, 9, 12, 18] and the references therein for an exhaustive list. If X and Y have continuous marginal distributions functions F_X and F_Y, respectively, and joint distribution function $F_{X,Y}$, the approach can also be applied to the joint distribution of the probability integral transformations $F_X(X)$ and $F_Y(Y)$, given by the copula of X and Y. In this case, by Sklar's theorem [15], there exists a unique copula C such that

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)). \tag{2}$$

This modified approach for the construction of a measure of dependence, introduced by Schweizer and Wolff [14] (see also [3]), has two important advantages. First, the measure is independent of the type of F_X and F_Y since, by Sklar's theorem, the joint distribution can be decomposed into the marginal distributions and the dependence structure, represented by the copula. Second, as shown in [14], it is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strictly increasing transformations of X and Y. Since X and Y are independent if and only if their connecting copula is the product copula $P(x, y) = xy$, Schweizer and Wolff [14] argued that *any* suitably normalized distance between C and P, in particular, any L^p-distance, should yield a symmetric nonparametric measure of dependence. Specifically, they studied the L^1-distance given by

$$\sigma(X,Y) = 12 \int_{I^2} |C - P| \, d\lambda \tag{3}$$

where $I = [0,1]$ denotes the closed unit interval and λ the two-dimensional Lebesgue measure.

It should be noted, however, that both construction methods, described above, yield, in general, a measure of independence only, since the measure always satisfy (i), but not necessarily (ii). In other words, while any distance guarantees that, at its lower bound, such a measure can capture independence in the variables, the type of the 'highest' dependence, detected at the upper bound, depends heavily on the type of the distance function employed. It follows that the choice of the distance function cannot be arbitrary, but is predetermined by the desired properties of a measure of dependence.

As mentioned in the beginning, we argue that mutual complete dependence is the opposite of stochastic independence and therefore, a measure of dependence should take this into account by satisfying (ii). This requirement is much stronger than both Renyi's original postulate [12] and its modification by Schweizer and Wolff [14]. In fact, there is a lack of measures satisfying (ii), probably due to the fact that mutual complete dependence seems incompatible with the concept of convergence in distribution. In particular, it has been shown in [10] (see also [7, 16, 17]) that the joint distribution of any two continuously distributed random variables X and Y can be approximated uniformly by the joint distribution of a pair of m.c.d. random variables, identically distributed as X and Y. This counterintuitive phenomenon led to the concept of monotone dependence, which corresponds to mutual complete dependence when in (1) the class of Borel measurable bijections f is restricted to a.s. monotone ones. Kimeldorf and Sampson [7] argued that monotone dependence could be interpreted as the opposite of stochastic independence because it is preserved under convergence in law.

For instance, Schweizer and Wolff's $\sigma(X,Y)$, defined in (3) is a measure of monotone dependence because it attains its maximum of 1 if and only if X and Y are monotone dependent. This is easily seen since any copula lies (pointwise) between the lower and upper Fréchet-Hoeffding bounds, which are copulas themselves and correspond precisely to monotone decreasing and increasing dependence, respectively [14]. However, if X and Y are m.c.d., $\sigma(X,Y)$ can attain any value in $(0,1]$. This follows from the above mentioned phenomenon, which implies that the set of copulas linking m.c.d. random variables is dense in the set of all copulas with respect to the L^∞-distance [10], and, since copulas are continuous functions, with respect to any L^p-distance, $p \geq 1$. In other words, none of the L^p-distances is capable of detecting mutual complete dependence, which emphasizes again that the choice of the metrical distance function used in the construction of a measure of dependence is crucial for its resulting properties.

We argue that the inconsistency between mutual complete dependence and the L^p-distance neither weakens the concept of mutual complete dependence as the opposite of independence, nor does it imply that a measure of dependence should be restricted to monotone dependence. It rather suggests that convergence in law, or, alternatively, uniform convergence of the corresponding copulas, is an inappropriate concept for the construction of measures of dependence.

Instead of the L^p-norm, we propose to measure the distance between two copulas by a modified Sobolev norm given by

$$\|C\| = \left(\int_{I^2} |\nabla C|^2 \, d\lambda \right)^{1/2} \tag{4}$$

where ∇ denotes the gradient of the copula. This norm derives from a scalar product which, among other things, allows a straightforward representation via the $*$-product for copulas, introduced by Darsow et al. [1]. Furthermore, this Sobolev norm turns out extremely advantageous since the degree of dependence between two continuous random variables X and Y, and, in particular, mutual complete dependence, can be determined by analytical properties of their copula. It follows that, in contrast to the L^p-distance, with respect to the Sobolev norm, mutual complete dependence cannot approximate any other kind of stochastic dependence.

Using this Sobolev norm we define a new nonparametric measure of dependence for two continuous random variables X and Y with copula C, given by

$$\omega(X,Y) = \left(3\|C\|^2 - 2\right)^{1/2} = \sqrt{3}\|C - P\| \,, \tag{5}$$

which represents the normalized Sobolev distance between C and the independence copula P. We show that $\omega(X,Y)$ has several appealing properties, e.g., its extremes are precisely at independence and mutual complete dependence.

The paper is organized as follows. Section 2 sets up the notation and briefly reviews some fundamental properties of copulas. Section 3 introduces the Sobolev scalar product for copulas and its corresponding norm and distance. We show that the scalar product allows a representation via the $*$-product. In Section 4 we turn to the statistical interpretation of the Sobolev norm for copulas, which leads naturally to a new nonparametric measure of dependence for two continuous random variables. Examples and comparisons are presented in Section 5.

2 Basic properties of copulas

Let $I = [0,1]$ be the closed unit interval and $I^2 = [0,1] \times [0,1]$ the closed unit square.

Definition 1 *A two-dimensional copula (or briefly, a copula) is a function* $C : I^2 \to I$ *satisfying the conditions:*

(i) $C(x,0) = C(0,y) = 0$ *for all* $x, y \in I$.
(ii) $C(x,1) = x$ *and* $C(1,y) = y$ *for all* $x, y \in I$.
(iii) $C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geq 0$ *for all rectangles* $[x_1,x_2] \times [y_1,y_2] \subset I^2$.

Let $\mathfrak{C}$ denote the set of all (two-dimensional) copulas. Denote by $\partial_i C$ the partial derivative of $C \in \mathfrak{C}$ with respect to the i-th variable. The conditions in Definition 1 imply the following key properties of copulas; for a proof see, e.g., [11].

Proposition 2 *(i)* C *is increasing in each argument.*
(ii) C *is Lipschitz (and hence uniformly) continuous.*
(iii) For $i \in \{1,2\}$, $\partial_i C$ *exists a.e. on* I^2 *with* $0 \leq \partial_i C(x,y) \leq 1$.
(iv) The functions $t \mapsto \partial_1 C(x,t)$ *and* $t \mapsto \partial_2 C(t,y)$ *are defined and increasing a.e. on* I.

There are three distinguished copulas, namely

$$\begin{aligned} C^-(x,y) &= \max(x+y-1,0), \\ C^+(x,y) &= \min(x,y), \\ P(x,y) &= xy\,. \end{aligned}$$

C^+ and C^- are called the Fréchet-Hoeffding upper and lower bound, respectively, since for any copula C and any $(x,y) \in I^2$ we have

$$C^-(x,y) \leq C(x,y) \leq C^+(x,y)\,. \tag{6}$$

The set $\mathfrak{C}$ can be equipped with the $*$-multiplication [1]. Let $A, B \in \mathfrak{C}$. For any $x, y \in I$, set

$$(A * B)(x,y) = \int_0^1 \partial_2 A(x,t)\, \partial_1 B(t,y)\, dt. \tag{7}$$

Theorem 3 $A * B$ *is in* $\mathfrak{C}$.

C^+ and P are the unit and null element, respectively, i.e., for any copula C, we have

$$C^+ * C = C * C^+ = C, \tag{8}$$
$$P * C = C * P = P. \tag{9}$$

Denote by $C^\top$ the transposed copula of C given by

$$C^\top(x, y) = C(y, x). \tag{10}$$

C is called symmetric if $C = C^\top$. It is easy to see that for any $A, B \in \mathfrak{C}$

$$(A * B)^\top = B^\top * A^\top. \tag{11}$$

3 The Sobolev scalar product for copulas

We denote by $\cdot$ the Euclidean scalar product, by $|\ |$ the Euclidean norm on $\mathbb{R}^2$, and by λ the 2-dimensional Lebesgue measure.

It follows immediately from Proposition 2 (iii), and has been noticed in [2], that

$$\mathfrak{C} \subset W^{1,p}(I^2, \mathbb{R})$$

for every $p \in [1, \infty]$ where $W^{1,p}(I^2, \mathbb{R})$ is the standard Sobolev space. However, it has not been exploited in this context that $W^{1,2}(I^2, \mathbb{R})$ is a Hilbert space with respect to the usual $W^{1,2}$-scalar product

$$\langle f, g \rangle_{W^{1,2}} = \int_{I^2} fg \, d\lambda + \int_{I^2} \nabla f \cdot \nabla g \, d\lambda \tag{12}$$

so that the set of copulas, $\mathfrak{C}$, comes equipped with a scalar product structure.

There is, however, an even simpler way to define a scalar product for copulas. Since copulas are continuous and satisfy $C(0, 0) = 0$ we can actually forgo the first term. Indeed,

$$\langle f, g \rangle = \int_{I^2} \nabla f \cdot \nabla g \, d\lambda \tag{13}$$

defines a scalar product on the subspace

$$W_0^{1,2}(I^2, \mathbb{R}) = \{f \in W^{1,2}(I^2, \mathbb{R}) \cap C(I^2, \mathbb{R}) \mid f(0, 0) = 0\}$$

which contains $\mathfrak{C}$. The restriction of $\langle\ ,\ \rangle$ to $\mathfrak{C}$ will be called the *Sobolev scalar product* for copulas. As usual, we define the corresponding Sobolev norm on $\mathfrak{C}$ by

$$\|C\| = \left(\int_{I^2} |\nabla C|^2 \, d\lambda \right)^{1/2} \tag{14}$$

and the Sobolev distance function on $\mathfrak{C} \times \mathfrak{C}$ by

$$d(A, B) = \Big(\int_{I^2} |\nabla A - \nabla B|^2 \, d\lambda \Big)^{1/2}. \tag{15}$$

We have seen that the Sobolev scalar product for copulas appears very naturally from an analytical point of view. However, it also allows a representation via the $*$-product, defined in (7).

Theorem 4 *For all $A, B \in \mathfrak{C}$ we have the identity*

$$\begin{aligned} \langle A, B \rangle &= \int_0^1 (A^\top * B + A * B^\top)(t,t) \, dt \\ &= \int_0^1 (A^\top * B + B * A^\top)(t,t) \, dt. \end{aligned}$$

PROOF. It follows from (10) that

$$\begin{aligned} \partial_1 A^\top(x,y) &= \partial_2 A(y,x) \\ \partial_2 A^\top(x,y) &= \partial_1 A(y,x) \end{aligned} \tag{16}$$

Using (7) and (16) we can write

$$\begin{aligned} \int_0^1 \int_0^1 \partial_1 A(x,y) \, \partial_1 B(x,y) \, dx \, dy &= \int_0^1 \Big(\int_0^1 \partial_2 A^\top(y,x) \, \partial_1 B(x,y) \, dx \Big) dy \\ &= \int_0^1 (A^\top * B)(y,y) \, dy \\ \int_0^1 \int_0^1 \partial_2 A(x,y) \, \partial_2 B(x,y) \, dx \, dy &= \int_0^1 \Big(\int_0^1 \partial_2 A(x,y) \, \partial_1 B^\top(y,x) \, dy \Big) dx \\ &= \int_0^1 (A * B^\top)(x,x) \, dx. \end{aligned}$$

Adding up both terms we obtain the first identity.

The second equation in Theorem 4 is equivalent to

$$\int_0^1 (A * B^\top)(t,t) \, dt = \int_0^1 (B * A^\top)(t,t) \, dt$$

which follows from $(A * B^\top)(t,t) = (A * B^\top)^\top(t,t) = (B * A^\top)(t,t)$, where we have used (11). □

The representation in Theorem 4 becomes particularly simple for symmetric copulas.

Corollary 5 *If $A, B \in \mathfrak{C}$ are symmetric, then*

$$\langle A, B \rangle = 2 \int_0^1 (A * B)(t,t)\, dt.$$

Theorem 4 yields upper and lower bounds for the scalar product of two copulas. More precisely, we have the following result.

Theorem 6 *Let $A, B \in \mathfrak{C}$. Then*

$$\frac{1}{2} \leq \langle A, B \rangle \leq 1,$$

where both bounds are sharp.

PROOF. Theorem 4, in connection with the bounds for copulas given in (6), implies that

$$2 \int_0^1 C^-(t,t)\, dt \leq \langle A, B \rangle \leq 2 \int_0^1 C^+(t,t)\, dt.$$

Simple calculations yield $\int_0^1 C^-(t,t)\, dt = 1/4$ and $\int_0^1 C^+(t,t)\, dt = 1/2$.

Finally, one easily computes that

$$\begin{aligned} \langle C^-, C^- \rangle &= \langle C^+, C^+ \rangle = 1 \\ \langle C^-, C^+ \rangle &= \frac{1}{2}\,. \end{aligned} \tag{17}$$

This shows that the bounds in the statement are sharp, and the proof is complete. □

Theorem 7 *For all $C \in \mathfrak{C}$, the following hold:*

(i) $\langle C, P \rangle = 2/3$.
(ii) $\|C - P\|^2 = \|C\|^2 - 2/3$.
(iii) $2/3 \leq \|C\|^2 \leq 1$.

PROOF. For (i), we remark that $P = P^\top$, so Theorem 4 and (9) imply

$$\langle P, C \rangle = \int_0^1 (P * C + C * P)(t,t)\, dt = 2 \int_0^1 P(t,t)\, dt = \frac{2}{3}\,.$$

This, in turn, proves (ii) because

$$\|C - P\|^2 = \|C\|^2 - 2\langle C, P \rangle + \|P\|^2 = \|C\|^2 - \frac{2}{3}.$$

Finally, (iii) is a consequence of (ii) and Theorem 6. □

4 A nonparametric measure of dependence

We now turn to the statistical interpretation of the Sobolev norm for copulas and the construction of a new nonparametric measure of dependence for two continuous random variables.

Lemma 8 ([1]) *Let X and Y be continuous random variables with copula C. The following statements are equivalent:*

(i) Y is completely dependent on X if and only if $\partial_1 C \in \{0,1\}$ a.e.
(ii) X is completely dependent on Y if and only if $\partial_2 C \in \{0,1\}$ a.e.
(iii) X and Y are m.c.d. if and only if $\partial_1 C, \partial_2 C \in \{0,1\}$ a.e.

PROOF. Darsow et al. [1, Theorem 7.1,Theorem 11.1] prove that Y is completely dependent on X if and only if, for each $y \in I$, one has $\partial_1 C(\cdot, y) \in \{0,1\}$ a.e. Actually, the proof shows that this is tantamount to assuming that $\partial_1 C(x,y) \in \{0,1\}$ a.e. This proves (i). Analogous statements hold for (ii), from which (iii) follows. □

The next theorem describes the main results of this paper.

Theorem 9 *Let X and Y be continuous random variables with copula C. The Sobolev norm for copulas satisfies $\|C\|^2 \in [2/3, 1]$, for all $C \in \mathfrak{C}$. Moreover, the following assertions hold, where the bounds are sharp:*

(i) $\|C\|^2 = 2/3$ if and only if X and Y are independent.
(ii) $\|C\|^2 \in [5/6, 1]$ if Y is completely dependent on X (or vice versa).
(iii) $\|C\|^2 = 1$ if and only if X and Y are m.c.d.

This result, together with the identity $\|C - P\|^2 = \|C\|^2 - 2/3$, expresses the astonishing fact that the Sobolev norm itself measures stochastic dependence, with extremes exactly at independence and mutual complete dependence. In addition, the Sobolev norm is able to detect that two random variables are not completely dependent.

PROOF. The foremost statement is contained in Theorem 7 (iii).

Assertion (i) is an immediate consequence of Theorem 7(ii). It follows from (14) that

$$\|C\|^2 = \int_0^1 \int_0^1 (\partial_1 C(x,y))^2 \, dx \, dy + \int_0^1 \int_0^1 (\partial_2 C(x,y))^2 \, dx \, dy. \tag{18}$$

If Y is completely dependent on X we know from Lemma 8 that $(\partial_1 C)^2 = \partial_1 C$ a.e., so the first summand in (18) is equal to

$$\int_0^1 \int_0^1 \partial_1 C(x,y)\,dx\,dy = \int_0^1 y\,dy = \frac{1}{2}\,. \tag{19}$$

To estimate the second term in (18) consider the inequality

$$\begin{aligned}
&\int_0^1 \int_0^1 (\partial_2 C(x,y) - x)^2\,dx\,dy \geq 0 \\
\Leftrightarrow &\int_0^1 \int_0^1 (\partial_2 C(x,y))^2\,dx\,dy - 2\int_0^1 \int_0^1 \partial_2 C(x,y)\,x\,dx\,dy + \int_0^1 \int_0^1 x^2\,dx\,dy \geq 0 \\
\Leftrightarrow &\int_0^1 \int_0^1 (\partial_2 C(x,y))^2\,dx\,dy \geq \frac{1}{3}\,.
\end{aligned}$$

To show that the bound is sharp set $C = P$. The case when X is completely dependent on Y can be shown analogously, which proves (ii).

Finally, in view of Proposition 2 (iii), we have $(\partial_i C)^2 \leq \partial_i C$ with equality if, and only if, $\partial_i C \in \{0,1\}$. Consequently, (18) implies that

$$\|C\|^2 \leq \int_0^1 \int_0^1 \partial_1 C(x,y)\,dx\,dy + \int_0^1 \int_0^1 \partial_2 C(x,y)\,dx\,dy = \frac{1}{2} + \frac{1}{2} = 1$$

with equality if and only if $\partial_i C \in \{0,1\}$ a.e. By Lemma 8, the latter is equivalent to X and Y being m.c.d. □

Corollary 10 *Let X and Y be continuous random variables with copula C. The following are equivalent:*

(i) X *and* Y *are m.c.d.*
(ii) $\|C\| = 1$.
(iii) $\partial_1 C, \partial_2 C \in \{0,1\}$ *a.e.*
(iv) $\int_0^1 (C * C^\top + C^\top * C)(t,t)\,dt = 1$.

PROOF. This follows immediately from Lemma 8, Theorem 9 and Theorem 4. □

Corollary 11 *Let (X,Y) and $\{(X_n,Y_n)\}$ be a pair and a sequence of pairs of continuous random variables, respectively, with respective copulas C and C_n. Then the following assertions hold:*

(i) If, for almost all n, X_n and Y_n are m.c.d. and $\lim_{n\to\infty} \|C_n - C\| = 0$, then X and Y are m.c.d.
(ii) If, for almost all n, X_n is completely dependent on Y_n, or Y_n on X_n, and $\lim_{n\to\infty} \|C_n - C\| = 0$, then X and Y are not independent, i.e., $C \neq P$.

PROOF. Part (i) is an immediate consequence of Theorem 9 since $\|C_n\| = 1$ and $\lim_{n\to\infty} \|C_n - C\| = 0$ implies $\|C\| = 1$. An analogous argument proves (ii). □

Corollary 11 emphasizes the advantage of the Sobolev distance over the L^p-distances, as mentioned in the Introduction. While, in the uniform sense, any copula, in particular, the independence copula P, can be approximated by copulas of m.c.d. random variables, the Sobolev convergence preserves the property of mutual complete dependence. Hence, with respect to the Sobolev distance, mutual complete dependence cannot approximate any other kind of stochastic dependence. In fact, independence cannot even be approximated by completely dependent random variables.

Therefore, measuring the distance between copulas with the Sobolev norm resolves the disturbing phenomenon observed in [7, 10].

These remarkable statistical properties of the Sobolev norm lead immediately to the following definition.

Definition 12 *Given two continuous random variables X, Y with copula C, we define*

$$\omega(X,Y) = \left(3\|C\|^2 - 2\right)^{1/2}. \tag{20}$$

In view of Theorem 7, the quantity $\omega(X,Y)$ represents a normalized Sobolev distance of C from the independence copula P:

$$\omega(X,Y) = \sqrt{3}\,\|C - P\| = \frac{\|C - P\|}{\|\widehat{C} - P\|}, \tag{21}$$

where $\widehat{C}$ is any copula of m.c.d. variables. The normalization guarantees that $\omega(X,Y) \in [0,1]$. Definition 12, however, makes clear that the Sobolev norm of C itself serves as a measure of dependence.

For symmetric C we may use Corollary 5 to write

$$\omega(X,Y) = \left(6\int_0^1 (C * C)(t,t)\,dt - 2\right)^{1/2}. \tag{22}$$

Theorem 13 *Let X and Y be continuous random variables with copula C. The quantity $\omega(X,Y)$ given by* (20) *has the following properties:*

(i) $\omega(X,Y)$ *is defined for any X and Y.*
(ii) $\omega(X,Y) = \omega(Y,X)$.
(iii) $0 \le \omega(X,Y) \le 1$.
(iv) $\omega(X,Y) = 0$ *if, and only if, X and Y are independent.*
(v) $\omega(X,Y) = 1$ *if, and only if, X and Y are m.c.d.*

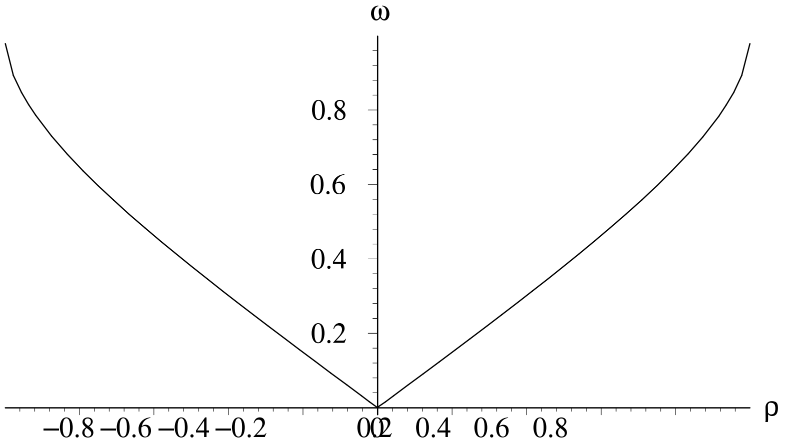

Fig. 1. $\omega(X, Y)$ as a function of ρ for jointly normal X, Y

(vi) $\omega(X, Y) \in [\sqrt{0.5}, 1]$ *if* Y *is completely dependent on* X *(or vice versa).*
(vii) *If* f *and* g *are a.s. strictly monotone functions on Range(*X*) and Range(*Y*), respectively, then* $\omega(f(X), g(Y)) = \omega(X, Y)$.
(viii) *If* $\{(X_n, Y_n)\}$ *is a sequence of pairs of continuous random variables with copulas* C_n, *and if* $\lim_{n\to\infty} \|C_n - C\| = 0$, *then* $\lim_{n\to\infty} \omega(X_n, Y_n) = \omega(X, Y)$.

PROOF. Everything is obvious by definition, or follows from Theorem 9, except for (vii). Here we distinguish four different cases. For the sake of clarity, let $C_{X,Y}$ denote the copula of X and Y.

If both f and g are increasing it is well known [11, Theorem 2.4.3] that $C_{f(X),g(Y)} = C_{X,Y}$ which implies $\omega(f(X), g(Y)) = \sqrt{3}\,\|C_{f(X),g(Y)} - P\| = \sqrt{3}\,\|C_{X,Y} - P\| = \omega(X, Y)$.

If f is increasing and g is decreasing then $C_{f(X),g(Y)}(x, y) = x - C_{X,Y}(x, 1-y)$; see [11, Theorem 2.4.4]. Therefore $(C_{f(X),g(Y)} - P)(x, y) = (P - C_{X,Y})(x, 1-y)$ which, by the transformation formula for the Lebesgue measure, again implies $\omega(f(X), g(Y)) = \omega(X, Y)$. If f is decreasing and g is increasing, the result follows from interchanging f and g in the previous case.

The case when f and g are both decreasing can be shown similarly. □

Remark 14 *If* X *and* Y *are jointly normal with correlation coefficient* ρ, *then* $\omega(X, Y)$ *is a strictly increasing function of* $|\rho|$ *whose graph is shown in Figure 1.*

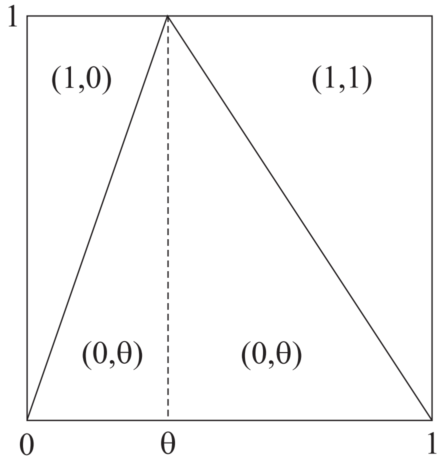

Fig. 2. The gradient ∇C of the copula C in Example 15

5 Examples and comparisons

We conclude the paper with some examples clarifying the relationship between the measure of dependence $\omega(X,Y)$ and the quantity $\sigma(X,Y)$, as defined in (3). We assume that the reader is familiar with the concept of singular copulas; for details we refer to [11].

Example 15 *Let $\theta \in [0,1]$, and consider the singular copula C whose support consists of two line segments in I^2, one joining $(0,0)$ and $(\theta,1)$, and the other joining $(\theta,1)$ and $(1,0)$ (see [11, Example 3.3]). It follows that*

$$C(x,y) = \begin{cases} x & \text{if } x \le \theta y \\ \theta y & \text{if } \theta y < x < 1-(1-\theta)y \\ x+y-1 & \text{if } 1-(1-\theta)y \le x. \end{cases}$$

Clearly, Y is completely dependent on X, but not vice versa. Since probability mass θ and $1-\theta$ is uniformly distributed on the first and second line segments, respectively, it is heuristically clear that the value $\theta = 1/2$ describes the least dependent situation, whereas the limiting cases $\theta = 0$ and $\theta = 1$, when $C = C^-$ and $C = C^+$, respectively, correspond to mutual complete dependence.

This is perfectly reflected in the behavior of $\omega(X,Y)$. Indeed, a straightforward calculation (compare Fig. 2) shows that

$$\|C\|^2 = \frac{1}{2}\left(\theta - \frac{1}{2}\right)^2 + \frac{7}{8} \in \left[\frac{7}{8}, 1\right]$$

with the lowest and highest values attained precisely for $\theta = 1/2$ and $\theta \in \{0,1\}$, respectively. Consequently, $\omega(X,Y)$ takes on its smallest value $\sqrt{10}/4 \approx .79$ for $\theta = 1/2$.

The quantity $\sigma(X,Y)$ shows the same qualitative behavior, however, its minimal value is $.5$.

Example 16 *Let $\theta \in [0,1]$, and consider the singular copula C whose support consists of the two segments $\{(x,1-x) \mid x \in [0,\theta] \cup [1-\theta,1]\}$ and the segment $\{(x,x) \mid x \in [\theta,1-\theta]\}$ (see [11, Exercise 3.15]). It follows that*

$$C(x,y) = \begin{cases} C^+(x,y) - \theta & \textit{if } (x,y) \in [\theta,1-\theta]^2 \\ C^-(x,y) & \textit{otherwise} \end{cases}$$

Now X and Y are mutually completely dependent so $\omega(X,Y) = 1$, regardless of the value of θ.

In contrast, $\sigma(X,Y)$ varies between 1 (for $\theta \in \{0,1\}$) and values around $.46$ (for $\theta \approx .12$), indicating a definite degree of independence when, actually, there is none. Note that the copula from Example 15 with $\theta = 1/2$ yields almost the same value for σ.

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