

**Mixing of generating functionals and
applications to (semi-)stability of
probabilities on groups**

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Abstract

Let (X_t) be a Lévy process on a simply connected nilpotent Lie group with corresponding continuous convolution semigroup (ν_t) . Assume (ν_t) to be semistable. Then a suitable mixing of (ν_t) resp. a random time substitution of (X_t) belongs to the domain of attraction of a stable Lévy process (U_t) , the infinitesimal generator resp. the generating functional of which is representable as mixing of semistable generating functionals. A similar result holds for random variables belonging to the domain of semistable attraction of (X_t) . These investigations generalize results to the case of probabilities on groups which were recently obtained for vector spaces in [1]. Furthermore, distributions of such stable Lévy processes are representable as limits of random products of semistable laws.

Introduction.

Investigations into stability and semistability of probabilities on groups are motivated by the rapid development of research of the behaviour of operator-normalized random variables on (finite-dimensional) vector spaces during the last decades. For infinitely divisible laws, Lévy processes and operator-semistability on vector spaces the reader is referred to e.g. [18], [16]. For a survey with emphasis on the parallel development for vector spaces and on locally compact groups see [9].

Semistable probabilities on groups are supported by a contractible subgroup. So w.l.o.g. here all groups are supposed to be contractible. In particular, stable probabilities exist iff the underlying supporting group admits a contractive continuous one-parameter group of automorphisms, hence it is

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a homogeneous Lie group, i.e. contractible, nilpotent and simply connected. (Cf. [9], §2.1, §3.1, §3.4.)

Furthermore, for this class of groups semistable probabilities are embeddable into a uniquely defined continuous convolution semigroup, hence – as for vector spaces – there is a 1–1-correspondence between a semistable law ν_1 – the distribution of X_1 – and the convolution semigroup $(\nu_t)_{t \geq 0}$ – the distributions of the semistable Lévy-process $(X_t)_{t \geq 0}$. (Cf. [9], 2.6.10 – 2.6.15; see also 2.16, R 2.6.)

So, since stability and semi-stability is involved, it turns out that the natural framework for the following investigations are *contractible simply connected and nilpotent Lie groups*

The first aim is to translate results obtained in [1] for vector spaces to the group case: There, in Theorem 2.1, P. Becker-Kern and H-P. Scheffler obtained for operator-(semi)stable laws on finite dimensional vector spaces a beautiful result showing that a suitable mixing of properly semistable laws leads to stable laws, and in Theorem 3.3 a similar result is obtained for laws belonging to the domain of semistable attraction. These results in [1] are obtained and proved in the context of operator (semi-)self-similar processes. This more general frame and some of the tools used in the proofs – in particular random integrals – have – up to now – no counterparts for group-valued Lévy processes. A further difference: in the case of (in general non-Abelian) groups, the restriction to *strict* (semi)stability is well motivated, whereas for vector spaces usually (as in [1]) normalization by affine transformations is admitted.

Here we present a new and direct approach to the above-mentioned results, simultaneously for groups and vector spaces. In Section 1, (A) – (H), we collect basic tools for investigations in probabilities on groups, which are needed in the sequel. Section 2, Theorems 2.1 and 2.2, contains the main results. In Section 3, Theorem 3.1, a related result is obtained: Convergence of random products of properly semistable laws to stable laws.

The proofs, based mainly on the use of *generating functionals* of continuous convolution semigroups (avoiding characteristic functions and random integrals), are found in Section 4. For reader, mainly interested in probabilities on vector spaces, the appendix contains outlines of the proofs for the vector space situation relying on characteristic functions and their logarithms only. In fact, logarithms of the Fourier transform (second characteristic functions) are just the Fourier transforms of the generating functionals.

1 Notations and basic facts.

In the following let \mathbb{G} denote a locally compact – mostly *contractible simply connected nilpotent Lie* – group. Let $(\nu_t)_{t \geq 0}$ be a *continuous convolution semigroup* of probabilities on \mathbb{G} with $\nu_0 = \varepsilon_e$. The *generating functional* of $(\nu_t)_{t \geq 0}$ is defined as

$$\langle A, f \rangle := \frac{d^+}{dt} \langle \nu_t, f \rangle |_{t=0} \quad \text{for } f \in \mathfrak{E}(\mathbb{G}) \quad (1.1)$$

where $\mathfrak{E}(\mathbb{G})$ denotes the space of bounded continuous functions which belong locally to the Bruhat test functions $\mathfrak{D}(\mathbb{G})$. Note that for Lie groups $\mathfrak{D}(\mathbb{G})$ is the space of C^∞ -functions with compact support and $\mathfrak{E}(\mathbb{G})$ is just the space of bounded C^∞ -functions.

We use the following notation (which will be justified in Theorem (A) below):

$$\nu_t =: \text{Exp}(t \cdot A) \quad \text{for } t \geq 0 \quad (1.2)$$

$\mathcal{GF}(\mathbb{G})$ denotes the set of generating functionals, $\mathcal{GF}(\mathbb{G}) \subseteq \mathfrak{E}(\mathbb{G})' \subseteq \mathfrak{D}(\mathbb{G})'$. Furthermore, $\mathcal{M}^b(\mathbb{G})$ and $\mathcal{M}^1(\mathbb{G})$, the sets of bounded measures resp. of probability measures, are always endowed with the weak* topology $\sigma(\mathcal{M}^b(\mathbb{G}), C_0(\mathbb{G}))$ and analogously, $\mathcal{GF}(\mathbb{G})$ is endowed with the weak* topology $\sigma(\mathfrak{E}(\mathbb{G})', \mathfrak{E}(\mathbb{G}))$. (If E is a locally convex vector space, the dual space is denoted by E' .) To simplify notations, *weak convergence* will mean convergence w.r.t. these weak*-topologies.

As mentioned afore, for the group case new methods of proofs are needed. Therefore, in this Section 1 we collect some well-known facts and repeat main definitions, so to say a tool box for investigations of limit theorems and continuous convolution semigroups on locally compact groups .

Let (ν_t) be defined as above, let $(T_t)_{t \geq 0}$ denote the corresponding C_0 -contraction semigroup of convolution operators, $T_t f := \nu_t * f$, acting on $C_0(\mathbb{G})$ or on $L^2(\mathbb{G})$ respectively, let N denote the infinitesimal generator with domain $D(N)$ and let A denote the generating functional of $(\nu_t)_{t \geq 0}$. Then we recall

Theorem (A) Uniqueness Theorem (F. Hirsch, J.P. Roth, [12], [13], [14], J. Faraut, Kh. Harzallah [2])

(a) $\mathfrak{D}(\mathbb{G}) \subseteq D(N)$, and the restriction to $\mathfrak{D}(\mathbb{G})$ is given by

$$N|_{\mathfrak{D}(\mathbb{G})} : f \mapsto A * f, \quad \text{in particular} \quad \langle A, f \rangle = Nf(e)$$

(b) $N|_{D(N)}$ and hence $(\nu_t)_{t \geq 0}$ are uniquely determined by $A|_{\mathfrak{D}(\mathbb{G})}$. In fact,

(c) $\mathfrak{D}(\mathbb{G})$ is a core for the infinitesimal generator N .

Hence there exists a common core for all generators of convolution semigroups.

[[a) follows from the Lévy-Khinchin-Hunt representation, see e.g. [10, 11, 9] and the references mentioned there. The last assertions follow as a corollary of a more general result which was proved in 1972 $\pm \epsilon$ independently by different authors, cf. [12, 13, 14], [2], Théorème 2.2. See also e.g. [3], §0.4, [9], Ch. II, 2.0.6]]

Theorem (B) Convergence Theorem (E. Siebert [20])

Let $A_n, n \geq 1$, and A be generating functionals of continuous convolution semigroups $(\nu_t^{(n)})_{t \geq 0}, n \geq 1$, and $(\nu_t)_{t \geq 0}$. Then

$$A_n \rightarrow A \quad \text{weakly on } \mathfrak{E}(\mathbb{G}) \quad (1.3)$$

iff

$$\nu_t^{(n)} = \text{Exp}(t \cdot A_n) \rightarrow \text{Exp}(t \cdot A) = \nu_t \text{ weakly for all } t \geq 0 \quad (1.4)$$

In this case convergence is uniform on compact subsets of \mathbb{R}_+ .

[[For a proof see [20, 8, 3, 17] or [9], Ch. II, 2.0.10 – 2.0.12, see also the hints to the literature there.]].

Theorem (C) Characterisation of generating functionals (E. Siebert [19])

A linear functional $A \in \mathfrak{D}(\mathbb{G})'$ is a generating functional, $A \in \mathcal{GF}(\mathbb{G})$, iff

A is almost positive

$$\text{i.e. } f \in \mathfrak{D}_+(\mathbb{G}) : f(e) = 0 \Rightarrow \langle A, f \rangle \geq 0$$

and

A is normalized

i.e. $\sup \{ \langle A, f \rangle : 1_U \leq f \leq 1_G : f \in \mathfrak{D}_+(\mathbb{G}) \} = 0$ for some (and hence for all) neighbourhoods U of the unit e .

Note that the group structure is not involved in this characterization, only the differentiable structure of the underlying topological space \mathbb{G} .

[[See e.g. [19, 11, 10], 4.4.18, 4.5.8, see also [9], Ch. II, 2.0.6.]]

Theorem (D) Convergence of discrete convolution semigroups (resp. of random walks) Let $\mu_n \in \mathcal{M}^1(\mathbb{G})$, $k_n \uparrow \infty$, and let $(\nu_t = \text{Exp}(tA))_{t \geq 0}$ be a continuous convolution semigroup. Then we have

$$\mu_n^{[k_n \cdot t]} \rightarrow \nu_t, \quad t \geq 0 \quad (1.5)$$

iff

$$A_n := k_n \cdot (\mu_n - \varepsilon_e) \rightarrow A \text{ weakly on } \mathfrak{E}(\mathbb{G}) \quad (1.6)$$

Again convergence in (1.5) is uniform in t on compact subsets of \mathbb{R}_+

Convergence in (1.5) is frequently called *functional convergence*.

[[See e.g. [8, 17, 9], 2.0.14.]]

Theorem (E) Mixing of probabilities and of generating functionals

Let $\mathbb{R} \ni t \mapsto \mu(t) \in \mathcal{M}^1(\mathbb{G})$ be a continuous function and let $\rho \in \mathcal{M}^1(\mathbb{R})$.

(a) Then

$$\int_{\mathbb{R}} \mu(t) d\rho(t) \in \mathcal{M}^1(\mathbb{G}). \quad (1.7)$$

And by Theorem (C) we obtain an analogous result for mixing of generating functionals:

(b) $\mathcal{GF}(\mathbb{G})$ is a closed convex cone. Let $\mathbb{R} \ni t \mapsto A(t) \in \mathcal{GF}(\mathbb{G})$ be continuous and let $\rho \in \mathcal{M}_+^b(\mathbb{R})$, the set of non-negative bounded measures. Then

$$\int_{\mathbb{R}} A(t) d\rho(t) \in \mathcal{GF}(\mathbb{G}) \quad (1.8)$$

[[(a) is obvious, (b) follows immediately by Theorem (C). For mixing of generating functionals and applications into a different direction, in particular to self-decomposable laws, see e.g. [4].]]

In the following we shall apply Theorem (E) in a particular form:

Theorem (E') Let $(\tau_t)_{t>0} \subseteq \text{Aut}(\mathbb{G})$ be a continuous one-parameter group of automorphisms with multiplicative parameterization, i.e. $\tau_s \tau_t = \tau_{s \cdot t}$ for $s, t > 0$. Let $(\nu_t)_{t \geq 0}$ be a continuous convolution semigroup, $(\nu_t := \text{Exp}(t \cdot A))_{t \geq 0}$, let further $c > 1$ and $\eta \in \mathcal{M}^1(\mathbb{G})$. Then

$$\mu := \int_0^1 \tau_{c^{-t}}(\nu_{c^t}) dt \in \mathcal{M}^1(\mathbb{G}), \quad (1.9)$$

and

$$\bar{\mu} := \int_0^1 \frac{\log c}{c-1} \cdot c^t \cdot \tau_{c^{-t}}(\eta) dt \in \mathcal{M}^1(\mathbb{G}) \quad (1.10)$$

Furthermore,

$$B := \int_0^1 c^t \cdot \tau_{c^{-t}}(A) dt \in \mathcal{GF}(\mathbb{G}) \quad (1.11)$$

Remark. As usual, $\tau(\mu)$ and $\tau(A)$ are defined by $\langle \tau(\mu), f \rangle := \langle \mu, f \circ \tau \rangle$, $f \in C_0(\mathbb{G})$ and $\langle \tau(A), f \rangle := \langle A, f \circ \tau \rangle$, $f \in \mathfrak{C}(\mathbb{G})$ for $\mu \in \mathcal{M}^b(\mathbb{G})$ and for $A \in \mathcal{GF}(\mathbb{G})$ respectively.

Definition (F) (Semi-)stability (See e.g. [9], §1.4, §2.4)

Let as before $(\nu_t := \text{Exp}(t \cdot A))_{t \geq 0}$ be a continuous convolution semigroup, let $c > 1$, let $(\tau_t)_{t>0} \subseteq \text{Aut}(\mathbb{G})$ be a continuous one-parameter group of automorphisms and $a \in \text{Aut}(\mathbb{G})$.

a) $(\nu_t)_{t \geq 0}$ resp. A is called (strictly) (a, c) -semistable if

$$a(\nu_t) = \nu_{c \cdot t} \quad \text{for all } t \geq 0 \quad (1.12)$$

equivalently, iff

$$a(A) = c \cdot A \quad (1.13)$$

b) $(\nu_t)_{t \geq 0}$ resp. A is called (strictly) $((\tau_s)_{s > 0})$ -stable if

$$\tau_s(\nu_t) = \nu_{s \cdot t} \text{ for all } s > 0, t \geq 0 \quad (1.14)$$

equivalently, iff

$$\tau_s(A) = s \cdot A \text{ for all } s > 0 \quad (1.15)$$

Remark. If \mathbb{G} is second countable and if $(X_t, t \geq 0)$ is a Lévy process corresponding to $(\nu_t)_{t \geq 0}$ starting in the origin, $X_0 = e$ a.s., then (strict) (a, c) -semistability resp. (τ_t) -stability are characterised by

$$a(X_t) \stackrel{D}{=} X_{c \cdot t} \text{ resp. } \tau_s(X_t) \stackrel{D}{=} X_{s \cdot t}, \quad s > 0, t \geq 0,$$

where $\stackrel{D}{=}$ means equality of finite dimensional distributions.

For a survey on semistability on groups see e.g. [9], §1.3 (for vector spaces), or [16], and [9], §2.3 ff resp. §3.4 for groups. See also the references mentioned there.

Definition (G) Domains of attraction (See e.g. [9], §1.6, §2.6)

Let $\eta \in \mathcal{M}^1(\mathbb{G})$. With the notations introduced above we define:

If (ν_t) is (a, c) -semistable:

η belongs to the normal domain of *semistable attraction*, $\eta \in \text{NDA}_{sst}((\nu_t))$, if with $k_n := [c^n]$

$$a^{-n}(\eta^{[k_n \cdot t]}) \rightarrow \nu_t \text{ for all } t \geq 0 \quad (1.16)$$

If (ν_t) is (τ_t) -stable:

η belongs to the normal domain of *(stable) attraction*, $\eta \in \text{NDA}((\nu_t))$, if

$$\tau_{1/n}(\eta^{[n \cdot t]}) \rightarrow \nu_t \text{ for all } t \geq 0 \quad (1.17)$$

For further references see also the hints in the Remark following Definition (F) above.

Remarks a) In the following we shall always assume that the underlying group admits a continuous group $(\tau_t) \subseteq \text{Aut}(\mathbb{G})$ and a (τ_t) -stable convolution semigroup $(\lambda_t)_{t \geq 0}$ with $\lambda_0 = \varepsilon_e$ and furthermore that \mathbb{G} is generated by the supports of λ_t , $t > 0$. In this case, as mentioned above, it is well known that \mathbb{G} is a contractible – hence in particular simply connected nilpotent – Lie group.

[[See e.g. [9], §3.4, §3.1 IV and the references mentioned there.]]

b) Furthermore, w.l.o.g. we shall always assume that $a = \tau_c$. This is justified by [9], Ch. II, 2.8.14, or [5], 2.4 resp. [6], appendix.

c) In the case of (τ_c, c) -semistability we observe in view of Theorem (D) replacing a by τ_c :

$$(1.16) \text{ is equivalent with } A_n := k_n \cdot (\tau_{c^{-n}}(\eta) - \varepsilon_e) \rightarrow A \quad (1.18)$$

and

$$(1.17) \text{ is equivalent with } A_n := n \cdot (\tau_{1/n}(\eta) - \varepsilon_e) \rightarrow A \quad (1.19)$$

(convergence in the weak* topology $\sigma(\mathfrak{E}(\mathbb{G})', \mathfrak{E}(\mathbb{G}))$).

Therefore we sometimes use the notation $\eta \in \text{NDA}_{sst}(A)$ synonymously with $\eta \in \text{NDA}_{sst}((\nu_t))$, and analogously, we define $\eta \in \text{NDA}(A)$ for the domain of stable attraction.

d) We defined (semi-)stability as a property of convolution semigroups resp. of generating functionals, not of a single measure as in the vector space case. In fact, in our case, i.e. for *simply connected nilpotent Lie groups*, these definitions are equivalent: As afore mentioned, in this case, for semistable laws the embedding into convolution semigroups is uniquely determined. \llbracket See e.g. [9], Ch. II, 2.6.11, 2.6.11* \rrbracket

(H) Transformation of Lévy processes and mixing

Let (X_t) be a (τ_c, c) -semistable Lévy process with continuous convolution semigroup $(\nu_t)_{t \geq 0}$. Let θ be a real random variable with uniform distribution in $[0, 1]$ which is independent from (X_t) . Then μ , defined in (1.9), is the distribution of the space-time transformation $Z := \tau_{c-\theta}(X_{c^\theta})$ of the process (X_t) . Analogously, the following construction leads to a random variable \bar{Z} with distribution $\bar{\mu}$ defined in (1.10):

Let Y be a \mathbb{G} -valued random variable with distribution η and let θ be as above, such that Y and θ are independent. Then the distribution of $\zeta := \frac{c^\theta - 1}{c - 1}$ has the density $x \mapsto 1_{[0,1]}(x) \cdot c^x \cdot \frac{\log(c)}{c-1}$, the density of the mixing measure of $\bar{\mu}$ (in (1.10)). Therefore, $\bar{Z} := \tau_\zeta(Y) = \tau_{\frac{c^\theta - 1}{c - 1}}(Y)$ has distribution $\bar{\mu}$.

2 Stable laws in a properly semistable context

In [1] Theorem 2.1 P. Becker-Kern and H-P. Scheffler obtained for operator-(semi)stable laws on finite dimensional vector spaces a beautiful result showing that a suitable mixing of properly semi stable laws leads to stable laws, and a similar result is obtained in [1] Theorem 3.3 for laws belonging to the domain of semistable attraction. We call the convolution semigroup $(\nu_t)_{t \geq 0}$ *properly semistable* if it is semistable but not stable. The above-mentioned results of Becker-Kern and Scheffler [1] are embedded into more general investigations in the context of operator (semi-)self-similar processes. This more general frame and some of the tools used in the proofs have no counterparts on groups. Furthermore, in the case of (in general non-Abelian) groups, it is natural to restrict the investigations to *strict* (semi)stability whereas for vector spaces usually (as in [1]) normalisation by affine transformations is admitted.

Here, in Section 2 we present a new and direct approach to the above-mentioned results, the proofs – in Section 4 – are based mainly on the use of

generating functionals. In the appendix we sketch proofs for the vector space situation relying on characteristic functions only.

In fact, with the notations introduced before we obtain for probabilities on a simply connected nilpotent Lie group \mathbb{G} with contractive automorphism group $(\tau_t)_{t>0}$:

Theorem 2.1 *Let $(\nu_t = \text{Exp}(tA))_{t \geq 0}$ be semi stable w.r.t. (τ_c, c) . Consider the mixing of probabilities resp. of generating functionals*

$$\mu := \int_0^1 \tau_{c^{-t}}(\nu_{c^t}) dt \in \mathcal{M}^1(\mathbb{G}) \quad (\text{cf. (1.9)})$$

$$B := \int_0^1 c^t \cdot \tau_{c^{-t}}(A) dt \in \mathcal{GF}(\mathbb{G}) \quad (\text{cf. (1.11)})$$

and put $(\lambda_t := \text{Exp}(t \cdot B))_{t \geq 0}$

(a) *Then*

$$(\lambda_t)_{t \geq 0} \text{ resp. } B \text{ is } (\tau_t) \text{-stable} \quad (2.1)$$

and

(b) μ belongs to the domain of stable attraction, $\mu \in \text{NDA}((\lambda_t)) (= \text{NDA}(B))$.

And for probabilities belonging to the domains of semistable attraction we obtain analogously

Theorem 2.2 *Let $\eta \in \text{NDA}_{sst}((\nu_t))$, i.e. $\tau_{c^{-n}}(\eta^{[k_n \cdot t]}) \rightarrow \nu_t$, $t \geq 0$, where we put again $k_n := [c^n]$. Consider in this case the mixing*

$$\bar{\mu} := \int_0^1 \frac{\log c}{c-1} \cdot c^t \cdot \tau_{c^{-t}}(\eta) dt \in \mathcal{M}^1(\mathbb{G}) \quad (\text{cf. (1.10)})$$

resp.

$$C := \frac{\log c}{c-1} \cdot B \in \mathcal{GF}(\mathbb{G}), \text{ with } B \text{ as in Theorem 2.1 above}$$

and put finally

$$\bar{\lambda}_t := \text{Exp}(t \cdot C) = \text{Exp}\left(t \cdot \frac{\log c}{c-1} \cdot B\right) = \lambda_{t \cdot \frac{\log c}{c-1}}, \quad t \geq 0. \quad (2.2)$$

Then

(a) $(\bar{\lambda}_t)_{t \geq 0}$ is (τ_t) -stable (according to Theorem 2.1)

and we have again

(b) $\bar{\mu}$ belongs to the domain of stable attraction, $\bar{\mu} \in \text{NDA}((\bar{\lambda}_t)) (= \text{NDA}(B))$.

Remarks. a) To reformulate the results in terms of random variables resp. processes let – as in Section 1 (H) – (X_t) be a \mathbb{G} -valued Lévy-process with corresponding convolution semigroup (ν_t) . Let θ be a real random variable with uniform distribution in $[0, 1]$ which is independent of (X_t) and consider the (random) space-time transformation $Z := \tau_{c^{-\theta}}(X_{c^\theta})$. Then, according to Theorem 2.1, Z has a distribution (called μ in Theorem 2.1) which belongs to the normal domain of stable attraction of $(\lambda_t)_{t \geq 0}$. Note however that for all *deterministic times* t the distributions of $\tau_{c^{-t}}(X_{c^{-t}})$, i.e. $\tau_{c^{-t}}(\nu_{c^t})$, are embedded into *properly* semistable convolution semigroups.

b) Note that the space-time transformation Z resp. the mixed probability law μ is in general *not* stable, however the ‘infinitesimal’ mixing (i.e., mixing of generating functionals resp. of infinitesimal generators) leads to stable laws $(\lambda_t)_{t \geq 0}$. As easily shown, if (ν_t) is already (τ_t) -stable then $\mu = \nu_1$ (in Theorem (2.1)), whereas in general, we have $\bar{\mu} \neq \nu_t$ for all $t > 0$ (in Theorem (2.2)).

c) Becker-Kern and Scheffler [1], 3.3 proved a slightly different version of Theorem 2.2. But I was unable to find an exact generalisation for the group case.

d) Since any semistable distribution belongs to its own domain of semistable attraction, Theorem 2.1 and 2.2 are closely related. To point out the differences, choose $\eta := \nu_1$, resp. $Y := X_1$. We obtain in this case (with the above notations) in Theorem 2.1 resp. in Theorem 2.2 the random variables

$$Z := \tau_{c^{-\theta}}(X_{c^\theta}) \quad \text{resp.} \quad \bar{Z} := \tau_{\frac{c^\theta - 1}{c - 1}}(X_1)$$

with distributions

$$\mu = \int_0^1 \tau_{c^{-t}}(\nu_{c^t}) dt \quad \text{resp.} \quad \bar{\mu} = \int_0^1 \frac{\log c}{c - 1} \cdot c^t \cdot \tau_{c^{-t}}(\nu) dt$$

According to Theorems 2.1 and 2.2 both sequences of generating functionals converge:

$$k_n \cdot (\tau_{c^{-n}}(\mu) - \varepsilon_e) \rightarrow B \quad \text{and} \quad k_n \cdot (\tau_{c^{-n}}(\bar{\mu}) - \varepsilon_e) \rightarrow C = \frac{\log c}{c - 1} \cdot B.$$

Semistability in a strictly stable context

In [1] the authors investigate *stability* in a (properly) *semistable* context, these investigations were translated to the group case in Theorem 2.1 and 2.2. The converse question, looking for *semistability* in a strictly *stable* context, is folklore:

In fact, let $(\nu_t)_{t \geq 0}$ be a continuous convolution semigroup of probabilities on \mathbb{G} which is strictly stable w.r.t. a group of automorphisms $(\tau_t)_{t > 0}$. Then random time substitution, equivalently subordination, defines a continuous affine convolution homomorphism

$$M^1(\mathbb{R}_+) \ni \xi \mapsto \int_0^\infty \nu_t d\xi(t) =: \nu_\xi \in M^1(\mathbb{G})$$

Therefore, as immediately verified, if $(\xi_s)_{s \geq 0}$ is a continuous convolution semigroup on \mathbb{R}_+ then $(\nu_{\xi_s})_{s \geq 0}$ is a continuous convolution semigroup on \mathbb{G} . Furthermore, if $(\xi_s)_{s \geq 0}$ is semistable w.r. to the automorphism $x \mapsto c \cdot x$ of \mathbb{R} for some $c > 0$ then $(\nu_{\xi_s})_{s \geq 0}$ is semistable w.r.t τ_c . (In a similar way this observation provides also a method to construct examples of (semi-)self-decomposable laws on groups.)

3 Random products and (semi)stability

A further related application of mixing of generating functionals in a semistable context is obtained in the following way: In [15] T. Kurtz proved a random Trotter product formula for C_0 -contraction semi groups of operators on a Banach space. The assumptions there are fulfilled for C_0 -semigroups of convolution operators on locally compact groups. Therefore we obtain – with the notations introduced above – as a particular case of Kurtz’s result the following limit theorem. (Note that Kurtz’s approach is more general, here we only consider a particular case of random products):

Theorem 3.1 *Let $(\nu_t = \text{Exp}(t \cdot A))_{t \geq 0}$ be semi stable w.r.t. (τ_c, c) . Define $A(t) := c^t \cdot \tau_{c^{-t}}(A) \in \mathcal{GF}(\mathbb{G})$ as before. Let $(t_n)_{n \in \mathbb{Z}_+} \subseteq [0, 1]$ be an equidistributed sequence, i.e. for all $f \in C[0, 1]$ we have*

$$\frac{1}{N} \sum_0^{N-1} f(t_i) \rightarrow \int_0^1 f dx \quad (3.1)$$

Put for $r, s > 0$:

$$\begin{aligned} \lambda(r, s) &:= \text{Exp} \left(\frac{1}{r} A(t_0) \right) * \cdots * \text{Exp} \left(\frac{1}{r} A(t_{[r \cdot s]}) \right) = \\ &= \tau_{c^{-t_0}} \left(\nu_{\frac{1}{r} c^{t_0}} \right) * \cdots * \tau_{c^{-t_{[r \cdot s]}}} \left(\nu_{\frac{1}{r} c^{t_{[r \cdot s]}}} \right). \end{aligned} \quad (3.2)$$

Then we obtain for all $s > 0$

$$\lambda(r, s) \xrightarrow{r \rightarrow \infty} \lambda_s = \text{Exp}(s \cdot B), \quad s \geq 0 \quad (3.3)$$

(with $(\lambda_s)_{s \geq 0}$ and B defined as before, in (1.11), resp. in Theorem 2.1.)

Note again that in this case the factors $\tau_{c^{-t_i}} \left(\nu_{\frac{1}{r} c^{t_i}} \right)$ in the random convolution products $\lambda(r, s)$ are properly semistable, whereas the limit is stable. Furthermore, if the underlying group is not Abelian, the measures $\lambda(r, s)$ will in general not be embeddable into continuous convolution semigroups, only into *hemigroups*. I.e., the corresponding approximating processes are additive processes but in general not Lévy processes. However, the limits (λ_t) resp. (X_t) form a continuous convolution semigroup resp. a Lévy process.

4 Proofs

Proof of Theorem 2.1.

1. Claim: μ belongs to the domain of *semistable* attraction of $(\lambda_t)_{t \geq 0}$:

⌈ We have to prove

$$\tau_{c^{-n}}(\mu^{[k_n]}) \xrightarrow{n \rightarrow \infty} \lambda_t, \quad t \geq 0, \quad \text{equivalently, by (1.16), (1.17)}$$

$$k_n \cdot (\tau_{c^{-n}}(\mu) - \varepsilon_e) \rightarrow B \quad (4.1)$$

In fact, we have

$$\begin{aligned} k_n \cdot (\tau_{c^{-n}}(\mu) - \varepsilon_e) &= k_n \cdot \left(\tau_{c^{-n}} \int_0^1 \tau_{c^{-t}}(\nu_{c^t}) dt - \varepsilon_e \right) = \\ &= \int_0^1 \tau_{c^{-t}}(k_n \cdot (\tau_{c^{-n}}(\nu_{c^t})) - \varepsilon_e) dt. \end{aligned} \quad (4.2)$$

$$\text{For all } t \geq 0 \text{ we observe } k_n \cdot (\tau_{c^{-n}}(\nu_{c^t}) - \varepsilon_e) \rightarrow c^t \cdot A. \quad (4.3)$$

To prove (4.3) we consider at first – in view of Theorem (D) in Section 1 – the equivalent formulation in terms of convolution powers resp. of discrete semigroups:

$$\begin{aligned} \tau_{c^{-n}}(\nu_{c^t})^{[k_n s]} &= \tau_{c^{-n}}(\nu_{[k_n s] \cdot c^t}) = (\text{since } (\nu_s) \text{ is } (\tau_c, c)\text{-semistable}) \\ &= \nu_{c^{-n} \cdot [k_n s] \cdot c^t} \xrightarrow{n \rightarrow \infty} \nu_{c^t \cdot s} \\ &\text{for } t, s \geq 0 \text{ (since } k_n = [c^n]) \end{aligned}$$

Switching back to the equivalent assertion for generating functionals (cf. (1.5), (1.6)) we obtain $k_n \cdot (\tau_{c^{-n}}(\nu_{c^t}) - \varepsilon_e) \rightarrow c^t \cdot A$ and hence (4.3). Furthermore, according to (1.6) convergence is uniform in t on compact subsets of \mathbb{R}_+ .

Hence the integrands in (4.2) above converge to $A(t) := c^t \cdot \tau_{c^{-t}}(A)$, uniformly on $[0, 1]$, whence (4.1) and therefore the claim follows. ⌋

2. Claim. B resp. $(\lambda_t := \text{Expt} \cdot B)_{t \geq 0}$ is (τ_t) -stable.

⌈ To prove this claim note first that the continuous function $t \mapsto A(t)$ is periodic with period 1: $A(t+1) = c^1 \cdot c^t \cdot \tau_{c^{-1}} \tau_{c^{-t}}(A) = (\text{since } A \text{ is } (\tau_c, c)\text{-semistable}) = c^1 \cdot c^t \cdot c^{-1} \cdot (\tau_{c^{-t}}(A)) = A(t)$.

Therefore, for all $r > 0$, $r = c^x$ with $x = x(r) = \log_c(r) \in \mathbb{R}$, we have $\tau_r(B) = \int_0^1 \tau_{c^x}(c^t \cdot \tau_{c^{-t}}(A)) dt = c^x \cdot \int_0^1 c^{t-x} \cdot \tau_{c^{-(t-x)}}(A) dt = (\text{by translation invariance of the Lebesgue measure on } [0, 1] \text{ and since } A(\cdot) \text{ is periodic}) = c^x \cdot \int_0^1 c^t \cdot \tau_{c^t}(A) dt = c^x \cdot B = r \cdot B$.

The claim follows. ⌋

3. We have proved: B is stable and μ belongs to the domain of *semistable* attraction of B resp. of $(\lambda_t)_{t \geq 0}$. Then μ belongs to the domain of *stable* attraction of B resp. of $(\lambda_t)_{t \geq 0}$. This follows e.g. by [9], 2.6.22 b).

⌈ In fact, the last assertion can be proved directly by standard methods: Put, for $n \in \mathbb{N}$, $n = c^m \cdot c^x$ with $m = m(n) := [\log_c(n)]$ and $x = x(n) \in [0, 1)$. Accumulation points of $\left\{ n \cdot \left(\tau_{\frac{1}{n}}(\mu) - \varepsilon_e \right) = c^x \cdot c^m \cdot \left(\tau_{c^{-x}} \tau_{c^{-m}}(\mu) - \varepsilon_e \right) \right\}_{n \in \mathbb{N}}$ are of the form $c^y \cdot \tau_{c^{-y}} \int_0^1 A(t) dt = \int_0^1 A(t+y) dt$. Hence, since $A(\cdot)$ is periodic, any accumulation point is equal to B . ⌋

Whence the proof of Theorem 2.1 follows. \square

Proof of Theorem 2.2.

$$\text{Let } \eta \in \text{NDA}_{sst}(A), \text{ i.e. } \tau_{c^{-n}}(\eta)^{[k_n \cdot t]} \rightarrow \nu_t = \text{Expt} \cdot A, \quad t \geq 0, \quad (4.4)$$

equivalently, $k_n \cdot (\tau_{c^{-n}}(\eta) - \varepsilon_e) \rightarrow A$. (cf. (1.17), (1.18))

Recall the definition (cf. (1.10)): $\bar{\mu} := \int_0^1 \frac{\log(c)}{c-1} \cdot c^t \cdot \tau_{c^{-t}}(\eta) dt$. (Mixing with respect to a probability density $t \mapsto \frac{\log(c)}{c-1} \cdot c^t \cdot 1_{[0,1]}(t)$.) We put furthermore, $C := \frac{\log(c)}{c-1} \cdot B$, the generating functional of $(\bar{\lambda}_t := \text{Expt} \cdot C = \lambda_{t, \frac{c-1}{\log(c)}})_{t \geq 0}$. Note that (τ_t) -stability of B implies that also $(\bar{\lambda}_t)_{t \geq 0}$ is (τ_t) -stable.

1. Claim. $\bar{\mu}$ belongs to the domain of semistable attraction of $(\bar{\lambda}_t)_{t \geq 0}$.

⌈ For $s > 0, t \in [0, 1]$ we obtain by (4.4), i.e. by the assumption $\eta \in \text{NDA}_{sst}(A)$:

$$\tau_{c^{-t}} \tau_{c^{-n}}(\eta)^{[k_n \cdot c^t \cdot s]} \xrightarrow{n \rightarrow \infty} \tau_{c^{-t}}(\nu_{c^t \cdot s})$$

and furthermore,

$$\tau_{c^{-t}}(\nu_{c^t \cdot s}) = \tau_{c^{-t}}(\text{Exp}(c^t \cdot s \cdot A)) = \text{Exp}(c^t \cdot s \cdot \tau_{c^{-t}}(A)).$$

Equivalently, by (1.6):

$$[k_n \cdot c^t] \cdot (\tau_{c^{-t}} \tau_{c^{-n}}(\eta) - \varepsilon_e) \xrightarrow{n \rightarrow \infty} c^t \cdot \tau_{c^{-t}}(A) =: A(t)$$

whence immediately

$$k_n \cdot c^t \cdot (\tau_{c^{-t}} \tau_{c^{-n}}(\eta) - \varepsilon_e) \xrightarrow{n \rightarrow \infty} A(t) \quad (4.5)$$

follows. Convergence in (4.5) is uniform on compact subsets of \mathbb{R}_+ , since $t \mapsto \tau_t$ is continuous, hence we obtain

$$\int_0^1 k_n \cdot c^t \cdot (\tau_{c^{-t}} \tau_{c^{-n}}(\eta) - \varepsilon_e) dt \xrightarrow{n \rightarrow \infty} \int_0^1 A(t) dt = B. \quad (4.6)$$

The left hand side of (4.6) equals

$$\begin{aligned}
& k_n \cdot \frac{c-1}{\log(c)} \cdot \tau_{c^{-n}} \int_0^1 \left(\frac{\log c}{c-1} \cdot c^t \cdot \tau_{c^{-t}}(\eta) - \varepsilon_e \right) dt = \\
& \text{(since } t \mapsto \frac{\log c}{c-1} \cdot c^t \cdot 1_{[0,1]}(t) \text{ is a probability density)} \\
& = \frac{c-1}{\log(c)} \cdot k_n \cdot \tau_{c^{-n}} \left(\int_0^1 \frac{\log c}{c-1} \cdot c^t \cdot \tau_{c^{-t}}(\eta) dt - \varepsilon_e \right) \\
& = \frac{c-1}{\log(c)} \cdot k_n \cdot (\tau_{c^{-n}}(\bar{\mu}) - \varepsilon_e) \xrightarrow{n \rightarrow \infty} B \text{ (according to (4.6)).}
\end{aligned}$$

Whence $k_n \cdot (\tau_{c^{-n}}(\bar{\mu}) - \varepsilon_e) \xrightarrow{n \rightarrow \infty} C = \frac{\log c}{c-1} \cdot B$ follows.

The claim is proved. \square

As in step 3 of the proof of Theorem 2.1 it follows by the stability of C , that $\bar{\mu}$ belongs to the domain of *stable* attraction of C resp. of $(\bar{\lambda}_t)_{t \geq 0}$. \square

Proof of Theorem 3.1.

The assertion follows as a particular case of T. Kurtz's random product formula [15], Theorem (2.1): Let $(t_n)_{n \in \mathbb{N}}$ be equi-distributed in $[0, 1]$. Then there exists a pure jump process taking values in $[0, 1]$ with jumps t_n at time n , and constant between n and $n + 1$.

Applying Kurtz's theorem to the C_0 -contraction semigroups of convolution operators of $\{(\text{Exp}(s \cdot A(t)))_{s \geq 0} : t \in [0, 1]\}$ with equi-distribution on $[0, 1]$ as mixing measure, the assertion follows:

Note that in Kurtz's random product formula for fixed $r > 0$ there appears an additional factor, $\tau_{c^{-t[r,s]}} \left(\nu_{r, s - [r,s], c^t [r,s]} \right)$. In the situation considered here these factors converge to ε_e with $r \rightarrow \infty$. \square

Appendix: The vector space case

A reader only interested in probabilities on vector spaces $\mathbb{G} = \mathbb{R}^d$ will probably prefer to have a proof avoiding generating functionals. We present sketches of such proofs. Let $(\nu_t)_{t \geq 0}$ be a continuous convolution semigroup of probabilities on \mathbb{R}^d with Fourier transform $\widehat{\nu}_t(\cdot) = e^{t \cdot L(\cdot)}$. L , the logarithm of $\widehat{\nu}_1$, often called second characteristic function, plays the role of the generating functional A of $(\nu_t)_{t \geq 0}$ in the group-case. In fact, the characters $\chi_{\vec{y}} : \vec{x} \mapsto e^{i \cdot \langle \vec{x}, \vec{y} \rangle}$ belong to $\mathfrak{C}(\mathbb{R}^d) = C_b^\infty(\mathbb{G})$, and obviously $\widehat{A}(\vec{y}) := A(\chi_{\vec{y}}) = L(\vec{y})$.

One parameter groups of automorphisms have the form $\{\tau_t = t^E\}_{t > 0}$ where $t^E := e^{\log(t) \cdot E}$ for some *exponent* $E \in \text{End}(\mathbb{R}^d)$. Semistability and stability may

be characterized by properties of L : (ν_t) resp. L is (strictly) (τ_c, c) -semistable iff

$$L(c^{E^*})(\vec{y}) = c \cdot L(\vec{y}) \quad \text{for all } \vec{y} \in \mathbb{R}^d.$$

The vector space version of the key Theorem (D) in Section 1 – convergence of discrete semigroups – can be formulated in the following way:

$$\text{(D)'} \quad \widehat{\mu}_n^{k_n} \rightarrow \widehat{\nu}_1 = e^L \quad \text{iff} \quad \widehat{\mu}_n^{[k_n \cdot s]} \rightarrow \widehat{\nu}_s = e^{s \cdot L} \quad \forall s \geq 0$$

(and convergence is uniform on compact subsets of $\mathbb{R}_+ \times \mathbb{R}^d$),

$$\text{furthermore, this is the case iff } k_n \cdot (\widehat{\mu}_n - 1) \rightarrow L. \quad (\text{A.1})$$

[[This result is folklore. For a proof see e.g. [9], 1.6.12. (For non identically distributed arrays see e.g. [16], Theorem 3.2.12.)]]

A.1 Now we are ready to sketch a **proof of Theorem 2.1** in terms of characteristic functions:

We have

$$\widehat{\mu}(\vec{y}) = \int_0^1 \widehat{\tau_{c^{-t}}(\nu_{c^t})}(\vec{y}) dt = \int_0^1 \exp(c^t \cdot L(c^{-t \cdot E^*}(\vec{y}))) dt$$

$$\text{and } \widehat{\lambda}_s(\vec{y}) = e^{s \cdot M(\vec{y})} \quad \text{with} \quad M(\vec{y}) = \int_0^1 c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) dt.$$

In view of (D)' – the vector space version of Theorem (D) in Section 1 – we have to show $k_n \cdot (\widehat{\tau_{c^{-n}}(\mu)} - 1) \rightarrow M$, i.e.

$$\int_0^1 k_n \cdot (\exp(c^t \cdot L(c^{-t \cdot E^*} c^{-n \cdot E^*}(\vec{y}))) - 1) dt \rightarrow M(\vec{y})$$

By semistability of L this is equivalent with

$$\int_0^1 k_n \cdot (\exp(c^t \cdot c^{-n} \cdot L(c^{-t \cdot E^*}(\vec{y}))) - 1) dt \rightarrow \int_0^1 c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) dt$$

In view of $k_n = [c^n]$ it easily follows that the integrands converge uniformly on $[0, 1]$, whence the first step is proved: μ belongs to the domain of semistable attraction of $(\lambda_t)_{t \geq 0}$.

(c^E, c) -semistability yields immediately that $t \mapsto c^t \cdot L(c^{-t \cdot E^*}(\vec{y}))$ is periodic with period 1 for every fixed $\vec{y} \in \mathbb{R}^d$, hence again translation invariance of the mixing measure implies stability of $M : \vec{y} \mapsto \int_0^1 c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) dt$.

Now the proof of Theorem 2.1 follows again by the observation that for (operator)-stable laws the domains of semistable attraction and of stable attraction coincide. This can be proved directly verbatim as in the last step in the group case. For a proof in the vector space case for generalized domains of attraction see e.g. [16], 7.5.11, or [9], 1.6.24 b). \square

A.2 The proof of Theorem 2.2 for vector spaces runs along the same lines: It is again sufficient to sketch a proof of the first step: $\bar{\mu} \in \text{NDA}_{sst}((\bar{\lambda}_t))$. According to (D)' we have (with the notations introduced above):

$$k_n \cdot (\widehat{\eta}(c^{-n \cdot E^*}(\vec{y})) - 1) \rightarrow L(\vec{y}) \text{ for all } \vec{y} \quad (\text{A.2})$$

By definition we have

$$\widehat{\mu}(\vec{y}) = \int_0^1 \frac{\log(c)}{c-1} \cdot c^t \cdot \widehat{\eta}(c^{-t \cdot E^*}(\vec{y})) dt \quad (\text{A.3})$$

$$\text{and } \widehat{\lambda}_t(\vec{y}) = e^{t \cdot \overline{M}(\vec{y})} \text{ with}$$

$$\overline{M}(\vec{y}) = \frac{\log(c)}{c-1} \cdot \int_0^1 c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) dt = \frac{\log(c)}{c-1} \cdot M(\vec{y}) \quad (\text{A.4})$$

(A.2) yields for fixed t and \vec{y} (cf. (4.5)):

$$k_n \cdot c^t \cdot (\widehat{\eta}(c^{-n \cdot E^*} c^{-t \cdot E^*}(\vec{y})) - 1) \rightarrow c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) \quad (\text{A.5})$$

Uniform convergence on compact subsets yields

$$\int_0^1 k_n \cdot c^t \cdot (\widehat{\eta}(c^{-n \cdot E^*} c^{-t \cdot E^*}(\vec{y})) - 1) dt \rightarrow \int_0^1 c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) dt = M(\vec{y})$$

Therefore, according to (A.3) and (A.4)

$$\begin{aligned} & k_n \cdot (\widehat{\mu}(c^{-n \cdot E^*}(\vec{y})) - 1) \\ &= k_n \cdot \left(\int_0^1 c^t \cdot \widehat{\eta}(c^{-n \cdot E^*} c^{-t \cdot E^*}(\vec{y})) \cdot \frac{\log(c)}{c-1} dt - 1 \right) \\ &= k_n \cdot \int_0^1 (c^t \cdot \widehat{\eta}(c^{-n \cdot E^*} c^{-t \cdot E^*}(\vec{y})) - 1) \cdot \frac{\log(c)}{c-1} dt \\ &= \frac{\log(c)}{c-1} \cdot k_n \cdot \int_0^1 (c^t \cdot \widehat{\eta}(c^{-n \cdot E^*} c^{-t \cdot E^*}(\vec{y})) - 1) dt \\ &\rightarrow \frac{\log(c)}{c-1} \cdot M(\vec{y}) = \overline{M}(\vec{y}). \end{aligned}$$

Hence $\bar{\mu} \in \text{NDA}_{sst}((\bar{\lambda}_t))$. Whence, as before in the proof of Theorem 2.1 $\mu \in \text{NDA}((\lambda_t))$ follows. \square

A.3 Finally we sketch a **proof of Theorem 3.1 for vector spaces**:

With the notations introduced above, $\frac{1}{r} \cdot L_t : \vec{y} \mapsto \frac{1}{r} \cdot c^t \cdot L(c^{-t \cdot E^*}(\vec{y}))$ is – for fixed t and $r > 0$ – the logarithm of the Fourier transform of $\text{Exp}(\frac{1}{r} \cdot A(t)) = c^{-t \cdot E}(\nu_{\frac{1}{r} \cdot c^t})$. Hence, since convolution is commutative in this case, $\lambda(r, s)$ is infinitely divisible and the logarithm of its Fourier transform is given as

$$\vec{y} \mapsto R(r, s)(\vec{y}) := \frac{1}{r} \cdot \sum_0^{[r \cdot s]} c^{tk} \cdot L(c^{-tk \cdot E^*}(\vec{y}))$$

Since $(t_k)_{k \in \mathbb{N}}$ is equi-distributed and $t \mapsto c^t \cdot L(c^{-t \cdot E^*}(\vec{y}))$ belongs to $C[0, 1]$, we observe

$$R(r, s) \xrightarrow{r \rightarrow \infty} s \cdot \int_0^1 c^t \cdot L(c^{-t \cdot E^*}(\vec{y})) dt = s \cdot M(\vec{y})$$

Hence Theorem 3 is proved: $\widehat{\lambda}(r, s) \xrightarrow{r \rightarrow \infty} \exp s \cdot M, s \geq 0$. □

References

- [1] **Becker-Kern, P., Scheffler, H-P.:** How to find stability in a purely semistable context. *Yokohama Math. J.* **51**, 75–88 (2005)
- [2] **Faraut, J., Harzallah, K.:** Semigroupes d’opérateurs invariants et opérateurs dissipatifs invariants. *Ann Math. Inst. Fourier* **22**, 147–164 (1972)
- [3] **Hazod, W.:** *Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmaßen und erzeugende Distributionen. Lecture Notes in Mathematics* **595** Springer (1977)
- [4] **Hazod, W.:** Stable hemigroups and mixing of generating functionals. *J. Math. Sci.* **111**, 3830–3840 (2002)
- [5] **Hazod, W.:** On normalizers and centralizers of compact Lie groups. Applications to structural probability theory. *Prob. Math. Stat.* **23**, 39–60 (2003)
- [6] **Hazod, W.:** Normalisers and centralisers of compact matrix groups. An elementary approach. *Linear Algebra Appl.* **410**, 96–111 (2005)
- [7] **Hazod, W.:** A generalization of E. Siebert’s theorem on convergence of convolution semigroups and accompanying laws. *Theory Prob. Appl.* **40**, 929–934 (1995)
- [8] **Hazod, W., Scheffler, H-P.:** The domains of partial attraction of probabilities on groups and on vector spaces. *J. Theoretical Probability* **6**, 175–186 (1993)

- [9] **Hazod, W., Siebert, E.:** *Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups. Structural Properties and Limit Theorems.* Mathematics and its Applications vol. **531**. Kluwer A.P. (2001)
- [10] **Heyer, H.:** *Probability Measures on Locally Compact Groups.* Berlin-Heidelberg-New York. Springer (1977)
- [11] **Heyer, H.:** Semi-groupes de convolution sur un groupe localement compact et applications à la théorie des probabilités. In: Ecole d'Eté de Probabilités de Saint Flour VII. *Lecture Notes Math.* **6** 173–236 Springer (1978)
- [12] **Hirsch, F.:** Opérateurs dissipatifs et codissipatifs invariants par translation sur les groupes localement compacts. *Séminaire de théorie de potentiel.* 15^{ième} année (1971/72)
- [13] **Hirsch, F.:** Sur les semi-groupes d'opérateurs invariants par translation. *CRAS Paris* **274**, 43–46 (1972)
- [14] **Hirsch, F., Roth, J.P.:** Opérateurs dissipatifs et codissipatifs invariants par translation sur un espace homogène. In: *Lecture Notes in Math.* **404**, 229–245, Springer (1974)
- [15] **Kurtz, T.:** A random Trotter product formula. *Proc. Amer. Math. Soc.* **35**, 147 – 154 (1973)
- [16] **Meerschaert, M.M., Scheffler, H-P.:** *Limit Theorems for Sums of Independent Random Vectors. Heavy Tails in Theory and Praxis.* J. Wiley Inc. (2001)
- [17] **Pap, G.:** Convergence of continuous convolution semigroups on Lie groups. *Techn. Report Univ. Debrecen* **92/49** (1992)
- [18] **Sato, K.:** *Lévy Processes and Infinitely divisible Distributions.* Cambridge Univ. Press (1999)
- [19] **Siebert, E.:** Über die Erzeugung von Faltungshalbgruppen auf beliebigen lokalkompakten Gruppen. *Math. Z.* **131**, 313–333 (1973)
- [20] **Siebert, E.:** Fourier Analysis and limit theorems for convolution semi-groups on a locally compact group. *Adv. Math.* **39**, 111–154 (1981)

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