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Multiple selfdecomposable laws on vector spaces and on groups: The existence of background driving processes

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MULTIPLE SELFDECOMPOSABLE LAWS ON VECTOR SPACES AND ON GROUPS: THE EXISTENCE OF BACKGROUND DRIVING PROCESSES

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Introduction

Self-decomposable laws or class $L-$laws were introduced by P. Lévy within the frame of limit distributions of normalized sums of independent (not necessarily identically distributed) random variables. In the past various types of distributions which are well-known in statistical applications turned out to be self-decomposable. See e.g. Z. Jurek [14], K. Sato[20] for a survey. Recently the self-decomposability property and the related additive processes – one- and multidimensional – turned out to be important for model building in Mathematical Finance. See e.g., [3] for a survey and for references.

K. Urbanik [25] extended the concept of self-decomposability to finite dimensional vector spaces $\mathbb{V}$ with operator normalization. See also [21] or the monograph [12], and the literature mentioned there.

Closely related to self-decomposability are generalized Ornstein–Uhlenbeck processes and Mehler semigroups of transition operators and, on the other hand, stable hemigroups and M-semigroups of probabilities. (For details and hints to the literature see e.g., [12, 5, 7, 9, 10, 22, 1, 19].) Let $(X_t)_{t\geq 0}$ be a stochastically continuous additive process taking values in $\mathbb{V}$ then the distributions $\nu(s,t)$ of the increments $X_s^{-1}X_t$, $s \leq t$, form a *continuous convolution hemigroup*, i.e. $(s,t) \mapsto \nu(s,t)$ is continuous and

$$\nu(s,t) \star \nu(t,r) = \nu(s,r) \quad \text{for } s \leq t \leq r \tag{H}$$

A hemigroup $(\nu(s,t))_{s\leq t}$ is called *stable* w.r.t. a continuous one-parameter group of vector space automorphisms $\mathbb{T} = (T_t)_{t\in\mathbb{R}} \subseteq \mathrm{GL}(\mathbb{V})$ if for all r, for all $s \leq t$

$$T_r(\nu(s,t)) = \nu(s+r,t+r), \quad \nu(s,t) = T_s\left(\nu(0,t-s)\right) \tag{S}$$

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Put $\nu(s) := \nu(0,s),\ s \geq 0$. Then, as easily verified,

$$\nu(s+t) = \nu(s) \star\ T_s(\nu(t)),\ 0 \leq s \leq t \qquad (M)$$

Continuous families $(\nu(s))_{s\geq 0}$ of probabilities satisfying (M) are called *M-semigroups* or skew semigroups. (The corresponding transition operators are generalized Mehler semigroups.)

$\mu \in M^1(\mathbb{V})$ is called (operator) self-decomposable w.r.t. $\mathbb{T}$ if $\forall t \geq 0$

$$\mu = T_t(\mu) \star \nu(t) \text{ for some } \nu(t) \in M^1(\mathbb{V}) \qquad (SD)$$

$\nu(t)$ is called *cofactor*. Stable hemigroups and M-semigroups are interesting objects of investigation in their own right. Furthermore, we have: If μ is self-decomposable (w.r.t. $\mathbb{T}$) then the cofactors $(\nu(t))_{t\geq 0}$ form a M-semigroup and $(T_s(\nu(t-s)))_{s\leq t}$ is a stable hemigroup. Hence in view of the above mentioned connections, self-decomposable laws with contracting $\mathbb{T}$ are limits of (generalized) Ornstein–Uhlenbeck processes (resp. of the corresponding M-semigroups). (Cf. [22], see also [10]).

K. Urbanik [26] introduced *multiple self-decomposability* defining nested classes of self-decompsable laws $L^{(m)}(\mathbb{T})$ inductively: $L^{(0)}(\mathbb{T}) := M^1(\mathbb{V})$, $L^{(1)}(\mathbb{T})$ the set of self-decomposable laws,

$$L^{(m+1)}(\mathbb{T}) := \left\{\mu : \mu = T_t(\mu) \star \nu(t) \text{ with } \nu(t) \in L^{(m)}(\mathbb{T}),\ t > 0\right\} \qquad (MSD)$$

See also e.g., [4, 21, 23]. The concepts of self-decomposability, M-semigroups, stable hemigroups generalize to contractible (hence simply connected nilpotent) Lie groups $\mathbb{G}$, where $\mathbb{T} \subseteq \mathrm{Aut}(\mathbb{G})$ denotes a subgroup of automorphisms. The afore mentioned defining equations $(H), (S), (SD)$ are used verbatim in this more general situation. See e.g., [5, 7, 10, 8, 19]. For self-decomposable laws on groups in connection with limit theorems see e.g. [24]. In particular, also multiply self-decomposability (MSD) and the classes $L^{(m)}(\mathbb{T})$ make sense in the group case.

For vector spaces $\mathbb{V}$, self-decomposable laws μ and their cofactors $\nu(t)$ are infinitely divisible. Hence for any fixed $s \geq 0$ there exists a Lévy process $\left(Z_t^{(s)}\right)_{t\geq 0}$ such that $Z_1^{(s)}$ is distributed according to $\nu(s)$. But the interesting objects are additive processes $(X_t)_{t\geq 0}$ with – in general non-stationary – increments $(X_s^{-1}X_t)_{s\leq t}$ distributed according to $\nu(s,t)$ with (H) and (S). There exist *hidden* Lévy processes $(Y_t)_{t\geq 0}$ (uniquely determined up to equivalence) *driving* $(X_t)_{t\geq 0}$, i.e., we have a random integral representation

$$X_t = \int_0^t T_u dY_u\ , \quad t \geq 0 \qquad (RI)$$

$(Y_t)_{t\geq 0}$ is called *background driving Lévy process.* See e.g. [14, 12, 20, 2].

For group-valued processes only weak versions of (RI) are known: Let $(\nu(s,t))_{s\leq t}$ be a stable hemigroup (with corresponding M-semigroup $\nu(t) := \nu(0,t) : t \geq 0)$ then there exists a uniquely determined continuous convolution semigroup $(\nu_t)_{t\geq 0}$ related to the M-semigroup $(\nu(t))_{t\geq 0}$ by Lie-Trotter formulas

$$\nu_t = \lim_n \nu(1/n)^{[nt]} = \lim_n \nu(t/n)^n \qquad (LT1)$$

and

$$\nu(t) = \lim_n \mathop{\bigstar}_{k=0}^{[nt]-1} T_{k/n}(\nu_{1/n}), \quad \nu(s,t) = \lim_n \mathop{\bigstar}_{k=[ns]}^{[nt]-1} T_{k/n}(\nu_{1/n}) \qquad (LT2)$$

By (slight) abuse of language we call in the sequel the continuous convolution semigroup $(\nu_t)_{t\geq 0}$ the background driving Lévy process of the M-semigroup $(\nu(t))_{t\geq 0}$ resp. of the stable hemigroup $(\nu(s,t))_{s\leq t}$.

Let $\mathbb{T}$ be contracting. Then $\lim_{t\to\infty} \nu(t) =: \mu$ exists (and is self-decomposable then) iff $\nu(t)$ possesses finite logarithmic moments for some – hence all – $t > 0$. The M-semigroup of cofactors $(\nu(t))_{t\geq 0}$ possesses finite logarithmic moments $(t > 0)$ iff the background driving process $(\nu_t)_{t\geq 0}$ shares this property $(t > 0)$. (For vector spaces see e.g., [12], for groups see e.g., [5, 8].)

The aim of this paper is to prove the existence of background driving processes for *multiple self-decomposable laws* $\mu \in L^{(m)}(\mathbb{T})$ and to investigate the correspondences between multiple-cofactors and background driving processes in this case. In particular, to obtain analogues of the Lie Trotter formulas $(LT1)$ and $(LT2)$ for the multiple self-decomposable case.

The main results are new even for vector spaces. They are formulated and proved for the group case (under a commutativity assumption). But the proofs are written in such a way that they can easily extended to other convolution structures, e.g., to matrix cone hypergroups, structures closely related to Wishart distributions. (See [6] and the literature mentioned there.) Wishart distributions are not infinitely divisible w.r.t the usual convolution on the vector space of Hermitean matrices, but w.r.t. the new convolution structures they are even stable, hence completely self-decomposable. So, even when the investigations here are motivated by purely mathematical questions, there might be statistical applications in the future.

1. Multiple self-decomposability

Let $\mathbb{G}$ be a contractible (hence simply connected nilpotent) Lie group, let $\mathbb{T} = (T_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group in $\mathrm{Aut}(\mathbb{G})$, $T_{t+s} = T_t T_s,\ t, s \in \mathbb{R}$. We defined in the introduction the classes $L^{(m)}(\mathbb{T})$ (cf. (MSD)).

Proposition 1.1. *Let $\mu \in L^{(m)}(\mathbb{T})$ for some m. Then for all $t_1, \dots, t_m \in \mathbb{R}_+$ and $1 \le k \le m$ there exist $\nu^{(k)}(t_1, \dots, t_k) \in L^{(m-k+1)}(\mathbb{T}) \subseteq M^1(\mathbb{G})$ such that*

$$\begin{aligned}\mu &= T_{t_1}(\mu) \star T_{t_2}\left(\nu^{(1)}(t_1)\right) \star \cdots \\ &\star\ T_{t_m}\left(\nu^{(m-1)}(t_1, \dots, t_{m-1})\right) \star \nu^{(m)}(t_1, \dots, t_m) \qquad (COF)\end{aligned}$$

The measures $\nu^{(k)}(t_1, \dots, t_k)$ are called $k-$cofactors.

$[\![$ $\mu = T_{t_1}(\mu) \star \nu^{(1)}(t_1) = T_{t_1}(\mu) \star T_{t_2}\left(\nu^{(1)}(t_1)\right) \star \nu^{(2)}(t_1, t_2)$, since $\nu^{(1)}(t_1)$ is T_{t_2}-decompsable. Per iteration we obtain finally

$$\begin{aligned}\mu &= T_{t_1}(\mu) \star (T_{t_2}\left(\nu^{(1)}(t_1)\right)) \star T_{t_3}\left(\nu^{(2)}(t_1, t_2)\right) \star \cdots \\ \cdots &\star T_{t_m}\left(\nu^{(m-1)}(t_1, \dots, t_{m-1})\right) \star \nu^{(m)}(t_1, \dots, t_m).\end{aligned}$$ $]\!]$

For the main result, the subsequent Theorem 1.2, we assume additional conditions: The convolution factors in (COF), i.e.,
the probabilities $\left\{T_t(\mu), \dots T_{t_k}(\nu^{(k-1)}(t_1, \dots, t_k)),\ \nu^{(m)}(t_1, \dots, t_m)\right\}$
$1 \le k \le m,\ t_i \in \mathbb{R}_+$ commute (CCF)
For all $\nu \in L^{(1)}(\mathbb{T})$ the convolution operator is injective, i.e.,
$\nu \star \rho = \nu \star \rho' \Rightarrow \rho = \rho'$ (I)
(hence in particular, the cofactors are uniquely determined)
and $\mathbb{T} = (T_t)$ is contracting, i.e., $\forall x \in (G)\ \lim_{t\to\infty} T_t(x) = e$ (CT)

Note that (CCF) and (I) are obviously true for vector spaces: In this case, the convolution semigroup is commutative, therefore (CCF) is trivial, and ν is infinitely divisible, hence the Fourier transform has no zeros. Whence (I) follows. For injectivity in the group case see e.g., the discussion in [18].

Theorem 1.2. *Let $\mu \in L^{(m)}(\mathbb{T})$. Assume (CCF), (I) and (CT). Then there exists a uniquely determined continuous convolution semigroup $\left(\nu_t^{(m)}\right)_{t\ge 0}$, the m^{th}-background driving Lévy process, such that the k-cofactors, $1 \le k \le m$, and μ are uniquely determined by $\left(\nu_t^{(m)}\right)_{t\ge 0}$. Furthermore, $\nu_t^{(m)}$ possesses finite $\log_+^m(\cdot)$–moments for all $t > 0$.*

Hence – under (CCF), (I) and (CT) – there exists a bijective mapping between Lévy processes with finite $\log_+^m(\cdot)$*–moments and* m*-self-decomposable laws. (To simplify notations we write* $\nu_t := \nu_t^{(m)}$*.)*

For $m = 2$ *the result can be formulated in the following way: For a continuous convolution semigroup* $\left(\nu_t = \nu_t^{(2)}\right)_{t\geq 0}$ *with finite* $\log_+^2(\cdot)$*–moments there exists a uniquely determined 2-self-decomposable law* μ *with cofactors* $\nu^{(1)}(s)$, $\nu^{(2)}(s,t)$, $s,t \geq 0$, *such that*

$$\nu^{(2)}(s,t) = \lim_N \lim_M \mathop{\bigstar}_{j=0}^{[Nt]-1} \mathop{\bigstar}_{k=0}^{[Ms]-1} T_{\frac{k}{M}+\frac{j}{N}} \left(\nu_{\frac{1}{N}\cdot\frac{1}{M}}\right)$$

and

$$\nu(s) = \nu^{(1)}(s) = \lim_{t\to\infty} \nu^{(2)}(s,t), \quad \mu = \lim_{s\to\infty} \nu(s)$$

Conversely, let μ *be 2-self-decomposable, let* $\left(\nu^{(2)}(s,t)\right)_{s,t\in\mathbb{R}_+}$ *be corresponding 2-cofactors, then there exists a continuous convolution semigroup* $\left(\nu_r = \nu_r^{(2)}\right)_{r\geq 0}$*, uniquely determined by* μ*, such that for* $r = s\cdot t$, $r,s,t \geq 0$

$$\nu_r = \nu_r^{(2)} = \nu_{s\cdot t}^{(2)} = \lim_N \lim_M \left(\nu^{(2)}(s/M, t/N)\right)^{N\cdot M}$$

The proof will be carried out only for $m = 2$, the general case follows along the same lines by induction. It relies on a space-time enlargement $\Gamma = \mathbb{G} \rtimes \mathbb{R}$, a semidirect extension of $\mathbb{G}$ by the real line via the automorphism group $\mathbb{T}$. The construction provides the means to investigate multi-parameter-analogues of M-semigroups $\left(\nu^{(m)}(t_1,\ldots,t_m)\right)_{t_i\geq 0}$, the m-cofactors of $\mu \in L^{(m)}(\mathbb{T})$. Multi-parameter M-semigroups are – via space-time continuous convolution semigroups and Lie-Trotter formulas – related to multi-parameter continuous convolution semigroups, *the* m^{th}*-background driving Lévy processes.*

2. The toolbox

We consider as afore mentioned the *space-time* group $\Gamma = \mathbb{G} \rtimes \mathbb{R}$, a semidirect product with group operation $(x,s)(y,t) = (xT_s(y), s+t)$, $x,y \in \mathbb{G}, s,t \in \mathbb{R}$. Let $M_*^1(\Gamma) := \{\rho \otimes \varepsilon_r : \rho \in M^1(\mathbb{G}), r \in \mathbb{R}\}$, a closed subsemigroup of $M^1(\Gamma)$. For probabilities in $M_*^1(\Gamma)$, convolution has a considerably simple form:

$$(\rho \otimes \varepsilon_s) * (\rho' \otimes \varepsilon_{s'}) = (\rho \star T_s(\rho')) \otimes \varepsilon_{s+s'}$$

where $*$ denotes convolution on Γ and $\star$ denotes convolution on $\mathbb{G}$.

If $(\mu(t))_{t\geq 0}$ is a M-semigroup on $\mathbb{G}$ then $(\lambda_t := \mu(t)\otimes\varepsilon_t)_{t\geq 0}$ is a continuous convolution semigroup, the *space-time semigroup*, and conversely, for a continuous convolution semigroup $(\lambda_t := \mu(t)\otimes\varepsilon_t)_{t\geq 0}$ of probabilities on Γ the space-component $(\mu(t))_{t\geq 0}$ is a M-semigroup. A continuous convolution semigroup $(\lambda_t)_{t\geq 0}$ is characterized by the *generating functional* $\mathcal{A} := \frac{d^+}{dt}\lambda_t|_{t=0}$ which has for $(\lambda(t))_{t\geq 0} \subseteq M^1_*(\Gamma)$ a pleasant form:

$$\mathcal{A} = B \oplus \varepsilon_0 + \varepsilon_e \oplus P \tag{LT}$$

where $B := \frac{d^+}{dt}\mu(t)|_{t=0}$ and P is a differential operator of 1^{st} order. In particular, $B := \frac{d^+}{dt}\mu(t)|_{t=0}$ exists for any M-semigroup, and B is the generating functional of a continuous convolution semigroup, $(\mu_t)_{t\geq 0}$ say. This semigroup is called background driving Lévy process, as afore mentioned. Applying the Lie-Trotter formula for generating functionals to the decomposition (LT) we obtain

$$\mu(t) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{n-1} T_{\frac{t}{n}k}\left(\mu_{\frac{t}{n}}\right) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{[nt]-1} T_{\frac{k}{n}}\left(\mu_{\frac{1}{n}}\right) \tag{$LT1$}$$

and conversely,

$$\mu_t = \lim_{n\to\infty} \mu\,(t/n)^n = \lim_{n\to\infty} \mu\,(1/n)^{[nt]} \tag{$LT2$}$$

Convergence is uniform on compact subsets of $\mathbb{R}_+$.

For the background of probabilities on groups the reader is referred to, e.g., [11, 5], for details concerning (LT), see e.g. [5, 10].

Putting things together we obtain

Proposition 2.1. ***a)*** *Let $(\mu(t))_{t\geq 0} \subseteq M^1(\mathbb{G})$ be a continuous M-semigroup. Then $(LT2)$ defines a (uniquely determined) continuous convolution semigroup $(\mu_t)_{t\geq 0} \subseteq M^1(\mathbb{G})$.*

b) *Conversely, let $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup then $(LT1)$ defines a (uniquely determined) continuous M-semigroup $(\mu(t))_{t\geq 0}$.*

In the sequel we shall tacitly make use of the following well-known result. (We formulate a version which is adapted to our situation):

Lemma 2.2. ***a)*** *Let $\mathbb{G}$ be a second countable locally compact group and let $\mathbb{R}_+ \ni t \mapsto \alpha_t^{(n)} \in M^1(\mathbb{G})$ be a sequence of functions which are assumed (1) to be weakly continuous, (2) $\forall t \geq 0$ there exists $\lim\limits_{n\to\infty} \alpha_t^{(n)} =: \alpha_t \in M^1(\mathbb{G})$, where (3) $(\alpha_t)_{t\geq 0}$ satisfies the semigroup condition $\alpha_{s+t} = \alpha_s \star \alpha_t,\ s,t \geq 0$.*

Then $(\alpha_t)_{t\geq 0}$ is a continuous convolution semigroup.

b) *As a corollary we obtain: Let* $\mathbb{G}$ *be a contractible Lie group, let* $\mathbb{T} = (T_t) \subseteq \mathrm{Aut}(\mathbb{G})$ *be contracting as before. Let (1)* $t \mapsto \alpha^{(n)}(t) \in M^1(\mathbb{G})$ *be continuous and (2) assume* $\lim_{n\to\infty} \alpha^{(n)}(t) =: \alpha(t) \in M^1(\mathbb{G})$ *to exist. Assume further (3)* $(\alpha(t))_{t\geq 0}$ *to satisfy the M-semigroup condition* $\alpha(s+t) = \alpha(s) \star T_s(\alpha(t))\,,\ s,t \geq 0$.

Then $(\alpha(t))_{t\geq 0}$ *is a continuous M-semigroup.*

$\llbracket$ To prove a) consider the convolution operators acting on $C_c(\mathbb{G}) \subseteq C_0(\mathbb{G}) \cap L^2(\mathbb{G})$: $R_\mu f(x) := \int f(xy)d\mu(y)$, $L_\mu f(x) := \int f(yx)d\mu(y)$. Let $f, g \in C_c(\mathbb{G})$. Then

$$\begin{aligned}\langle R_\mu f, g\rangle &= \int R_\mu f(x)\overline{g(x)}d\omega_{\mathbb{G}}(x) = \int\int f(xy)d\mu(y)\overline{g(x)}d\omega_{\mathbb{G}}(x)\\ &= \int\int f(xy)\overline{g(x)}d\omega_{\mathbb{G}}(x)d\mu(y) =: \langle L_\nu f, \mu\rangle\end{aligned}$$

where $\omega_{\mathbb{G}}$ denotes a Haar measure and $\nu := \overline{g}\cdot\omega_{\mathbb{G}}$ denotes the measure with density $\overline{g}$. Applying this formula to $\mu = \alpha_t^{(n)}$ and to α_t, we obtain that $t \mapsto \langle R_{\alpha_t} f, g\rangle$ is measurable for all $f, g \in C_c(\mathbb{G})$. A density argument shows that $(R_{\alpha_t})_{t\geq 0}$ is a C_0 contraction semigroup on $L^2(\mathbb{G})$, measurable w.r.t. the weak operator topolgy. Since $L^2(\mathbb{G})$ is separable by assumption, continuity (in the strong operator topology) follows. Then, as well known and easily verified, weak continuity of $t \mapsto \alpha_t$ follows.

To prove b) we notice that $(\beta_t := \alpha(t) \otimes \varepsilon_t)_{t\geq 0} \subseteq M^1(\Gamma)$ satisfies the assumptions of a). Hence continuity of $t \mapsto \beta_t$ follows, and therefore $t \mapsto \alpha(t)$ is continuous. $\rrbracket$

Definition 2.3. ***a)*** *A family* $(\nu(s,t))_{s,t\geq 0} \subseteq M^1(\mathbb{G})$ *is called 2-M-semigroup if for fixed* $s \geq 0$ *resp.* $t \geq 0$, $t \mapsto \nu(s,t)$ *resp.* $s \mapsto \nu(s,t)$ *are continuous M-semigroups. (Analogously, m-M-semigroups are defined for* $m \geq 2$*.)*

b) *A family* $(\nu_{s,t})_{s,t\geq 0} \subseteq M^1(\mathbb{G})$ *is called continuous 2-semigroup if for fixed* $s \geq 0$ *resp.* $t \geq 0$, $t \mapsto \nu_{s,t}$ *resp.* $s \mapsto \nu_{s,t}$ *are continuous convolution semigroups.*

In the following we assume throughout (in view of (CCF)) that

$$\{T_r(\nu(s,t))\,,\ r,s,t \geq 0\} \text{ commute} \qquad (C)$$

Applying Proposition 2.1 for fixed s resp. for fixed t we obtain

Proposition 2.4. *Let* $(\nu(s,t))_{s,t\geq 0}$ *be a 2-M-semigroup. Then for fixed* $s \geq 0$ *resp.* $t \geq 0$*, there exist continuous convolution semigroups*

$\left(\rho_t^{(s)}\right)_{t\geq 0}$ *resp.* $\left(\sigma_s^{(t)}\right)_{s\geq 0}$ *such that for fixed* $t \geq 0$ *resp.* $s \geq 0$ $s \mapsto \rho_t^{(s)}$ *and* $t \mapsto \sigma_s^{(t)}$ *are continuous M-semigroups. The correspondence is given by the Lie-Trotter formulas* $(LT1)$ *and* $(LT2)$*:*

$$\rho_t^{(s)} = \lim_n \nu(s, t/n)^n, \quad \sigma_s^{(t)} = \lim_m \nu(s/m, t)^m$$

and conversely (MS)

$$\nu(s,t) = \lim_n \mathop{\bigstar}_{k=0}^{[nt]-1} T_{k/n}\left(\rho_{1/n}^{(s)}\right) = \lim_m \mathop{\bigstar}_{j=0}^{[ms]-1} T_{j/m}\left(\sigma_{1/m}^{(t)}\right)$$

⟦ Continuity follows since convergence in $(LT1)$ and $(LT2)$ is uniform on compact subsets. Alternatively, this follows by Lemma 2.2. To prove the M-semigroup property of e.g., $\left(\rho_t^{(s)}\right)_{s\geq 0}$, note that for $t \geq 0, s_1, s_2 \geq 0$ we have

$$\begin{aligned}\rho_t^{(s_1+s_2)} &= \lim_n \nu(s_1+s_2, t/n)^n = \lim_n \left(\nu(s_1, t/n) \star T_{s_1}\left(\nu(s_2, t/n)\right)\right)^n \\ &\overset{(C)}{=} \lim_n \nu(s_1, t/n)^n \star T_{s_1}\left(\lim_n \nu(s_2, t/n)^n\right) = \rho_t^{(s_1)} \star T_{s_1}\left(\rho_t^{(s_2)}\right)\end{aligned}$$

The other assertions are proved analogously. ⟧

Proposition 2.5. *Let, as in Proposition 2.4,* $(\nu(s,t))_{s,t\geq 0}$ *be a 2-M-semigroup. Define for* $s, t \geq 0$*:*

$$\nu_{s,t} := \lim_n \left(\sigma_s^{(t/n)}\right)^n = \lim_n \lim_m \left(\nu\left(s/m, t/n\right)\right)^{m\cdot n}$$

and $(2SG)$

$$\overline{\nu}_{s,t} := \lim_n \left(\rho_s^{(t/n)}\right)^n = \lim_m \lim_n \left(\nu\left(s/m, t/n\right)\right)^{n\cdot m}$$

Then we have:

$(s,t) \mapsto \nu_{s,t}$ *and* $(s,t) \mapsto \overline{\nu}_{s,t}$ *are continuous 2-semigroups (cf. Definition 2.3).*

⟦ Continuity follows by Lemma 2.2. We have to show the 2-semigroup property:

$s \mapsto \sigma_s^{(u)}$ is a continuous convolution semigroup for all u, therefore also $s \mapsto \nu_{s,t}$ is a continuous convolution semigroup for all fixed t. (Recall that we assumed that all convolution factors commute (C)).

For fixed $s \geq 0$, $t \mapsto \sigma_s^{(t)}$ is a M-semigroup. Hence by $(LT1)$ and $(LT2)$, $t \mapsto \nu_{s,t}$ is a continuous convolution semigroup. The other assertions are proved analogously. ⟧

Conversely, we obtain with a similar proof:

Proposition 2.6. *Let $(\nu_{s,t})_{s,t\geq 0}$ be a continuous 2-semigroup. Define*

$$\begin{aligned}\nu(s,t) &:= \lim_n \lim_m \mathop{\bigstar}_{k=0}^{[nt]-1} \mathop{\bigstar}_{j=0}^{[ms]-1} T_{\frac{k}{n}+\frac{j}{m}}\left(\nu_{1/m,1/n}\right) \\ &= \lim_n \lim_m \mathop{\bigstar}_{k=0}^{n-1} \mathop{\bigstar}_{j=0}^{m-1} T_{\frac{t}{n}k+\frac{s}{m}j}\left(\nu_{s/m,t/n}\right)\end{aligned}$$

for $s,t \geq 0$. Then $(\nu_{s,t})_{s,t\geq 0}$ is a continuous 2-M-semigroup.

⟦ Continuity follows by Lemma 2.2. Furthermore, for fixed $s \geq 0$, $t \mapsto \sigma_s^{(t)} = \lim_m \mathop{\bigstar}_{j=0}^{[ms]-1} T_{j/m}\left(\nu_{1/m,t}\right)$ is a M-semigroup, and for fixed $t \geq 0$, $s \mapsto \sigma_s^{(t)}$ is a continuous convolution semigroup. Moreover, $\left(\nu(s,t) = \lim_n \mathop{\bigstar}_{k=0}^{[nt]-1} T_{k/n}\left(\sigma_s^{(1/n)}\right)\right)_{s,t\geq 0}$ is a 2-M-semigroup. ⟧

Finally, for continuous 2-semigroups we obtain the following representation:

Proposition 2.7. *Let $(\mu_{s,t})_{s,t\geq 0}$ be a continuous 2-semigroup. Then there exists a uniquely determined continuous convolution semigroup $(\alpha_r)_{r\geq 0} \subseteq M^1(\mathbb{G})$ such that $\mu_{s,t} = \alpha_{s\cdot t}$, $s,t \geq 0$. In fact, $\alpha_r = \mu_{r,1} = \mu_{1,r}$, $r \geq 0$.*

Conversely, to any continuous convolution semigroup $(\alpha_r)_{r\geq 0}$ the mapping $(s,t) \mapsto \mu_{s,t} := \alpha_{s\cdot t}$ defines a continuous 2-semigroup.

⟦ For fixed $t \geq 0$, $s \mapsto \mu_{s,t}$ is a continuous convolution semigroup. Let $B(t) := \frac{d^+}{ds}\mu_{s,t}|_{s=0}$ denote the generating functional. Hence for all test functions $f \in \mathcal{D}(\mathbb{G})$, $\mathbb{R}_+ \ni t \mapsto \langle B(t), f\rangle$ is measurable. Furthermore, the semigroup property $\mu_{s,t_1+t_2} = \mu_{s,t_1} \star \mu_{s,t_2}$ yields $\langle B(t_1+t_2), f\rangle = \langle B(t_1), f\rangle + \langle B(t_2), f\rangle$. Whence $\langle B(t), f\rangle = t \cdot \langle B(1), f\rangle$ follows. This holds for any f, whence, with $B := B(1)$ we obtain: $B(t) = t \cdot B$.

Put $\beta_s^{(t)} := \mu_{s,t}$ and $\alpha_s := \beta_s^{(1)} = \mu_{s,1}$. The continuous convolution semigroup $\left(\beta_s^{(t)}\right)_{s\geq 0}$ is generated by $B(t) = t\cdot B$. Whence $\beta_s^{(t)} = \beta_{s\cdot t}^{(1)} = \alpha_{s\cdot t}$ follows. Hence, $\mu_{s,t} = \alpha_{s\cdot t}$ as asserted.

The converse is obvious. ⟧

3. Proof of Theorem 1.2

As afore announced, to simplify notations we shall prove Theorem 1.2 for $m = 2$ only. Let $\mathbb{T}$ be a contracting group of automorphisms, let $\mu \in L^{(2)}(\mathbb{T})$. For $s, t \geq 0$ we have $\mu = T_t(\mu) \star \nu^{(1)}(t) = T_s\left(T_t(\mu) \star \nu^{(1)}(t)\right) \star \nu(s) = T_{t+s}(\mu) \star T_s\left(\nu^{(1)}(t)\right) \star \nu^{(1)}(s)$. On the other hand, $\mu = T_{t+s}(\mu) \star \nu^{(1)}(s+t)$. By the injectivity assumption (I) and commutativity (CCF), we obtain $\nu^{(1)}(s+t) = \nu(s) \star T_s(\nu^{(1)}(t))$, i.e., the 1-cofactors form a M-semigroup. (Note that independently from the the injectivity assumption, 1-cofactors $\left(\nu^{(1)}(s)\right)_{s\geq 0}$ may be chosen in such a way. Cf. [7]).

Applying these considerations to the 1-cofactors $\nu^{(1)}(s)$ instead of μ we obtain for fixed s: $\nu^{(1)}(s) = T_t(\nu^{(1)}(s)) \star \nu^{(2)}(s,t), \forall t \geq 0$, and $t \mapsto \nu^{(2)}(s,t)$ is a continuous M-semigroup.

Claim: For fixed $t \geq 0$, $s \mapsto \nu^{(2)}(s,t)$ is a M-semigroup. Hence the 2-cofactors $\left(\nu^{(2)}(s,t)\right)_{s,t\geq 0}$ form a 2-M-semigroup (cf. Definition 2.3).

⟦ Let $s_1, s_2, r \geq 0$. The injectivity assumption (I) yields uniqueness of the cofactors, hence

$$\begin{aligned}\nu^{(1)}(s_1+s_2) &= T_r\left(\nu^{(1)}(s_1+s_2)\right) \star \nu^{(2)}(s_1+s_2, r)\\ &\overset{(C)}{=} \nu^{(2)}(s_1+s_2, r) \star T_r\left(\nu^{(1)}(s_1+s_2)\right)\end{aligned}$$

On the other hand, 1-cofactors being M-semigroups,

$$\begin{aligned}&\nu^{(1)}(s_1+s_2) = \nu^{(1)}(s_1) \star T_{s_1}\left(\nu^{(1)}(s_2)\right)\\ = \quad &\text{(by self-decomposability of 1-cofactors)}\\ \overset{\forall r\geq 0}{=} \quad &T_r\left(\nu^{(1)}(s_1)\right) \star \nu^{(2)}(s_1,r) \star T_{s_1}\left(T_r\left(\nu^{(1)}(s_2)\right) \star \nu^{(2)}(s_2,r)\right)\\ \overset{(C)}{=} \quad &\left(\nu^{(2)}(s_1,r) \star T_{s_1}\left(\nu^{(2)}(s_2,r)\right)\right) \star T_r\left(\nu^{(1)}(s_1) \star T_{s_1}\left(\nu^{(1)}(s_2)\right)\right)\\ = \quad &\nu^{(2)}(s_1,r) \star T_{s_1}\left(\nu^{(2)}(s_2,r)\right) \star T_r\left(\nu^{(1)}(s_1+s_2)\right)\end{aligned}$$

Again by the injectivity assumption (I) we may identify the cofactors to obtain $\nu^{(2)}(s_1+s_2, r) = \nu^{(2)}(s_1,r) \star T_{s_1}\left(\nu^{(2)}(s_2,r)\right), \ r, s_1, s_2 \geq 0$.

The claim is proved. ⟧

Applying the tools in Section 2 (Propositions 2.5 – 2.7) we obtain : There exists a uniquely determined continuous convolution semigroup $(\nu_r)_{r\geq 0} \doteq \left(\nu_r^{(2)}\right)_{r\geq 0}$ such that for all $r, s, t \geq 0, \ r = s \cdot t$

$$\nu_r = \nu_{s\cdot t} = \lim_N \lim_M \left(\nu^{(2)}\left(s/M, t/N\right)\right)^{N\cdot M}$$

and conversely (cf. Proposition 2.6)

$$\nu^{(2)}(s,t) = \lim_N \lim_M \mathop{\bigstar}_{k=0}^{[Nt]-1} \mathop{\bigstar}_{j=0}^{[Ms]-1} T_{\frac{k}{N}+\frac{j}{M}} \left(\nu_{\frac{1}{N}\cdot\frac{1}{M}}\right)$$

By assumption, $\mathbb{T}$ is contracting. Hence $\nu^{(2)}(s,t) \xrightarrow{t\to\infty} \nu^{(1)}(s), \ \forall s \geq 0$, furthermore, $\nu^{(1)}(s) \xrightarrow{s\to\infty} \mu$ and thus $\lim_{s\to\infty} \lim_{t\to\infty} \nu^{(2)}(s,t) = \mu$.

Note that in view of the 2-M-semigroup property this yields

$$\nu^{(2)}(M\cdot s, N\cdot t) = \mathop{\bigstar}_{k=0}^{[Nt]-1} \mathop{\bigstar}_{j=0}^{[Ms]-1} T_{kt+js}\left(\nu^{(2)}(s,t)\right) \xrightarrow{M,N\to\infty} \mu$$

These convolution products converge iff $\nu^{(2)}(s,t)$ has finite $\log_+^2(\cdot)$–moments, i.e., iff $\int_{\mathbb{G}} \left(\log_+(||x||)\right)^2 d\nu^{(2)}(s,t)(x) < \infty$. (For vector spaces see e.g., [4, 12, 21], for groups see [17]).

Claim: $\int_{\mathbb{G}} \left(\log_+(||x||)\right)^2 d\nu^{(2)}(s,t)(x)$ is finite iff the 2^{nd}-background driving Lévy process shares this property, i.e., $\int_{\mathbb{G}} \left(\log_+(||x||)\right)^2 d\nu_r^{(2)}(x)$ is finite for $r > 0$.

We sketch a proof in complete analogy to [5, 8] (for the case $m = 1$):

⟦ Let $\varphi : \mathbb{G} \to \mathbb{R}_+$ be a continuous sub-multiplicative function equivalent with $\log_+^2(||\cdot||)$ and let $\psi : \Gamma \to \mathbb{R}_+$ be an analogous function on the space-time group.

For fixed $t > 0$ let $\left(\lambda_s^{(t)} := \nu^{(2)}(s,t) \otimes \varepsilon_s\right)_{s\geq 0}$ be the space-time continuous convolution semigroup. Since $\lambda_s^{(t)} \in M_*^1(\Gamma)$, $\int_{\mathbb{G}} \varphi d\nu(s,t) < \infty$ iff $\int_\Gamma \psi d\lambda_s^{(t)} < \infty$. This is the case iff the Lévy measure $\gamma^{(t)}$ of $\left(\lambda_s^{(t)}\right)_{s\geq 0}$ fulfills $\int_{\complement U} \psi d\gamma^{(t)} < \infty$ for all neighbourhoods U of the unit in Γ.

Since $\lambda_t^{(s)} \in M_*^1(\Gamma)$ it follows easily that this is again equivalent with $\int_{\complement V} \varphi d\eta^{(t)} < \infty$ for all neighbourhoods V of the unit in $\mathbb{G}$, where $\eta^{(t)}$ denotes the Lévy measure of $B(t) := \frac{\partial^+}{\partial s}\nu^{(2)}(s,t)\,|_{s=0}$.

But $B(t)$ is the generating functional of the continuous convolution semigroup $\left(\sigma_s^{(t)}\right)_{s\geq 0}$. Hence the above integrals are finite iff $\int_{\mathbb{G}} \varphi d\sigma_s^{(t)} < \infty$, $s > 0$, hence iff $\int_{\mathbb{G}} \left(\log_+(||x||)\right)^2 d\sigma_s^{(t)} < \infty$ for all $t > 0$.

Repeating these arguments and replacing $t \mapsto \nu(s,t)$ by $t \mapsto \sigma_s^{(t)}$ we obtain finally:

$$\int \left(\log_+(||x||)\right)^2 d\nu^{(2)}(s,t) < \infty \quad \text{iff} \quad \int \left(\log_+(||x||)\right)^2 d\nu_{s\cdot t}^{(2)} < \infty$$

($\forall s,t > 0$), as asserted. ⟧

Theorem 1.2 is proved. □

Concluding Remark. At a first glance the foregoing construction appears asymmetric: The Lie Trotter formula is applied first to s then to t, consequently the 2^{nd} background process was constructed via the family of continuous convolution semigroups $\sigma_s^{(t)}$. Switching to the space-time semigroups we obtained differentiability of $\left(\nu^{(2)}(s,t)\right)_{s,t\in\mathbb{R}_+}$ (evaluated at test functions). In particular, for fixed $t \geq 0$ and for $s = 0$, $B(t) := \frac{\partial^+}{\partial s}\nu^{(2)}(s,t)\,|_{s=0}$, $t \geq 0$, is the generating functional of the continuous convolution semigroup$\left(\sigma_s^{(t)}\right)_{s\geq 0}$, i.e., $\frac{\partial^+}{\partial s}\sigma_s^{(t)}\,|_{s=0} = B(t)$. Adopting the notation $\left(\sigma_s^{(t)} =: \operatorname{Exp}(s\cdot B(t))\right)_{s\geq 0}$, for $t \geq 0$, this yields $\frac{\partial^+}{\partial t}\sigma_s^{(t)}|_{t=0} = \frac{\partial^+}{\partial t}\operatorname{Exp}(sB(t))|_{t=0} =: s\cdot C$ where C is the generating functional of the background driving process $(\nu_r)_{r\geq 0}$, i.e., $\nu_r = \operatorname{Exp}(r\cdot C)$. In other words, – explaining the afore mentioned asymmetry – we obtain

$$C = \frac{\partial^+}{\partial t}\operatorname{Exp}\left(\frac{\partial^+}{\partial s}\nu^{(2)}(s,t)\big|_{s=0}\right)|_{t=0}$$

Interchanging the role of s and t , $\left(\sigma_s^{(t)}\right)_{s,t\geq 0}$ and $\left(\rho_t^{(s)}\right)_{t,s\geq 0}$ and M and N, we obtain analogously $\frac{\partial^+}{\partial s}\rho_t^{(s)}|_{s=0} = t\cdot\overline{C}$, the generating functional of a Lévy process $(\overline{\nu}_r)_{r\geq 0}$, and moreover

$$\nu^{(2)}(s,t) = \lim_M \lim_N \mathop{\bigstar}_{j=0}^{[Ms]-1} \mathop{\bigstar}_{k=0}^{[Nt]-1} T_{\frac{k}{N}+\frac{j}{M}}\left(\overline{\nu}_{\frac{1}{N}\cdot\frac{1}{M}}\right)$$

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