Regularization of outflow problems in unsaturated porous media with dry regions

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Abstract: We study a porous medium with saturated, unsaturated, and dry regions, described by Richards' equation for the saturation s and the pressure p. Due to a degenerate permeability coefficient $k(x, s)$ and a degenerate capillary pressure function $p_c(x, s)$, the equations may be of elliptic, parabolic, or of ODE-type. We construct a parabolic regularization of the equations and find conditions that guarantee the convergence of the parabolic solutions to a solution of the degenerate system. An example shows that the convergence fails in general. Our approach provides an existence result for the outflow problem in the case of x-dependent coefficients and a method for a numerical approximation.

1 Introduction

We study the motion of fluids in porous materials, e.g. the flow of water in soil or in artificial porous media. We are interested in the case that a second fluid, e.g. air, is present and that the two fluids do not mix. In this situation, water occupies one part of the pore space and air occupies the remaining pore space. Modelling the flow of both fluids leads to the two-phase flow equations, neglecting the motion of air by assuming a constant air pressure leads to the unsaturated flow or Richards' equation that we study here. To fix notations we denote the region occupied by the porous material by $\Omega \subset \mathbb{R}^n$ and describe the physical situation in the medium at a point $x \in \Omega$ at time $t \in [0, T)$ with two variables, the saturation $s(x, t)$ and the pressure $p(x, t)$. Here, $s : \Omega \times [0, T) \to [0, 1]$ is the volume fraction

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of pore space occupied by water, $p : \Omega \times [0,T) \to \mathbb{R}$ is the pressure of the water. One assumes that the (macroscopic) velocity v of the water is given by Darcy's law as $v(x,t) = -k(x,s(x,t))\nabla p(x,t)$ for some permeability $k(x,s)$, and that pressure and saturation are coupled through the capillary pressure as $p(x,t) = \tilde{p_c}(x,s(x,t))$. We recall that, once the maximal saturation $s = 1$ is achieved, also any higher pressure can be realized with the same saturation $s = 1$. We therefore regard p_c as the multi-valued graph with $p_c(s) = \{\tilde{p_c}(s)\}$ for $s < 1$, and $p_c(1) = [\tilde{p_c}(1), \infty)$. Normalizing physical coefficients as the density and assuming the incompressibility of water, the law of conservation of mass with sources f reads $\partial_t s + \operatorname{div} v = f$. Inserting from above, the problem takes the form

$$\partial_t s = \nabla \cdot (k(s)\nabla p) + f, \qquad p \in p_c(s). \tag{1.1}$$

In this equation we regard k and p_c as given coefficient functions, k non-negative and p_c monotone, and have thus, at least formally, a single evolution problem for s. The boundary conditions for the equation are described below. The first difficulty in the analytical treatment of (1.1) is that both coefficient functions are degenerate. We refer to Figure 1 for typical shapes of the coefficient functions, the graphs on the left correspond to a hydrophilic material, the graphs on the right to a material with hydrophilic and hydrophobic parts. The function $p_c(s)$ may as well remain finite at $s = a$ or have finite slope at $s = 1$. The number $a \geq 0$ is the residual saturation. Since the permeability vanishes below saturation $s = a$, flow processes are interrupted and no further water extraction is possible. We call a subset $\{x \in \Omega | s(x,t) < a\}$ a dry region at time t. We emphasize that, in this terminology, a dry region does contain water, but not enough to induce a positive permeability.

The second difficulty lies in the outflow boundary condition. In order to model the situation that the porous material is in contact with open space (occupied by air), one imposes boundary conditions in the form of variational inequalities. In the easiest case one imposes

$$v \cdot n \geq 0 \text{ and } p \leq 0 \text{ and } (v \cdot n) \cdot p = 0. \tag{1.2}$$

In words: (i) Water cannot enter, since outside there is no water (ii) the capillary pressure cannot be positive (iii) water can exit only if the capillary pressure is $p = 0$. For further details on the derivation of these equations we refer to [9]. Regarding the analytical treatment of (1.2) we note that the pressure is not defined in dry regions, i.e. in points (x,t) with $s(x,t) < a$. This fact already demands for a modification of the boundary condition. Furthermore, it is a difficult task to give sense to the product of traces in the last equality.

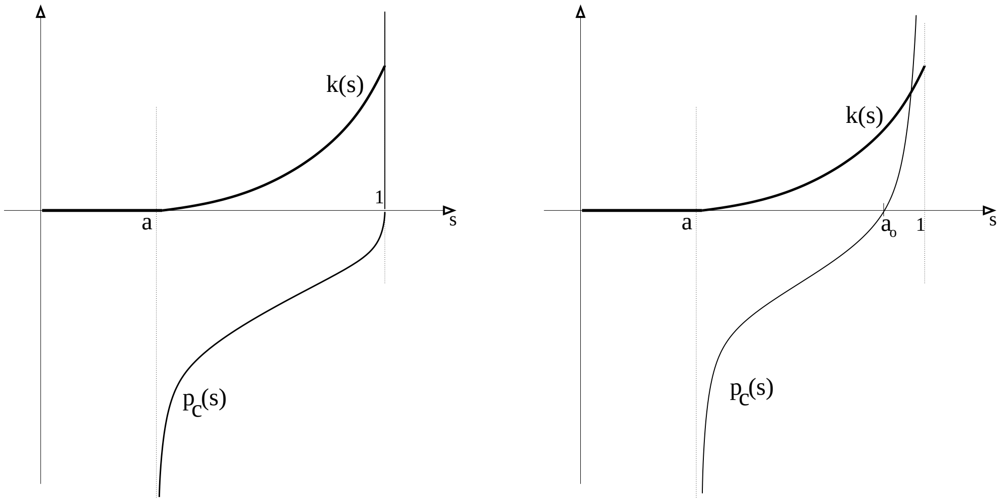

Figure 1: Possible shapes of the functions $p_c(s)$ and $k(s)$. On the left, a purely hydrophilic medium, on the right, a medium with hydrophilic and hydrophobic parts. On the left, p_c is multivalued, $p_c(1) = [0, \infty)$.

Outline of this contribution. In this article we analyze a regularization procedure for (1.1), (1.2) and derive the existence of weak solutions. We replace the coefficient functions k and p_c by smooth functions k_δ and ρ_δ with k_δ strictly positive and ρ_δ strictly increasing. Furthermore, we replace the outflow condition by a Dirichlet-to-Neumann condition. This defines a regularized problem which is a standard parabolic boundary value problem with a unique solution (s_δ, p_δ).

One checks easily that the approximation of the coefficients must confirm to certain conditions, for example that the convergence $k_\delta \to 0$ must be faster than $\rho_\delta \to -\infty$ on $(0, a)$, since otherwise the approximate solutions (s_δ, p_δ) will, in general, not converge to a solution of problem (1.1). We find conditions on the approximations which, instead, guarantee the convergence of the approximate solutions to a solution of the original problem. This, in particular, implies a new existence result for the doubly degenerate equation. We thus transfer known existence results to the case of x-dependent coefficient functions.

The most intricate part in the proof is the verification of the weak counterpart of the boundary condition (1.2). We use the method of compensated compactness to show that $(v \cdot n) \cdot p$ coincides with a non-negative defect measure. Due to the first two conditions of (1.2), this is formally equivalent with the equality.

1.1 Comparison with existing literature

Most articles in the field consider the case of x-independent coefficient functions k and p_c. This simplifies the system considerably since, after a suitable transformation of the problem, the elliptic operator becomes linear.

The global pressure. The Baiocchi transformation (or Kirchhoff transformation) introduces a global pressure function as

$$\tilde{\Phi}(s) := \int_0^{p_c(s)} k({p_c}^{-1}(q))\, dq, \quad \Phi(s) := \begin{cases} \{\tilde{\Phi}(s)\} & \text{if } s < 1, \\ [\tilde{\Phi}(1), \infty) & \text{if } s = 1. \end{cases}$$

In the case of x-independent coefficients $k = k(s)$ and $p_c = p_c(s)$, given a

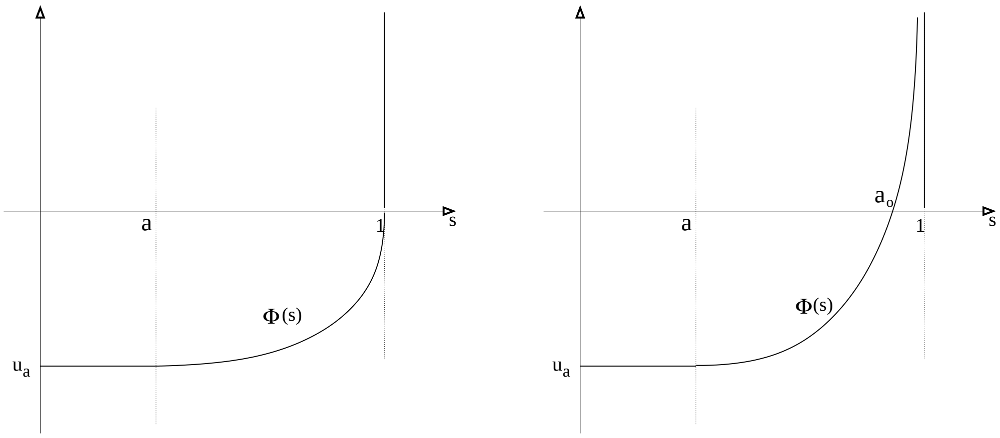

Figure 2: Possible shapes of Φ. On the left, Φ is multivalued, $\Phi(1) = [u_1, \infty)$.

sufficiently smooth solution (s, p) of (1.1), the global pressure

$$u(x,t) := \begin{cases} \tilde{\Phi}(s(x,t)) & \text{if } s(x,t) < 1, \\ \tilde{\Phi}(1) + k(1)(p(x,t) - \tilde{p_c}(1)) & \text{if } s(x,t) = 1, \end{cases}$$

satisfies $\nabla u = k(s)\nabla p$. Hence equation (1.1) now reads

$$\partial_t s = \Delta u + f, \qquad u \in \Phi(s). \tag{1.3}$$

Results for unsaturated porous media. A fundamental theorem that initiated much of the later research is due to Alt, Luckhaus, and Visintin

[3]. It provides the existence of a weak solution (s, u) of equation (1.3) with outflow boundary conditions on $\Omega \times [0, T)$, for $f = 0$. The result allows quite general coefficient functions, in particular, Φ can have vanishing slope on $(0, a)$ and can be multi-valued in $s = 1$. An approximate solution sequence is constructed with a time-discretization of (1.3) in which a variational inequality is solved in each time step. The proof of convergence of the approximate solution sequence exploits the idea of compensated compactness and uses a very weak solution concept. The authors were interested in applications to groundwater flow where no sources are present, hence $f = 0$.

More is known in the case without an outflow condition: In [5] dry regions are studied, the existence of solutions and the continuity of the free boundaries saturated/unsaturated and unsaturated/dry is shown. Many results are known on numerical approximations in this case (see [12] and references therein). All the above results regard x-independent coefficients.

The fundamental contribution of [11], obtained by adapting methods of [10], is a uniqueness result for the outflow problem. It provides the uniqueness of the weak solution in the sense of [3], and does not assume the additional regularity $\partial_t s \in L^1(\Omega_T)$. It is not necessarily applicable to our problem, since, e.g., $b = \Phi^{-1}$ can be extended to a continuous function on $\mathbb{R}$ only in the case $a = 0$. We nevertheless note that in our contribution $\partial_t s \notin L^1(\Omega_T)$ in general, and that, for x-independent coefficients, our solutions are also solutions in the sense of [3].

Results for two-phase flow. One of the first existence results appeared in [8]. The restriction of this result is that the initial saturation is assumed to be bounded away from the critical values, and it is exploited that this property remains valid for all times. Another existence result is that of [1], where the capillary pressure function is assumed to be non-singular. Both restrictions are removed in [6], where also x-dependent coefficients are studied. We note that in [6] the situation with $a = 0$ is studied and that the outflow condition is not included.

Our research was motivated also by questions of homogenization. We refer to [4] for some results concerning the averaging of two-phase flow equations and further references. Again, it must be assumed that the saturation is bounded away from the critical values.

1.2 Concepts and results of this contribution.

We are interested in including $f \neq 0$ and x-dependent coefficients. The first is interesting if drying or condensation becomes important as e.g. in fuel cells. It adds a new quality to the system, since, for $f \neq 0$, in dry regions we have to solve an infinite family of ordinary differential equations $\partial_t s(x,t) = f(x,t,s(x,t))$. Allowing for x-dependent coefficients is important in applications and necessary in order to tackle homogenization questions. The generalization is non-trivial since the description with the help of the global pressure fails. Another reason for working in the primary variables would be the inclusion of capillary hysteresis as in [4], [13].

Another aim was the construction of a regularized equation. This leads to a new existence result, but it is also desirable for the design of a numerical scheme. With the help of the global pressure one easily sees that not every regularization of the physical coefficients provides a correct approximation of solutions. Our goal was to give sufficient conditions on the approximations k_δ and p_δ that assure the correct limit. The compactness results for the regularized sequence are derived with methods inspired by [2].

Our solution concept uses the primal variables s and p, with the difficulty that p is not defined in dry regions. We must understand $v = -k(s)\nabla p$ in the sense that $v = 0$ wherever $s \leq a$. We demand $p(x,t) \in p_c(s(x,t))$ almost everywhere on $\{s > a\}$, and interpret the evolution equation in (1.1) in the distributional sense. In the outflow condition (1.2) we introduce an artificial factor $k(x,t)$ in order to deal with functions with well-defined traces.

Note on the proofs. Testing the equation yields estimates for ∇p, but only with a weight. To be precise, we can expect from the equations estimates for the integrals

$$\int_\Omega k(s)\nabla p \cdot \nabla s \sim \int_\Omega k(s){p_c}'(s)|\nabla s|^2.$$

In particular, we cannot read off compactness of sequences s_δ or of sequences p_δ from this estimate. Furthermore, for $p_\delta \to p$, $k_\delta(s_\delta) \to k$, and $v_\delta \to v$, with the convergence of p_δ in the sense of the above estimate, it is not clear how to identify the limiting relation $v = -k\nabla p$. Again, we will introduce an additional factor k and derive the relation in the distributional sense.

Gravity. Our model allows to include gravity. For a constant porosity of the medium, up to a factor, Darcy's law with gravity reads

$$v = -k(s)(\nabla p + g e_N).$$

Since we allow for an x-dependent coefficient function p_c, it suffices to set $\bar{p_c}(x,s) := p_c(x,s) + gx \cdot e_N$.

Notation. Constants C may change from one line to the next, we write ∇ for spatial gradients, e.g. $\nabla[k(s)]$ for the gradient of the function $x \mapsto k(x, s(x))$ and $\nabla_x k(s)$ for the evaluation of the partial x-derivatives of k in the points $s(x)$. The lower indices $\pm$ denote positive and negative parts of a function, $f = (f)_+ + (f)_-$.

2 Regularization of the outflow problem in the case of constant coefficients

In this section we prove an existence and a convergence result for the doubly degenerate evolution equation in the case of constant coefficients; the boundary conditions are verified in section 3. The methods carry over to the case of non-constant coefficients, as we will show in section 4. The proof is based on a careful analysis of regularized problems. We collect assumptions on how the regularized problems must be constructed in order to have the convergence of the solutions to a solution of the original problem. Loosely speaking, we will see the following: if k_δ and ρ_δ generate a global pressure function Φ_δ that approximates the degenerate global pressure function Φ, then also the corresponding solutions converge.

Assumptions on the coefficient functions

The precise assumptions on the degenerate coefficients k and p_c are as follows.

Assumption 1 (Degenerate coefficients)**.** *There exist $a \in (0,1)$ and $c_0 > 0$ such that the following holds. The permeability*

$$k \in C^1([0,1],[0,\infty)) \text{ is monotonically non-decreasing}$$

with $k(s) = 0$ iff $s \in [0,a]$. The capillary pressure $p_c : (a,1] \to \{0,1\}^{\mathbb{R}}$ is a monotone graph given by a function

$$\tilde{p}_c \in C^1((a,1),\mathbb{R}), \text{ monotonically increasing.}$$

In the case $\tilde{p}_c(s) \to \infty$ for $s \to 1$ we identify p_c with $\tilde{p}_c$. In the opposite case we extend $\tilde{p}_c$ continuously to $(a,1]$ and set $p_c(s) = \{\tilde{p}_c(s)\}$ for $s \in (a,1)$ and $p_c(1) = [\tilde{p}_c(1),\infty)$. We assume that $\tilde{p}_c$ has a zero $a_0 \in (a,1]$, $\tilde{p}_c(a_0) = 0$.

With the intermediate value $\bar a = (a+1)/2$ we make the following quantitative assumption. For some $c_0 > 0$ there holds

$$
\begin{aligned}
\partial_s p_c(s) &\geq 1/c_0 && \forall s \in (a,1),\\
|\partial_s k(s)|^2 &\leq c_0 k(s) && \forall s \in (a,1),\\
k(s)|p_c(s)| + \sqrt{k(s)}\partial_s p_c(s) &\leq c_0 && \forall s \in (a,\bar a),\\
(1-s)\sqrt{\partial_s p_c(s)} &\leq c_0 && \forall s \in (\bar a,1),\\
\int_{\bar a}^1 p_c(s)\, ds &\leq c_0.
\end{aligned}
$$

Regarding the generality of our assumptions we emphasize that: (i) we allow finite and infinite capillary pressure $p_c(1)$, (ii) we allow finite and infinite derivative ${p_c}'(1)$, (iii) we allow finite $p_c(a)$ and $p_c(a) = -\infty$. The quantitative assumptions are all satisfied for a quadratic permeability $k(s) \sim (s-a)_+^2$, if the capillary pressure function does not degenerate too fast.

Choice of regularized coefficient functions

In order to replace the degenerate system by a family of regular parabolic problems, we approximate, for a sequence $\delta \searrow 0$, the degenerate coefficients k and p_c by functions k_δ and ρ_δ.

Assumption 2 (Regularized coefficients)**.** *The regularized coefficients satisfy*

$$k_\delta \in C^1([0,1],(0,\infty)), \quad \rho_\delta \in C^0([0,1],\mathbb{R}) \text{ piecewise } C^1,$$

both monotonically increasing. For $\delta \to 0$ we have the convergences $k_\delta \searrow k$ uniformly on $[0,1]$, $\rho_\delta \to p_c$ uniformly on compact subsets of $(a,1)$, and $k_\delta = \delta^2$ on $[0,a]$.

We assume $\bigcup_\delta \rho_\delta([0,1]) = \mathbb{R}$ and, for simplicity, $p_c(\bar a) = \rho_\delta(\bar a)$. Furthermore, all the quantitative statements of Assumption 1 shall hold with k and p_c replaced by k_δ and ρ_δ and with the point $a \in [0,1)$ replaced by 0.

Example 1. *Let Assumption 1 be satisfied and assume $c_1(s-a)^2 \leq k(s) \leq c_2(s-a)^2$ on $(a,1)$ for constants $0 < c_1 \leq c_2$. Then, for small $\delta > 0$, a regularization satisfying Assumption 2 is given by*

$$
\begin{aligned}
k_\delta(s) &:= \delta^2 + k(s) \qquad \forall s \in [0,1],\\
\rho_\delta(s) &:= \begin{cases} p_c(a+\delta) + (s-(a+\delta))/\delta & \forall s \in [0,a+\delta],\\ p_c(s) & \forall s \in (a+\delta, 1-\delta],\\ p_c(1-\delta) + (s-(1-\delta))/\delta^2 & \forall s \in (1-\delta,1]. \end{cases}
\end{aligned}
$$

The approximations allow to introduce a regularized global pressure function,

$$\Phi_\delta(s) = \int_0^{\rho_\delta(s)} k_\delta(\rho_\delta^{-1}(q))\, dq.$$

Assumptions 1 and 2 guarantee that Φ_δ is bounded from below and that $\Phi_\delta \to \Phi$ uniformly on compact subsets of $[0, 1)$. Furthermore, $\Psi_\delta \to \Psi$ uniformly on compact subsets of $(p_c(a), \infty)$ for

$$\Psi(p) = \int_0^{p} k({p_c}^{-1}(q))\, dq, \quad \Psi_\delta(p) = \int_0^{p} k_\delta(\rho_\delta^{-1}(q))\, dq.$$

The Ψ-functions will be used in the inflow boundary condition.

For later use we set $u_a := \Phi(0)$, the minimal global pressure for the degenerate system. Typical shapes of ρ_δ and Φ_δ are depicted in Figure 3.

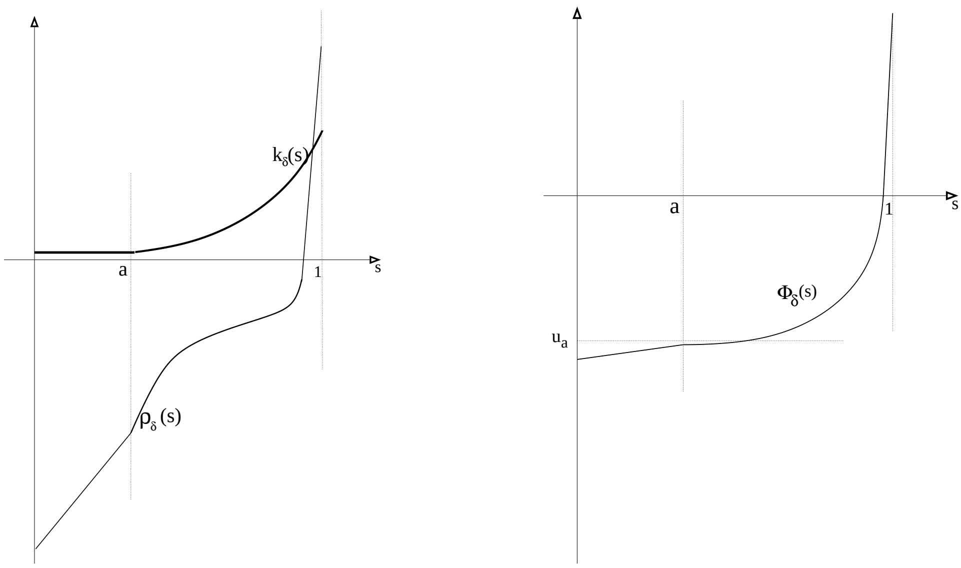

Figure 3: Typical shapes of ρ_δ, k_δ, and Φ_δ.

Geometry and boundary conditions

We assume that the porous material occupies a bounded set $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1, with exterior normal n. Let $\Sigma_{in}, \Sigma_N, \Sigma_{out} \subset \partial\Omega$ be three relatively open, pairwise disjoint $N-1$-dimensional C^1-manifolds such that $\partial\Omega$ is the union of the closure of the three manifolds. Here, $\Sigma_{in} \neq \emptyset$

is an inflow boundary on which we prescribe the pressure, $p = p_{in}$, Σ_N is an impenetrable boundary with Neumann condition, $v \cdot n = 0$. Along Σ_{out}, the porous medium is in contact with free space occupied by the gas phase and we impose the above outflow boundary condition (1.2). We assume that inflow and outflow boundary are nowhere in contact, $\bar{\Sigma}_{out} \cap \bar{\Sigma}_{in} = \emptyset$. For $\Omega, \Sigma_i \subset \mathbb{R}^N$ we write Ω_T and $\Sigma_{i,T}$ for $\Omega \times (0,T)$ and $\Sigma_i \times (0,T)$.

The initial condition is given by the initial saturation $s_0 \in L^2(\Omega)$. We demand

$$s(x,0) = s_0(x) \quad \text{a.e. in } \Omega.$$

Assumptions on the data

We assume that $f : \Omega_T \times [0,1] \to \mathbb{R}$, $f : ((x,t),s) \mapsto f(x,t,s)$, is bounded and Lipschitz continuous, and that f satisfies $f(x,t,1) \le 0$, $f(x,t,0) \ge 0$ for all $(x,t) \in \Omega_T$. We furthermore assume that $s \mapsto f(x,t,s)$ is affine on $(0,a)$ for all $(x,t) \in \Omega_T$. With minor changes also fixed $f : \Omega_T \to \mathbb{R}$ may be considered; the results then hold on the time interval where $0 < s < 1$ holds.

The initial data shall be given by a function $s_0 : \Omega \to [0,1]$. We emphasize that there are two problems concerning the initial values. (i) $s_0 \in H^1$ does not imply $\Phi(s_0) \in H^1$, since Φ has an unbounded derivative. (ii) $s_0 = 1$ does not specify uniquely a corresponding pressure. We impose the following compatibility condition: Let the initial saturation $s_0 \in H^1(\Omega)$ satisfy

$$\begin{aligned} &\exists p_0 : \Omega \to \mathbb{R}, \quad p_0 \in p_c(s_0) \text{ a.e. on } \{k(s_0) > 0\}, \quad (p_0)_+ \text{ bounded}, \\ &k(s_0)^2 p_0 \in H^1(\Omega), \quad s_0|_{\Sigma_{out}} \le a_0, \qquad (k(s_0)^2 p_0)|_{\Sigma_{in}} = k(s_0)^2\, p_{in}. \end{aligned}$$

We assume that the boundary data p_{in} are continuous and that $p_{in} \ge p_c(\bar{a})$. We furthermore assume that p_{in} and $\partial_t p_{in}$ are traces of $L^2((0,T), H^1(\Omega)) \cap L^\infty(\Omega_T)$-functions.

The regularized problem

We consider the regularized problem

$$\partial_t s_\delta = \Delta u_\delta + f_\delta, \qquad u_\delta = \Phi_\delta(s_\delta) \tag{2.1}$$

with the pressure $p_\delta = \rho_\delta(s_\delta)$ and the right hand side $f_\delta(x,t) = f(x,t,s_\delta(x,t))$. On the boundary Σ_{in} we impose $u_\delta = \Psi_\delta(p_{in})$, on Σ_N the Neumann condition $\nabla u_\delta \cdot n = 0$. On the outflow boundary Σ_{out} we impose, with $v_\delta = -\nabla u_\delta$, the mixed boundary condition

$$v_\delta \cdot n = \frac{1}{\delta} k_\delta(s_\delta)(p_\delta)_+\,, \tag{2.2}$$

where we recall that $(p_\delta)_+$ vanishes for $p_\delta \leq 0$ and otherwise coincides with p_δ. The initial condition is replaced by

$$s_\delta(x,0) = s_0^\delta(x) := \begin{cases} \rho_\delta^{-1}(p_0(x)) & \text{if } s_0(x) > \bar{a}, \\ s_0(x) & \text{if } s_0(x) \leq \bar{a}, \end{cases} \tag{2.3}$$

for all $x \in \Omega$. Our assumptions on the initial values s_0 guarantee the uniform boundedness of $\Phi_\delta(s_0^\delta) \in H^1(\Omega)$ since, for $s_0^\delta \geq \bar{a}$, we have $\Phi_\delta(s_0^\delta) = \Psi_\delta(p_0)$.

Our first theorem shows that the solutions of the regularized problems approximate a solution of the degenerate system. With the existence part of this theorem we essentially rediscover the Theorem of Alt, Luckhaus and Visintin, in our case allowing for $f \neq 0$. Our regularity assumptions on the data are stronger, hence we can also use a stronger solution concept. We add the information that the solutions of the regularized problems (instead of time-discrete solutions) approximate the solution of the degenerate system.

Theorem 1. *Let $T > 0$, p_{in} and s_0 as above, let $k = k(s)$ and $p_c = p_c(s)$ satisfy Assumption 1 and let k_δ and ρ_δ satisfy Assumption 2. Let (s_δ, p_δ) be the solutions of the regularized problems. Then, for a subsequence $\delta \to 0$ and appropriate limiting functions $u : \Omega_T \to \mathbb{R}$ and $s : \Omega_T \to [0,1]$, there holds*

$$s_\delta \to s \text{ weakly-* in } L^\infty(\Omega_T), \tag{2.4}$$

$$u_\delta \to u \text{ weakly in } L^2((0,T), H^1(\Omega)). \tag{2.5}$$

The limits satisfy with $v = -\nabla u$

$$\partial_t s = -\mathrm{div}\, v + f(.,s) \text{ in } \mathcal{D}'(\Omega_T), \tag{2.6}$$

$$u \in \Phi(s) \text{ a.e. in } \Omega_T, \tag{2.7}$$

and $s(t=0) = s_0$ in the weak sense. On the boundary $\partial\Omega$ with normal vector n there holds $u = \Psi(p_{in})$ on $\Sigma_{in,T}$, $v \cdot n = 0$ on $\Sigma_{N,T}$, and

$$v \cdot n \geq 0 \text{ on } \bar{\Sigma}_{out,T} \cup \Sigma_{N,T} \tag{2.8}$$

$$u \leq \tilde{\Phi}(a_0) \text{ and } k^2(s)s - k^2(a_0)a_0 \leq 0 \text{ on } \Sigma_{out,T} \tag{2.9}$$

$$(v \cdot n) \cdot (k^2(s)s - k^2(a_0)a_0) \geq 0 \text{ on } \Sigma_{out,T}\,. \tag{2.10}$$

The traces in (2.8)–(2.10) exist in the sense of distributions.

We recall that $p_c(a_0) = 0$ and that $s \mapsto k^2(s)s$ and $s \mapsto p_c(s)$ are monotone functions. Therefore (2.8)–(2.10) is formally equivalent with (1.2), since either $a_0 < 1$ or $\tilde{p}_c(1) = 0$.

Proof. We have to study the approximate solutions s_δ, $p_\delta = \rho_\delta(s_\delta)$, $u_\delta = \Phi_\delta(s_\delta)$, $v_\delta = -\nabla u_\delta = -k_\delta(s_\delta)\nabla p_\delta$. These solutions exist on $(0,T)$ and they satisfy

$$s_\delta : \Omega_T \to [0,1], \quad \|u_\delta\|_{L^\infty(\Omega_T)} + \|u_\delta\|_{L^2((0,T),H^1(\Omega))} \le C,$$
$$\int_{\Omega_T} k_\delta(s_\delta)\,|\nabla s_\delta|^2 \le C,$$

with a constant C independent of δ. We refer to section 5 for these a priori estimates for the regular parabolic problem.

Choosing a subsequence $\delta \to 0$ we find $s \in L^\infty(\Omega_T)$ and $u \in L^2((0,T),H^1(\Omega))$, such that $s_\delta \to s$ weakly-* in L^∞ and $u_\delta \to u$ weakly in L^2H^1, hence (2.4) and (2.5) and, in particular, $s : \Omega_T \to [0,1]$. The sequence f_δ is bounded in $L^\infty(\Omega_T)$, we can therefore assume weak $L^2(\Omega_T)$-convergence to a limit f and find (2.6) as the distributional limit of (2.1). Here, the fact that $f(x,t,.)$ is affine on $(0,a)$ assures the convergence $\chi_{\{s\le a\}} f(s_\delta) \rightharpoonup \chi_{\{s\le a\}} f(s)$. On the remaining set $\{s > a\} \subset \Omega_T$, the convergence $\chi_{\{s>a\}} f(s_\delta) \rightharpoonup \chi_{\{s>a\}} f(s)$ is a consequence of the compactness of s_δ on this set (see below).

The last of the a priori estimates can be used to find bounds for gradients of $k_\delta(s_\delta)$ or its products with s_δ. Indeed, by Assumption 2,

$$|\nabla k_\delta|^2 = |k_\delta'(s_\delta)\,\nabla s_\delta|^2 \le c_0 k_\delta(s_\delta)|\nabla s_\delta|^2$$

is uniformly bounded in $L^1(\Omega_T)$. We can therefore assume $\nabla[k_\delta(s_\delta)] \rightharpoonup \nabla[k(s)]$ in $L^2(\Omega_T)$, and similarly $\nabla[k_\delta(s_\delta)^j s_\delta] \rightharpoonup \nabla[k(s)^j s]$ for $j = 1$ and $j = 2$. Here, the identification of the limit functions exploits that $k = 0$ holds on $\{s \le a\}$ and the compactness of s_δ on sets $\{s \ge a + \varepsilon\}$.

Compactness. In order to verify (2.7) we need compactness results for the families u_δ and s_δ. We start with the sequence u_δ and note that the unboundedness of Φ_δ' does not allow to conclude from estimates for $\partial_t s_\delta$ estimates for $\partial_t u_\delta$. For this reason, we only aim at compactness away from regions with maximal saturation. We assume in the following that $\sup_{\{s\in(0,1)\}} \Phi(s) = u_1 < \infty$, the other case is simpler by the a priori estimate for the pressure.

We use a sequence $\varepsilon \to 0$ and a corresponding sequence of cut-off functions $\alpha_\varepsilon \in C_0^\infty(\Omega,[0,1])$ with $\alpha_\varepsilon(x) = 1$ for all $x \in \Omega$ with $\mathrm{dist}(x,\partial\Omega) \ge \varepsilon$. We claim that, for fixed $\varepsilon > 0$ and with $\eta_\varepsilon(\xi) = (\xi - u_1 + \varepsilon)_-$, the family $\alpha_\varepsilon \cdot \eta_\varepsilon(u_\delta)$ is compact in $L^2(\Omega_{T-\varepsilon})$, and that, for a subsequence $\delta \to 0$, there holds

$$\alpha_\varepsilon \cdot \eta_\varepsilon(u_\delta) \to \alpha_\varepsilon \cdot \eta_\varepsilon(u) \text{ in } L^2(\Omega_{T-\varepsilon}) \text{ as } \delta \to 0. \tag{2.11}$$

As a consequence, by taking successively subsequences, we may assume that the above convergence holds for all ε with a single sequence $\delta \to 0$. In particular we may assume the convergence pointwise almost everywhere $\eta_\varepsilon(u_\delta) \to \eta_\varepsilon(u)$ for $\delta \to 0$ and any $\varepsilon > 0$.

In order to prove (2.11), we study finite time differences. For a function $t \mapsto w(t)$ and $h > 0$ we introduce the expression $\Delta_h w(t) = w(t+h) - w(t)$. We integrate, for fixed t, the equation $\partial_t s_\delta = \Delta u_\delta + f_\delta$ over the interval $(t, t+h)$ to find

$$\Delta_h s_\delta(t) = \int_t^{t+h} \{\Delta u_\delta(\tau) + f_\delta(\tau)\}\, d\tau.$$

Multiplication with $\alpha_\varepsilon\, \Delta_h u_\delta(t)$ and an integration over space and time yields

$$\begin{aligned}
&\int_0^{T-h} \int_\Omega \Delta_h s_\delta(t) \cdot \alpha_\varepsilon\, \Delta_h u_\delta(t) \\
&\quad = \int_0^{T-h} \int_\Omega \int_t^{t+h} \{\Delta u_\delta(\tau) + f_\delta(\tau)\}\; \alpha_\varepsilon\, \Delta_h u_\delta(t)\, d\tau\, dx\, dt \\
&\quad \le \int_0^{T-h} \int_\Omega \int_t^{t+h} |\nabla u_\delta(\tau)|\; (|\nabla(\alpha_\varepsilon u_\delta(t))| + |\nabla(\alpha_\varepsilon u_\delta(t+h))|)\, d\tau\, dx\, dt \\
&\qquad + \int_0^{T-h} \int_\Omega \int_t^{t+h} |f_\delta(\tau)|\; (|\alpha_\varepsilon u_\delta(t)| + |\alpha_\varepsilon u_\delta(t+h)|)\, d\tau\, dx\, dt \le Ch.
\end{aligned}$$

The monotonicity of Φ_δ implies that the expression $\Delta_h s_\delta \cdot \Delta_h u_\delta$ is non-negative, hence we conclude the $L^1(\Omega_{T-\varepsilon})$-convergence $\alpha_\varepsilon \Delta_h s_\delta \cdot \Delta_h u_\delta \to 0$ for $h \to 0$, uniformly in $\delta > 0$.

We now restrict our attention to u_δ-values away from u_1. The uniform, strict monotonicity of Φ_δ^{-1} in the interval $\xi \in (\Phi_\delta(0), u_1 - \varepsilon)$ allows to find a number $\kappa_\varepsilon > 0$ such that

$$|\eta_\varepsilon(\xi_2) - \eta_\varepsilon(\xi_1)|^2 \le \kappa_\varepsilon\; (\xi_2 - \xi_1)\, (\Phi_\delta^{-1}(\xi_2) - \Phi_\delta^{-1}(\xi_1)). \tag{2.12}$$

In particular, we have the uniform (in δ) L^2-convergence $\alpha_\varepsilon \Delta_h[\eta_\varepsilon(u_\delta)] \to 0$ for $h \to 0$.

For spatial finite differences we find a similar bound by the uniform $L^2((0,T), H^1(\Omega))$-estimate for u_δ. The Riesz characterization of compact sets in L^2 implies the compactness of the family $\alpha_\varepsilon \eta_\varepsilon(u_\delta)$ in $L^2(\Omega_T)$ and thus (2.11).

Similar to the above reasoning, we can conclude compactness for s_δ in regions with $s_\delta > a$. This time, we use the cut-off function $\sigma_\varepsilon : \mathbb{R} \to \mathbb{R}$,

$\sigma_\varepsilon(\zeta) = (\zeta - a - \varepsilon)_+$. Regarding temporal differences of the family of functions $\alpha_\varepsilon\sigma_\varepsilon(s_\delta)$ we exploit that, by $\Phi'_\delta \geq \bar\kappa_\varepsilon > 0$ on $s \in [a+\varepsilon, 1)$,

$$|\sigma_\varepsilon(\zeta_2) - \sigma_\varepsilon(\zeta_1)|^2 \leq \bar\kappa_\varepsilon^{-1}\,(\zeta_2 - \zeta_1)\,(\Phi_\delta(\zeta_2) - \Phi_\delta(\zeta_1)). \tag{2.13}$$

For spatial gradients we find a bound exploiting $\nabla\sigma_\varepsilon(s_\delta) = \nabla s_\delta\ \chi_{\{s_\delta > a+\varepsilon\}}$. Hence we can conclude for a subsequence $\delta \to 0$ that

$$\alpha_\varepsilon\sigma_\varepsilon(s_\delta) \to \alpha_\varepsilon\sigma_\varepsilon(s) \tag{2.14}$$

in $L^2(\Omega_{T-\varepsilon})$ and pointwise almost everywhere.

Relation (2.7). Based on the above compactness results we can now verify the constitutive relation. We choose $\beta > 0$ and consider first the "good" set $G^\beta := \{u_a + \beta \leq u \leq u_1 - \beta\} \subset \Omega_T$. On almost all points (x,t) of G^β we have the convergence $u_\delta(x,t) \to u(x,t)$. Furthermore, uniform positivity of Φ'_δ allows, for small $\delta > 0$, to find uniformly continuous inverse maps $\Phi_\delta^{-1} : [u_a + \beta, u_1 - \beta] \to (0,1)$, which converge uniformly. We conclude that, pointwise a.e. in G^β, also $s_\delta(x,t) \to s(x,t)$. The uniform convergence $\Phi_\delta \to \Phi$ on compact subsets of $(0,1)$ yields $u = \Phi(s)$ on G^β. Since $\beta > 0$ is arbitrary, we have $u = \Phi(s)$ for almost every (x,t) with $u_a < u(x,t) < u_1$.

In order to study points (x,t) with $u(x,t) = u_a$, we consider the set $E^\beta := \{u < u_a + \beta\} \subset \Omega_T$. The uniform strict monotonicity of Φ_δ on compact subsets of $(a,1)$ implies that for some $\omega_\beta > 0$, $\omega_\beta = o(1)$ and $\delta_\beta > 0$, $\delta_\beta = o(1)$ for $\beta \to 0$, for all $\delta \leq \delta_\beta$

$$s_\delta > a + \omega_\beta \quad \Rightarrow \quad u_\delta > u_a + 2\beta.$$

We can therefore calculate for $E_\delta^\beta := \{(x,t) \in E^\beta : s_\delta > a + \omega_\beta\}$

$$|E_\delta^\beta| \leq |\{(x,t) \in E^\beta : u_\delta > u_a + 2\beta\}| \to 0 \text{ for } \delta \to 0,$$

the latter by the strong convergence $\alpha_\varepsilon\eta_\varepsilon(u_\delta) \to \alpha_\varepsilon\eta_\varepsilon(u)$ on E^β. The weak L^2-convergence $s_\delta \rightharpoonup s$ yields, with characteristic functions χ_E of E^β and χ_{E_δ} of E_δ^β, the L^2-weak convergences

$$s\chi_E \leftharpoonup s_\delta\chi_E = s_\delta(\chi_E - \chi_{E_\delta}) + s_\delta\chi_{E_\delta} \leq a + \omega_\beta + \chi_{E_\delta} \rightharpoonup a + \omega_\beta$$

for $\delta \to 0$. Since $\beta > 0$ is arbitrary, we conclude that, almost everywhere, $u(x,t) \leq u_a$ implies $s \leq a$, hence $u = \Phi(s)$.

With the same methods we finally study the set $F^\beta := \{u > u_1 - \beta\} \subset \Omega_T$. Again, for some $\omega_\beta > 0$, $\omega_\beta = o(1)$ for $\beta \to 0$, we find that the exceptional set $F_\delta^\beta := \{(x,t) \in F^\beta : s_\delta < 1 - \omega_\beta\}$ satisfies

$$|F_\delta^\beta| \leq |\{(x,t) \in F^\beta : u_\delta < u_1 - 2\beta\}| \to 0 \text{ for } \delta \to 0,$$

the latter by the strong convergence $\alpha_\varepsilon\eta_\varepsilon(u_\delta) \to \alpha_\varepsilon\eta_\varepsilon(u) = 0$ for $\varepsilon \leq \beta$. We calculate with the characteristic functions χ_F of F^β and χ_{F_δ} of F_δ^β,

$$s\chi_F \leftharpoonup s_\delta\chi_F = s_\delta(\chi_F - \chi_{F_\delta}) + s_\delta\chi_{F_\delta} \geq (1-\omega_\beta)(\chi_F - \chi_{F_\delta}) \rightharpoonup (1-\omega_\beta)\chi_F$$

for $\delta \to 0$. We conclude that $u(x,t) \geq u_1$ implies $s(x,t) = 1$ for almost every $(x,t) \in \Omega_T$, hence $u = \Phi(s)$ also in this case. We have thus verified (2.7).

Initial condition. The weak convergences allow to calculate for a test-function $\varphi \in C_0^\infty(\Omega \times [0,T))$

$$\begin{aligned} 0 &= \int_{\Omega_T} \{s_\delta\, \partial_t\varphi + v_\delta \cdot \nabla\varphi + f_\delta\, \varphi\} + \int_\Omega s_0^\delta\, \varphi(.,0) \\ &\to \int_{\Omega_T} \{s\, \partial_t\varphi + v \cdot \nabla\varphi + f\, \varphi\} + \int_\Omega s_0\, \varphi(.,0). \end{aligned}$$

The initial condition is satisfied in this weak sense.

Boundary conditions. Along the Dirichlet boundary Σ_{in} we can take the weak $L^2(\Sigma_{in,T})$ limit

$$0 = u_\delta - \Psi_\delta(p_{in}) \rightharpoonup u - \Psi(p_{in}),$$

since u_δ converges weakly together with its trace, and Ψ_δ converges uniformly to Ψ on compact subsets of $\mathbb{R}$.

Regarding the normal velocity at the boundary we find the convergence $v_\delta \cdot n \to v \cdot n$ in the sense of distributions on $\partial\Omega \times (0,T)$ with the help of the equation as in (3.1). In particular, on the Neumann boundary Σ_N we can take the distributional limit $0 = v_\delta \cdot n \to v \cdot n$.

Inequality (2.8) is satisfied by the non-negativity of $v_\delta \cdot n$ in the regularized boundary condition (2.2). Concerning (2.9) we can calculate with $C > 0$ independent of δ on Σ_{out}

$$\begin{aligned} (k_\delta^2(s_\delta)s_\delta - k_\delta^2(a_0)a_0)_+ &\leq Ck_\delta(s_\delta) \cdot (s_\delta - a_0)_+ \\ &\leq Ck_\delta(s_\delta)(p_\delta)_+ + O(\delta) \\ &= C\delta\, v_\delta \cdot n + O(\delta) \to 0 \text{ in } \mathcal{D}'(\Sigma_{out,T}). \end{aligned}$$

We conclude that the $H^{1/2}$-weak limit of the left hand side vanishes, which is the desired non-positivity result. A similar calculation can be performed with $(u_\delta - \tilde{\Phi}(a_0))_+ \leq C(p_\delta)_+ + O(\delta)$.

It remains to verify (2.10). This inequality is shown in Proposition 1 of the next section. □

Failure of the approximation process

With the above theorem we show that, if Assumption 2 is satisfied, the approximate solutions converge to solutions of the degenerate problem. We wish to mention at this point what can be said if the assumption fails in one of the estimates. If the regularizations do not satisfy $\partial_s p_c \geq 1/c_0$, we have no uniform estimate for $\operatorname{div} v_\delta$. If the regularizations do not satisfy $|\partial_s k|^2 \leq c_0 k$, we have no uniform H^1-bound for $k_\delta(s_\delta)$. In both cases we cannot derive the limiting equations and even the formulation of boundary conditions becomes a problem.

The most interesting case is that the regularizations fail to confirm to the third estimate which imposes, in a rigorous sense, that the convergence $k_\delta \to 0$ is faster than that of $-\rho_\delta \to \infty$. The assumption fails, e.g., if $\rho_\delta(0) \sim -1/k_\delta(0)$. In this case we may still consider limits (s, u) of the approximate solutions, and we may still derive

$$u \in \hat{\Phi}(s), \quad \hat{\Phi} = \lim_{\delta \to 0} \Phi_\delta.$$

But, in general, $\hat{\Phi}$ will be different from Φ. In this case, we approximate a solution of the wrong equation.

3 Boundary conditions: compensated compactness and defect measures

In this section we analyze boundary values such as $v \cdot n$ or $k(s) v \cdot n$ on $\Sigma_T = \partial\Omega \times (0, T)$, where n is the exterior normal vector to Ω. Before analyzing the limiting relations, we note that the boundary values $v \cdot n$ are a well-defined distribution on Σ_T. Indeed, on the basis of the limiting equation we may define the boundary values by setting, for arbitrary $\varphi \in C_0^\infty(\bar{\Omega} \times (0, T))$,

$$\int_{\Sigma_T} v \cdot n \, \varphi \stackrel{Def}{=} \int_{\Omega_T} \{ s \, \partial_t \varphi + f \, \varphi + v \, \nabla \varphi \} .$$

Additionally, by the analogous calculation, in the sense of distributions

$$v_\delta \cdot n \rightharpoonup v \cdot n \text{ in } \mathcal{D}'(\partial\Omega_T). \tag{3.1}$$

Lemma 1 (Divergence estimate)**.** *Let (s_δ, u_δ) be a sequence of approximate solutions, k and p_c independent of x and let the assumptions of Theorem 1 be* *satisfied. Then, for some $C > 0$ independent of δ, the sequence $\partial_t s_\delta$ satisfies the uniform bound*

$$\int_0^T \int_\Omega k_\delta \rho'_\delta |\partial_t s_\delta|^2 \le C. \tag{3.2}$$

As a consequence, the sequence v_δ has its divergence bounded in a weighted L^2-space,

$$\int_0^T \int_\Omega k_\delta |div\ v_\delta|^2 \le C. \tag{3.3}$$

Proof. The lemma concerns only the approximate solutions, we therefore omit the index δ in the expressions s_δ, k_δ, ρ_δ, Φ_δ, Ψ_δ, etc. None of the degenerate limiting functions is meant in this proof. We start by a multiplication of the equation with $\partial_t u$,

$$\partial_t s = \Delta u + f \qquad / \partial_t u = \partial_s \Phi(s)\ \partial_t s$$

An integration over Ω_T yields, with $v_n = v \cdot n = -\nabla u \cdot n$,

$$\begin{aligned}\int_{\Omega_T} \partial_s \Phi(s)\ |\partial_t s|^2 + \int_{\Omega_T} \frac{1}{2} \partial_t |\nabla u|^2 \\ = \int_{\Omega_T} f \cdot \partial_t u - \int_{\Sigma_{in,T}} v_n\ \partial_t \Psi(p_{in}) - \int_{\Sigma_{out,T}} v_n\ \partial_t u.\end{aligned}$$

The left hand side is non-negative and contains the expression of (3.2). We have to analyze the right hand side. Exploiting the Lipschitz assumption on f and Assumption 2 we write

$$\begin{aligned}\int_{\Omega_T} f \cdot \partial_t u &\le \int_{\Omega_T} f(x,t,1) \cdot \partial_t u(x,t)\, dx\, dt + C \int_{\Omega_T} |(1-s) \partial_s \rho(s)\, k(s) \partial_t s| \\ &\le C + C \int_{\Omega_T} |\partial_t f(.,1) \cdot u| + C \int_{\Omega_T} |\sqrt{\partial_s \rho(s) k(s)} \partial_t s| \\ &\le \varepsilon \int_{\Omega_T} \partial_s \Phi(s)\ |\partial_t s|^2 + C_\varepsilon,\end{aligned}$$

for arbitrary $\varepsilon > 0$ and C_ε independent of δ. This allows to absorb the first term into the left hand side of our estimate.

In the first boundary integral, by the assumption on p_{in}, we can find a bounded function $q \in L^2((0,T), H^1(\Omega))$ which takes the values $(\partial_t [\Psi(p_{in})])/k(\rho^{-1}(p_{in})) = \partial_t p_{in}$ on $\Sigma_{in,T}$ and vanishes on $\Sigma_{out,T}$. We can write

$$\begin{aligned}-\int_{\Sigma_{in,T}} v_n\, \partial_t \Psi(p_{in}) &= -\int_{\Omega_T} \mathrm{div}(k(s) v\, q)\\ &= -\int_{\Omega_T} \nabla[k(s)] v\, q + k(s) \mathrm{div}\, v\, q + k(s) v\, \nabla q\\ &\le \varepsilon \int_{\Omega_T} k(s) |\mathrm{div}\, v|^2 + C_\varepsilon,\end{aligned}$$

for arbitrary $\varepsilon > 0$. Again, the first term can be absorbed in the left hand side of our inequality. We finally study

$$-\int_{\Sigma_{out,T}} v_n\, \partial_t u = -\int_{\Sigma_{out,T}} \frac{k(s)}{\delta} (p)_+ \partial_s \Phi(s)\, \partial_t s.$$

In order to write the integrand as a total time derivative we define a function $H : [0, 1] \to \mathbb{R}$ by setting $H(a_0) = 0$ and $\partial_s H(s) = k(s)(\rho(s))_+ \partial_s \Phi(s)$. The derivative $\partial_s H$ is non-negative and vanishes for $s \in [0, a_0]$, hence also H has these properties. We conclude

$$-\int_{\Sigma_{out,T}} v_n\, \partial_t u = -\int_{\Sigma_{out,T}} \frac{1}{\delta}\, \partial_t [H(s)] = -\int_{\Sigma_{out}} \frac{1}{\delta}\, H(s(t)) \Big|_{t=0}^{T}.$$

Initially, i.e. for $t = 0$, we have $s_0^\delta \le a_0$ on Σ_{out}, hence H vanishes in $t = 0$. The right hand side is therefore non-positive. This concludes the proof. □

The aim of this section is the derivation of the boundary condition (2.10). We have to analyze the product of two limit functions, which is a severe problem for the following reason. For the term $k_\delta(s_\delta)^2 s_\delta \to k(s)^2 s$ we can, based on the estimates, expect the weak convergence of the traces in the space $H^{1/2}(\Sigma)$. Unfortunately, the other factor, $v_n|_\Sigma$, converges only as a distribution. If we had the divergence estimate without the degenerate factor k, we could hope for the convergence $v_\delta \cdot n \to v \cdot n$ weakly in $H^{-1/2}$ (the classical estimate). But we do not have the divergence estimate in L^2. Furthermore, even if we had the estimate, we would still have a product of two weakly convergent sequences in dual spaces. This does not allow a conclusion for the product of the limit functions. We circumvent both problems by exploiting the equation in a compensated compactness argument using defect measures.

The subsequent calculations are almost identical in the case of x-dependent coefficient functions $k = k(x, s)$ and $p_c = p_c(x, s)$. We therefore allow this dependence in the sequel. To shorten notation, we set

$$K : \Omega \times [0, 1] \to \mathbb{R}, \quad K(x, s) = k(x, s)^2\, s.$$

We emphasize that we do not consider a family K_δ based on k_δ, but only a single function. Lemma 1 provides a uniform $L^2(\Omega_T)$-bound for the sequence $K(s_\delta)\,\mathrm{div}\,v_\delta$ and we may assume the weak convergence to a limit function g. The next lemma characterizes the limit.

Lemma 2. *Under the assumptions of either Theorem 1 or Theorem 2, there exists a subsequence $\delta \to 0$ such that*

$$K(s_\delta)\ div v_\delta\ = g_\delta \rightharpoonup g = K(s)\ div v \quad \text{weakly in } L^2(\Omega_T), \tag{3.4}$$

where $K(s) div v$ is interpreted as a distribution.

The lemma is shown below. We exploit it here to *define* the boundary values of $K(s)v \cdot n$ by the integral

$$\int_{\Sigma_T} (K(s)\, v\cdot n)|_{\Sigma_T}\ \varphi \stackrel{Def}{=} \int_{\Omega_T} \nabla[K(s)]\cdot v\ \varphi + \int_{\Omega_T} g\ \varphi + \int_{\Omega_T} K(s)\, v\,\nabla\varphi, \tag{3.5}$$

for all $\varphi \in C_0^\infty(\bar{\Omega} \times (0,T))$. With the following proposition we derive the product outflow condition (2.10).

Proposition 1. *Let $v_\delta = -k_\delta(s_\delta)\nabla[\rho_\delta(s_\delta)]$ with $k_\delta, \partial_s\rho_\delta \geq 0$. Let*

$$K(s_\delta) \to K(s), \quad \nabla[K(s_\delta)] \rightharpoonup \nabla[K(s)], \quad v_\delta \rightharpoonup v \quad \text{in } L^2(\Omega_T), \tag{3.6}$$

and let (3.4) hold. In the case of x-dependent coefficients we additionally assume

$$(\nabla_x K)(s_\delta) \to (\nabla_x K)(s) \qquad \text{in } L^2(\Omega_T), \tag{3.7}$$

$$k_\delta(s_\delta)(\nabla_x\rho_\delta)(s_\delta) \to k(s)(\nabla_x p_c)(s) \qquad \text{in } L^2(\Omega_T). \tag{3.8}$$

Then, for a signed measure $\mu \in \mathcal{M}(\Sigma_T)$, $\mu \leq 0$, and a subsequence $\delta \to 0$, we find

$$(K(s_\delta)\, v_\delta \cdot n)|_{\Sigma_T} \rightharpoonup (K(s)\, v\cdot n)|_{\Sigma_T} + \mu \text{ in } \mathcal{D}'(\Sigma_T). \tag{3.9}$$

As a consequence, if the approximation satisfies $(K(s_\delta) - K(a_0))\, v_\delta \cdot n \geq 0$ on $\Sigma_{out,T}$, then we also have

$$(K(s) - K(a_0))\, v\cdot n \geq 0 \text{ on } \Sigma_{out,T} \tag{3.10}$$

in the sense of distributions.

Proof. We have to study limits of products, where both factors converge weakly. The sequences $\nabla[K(s_\delta)]$ and v_δ are bounded in $L^2(\Omega_T)$, hence the product is bounded in $L^1(\Omega_T)$. For a subsequence $\delta \to 0$ and some measure ν we find

$$\nabla[K(s_\delta)] \cdot v_\delta \rightharpoonup \nu \text{ in } \mathcal{M}(\bar{\Omega}_T). \tag{3.11}$$

The limiting measure coincides in the interior of Ω_T with the formal limit. Indeed, with the help of (3.4), we calculate for functions $\varphi \in C_0^\infty(\Omega_T)$

$$\begin{aligned} \int_{\Omega_T} \varphi \, d\nu \leftarrow \int_{\Omega_T} v_\delta \, \nabla[K(s_\delta)] \, \varphi = -\int_{\Omega_T} K(s_\delta) \operatorname{div} v_\delta \, \varphi - \int_{\Omega_T} v_\delta \, K(s_\delta) \, \nabla\varphi \\ \to -\int_{\Omega_T} g\varphi - \int_{\Omega_T} v \cdot K(s) \, \nabla\varphi \overset{(3.4)}{=} \int_{\Omega_T} v \cdot \nabla[K(s)] \, \varphi. \end{aligned}$$

We write $\nu = \nabla[K(s)] \cdot v + \mu$ for some defect measure $\mu \in \mathcal{M}(\bar{\Omega}_T)$. The above shows that μ is concentrated on the boundary, $\mu = 0$ on Ω_T. The measure ν is generated by

$$\begin{aligned} &\nabla[K(s_\delta)] \cdot v_\delta \\ &\quad = (\partial_s K(s_\delta) \, \nabla s_\delta + \nabla_x K(s_\delta)) \cdot (-k_\delta(s_\delta)\partial_s \rho_\delta(s_\delta) \nabla s_\delta - k_\delta(s_\delta) \nabla_x \rho_\delta(s_\delta)). \end{aligned}$$

In both factors, the second term converges strongly in $L^2(\Omega_T)$ by assumption. Therefore the singular part μ of the measure ν is generated by $-\partial_s K(s_\delta) \, k_\delta(s_\delta) \partial_s \rho_\delta(s_\delta) |\nabla s_\delta|^2 \leq 0$, which provides $\mu \leq 0$.

We can now derive equation (3.9) for the boundary values with a function $\varphi \in C_0^\infty(\bar{\Omega} \times (0,T))$.

$$\begin{aligned} &\int_{\Sigma_T} K(s_\delta) \, v_\delta \cdot n \, \varphi \\ &= \int_{\Omega_T} \nabla[K(s_\delta)] \cdot v_\delta \, \varphi + \int_{\Omega_T} K(s_\delta) \operatorname{div} v_\delta \, \varphi + \int_{\Omega_T} K(s_\delta) \, v_\delta \, \nabla\varphi \\ &\to \int_{\Omega_T} \nabla[K(s)] \cdot v \, \varphi + \int_{\bar{\Omega}_T} \varphi \, d\mu + \int_{\Omega_T} g \, \varphi + \int_{\Omega_T} K(s) \, v \, \nabla\varphi \\ &= \int_{\Sigma_T} (K(s) \, v \cdot n)|_{\Sigma_T} \varphi + \int_{\Sigma_T} \varphi \, d\mu. \end{aligned}$$

This proves (3.9). Inequality (3.10) follows immediately upon taking distributional limits,

$$\begin{aligned} 0 &\leq (K(s_\delta) - K(a_0)) \, v_\delta \cdot n \rightharpoonup K(s) v \cdot n + \mu - K(a_0) v \cdot n \\ &\leq (K(s) - K(a_0)) \, v \cdot n. \end{aligned}$$

This was the claim in (3.10). $\square$

Identification of limits with compensated compactness.

It remains to verify Lemma 2, i.e. to identify the limit in the weak convergence $K(s_\delta)\,\mathrm{div}\, v_\delta \rightharpoonup K(s)\,\mathrm{div}\, v$. This can not be done on the basis of the convergences alone, but we must exploit the differential equation.

Proof of Lemma 2. We start by observing that $g_\delta = K(s_\delta)\,\mathrm{div}\, v_\delta$ is bounded in $L^2(\Omega_T)$ by Lemma 1 and assumption (4.2) of Theorem 2, respectively. We can therefore extract a weakly convergent subsequence $g_\delta \rightharpoonup g$. We have to verify

$$\int_{\Omega_T} g\,\varphi = -\int_{\Omega_T} \nabla[K(s)]\cdot v\,\varphi - \int_{\Omega_T} K(s)\, v\,\nabla\varphi \qquad (3.12)$$

for all $\varphi \in C_0^\infty(\Omega_T)$. In the following we perform all calculations for x-independent coefficients and note that they remain valid under the assumptions of Theorem 2.

We use a primitive $\bar{K} : \Omega\times[0,1] \to \mathbb{R}$ with $\partial_s \bar{K}(x,s) = K(x,s)$ and $\bar{K}(x,0) = 0$. This allows to calculate

$$\begin{aligned} g \leftharpoonup K(s_\delta)\mathrm{div}\, v_\delta &= K(s_\delta)[-\partial_t s_\delta + f_\delta] = -\partial_t[\bar{K}(s_\delta)] + K(s_\delta) f_\delta \\ &\rightharpoonup -\partial_t[\bar{K}(s)] + K(s) f. \end{aligned}$$

Here, the convergences $K(s_\delta) \to K(s)$ and $\bar{K}(s_\delta) \to \bar{K}(s)$ follow from the convergence almost everywhere $\sigma_\varepsilon(s_\delta) \to \sigma_\varepsilon(s)$ for all $\varepsilon > 0$. We have thus identified $g = -\partial_t[\bar{K}(s)] + K(s) f$.

We now show that $\partial_t[\bar{K}(s)] = K(s)\partial_t s$ in the distributional sense, i.e. that for all $\varphi \in C_0^\infty(\Omega_T)$ holds

$$\int_{\Omega_T} \bar{K}(s)\,\partial_t\varphi = \int_{\Omega_T} \partial_t[K(s)]\, s\,\varphi + \int_{\Omega_T} K(s)\, s\,\partial_t\varphi. \qquad (3.13)$$

This is derived by approximating the function $\xi \mapsto K(\xi)$ by $K^\varepsilon(\xi) := K(\xi - \varepsilon)$, with the corresponding primitive $\bar{K}^\varepsilon$. For the smooth solutions s_δ we have the chain rule in the ordinary sense, hence

$$\int_{\Omega_T} \bar{K}^\varepsilon(s_\delta)\,\partial_t\varphi = \int_{\Omega_T} \partial_t[K^\varepsilon(s_\delta)]\, s_\delta\,\varphi + \int_{\Omega_T} K^\varepsilon(s_\delta)\, s_\delta\,\partial_t\varphi.$$

In this equation we first send $\delta \to 0$ and find

$$\int_{\Omega_T} \bar{K}^\varepsilon(s)\,\partial_t\varphi = \int_{\Omega_T} \partial_t[K^\varepsilon(s)]\, s\,\varphi + \int_{\Omega_T} K^\varepsilon(s)\, s\,\partial_t\varphi.$$

In this limit we exploited the compactness of $\sigma_\varepsilon(s_\delta)$ and the uniform $H^1(\Omega_T)$-estimate for $K^\varepsilon(s_\delta)$ which follows from (3.2). We now send $\varepsilon \to 0$ and find (3.13).

After this preparation, we can now show (3.4). We mollify the limit functions $K = K(s)$ and v by convolution with a Dirac sequence (for any continuation across the boundary), and find smooth functions $K_\varepsilon \to K$ locally in $H^1(\Omega_T)$ and $v_\varepsilon \to v$ locally in $L^2(\Omega_T)$. For arbitrary $\varphi \in C_0^\infty(\Omega_T)$ we find, for $\varepsilon \to 0$,

$$
\begin{aligned}
-\int_{\Omega_T} \nabla K \cdot v\,\varphi &\leftarrow -\int_{\Omega_T} \nabla K_\varepsilon \cdot v_\varepsilon\,\varphi \\
&= \int_{\Omega_T} K_\varepsilon \operatorname{div} v_\varepsilon\,\varphi + \int_{\Omega_T} K_\varepsilon\, v_\varepsilon \cdot \nabla\varphi \\
&= \int_{\Omega_T} K_\varepsilon\,(-\partial_t s + f)_\varepsilon\,\varphi + \int_{\Omega_T} K_\varepsilon\, v_\varepsilon \cdot \nabla\varphi \\
&= \int_{\Omega_T} \partial_t K_\varepsilon\, s_\varepsilon\,\varphi + \int_{\Omega_T} K_\varepsilon\, s_\varepsilon\,\partial_t\varphi + \int_{\Omega_T} K_\varepsilon\, f_\varepsilon\,\varphi + \int_{\Omega_T} K_\varepsilon\, v_\varepsilon \cdot \nabla\varphi \\
&\to \int_{\Omega_T} \partial_t K\, s\,\varphi + \int_{\Omega_T} K\, s\,\partial_t\varphi + \int_{\Omega_T} K\, f\,\varphi + \int_{\Omega_T} K\, v \cdot \nabla\varphi \\
&= -\int_{\Omega_T} \partial_t \bar K\,\varphi + \int_{\Omega_T} K\, f\,\varphi + \int_{\Omega_T} K\, v \cdot \nabla\varphi \\
&= \int_{\Omega_T} g\,\varphi + \int_{\Omega_T} K\, v \cdot \nabla\varphi.
\end{aligned}
$$

This proves the claim. □

4 Outflow problem for non-constant coefficients

In this section we transfer the previous results to x-dependent coefficient functions $k(x,s)$ and $p_c(x,s)$. The precise assumptions on k, p_c and their regularizations are collected in Assumption 3. They are not optimized with respect to regularity properties.

Assumption 3. *We assume that for some function $a \in C^1(\bar\Omega, (0,\bar a))$, $\bar a \in (0,1)$, the coefficients $k(x,s)$ and $\tilde p_c(x,s)$ satisfy*

$$k \in C^1(\bar\Omega \times [0,1], \mathbb{R}), \quad \tilde{p_c} \in C^1(\{(x,s) | x \in \Omega, a(x) < s < 1\}, \mathbb{R}),$$

and we set $p_c(1) = [\tilde{p_c}(1), \infty)$ *if* $\tilde{p}(x,.)$ *can be continued continuously to* $(a(x), 1]$*. We assume*

$$k \geq 0, \quad \partial_s k \geq 0, \quad k(x, a(x)) = 0, \quad k\nabla_x \tilde{p_c} \text{ bounded,}$$

and that the estimates of Assumption 1 hold pointwise for all s *and all* x*. On the approximations we assume* $k_\delta \searrow k$ *in* C^1 *and* $k_\delta = \delta^2$ *for* $s \leq a(x)$*,* $\rho_\delta \to \tilde{p_c}$ *in* C^1 *on compact subsets of* $\{(x,s)|x \in \Omega, a(x) < s < 1\}$*, and* $\sqrt{k_\delta}\nabla_x \rho_\delta \to \sqrt{k}\nabla_x p_c$ *uniformly on* $\{(x,s)|x \in \bar{\Omega}, a(x) \leq s \leq 1\}$*. Furthermore, the estimates of Assumption 1 shall hold pointwise for* k_δ *and* ρ_δ*.*

To simplify the notations we additionally assume that we have globally only one bahavior of the capillary pressure curve: either $a(x) < 1$ *for all* x*, or* $\partial_s \rho_\delta$ *uniformly bounded for* $s \to 1$*, or* $a \equiv 1$ *and* $\partial_s \rho_\delta \to \infty$ *for* $s \to 1$*, uniformly on* Ω*.*

The assumptions on f, s_0, and $p_{in} \geq p_c(., \bar{a})$ are as in section 2. We formulate the theorem with an assumption concerning estimates of the divergence of v_δ. The assumption is verified in Lemma 1 for constant coefficients, and, under different assumptions, in Lemma 5 for non-constant coefficients.

We use here again the regularized outflow condition with pressure driven velocity (2.2) which was

$$v_\delta \cdot n = \frac{1}{\delta} k_\delta(s_\delta)(p_\delta)_+ \,. \tag{4.1}$$

Theorem 2. *Let* $T > 0$*, let Assumption 3 hold, and let* (s_δ, p_δ) *be solutions of the regularized problems with* $v_\delta = -k_\delta \nabla p_\delta$ *and boundary condition (4.1). We assume the divergence estimate*

$$\int_0^T \int_\Omega k_\delta \partial_s \rho_\delta |\partial_t s_\delta|^2 \leq C \tag{4.2}$$

with C *independent of* δ*. Then, for a subsequence* $\delta \to 0$ *and appropriate limiting functions, there holds*

$$s_\delta \rightharpoonup s \qquad \text{weakly-* in } L^\infty(\Omega_T), \tag{4.3}$$

$$v_\delta \rightharpoonup v \qquad \text{weakly in } L^2(\Omega_T), \tag{4.4}$$

$$k_\delta(s_\delta) \to k(s) \qquad \text{weakly in } H^1(\Omega_T), \text{ strongly in } L^2(\Omega_T), \tag{4.5}$$

$$k_\delta^2(s_\delta)\, p_\delta \rightharpoonup k^2(s)\, p \qquad \text{weakly in } L^2(\Omega_T), \tag{4.6}$$

and $p(x,t) \in p_c(x, s(x,t))$ *almost everywhere on* $\{(x,t)|k(x, s(x,t)) > 0\}$*. The limits satisfy*

$$\partial_t s = -div\, v + f \qquad \text{in } \mathcal{D}'(\Omega_T), \tag{4.7}$$

$$k^2\, v = -k^3\, \nabla p \qquad \text{in } \mathcal{D}'(\Omega_T), \tag{4.8}$$

and $v = 0$ almost everywhere on $\{k = 0\}$. On the boundary $\partial\Omega$ with normal vector n the limiting functions satisfy $v \cdot n = 0$ on $\Sigma_{N,T}$ and $k^3\, p = k^3\,(p_c^{-1}(p_{in}))\, p_{in}$ on $\Sigma_{in,T}$. On Σ_{out} holds

$$v \cdot n \geq 0 \text{ on } \bar{\Sigma}_{out,T} \cup \Sigma_{N,T}, \tag{4.9}$$

$$k^2(s)p \leq 0 \text{ and } k^2(s)s - k^2(a_0)a_0 \leq 0 \text{ on } \Sigma_{out,T}, \tag{4.10}$$

$$(v \cdot n) \cdot (k^2(s)s - k^2(a_0)a_0) \geq 0 \text{ on } \Sigma_{out,T}. \tag{4.11}$$

The traces above exist in the sense of distributions.

Proof. Estimates, convergences, and definition of limit functions. The fundamental a priori estimates are shown in section 5. They provide boundedness of the saturation s_δ, an upper bound for the pressure p_δ, and an L^2 estimate for the velocity v_δ. In particular, we find limits s and v and a subsequence with (4.3) and (4.4). In order to derive (4.5), it suffices to calculate for the gradients

$$\nabla k_\delta = \partial_s k_\delta \cdot \nabla s_\delta + \nabla_x k_\delta.$$

The assumption $\partial_s k_\delta \leq C\sqrt{k_\delta}$ together with the L^2-bound for $\sqrt{k_\delta}|\nabla s_\delta|$ of the energy estimate yield the boundedness of $\nabla k_\delta \in L^2(\Omega_T)$. Additionally, time derivatives of k_δ are bounded due to (4.2). Together, we can assume $k_\delta \rightharpoonup k$ in $H^1(\Omega_T)$ and (4.5). For the proof we additionally assume the convergence pointwise almost everywhere. The identification of the limit $k = k(s)$ relies on the compactness result below.

The uniform upper bound for p_δ together with the uniform bound for $k_\delta \rho_\delta$ implies that the family $\psi_\delta := k_\delta^2(s_\delta)p_\delta$ is uniformly bounded. We may therefore assume $\psi_\delta = k_\delta^2 p_\delta \rightharpoonup \psi$ in $L^2(\Omega_T)$ for some function ψ. We now define the limiting pressure by $p(x,t) := \psi(x,t)/k(x,t)^2$ wherever $k(x,t)$ is positive. Since $k_\delta p_\delta$ is bounded and $k_\delta \to k$ pointwise almost everywhere, $k_\delta^2 p_\delta \to 0$ on $\{k = 0\}$. Therefore, this construction of p implies (4.6). Note that we have not defined a pressure on $\{k = 0\}$, and for the sequel we set $k^2 p = 0$ on this set.

Compactness. In order to derive the constitutive relation $p(x,t) \in p_c(x, s(x,t))$ we need a compactness result. Loosely speaking, we want that the convergence $\psi_\delta \to \psi$ is strong. For spatial derivatives of ψ_δ we calculate

$$\nabla \psi_\delta = k_\delta^2\, \nabla p_\delta + 2k_\delta \partial_s k_\delta\, \nabla s_\delta\; p_\delta + 2k_\delta \nabla_x k_\delta\; p_\delta,$$

and find a uniform estimate in $L^2(\Omega_T)$. In order to control temporal variations of ψ_δ we integrate the $\partial_t s_\delta$-equation over a time interval $(t, t+h)$ to find

$$s_\delta(t+h) - s_\delta(t) = \int_t^{t+h} \nabla \cdot (k_\delta \nabla p_\delta) + f_\delta.$$

In order to avoid the boundary integrals we use again a sequence of cut-off functions $\alpha_\varepsilon \in C_0^\infty(\Omega, [0,1])$ with $\alpha_\varepsilon(x) = 1$ for all x with $\mathrm{dist}(x, \partial\Omega) > \varepsilon$. We multiply with $(k_\delta \cdot s_\delta)(t+h) - (k_\delta \cdot s_\delta)(t)$ and α_ε and integrate over $\Omega \times (0, T-h)$ to find

$$\begin{aligned}\int_{\Omega\times(0,T-h)} &\alpha_\varepsilon [s_\delta(t+h) - s_\delta(t)] \cdot [(k_\delta \cdot s_\delta)(t+h) - (k_\delta \cdot s_\delta)(t)] \\ &\leq \int_{\Omega\times(0,T-h)} \int_t^{t+h} k_\delta(\tau) \nabla p_\delta(\tau) \cdot \nabla [\alpha_\varepsilon (k_\delta \cdot s_\delta)]_t^{t+h} + Ch \leq C'h.\end{aligned}$$

In particular, we have uniform interior bounds for finite differences of $k_\delta s_\delta$.

We now have to distinguish two cases. The case that either $p_c(x, s) \to \infty$ for $s \to 1$ or that $\partial_s \rho_\delta$ is uniformly bounded for $s \to 1$ is the easy case. By the uniform upper bound for the pressure functions p_δ, in this case, the functions $k_\delta(s_\delta)\partial_s \rho_\delta(s_\delta)$ are uniformly bounded. The reasoning below remains valid without the cut-off argument, i.e. for $\eta_\varepsilon = \mathrm{id}$.

The second possibility is that $p_c(.,1) = 0$ with $\partial_s \rho_\delta(s) \to \infty$ for $s \to 1$, uniformly on Ω. For a sequence $C_\varepsilon \to +\infty$ for $\varepsilon \to 0$ we introduce a limiting value function

$$\begin{aligned}&m_\varepsilon \in C^1(\Omega, \mathbb{R}), \quad m_\varepsilon(x) \leq 0, \\ &k_\delta \partial_s \rho_\delta \leq 2C_\varepsilon \text{ if } k_\delta^2 \rho_\delta \leq m_\varepsilon, \quad k_\delta \partial_s \rho_\delta \geq C_\varepsilon \text{ if } k_\delta^2 \rho_\delta \geq m_\varepsilon,\end{aligned}$$

and the nonlinear cut-off function

$$\eta_\varepsilon(x, \zeta) := (\zeta - m_\varepsilon(x))_- \,.$$

We claim that the sequence $\alpha_\varepsilon \eta_\varepsilon(\psi_\delta)$ is compact in L^2 and that for a subsequence $\delta \to 0$

$$\alpha_\varepsilon \eta_\varepsilon(\psi_\delta) = \alpha_\varepsilon \eta_\varepsilon(k_\delta^2 p_\delta) \to \alpha_\varepsilon \eta_\varepsilon(k^2 p) \text{ in } L^2(\Omega_T) \tag{4.12}$$

for all $\varepsilon > 0$ from a sequence $\varepsilon \to 0$. Indeed, the partial derivative

$$\frac{\partial[\eta_\varepsilon(k_\delta^2 p_\delta)]}{\partial s_\delta} \cdot \left(\frac{\partial[k_\delta s_\delta]}{\partial s_\delta} \right)^{-1} = \chi_{\{k_\delta^2 p_\delta < m_\varepsilon\}} \frac{2k_\delta \partial_s k_\delta p_\delta + k_\delta^2 \partial_s \rho_\delta}{k_\delta + \partial_s k_\delta s_\delta}$$

is uniformly bounded by the assumptions. Therefore the above interior bound for temporal finite differences of $k_\delta s_\delta$ implies the analogous bound for $\eta_\varepsilon(k_\delta^2 p_\delta)$. This implies the strong convergence of the left hand side of (4.12), since the partial derivatives $\nabla_x \eta_\varepsilon$ are bounded.

In order to conclude (4.12), it remains to identify the strong limit q_ε of the sequence $\alpha_\varepsilon \eta_\varepsilon(\psi_\delta)$. On the set $\{q_\varepsilon < 0\}$ the function q_ε is also a pointwise a.e. limit for a subsequence; this allows to conclude the pointwise a.e. convergence $k_\delta^2 p_\delta \to \eta_\varepsilon^{-1}(q_\varepsilon/\alpha_\varepsilon)$ and the weak limit must coincide with this limit. On the set $\{q_\varepsilon = 0\}$ we have, by convexity of $-\eta_\varepsilon$ and the weak convergence $k_\delta^2 p_\delta \to k^2 p$ the inequality $0 = q_\varepsilon \le \alpha_\varepsilon \eta_\varepsilon(k^2 p) \le 0$, hence $q_\varepsilon = \alpha_\varepsilon \eta_\varepsilon(k^2 p)$ also on this set and (4.12) is shown.

Bulk equations. Based on the compactness result it is now easy to verify the constitutive relation $p \in p_c(s)$ for almost all (x,t) with $k(x,t) > 0$. Indeed, on almost all points (x,t) with $k(x,t) > 0$ and $k^2 p \le m_\varepsilon$ for some ε, we have the pointwise convergences of $s_\delta(x,t)$ and of $p_\delta(x,t)$. This implies $p(x,s) \in p_c(s(x,t))$, since $\rho_\delta \to p_c$ uniformly on compact sets. On the other hand, for points (x,t) with $k^2 p \ge m_\varepsilon$ for all ε, we have $p(x,t) \ge \tilde{p}_c(x,1)$ and thus, again, $p(x,s) \in p_c(x, s(x,t))$.

Regarding the limiting equation (4.7) it suffices to take the distributional limit on both sides of $\partial_t s_\delta = -\operatorname{div} v_\delta + f_\delta$.

Concerning relation (4.8) we claim that, for an arbitrary vector field $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$, there holds

$$\int_{\Omega_T} k_\delta^2 p_\delta \, \nabla k_\delta \cdot \varphi \to \int_{\Omega_T} k^2 p \, \nabla k \cdot \varphi. \tag{4.13}$$

Once this is shown we can calculate

$$\begin{aligned}\int_{\Omega_T} k^2 \, v \cdot \varphi \leftarrow -\int_{\Omega_T} k_\delta^2 \, k_\delta \nabla p_\delta \cdot \varphi &= \int_{\Omega_T} p_\delta \, \left\{ 3k_\delta^2 \nabla k_\delta \cdot \varphi + k_\delta^3 \nabla \cdot \varphi \right\} \\ \to \int_{\Omega_T} p \, \left\{ 3k^2 \nabla k \cdot \varphi + k^3 \nabla \cdot \varphi \right\} &\overset{Def}{=} -\int_{\Omega_T} k^3 \nabla p \, \varphi\end{aligned}$$

and find (4.8). In order to prove (4.13) we decompose, for arbitrary $\varepsilon > 0$, the integral as

$$\begin{aligned}&\int_{\Omega_T} k_\delta^2 p_\delta \, \nabla k_\delta \cdot \varphi = \int_{\Omega_T} k_\delta^2 p_\delta \, \nabla k_\delta \cdot \varphi \chi_{\{k_\delta^2 p_\delta < m_\varepsilon\}} + \int_{\Omega_T} k_\delta^2 p_\delta \, \nabla k_\delta \cdot \varphi \chi_{\{k_\delta^2 p_\delta \ge m_\varepsilon\}} \\ &\to \int_{\Omega_T} k^2 p \, \nabla k \cdot \varphi \chi_{\{k^2 p < m_\varepsilon\}} + \int_{\Omega_T} k^2 p \, \nabla_x k \cdot \varphi \chi_{\{k^2 p \ge m_\varepsilon\}} + o_\varepsilon(1),\end{aligned}$$

for $\delta \to 0$. In the convergence of the first integral we used the strong convergence of $\alpha_\varepsilon \eta_\varepsilon(\psi_\delta)$, in the convergence of the second integral we used the strong convergence $\nabla_x k_\delta(s_\delta) \to \nabla_x k(s)$, which follows from the strong convergence $s_\delta \to s$ (on this set) and the uniform convergence $\nabla_x k_\delta \to \nabla_x k$. The error term $o_\varepsilon(1) \to 0$ for $\varepsilon \to 0$ is induced by boundary layer integrals (factor α_ε), and the term

$$\int_{\Omega_T} k_\delta^2 p_\delta \, \partial_s k_\delta \nabla s_\delta \cdot \varphi \chi_{\{k_\delta^2 p_\delta \geq m_\varepsilon\}}.$$

In this integral, $k_\delta^2 p_\delta$ and $\partial_s k_\delta$ are uniformly bounded, and $\chi \, \nabla s_\delta = \chi \, (\nabla p_\delta - \nabla_x \rho_\delta)/\partial_s \rho_\delta$ is small in $L^2(\Omega_T)$ by the energy bound (5.1) for ∇p_δ and by $1/\partial_s \rho_\delta \leq 1/C_\varepsilon$. In the above expression we now take the limit $\varepsilon \to 0$ and find

$$\int_{\Omega_T} k_\delta^2 p_\delta \, \nabla k_\delta \cdot \varphi \to \int_{\Omega_T} k^2 p \, \nabla k \cdot \varphi \chi_{\{s<1\}} + \int_{\Omega_T} k^2 p \, \nabla_x k \cdot \varphi \chi_{\{s=1\}}.$$

Since $\nabla k \; \chi_{\{s=1\}} = (\nabla_x k)(s) \; \chi_{\{s=1\}}$ by the chain rule for Sobolev functions, we have shown (4.13) and thus (4.8).

Regarding $v = 0$ on $\{k = 0\}$ it suffices to use the energy estimate (5.1) in the form

$$\int_{\Omega_T} \frac{1}{\sqrt{k_\delta}} |v_\delta|^2 \leq \int_{\Omega_T} \frac{k_\delta}{\sqrt{k_\delta}} k_\delta(s_\delta) |\nabla p_\delta|^2 \chi_{\{s_\delta \geq \bar{a}\}} + \int_{\Omega_T} \frac{k_\delta \partial_s \rho_\delta}{\sqrt{k_\delta}} k_\delta(s_\delta) \partial_s \rho_\delta(s_\delta) |\nabla s_\delta|^2 \chi_{\{s_\delta < \bar{a}\}} + C \leq C',$$

where in C we collected terms generated by $k_\delta \nabla_x \rho_\delta$. The estimate implies that every weak L^2 limit of $|v_\delta|$ vanishes on the set, where the strong L^2 limit of $(k_\delta)^{1/4}$ vanishes. This yields the result.

Boundary relations. Based on the equation, the velocity v has a trace in the distributional sense, and, additionally, $v_\delta \cdot n \to v \cdot n$ in the distributional sense on the boundary. This provides the Neumann condition along $\Sigma_{N,T}$. Concerning the inflow condition on $\Sigma_{in,T}$, it suffices to check the distributional convergence of $k_\delta^3 p_\delta$. Let φ be a vector field, $\varphi \in C^\infty(\bar{\Omega}_T, \mathbb{R}^N)$,

$$\int_{\Sigma_T} k_\delta^3 p_\delta \, \varphi \cdot n = \int_{\Omega_T} \left\{ 3k_\delta^2 \nabla k_\delta \, p_\delta \cdot \varphi + k_\delta^3 \, \nabla p_\delta \cdot \varphi + k_\delta^3 p_\delta \, \nabla \cdot \varphi \right\} \to \int_{\Omega_T} \left\{ 3k^2 \nabla k \, p \cdot \varphi - k^2 \, v \cdot \varphi + k^3 p \, \nabla \cdot \varphi \right\} = \int_{\Sigma_T} k^3 p \, \varphi \cdot n,$$

where we exploited again (4.13).

The outflow boundary condition (4.10) is verified with a calculation as in Theorem 1. The product inequality (4.11) was derived in Proposition 1 of section 3 for general coefficient functions. It suffices to check the assumptions of Proposition 1. The convergences of (3.6) and (3.7) follow for a subsequence from the pointwise a.e. convergence of s_δ on $\{a(x)+\varepsilon < s_\delta(x,t) < 1-\varepsilon\}$ for all $\varepsilon > 0$. The limit (3.8) is a consequence of our assumption on the uniform convergence of $k_\delta \nabla_x \rho_\delta$. □

5 Estimates

In this section we collect, for x-dependent coefficients k_δ and ρ_δ, the fundamental estimates for solutions of the regularized equation

$$\partial_t s_\delta = \nabla \cdot (k_\delta(s_\delta)\nabla[\rho_\delta(s_\delta)]) + f_\delta,$$

with boundary condition (2.2) and initial data s_0^δ. We impose the general assumptions of subsection 2 and Assumption 3 for the coefficients. We find a solution $s_\delta : \Omega_T \to [0,1]$ in two steps. 1) Extending the coefficient functions to all $s \in \mathbb{R}$, we find a solution s_δ of the parabolic problem by local existence theory and the energy estimate below. 2) The parabolic maximum principle provides the bounds $s(x,t) \in [0,1]$.

Another application of a maximum principle yields additionally an upper bound for p_δ.

Lemma 3 (Maximum principle)**.** *There exists $p_{MAX} < \infty$ independent of $\delta > 0$ such that $p_\delta(x,t) \le p_{MAX}$ for almost all $(x,t) \in \Omega_T$ and all $\delta > 0$.*

Proof. We recall that $p_{max} = \max\{p_{in}(x,t) | x \in \Sigma_{in}, t \in [0,T]\}$ is a finite number. Furthermore, by the compatibility condition on the initial values and the construction of the initial date for the regularized problem, we have bounded initial values. For some $p_M > 0$ we find $\rho_\delta(s_0^\delta) \le p_0 \le p_M$.

For bounded coefficient $x \mapsto k_\delta(x,\bar{s})$ we may solve the following elliptic problem for $H : \Omega \to \mathbb{R}$, $H = H_{\delta,\bar{s}}$,

$$-\nabla \cdot (k_\delta(.,\bar{s})\nabla H) = 1, \quad H = 0 \text{ on } \Sigma_{in}, \quad \nabla H \cdot n = -1 \text{ on } \Sigma_N \cup \Sigma_{out}.$$

We vary the parameter $\bar{s}$ in the set $[\bar{a},1]$ so that the coefficients k_δ are uniformly non-degenerate. $H_{\delta,\bar{s}}(x)$ and its derivatives depend continuously on $\delta \in [0,\delta_0]$ and $\bar{s}$.

We choose $\varepsilon > 0$ so small that $\partial_s k_\delta(x,s)(\partial_s \rho_\delta(x,s))^{-1}|\nabla H_{\delta,\bar{s}}(x)|^2 \le (2\varepsilon)^{-1}$ for all x, s, $\bar{s}$, and δ, and then choose $\bar{s} \in (0,1)$ close to 1 (specified below, independent of δ) and enlarge p_M such that $\rho_\delta(x,s) > p_M$ implies $s > \bar{s}$ independent of x and δ.

We compare the solution p_δ of the regularized problem with $p_M + \varepsilon H_{\delta,\bar{s}}(x)$. Let t be the first time instance such that $p_\delta(x,t) = p_M + \varepsilon H(x)$ for some $x \in \bar{\Omega}$. Necessarily, (x,t) is an inner point of Ω_T. Exploiting $\nabla p_\delta(x,t) = \varepsilon \nabla H(x)$ and $\Delta p_\delta(x,t) \le \varepsilon \Delta H(x)$, and thus

$$\nabla_x k_\delta(x,\bar{s})\nabla p_\delta(x,t) + k_\delta(x,\bar{s})\Delta p_\delta(x,t) \le -\varepsilon.$$

We can calculate in the point (x,t)

$$\begin{aligned}
\partial_t s_\delta &= \nabla \cdot (k_\delta(s_\delta)\nabla p_\delta) + f_\delta \\
&= (\nabla_x k_\delta)(x,s_\delta)\nabla p_\delta + (\partial_s k_\delta)(x,s_\delta)\nabla s_\delta \,\nabla p_\delta + k_\delta(s_\delta)\Delta p_\delta + f_\delta \\
&= [(\nabla_x k_\delta)(x,s_\delta) - (\nabla_x k_\delta)(x,\bar{s})]\nabla p_\delta + [k_\delta(s_\delta) - k_\delta(\bar{s})]\Delta p_\delta \\
&\quad + (\partial_s k_\delta)(x,s_\delta)\nabla s_\delta \,\nabla p_\delta - \varepsilon + f_\delta \\
&\le C\varepsilon(1-\bar{s}) + (\partial_s k_\delta)\cdot(\partial_s \rho_\delta)^{-1}\varepsilon^2|\nabla H(x)|^2 - \varepsilon + C_L|1-\bar{s}|,
\end{aligned}$$

where C_L is the Lipschitz constant of f. Choosing $\bar{s} < 1$ such that $1 - \bar{s}$ is small compared to ε, we find that the time derivative is negative, a contradiction.

The result is obtained with $p_{MAX} = \max\{p_M + \varepsilon H\}$. □

Lemma 4 (Energy estimate)**.** *There exists $C = C(T) < \infty$ independent of $\delta > 0$ such that, for all $\delta > 0$,*

$$\int_0^T \int_\Omega k_\delta(s_\delta)|\nabla p_\delta|^2 \chi_{\{s_\delta \ge \bar{a}\}} + k_\delta(s_\delta)\partial_s \rho_\delta(s_\delta)|\nabla s_\delta|^2 \chi_{\{s_\delta < \bar{a}\}} \le C. \tag{5.1}$$

In particular, the family of velocity fields is bounded, $\|v_\delta\|_{L^2(\Omega_T)} \le C$.

Proof. For notational convenience in the proof we assume the existence of a number $\bar{p} \in \mathbb{R}$ is such that $\rho_\delta(x,a(x)) \le \bar{p} \le \rho_\delta(x,\bar{a})$ for all $x \in \Omega$.

The estimate (5.1) is of energy type and can be obtained by a testing procedure. Multiplication of the equation with a function $\varphi \in H^1(\Omega_T)$ and an integration yields

$$\begin{aligned}
&\int_0^t \int_\Omega \partial_t s_\delta \,\varphi + \int_0^t \int_\Omega k_\delta(s_\delta)\nabla p_\delta \cdot \nabla \varphi \\
&\quad + \int_0^t \int_{\Sigma_{in}} v_\delta \cdot n \,\varphi + \int_0^t \int_{\Sigma_{out}} v_\delta \cdot n \,\varphi = \int_0^t \int_\Omega f_\delta \,\varphi,
\end{aligned}$$

with $\nabla p_\delta = \partial_s\rho_\delta(s_\delta)\nabla s_\delta + \nabla_x \rho_\delta(s_\delta)$.

Loosely speaking, to find estimates for the gradient of p_δ, we must multiply the equation with p_δ. This works in regions where the saturation is large enough. To make the method rigorous, we set $\eta_+ : \mathbb{R} \to \mathbb{R}$, $\eta_+(\zeta) = (\zeta - \bar{p})_+$ and use the bounded function $\tilde{p}_{in} \in H^1(\Omega_T)$ that continues the boundary values on Σ_{in} and vanishes on the outflow boundary Σ_{out}, existing by the assumption on p_{in}. We insert above $\varphi = \eta_+(p_\delta) - \tilde{p}_{in}$ and exploit the uniform boundedness of $|\varphi|$ (due to the upper bound for p_δ), $v_\delta \cdot n \geq 0$ and $\varphi \geq 0$ on Σ_{out}, $\varphi = 0$ on Σ_{in}, and $f_\delta \leq C_f$, to find

$$\int_0^t \int_\Omega \partial_t s_\delta\ \eta_+(p_\delta) + \int_0^t \int_\Omega k_\delta(s_\delta)\nabla p_\delta\ \left(\nabla p_\delta\, \chi_{\{p_\delta \geq \bar{p}\}} - \nabla \tilde{p}_{in}\right) \leq C.$$

In order to analyze the first integral we introduce the function $H_\delta : \mathbb{R}\times\Omega \to \mathbb{R}$ such that, for all $x \in \Omega$, $H_\delta(\bar{p}, x) = 0$ and

$$\partial_\zeta H_\delta(\zeta, x) = \frac{\eta_+(\zeta)}{\partial_s\rho_\delta(x, \rho_\delta^{-1}(x,\zeta))}.$$

The function H_δ vanishes on $(-\infty, \bar{p}) \times \Omega$ and is uniformly bounded on $(\bar{p}, p_{MAX}] \times \Omega$. Since we can write

$$\eta_+(p_\delta)\partial_t s_\delta = \frac{\eta_+(p_\delta)}{\partial_s\rho_\delta(\rho_\delta^{-1}(p_\delta))}\partial_t p_\delta = \partial_t[H_\delta(p_\delta)],$$

the first integral is bounded. We arrive at

$$\int_0^t \int_\Omega k_\delta(s_\delta)|\nabla p_\delta|^2\ \chi_{\{s_\delta \geq \bar{a}\}} \leq C + \int_0^t \int_\Omega k_\delta(s_\delta)\nabla p_\delta \cdot \nabla \tilde{p}_{in}.$$

We next want to study the region with low saturation. To this end we set $\eta_-(s) = (s - \bar{a})_- + \bar{a}$ and choose $\varphi = \eta_-(s_\delta)$. Exploiting $0 \leq \varphi \leq s_\delta \leq 1$, $v_\delta \cdot n \geq 0$ on Σ_{out}, $s_\delta \geq \bar{a}$ and hence $\varphi = \bar{a}$ on Σ_{in}, and $f_\delta \leq C_f$, we find

$$\int_0^T \int_\Omega k_\delta(s_\delta)\partial_s\rho_\delta(s_\delta)|\nabla s_\delta|^2 \chi_{\{s_\delta < \bar{a}\}} + \bar{a}\int_0^T \int_{\Sigma_{in}} v_\delta \cdot n$$
$$\leq C - \int_0^T \int_\Omega k_\delta(s_\delta)\nabla_x\rho_\delta \cdot \nabla s_\delta \chi_{\{s_\delta < \bar{a}\}}.$$

By the uniform boundedness of $\sqrt{k_\delta}\nabla_x\rho_\delta$ we can absorb the last integral into the left hand side. It remains to control the net inflow through Σ_{in}. We choose a smooth function $\alpha \in C^\infty(\bar\Omega)$ with $\alpha = 1$ on Σ_{in} and $\alpha = 0$ on Σ_{out}. We calculate

$$\int_0^T \int_{\Sigma_{in}} v_\delta \cdot n = \int_0^T \int_\Omega \operatorname{div}(v_\delta\, \alpha) = \int_0^T \int_\Omega v_\delta \cdot \nabla\alpha + \alpha\,(-\partial_t s_\delta + f_\delta)$$
$$\leq C\ (1 + \|v_\delta\|_{L^2(\Omega_T)}).$$

We collect the estimates and find

$$\int_0^t \int_\Omega k_\delta(s_\delta)|\nabla p_\delta|^2\ \chi_{\{s_\delta \geq \bar a\}} + \int_0^T \int_\Omega k_\delta(s_\delta)\partial_s\rho_\delta(s_\delta)|\nabla s_\delta|^2 \chi_{\{s_\delta < \bar a\}}$$
$$\leq C(1 + \|v_\delta\|_{L^2(\Omega_T)}).$$

We can write $v_\delta = -k_\delta \nabla p_\delta = -k_\delta \partial_s\rho_\delta \nabla s_\delta - k_\delta \nabla_x \rho_\delta$ and use the first expression for $s_\delta \geq \bar a$ and the second expression for $s_\delta < \bar a$. Since k_δ is bounded, and $k_\delta\, \partial_s\rho_\delta$ is bounded on $s \in [0, \bar a]$, the squared L^2-norm of v_δ is bounded by the above integral and an additive constant. □

Lemma 5 (Divergence estimate for non-constant coefficients)**.** *We study $k(x,s)$ and $p_c(x,s)$ as described in the beginning of section 4, assumptions on initial and boundary value as before, and we additionally assume that for some C_0 the partial derivatives satisfy*

$$|\nabla_x k_\delta(x,s)| \leq C_0(1-s)k_\delta,$$
$$|\nabla_x \rho_\delta(x,s)| \leq C_0, \quad \sqrt{k_\delta\, \partial_s\rho_\delta}(1-s) \leq C_0.$$

Then, for some $C > 0$ independent of δ, the sequence $\partial_t s_\delta$ satisfies the uniform bound

$$\int_0^T \int_\Omega k_\delta \partial_s\rho_\delta |\partial_t s_\delta|^2 \leq C.$$

Proof. Once more, in this proof we omit the index δ in the expressions $k_\delta(x,)$, $\rho_\delta(x,.)$, etc. We define a function $\Psi(x,\xi)$ through $\partial_\xi \Psi(x,\xi) = k(x, (\rho(x,.))^{-1}(\xi))$ and $\Psi(x,0) = 0$. By the uniform bound $p \leq p_{MAX}$ and the assumption on $k\partial_s\rho$, the functions $x \mapsto \Psi(x, p(x,t))$ are uniformly bounded.

Multiplication of the equation

$$\partial_t s = \nabla\cdot(k\nabla p) + f \qquad \Big/ \frac{d}{dt}[\Psi(p)] = k(x, (\rho(x,p))^{-1}(\xi))\, \partial_t p(x,t),$$

and integration over Ω_T yields

$$\int_{\Omega_T} \left\{ k\,\partial_t s\;\partial_t p - f\,\frac{d}{dt}\Psi \right\} + \int_{\Sigma_{in,T}} v_n\;\frac{d}{dt}[\Psi(p_{in})] + \int_{\Sigma_{out,T}} v_n\;k\partial_t p$$
$$= -\int_{\Omega_T} (k\nabla p)\frac{d}{dt}\nabla[\Psi(p)]$$
$$= -\int_{\Omega_T} \partial_t \frac{1}{2}|k\,\nabla p|^2 - \int_{\Omega_T} k\,\nabla p \frac{d}{dt}[\nabla_x \Psi(p)].$$

We write the last integral as

$$\int_{\Omega_T} k\nabla p\,\frac{d}{dt}[\nabla_x\Psi(p)] = \int_{\Omega_T} k\nabla p\,(\partial_\xi \nabla_x \Psi)(p)\,\partial_t p$$
$$= \int_{\Omega_T} k\nabla p\,\nabla_x[k(x,(\rho(x,.))^{-1}(\xi))]|_{\xi=p}\;\partial_t p$$
$$= \int_{\Omega_T} k\nabla p\;[\nabla_x k(x,s) - \partial_s k(x,s)\cdot(\partial_s\rho(x,s))^{-2}\nabla_x\rho(x,s)]\,\partial_t p$$
$$\le C\int_{\Omega_T} |k\nabla p|\;[(1-s)k + \sqrt{k}(\partial_s\rho(x,s))^{-2}]\,|\partial_t p|$$
$$\le C\int_{\Omega_T} |k\nabla p|\;\sqrt{k}(\partial_s\rho(x,s))^{-1/2}\,|\partial_t p|$$
$$= C\int_{\Omega_T} |k\nabla p|\cdot(k\partial_t s\,\partial_t p)^{1/2}\,.$$

The first factor under the integral is the velocity and uniformly bounded in L^2, the second factor appears squared on the left hand side of the inequality. We can absorb the term.

Exploiting the Lipschitz assumption on f, we write

$$\int_{\Omega_T} f\,\frac{d}{dt}\Psi \le \int_{\Omega_T} f(x,t,1)\,\frac{d}{dt}\Psi + C\int_{\Omega_T} |(1-s)k(s)\partial_s\rho(s)\,\partial_t s|$$
$$\le C + \int_\Omega f(x,t,1)\,\Psi(x,t)\,dx\bigg|_{t=0}^{T} + C\int_{\Omega_T} |\sqrt{\partial_s\rho(s)k(s)}\partial_t s|,$$

which can be absorbed in the left hand side.

The boundary integrals are treated precisely as in Lemma 1. By the assumption on p_{in}, we can find a bounded function $q \in L^2H^1$ which takes the values $\partial_t p_{in} = (\partial_t[\Psi(p_{in})])/k(\rho^{-1}(p_{in}))$ on $\Sigma_{in,T}$ and vanishes on $\Sigma_{out,T}$, and write

$$-\int_{\Sigma_{in,T}} v_n \, \frac{d}{dt}\Psi(p_{in}) = -\int_{\Omega_T} \operatorname{div}(k(s)v\, q)$$
$$= -\int_{\Omega_T} \nabla[k(s)]v\, q + k(s)\operatorname{div} v\, q + k(s)v\,\nabla q \le \varepsilon \int_{\Omega_T} k(s)|\operatorname{div} v|^2 + C_\varepsilon.$$

The outflow boundary integral is written as

$$\int_{\Sigma_{out,T}} v_n \, k\partial_t p = \int_{\Sigma_{out,T}} \frac{k(s)}{\delta}\,(p)_+ k\,\rho'\,\partial_t s$$
$$= \int_{\Sigma_{out,T}} \frac{1}{\delta}\,\partial_t[H(s)] = \int_{\Sigma_{out}} \frac{1}{\delta} H(s)\bigg|_{t=0}^{T},$$

where $H : [0,1]\times\Omega \to \mathbb{R}$ satisfies $H(x,a_0) = 0$ and $\partial_s H(x,s) = k(x,s)(\rho(x,s))_+ k(x,s)\rho'(x,s)$. Thus H is non-negative and vanishes $[0,a_0]$, hence, initially. We conclude that the integral is non-negative. □

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