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Wilfried Hazod

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Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

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Abstract

Recent investigations of M. Rösler [13] and M. Voit [16] provide examples of hypergroups with properties similar to the group- or vector space case and with a sufficiently rich structure of automorphisms, providing thus tools to investigate the theory limits of normalized random walks and the structure of the corresponding limit laws. The investigations are parallel to corresponding investigations for vector spaces and simply connected nilpotent Lie groups.

Introduction

Let Π_d denote the cone of positive semidefinite linear operators on $\mathbb{K}^d$, with coefficient field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The locally compact space $\mathcal{K} := \Pi_d \subseteq \mathbb{K}^{d^2}$ is endowed with a convolution structure $\star_\mu : M^b(\mathcal{K}) \times M^b(\mathcal{K}) \to M^b(\mathcal{K})$ such that $(\mathcal{K}, \star_\mu)$ is a hypergroup. These convolution structures $(\mathcal{K}, \star_\mu)\,,\ \mu > \rho - 1$, were investigated by M. Rösler [13] and M. Voit [16]. (There μ, ρ denote real parameters. We omit details.) In the following we do not need the analytical detailes, which are found in [13], [16]. We only use certain particular *properties* of these hypergroup structures listed up in Section 0.1 below. Throughout we fix a convolution $(\mathcal{K}, \star_\mu)$ and use the abbreviation $(\mathcal{K}, \star)$. For $d = 1$ these hypergroups are just the Bessel-Kingman hypergroups, therefore we mostly assume w.l.o.g. $d \geq 2$. For basic facts on hypergroups the reader is referred e.g. to the monograph W. Bloom, H. Heyer [2].

In fact, in analogy to Bessel-Kingman hypergroups on $\mathbb{R}_+$, these convolution structures on the matrix cones Π_d have *group like* properties, and therefore many well-known features of probabilities on vector spaces and on (certain) groups generalize to to these convolution structures.

Our aim is to sketch a survey of investigations of limit distributions of normalized random walks on these hypergroups parallel to the well-known results in the 'classical' situation. Also the proofs are – as far as possible – close to the classical ones. This survey is of course far from being complete: it just shows the power of the methods. In fact, drawing a cross-section of the theory

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of limit distributions of operator-normalized random walks on finite dimensional vector spaces (in short: operator limit distributions), we concentrate on results and structures which were already generalized in the past to locally compact groups with sufficiently nice automorphism groups, in particular to homogeneous nilpotent Lie groups. Therefore, the emphasis is laid on characterizations of stable, semistable and (semi-) self-decomposable laws. Taking into account that the matrix cone hypergroups $(\mathcal{K}, \star)$ possess a sufficiently rich structure of automorphisms we have the means to investigate the analogues of operator limit distributions on these hypergroups.

Let us mention that for one-dimensional hypergroups, in particular for Sturm Liouville hypergroups, the structure of automorphisms is considerably poor: Either $\mathrm{Aut}(\mathcal{K})$ is trivial or – for Bessel-Kingman structures – isomorphic to homothetical transformations, hence $\mathrm{Aut}(\mathcal{K}) \equiv \mathbb{R}_+^*$. In this case limit theorems had been investigated e.g. by Hm. Zeuner, see in particular [18], [19], [15], see also the monograph [2] and the literature mentioned there. In a more general context, limit laws, in particular semistability, had been investigated by S. Menges [11], [12], however at this time only a few concrete examples beyond the one-dimensional case were available. The afore mentioned results [16], [13] provide now examples of hypergroups and their automorphisms which motivate systematic investigations of the structure of operator limit laws.

The paper is organized as follows: At the beginning we collect for the hypergroups under consideration the main features which are important for the following investigations. Then, in Section 1 we collect properties of *automorphisms*, *endomorphisms* and *subhypergroups*. Defining a suitable class of *full measures* we prove a *convergence of types theorem*: A crucial tool for all investigations of limit behaviour of normalized random walks.

In Section 2 we investigate continuous convolution semigroups, in particular *(semi-)stability* and T_a*-decomposability* for $T_a \in \mathrm{Aut}(\mathcal{K})$, following as close as possible the 'classical setup': Characterization of semistability by the decomposability group $\mathfrak{Dec}(\mu)$ and by domains of attraction and describing self-decomposability by the Urbanik semigroup $\mathfrak{D}(\mu)$ and by Lévy processes on space-time hypergroups. Furthermore, again as in the classical situation, it is shown that T_a-decomposable laws are representable as infinite convolution products of probabilities with finite logarithmic moments.

In Section 3 we investigate, following [16], squared Wishart- resp. Gaussian distributions on $\mathcal{K}$, showing that these laws are stable in the afore mentioned sense. Firstly we examen these laws in details and construct new semistable and self-decomposable laws by the method of subordination. In fact, the features of Gaussian laws on $\mathcal{K}$ motivate to introduce *cone-semigroups*, i.e. continuous convolution semigroups with time parameter taking values in a cone, and to generalize the concept of subordination to this more general class of continuous convolution semigroups in order to obtain more examples of limit laws.

All these examples are located in the convex hull of the Gaussian laws. To show the power of the afore mentioned tools, we construct in addition some explicit examples of semistable, stable and decomposable laws beyond the Gaussian case.

0.1 Basic properties of of the hypergroups $(\mathcal{K}, \star)$

As shown in [16] the hypergroups under consideration share the following properties 0.1–0.8:

Property 0.1 *$(\mathcal{K}, \star)$ is Hermitean, i.e. the identity* $id : x \mapsto \tilde{x} := x$ *is the involution. In particular, $(\mathcal{K}, \star)$ is Abelian.*

Furthermore,

Property 0.2 *$(\mathcal{K}, \star)$ is self dual, i.e.* $\widehat{\mathcal{K}} \cong \mathcal{K}$

$[\![$ Cf. [16], Theorem 2.1 $]\!]$

Property 0.3 *The operator semigroup of vector space endomorphisms* $\mathbb{K}^{d^2} \cong \mathrm{End}(\mathbb{K}^d)$ *operates on the hypergroup* $(\mathcal{K}, \star)$ *as homomorphisms w.r.t. convolution: In fact, for $a \in \mathrm{End}(\mathbb{K}^d)$, T_a defined by*

$$\Pi_d \ni A \mapsto T_a(A) := (aAA^*a^*)^{1/2} = \left(aA^2a^*\right)^{1/2} = ((aA)(aA)^*)^{1/2} \in \Pi_d,$$

is a homomorphism of the underlying convolution structures $(\mathcal{K}, \star)$. In short, we write $T_a \in \mathrm{End}(\mathcal{K})$.
Furthermore, the map $\mathfrak{T} : \mathrm{End}(\mathbb{K}^d) \ni a \mapsto T_a \in \mathrm{End}(\mathcal{K})$ *is a semigroup homomorphism.*

$[\![$ Cf. [16], Section 4, Proposition 4.3, 4.7. Note that $\mathrm{GL}(\mathbb{K}^d)$ is dense in $\mathrm{End}(\mathbb{K}^d)$, hence $T_a \in \mathrm{End}(\mathcal{K}, \star)$ for all $a \in \mathrm{End}(\mathbb{K}^d)$. $]\!]$
A more detailed description is given below in Section 1, cf. 1.1–1.5.

Property 0.4 $\mathrm{Aut}(\mathcal{K}) = \mathrm{im}(\mathfrak{T}) = \left\{T_a : a \in \mathrm{GL}(\mathbb{K}^d)\right\}$ *in case* $\mathbb{K} = \mathbb{R}$, *resp.* $\mathrm{Aut}(\mathcal{K}) \supseteq \{\mathrm{im}(\mathfrak{T}) \cup \{\tau\}\}$, *and* $[\mathrm{Aut}(\mathcal{K}) : \mathrm{im}(\mathfrak{T})] = 2$ *in case* $\mathbb{K} = \mathbb{C}$, *τ denoting the involutive mapping defined by complex conjugation.*

$[\![$ Cf. [16], Theorem 4.11, 4.12. $]\!]$ In particular, the connected component $\mathrm{Aut}(\mathcal{K})_0 \cong \mathrm{GL}(\mathbb{K}^d)_0$ is of finite index in $\mathrm{Aut}(\mathcal{K})$. Furthermore, there exists $k_0 \in \mathbb{N}$ such that for any $\tau \in \mathrm{Aut}(\mathcal{K})$ the power τ^{k_0} belongs to the range of the exponential map. $[\![$ See [3], [4]. $]\!]$

Property 0.5 *The proper subhypergroups H of $\mathcal{K}$ are* $H = \ker(T_a)$ *– equivalently:* $H = \mathrm{im}(T_b)$ *– for some a, resp. $b \in \mathrm{End}(\mathbb{K}^d)$.*
In particular, the hypergroups $\mathcal{K}$ under consideration are aperiodic, i.e. there exist no non-trivial compact subhypergroups.

$[\![$ Cf. [16], Section 4, Proposition 4.6 $]\!]$
We describe the subhypergroups in more details in 1.1, 1.2 below.

Property 0.6 *$\mathcal{K}$ is a Godement hypergroup, i.e.,* $\mathbf{1} \in \widehat{\mathcal{K}}$ *belongs to* $\mathrm{supp}(\pi_{\mathcal{K}})$, *$\pi_{\mathcal{K}}$ denoting the Plancherel measure.*
In fact, Haar measure and Plancherel measure on $\mathcal{K}$ resp. on $\widehat{\mathcal{K}}\,(\cong \mathcal{K})$ are equivalent to the d^2-dimensional Lebesgue measure restricted to Π_d.

$[\![$ Cf. [16], Theorem 2.1 $]\!]$

The main part of these investigations relies on the above-mentioned properties 0.1 – 0.6. For investigations of examples we need further particular properties of the matrix-cone hypergroups:

Let $||\cdot||$ denote the Euclidean norm on $\mathbb{K}^d$ and let $|||\cdot|||$ denote the corresponding operator norm on $\mathrm{End}(\mathbb{K}^d)$, hence defined on $\Pi_d \cong \mathcal{K}$. Therefore, $|||\cdot|||$ fulfills the C^*-condition $|||aa^*||| = |||a|||^2$, hence in particular we have for $x \in \Pi_d$: $|||x^2||| = |||x|||^2$.
Note that the Hilbert space norm $||x||_* := (\mathrm{tr}\,(x^2))^{1/2}$ used in [16] is different, but of course equivalent. In the sequel we write $||\cdot||$ for an arbitrary norm, and $|||\cdot|||$ resp. $||\cdot||_*$ if we want to emphasize the particular chosen norm.

Property 0.7 *For $x, y \in \mathcal{K}$ we have $||\cdot||_*$ we have $||z||_* \leq ||x||_* + ||y||_*$. $[\![$ Cf. [16], Theorem 2.1 a). $]\!]$ Hence, for some constant $C > 0$,*

$$\mathrm{supp}(\varepsilon_x \star \varepsilon_y) \subseteq \{z : |||z||| \leq C \cdot (|||x||| + |||y|||)\}$$

Therefore, for independent random variables $X, Y : \Omega \to \mathbb{K}$ we obtain

$$||X \overset{\Lambda}{+} Y||_* \leq ||X||_* + ||Y||_* \text{ resp. } |||X \overset{\Lambda}{+} Y||| \leq C \cdot (|||X||| + |||Y|||),$$

where Λ denotes a concretization of the hypergroup operation: If μ, ν are the distributions of X, Y then $\mu \star \nu$ is the distribution of $X \overset{\Lambda}{+} Y$. $[\![$ Cf. [2] $]\!]$

In Section 3 we make use of particular properties of the dual of the underlying hypergroups: For fixed parameter μ and dimension d, we consider the convolution structure $(\mathcal{K}, \star) = (\Pi_d, \star_\mu)$ on Π_d. (Cf. [16] Theorem 2.1 c), Lemma 4.1, Proposition 4.3, [13], (3.15)).

Property 0.8 *The elements of the dual hypergroup $\widehat{\mathcal{K}}$ are representable by matrix-Bessel functions $\mathcal{J}_\mu$. In fact, the characters φ_κ induced by $\kappa \in \Pi_d \cong \widehat{\Pi_d}$ are*

$$\varphi_\kappa(x) = \mathcal{J}_\mu\left(\frac{1}{4}\kappa x^2 \kappa\right) = \mathcal{J}_\mu\left(\frac{1}{4}x \kappa^2 x\right) = \varphi_x(\kappa) \tag{0.1}$$

and we have the asymptotics (for $x \to 0$)

$$\mathcal{J}_\mu(x) = 1 - \frac{1}{\mu}\mathrm{tr}(x) + O(||x||^2) \tag{0.2}$$

Furthermore, the endomorphisms T_a act canonically on the dual hypergroup

$$\varphi_\kappa(T_a(x)) = \varphi_{T_{a^*}(\kappa)}(x) \quad \forall\, x \in \Pi_d,\ \kappa \in \widehat{\Pi}_d \tag{0.3}$$

0.2 Convolutions

Recall first the definitions of convolution of measures and functions on hypergroups: For $\mu, \nu \in M^1(\mathcal{K})$ and for bounded measurable f we define

$$\langle f, \mu \star \nu \rangle := \int_{\mathcal{K}} \int_{\mathcal{K}} f(x \star y) d\mu(x) d\nu(y) \tag{0.4}$$

$$\text{where } f(x \star y) := \int_{\mathcal{K}} f(z) d\varepsilon_x \star \varepsilon_y(z)$$

With the notations $f_y : z \mapsto f(z \star y)$ resp. ${}_x f : z \mapsto f(x \star z)$ we obtain

$$\langle f, \mu \star \nu \rangle = \int_{\mathcal{K}} \int_{\mathcal{K}} f_y(x) d\mu(x) d\nu(y) = \int_{\mathcal{K}} \int_{\mathcal{K}} {}_x f(y) d\mu(x) d\nu(y) \tag{0.5}$$

Sometimes it is convenient to use convolution operators

$$L_\lambda(f)(x) := \langle {}_xf, \lambda\rangle \text{ resp. } R_\lambda(f)(y) := \langle f_y, \lambda\rangle.$$

Note that Hermitean hypergroups are Abelian and the involution is the identity. Hence $R_\lambda = L_\lambda$ and the definition coicides with the usual one. (Cf [2]).

For Borel-measurable $B \subseteq \mathcal{K}$ we obtain therefore

$$\begin{aligned} \mu \star \nu(B) &= \langle 1_B, \mu \star \nu\rangle \qquad (0.6)\\ &= \int_{\mathcal{K}}\int_{\mathcal{K}} 1_B(x \star y) d\mu(x) d\nu(y) = \int_{\mathcal{K}}\int_{\mathcal{K}} 1_{B_y}(x) d\mu(x) d\nu(y)\\ &= \int_{\mathcal{K}}\int_{\mathcal{K}} {}_x 1_B(x) d\mu(x) d\nu(y) = \int_{\mathcal{K}}\int_{\mathcal{K}} L_{\varepsilon_y}(1_B)(x) d\mu(x) d\nu(y) \end{aligned}$$

Furthermore, sometimes we make use of the following abbreviations (cf. e.g., [8]): Let $\{\mu\} := \mathrm{supp}(\mu)$, in particular, $\{\varepsilon_a\} = \{a\}$. Furthermore, we put $x \star y := \{(\varepsilon_x \star \varepsilon_y)\}$ and more generally, $y \star A := \bigcup_{a\in A} \{(\varepsilon_y \star \varepsilon_a)\}$ and finally $A \star B := \bigcup_{a\in A,\ b\in B} \{(\varepsilon_a \star \varepsilon_b)\}$. By (0.6) we obtain immediately

Proposition 0.9 *For $\mu, \nu \in M^1(\mathcal{K})$ and Borel sets B we have*

$$\mu \star \nu(B) \le \int \mu(y \star B) d\nu(y)$$

There exist no divisors of the unit in $M^1(\mathcal{K})$:

Proposition 0.10 *(a) Let $\mu, \nu \in M^1(\mathcal{K})$ then $\mu \star \nu = \varepsilon_0 \Rightarrow \mu = \nu = \varepsilon_0$.*
(b) In particular, $\mu^k = \varepsilon_0 \ \Rightarrow \mu = \varepsilon_0,\ k \in \mathbb{N}$

⟦ For $x \in \mathrm{supp}(\mu),\ y \in \mathrm{supp}(\nu)$ we have $\{x \star y\} \subseteq \{0\}$, hence $\varepsilon_x \star \varepsilon_y = \varepsilon_0$. Whence – since $y = \tilde{y}$ – $\{x\} \cap \{y\} \neq \emptyset$. Hence $x = y$ and therefore $\varepsilon_x^2 = \varepsilon_0$ follows. I.e., $\{x, 0\}$ is a compact subhypergropup. But $\mathcal{K}$ is aperiodic according to Property 0.5, therefore $x = y = 0$. Thus $\mu = \nu = \varepsilon_0$. ⟧

In fact, we prove a generalization of 0.10 (b): No non-trivial point-measure is divisible.

Proposition 0.11 *Let $k \in \mathbb{N}\backslash\{1\}$ and assume $\mu^k = \varepsilon_z$. Then $z = 0$ and $\mu = \varepsilon_0$*

⟦ Let $k = 2$.
Hence $\mu^2 = \varepsilon_z$. Hence for all $x \in \mathrm{supp}(\mu)$ we observe – since $x = \tilde{x}$ – that $0 \in \{x \star x\} = \{z\}$ and therefore $z = 0$. Hence $\mu = \varepsilon_0$ according to Proposition 0.10(b).
Let $k = 2l,\ l \in \mathbb{N}$.
$\mu^{2l} = \varepsilon_z$, hence $\nu^2 = \varepsilon_z$ where $\nu := \mu^l$. According to the first step we obtain $\nu = \varepsilon_0$, i.e., $\mu^l = \varepsilon_0$. Therefore, again using Proposition 0.10 above, $\mu = \varepsilon_0$ follows.
Let $k = 2l + 1,\ l \in \mathbb{N}$.
For all $x \in \mathrm{supp}(\mu)$ we have: $\{x\} \star \{x \star x\} \ldots \{x \star x\} \subseteq \{z\}$. Since $0 \in \{x \star x\}$ we conclude $\{x\} \subseteq \{z\}$, hence $x = z$. I.e., $\{x\} \star \{x \star x\} \ldots \{x \star x\} = \{x\}$
Hence $\{x \star x\} \star \{x \star x\} \ldots \{x \star x\} \subseteq \{x \star x\}$ for k factors, $k \le l + 1$. Therefore, $\{x \star x\}$ is a compact subhypergroup, hence trivial. Whence $x = 0$ follows. ⟧

0.3 Uniform tightness and shift compactness

Shift compactness is an essential tool for investigations in limit theorems on vector spaces and on groups. Therefore we recall shift-compactness properties for hypergroups:

Proposition 0.12 *(Shift-compactness)*
Let $\{\mu_n,\ \nu_n,\ \lambda_n := \mu_n \star \nu_n,\ n \in \mathbb{N}\}$ *be sequences of probabilities on the hypergroup* $\mathcal{K}$.
(a) *Assume* $\{\lambda_n\}$ *to be uniformly tight. Then there exist sequences* $\{x_n\}$, $\{y_n\} \subseteq \mathcal{K}$ *such that* $\{\mu_n \star \varepsilon_{x_n}\}$ *and* $\{\varepsilon_{y_n} \star \nu_n\}$ *are uniformly tight. Furthermore – since the hypergroup is Abelian –* $\{\varepsilon_{x_n} \star \varepsilon_{y_n}\}$ *is uniformly tight too.*
(b) *If any two of the sequences* $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda_n\}$ *are uniformly tight, then so is the third one.*
(c) *If* $\{\lambda_n\}$ *is uniformly tight then* $\{\mu_n^2\}$ *and* $\{\nu_n^2\}$ *are uniformly tight.*
(d) *Let* $\{\lambda_n\}$ *be uniformly tight. Then – with the the notations of (a) –* $\left\{\varepsilon_{x_n}^2\right\}$ *and* $\left\{\varepsilon_{y_n}^2\right\}$ *are uniformly tight.*

For a **Proof** of (a) and (b) see [2], 5.1.14 ff.
Proof of (c):
$\mathcal{K}$ is Hermitean, hence the Fourier-transforms $\left\{\widehat{\mu_n^2} = \widehat{\mu_n}^2\right\}$ and $\left\{\widehat{\nu_n^2} = \widehat{\nu_n}^2\right\}$ are non-negative. Therefore the Godement-property 0.5 yields that $\{\mu_n^2\}$ and $\{\nu_n^2\}$ are uniformly tight. (Cf. [2], 5.1.1)
(d) follows immediately by (a) and (c). □

We need the following Corollary.

Corollary 0.13 ***(a)*** *Let* $\{\mu_n\}$ *be a sequence of probabilities with* $\mu_n \to \mu$ *and* $\lambda_n := \mu_n \star \varepsilon_{x_n} \to \lambda$ *for some sequence* $\{x_n\} \subseteq \mathcal{K}$. *Then* $\{\mu_n\}$ *is relatively compact. And for all accumulation points* x *of* $\{x_n\}$ *we have* $\lambda = \mu \star \varepsilon_x$.
(b) *Let* $\{\nu_n\}$, $\{\lambda_n\}$ *be sequences of probabilities with* $\nu_n \to \varepsilon_0$. *Assume that* $\nu_n \star \lambda_n \to \lambda$. *Then* $\lambda_n \to \lambda$.

⟦ Shift-compactness, 0.12, applied to $\nu_n := \varepsilon_{x_n}$ implies uniform tightness of $\{\varepsilon_{x_n}\}$, whence (a) follows.
To prove (b) note that again shift compactness (cf. Proposition 0.12(b)) implies uniform tightness of $\{\lambda_n\}$. For all converging subsequences $\lambda_n \to \lambda'$ therefore continuity of convolution yields $\nu_n \star \lambda_n \to \lambda'$. Hence by assumption, $\lambda = \lambda'$ follows. ⟧

1 Hypergroup automorphisms and Convergence of types

1.1 $\mathrm{Aut}(\mathcal{K})$ and $\mathrm{End}(\mathcal{K})$

In 1.1 we study in more details the underlying matrix cones $\Pi_d \cong \mathcal{K}$ and the corresponding endomorphisms and automorphisms: As already mentioned (Properties 0.3– 0.5), for $a \in \mathrm{End}(\mathbb{K}^d)$ the operator $T_a : A \mapsto (aAA^*a^*)^{1/2} = ((aA)(aA)^*)^{1/2}$ is a convolution homomorphism of $(\mathcal{K}, \star)$, the kernels resp. images of which determine the subhypergroups. We now describe more explicitly $\ker(T_a)$ and $\mathrm{im}(T_a)$:

Consider the polar decomposition $a = up$ with unitary u and $p \in \Pi_d$. We obtain an orthogonal decomposition $\mathbb{K}^d = U \oplus V$ with $\ker(p) = V$, hence $p = p_1 \oplus 0$ with positive definite $p_1 : U \to U$. Recall the notation $\mathfrak{T} : a \mapsto T_a$. We have (cf. [16], Section 4, in particular Proposition 4.6, Remark 4.10):

Proposition 1.1 ***(a)*** $\ker(T_a) = \ker(T_p) = \{A = 0 \oplus \gamma$ *with γ positive semidefinite on $V\}$. In other words, with respect to the decomposition $\mathbb{K}^d = U \oplus V$ (defined by the positive semidefinite part p of a) we obtain*

$$\ker(T_a) = \left\{ A = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}, \gamma \textit{ positive semidefinite} \right\}$$

(b) $a \in \mathrm{GL}(\mathbb{K}^d)$ *iff* $\ker(T_a) = \{0\}$. *Hence* $\mathfrak{T}(\mathrm{GL}(\mathbb{K})) \subseteq \mathrm{Aut}(\mathcal{K})$.
(In fact, as afore mentioned in 0.4, in [16] it is shown that equality holds for $\mathbb{K} = \mathbb{R}$).

⟦ (a) For $a = up$ and $A \in \ker(T_a)$ we have:
$0 = T_a(A)^2 = aA^2a^* = upA^2pu^* = u(pA)(pA)^*u^* \Leftrightarrow 0 = (pA)(pA)^* = T_p(A)^2$.
Hence $\ker(T_a) = \ker(T_p)$.

Let, with respect to the above mentioned decomposition, $p = p_1 \oplus 0$ and $A = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$. Then, considering $pA^2p = 0$ we obtain $p_1\alpha^2p_1 + p_1\beta^*\beta p_1 = 0$. Both summands are positive semidefinite, whence $p_1\alpha^2p_1 = (p_1\alpha)(p_1\alpha)^* = 0$ and $p_1\beta^*\beta p_1 = (p_1\beta^*)(p_1\beta^*)^* = 0$ follow. This yields $p_1\alpha = 0$ and $p_1\beta^* = 0$. But $p_1 : U \to U$ is injective by assumption, whence $\alpha = \beta = 0$.
(b) is now obvious. ⟧

With the afore mentioned notations we obtain analogously the following description of $\mathrm{im}(T_b)$:

Proposition 1.2 *Let $b \in \mathrm{End}(\mathbb{K}^d)$, consider the polar decomposition $b = uq$ with corresponding orthogonal decomposition $\mathbb{K}^d = U_1 \oplus V_1$, $q = 0 \oplus q_1$ with $q_1 : V_1 \to V_1$ positive definite.*
Then $\mathrm{im}(T_b) = \left\{ B = u \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix} u^* : \delta \textit{ positive semidefinite on } V_1 \right\}$.

⟦ Since $\mathrm{im}(T_b)$ is a subhypergroup it has the above mentioned representation according to Proposition 1.1.
To prove the converse, let $b = u(0 \oplus q_1)$. u defines an isometry $u : V_1 \to \mathrm{im}(q) = u(V_1) =: V$.

Let $A \in \mathrm{im}(T_b)$, $A = A^* = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$ w.r.t. the orthogonal decomposition $\mathbb{K}^d = U_1 \oplus V_1$. Let $y = y_1 + y_2$, $y_1 \in U, y_2 \in V$, let $z := u^*(y) = z_1 + z_2$ with $z_1 \in U_1, z_2 \in V_1$. Then $T_b(A)^2(y) = (bA)(bA)^*(y) = uqA^2qu^*(y) = u\left((qA)(qA)^*\right)(z) = u\left((qA)(qA)^*\right)(z_1+z_2) = u\left(\left((q_1\beta\beta^*q_1) + (q_1\gamma^2q_1)\right)(z_2)\right) =: u\delta u^*(y_2) \in u(V_1)$. Hence, $B = T_b(A)$ has the asserted representation. ⟧

Proposition 1.3 *The above mentioned map $\mathfrak{T} : a \mapsto T_a$, $\mathrm{End}(\mathbb{K}^d) \to \mathrm{End}(\mathcal{K})$ is an isometry with respect to the above mentioned norms. In fact,*

$$|||a||| := \sup_{x \in \mathbb{K}^d:\ ||x|| \leq 1} ||a(x)|| = \sup_{A \in \Pi_d:\ |||A||| \leq 1} |||T_a(A)||| =: |||T_a|||$$

Proof: Observe first that $|||T_a(A)||| = |||\left((aA)(aA)^*\right)^{1/2}||| = |||aA|||$ according to the C^*-property of $|||\cdot|||$. Hence $|||T_a(A)||| \leq |||a||| \cdot |||A|||$. Whence $|||T_a||| \leq |||a|||$.

To prove the converse inequality, define for $x \in \mathbb{K}^d \backslash \{0\}$ the operator P_x by $P_x(y) := \langle y, x\rangle \cdot x$. For normalized x, P_x is an orthogonal projection, hence for all $x \neq 0$ we observe that $P_x = ||x||^2 \cdot P_{x/||x||}$ belongs to Π_d. For $a \in \mathrm{End}(\mathbb{K}^d)$ we have $T_a(P_x) = (P_{(a(x))})^{1/2}$, therefore for $||x|| = 1$ we have $T_a(P_x) = ||a(x)|| \cdot P_{\frac{a(x)}{||a(x)||}}$. Note that $|||P_{\frac{a(x)}{||a(x)||}}||| = 1$. Hence $|||T_a(P_x)||| = ||a(x)||$.

Whence by definition, $|||a||| \leq |||T_a|||$ follows. □

Definition 1.4 *A linear operator $a \in \mathrm{End}(\mathbb{K}^d)$ is called contracting if $a^n(x) \to 0$ for all $x \in \mathbb{K}^d$. Analogously, $T_a \in \mathrm{End}(\mathcal{K})$ is contracting if $T_a^n(A) \to 0$ for all $A \in \Pi_d \cong \mathcal{K}$, i.e., $T_a^n \to T_0$.*

Clearly, a is contracting iff $\rho(a) < 1$, ρ denoting the spectral radius. Furthermore, by Proposition 1.3 we have

Corollary 1.5 *$a \in \mathrm{End}(\mathbb{K}^d)$ is contracting iff $T_a \in \mathrm{End}(\mathcal{K})$ is contracting.*

1.2 Properties of full measures

Next we define as in the vector space- or group case the *class of full probabilities.* (Full measures are sometimes also called non-degenerate).

Definition 1.6 *Full probabilities are defined as*
$\mathcal{F} := \{\mu \in M^1(\mathcal{K}) :$ *not concentrated on a proper subhypergroup*$\}$.

Obviously, we have in view of Property 0.4 resp. Proposition 1.1, 1.2

$$\begin{aligned} \mathcal{F} &= \left\{\mu \in M^1(\mathcal{K}) : \mu\left\{\ker(T_a)\right\} \neq 1 \ \forall\ T_a \in \mathrm{End}(\mathcal{K}) \backslash \left\{T_0\right\}\right\} \\ &= \left\{\mu \in M^1(\mathcal{K}) : \mu\left\{\mathrm{im}(T_b)\right\} \neq 1 \ \forall\ T_b \in \mathrm{End}(\mathcal{K}) \backslash \left\{\mathrm{Aut}(\mathcal{K})\right\}\right\} \\ &= \left\{\mu \in M^1(\mathcal{K}) : T_a(\mu) \neq \varepsilon_0 \ \forall\ T_a \in \mathrm{End}(\mathcal{K}) \backslash \left\{T_0\right\}\right\} \end{aligned}$$

As for vector spaces and groups we observe

Proposition 1.7 *$\mathcal{F}$ is open in $M^1(\mathcal{K})$.*

Proof: (Cf. [5], p. 126). The subset $\mathbb{H} := \{T_a \in \mathrm{End}(\mathcal{K}) : |||T_a||| = 1\}$ is compact. $\mathfrak{W} := M^1(\mathcal{K}) \backslash \{\varepsilon_0\}$ is open. $F : M^1(\mathcal{K}) \times \mathbb{H} \to M^1(\mathcal{K}),\ F(\mu, T_a) := T_a(\mu)$ is continuous. Consequently, $T_a^{-1}(\mathfrak{W})$ is open for all T_a.

Hence $\mathcal{F} = \bigcap\limits_{T_a \in \mathbb{H}} T_a^{-1}(\mathfrak{W})$ is open. □

The last step of the proof relies on the following well-known Lemma:

Lemma 1.8 *Let H, K, S denote Hausdorff spaces, H compact. Let $\mathcal{O} \subseteq S$ be open. Let $F : H \times K \to S$ be continuous. For $h \in H$ define $F_h : K \to S,\ k \mapsto F(h, k)$.*

Then $U := \bigcap\limits_{h \in H} F_h^{-1}(\mathcal{O})$ is open in K.

⟦ We sketch a **proof:**
Let $C(H,S)$ denote the space of continuous functions $H \to S$ endowed with the compact-open topology. Hence a neighbourhood basis is given by $U_{L,W} := \{\phi : H \to S : \text{ with } \phi(L) \subseteq W\}$, where L and W denote the compact subsets in H and open subsets in S respectively.

For $L = H$, $W = \mathcal{O}$ therefore $\mathfrak{W} := \{\phi \in C(H,S) : \phi(h) \in \mathcal{O} \ \forall \ h \in H\}$ is open in $C(H,S)$.

The map $\Phi : K \ni k \mapsto F^k \in C(H,S)$, $F^k : H \ni h \mapsto F(h,k)$ is continuous, since F is simultaneously continuous. Hence $\Phi^{-1}(\mathfrak{W})$ is open, i.e. $\{k \in K : F(h,k) \in \mathcal{O} \ \forall \ h \in H\}$ is open in K. ⟧

Proposition 1.9 ***(a)*** *$\mathcal{F}$ is an ideal in the convolution semigroup $M^1(\mathcal{K})$, i.e., for $\mu \in \mathcal{F}$ and all $\nu \in M^1(\mathcal{K})$ we have $\mu \star \nu \in \mathcal{F}$.*
(b) *If $\mu^k \in \mathcal{F}$ for some $k \in \mathbb{N}$ then $\mu \in \mathcal{F}$.*

⟦ Assume $\mu \star \nu \notin \mathcal{F}$. Then by definition, we have $T_a(\mu \star \nu) = \varepsilon_0$ for some $T_a \neq T_0$. Hence $T_a(\mu) \star T_a(\nu) = \varepsilon_0$. Therefore, according to Proposition 1.3, $T_a(\mu) = T_a(\nu) = \varepsilon_0$ follows. In particular, $\mu \notin \mathcal{F}$. Hence (a) is proved.

To prove (b), assume $\mu \notin \mathcal{F}$. Hence again $T_a(\mu) = \varepsilon_0$ for some $T_a \neq T_0$. Then $T_a(\mu^k) = T_a(\mu)^k = \varepsilon_0$ for all k. Whence (b) follows. ⟧

Now we are ready to prove a *convergence of types theorem* for matrix cone hypergroups $\mathcal{K}$:

Theorem 1.10 *Let $\{\mu_n, \ \nu_n, \ n \in \mathbb{N}, \ \mu, \nu\}$ be probabilities, let $\{a_n, \ n \in \mathbb{N}\} \subseteq$ $\mathrm{End}(\mathcal{K})$, such that $\nu_n = T_{a_n}(\mu_n)$. (In short, ν_n belongs to the type of μ_n.)*
(a) *Assume (1) $\mu_n \to \mu$ (2) $\nu_n = T_{a_n}(\mu_n) \to \nu$ and (3) $\mu \in \mathcal{F}$.*
Then $\{T_{a_n}\}$ is relatively compact (equivalently: $\{a_n\}$ is relatively compact in $\mathrm{End}(\mathbb{K}^d)$) and for any accumulation point T_a we have: $\nu = T_a(\mu)$.
(b) *Assume (1) $\mu_n \to \mu$ (2) $\nu_n = T_{a_n}(\mu_n) \to \nu$ (3') $\nu \in \mathcal{F}$ and (4) $a_n \in \mathrm{GL}(\mathbb{K}^d)$, $n \in \mathbb{N}$.*
Then $\{T_{a_n}^{-1}\}$ is relatively compact (equivalently: $\{a_n^{-1}\}$ is relatively compact in $\mathrm{End}(\mathbb{K}^d)$) and for any accumulation point T_b of $\{T_{a_n}^{-1}\}$ we have: $\mu = T_b(\nu)$.
(c) *Together: Assume (1), (2), (3") $\mu, \nu \in \mathcal{F}$ and (4).*
Then $\{T_{a_n}\}$ is relatively compact in $\mathrm{Aut}(\mathcal{K})$ (equivalently: $\{a_n\}$ is relatively compact in $\mathrm{GL}(\mathbb{K}^d)$) and for any accumulation point T_a of $\{T_{a_n}\}$ we have: $\mu = T_a(\nu)$ and $\nu = T_a^{-1}(\mu)$.

Proof. We adapt a standard proof for vector spaces (cf. e.g., [5], § 1.13) :
To show that $\{|||a_n|||\}$ is uniformly bounded, assume w.l.o.g. that $|||a_n||| \to \infty$. Decompose $a_n = (|||a_n||| \cdot I)\left(\frac{1}{|||a_n|||} \cdot a_n\right) =: \alpha_n \beta_n$. There exists a subsequence (n') such that $\beta_n \to \beta \in \mathrm{End}(\mathbb{K}^d)$ with $|||\beta||| = 1, n \in (n')$. Therefore, $\lambda_n := T_{\beta_n}(\mu_n) \to T_\beta(\mu) =: \lambda$, $n \in (n')$. On the other hand, $\nu_n = T_{\alpha_n}(\lambda_n) \to \nu$ by assumption.

But $\alpha_n^{-1} \to 0$, therefore $T_{\alpha_n^{-1}}(\lambda_n) \to \varepsilon_0$, hence $\lambda = \varepsilon_0$ follows. Consequently, $T_\beta(\mu) = \varepsilon_0$, whence $\mu \notin \mathcal{F}$, a contradiction.
(a) is proved.

To prove (b) apply the preceding step to $\{b_n := a_n^{-1}\}$, $\{\mu_n = T_{b_n}(\nu_n)\}$. Then uniform boundedness of $\{|||a_n^{-1}|||\}$ follows.

(c) follow immediately by (a) and (b) □

Again as in the vector space- or group-case we obtain a characterization of full measures via the *invariance group.* (Cf. e.g., [5], e.g., 2.2.5 ff.)

Definition 1.11 *The invariance group of $\mu \in M^1(\mathcal{K})$ is defined as*

$$\mathfrak{Inv}(\mu) := \{T_a \in \mathrm{End}(\mathcal{K}) : \ T_a(\mu) = \mu\}$$

Obviously $\mathfrak{Inv}(\mu)$ is a closed subsemigroup of $\mathrm{End}(\mathcal{K})$. For full measures $\mu \in \mathcal{F}$ it is immediately seen that $\mathfrak{Inv}(\mu)$ is a closed subgroup of $\mathrm{Aut}(\mathcal{K})$. In fact, we obtain the following characterization of full measures:

Proposition 1.12 *$\mu \in \mathcal{F}$ iff $\mathfrak{Inv}(\mu)$ is a compact subgroup of* $\mathrm{Aut}(\mathcal{K})$.

Proof: As afore mentioned, for full measures, $\mathfrak{Inv}(\mu)$ is a closed subgroup of $\mathrm{Aut}(\mathcal{K})$. The convergence of types theorem 1.10 (c) yields compactness of $\mathfrak{Inv}(\mu)$ in $\mathrm{Aut}(\mathcal{K})$.

Conversely, assume $\mu \notin \mathcal{F}$. Hence $\mu(\ker(T_a)) = 1$ for some $a \in \mathrm{End}(\mathbb{K}^d)$. According to Proposition 1.1 for $a = p \in \Pi_d$, and $\ker(T_a) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \right\}$ (with respect to an orthogonal decomposition $\mathbb{K}^d = U \oplus V$, and positive semidefinite $c \in \mathrm{End}(V)$) we obtain $\left\{ B = \begin{pmatrix} \alpha & 0 \\ 0 & Id_V \end{pmatrix} : \ \alpha \in \mathrm{End}(U) \right\} \subseteq \mathfrak{Inv}(\mu)$. This set is not bounded, hence $\mathfrak{Inv}(\mu)$ is not compact. □

1.3 S-full measures and convergence of types theorems with shifts

On vector spaces full measures are usually defined as measures not concentrated on proper *hyperplanes*, hence not concentrated on kernels of proper *affine maps*. Already in the case of (non Abelian) groups it turned out to be necessary to distinguish between fullness w.r.t. endomorphisms and w.r.t. affine transformations respectively. Therefore we sketch briefly this concept in the case of hypergroups, though it will not be needed in the sequel.

Note that for vector spaces (and for groups) there exist various equivalent definitions of full measures. (See e.g. [5], § 1.13, 2.2.5 ff, [10]). Generalizing the vector space case, it seems at first natural to define S-full measures as

$\mathcal{F}_1 := \{\mu : \mu \star \varepsilon_z \in \mathcal{F}$ for all $z \in \mathcal{K}\}$

However it is easily shown that for the class of hypergroups considered here we have $\mathcal{F} = \mathcal{F}_1$.

⟦ Obviously, $\mathcal{F}_1 \subseteq \mathcal{F}$. On the other hand, $\mathcal{F}$ is an ideal (cf. 1.9), hence, if $\mu \in \mathcal{F}$ then $\mu \star \varepsilon_z \in \mathcal{F}$ for all z. Whence $\mathcal{F} \subseteq \mathcal{F}_1$ follows. ⟧

Hence we use an other definition, which is equivalent to the previous one in the vector space case:

Definition 1.13 *S-full measures are defined as*

$$\mathcal{SF} := \{\mu : \forall \ z \in \mathcal{K}, \ \forall T_a \in \mathrm{End}(\mathcal{K}) \setminus \{T_0\} : \ T_a(\mu) \neq \varepsilon_z\}$$

We observe

Proposition 1.14 ***(a)*** $\mathcal{SF} \subseteq \mathcal{F}$
(b) *$\mathcal{SF}$ is open in $M^1(\mathcal{K})$*

$[\![$ Let $\mathcal{F}_* := M^1(\mathcal{K}) \backslash \{\varepsilon_0\}$ and $\mathcal{SF}_* := M^1(\mathcal{K}) \backslash \{\varepsilon_z,\ z \in \mathcal{K}\}$. Then $\mathcal{F} = \bigcap\limits_{|||a|||=1} T_a^{-1}(\mathcal{F}_*) \supseteq \bigcap\limits_{|||a|||=1,\ z\in\mathcal{K}} T_a^{-1}(M^1(\mathcal{K}) \backslash \{\varepsilon_z\}) = \mathcal{SF}$. Whence (a) follows.
To prove (b) represent $\mathcal{SF}$ as $\bigcap\limits_{|||a|||=1} T_a^{-1}(\mathcal{SF}_*)$ and note that $\mathcal{SF}_*$ is open.
According to Lemma 1.8, $\mathcal{SF}$ is open. $]\!]$

Proposition 1.15 ***(a)*** $\mu \in \mathcal{F} \Rightarrow \mu^k \in \mathcal{SF},\ k \in \mathbb{N} \backslash \{1\}$
(b) *However, $\mu^k \in \mathcal{SF}$ does in general not imply $\mu \in \mathcal{SF}$*
(c) *In general, $\mathcal{SF} \overset{\subset}{\neq} \mathcal{F}$*

$[\![$ Assume $\mu^k \notin \mathcal{SF}$, hence $T_a(\mu^k) = T_a(\mu)^k = \varepsilon_z$ for some $z \in \mathcal{K}$, $a \neq 0$ and $k \geq 2$. Therefore, according to 0.10, $T_a(\mu) = \varepsilon_0 = \varepsilon_z$. Hence $\mu \notin \mathcal{F}$.

Concerning (b) and (c) consider the case $\dim(\mathcal{K}) = d = 1$. (Note that in this case, $\mathrm{End}(\mathcal{K}) \cong \mathbb{K}$.)

Let $\mu = \varepsilon_z$ for some $z \neq 0$. Hence $T_a(\mu) \neq \varepsilon_0\ \forall a \neq 0$, hence $\mu \in \mathcal{F}$ but $\mu \notin \mathcal{SF}$. This proves (c).

But for $k \geq 2$, $\mu^k = \varepsilon_z^k \neq \varepsilon_u$ for all $u \in \mathcal{K}$ (cf. 0.11). It follows, $\mu \notin \mathcal{F}$, but $\mu^k \in \mathcal{SF}$. Whence (b) is proved. $]\!]$

Next we prove a convergence-of-types-theorem for S-full measures:

Theorem 1.16 *Let $\{\mu_n,\ \nu_n,\ n \in \mathbb{N},\ \mu, \nu\}$ be probabilities, let $\{a_n,\ n \in \mathbb{N}\} \subseteq \mathrm{End}(\mathcal{K})$, $\{x_n\} \subseteq \mathcal{K}$ such that $\nu_n = T_{a_n}(\mu_n) \star \varepsilon_{x_n}$. (In short, ν_n belongs to the S-type of μ_n.)*
(a) *Assume (1) $\mu_n \to \mu$ (2) $\nu_n = T_{a_n}(\mu_n) \star \varepsilon_{x_n} \to \nu$ and (3) $\mu \in \mathcal{F}$. Then $\{T_{a_n}\}$ and $\{x_n\}$ are relatively compact and for any accumulation point $(T_a,\ x)$ we have: $\nu = T_a(\mu) \star \varepsilon_x$.*
(b) *Assume (1) $\mu_n \to \mu$ (2) $\nu_n = T_{a_n}(\mu_n) \star \varepsilon_{x_n} \to \nu$ (3') $\nu \in \mathcal{SF}$ and (4) $a_n \in \mathrm{GL}(\mathbb{K}^d)$, $n \in \mathbb{N}$.*
Then $\{T_{a_n}^{-1}\}$ and $\{y_n := T_{a_n}^{-1}(x_n)\}$ are relatively compact and for any accumulation point $(T_b,\ y)$ of $\{(T_{a_n}^{-1},\ y_n)\}$ we have: $\mu \star \varepsilon_y = T_b(\nu)$.
(c) *Together: Assume (1), (2), (3") $\mu \in \mathcal{F}$, $\nu \in \mathcal{SF}$ and (4).*
Then $\{T_{a_n}\}$, $\{x_n\}$, $\{y_n := T_{a_n^{-1}}(x_n)\}$ are relatively compact and for any accumulation point $(T_a,\ x,\ y)$ of $\{(T_{a_n},\ x_n,\ y_n)\}$ in $\mathrm{Aut}(\mathcal{K}) \times \mathcal{K} \times \mathcal{K}$ we have: $\nu = T_a(\mu) \star \varepsilon_x$ and $\mu \star \varepsilon_y = T_a^{-1}(\nu)$.

Sketch of the **Proof**. It is similar to the proof of the corresponding result Theorem 1.10 without shifts:
(a) Assume $|||a_n||| \to \infty$. Decompose again as in the proof of Theorem 1.10 $T_{a_n} = T_{\alpha_n} T_{\beta_n}$ with $\beta_n \to \beta \neq 0$ (along a subsequence) and $\alpha_n^{-1} \to 0$.

Hence $\lambda_n := T_{\beta_n}(\mu_n) \to T_\beta(\mu) =: \lambda$. Therefore, $\nu_n = T_{\alpha_n}(T_{\beta_n}(\mu_n) \star \varepsilon_{y_n})$ with y_n defined as above. $\nu_n \to \nu$ (by assumption) yields $T_{\alpha_n}^{-1}(\nu_n) \to \varepsilon_0$, hence $\lambda_n \star \varepsilon_{y_n} \to \varepsilon_0$, and $\lambda_n \to \lambda$.
Shift-compactness (cf. 0.12) yields relative compactness of $\{y_n\}$. Hence, according to Corollary 0.13 , $\lambda \star \varepsilon_y = \varepsilon_0$ for all accumulation points y of $\{y_n\}$. But there are no divisors of ε_0 (see 0.10) , hence $\lambda = \varepsilon_0$ and $y = 0$ follow. A contradiction.

Hence, $|||a_n|||$ is bounded and thus $\{T_{a_n}\}$ and hence $\{T_{a_n}(\mu_n)\}$ are relatively compact.
Since in addition $T_{a_n}(\mu_n) \star \varepsilon_{x_n} \to \nu$, again by shift-compactness, $\{x_n\}$ is relatively compact. (a) is proved.

To prove (b), assume $|||a_n^{-1}||| \to \infty$. Decompose
$a_n^{-1} =: b_n = (|||a_n^{-1}||| \cdot I) \left(\frac{1}{|||a_n^{-1}|||} \cdot a_n^{-1} \right) =: \widetilde{\alpha}_n \widetilde{\beta}_n$, where $\widetilde{\beta}_n \to \widetilde{\beta} \neq 0$ and $\widetilde{\alpha}_n^{-1} \to 0$.

$\nu_n = T_{a_n}(\mu_n) \star \varepsilon_{x_n}$ yields $T_{a_n^{-1}}(\nu_n) = T_{\widetilde{\alpha}_n^{-1}}(\mu_n) \star \varepsilon_{y_n}$, $y_n = a_n^{-1}(x_n)$. Hence $T_{\widetilde{\beta}_n}(\nu_n) = T_{\widetilde{\alpha}_n^{-1}}(\mu_n) \star \varepsilon_{z_n} =: \lambda_n \star \varepsilon_{z_n}$ (with $z_n = \widetilde{\alpha}_n^{-1}(y_n) = \widetilde{\beta}_n(x_n)$).
We have $T_{\widetilde{\beta}_n}(\nu_n) \to T_{\widetilde{\beta}}(\nu)$ along a subsequence and $\lambda_n \to \varepsilon_0$ (since $\widetilde{\alpha}_n^{-1} \to 0$), hence $\widetilde{\beta}_n(x_n) \to z$ for some z.
Hence we conclude $T_{\widetilde{\beta}}(\nu) = \varepsilon_z$, i.e., $\nu \notin \mathcal{SF}$, a contradiction.

Hence $\{a_n^{-1}\}$ is relatively compact, therefore $T_{a_n^{-1}}(\nu_n) \to T_b(\nu)$ for any accumulation point b. (Convergence along a subsequence.) Shift-compactness yields relative compactness of $\left\{y_n = T_{a_n^{-1}}(x_n)\right\}$ and therefore $T_b(\nu) = \mu \star \varepsilon_y$ for any accumulation point.

(c) follows immediately by (a) and (b). □

Remarks 1.17 *Note the crucial differences between the hypergroups $(\mathcal{K}, \star)$ and the vector space case:*
(1) *Shifts are not invertible, hence the condition $\nu = T_a(\mu) \star \varepsilon_x$ (for some $x \in \mathcal{K}$) is no more equivalent to $\mu = T_{a^{-1}}(\nu) \star \varepsilon_z$ (for some $z \in \mathcal{K}$)*
(2) *In Theorem 1.16 the assumptions are not symmetric: We assume $\mu \in \mathcal{F}$, $\nu \in \mathcal{SF}$.*
(3) *Note that for vector spaces $\mathbb{V}$ and simply connected nilpotent Lie groups the results 1.15 a), b) are not valid.*
(4) *Furthermore, in contrast to Proposition 1.9, for vector spaces $\mathcal{F}$ is not an ideal. On the other hand, the characterization of $S-$full measures on vector spaces $\mu \in \mathcal{SF}$ iff $\mu \star \widetilde{\mu} \in \mathcal{F}$ is not valid for the hypergroups $\mathcal{K}$.*

2 Semistability and Selfdecomposability

2.1 Continuous convolution semigroups

Continuous convolution semigroups are defined as usual:

Definition 2.1 ***(a)*** *A family $\{\mu_t,\ t \geq 0\} \subseteq M^1(\mathcal{K})$ is called continuous convolution semigroup if $t \mapsto \mu_t$ is weakly continuous and $\mu_t \star \mu_s = \mu_{t+s}\ \forall\, t, s \geq 0$. A probability measure $\mu \in M^1(\mathcal{K})$ is called embeddable if there exists a continuous convolution semigroup $(\mu_t)_{t \geq 0}$ with $\mu_1 = \mu$.*
(b) *A sequence of convolution powers $\left(\mu^k\right)_{k \in \mathbb{Z}_+}$ with $\mu^0 := \varepsilon_0$ is called discrete convolution semigroup.*

Remarks 2.2 ***(a)*** *μ_0 is an idempotent, hence, since $\mathcal{K}$ is aperiodic, we always have $\mu_0 = \varepsilon_0$.*
(b) *Let μ be embeddable. Then, since the Fourier-transforms $\widehat{\mu}_t$ are non-negative, the continuous convolution semigroup $(\mu_t)_{t \geq 0}$ is uniquely determined by $\mu = \mu_1$.*

(c) *Discrete convolution semigroups are the distributions of random walks on $\mathcal{K}$, whereas continuous convolution semigroups are the distributions of Lévy processes on $\mathcal{K}$.*

Lemma 2.3 *Let (μ_t) be a continuous convolution semigroup. Assume $\mu_{t_0} \in \mathcal{F}$ for some $t_0 > 0$. Then $\mu_t \in \mathcal{SF}\ \forall\ t > 0$.*

⟦ Let $n, k \in \mathbb{N}$. We have $\mu_{t_0} = \mu_{t_0/n}^n \in \mathcal{F}$, whence by Proposition 1.9 (b) $\mu_{t_0/n} \in \mathcal{F}$. But then $\mu_{t_0} \in \mathcal{SF}$ according to 1.15 (a), and again by 1.15(a) $\mu_{t_0/n}^k = \mu_{\frac{k}{n}t_0} \in \mathcal{SF}$ for all $k \geq 2$. Hence $\mu_r \in \mathcal{SF}$ for all $r = \frac{k}{N}t,\ k, N \in \mathbb{N}$, a dense subset of $\mathbb{R}_+ \backslash \{0\}$.

Hence for all $t > 0$ there exists a $r < t$ with $\mu_r \in \mathcal{SF} \subseteq \mathcal{F}$. But $\mathcal{F}$ is an ideal (Propositon 1.9), hence $\mu_t = \mu_r \star \mu_{t-r} \in \mathcal{F}$. I.e., $\mu_t \in \mathcal{F}\ \forall\ t > 0$.

Then, arguing as before, $\mu_t \in \mathcal{SF}$ follows. ⟧

Next we recall a limit theorem for random walks on Hermitean hypergroups (in short *functional limit theorem*), i.e., convergence of discrete convolution semigroups to continuous convolution semigroups.

Proposition 2.4 *Let $\nu_n,\ n \in \mathbb{N}$, be a sequence of probabilities on $\mathcal{K}$. Let $k_n \in \mathbb{N}, k_n \to \infty$.*

Then $\nu_n^{k_n} \to \mu$ implies that $\nu_n \to \varepsilon_0$ (infinitesimality) and there exists a continuous convolution semigroup $(\mu_t)_{t \geq 0}$ with $\mu = \mu_1$. I.e., μ is embeddable.

Furthermore, (μ_t) is uniquely determined by μ and we have $\nu_n^{[k_n t]} \to \mu_t$ uniformly for t in compact subsets of $\mathbb{R}_+$. (In short, 'functional convergence')

⟦ See e.g. [17], [11], [12], Theorem 2.4: There it is proved uniform convergence on compact subsets of $(0, \infty)$. But $\mathcal{K}$ is aperiodic and $\nu_n \to \varepsilon_0$. Then the assertion follows by [12], Theorem 3.5 ⟧

In the following we consider continuous one-parameter groups $(T_{a_t})_{t>0}$ of automorphisms. In the context of *stability* we assume throughout *multiplicative parametrization*, i.e.

$$t \mapsto a_t \in \mathrm{GL}(\mathbb{K}^d) \text{ is continuous and } a_t a_s = a_{t \cdot s} \text{ for } t, s > 0$$

Definition 2.5 ***(a)*** *A probability $\mu \in M^1(\mathcal{K})$ is stable w.r.t. a group of automorphisms $(T_{a_t})_{t>0}$ if μ is embeddable into a continuous convolution semigroup (μ_t) with $\mu = \mu_1$ and*

$$T_{a_t}(\mu_1) = \mu_t \text{ for all } t > 0$$

(b) *A probability $\mu \in M^1(\mathcal{K})$ is semistable w.r.t. an automorphisms T_a and $c \in (0, 1)$ if μ is embeddable into a continuous convolution semigroup (μ_t) with $\mu = \mu_1$ and*

$$T_a(\mu_t) = \mu_{c \cdot t} \text{ for all } t > 0$$

μ is called stable resp. semistable if μ is stable resp. semistable w.r.t. some one-parameter group (T_{a_t}) resp. some $(T_a,\ c)$

Note that in the case of stability $T_{a_t}(\mu_s) = \mu_{t \cdot s}$ follows for all $t > 0, s \geq 0$. Analogously, for semistable laws we have $T_{a^k}(\mu_t) = \mu_{c^k \cdot t}$ for all $t \geq 0$ and all $k \in \mathbb{Z}$.

Note that for vector spaces usually stability and semistability are defined in a more general way by affine normalizations:

$$T_a(\mu_t) = \mu_{c\cdot t} \star \varepsilon_{x(t)} \text{ or } \mu_{c\cdot t} = T_a(\mu_t) \star \varepsilon_{x(t)} \tag{2.1}$$

In analogy to the vector space situation the definitions afore should therefore be called *strict* stability resp. *strict* semistability. However, for the hypergroups $\mathcal{K}$ no examples are known fulfilling the more general defining properties (2.1). Hence we prefer to suppress the notation *strict*.

2.2 Algebraic characterization of semistability

Definition 2.6 *Let $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup with $\mu = \mu_1$. The decomposability group of μ is defined as*

$$\mathfrak{Dec}(\mu) := \{T_a \in \mathrm{End}(\mathcal{K}) : \exists\ c > 0,\ T_a(\mu_t) = \mu_{c\cdot t}\ \forall\ t \geq 0\}$$

Note that μ_t is uniquely determined by $\mu = \mu_1$. Therefore, the definition of the decomposability group depends on μ only.

Definition 2.7 *The map φ defined by*

$$\varphi : \mathfrak{Dec}(\mu) \ni T_a \mapsto c\ \in \mathbb{R}_+, \quad \textit{with} \quad T_a(\mu_t) = \mu_{\varphi(a)\cdot c} =: \mu_{c\cdot t}$$

is called canonical homomorphism.

Proposition 2.8 *The canonical homomorphism φ is a continuous homomorphism $\mathfrak{Dec}(\mu) \to \mathbb{R}_+^*$ with kernel* $\ker(\varphi) = \mathfrak{Inv}(\mu)$*. ($\mathbb{R}_+^*$ denoting the multiplicative group $((0,\infty), \cdot)$.)*
In particular, $\mathfrak{Inv}(\mu) \lhd \mathfrak{Dec}(\mu)$, and we have $\mathfrak{Dec}(\mu)/\mathfrak{Inv}(\mu) \cong \mathrm{im}(\varphi)$.

If $\mu \in \mathcal{F}$ (hence $\mu_t \in \mathcal{SF}$ for all $t > 0$ according to Lemma 2.3), then φ is a closed map with compact $\varphi^{-1}(c)$ for all $c > 0$. Hence φ is a perfect map.

⟦ Obviously, φ is continuous and $\ker(\varphi) = \mathfrak{Inv}(\mu)$, hence $\mathfrak{Inv}(\mu)$ is a normal subgroup of $\mathfrak{Dec}(\mu)$. For all $c \in \mathrm{im}(\varphi)$ and $T_a \in \mathfrak{Dec}(\mu)$ with $\varphi(T_a) = c$ we have $\varphi^{-1}(\{c\}) = T_a \cdot \mathfrak{Inv}(\mu)$.

Let $\mu \in \mathcal{F}$. Let $c_n \in \mathrm{im}(\varphi)$, $c_n = \varphi(T_{a_n})$ and assume $c_n \to c$. Then, by continuity of convolution, $\mu_{c_n\cdot t} \to \mu_{c\cdot t}$. On the other hand, $T_{a_n}(\mu_t) = \mu_{c_n\cdot t} \to \mu_{c\cdot t}$. The convergence of types theorem 1.10 yields relative compactness of $\{a_n\}$ in $\mathrm{GL}(\mathbb{K}^d)$. And for all accumulation points a of $\{a_n\}$ we conclude $T_a(\mu_t) = \mu_{c\cdot t}$. I.e., $T_a \in \mathfrak{Dec}(\mu)$ with $\varphi(T_a) = c$. ⟧

Theorem 2.9 *Let μ, μ_t be as above.*
(a) *μ is semistable iff $\mathfrak{Dec}(\mu) \neq \mathfrak{Inv}(\mu)$. This is the case, iff* $\mathrm{im}(\varphi) \neq \{1\}$
(b) *μ is stable (w.r.t. some group (T_{a_t})) iff* $\mathrm{im}(\varphi) = \mathbb{R}_+$. ⟦ *Note that φ is a continuous surjective Lie group homomorphism.* ⟧

Assume now that $\mu \in \mathcal{F}$. Then we have in addition:
(c) *μ is stable iff* $\mathrm{im}(\varphi)$ *is dense in* $\mathbb{R}_+$.
(d) *μ is properly semistable, i.e. semistable but not stable, iff* $\mathrm{im}(\mu)$ *is a discrete subsemigroup of* $(\mathbb{R}_+, \cdot)$.
(e) *In particular, let $c_1,\ c_2 \in \mathrm{im}(\varphi)$ such that* $\log(c_1),\ \log(c_2)$ *are not commensurable. Then μ is stable.*

Proof: (a) is obvious since $\varphi(\mathfrak{Inv}(\mu)) = \{1\}$. Analogously, (b) is obvious.

Let μ be full. Then closedness of φ yields that $\mathrm{im}(\varphi)$ is closed. Whence (c) follows. Now (d) and (e) are immediate consequences. □

2.3 Domains of attraction and semistability

Next we characterize semistable laws as limit laws. First we define domains of attraction:

Definition 2.10 *Let $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup with $\mu = \mu_1$. The domain of attraction is defined as*

$$\mathcal{DA}(\mu) := \left\{ \nu \in M^1(\mathcal{K}) : \ \exists \ T_{a_n} \in \operatorname{Aut}(\mathcal{K}), n \in \mathbb{N}, \textit{with } T_{a_n}(\nu)^n \to \mu \right\}$$

(equivalently, according to Proposition 2.4, $T_{a_n}(\nu)^{[n \cdot t]} \to \mu_t$ uniformly on compact subsets of $\mathbb{R}_+$.)

The (partial) domain of semistable attraction (for some $c \in (0,1)$) is defined as

$$\mathcal{DA}_{sst,\,c}(\mu) := \left\{ \nu \in M^1(\mathcal{K}) : \ \exists \ \{T_{a_n}\} \subseteq \operatorname{Aut}(\mathcal{K}) \ \textit{with } T_{a_n}(\nu)^{k_n} \to \mu \right\}$$

for some sequence $\{k_n\} \subseteq \mathbb{N}$, $k_n \to \infty$, with $k_n/k_{n+1} \to c$. (Again this is equivalent to $T_{a_n}(\nu)^{[k_n \cdot t]} \to \mu_t$, uniformly on compact subsets of $\mathbb{R}_+$.)

Note again, that μ_t is uniquely determined by $\mu_1 = \mu$. Hence the domains of attraction are determined by μ. As in the group- or vector space situation (cf. e.g., [5], § 1.6, § 2.6) we obtain (with the notations defined before):

Theorem 2.11 ***(a)*** *If μ is semistable w.r.t. $c \in \operatorname{im}(\varphi)$, then we have $\mathcal{DA}_{sst,\,c}(\mu) \neq \emptyset$. Analogously, if μ is stable, then $\mathcal{DA}(\mu) \neq \emptyset$.*
(b) *Conversely, let $\mu \in \mathcal{F}$. Then we have:*

$$\mathcal{DA}_{sst,\,c}(\mu) \neq \emptyset \quad \Rightarrow \quad \mu \textit{ is semistable}$$

$$\mathcal{DA}(\mu) \neq \emptyset \quad \Rightarrow \quad \mu \textit{ is stable}$$

The **Proof** is almost verbatim as in the vector space case:
(a) Let μ be semistable w.r.t. $(T_a,\ c)$. Put $a_n := a^n$, $k_n := [c^{-n}]$, $n \in \mathbb{N}$. Continuity of $(a, \nu) \mapsto T_a(\nu)$ and $T_{a^n}\left(\mu^{[c^{-n}]}\right) = \mu_{c^n \cdot [c^{-n}]}$ yield $T_{a^n}(\mu^{k_n}) \to \mu$. I.e., $\mu \in \mathcal{DA}_{sst,\,c}(\mu)$.

(b) Let $\varphi : \mathfrak{Dec}(\mu) \to \mathbb{R}_+$ be the canonical homomorphism. We have to show: $\mathcal{DA}_{sst,\ c} \neq \emptyset \ \Rightarrow \ c \in \operatorname{im}(\varphi)$.

We have

$$T_{a_n} \nu^{[k_n t]} = \left(T_{a_n} T_{a_{n+1}}^{-1}\right) \left(T_{a_{n+1}} \left(\nu^{[k_{n+1} \cdot \frac{k_n}{k_{n+1}}]}\right)\right) \to \mu_t$$

According to Lemma 2.3 we have $\mu_t \in \mathcal{SF} \subseteq \mathcal{F}$ for all $t > 0$, and furthermore, compact-uniform convergence yields $T_{a_{n+1}}\left(\nu^{[k_{n+1} \cdot \frac{k_n}{k_{n+1}}]}\right) \to \mu_{c \cdot t}$. Applying the convergence of types theorem 1.10 we conclude that $\left\{b_n := a_n a_{n+1}^{-1}\right\}$ is relatively compact, and for any accumulation point $b =: a^{-1}$ we have $T_b(\mu_{ct}) = \mu_t$ resp. $T_a(\mu_t) = \mu_{ct}$.

The proof for the *stable case* is analogous. □

Remarks 2.12 *Note that in complete analogy to the group- or vector space situation we observe the following additional features of the set of norming automorphisms:*

(a) *There exists some $k \in \mathbb{N}$ such that for $T_a \in \mathrm{Aut}(\mathcal{K})$ there exists a one-parameter group $(T_{a_t})_{t>0}$ and some $c > 0$ such that $T_a = T_{a_c}$. Hence in investigations of semistability we may assume w.l.o.g. the norming automorphisms to be located on one-parameter groups.*
(b) *If μ is full and $T_a \in \mathfrak{Dec}(\mu)$, then for some $k \in \mathbb{N}$, $T_b := T_a^k$ centralizes the invariance group, i.e., $T_b T_u = T_u T_b$ for all $T_u \in \mathfrak{Inv}(\mu)$.*
(c) *Analogously, let μ be full. Let $(T_{a_t}) \subseteq \mathfrak{Dec}(\mu)$. Then there exists a group $(T_{b_t}) \subseteq \mathfrak{Dec}(\mu)$ centralizing $\mathfrak{Inv}(\mu)$, with $b_t = a_t u_t$ for $u_t \in \mathfrak{Inv}(\mu)$.*

⟦ a) follows immediately by Property 0.4. To prove b) c), see e.g. [5], 1.8.11, 1.8.16, 2.8.8, 2.8.16; see also [3], [4]. ⟧

2.4 Contraction Properties

Let $(\mu_t)_{t\geq 0} \subseteq M^1(\mathcal{K})$ be semistable. Then, as for groups or vector spaces (cf. e.g., [5], 3.4.3) we observe that the measures are concentrated on the T_a-*contactible part* of $\mathcal{K}$:

Definition 2.13 *Let $T_a \in \mathrm{Aut}(\mathcal{K}) = \mathrm{Aut}(\Pi_d)$. The T_a-contactible part is defined as $C_+(T_a) := \left\{p \in \Pi_d :\ T_{a^k}(p) \overset{t\to\infty}{\longrightarrow} 0\right\}$.*

Proposition 2.14 *Let $(\mu_t)_{t\geq 0} \subseteq M^1(\mathcal{K})$ be (T_a, c)-semistable. Then*

$$\bigcup_{t\geq 0} \mathrm{supp}(\mu_t) \subseteq C_+(T_a).$$

Proof: Let $C(T_a) := \left\{x \in \mathbb{H} \subseteq \mathbb{K}^{d^2} :\ T_{a^k}(x) \overset{t\to\infty}{\longrightarrow} 0\right\}$ denote the contractible part, where T_a is considered as linear operator acting on the vector space $\mathbb{H} := \Pi_d - \Pi_d$ of Hermitean matrices. Hence, $C_+(T_a) = C(T_a) \cap \Pi_d$. In particular, the cone $C_+(T_a)$ is closed.

The subsequent Lemma 2.15 will show that there exists $g \in C^b(\mathcal{K})$ with following properties:
(1) $0 \leq g \leq 1$, (2) $g(0) = 0$, $g(p) \neq 0\ \forall p \neq 0$, (3) $\lim\limits_{t\to 0} \frac{1}{t}\langle \mu_t, g\rangle$ exists and furthermore, (4) $p_n \to 0$ iff $g(p_n) \to 0$.

Hence there exists some $\alpha > 0$, such that $\langle \mu_t, g\rangle \leq \alpha \cdot t$ for $t \geq 0$.

For fixed $t > 0$ we have therefore $\langle \mu_{c^k \cdot t}, g\rangle = \langle T_{a^k}(\mu_t), g\rangle = \langle \mu_t,\ g \circ T_{a^k}\rangle \leq \alpha \cdot c^k \cdot t$. (Where we use again the abbreviation $\langle \mu, f\rangle := \int f d\mu$.)

Therefore, $\sum_k \langle \mu_t,\ g \circ T_{a^k}\rangle < \infty$, whence $\sum_k g \circ T_{a^k} < \infty$ μ_t-a.e. Consequently, $g \circ T_{a^k} \to 0$ and therefore, according to property (4) $T_{a^k} \to 0$, μ_t-a.e. follows. Hence, since $C_+(T_a)$ is closed, $T_{a^k}(p) \to 0$ for all $p \in \mathrm{supp}(\mu_t)$. □

Lemma 2.15 *There exists a function $g \in C^b(\mathcal{K})$ with the afore mentioned properties (1)–(4)*

⟦ Let $h \in C_c(\widehat{\mathcal{K}}), 0 \leq h \leq 1$, $h \neq 0$. Then $h \star h \in C_c(\widehat{\mathcal{K}})_+$ and $k := (h \star h)^\vee = |\overset{\vee}{h}|^2$ is positive definite and positiv. (Recall that $\mathcal{K}$ is Hermitean.)

Therefore, $k(0) \geq k(p)$ for all p (cf. [2], 4.1.3.) Assume w.l.o.g. that $k(0) = 1$ and put $g_1 := 1 - k$.

Since for all characters $\kappa \in \widehat{\mathcal{K}}$ the limit $\frac{d^+}{dt}\widehat{\mu}_t(\kappa)|_{t=0} = \lim\limits_{t\to 0} \frac{1}{t}\langle \mu_t - \varepsilon_0, \kappa\rangle$ exists, as easily seen, this holds true for g_1 too. And finally, according to the Riemann-Lebesgue Lemma (cf. [2], 2.2.4 (vii), (viii)), $k \in C_0(\mathcal{K})$, hence $g_1 > 1/2$ outside some compact neighbourhood V_0 of 0.

Next, for all $x \in V_0$ there exists a character $\kappa_x \in \widehat{\mathcal{K}}$ such that $\langle \kappa_x, x \rangle > 0$, and for all $x \in V_0$ there exists a neighbourhood V_x of x such that $\kappa|_{V_x \setminus \{0\}} > 0$. There exist $\{x_1, \dots x_N\}$ such that $\bigcup_1^N V_{x_i} \supseteq V_0$. Hence $g_2 := 1 - \frac{1}{N} \sum_1^N \kappa_{x_i}$ has the properties (1)–(3).
Putting finally $g := \frac{1}{2}(g_1 + g_2)$ we obtain a function with properties (1)–(3). Furthermore, $g|_{\complement V_0} \geq 1/4$ and $g|_{V_0 \setminus \{0\}} \overset{>}{\neq} 0$ show, that also (4) is fulfilled.]]

2.5 (Semi-)Selfdecomposability

We show briefly that our tools are sufficient to investigate also (semi-)self-decomposability on the hypergroups $(\mathcal{K}, \star)$. For $(T_a,\ c)$-semistable laws we obviously have: $\mu = T_{a^k}(\mu) \star \mu_{1-c^k},\ k \in \mathbb{N}$. (Recall that we assumed $0 < c < 1$.) This motivates the folowing definition of decomposability:

Definition 2.16 (Urbanik semigroup)
Let $\mathfrak{D}(\mu) := \{T_a \in \mathrm{End}(\mathcal{K}) :\ \mu = T_a(\mu) \star \nu(a) \ \ \textit{for some}\ \nu(a) \in M^1(\mathcal{K})\}$.
$\nu(a)$ is called cofactor, $\mathfrak{Cof}(\mu,\ T_a)$ denotes the set of cofactors.

Proposition 2.17 *If $\mu \in \mathcal{F}$ then $\mathfrak{D}(\mu)$ is a compact semigroup in* $\mathrm{End}(\mathcal{K})$*. In fact,* $\{(T_a, \mathfrak{Cof}(\mu, T_a)) :\ T_a \in \mathfrak{D}(\mu)\}$ *is compact in* $\mathrm{End}(\mathcal{K}) \times M^1(\mathcal{K})$.

Proof: We have for $T_a, T_b \in \mathfrak{Dec}(\mu)$:

$$\mu = T_a(\mu) \star \nu(a) = T_a\left(T_b\left(\mu\right) \star \nu(b)\right) \star \nu(a) = T_{ab}(\mu) \star \left(T_a(\nu(b)) \star \nu(a)\right)$$

Hence $T_a(\nu(b)) \star \nu(a) \in \mathfrak{Cof}(\mu, T_a)$.
In particular, $\mathfrak{D}(\mu)$ is a semigroup.

Let $\mu \in \mathcal{F}$. Let $\{a_n\} \subseteq \mathfrak{D}(\mu)$, hence $\mu = T_{a_n}(\mu) \star \nu(a_n)$. Therefore, $\mu^2 = T_{a_n}(\mu^2) \star \nu(a_n)^2$. The Godement property 0.6 implies that $\{T_{a_n}(\mu^2)\}$ and $\{\nu(a_n)^2\}$ are relatively compact (cf. 0.12.c)). $\mu^2 \in \mathcal{F}$ implies – according to the convergence of types theorem 1.10 – that $\{T_{a_n}\}$ is relatively compact in $\mathrm{End}(\mathcal{K})$. Hence, along a subsequence, $T_{a_n}(\mu) \to T_a(\mu)$.

Shift compactness 0.12 yields that $\{\nu(a_n)\}$ is relatively compact, hence $\nu(a_n) \to \nu$ along a subsequence. Therefore, since convolution is continuous, $\mu = T_a(\mu) \star \nu$ follows. I.e., $T_a \in \mathfrak{D}(\mu)$ and $\nu \in \mathfrak{Cof}(\mu,\ T_a)$. □

Let $\left(T_{b(t)}\right)_{t \geq 0} \subseteq \mathrm{End}(\mathcal{K})$ be a continuous one-parameter semigroup. Here, for investigations in self-decomposability, we use *additive parametrization*, i.e. we assume $b(t)b(s) = b(t+s)$ for $s, t \geq 0$.

Definition 2.18 ***(a)*** *Assume $\left(T_{b(t)}\right)_{t \geq 0} \subseteq \mathfrak{D}(\mu)$. (In other words, there exist cofactors $\nu(t) \in M^1(\mathcal{K})$ such that for all $t \geq 0$ we have $\mu = T_{b(t)}(\mu) \star \nu(t)$.) Then μ is called self-decomposable.*
(b) *μ is called T_a-decomposable if $\mu = T_a(\mu) \star \nu(a)$ for some cofactor $\nu(a) \in M^1(\mathcal{K})$, i.e., if $T_a \in \mathfrak{D}(\mu)$.*

For investigations in self-decomposability on vector spaces see e.g., [9] Chap. 3 – note that $\mathfrak{D}(\mu)$ is called *Urbanik semigroup* there – and for (nilpotent) groups see e.g., [5], § 2.14. See also the references mentioned mentioned in these monographs.

If T_a is contracting we obtain the following representation of T_a-decomposable laws:

Proposition 2.19 *Assume T_a to be contracting. Then, if $T_a \in \mathfrak{D}(\lambda)$ with cofactor $\nu \in \mathfrak{Cof}(\lambda, T_a)$, we obtain $\lambda = \mathop{\bigstar}\limits_{k=0}^{\infty} T_a^k(\nu)$.*
Conversely, any such probability λ is T_a-decomposable.

$\llbracket$ We have by assumption $\lambda = T_a(\lambda) \star \nu = \ldots = T_{a^{k+1}}(\lambda) \star \mathop{\bigstar}\limits_{j=0}^{k} T_{a^j}(\nu)$. The first factor converges to ε_0, whence the assertion follows.
Conversely, assume $\lambda = \mathop{\bigstar}\limits_{j=0}^{\infty} T_{a^j}(\nu)$. Then $T_a(\lambda) = \mathop{\bigstar}\limits_{j=1}^{\infty} T_{a^j}(\nu)$, hence $\lambda = \nu \star T_a(\lambda)$, i.e., $\nu \in \mathfrak{Cof}(\lambda, T_a)$. $\rrbracket$

In order to construct *random variables* with given distribution it is convenient to to have tools like P. Lévy's equivalence theorem, saying that for series of independent random variables convergence in distribution, stochastic convergence and almost sure convergence are equivalent. Usually, for vector spaces and groups, the proof relies on Kolmogorov, Skorohod- or Otiviani inequalities which are not known for hypergroups. For Sturm-Liouville hypergroups Zeuner [19], Corollary 2.7, obtained an equivalence theorem as a by-product of a version of Kolmogorov's three series theorem. The proof relies on moment functions and on the local behaviour of characters. Here we present for Pontryagin hypergroups $\mathcal{H}$ a similar result, which is however slightly weaker since a technical condition is involved.

Let Y_k, $k \geq 0$, be a sequence of independent $\mathcal{H}-$valued random variables with distributions $\xi_k \in M^1(\mathcal{H})$. Define $\mathop{\bigstar}\limits_{k=0}^{N} \xi_k =: \lambda_N$ and let $S_N := \Lambda - \sum_{k=0}^{N} Y_k$ denote the corresponding random walk. (S_n) is a $\mathcal{K}-$valued increment process with discrete time, i.e. a Markov chain with transition kernels $P\left(S_{n+1} \in A | S_n\right)(\cdot) = \left(\xi_n \star \varepsilon_{S_n(\cdot)}\right)(A)$.
In particular, S_N is distributed according to λ_N.

The randomized sums $\Lambda - \sum_{k=0}^{N} Y_k$ are defined via *concretisations*, see e.g. [18], or the monograph [2]. In short we call (S_n) *random walk* though the Markov chain need not to be time-homogeneous.

Theorem 2.20 *Let $(\mathcal{H}, \star)$ be a (w.l.o.g. second countable) commutative hypergroup with dual hypergroup $\widehat{\mathcal{H}}$ and bidual $\widehat{\widehat{\mathcal{H}}} \cong \mathcal{H}$.*
(a) *If $S_n \to S$ almost surely, then the distributions converge weakly, $\lambda_n \to \lambda$, the distribution of S.*
(b) *Conversely, assume that $\lambda_n \to \lambda \in M^1(\mathcal{H})$ and assume in addition that the complement of the set $\mathcal{N}(\lambda) := \left\{\kappa \in \widehat{\mathcal{H}} : \widehat{\lambda}(\kappa) = 0\right\}$ is dense in $\widehat{\mathcal{H}}$. Then S_n converges almost surely to a random variable with distribution λ.*

Proof: The first assertion is obvious.
To prove (b), put $\complement\mathcal{N}(\lambda) =: \mathcal{P}(\lambda)$. Note that $\widehat{\lambda}_n \to \widehat{\lambda}$ and that for $\kappa \in \mathcal{P}(\lambda)$ we have $\widehat{\lambda}_n(\kappa) \neq 0$, at least for sufficiently large n.

Step 1. For all $\kappa \in \mathcal{P}(\lambda)$ the sequence $\varphi_\kappa(S_n)$ converges a.s., where φ_κ denotes the character induced by $\kappa \in \widehat{\mathcal{H}}$.

$\llbracket$ Fix $\kappa \in \mathcal{P}(\lambda)$. The sequence of random variables $\left(\Phi_n := \frac{1}{\widehat{\lambda_n}(\kappa)} \cdot \varphi_\kappa(S_n)\right)_{n\geq 0}$ is a martingale w.r.t. the canonical filtration defined by (Y_n) (cf. e.g., [16], Lemma 6.6). By assumption, the sequence $|\widehat{\lambda_n}(\kappa)|$ is bounded from below (at least for sufficiently large n), hence the martingale is uniformly bounded. Therefore, according to the martingale convergence theorem, Φ_n converges a.s. Whence a.s. convergence of $\left(\varphi_\kappa(S_n) = \widehat{\lambda}_n(\kappa) \cdot \Phi_n\right)_{n\geq 0}$ follows. $\rrbracket$

Step 2. Let $(x_n) \subseteq \mathcal{H}$ be a sequence. Let D be dense in $\widehat{\mathcal{H}}$. Then $x_n \to x \in \mathcal{H}$ iff $\varphi_\kappa(x_n) \to \varphi_\kappa(x)$ for all $\kappa \in D$.

$\llbracket$ Obviously we have: $x_n \to x \in \mathcal{H}$ iff $\varphi_\kappa(x_n) \to \varphi_\kappa(x)$ for all $\kappa \in \widehat{\mathcal{H}}$. Since $\widehat{\widehat{\mathcal{H}}} = \mathcal{H}$, the sets $U_{L,\delta} := \{\ z \in \mathcal{H} : |\varphi_\kappa(x) - \varphi_\kappa(z)| \leq \delta\ \forall \kappa \in L\}$ is a basis of neighbourhoods of x, where $\delta > 0$ and $L \subseteq \widehat{\mathcal{H}}$ is compact. If D is dense in $\widehat{\mathcal{H}}$ then obviously we have $U_{L,\delta} = \{z \in \mathcal{H} : |\varphi_\kappa(x) - \varphi_\kappa(z)| \leq \delta\ \forall \kappa \in L \cap D\}$. Whence the assertion immediately follows. $\rrbracket$

(Note that the result in step 2 is close to Lévy's continuity theorem for hypergroups, [2], 4.2.4, 4.2.5., if we identify $x \in \mathcal{K}$ with $\varepsilon_x \in M^1(\mathcal{K})$.)

Step 3. According to step 1, $\varphi_\kappa(S_n)(\omega)$ converges for all $\omega \notin N_\kappa$, where N_κ denotes some set of probability 0. Let D be some countable subset of $\mathcal{P}(\lambda)$ which is dense in $\widehat{\mathcal{H}}$. $\varphi_\kappa(S_n)(\omega)$ converges for all $\kappa \in D$ and for all $\omega \notin \bigcup_{\kappa\in D} N_\kappa$.

According to step 2 therefore $S_n(\omega)$ converges for all $\omega \notin \bigcup_{\kappa\in D} N_\kappa$. The exception set is a set of probability zero. □

Applying Theorem 2.20 to T_a-decomposable laws we obtain e.g. a method to construct random variables with given T_a-decomposable distribution λ:

Corollary 2.21 ***(a)*** *Let – as before – $(\mathcal{K}, \star)$ be a hypergroup structure on Π_d, and let T_a denote a contracting automorphism. Let $\nu \in M^1(\mathcal{K})$. Let (X_k) be iid random variables with distribution ν, and let $S_N := \Lambda - \sum_0^N T_{a^k}(X_k)$ denote a corresponding random walk. Assume $\bigstar_{n=0}^{N} T_{a^n}(\nu) =: \lambda_N \to \lambda \in M^1(\mathcal{K})$. Assume in addition that the Fourier-transform $\widehat{\nu}$ never vanishes. Then S_n converges a.s. to a random variable S with distribution $\bigstar_{k=0}^{\infty} T_{a^k}(\nu) = \lambda$. This distribution is T_a-decomposable according to Proposition 2.19.*

(b) *The assertion of a) remains true under the weaker condition that $\mathcal{N}(\nu)$ is of first category.*

Proof: a) Note that for the T_a-decomposable law λ we obtain for $k \geq 1$: $\lambda = T_{a^{k+1}}(\lambda) \star \bigstar_{j=0}^{k} T_{a^j}(\nu)$, whence $\mathcal{N}(\lambda) = {T_{a^*}}^{k+1}(\mathcal{N}(\lambda)) \cup \bigcup_0^k (\mathcal{N}(T_{a^j}(\nu)))$. Since by assumption, $\mathcal{N}(T_{a^j}(\nu)) = \emptyset$, $\mathcal{N}(\lambda) = \mathcal{N}\left(T_{a^{k+1}}(\lambda)\right) = {T_{a^*}}^{k+1}(\mathcal{N}(\lambda))$ follows. If $\kappa \in \mathcal{N}(\lambda)$, $\left({T_{a^*}}^k(\kappa)\right)_{k\geq 0} \subseteq \mathcal{N}(\lambda)$, a contradiction, since ${T_{a^*}}^k(\kappa) \to 0$ and $\widehat{\lambda}(0) = 1$. Hence $\mathcal{N}(\lambda) = \emptyset$ and Theorem 2.20 applies.

b) Following the lines of the proof of a) we obtain : $\kappa \in \mathcal{N}(\lambda)$ iff $\kappa \in \bigcup_0^N \mathcal{N}\left(T_a^k(\nu)\right)$ or $T_{a^*}^{N+1}(\kappa) \in \mathcal{N}(\lambda)$ for all $N \in \mathbb{N}$. Put $\mathcal{N}^* := \bigcup_0^\infty \mathcal{N}\left(T_a^k(\nu)\right)$ and $D := \complement\mathcal{N}^*$. If $\kappa \in \mathcal{N}(\lambda) \cap D$ then $\widehat{\lambda}\left(T_{a^*}^{N+1}(\kappa)\right) = 0$ for all N, a contradiction. Hence $\mathcal{N}(\lambda) = \mathcal{N}^*$.

Furthermore, $\mathcal{N}\left(T_a^k(\nu)\right)$ is closed for all $k \geq 0$, and as by assumption these sets have no inner points, $D = \complement\mathcal{N}(\lambda)$ is dense in $\mathcal{K}$. (See e.g. [7], Ch. II, Lemma (5.28).) Therefore again Theorem 2.20 applies. □

Note that the condition of Corollary 2.21 a) is fulfilled if λ is self-decomposable (cf. Definition 2.18): In this case, as easily verified, λ is representable as a limit of an infinitesimal triangular array. Hence λ is infinitely divisible and therefore the Fourier transform is without zeros. (See also Remarks 2.26 below.) But this is not true true for general T_a-decomposable laws. Even in the classical situation, on the real line, the Fourier transforms of "nouvelles lois limites" de M. Loève may have zeros.

Next we shall apply these tools to obtain criteria for T_a-decomposability, criteria similar to the well known characterizations of operator-decomposability for vector spaces and groups (compare e.g., [5], § 2.14). However, in the hypergroup situation we have to use extra conditions, therefore our results are considerably weak.

Let in the following $||\ \cdot\ ||$ be a Hilbert space norm on Π_d, and $|||\ \cdot\ |||$ the corresponding operator norm. Let $T_a \in \mathrm{Aut}(\mathcal{K})$ be contracting. To avoid technical details we assume $0 < \beta \leq |||T_{a^{-1}}|||^{-1}, |||T_a||| \leq \alpha < 1$. (Such a condition is always satisfied for a suitable power T_a^k). Hence we obtain for $x \in \mathcal{K} = \Pi_d$:

$$\beta^n \cdot ||x|| \leq ||T_a^n x|| \leq \alpha^n \cdot ||x|| \tag{2.2}$$

Let $\nu \in M^1(\mathcal{K})$ and let $X_k, k \geq 0$, be an iid sequence distributed according to ν. Put $Y_k := T_a^k(X_k)$ and denote by $S_n := \Lambda - \sum_0^n Y_k$ the corresponding random walk. Note that (ii') $Y_n = T_a^n X_n \to 0$ in distribution since T_a is contracting.

Proposition 2.22 *Assume* (i) $\quad \widehat{\nu}(\kappa) \neq 0,\ \forall \kappa \in \widehat{\mathcal{K}}$, *and* (ii) $\quad T_a^n(X_n) \to 0$ *almost surely. Furthermore, assume* (iii) $\quad \lambda_n := \bigstar_{n=0}^N T_a^n(\nu) \to \lambda \in M^1(\mathcal{K})$. *Then ν possesses logarithmic moments, i.e.*

$$(iv) \quad \mathbb{E}(\log^+(||X_0||) = \int_{\mathcal{K}} \log^+(||x||) d\nu(x) < \infty$$

The **Proof** runs along the steps of the 'classical' proof for groups or vector spaces. (However, there condition (i) is superfluous and (ii) follows by the other assumptions.)

Condition (ii) yields that for all $\delta > 0$ $P\left(\mathrm{Limsup}\left\{||T_a^n X_n|| > \delta\right\}\right) = 0$, hence we obtain by (2.2) e.g. for $\delta = 1$: $P\left(\mathrm{Limsup}\left\{||X_n|| > \beta^{-n}\right\}\right) = 0$. Whence by the Borel-Cantelli Lemma

$$\sum_{n\geq 0} P\left(\left\{||X_n|| > \beta^{-n}\right\}\right) = \sum_{n \geq 0} P\left(\left\{\log^+ ||X_n|| > n \cdot (-\log \beta)\right\}\right) < \infty$$

The iid–assumption implies (with $\gamma := -\log\beta$) that

$$\sum_{n\geq 0} P\left(\left\{\log^+(||X_0||) > n\cdot\gamma\right\}\right) < \infty$$

whence $\mathbb{E}(\log^+(||X_0||)) < \infty$ follows. □

Remark. Note that according to Corollary 2.21 we have almost sure convergence $S_n \to S$. In the 'classical situation' this implies condition (ii). Here we had to assume (ii) as additional condition.

A partial converse result is contained in

Proposition 2.23 *Assume as before* (i) $\quad\widehat{\nu} \neq 0$ *and furthermore assume the existence of logarithmic moments* (iv) $\quad\mathbb{E}(\log^+(||X_0||)) < \infty$.

Then we obtain convergence (iii) $\quad \lambda_n := \mathop{\bigstar}\limits_{k=0}^{n} T_a^k(\nu) \to \lambda \in M^1(\mathcal{K})$, *i.e., the distribution* λ *is* T_a-*decomposable.*

Proof: $\mathbb{E}(\log^+||X_0||) < \infty$ implies $\sum_0^\infty P(\log^+(||X_0||) > n\delta) < \infty$ for any $\delta > 0$, whence by the iid assumption

$$\sum_0^\infty P(\log^+||X_n|| > n\delta) = \sum_0^\infty P(||X_n||^{1/n} > \delta^{1/n}) < \infty$$

follows. Hence $\sum_0^\infty P\left((\alpha^n||X_n||)^{1/n} > \alpha\delta^{1/n}\right) < \infty$ and therefore, according to (2.2) with $\delta = 1$ we conclude $\sum_0^\infty P\left(||T_a^n(X_n)|| > \alpha^n\right) < \infty$.

Applying again the Borel-Cantelli Lemma, we obtain

$$P\left(\text{Liminf}\left\{||T_a^n(X_n)|| \leq \alpha^n\right\}\right) = 1$$

Hence we obtain a.s. *absolute* convergence

$$\sum_0^\infty ||T_a^n(X_n)|| < \infty \quad \text{a.s.} \tag{2.3}$$

In the 'classical situation' (2.3) implies a.s. convergence $S_n \to S$. Here, for hypergroups, we have to argue in a different way:

Let $U := \sum_0^\infty ||T_a^n(X_n)||_*$. (Cf. Property 0.7). Then we obtain for all n for the (randomized) partial sums $S_n = \Lambda - \sum_0^n T_a^k(X_k)$ upper bounds $||S_n||_* \leq U$. Therefore in particular, for all n and $K > 0$ we have $P(||S_n||_* > K) \leq P(U > K) \to 0$ with $K \to \infty$. Hence the distributions λ_n of S_n, $n \in \mathbb{Z}_+$, are uniformly tight. Let λ be an accumulation point of $\{\lambda_n\}$.

(2.3) implies in particular that for all sequences $m_n \to \infty$, and $M_n > m_n$

$||\Lambda - \sum\limits_{m_n+1}^{M_n} T_a^k(X_k)||_* \leq \sum\limits_{m_n+1}^{M_n} ||T_a^k(X_k)||_* \to 0$ a.s., whence $\mathop{\bigstar}\limits_{j=m_n+1}^{M_n} T_a^k(\nu) \to \varepsilon_0$.

Equivalently, the sequence of Fourier transforms $\left(\widehat{\lambda_n}\right)_{n\geq 0}$ satisfies the Cauchy condition. On the other hand, $\widehat{\lambda}_n \to \widehat{\lambda}$ along a subsequence.

Whence $\widehat{\lambda}_n \stackrel{n\to\infty}{\longrightarrow} \widehat{\lambda}$, equivalently, $\lambda_n \stackrel{n\to\infty}{\longrightarrow} \lambda$ follows. □

2.6 Space-time processes and self-decomposability

Let μ be self-decomposable with a contracting semigroup $\left(T_{a(t)}\right)_{t\geq 0} \subseteq \mathfrak{D}(\mu)$. (Additive parametrization). Then, as noted above, $\mu = T_{a(t)}(\mu) \star \nu(t) = T_{a(t+s)}(\mu) \star T_{a(s)}(\nu(t)) \star \nu(s)$. Repeating this decomposition with $\frac{k}{N} \cdot t$, $\frac{t}{N}$ instead of s, t, we obtain in particular for all $t > 0$, $N, M \in \mathbb{N}$: $\mu = T_{a([Nt]+M)}(\mu) \star \bigstar_{k=0}^{[Nt]+M} T_{a(\frac{k}{N}t)}\left(\nu(\frac{t}{N})\right)$. Since the automorphisms are contracting all factors converge to ε_0. Hence in particular, $\widehat{\mu}(\cdot) > 0$, and therefore the cofactors are uniquely determined: $\widehat{\nu(t)}(\cdot) = \widehat{\mu}(\cdot)/\widehat{T_{a(t)}(\mu)}(\cdot)$. Furthermore, $\nu(t) \overset{t\to\infty}{\longrightarrow} \mu$. And in addition, the cofactors depend continuously on t and fulfil the cocycle equation

$$\nu(s+t) = T_{a(t)}(\nu(s)) \star \nu(t) \tag{2.4}$$

Put for $s < t$: $\nu(s,t) := T_{a(s)}\left(\nu(t-s)\right)$. Then

$$(s,t) \mapsto \nu(s,t) \text{ is continuous} \tag{2.5}$$
$$\nu(s,t) \star \nu(t,r) = \nu(s,r) \quad \text{for } s \leq t \leq r \tag{2.6}$$
$$T_{a(h)}(\nu(s,t)) = \nu(s+h, t+h) \quad \text{for } s \leq t \; h \geq 0 \tag{2.7}$$

As in the group- or vector space case we define:

Definition 2.24 ***(a)*** *A family of probabilities $\nu(t)$, $t \geq 0$, satisfying the relation (2.4) is called M-semigroup.*
(b) *A family of probabilities $\nu(s,t)$, $0 \leq s \leq t$, satisfying the relations (2.5), (2.6), (2.7) is called stable hemigroup.*
(c) *A family of probabilities $\nu(s,t)$, $0 \leq s \leq t$, satisfying the relations (2.5), (2.6) and*

$$T_a(\nu(s,t)) = \nu(s+c, t+c) \quad \textit{for } s \leq t \tag{2.8}$$

for some contracting T_a and $c > 0$ is called $(T_a, c)-$semistable hemigroup.

Obviously, $\nu(t)$, $t \geq 0$, is a M-semigroup iff $\nu(s,t) := T_{a(s)}(\nu(t-s))$, $s \leq t$, is a stable hemigroup.

The *space-time hypergroup* is defined as semi-direct product of hypergroups, i.e., as Cartesian product $\Gamma := \mathcal{K} \times \mathbb{R}$ endowed with the convolution structure

$$(\varepsilon_x \otimes \varepsilon_s) * (\varepsilon_y \otimes \varepsilon_t) := \left(\varepsilon_x \star T_{a(s)}\left(\varepsilon_y\right)\right) \otimes \varepsilon_{t+s}$$

($*$ denoting convolution on Γ, and $\star$ convolution on $\mathcal{K}$). As easily seen, $(\Gamma, *)$ defines a hypergroup.

For a fixed group of automorphisms $\left(T_{a(t)}\right)_{t\in\mathbb{R}}$ we have:

Proposition 2.25 $(\nu(t))_{t\geq 0} \subseteq M^1(\mathcal{K})$ *is a M-semigroup on $\mathcal{K}$ w.r.t.* $\left(T_{a(t)}\right)_{t\geq 0}$ *iff*
$(\lambda(t) := \nu(t) \otimes \varepsilon_t)_{t\geq 0} \subseteq M^1(\Gamma)$ *is a convolution semigroup on the space-time hypergroup Γ defined by* $\left(T_{a(t)}\right)_{t\geq 0}$.

⟦ For groups and vector spaces compare e.g., [5], § 2.14 V. ⟧

Remarks 2.26 ***(a)*** *Let* $(\nu(s,t))_{0\leq s\leq t}$ *satisfy (2.5) and (2.6) (hence a 'convolution hemigroup'), then any probability* $\nu(s,t)$ *is embeddable into a continuous convolution semigroup.*
$[\![$ *Obviously* $\nu(s,t)$ *is a limit of an infinitesimal triangular array, hence infinitely divisible and therefore embeddable. Cf. [2], 5.3.4, 5.3.11.* $]\!]$

(b) *Let the convolution hemigroup* $(\nu(s,t))_{0\leq s\leq t}$ *be* $(T_a, c)-$*semistable (cf. Definition 2.24 (c)). Assume moreover that* $\mu(s) := \lim_{t\to\infty} \nu(s,t)$ *exists. Then* $\mu(s)$ *is* T_a-*decomposable with cofactor* $\nu(s, s+c)$.
If in addition $(\nu(s,t))_{0\leq s\leq t}$ *is* $\left(T_{b(r)}\right)-$*stable then* $\mu(s)$ *is self-decomposable with cofactors* $\nu(s, s+r) \in \mathfrak{Cof}(\mu(s), T_{b(r)}),\ r \geq 0$. $[\![$ *Immediately verified.* $]\!]$

(c) *In case b) the cofactors are continuously emeddable according to a). Hence the* T_a-*decomposable limit laws* $\mu = \mu(s)$ *defined by a semistable (resp. stable) hemigroup fulfill the assumptions of Corollary 2.21, i.e. the Fourier transform have no zeros. Therefore, the corresponding random walks converge almost surely.*

3 Gaussian laws, subordination and examples of (semi-)stable and decomposable laws

3.1 Gaussian Laws

In [16], Section 5, the class of *squared Wishart distributions* is investigated, playing the role of *Gaussian* distributions on the hypergroup $\mathcal{K}$. For more informations about Wishart distributions and different approaches the reader is referred to the literature mentioned in [16].

The *standard Gaussian* is denoted by $W = W(I)$, characterized by the Fourier-transform $\widehat{W(I)}(\kappa) = e^{-\mathrm{tr}\left(\kappa^2\right)/2},\ \kappa \in \Pi_d$. (Note that $\widehat{\mathcal{K}} \cong \mathcal{K}$.) In general, the set of Gaussian laws $\{W(p^2),\ p \in \Pi_d\}$, is defined as the orbit of $W(I)$ under the action of $T_p,\ p \in \Pi_d$. They are characterized by the Fourier-transform $\widehat{W(p^2)}(\kappa) = e^{-\mathrm{tr}\left(p\kappa^2 p\right)/2} = e^{-\mathrm{tr}\left(p^2\kappa^2\right)/2},\ \kappa \in \Pi_d$. $[\![$ [16], Lemma 5.4. $]\!]$
Therefore, for $a = up \in \mathrm{End}(\mathbb{K}^d),\ p \in \Pi_d$ and u unitary, we have $W(p^2) = W(aa^*)$, furthermore, $W(p^2) = T_p(W(I))$ and for $p,\ q \in \Pi_d$, $T_p(W(q^2)) = W(r^2)$ with $r = (pq^2p)^{1/2}$. In fact, $W(pq^2p) = W(qp^2q)$. Whence

$$W(p^2) \star W(q^2) = W(p^2+q^2) \text{ for } p,\ q \in \Pi_d \tag{3.1}$$

follows. Therefore, for fixed $p^2 \in \Pi_d$, $(\mu_t := W(t\cdot p^2))_{t\geq 0}$ is a continuous convolution semigroup with $\mu_1 = W(p^2)$. (μ_t) is called Gaussian continuous convolution semigroup. $[\![$ [16], Lemma 5.5. $]\!]$

Consequently, for fixed $p \in \Pi_d$, let $\left(c_s := s^{1/2}\cdot I\right)_{s>0} \subseteq \mathrm{GL}(\mathbb{K}^d)$. Then

$$T_{c_s}(\mu_1) = T_{c_s}(W(p^2)) = W(s\cdot p^2) = \mu_s \quad \forall s > 0$$

In other words,

3.1 *The Gaussian laws (resp. squared Wishart laws)* $W(q^2)$ *are stable in the afore defined sense with* $(T_{c_s}) \subseteq \mathfrak{Dec}(W(q^2))$.

Next we determine the invariance groups of Gaussian laws:

3.2 ***(a)*** *$\mathfrak{Inv}(W(I)) = \mathfrak{U}_d$, the group of orthogonal (resp. unitary) matrices.*
(b) *Let $q \in \Pi_d \cap \mathrm{GL}(\mathbb{K}^d)$. Then $\mathfrak{Inv}(W(q^2)) = q^{-1}\mathfrak{U}_d q$.*
In particular we have: $\mathfrak{Inv}(W(q^2))$ is a compact group if q is positive definite. Hence $W(q^2) \in \mathcal{F}$ for $q \in \mathrm{GL}(\mathbb{K}^d)$.
(c) *If $q \in \Pi_d$ is not invertible then $W(q^2)$ is not full and hence $\mathfrak{Inv}(W(p^2))$ is not compact.*

⟦ (a) Let $a = up$ be the polar decomposition of $a \in \mathrm{End}(\mathbb{K}^d)$. As afore mentioned, $T_a(W(I)) = W(aa^*) = W(up^2u^*) = W(p^2)$, hence $T_a(W(I)) = W(I)$ iff $aa^* = I$, i.e., iff $a \in \mathfrak{U}_d$.

(b) follows easily since $W(q^2) = T_q(W(I))$: In fact, we have $T_a(W(q^2)) = T_q\left(T_{q^{-1}aq}(W(I))\right)$. Hence $T_a(W(q^2)) = W(q^2)$ iff $T_{q^{-1}aq}(W(I)) = W(I)$, i.e., iff $q^{-1}aq \in \mathfrak{U}_d$. Whence the assertion.

(c) Let $V := \ker(q)$ and $U := V^{\perp}$. Then $b := Id_U \oplus \alpha \in \mathfrak{Inv}(W(q^2))$ for all $\alpha \in \mathrm{End}(V)$. Whence the assertion. ⟧

Analogously we describe the decompsability group of full Gaussian laws:

3.3 *Let q be invertible, hence $W(q^2) \in \mathcal{F}$. Then*

$$\begin{aligned}
\mathfrak{Dec}(W(q^2)) &:= \left\{T_a \in \mathrm{Aut}(\mathcal{K}) :\ T_a(W(q^2)) = W(c\cdot q^2) \textit{ for some } c > 0\right\}\\
&= \left\{\sqrt{c}\cdot \mathfrak{Inv}(W(q^2)) \ \textit{ for } c \in \mathrm{im}(\varphi) = (0,\ \infty)\right\}\\
&= \left\{\sqrt{c}\cdot q^{-1}\mathfrak{U}_d q \ \textit{ for } c \in \mathrm{im}(\varphi) = (0,\ \infty)\right\}
\end{aligned}$$

The canonical homomorphism $\varphi :\ \mathfrak{Dec}(W(q^2)) \to (0,\infty)$ is given by $\varphi(\sqrt{c}\cdot q^{-1}uq) = c$ for unitary u.

⟦ In fact, $T_a(W(q^2)) = W(c\cdot q^2)$ iff $T_{\frac{1}{\sqrt{c}}a}(W(q^2)) = W(q^2)$. This is the case iff $\frac{1}{\sqrt{c}}a \in \mathfrak{Inv}(W(q^2)) = q^{-1}\mathfrak{U}_d q$. Whence the assertion follows. ⟧

Concerning domains of attraction, note that in [16], Theorem 6.4, Voit proved a central limit theorem. Hence we describe with the notations introduced before:

3.4 *For the domain of attraction of $W(q^2)$ we have*

$$\mathcal{DA}(W(q^2)) \ \supseteq \ \left\{\nu \in M^1(\mathcal{K}) \textit{ with second moment } \int_{\Pi_d} ||r^2|| d\nu(r) < \infty\right\}$$

In this case, $\int_{\Pi_d} r^2 d\nu(r) =: q^2 \in \Pi_d$ exists and moreover, $T_{\frac{1}{\sqrt{n}}\cdot I}(\nu^n) \to W(q^2)$.

Finally, any Gaussian distribution $W(q^2)$ is – as a stable law – self-decomposable. We describe the decomposability semigroup resp.

3.5 *Urbanik semigroup (cf Definition 2.16): Let $a = up$ denote the polar decomposition, then*

$$\mathfrak{D}(W(q^2)) = \{T_a \in \mathrm{End}(\mathcal{K}) :\ q\,(I-p)\,q \in \Pi_d\}$$

⟦ $a \in \mathfrak{D}(W(q^2))$ iff $W(q^2) = W(aq^2a^*) \star \nu(a)$ for some cofactor $\nu(a) \in M^1(\mathcal{K})$. This is the case iff for all $\kappa \in \Pi_d$

$$\begin{aligned}\widehat{\nu(a)}(\kappa) &= \widehat{W(q^2)}(\kappa)/\widehat{W(aq^2a^*)}(\kappa) = \exp\left(-\frac{1}{2}\left(\operatorname{tr}\left(q\kappa^2 q\right) - \operatorname{tr}\left(aq\kappa^2 qa^*\right)\right)\right)\\ &= \exp\left(-\frac{1}{2}\left(\operatorname{tr}\left(q\kappa^2 q\right) - \operatorname{tr}\left(pq\kappa^2 qp\right)\right)\right).\end{aligned}$$

Hence iff $\operatorname{tr}(q\kappa^2 q) - \operatorname{tr}(aq\kappa^2 qa^*) \geq 0$. But $\operatorname{tr}(q\kappa^2 q) - \operatorname{tr}(pq\kappa^2 qp) = \operatorname{tr}(q^2\kappa^2) - \operatorname{tr}(qp^2q\kappa^2) = \operatorname{tr}((q^2 - qp^2q)\kappa^2)$. Whence the assertion follows. ⟧

3.2 Further examples

A standard way to construct semistable resp. selfdecomposable laws is the method of *subordination.* Let (μ_t) be a continuous convolution semigroup, i.e. a continuous semigroup homomorphism $(\mathbb{R}_+, +) \to (M^1(\mathcal{K}), \star)$, then this homomorphism may be extended to a continuous affine convolution homomorphism $(M^1(\mathbb{R}_+), *) \to (M^1(\mathcal{K}), \star)$, where $*$ on the left side denotes convolution on $\mathbb{R}$ whereas on the right side $\star$ denotes convolution on the hypergroup $\mathcal{K}$. This affine homomorphism is given by

$$\Phi: \; M^1(\mathbb{R}_+) \ni \xi \mapsto \int_{\mathbb{R}_+} \mu_t d\xi(t). \tag{3.2}$$

(See e.g., [5], § 1.5 II, § 2.5 III.)

⟦ As well known and easily verified, we have $\Phi(\sum \alpha_i \xi_i) = \sum \alpha_i \Phi(\xi_i)$ for $\alpha_i \geq 0, \sum \alpha_i = 1$ and $\Phi(\xi * \eta) = \Phi(\xi) \star \Phi(\eta)$. ⟧

As in the group- or vector space case we observe:

Proposition 3.6 *Let $(\mu_t) \subseteq M^1(\mathcal{K})$ be a continuous convolution semigroup which is stable w.r.t. $(T_{a_t})_{t>0}$. Φ will denote the corresponding subordination homomorphism. Let $\xi, (\xi_t) \subseteq M^1(\mathbb{R}_+)$, (ξ_t) a continuous convolution semigroup. Let $H_\alpha : x \mapsto \alpha \cdot x$ denote the homothetical transformation on $\mathbb{R}_+$, $\alpha > 0$. Let $\nu_s := \Phi(\xi_s), s \geq 0$ and $\nu := \Phi(\xi)$.*
(a) *If (ξ_t) is semistable w.r.t. (H_α, c) for $\alpha > 0, c \in (0,1)$, then $(\nu_s) \subseteq M^1(\mathcal{K})$ is semistable w.r.t. (T_{a_α}, c)*
(b) *If (ξ_t) is stable w.r.t. $(H_{t^\alpha})_{t>0}$ for some $\alpha > 0$. Then $(\nu_s) \subseteq M^1(\mathcal{K})$ is stable w.r.t. (T_{b_t}), with $b_t := a_{t^\alpha}$.*
(c) *If ξ is semi-self-decomposable with $H_\alpha \in \mathfrak{D}(\xi)$ and cofactors $\eta(H_\alpha) \in \mathfrak{Cof}(\xi, H_\alpha)$. Then ν is semi-self-decomposable in $M^1(\mathcal{K})$ with $T_{a_\alpha} \in \mathfrak{D}(\xi)$ and cofactors $\Phi(\eta(H_\alpha)) \in \mathfrak{Cof}(\xi, H_\alpha)$. An analogous assertion holds for self-decomposable laws.*

Sketch of a **Proof:** (a) We have

$$\begin{aligned}T_{a_\alpha}(\Phi(\xi_s)) &= \int_{\mathbb{R}_+} T_{a_\alpha}(\nu_t)\, d\xi_s(t) = \int_{\mathbb{R}_+} \nu_{\alpha \cdot t}\, d\xi_s(t)\\ &= \int_{\mathbb{R}_+} \nu_t\, dH_\alpha(\xi_s)(t)) = \int_{\mathbb{R}_+} \nu_t\, d\xi_{c\cdot s}(t) = \Phi(\xi_{c\cdot s})\end{aligned}$$

I.e., $T_{a_\alpha}(\nu_s) = \nu_{c\cdot s}$

(b) Assume $H_{t^\alpha}(\xi_s) = \xi_{s\cdot t},\ s \geq 0, t > 0$. Then analogously, $T_{a_{t^\alpha}}(\Phi(\xi_s)) = \Phi(\xi_{s\cdot t})$. I.e., defining $b_t := a_{t^\alpha}$, we have proved that (ν_s) is stable w.r.t. (T_{b_t}).

(c) Assume $\xi = H_\alpha(\xi) * \eta(H_\alpha)$ for $\alpha > 0$, and cofactor $\eta(H_\alpha) \in M^1(\mathbb{R}_+)$. Then, as before, $\Phi(\xi) = T_{a_\alpha}(\Phi(\xi)) \star \Phi(\eta(H_\alpha))$. Hence ν is $T_{a_\alpha}-$decomposable with cofactor $\Phi(\eta(H_\alpha))$. □

As an example we consider Wishart resp. Gaussian distributions on $\mathcal{K}$ introduced above:

Example 3.7 *$(\mu_t := W(tq^2))_{t\geq 0}$ is stable w.r.t. $(T_{t^{1/2}\cdot I})_{t>0}$. (In fact, stable w.r.t. any group (T_{a_t}) with $a_t = t^{1/2}\cdot I\ u(t)$, $u(\cdot)$ denoting a group of transformations in $\mathfrak{Inv}(W(q^2))$.)*

The Fourier-transform is given as $\widehat{\mu}_t(\kappa) = e^{-t\cdot \mathrm{tr}(\kappa^2 q^2)}$. Let (ξ_s) denote a semigroup of one-sided stable distributions on the half line $\mathbb{R}_+$, defined by the Laplace transform $\mathcal{L}_{\xi_s}(u) = e^{-su^\alpha},\ u \geq 0$, for some (fixed) $\alpha \in (0,1)$. Let Φ denote the subordination map (w.r.t. (μ_t)). Then $(\Phi(\xi_s))$ is a stable continuous convolution semigroup. In fact, the Fourier-transform is given by $\kappa \mapsto \widehat{\Phi(\xi_s)}(\kappa) = e^{-s\cdot\left(\mathrm{tr}(q^2\kappa^2)\right)^\alpha}$. This is proved by simple calculations: $\Phi(\xi_s)(\kappa) = \int_{\mathbb{R}_+}\widehat{\mu}(\kappa)d\xi_s(t) = \int e^{-t\cdot \mathrm{tr}\left(q^2\kappa^2\right)/2}d\xi_s(t) = exp\left(-s\left(\mathrm{tr}(q^2\kappa^2)/2\right)^\alpha\right)$. Hence $(\Phi(\xi_s))$ is stable w.r.t. the group $(T_{t^{\alpha/2}\cdot I})_{t>0}$.

The structure of the Gaussian resp. Wishart distributions on $\mathcal{K}$ is a motivation to generalize the concept of subordination in order to obtain a wider class of examples. First we generalize the concept of continuous convolution semigroups:

Definition 3.8 *Let $\mathcal{P}$ denote a closed cone in a vector space $\mathbb{K}^r$. A **cone semigroup** is a family of probabilities $(\mu_{\vec{t}})_{\vec{t}\in\mathcal{P}}$ on a vector space $\mathbb{V}$, on a group $\mathbb{G}$ or on a hypergroup $\mathcal{K}$ respectively, such that $\mathcal{P} \ni \vec{t} \mapsto \mu_{\vec{t}} \in M^1(\mathcal{K})$ is a continuous semigroup homomorphism. (The definition is analogous for vector spaces or groups.)*

The cone semigroup is called semistable w.r.t. $(T_a, c) \in \mathrm{Aut}(\mathcal{K}) \times \mathrm{GL}(\mathcal{P})$ if $T_a(\mu_{\vec{t}}) = \mu_{c(\vec{t})}$. There we define $\mathrm{GL}(\mathcal{P}) := \{c \in \mathrm{GL}(\mathbb{K}^r)$ with $c(\mathcal{P}) = \mathcal{P}\}$.

Analogously, the cone semigroup is called stable if there exists a continuous homomorphism $\phi : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{K}),\ \alpha \mapsto \phi(\alpha)$, such that $\phi(\alpha)(\mu_{\vec{t}}) = \mu_{\alpha(\vec{t})}$.

If we want to emphasize the homomorphism, we call the semigroup $\phi-$stable. (Note that for $\mathcal{P} = \mathbb{R}_+$ we have $\mathrm{Aut}(\mathbb{R}_+) \cong ((0,\infty),\cdot)$, hence ϕ defines a one-parameter group of operators $t \mapsto T_{a_t}$. Hence in this case we obtain just the usual definition of stability.)

If the hypergroup $\mathcal{K}$ is replaced by a group or by a vector space we define (semi-) stability of cone-semigroups in an analogous way.

We continue the example 3.7 of Wishart distributions:

3.9 *Let $\mathcal{P}$ be a subcone of $\Pi_d \subseteq \mathbb{K}^{d^2}$. Then $\mathcal{P} \ni q^2 =: p \mapsto \mu_p := W(q^2)$ is a cone semigroup (according to (3.1)). Furthermore, the relation*

$$\begin{aligned} T_a(\mu_p) &= T_a(W(q^2)) = W(aq^2a^*) = W\left(T_a(q^2)\right) = \mu_{T_a(p)} \\ & \quad q \in \mathcal{P},\ T_a \in \mathrm{Aut}(\Pi_d) \text{ with } T_a(\mathcal{P}) = \mathcal{P} \end{aligned}$$

shows that the cone semigroup $(\mu_p)_{p\in\mathcal{P}}$ is stable w.r.t. the identity (restricted to $\mathcal{P}$): $\mathrm{Aut}(\mathcal{P}) \ni T_a \mapsto T_a \in \mathrm{Aut}(\Pi_d)$.

The concept of subordination is easily generalized to cone semigroups. (For a different approach to subordination of cone semigroups see e.g., [1]).

Definition 3.10 *Let – with the notations introduced above – $\xi \in M^1(\mathcal{P})$. Then the (cone-)* ***subordination map*** *is defined as*

$$\Phi : \ M^1(\mathcal{P}) \ni \xi \mapsto \int_{\mathcal{P}} \mu_{\vec{t}} \, d\xi(\vec{t}) \in M^1(\mathcal{K}). \tag{3.3}$$

As easily seen, again Φ is a continuous affine semigroup homomorphism.

Proposition 3.11 *Let Π be a closed subcone of Π_d. $M^1(\Pi)$ is endowed with the convolution $*$ induced by the additive structure $(\Pi, +)$. We assume throughout $(\mu_{\vec{t}})_{\vec{t}\in\Pi} \subseteq M^1(\mathcal{K})$ to be stable w.r.t. a fixed homomorphism $\phi : \mathrm{Aut}(\Pi) \to \mathrm{Aut}(\mathcal{K})$. Let Φ denote the corresponding subordination map. Let as above, $\mathcal{P}$ denote a closed cone in a vector space $\mathbb{K}^r$.*

(a) *Let $(\xi_{\vec{s}})_{\vec{s}\in\mathcal{P}} \subseteq M^1(\Pi)$ be semistable w.r.t. $(A, b) \in \mathrm{Aut}(\Pi) \times \mathrm{Aut}(\mathcal{P})$. Then $(\nu_{\vec{s}} := \Phi(\xi_{\vec{s}}))_{\vec{s}\in\mathcal{P}} \subseteq M^1(\mathcal{K})$ is semistable w.r.t $(\phi(A), b)$, i.e.,*

$$\phi(A)(\nu_{\vec{s}}) = \nu_{b(\vec{s})}, \ \ \vec{s} \in \mathcal{P}. \tag{3.4}$$

(b) *Let $(\xi_{\vec{s}})_{\vec{s}\in\mathcal{P}} \subseteq M^1(\Pi)$ be stable w.r.t. $\psi : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\Pi)$. Then $(\nu_{\vec{s}} := \Phi(\xi_{\vec{s}}))_{\vec{s}\in\mathcal{P}} \subseteq M^1(\mathcal{K})$ is stable w.r.t. $\gamma := \psi \circ \phi$, i.e.,*

$$\gamma(b)(\nu_{\vec{s}}) = \nu_{b(\vec{s})}, \ \textit{for all } \vec{s} \in \mathcal{P}, \ b \in \mathrm{Aut}(\mathcal{P}). \tag{3.5}$$

(c) *Analogous assertions hold true for (semi-)self-decomposable laws. (We omit the details.)*

⟦ (a) Obviously we obtain again

$$\begin{aligned} \phi(A)\,(\nu_{\vec{s}}) &= \int_{\Pi} \phi(A)(\mu_{\vec{t}}) \, d\xi_{\vec{s}} = \int_{\Pi} \mu_{A(\vec{t})} \, d\xi_{\vec{s}}(\vec{t}) \\ &= \int_{\Pi} \mu_{\vec{t}} \, dA\,(\xi_{\vec{s}})\,(\vec{t}) = \int_{\Pi} \mu_{\vec{t}} \, d\xi_{b(\vec{s})}\left(\vec{t}\right) = \nu_{b(\vec{s})} \end{aligned}$$

I.e., $\phi(A)\,(\Phi(\xi_{\vec{s}})) = \Phi\left(\xi_{b(\vec{s})}\right)$ as asserted.
(b) Let $b \in \mathrm{Aut}(\mathcal{P})$, $A := \psi(b) \in \mathrm{Aut}(\Pi)$.

$$\begin{aligned} \gamma(b)\,(\nu_{\vec{s}}) &= \int_{\Pi} \phi(A)(\mu_{\vec{t}}) \, d\xi_{\vec{s}}(\vec{t}) = \int_{\Pi} \mu_{A(\vec{t})} \, d\xi_{\vec{s}}(\vec{t}) \\ &= \int_{\Pi} \mu_{\vec{t}} \, d\psi(b)\,(\xi_{\vec{s}})\,(\vec{t}) = \int_{\Pi} \mu_{\vec{t}} \, d\xi_{b(\vec{s})}\left(\vec{t}\right) = \nu_{b(\vec{s})} \end{aligned}$$

I.e., $\gamma(b)\,(\Phi(\xi_{\vec{s}})) = \Phi\left(\xi_{b(\vec{s})}\right)$, for all $\vec{s} \in \mathcal{P}, b \in \mathrm{Aut}(\mathcal{P})$ as asserted. ⟧

As we are mainly interested in examples of continuous convolution semigroups, we note the following result for the particular cone $\mathcal{P} = \mathbb{R}_+$:

Corollary 3.12 *Let as above the cone semigroup $(\mu_{\vec{t}})_{\vec{t}\in\Pi} \subseteq M^1(\mathcal{K})$ be stable w.r.t. a homomorphism $\phi : \mathrm{Aut}(\Pi) \to \mathrm{Aut}(\mathcal{K})$ and consider $\mathcal{P} := \mathbb{R}_+$. Let $(\xi_s)_{s\geq 0}$ be a continuous convolution semigroup in $M^1(\Pi)$. Let Φ denote the corresponding subordination map. Then we have:*

(a) *If* $(\xi_s)_{s\geq 0} \subseteq M^1(\Pi)$ *is a* (A,c) *semistable continuous convolution semigroup, then the continuous convolution semigroup* $(\nu_s := \Phi(\xi_s))_{s\geq 0} \subseteq M^1(\mathcal{K})$ *is semistable w.r.t* $(\phi(A),c)$*, i.e.,* $\phi(A)(\nu_s) = \nu_{c\cdot s},\ s \geq 0$*.*
(b) *Let* $(\xi_s)_{s\geq 0} \subseteq M^1(\Pi)$ *be an* (A_t) *stable continuous convolution semigroup w.r.t. an one-parameter group* $(A_t)_{t>0} \subseteq \mathrm{Aut}(\Pi)$*. Then the continuous convolution semigroup* $(\nu_s := \Phi(\xi_s))_{s\geq 0} \subseteq M^1(\mathcal{K})$ *is stable w.r.t* $(\phi(A_t))_{t>0}$*, i.e.,* $\phi(A_t)(\nu_s) = \nu_{t\cdot s}$ *for all* $s \geq 0, t > 0$*.*

To apply this result for the construction of semistable laws on the *hypergroup* $\mathcal{K}$ we only have to show that non-trivial examples of semistable continuous convolution semigroups on *cones* Π of vector spaces exist. This is easily shown by a standard construction:

Proposition 3.13 *Let* $0 < c < 1$ *and let* $A \in \mathrm{Aut}(\Pi)$ *be contracting (on the vector space* $\mathbb{H} := \Pi - \Pi$*), hence* $|||A^k||| \leq C \cdot \beta^k$ *for some* $C > 0,\ 0 < \beta < 1,\ k \in \mathbb{N}$*. Assume* $\beta/c < 1$*.*

As A *is contracting, there exists a relatively compact Borel cross section w.r.t. the discrete action* $A^k, k \in \mathbb{Z}$*, hence there exists a relatively compact Borel set* $\Gamma \subseteq \{x \in \Pi : r < ||x|| \leq 1\}$ *for some* $r > 0$ *and* $\Pi_d\backslash\{0\} = \dot{\bigcup}_{k\in\mathbb{Z}} A^k(\Gamma)$*.*

Then for any $\lambda \in M^1(\Pi)$ *with* $\lambda(\complement\Gamma) = 0$ *there exists an* $(A,c)-$*semistable continuous convolution semigroup supported by* Π*. The Fourier-transform is given by*

$$\kappa \mapsto \exp\left(t \cdot \sum_{k\in\mathbb{Z}} c^{-k}\left(\widehat{\lambda}\left(A^{*k}(\kappa)\right) - 1\right)\right)$$

Proof: Since $\Pi \subseteq \mathbb{H}$ we apply the standard methods for probabilities on vector spaces:

Put $\eta := \sum\limits_{k\in\mathbb{Z}} c^{-k} A^k(\lambda)$. As easily seen, η is a Lévy measure and the bounded measures $\eta_N := \sum\limits_{k\leq N} c^{-k} A^k(\lambda)$ converge vaguely to η. Denote by $\pi_t^{(N)} := \exp\left(t \cdot (\eta_N - ||\eta_N||)\right)$ the Poisson measures. Obviously these measures are concentrated on $\bigcup\limits_{k\in\mathbb{Z}} A^k(\Gamma) \subseteq \Pi$. Furthermore, note that

$$A(\eta) = c \cdot \eta \tag{3.6}$$

If we can show that $\lim \pi_t^{(N)} =: \pi_t,\ t \geq 0$, exists, then, in view of the properties (3.6) of η we conclude that $A(\pi_t) = \pi_{c\cdot t}$, whence the proof follows. (Then η is the Lévy measure of (π_t).)

In order to show $\lim \pi_t^{(N)} =: \pi_t,\ t \geq 0$, exists, we have to prove that the function $x \mapsto ||x||$ is locally integrable w.r.t. the Lévy measure:
In fact, this follows by

$$\begin{aligned}
\int_{\{||\cdot||\leq 1\}} ||x||\ d\eta(x) &= \sum_{k\in\mathbb{Z}} c^{-k} \int_{A^k(\Gamma)\cap\{||\cdot||\leq 1\}} ||A^k(x)||\ d\lambda(x) \\
&= \sum_{k\leq -1} + \sum_{k\geq 0} \int_{A^k(\Gamma)\cap\{||\cdot||\leq 1\}} ||A^k(x)||\ d\lambda(x) \\
&\quad \text{(note that the first sum vanishes)} \\
&\leq \sum_{k\geq 0} \int_\Gamma C c^{-k} \beta^k\ d\lambda(x) = C \cdot \sum_0^\infty (\beta/c)^k < \infty
\end{aligned}$$

□

As mentioned, examples of (semi-)self-decomposable laws in $M^1(\mathcal{K})$ can easily be constructed in the same way via subordination of a fixed stable continuous convolution semigroup.

E.g., let (μ_t) be (T_{a_s})-stable. Let ρ denote the uniform distribution on the unit interval, $\rho = \lambda^1|_{[0,1]}$. As well known, and easily verified, $\rho = \lambda^1|_{[0,1]}$ is $H_{1/2}-$decomposable, $\rho = \bigstar_{n=0}^{\infty} \left(\frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_{1/2^n}\right) = \bigstar_{n=0}^{\infty} H_{\frac{1}{2^n}}\left(\beta(\frac{1}{2},1)\right)$, $\beta(p,n)$ denoting the binomial distribution.

Example 3.14 *Therefore by subordination, $\lambda := \int_0^1 \mu_t dt$ is $T_{a_{1/2}}-$decomposable, $\lambda = \bigstar_{n=0}^{\infty} T_{a_{1/2^n}}\left(\frac{1}{2}\mu_0 + \frac{1}{2}\mu_1\right) = \bigstar_{n=0}^{\infty} \left(\frac{1}{2}\mu_0 + \frac{1}{2}\mu_{1/2^n}\right)$.*

To obtain further examples, note that convolution products of stable laws are self-decomposable. Precisely:

3.15 *Let (T_{a_t}) be a one-parameter group of automorphisms. Let the continuous convolution semigroups (μ_t) and (ν_t) be stable w.r.t. the groups $(T_{a_{t^\alpha}})_{t>0}$ and $\left(T_{a_{t^\beta}}\right)_{t>0}$ respectively, where α, $\beta > 0$. Then, as easily verified, $\mu_t \star \nu_t$ is self-decomposable (for any $t > 0$), but in general not stable.*

Up to now all our examples, in particular, all examples of (semi)stable laws, are constructed via subordination of Gaussian laws, and as in 3.15, as convolution products of such laws. Nevertheless, this method already allows to construct a variety of new examples: Let $(T_{a_t})_{t>0}$ be a fixed group in $\mathrm{Aut}(\mathcal{K})$. The set of $(T_{a_t})_{t>0}-$stable laws is a sub-semigroup of $M^1(\mathcal{K})$. Hence

3.16 *Convolutions $(\mu_t \star \nu_t)$ of $(T_{a_t})_{t>0}-$stable laws are $(T_{a_t})_{t>0}-$stable. In particular, convolution products of stable laws obtained by subordination are stable. Applying again subordination to these new stable continuous convolution semigroups we obtain e.g. that for $0 < \alpha, \beta < 1$*

$$\kappa \mapsto \exp\left(-s \cdot \left\{\left(\mathrm{tr}(p^2\kappa^2)/2\right)^\alpha + \left(\mathrm{tr}(q^2\kappa^2)/2\right)^\alpha\right\}^\beta\right) \ , \ s \geq 0$$

are Fourier-transforms of a $(T_{b_t})_{t>0}-$stable continuous convolution semigroup, where $b_t := a_{t^{\alpha\beta}}$.

Here we supposed both Gaussians $W(p^2)$ and $W(q^2)$ to be stable w.r.t. the same group $(T_{a_t})_{t>0}$. (Note that this is also a restriction on the admissible automorphism groups.)

In order to obtain more examples beyond the Gaussian resp. Wishart distributions, we use the particular structure of the dual $\widehat{\mathcal{K}}$ of underlying hypergroups mentioned in Properties 0.8. First we construct semistable laws:

Example 3.17 *Let $a \in \mathrm{GL}(\mathbb{K}^d)$ be contracting, hence $|||a^n||| \leq C \cdot \rho^n$ for some $C > 0$ and $0 < \rho < 1$. Choose $0 < \rho, c < 1$, such that $\rho^2/c < 1$. Then there exist $(T_a, c)-$semistable laws.*

For the **Proof** we repeat the construction already used in 3.13:

Let $\lambda \in M^1(\mathcal{K})$ and define the positive measure $\eta := \sum\limits_{k\in\mathbb{Z}} c^{-k} T_{a^k}(\lambda)$. Obviously, $T_a(\eta) = c \cdot \eta$. We put $\pi_t^{(N)} := \exp\left(t\left(\eta_N - ||\eta_N|| \cdot \varepsilon_0\right)\right)$. For $N \in \mathbb{N}$ the measure $\eta_N := \sum\limits_{k\le N} c^{-k} T_{a^k}(\lambda)$ is bounded and we have $\eta_N \to \eta$ vaguely on $\mathcal{K}\backslash\{0\}$.

We have to show that for suitable λ the Fourier transforms $\widehat{\eta}_N - ||\eta_N|| \cdot 1$ converge to a continuous function, $\widehat{\eta}$ say. Then, according to Lévy's continuity theorem (for hypergroups), $\pi_t^{(N)} \to \pi_t,\ t \ge 0$, a (T_a, c) −semistable continuous convolution semigroup.

Here we prove this result for the particular law $\lambda = \varepsilon_x$. (Note that this yields semistable laws which are known – for groups and vector spaces – as the laws with extremal (semistable) Lévy measures. Cf. [5], 2.4 I.)

In fact, using the concrete representation of characters (cf. (0.1)), and in view of the mentioned asymptotic behaviour of the characters near 0 (cf. (0.2)), the assertion easily follows:

$$\begin{aligned}
\widehat{\eta}_N(\kappa) &= \sum_{k\le N} c^{-k}\left(\widehat{\varepsilon_{T_{a^k}(x)}}(\kappa) - 1\right) \\
&= \sum_{k\le N} c^{-k}\left\{\mathcal{J}_\mu\left(-\frac{1}{4}\left(\kappa\left(a^k x^2 a^{*k}\right)\kappa\right)\right) - 1\right\} \\
&= \sum_{k\le N} c^{-k}\left(-\frac{1}{4\mu}\left(\kappa a^k x^2 a^{*k}\kappa\right) + R_k\right)
\end{aligned}$$

with remainder term $R_k = O(||\kappa a^{*k} x^2 a^k \kappa||^2)$.

Considering the power-bounded sequence $\widetilde{a}^k = \frac{1}{\rho^k} \cdot a^k,\ k \ge 0$, we obtain $\widehat{\eta}_N(\kappa) = \sum\limits_{k\le N}\left(\frac{\rho^2}{c}\right)^k\left(-\frac{1}{4\mu}\left(\kappa\widetilde{a}^k x^2 \widetilde{a}^{*k}\kappa\right) + \widetilde{R}_k\right)$, with $||\widetilde{R}_k|| = \frac{1}{\rho^{2k}} \cdot ||R_k|| \le C_1||x||^4||\kappa||^4\rho^{2k}$. Hence the limit $\widehat{\eta}(\cdot) := \lim_N \widehat{\eta}_N(\cdot)$ exists as a continuous function. Hence, as mentioned, according to Lévy's continuity theorem for hypergroups, $\lim \pi_t^{(N)} =: \pi_t$ exists in $M^1(\mathcal{K})$.
By construction, this limit is (T_a, c) −semistable. □

In a similar way we can prove that existence of stable laws w.r.t. an arbitrary contracting one-parameter group $(T_{a_t})_{t>0}$. ⟦ Cf. [5], 2.4 II. ⟧

Example 3.18 *Assume w.l.o.g. that* $a_t = t^E$ *with* $E \in \mathrm{GL}(\mathbb{K}, d)$, $Re(\alpha) > 1/2$ *for all* $\alpha \in \mathrm{Spec}(E)$. *(Else replace* a_t *by* a_{t^α} *for suitable real* α.*) Let* $y_0 \in \Pi_d$ *and define the measure* $\eta := \int\limits_0^\infty \varepsilon_{T_{a_s}(y_0)} \cdot s^{-2}ds$.

Claim: $\kappa \mapsto \widehat{\mu}_t(\kappa) := \exp t\int_{\Pi_d}\left(\varphi_\kappa(x) - 1\right)d\eta(x)$ *is the Fourier transform of a* (T_{a_t}) *−stable continuous convolution semigroup* (μ_t).

⟦ Obviously, $T_{a_t}(\eta) = t \cdot \eta$ for all $t > 0$. Furthermore, we have $\psi(\kappa) := \int_{\Pi_d}\left(\varphi_\kappa(x) - 1\right)d\eta(x) = \int_0^\infty\left(\varphi_\kappa(T_{a_s}(y_0)) - 1\right)s^{-2}ds$. This integral converges since the integrand is bounded by $const \cdot ||T_{a_s}(y_0)||^2$ according to (0.8), hence $\le const_1 \cdot s^{2\cdot(\alpha+\varepsilon)}$ for suitable small ε.

Hence $\kappa \mapsto \exp t\psi(\kappa)$ is a limit of Fourier transforms of Poisson measures (with Fourier transforms $\kappa \mapsto \exp t \cdot \psi_\varepsilon(\kappa)$, where the exponent is defined as $\psi_\varepsilon(\kappa) := \int_\varepsilon^\infty\left(\varphi_\kappa(T_{a_s}(y_0)) - 1\right)\frac{ds}{s^2}$. Therefore, according to the continuity theorem, $\kappa \mapsto \widehat{\mu}_t(\kappa) := \exp t\psi(\kappa)$ is the Fourier transform of a probability measure μ_t, and since by construction $\psi(T^*_{a_t}(\kappa)) = t \cdot \psi(\kappa)$, we obtain $T_{a_t}(\mu_s) = \mu_{s\cdot t}$, i.e. (μ_s) is $(T_{a_t})_{t>0}$ −stable as asserted. ⟧

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Wilfried Hazod,
Faculty of Mathematics, Dortmund University of Technology,
D-44221 Dortmund, Germany
E-mail: `Wilfried.Hazod@math.uni-dortmund.de`
or
`Wilfried.Hazod@tu-dortmund.de`

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