

**Limit theorems for radial random
walks on homogeneous spaces with
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Preprint 2008-18

Oktober 2008

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Abstract

Let $X_p = G_p/K_p$ be homogeneous spaces (K_p a compact subgroup of a locally compact group G_p) with dimension parameter p such that the double coset spaces $G_p//K_p$ can be identified with some fixed space X . Then we obtain projections $T_p : X_p \rightarrow X$, and for a given probability measure $\nu \in M^1(X)$ there exist unique "radial", i.e. K_p -invariant measures $\nu_p \in M^1(X_p)$ with $T_p(\nu_p) = \nu$ and associated radial random walks $(S_n^p)_n$ on the X_p . We generally ask for limit theorems for $T_p(S_n^p)$ for $n, p \rightarrow \infty$. In particular we give a survey about existing results for the Euclidean spaces $X_p = \mathbb{R}^p$ (with $K_p = SO(p)$ and $X = [0, \infty[)$) and matrix spaces, and we derive a new central limit theorem for the hyperbolic spaces X_p of dimensions p over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

KEYWORDS: Radial random walks, hyperbolic spaces, Jacobi functions, central limit theorems, large dimensions, Bessel convolution, Jacobi convolution

1 Introduction

In the theory of classical limit theorems for random walks, one usually considers a (say time-homogeneous) Markov chain $(S_n)_{n \geq 0}$ on a fixed state space and is interested in limit theorems for functionals of S_n for $n \rightarrow \infty$. On the other hand, e.g., in random matrix theory, often different limit theorems are considered. Here one has a sequence $(X_p)_{p \geq 1}$ of state spaces, a sequence of probability measures $\nu_p \in M^1(X_p)$, and a sequence of mappings $T_p : X_p \rightarrow X$ into a fixed space X of characteristics. Here one may ask for limit theorems for the distributions $T_p(\nu_p) \in M^1(X)$ for $p \rightarrow \infty$. A famous typical example is given by the the largest eigenvalues of a Gaussian unitary ensemble where the $\nu_p \in M^1(X_p)$ are the distributions on the spaces X_p of all $p \times p$ complex Hermitian matrices whose entries are independent centered Gaussians with suitable normalizations, where the Tracy-Widom distribution appears as limit, see e.g. [18] for a survey.

It is an interesting general task to combine both problems: Let $(X_p)_{p \geq 1}$ be a sequence of state spaces with a sequence of measurable mappings $T_p : X_p \rightarrow X$ into a fixed space X of characteristics. For each dimension parameter p consider a time-homogeneous Markov-chain $(S_n^p)_{n \geq 0}$ on X_p such that the S_n^p are coupled in a suitable way. We now ask for limit theorems for $T_p(S_n^p)$ for $n, p \rightarrow \infty$ in a suitably coupled way.

This is particularly interesting for series of homogeneous spaces $X_p = G_p/K_p$ with compact subgroups K_p of locally compact groups G_p , where the double coset spaces $G_p//K_p$ may be identified with a space X independent of p . Then one has canonical mappings

$T_p : X_p = G_p/K_p \rightarrow G_p//K_p \simeq X$, and for a fixed probability measure $\nu \in M^1(X)$ one has a unique radial (i.e., K_p -invariant) measure $\nu_p \in M^1(X)$ with $T_p(\nu_p) = \nu$. In this way one can introduce time-homogeneous radial (i.e., K_p -invariant) random walks $(S_n^p)_{n \geq 0}$ associated with the ν_p by $S_0^p := K_p \in X_p$ and

$$P(S_{n+1}^p \in AK_p | S_n^p = xK_p) = \nu_p(x^{-1}AK_p) = \nu(K_px^{-1}AK_p) \quad (1.1)$$

for $n \geq 0, x \in G_p$, and $A \subset G_p$ a Borel set. In this setting, one now may ask for limit theorems for the X -valued random variables $T_p(S_n^p)$ for $n, p \rightarrow \infty$. It turns out that one obtains fundamentally different results for the compact and non-compact case.

In the compact case, the Kawada-Ito theorem yields under natural assumptions that for fixed p and $n \rightarrow \infty$, the distributions of S_n^p tend the uniform distribution ω_p on X_p , and it is an interesting task to study the total variation distances $d(p, n) := \|P_{S_n^p} - \omega_p\|$ for $n, p \rightarrow \infty$. It is well-known that in many cases so-called cutoff-phenomena appear (see [3] for a survey), i.e., one finds an increasing sequence $n(p)$ such that roughly for $p \rightarrow \infty$, $d(p, n) \rightarrow 0$ for $n > n(p)$ and $d(p, n) \rightarrow 1$ for $n < n(p)$. For particular examples, this cutoff phenomenon can be explained in a precise quantitative way via limit theorems for $T_p(S_{n(p)}^p)$ (for particular sequences $(n(p))_p$ and $T_p(\omega_p)$ and comparing the different limits. This was done, for instance, for random walks associated the Ehrenfest urn and Hamming schemes in [4], [9], [21], for random walks associated with the Johnson scheme in [10], and for the spheres $S^p \subset \mathbb{R}^{p+1}$ and projective spaces over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions \mathbb{H} in [20], [22], and [23]. We also mention that for finite graphs X_p such limit theorems are related with the spectral theoretic point of view in [11].

In this paper we focus on the noncompact case where only a few examples are discussed up to now. We consider the following three examples:

- (1) Let $X_p = \mathbb{R}^p$, $K_p = SO(p)$, and $G_p := SO(p) \times \mathbb{R}^p$ the motion group acting on \mathbb{R}^p . Then $X_p \simeq G_p/K_p$ and $G_p//K_p \simeq (\mathbb{R}^p)^{SO(p)} \simeq [0, \infty[$ in a canonical way. Let $\nu \in M^1([0, \infty[)$ be a fixed probability measure. Then for each dimension $p \in \mathbb{N}$ there is a unique radial (i.e., $SO(p)$ -invariant) probability measure $\nu_p \in M^1(\mathbb{R}^p)$ with ν as its radial part. Consider i.i.d. \mathbb{R}^p -valued random variables X_k^p , $k \in \mathbb{N}$, with law ν_p and the associated radial random walks $(S_n^p := \sum_{k=1}^n X_k^p)_{n \geq 0}$ on \mathbb{R}^p . One now might ask for limit theorems for the $[0, \infty[$ -valued random variables $\|S_n^p\|_2$ for $n, p \rightarrow \infty$. Some answers (laws of large numbers, central limit theorems) are given in [17], [19], [25] and will be reviewed in Section 2.
- (2) The examples in (1) admit a higher rank extension: Fix the field $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} and a dimension $q \geq 1$ (where case (1) appears for $q = 1$). Consider the vector spaces $X_p = M_{p,q}(\mathbb{F})$ of $p \times q$ -matrices over \mathbb{F} together with the mappings $\varphi_p : X_p \rightarrow \Pi_q$, $x \mapsto (x^*x)^{1/2}$ into the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$ -matrices over \mathbb{F} . We fix a probability measure $\nu \in M^1(\Pi_q)$. It can be checked by uniqueness of polar decomposition that for each p there is a unique U_p -invariant distribution $\nu_p \in M^1(X_p)$ with $\varphi_p(\nu_p) = \nu$. We now again consider i.i.d. \mathbb{R}^p -valued random variables X_k^p , $k \in \mathbb{N}$, with law ν_p as well as the associated radial random walks $(S_n^p := \sum_{k=1}^n X_k^p)_{n \geq 0}$ on \mathbb{R}^p , and one might ask for limit theorems for the Π_q -valued random variables $\varphi_p(S_n^p)$ for $n, p \rightarrow \infty$. Similar answers as in (1) are given in [17], [25], and will be reviewed in Section 3 below.
- (3) Let again $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or \mathbb{H} . For $p \geq 2$ consider the hyperbolic space $X_p = G_p/K_p$ of

dimension p over \mathbb{F} with

$$\begin{aligned} \mathbb{F} = \mathbb{R} : & \quad G_p = SO_o(1, p), \quad K_p = SO(p) \\ \mathbb{F} = \mathbb{C} : & \quad G_p = SU(1, p), \quad K_p = S(U(1) \times U(p)) \\ \mathbb{F} = \mathbb{H} : & \quad G_p = Sp(1, p), \quad K_p = Sp(1) \times Sp(p). \end{aligned}$$

For these noncompact symmetric spaces of rank 1, the hyperbolic distances lead to canonical homeomorphisms between $\varphi : G_p/K_p$ and $[0, \infty[$. If we fix a probability measure $\nu \in M^1([0, \infty[)$ as well as p , we obtain a unique associated radial distribution $\nu_p \in M^1(X_p)$ as well as an associated radial random walk $(S_n^p)_{n \geq 0}$ on X_p as described above. In Section 4 of this paper we shall derive a central limit theorem for $n \rightarrow \infty$ and $n^2/p \rightarrow 0$.

It is clear that there exist many further interesting examples.

We briefly turn to the proofs. All proofs (in the compact and noncompact case) are based on asymptotic results for the spherical functions on the spaces X_p or asymptotic results on the double coset convolutions on $G_p/K_p = X$ for $p \rightarrow \infty$ (or a combination of both methods). For instance, in case (1) above, the spherical functions are Bessel functions, and the double coset convolutions are the well-known Bessel convolutions (see e.g. [14]). Moreover, in case (2), Bessel functions of matrix argument and the associated Bessel convolutions on matrix cones appear (see [6], [8], [16]), and in case (3), Jacobi functions and Jacobi convolutions on $[0, \infty[$ appear (see the survey [15]). Finally, in the compact case, spheres and projective spaces lead to Jacobi polynomials and Jacobi convolutions on $[-1, 1]$, and the finite examples associated with Hamming schemes or Johnson schemes lead to Krawtchouk and Meixner polynomials respectively.

All examples have in common that the associated spherical are special functions depending on a discrete series of dimension parameters, which admit a product formula associated with the double coset convolution, and that these product formulas and the associated convolution may be extended to a continuous family of parameters. In this way one obtains continuous families of commutative hypergroup structures $(X; *_{\rho})$ with these special functions are characters, which generalize discrete families of double coset hypergroups; see [2] and [13] for the theory and notion of hypergroups. The hypergroup convolution $*_{\rho}$ on X allows to introduce the notion of random walks on $(X, *)$ by saying that a (time-homogeneous) Markov $(Z_n^{\rho})_{n \geq 0}$ is a random walk on $(X; *_{\rho})$ with law $\nu \in M^1(X)$ if

$$P(Z_{n+1}^{\rho} \in A | Z_n^{\rho} = x) = (\delta_x * \nu)(A) \tag{1.2}$$

for all $n \geq 0$, $x \in X$, and Borel sets $A \subset X$. It can be checked easily that for a parameter p belonging to the homogeneous space $X_p = G_p/K_p$, and for a radial random walk $(S_n^p)_{n \geq 0}$ on X_p as discussed in Eq. (1.1), the projection $(T_p(S_n^p))_{n \geq 0}$ becomes a random walk on $(X, *_{\rho})$ as in ((1.2)). In this way, many limit theorems for the $T_p(S_n^p)$ can be embedded into limit theorems for the random variables Z_n^{ρ} on the hypergroups for $n, p \rightarrow \infty$ without additional effort. We thus state most results in this context.

2 Radial limit theorems on \mathbb{R}^p for $p \rightarrow \infty$

Let $\nu \in M^1([0, \infty[)$ be a fixed probability measure. Then for each dimension $p \in \mathbb{N}$ there is a unique radial probability measure $\nu_p \in M^1(\mathbb{R}^p)$ with ν as its radial part, i.e., ν is the

image of ν_p under the norm mapping $\varphi_p(x) := \|x\|_2$. For each $p \in \mathbb{N}$ consider i.i.d. \mathbb{R}^p -valued random variables X_k^p , $k \in \mathbb{N}$, with law ν_p as well as the associated radial random walks

$$(S_n^p := \sum_{k=1}^n X_k^p)_{n \geq 0}$$

on \mathbb{R}^p . According to the philosophy of the introduction, we are interested in limit theorems for the $[0, \infty[$ -valued random variables $\|S_n^p\|_2$ for $n, p \rightarrow \infty$. In [19] we proved that for all sequences $p_n \rightarrow \infty$,

$$\|S_n^{p_n}\|_2^2/n \rightarrow \sigma^2 := \sigma^2(\nu) := \int_0^\infty x^2 d\nu(x)$$

in law under the condition $\sigma^2 < \infty$. Moreover, in [17] an associated strong law and a large deviation principle were derived under the condition that p_n grows fast enough. As the precise conditions are a little bit complicated, we here omit details and turn to central limit theorems. It turns out that two different results appear for $p \gg n$ and $n \gg p$. For $n \gg p$, the multidimensional Berry-Esseen inequality on \mathbb{R}^p of Bentkus [1] with an explicit constant depending on p leads to the following result (see [25]):

2.1 Theorem. *Assume that $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ admits a finite third moment $m_3(\nu) := \int_0^\infty x^3 d\nu(x) < \infty$, and that $\lim_{n \rightarrow \infty} n/p_n^3 = \infty$. Then*

$$\frac{\sqrt{p_n}}{n\sigma^2\sqrt{2}}(\|S_n^{p_n}\|_2^2 - n\sigma^2)$$

tends in distribution for $n \rightarrow \infty$ to the standard normal distribution $N(0, 1)$.

For $p \gg n$, the following result was established in [25]:

2.2 Theorem. *Assume that $\nu \in M^1([0, \infty[)$ admits a finite fourth moment $m_4(\nu) := \int_0^\infty x^4 d\nu(x) < \infty$, and that $\lim_{n \rightarrow \infty} n^2/p_n \rightarrow 0$. Then*

$$\frac{\|S_n^{p_n}\|_2^2 - n\sigma^2}{\sqrt{n}}$$

tends in distribution for $n \rightarrow \infty$ to $N(0, m_4(\nu) - \sigma^4)$.

The proof of this theorem in [25] is based on the observation that for fixed p , $(\|S_n^p\|_2)_{n \geq 0}$ is a random walk on $[0, \infty[$ associated with the Bessel-Kingman hypergroup structure $([0, \infty[, *_\alpha)$ of index $\alpha := p/2 - 1$, where the convolution is generated by

$$\delta_x *_\alpha \delta_y(f) := \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1/2)\sqrt{\pi}} \int_{-1}^1 f(\sqrt{x^2 + y^2 + 2xyt}) \cdot (1 - t^2)^{\alpha - 1/2} dt$$

for $f \in C_b([0, \infty[)$, $\alpha > -1/2$, $x, y \geq 0$, and that for $\alpha \rightarrow \infty$,

$$\delta_x *_\alpha \delta_y(f) \rightarrow f(\sqrt{x^2 + y^2}).$$

this limit is the key for the proof of Theorem 2.2 and works for all random walks on the Bessel-Kingman hypergroups $([0, \infty[, *_\alpha)$ and not just for the group parameters $\alpha := p/2 - 1$.

3 Radial limit theorems on the matrix spaces $M_{p,q}$ for $p \rightarrow \infty$

For fixed dimensions $p, q \in \mathbb{N}$ let $M_{p,q} = M_{p,q}(\mathbb{F})$ be the space of $p \times q$ -matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions \mathbb{H} with real dimension $d = 1, 2$ or 4 respectively. This is a Euclidean vector space of real dimension dpq with scalar product $\langle x, y \rangle = \Re \text{tr}(x^*y)$ where $x^* := \bar{x}^t$, $\Re t := \frac{1}{2}(t + \bar{t})$ is the real part of $t \in \mathbb{F}$, and tr is the trace in $M_q := M_{q,q}$. A measure on $M_{p,q}$ is called radial if it is invariant under the action of the unitary group $U_p = U_p(\mathbb{F})$ by left multiplication, $U_p \times M_{p,q} \rightarrow M_{p,q}$, $(u, x) \mapsto ux$. This action is orthogonal w.r.t. the scalar product above, and, by uniqueness of the polar decomposition, two matrices $x, y \in M_{p,q}$ belong to the same U_p -orbit if and only if $x^*x = y^*y$. Thus the space $M_{p,q}^{U_p}$ of U_p -orbits in $M_{p,q}$ is naturally parameterized by the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$ -matrices over \mathbb{F} . We identify $M_{p,q}^{U_p}$ with Π_q via $U_q x \simeq (x^*x)^{1/2}$, i.e., the canonical projection $M_{p,q} \rightarrow M_{p,q}^{U_p}$ will be realized as the mapping

$$\varphi_p : M_{p,q} \rightarrow \Pi_q, \quad x \mapsto (x^*x)^{1/2}.$$

The square root is used here in order to ensure for $q = 1$ and $\mathbb{F} = \mathbb{R}$ that the setting above with $\Pi_1 = [0, \infty[$ and $\varphi_p(x) = \|x\|$ appears. By taking images of measures, φ_p induces a Banach space isomorphism between the space $M_b^{U_q}(M_{p,q})$ of all bounded radial Borel measures on $M_{p,q}$ and the space $M_b(\Pi_q)$ of bounded Borel measures on the cone Π_q . In particular, for each $\nu \in M^1(\Pi_q)$ there is a unique radial probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$.

As in the case $q = 1$, we now consider for each $p \in \mathbb{N}$ i.i.d. $M_{p,q}$ -valued random variables X_k^p , $k \in \mathbb{N}$, with law ν_p and the associated radial random walks $(S_n^p := \sum_{k=1}^n X_k^p)_{n \geq 0}$.

3.1 Definition. We say that $\nu \in M^1(\Pi_q)$ admits a k -th moment ($k \in \mathbb{N}$) if

$$m_k(\nu) := \int_{\Pi_q} \|s\|^k d\nu(s) < \infty$$

where $\|s\| = (\text{tr} s^2)^{1/2}$ is the Hilbert-Schmidt norm. If the second moment exists, the second moment of ν is defined as the matrix-valued integral

$$\sigma^2 := \sigma^2(\nu) := \int_{\Pi_q} s^2 d\nu(s) \in \Pi_q.$$

With these notions, the following generalizations of Theorems 2.1 and 2.2 hold; see [25]:

3.2 Theorem. Assume that $m_4(\nu) < \infty$. Moreover, let $\lim_{n \rightarrow \infty} n/p_n^4 = \infty$. Then

$$\frac{\sqrt{p_n}}{n} (\varphi_{p_n}(S_n^{p_n})^2 - n\sigma^2)$$

tends in distribution to some normal distribution $N(0, T^2)$ on the vector space H_q of hermitian $q \times q$ -matrices over \mathbb{F} (with a covariance matrix $T^2 = T^2(\sigma^2)$ described in the proof below for $\mathbb{F} = \mathbb{R}$).

3.3 Theorem. Assume that $m_4(\nu) < \infty$ and $\lim_{n \rightarrow \infty} n^2/p_n = 0$. Then

$$\frac{1}{\sqrt{n}} (\varphi_{p_n}(S_n^{p_n})^2 - n\sigma^2)$$

tends in distribution to the normal distribution $N(0, \Sigma^2)$ on H_q where Σ^2 is the covariance matrix of $\varphi_{p_n}(X_1^{p_n})$ (which is independent of p_n).

Again, the proof of Theorem 3.2 is based on a Berry-Esseen inequality on the Eukclidean space $M_{p,q} \simeq \mathbb{R}^{dpq}$ of [1], while the proof of Theorem 3.3 relies on an explicit formula for the Bessel convolution on Π_q due to Rösler [16]. With this convolution, the result follows Similar as for the case $q = 1$ described in Section 2.

4 A central limit theorem for hyperbolic spaces and Jacobi convolutions on $[0, \infty[$

Let $k \geq 2$, and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or \mathbb{H} . The hyperbolic space of dimension k over \mathbb{F} may be regarded as the symmetric space G/K with

$$\begin{aligned} \mathbb{F} = \mathbb{R} : & \quad G = SO_o(1, k), \quad K = SO(k) \\ \mathbb{F} = \mathbb{C} : & \quad G = SU(1, k), \quad K = S(U(1) \times U(k)) \\ \mathbb{F} = \mathbb{H} : & \quad G = Sp(1, k), \quad K = Sp(1) \times Sp(k). \end{aligned}$$

In all cases, the double coset $G//K$ can be regarded as the interval $[0, \infty[$ by identifying $t \geq 0$ with the double coset

$$K a_t K \quad \text{with} \quad a_t = \begin{pmatrix} \text{ch } t & 0 & \dots & 0 & \text{sh } t \\ 0 & & & & 0 \\ \vdots & & I_{k-1} & & \vdots \\ 0 & & & & 0 \\ \text{sh } t & 0 & \dots & 0 & \text{ch } t \end{pmatrix};$$

see [5]. Using this homeomorphism $\varphi : G//K \rightarrow [0, \infty[$, the hyperbolic distance on G/K is given by $d(xK, yK) = \varphi(Ky^{-1}xK)$. We derive the following central limit theorem for G -invariant time-homogeneous random walks $(S_n^k)_{n \geq 0}$ on $G//K$.

4.1 Theorem. *Let $(k_n)_{n \geq 1} \subset \mathbb{N}$ be an increasing sequence of dimensions with $\lim_{n \rightarrow \infty} n/\sqrt{k_n} = 0$, and let $\nu \in M^1([0, \infty[)$ with a finite second moment $\int_0^\infty x^2 d\nu(x) < \infty$. For each dimension $k \geq 2$ consider a G -invariant time-homogeneous random walk $(S_n^k)_{n \geq 0}$ on G/K such that for all n, k , the random variables $d(S_n^{k+1}, S_n^k)$ have distribution ν . Then, $\rho := \int_0^\infty \ln(\text{ch } x) d\nu(x) < \infty$ and $\sigma^2 := \int_0^\infty (\ln(\text{ch } x))^2 d\nu(x) < \infty$ exist, and*

$$\frac{d(S_n^{k_n}, S_0^{k_n}) - n\rho}{\sqrt{n}}$$

tends in distribution for $n \rightarrow \infty$ to the normal distribution $N(0, \sigma^2 - \rho^2)$.

We shall derive this CLT by considering the distributions of the random walks $(d(S_n^k, S_0^k))_{n \geq 0}$ on the spaces $G//K \simeq [0, \infty[$ equipped with the associated double coset convolutions. These convolutions may be regarded as special cases of Jacobi convolutions on $[0, \infty[$ depending on indices $\alpha \geq \beta \geq -1/2$ which were investigated mainly by Flensted-Jensen and Koornwinder. In the following we refer to the survey [15] on this subject. For $\alpha > \beta > -1/2$, this convolution is given by

$$\begin{aligned} \delta_r *_{(\alpha, \beta)} \delta_s(f) := c_{(\alpha, \beta)} \int_0^1 \int_0^\pi f(\text{arch} |\text{ch } r \cdot \text{ch } s + t e^{i\varphi} \text{sh } r \cdot \text{sh } s|) \\ \cdot (1 - t^2)^{\alpha - \beta - 1} (t \sin \varphi)^{2\beta} t dt d\varphi \end{aligned} \quad (4.1)$$

with

$$c_{(\alpha,\beta)} = \frac{2\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha-\beta)\Gamma(\beta+1/2)} = \left(\int_0^1 \int_0^\pi (1-t^2)^{\alpha-\beta-1} (t \sin \varphi)^{2\beta} t \, dt \, d\varphi \right)^{-1}.$$

For $\alpha > \beta = -1/2$, the degenerate formula

$$\delta_r *_{(\alpha,-1/2)} \delta_s(f) := c_{(\alpha,-1/2)} \int_{-1}^1 f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + t \cdot \operatorname{sh} r \cdot \operatorname{sh} s|) \cdot (1-t^2)^{\alpha-1/2} \, dt$$

appears with

$$c_{(\alpha,-1/2)} = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+1/2)}.$$

The case $\alpha = \beta$ can be transformed into the case $\beta = -1/2$ by some quadratic transformation (see Section 5.3 of [15]) and will not be considered here separately.

With this notion, the hyperbolic spaces G/K and their associated double coset convolutions are related with the indices α, β by

$$\alpha = dk/2 - 1, \quad \beta = d/2 - 1, \quad (4.2)$$

where $d = 1, 2, 4$ is the dimension of \mathbb{F} over \mathbb{R} , and k the dimension of G/K as before.

Now fix $\alpha \geq \beta \geq -1/2$. It is well-known that the Jacobi convolution of point measures above can be extended uniquely in a weakly continuous, bilinear way to a probability-preserving convolution $*_{(\alpha,\beta)}$ on $M_b([0, \infty[)$, and that one obtains the so-called Jacobi-type hypergroup on $[0, \infty[$ in this way.

Now fix a probability measure $\nu \in M^1([0, \infty[)$, and consider a time-homogeneous random walk $(S_n^{(\alpha,\beta)})_{n \geq 0}$ on $[0, \infty[$ with law ν of index (α, β) , i.e., a time-homogeneous Markov process starting at the identity 0 with transition probability

$$P(S_{n+1}^{(\alpha,\beta)} \in A | S_n^{(\alpha,\beta)} = x) = (\delta_x *_{(\alpha,\beta)} \nu)(A) \quad (x \geq 0, A \subset [0, \infty[\text{ a Borel set})$$

as described in the introduction. Theorem 4.1 above then can be regarded as a special case of the following central limit theorem:

4.2 Theorem. *Let $\beta \geq -1/2$, and let $(\alpha_n)_{n \in \mathbb{N}} \subset [\beta, \infty[$ be an increasing sequence of indices with $\lim_{n \rightarrow \infty} n/\sqrt{\alpha_n} = 0$. Let $\nu \in M^1([0, \infty[)$ with a finite second moment $\int_0^\infty x^2 \, d\nu(x) < \infty$. Then, $\rho := \int_0^\infty \ln(\operatorname{ch} x) \, d\nu(x) < \infty$ and $\sigma^2 := \int_0^\infty (\ln(\operatorname{ch} x))^2 \, d\nu(x) < \infty$ exist, and for the random walks $(S_n^{(\alpha_n,\beta)})_{n \geq 0}$ on $[0, \infty[$,*

$$\frac{S_n^{(\alpha_n,\beta)} - n\rho}{\sqrt{n}}$$

tends in distribution for $n \rightarrow \infty$ to $N(0, \sigma^2 - \rho^2)$.

The proof is based on the observation that for $\alpha \rightarrow \infty$ and $f \in C([0, \infty[)$,

$$\delta_r *_{(\alpha,\beta)} \delta_s(f) \quad \rightarrow \quad f(\operatorname{arch}(\operatorname{ch} r \cdot \operatorname{ch} s)).$$

The right hand side motivates to study a limit semigroup structure on $[0, \infty[$. We next introduce this limit structure and make the convergence above more precise. This then finally will lead to the proof of Theorem 4.2.

4.3 Definition. (1) We equip the set $[0, \infty[$ with the commutative semigroup operation

$$s \bullet r := \operatorname{arch}(\operatorname{ch} r \cdot \operatorname{ch} s) \quad (r, s \geq 0),$$

and extend this operation in a linear, weakly continuous way to the space of all bounded Borel measures by

$$\nu_1 \bullet \nu_2(f) := \int_0^\infty \int_0^\infty f(r \bullet s) d\nu_1(r) d\nu_2(s).$$

We here notice that the mapping $Q(x) := \ln \operatorname{ch} x$, establishes a semigroup isomorphism from $([0, \infty[, \bullet)$ onto the semigroup $([0, \infty[, +)$.

(2) A function $f \in C([0, \infty[)$ is called arche-Lipschitz continuous with constant L , if for all $x, y \geq 0$,

$$|f(\operatorname{arch}(e^x)) - f(\operatorname{arch}(e^y))| \leq L|x - y|.$$

This notion is motivated by the fact, that for a function $f \in C_0([0, \infty[)$, which is Lipschitz continuous in the usual sense with constant L , the function $\tilde{f}(x) := f(Q(x))$ is arche-Lipschitz with the same L .

4.4 Lemma. *Let $f \in C([0, \infty[)$ be arche-Lipschitz continuous with constant L . Then, for all $r, s \geq 0$, $t \in [0, 1]$ and $\varphi \in [0, \pi]$,*

$$\begin{aligned} & \left| f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + te^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s|) - f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s|) \right| \\ & \leq 2L \frac{t}{1 - t^2}. \end{aligned} \quad (4.3)$$

Proof. By Definition 4.3(2), the left hand side of Eq. (4.3) is bounded above by

$$\begin{aligned} R &:= L \cdot \left| \ln(|\operatorname{ch} r \cdot \operatorname{ch} s + te^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s|) - \ln(\operatorname{ch} r \cdot \operatorname{ch} s) \right| \\ &= L \cdot \left| \ln(|1 + x|) \right| \end{aligned} \quad (4.4)$$

with

$$x := te^{i\varphi} \frac{\operatorname{sh} r \cdot \operatorname{sh} s}{\operatorname{ch} r \cdot \operatorname{ch} s}$$

which satisfies $|x| \leq t$. Moreover, for $|1 + x| \geq 1$, we have

$$|\ln |1 + x|| \leq |x| \leq t. \quad (4.5)$$

On the other hand, for $|1 + x| \leq 1$, we first observe that by the mean value theorem,

$$|\ln |1 + x|| \leq |x|/|1 + x|.$$

Therefore, using $\operatorname{ch} r \cdot \operatorname{ch} s - \operatorname{sh} r \cdot \operatorname{sh} s = \operatorname{ch}(r - s)$, we obtain

$$|\operatorname{ch} r \cdot \operatorname{ch} s + te^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s| \geq \operatorname{ch} r \cdot \operatorname{ch} s - t \cdot \operatorname{sh} r \cdot \operatorname{sh} s = 1 + (1 - t)\operatorname{sh} r \cdot \operatorname{sh} s,$$

and thus

$$|\ln |1 + x|| \leq t \cdot \frac{\operatorname{sh} r \cdot \operatorname{sh} s}{1 + (1 - t)\operatorname{sh} r \cdot \operatorname{sh} s} \leq t/(1 - t) \leq 2t/(1 - t^2). \quad (4.6)$$

Combining (4.4), (4.5), and (4.6), we obtain the estimate in the lemma. \square

4.5 Lemma. *There is a constant $C = C(\beta)$ such that for all $r, s \in [0, \infty[$, all sufficiently large α , and all arche-Lipschitz continuous functions f on $[0, \infty[$ with constant L ,*

$$|\delta_r *_{(\alpha, \beta)} \delta_s(f) - \delta_r \bullet \delta_s(f)| \leq CL/\sqrt{\alpha}.$$

Proof. Assume that $\beta > -1/2$ holds. We use the definition of the Jacobi convolution above together with Lemma 4.4 and obtain

$$\begin{aligned} & |\delta_r *_{(\alpha, \beta)} \delta_s(f) - \delta_r \bullet \delta_s(f)| \\ & \leq c_{(\alpha, \beta)} \int_0^1 \int_0^\pi \left| f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + t e^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s|) - f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s|) \right| \\ & \quad \cdot (1-t^2)^{\alpha-\beta-1} (t \sin \varphi)^{2\beta} t \, dt \, d\varphi \\ & \leq 2Lc_{(\alpha, \beta)} \int_0^1 \int_0^\pi (1-t^2)^{\alpha-\beta-2} t^{2\beta+2} \sin^{2\beta} \varphi \, dt \, d\varphi \\ & \leq C_1 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)} \int_0^1 (1-t^2)^{\alpha-\beta-2} t^{2\beta+2} \, dt \end{aligned}$$

for some constant C_1 . If we now use the beta integral

$$\int_0^1 (1-t^2)^{\alpha-\beta-2} t^{2\beta+2} \, dt = \frac{\Gamma(\alpha-\beta-1)\Gamma(\beta+3/2)}{2\Gamma(\alpha+1/2)}.$$

as well as $\Gamma(\alpha+1)/\Gamma(\alpha+3/2) = O(\alpha^{-1/2})$ for $\alpha \rightarrow \infty$, the statement of the lemma follows for $\beta > -1/2$. The proof for $\beta = -1/2$ is similar. \square

4.6 Lemma. *There is a constant $C = C(\beta)$ such that for all arche-Lipschitz continuous functions f on $[0, \infty[$ with constant L , all $\nu_1, \nu_2 \in M^1([0, \infty[)$, and all sufficiently large α ,*

$$|\nu_1 *_{(\alpha, \beta)} \nu_2(f) - \nu_1 \bullet \nu_2(f)| \leq CL/\sqrt{\alpha}.$$

Proof. By Lemma 4.5,

$$\begin{aligned} & |\nu_1 *_{(\alpha, \beta)} \nu_2(f) - \nu_1 \bullet \nu_2(f)| \\ & \leq \int_0^\infty \int_0^\infty |\delta_r *_{(\alpha, \beta)} \delta_s(f) - \delta_r \bullet \delta_s(f)| \, d\nu_1(r) \, d\nu_2(s) \\ & \leq \frac{CL}{\sqrt{\alpha}}. \end{aligned}$$

\square

4.7 Lemma. *There is a constant $C = C(\beta)$ such that for all arche-Lipschitz continuous functions f on $[0, \infty[$ with constant L , all $\nu_1, \nu_2, \nu_3 \in M^1(\Pi_q)$ and all sufficiently large α ,*

$$|(\nu_1 *_{(\alpha, \beta)} \nu_2) \bullet \nu_3(f) - (\nu_1 \bullet \nu_2) \bullet \nu_3(f)| \leq CL/\sqrt{\alpha}.$$

Proof. For $y \geq 0$, consider the function $f_y(x) := f(x \bullet y)$ on $[0, \infty[$. By the definition of \bullet and our definition of arche-Lipschitz continuity, these translated functions f_y are also arche-Lipschitz continuous with the same constant L . Therefore, by Lemma 4.6,

$$\begin{aligned} & |(\nu_1 *_{(\alpha, \beta)} \nu_2) \bullet \nu_3(f) - (\nu_1 \bullet \nu_2) \bullet \nu_3(f)| \\ & \leq \int_0^\infty \left| \int_0^\infty f_y(x) \, d(\nu_1 *_{(\alpha, \beta)} \nu_2)(x) - \int_0^\infty f_y(x) \, d(\nu_1 \bullet \nu_2)(x) \right| \, d\nu_3(y) \\ & \leq CL/\sqrt{\alpha}. \end{aligned}$$

\square

For $\nu \in M^1([0, \infty[)$ and $n \in \mathbb{N}$, we denote the n -fold convolution powers of ν w.r.t. the convolutions $*_{(\alpha, \beta)}$ and \bullet by $\nu^{(n, *_{(\alpha, \beta)})}$ and $\nu^{(n, \bullet)}$ respectively.

4.8 Proposition. *Let $\nu \in M^1([0, \infty[)$. Then there is a constant $C = C(\beta)$ such that for all arche-Lipschitz continuous functions f on $[0, \infty[$ with constant L , all $n \in \mathbb{N}$, and all sufficiently large α ,*

$$|\nu^{(n, *_{(\alpha, \beta)})}(f) - \nu^{(n, \bullet)}(f)| \leq CL \frac{n}{\sqrt{\alpha}}.$$

Proof. We observe that

$$\begin{aligned} & |\nu^{(n, *_{(\alpha, \beta)})}(f) - \nu^{(n, \bullet)}(f)| \\ & \leq |\nu^{(n, *_{(\alpha, \beta)})}(f) - \nu^{(n-1, *_{(\alpha, \beta)})} \bullet \nu(f)| \\ & \quad + |(\nu^{(n-2, *_{(\alpha, \beta)})} *_{(\alpha, \beta)} \nu) \bullet \nu(f) - \nu^{(n-2, *_{(\alpha, \beta)})} \bullet \nu \bullet \nu(f)| \\ & \quad + \dots + |(\nu *_{(\alpha, \beta)} \nu) \bullet \nu^{(n-2, \bullet)}(f) - \nu^{(n, \bullet)}(f)| \end{aligned}$$

where by Lemma 4.7, each summand of the right hand side is bounded by $CL/\sqrt{\alpha}$. This implies the proposition. \square

Proof of Theorem 4.2. Consider the isomorphism $Q(x) := \ln \operatorname{ch} x$ from $([0, \infty[, \bullet)$ onto $([0, \infty[, +)$. Let $f \in C_0(\mathbb{R})$ be Lipschitz-continuous in the usual sense on \mathbb{R} with constant L . Then, for $n \in \mathbb{N}$, the function

$$f_n(x) := f\left(\frac{Q(x) - n\rho}{\sqrt{n}}\right)$$

on $[0, \infty[$ satisfies

$$|f_n(\operatorname{arch} e^x) - f_n(\operatorname{arch} e^y)| = \left| f\left(\frac{x - n\rho}{\sqrt{n}}\right) - f\left(\frac{y - n\rho}{\sqrt{n}}\right) \right| \leq \frac{L|x - y|}{\sqrt{n}},$$

i.e., f_n is arche-Lipschitz with constant $n^{-1/2}L$. Therefore, by Proposition 4.8 and the assumptions of the theorem,

$$\left| \int_0^\infty f_n d\nu^{(n, *_{(\alpha_n, \beta)})} - \int_0^\infty f_n d\nu^{(n, \bullet)} \right| \leq CL \frac{n}{\sqrt{\alpha_n}} \rightarrow 0 \quad (4.7)$$

for $n \rightarrow \infty$ with

$$\int_0^\infty f_n d\nu^{(n, *_{(\alpha_n, \beta)})} = \int_0^\infty f_n dP_{S_n^{(\alpha_n, \beta)}} = \int_{\mathbb{R}} f dP_{(Q(S_n^{(\alpha_n, \beta)}) - n\rho)/\sqrt{n}}, \quad (4.8)$$

where P_X denotes the law of a random variable X . Moreover, as Q is a isomorphism from $([0, \infty[, \bullet)$ onto $([0, \infty[, +)$, and denoting the classical convolution of measures on $([0, \infty[, +)$ by $*$, we see that

$$\begin{aligned} \int_0^\infty f_n d\nu^{(n, \bullet)} &= \int_0^\infty f\left(\frac{Q(x) - n\rho}{\sqrt{n}}\right) d\nu^{(n, \bullet)}(x) \\ &= \int_0^\infty f\left(\frac{x - n\rho}{\sqrt{n}}\right) dQ(\nu^{(n, \bullet)})(x) \\ &= \int_0^\infty f\left(\frac{x - n\rho}{\sqrt{n}}\right) d(Q(\nu)^{(n, *)})(x) \end{aligned} \quad (4.9)$$

where the latter tends by the classical central limit theorem on \mathbb{R} to $\int_{\mathbb{R}} f dN(0, \sigma^2 - \rho^2)$. Taking (4.7) and (4.8) into account, we obtain that

$$\int_{\mathbb{R}} f dP_{(Q(S_n^{(\alpha_n, \beta)})) - n\rho / \sqrt{n}} \rightarrow \int_{H_q} f dN(0, \sigma^2 - \rho^2) \quad (4.10)$$

for $n \rightarrow \infty$ and all Lipschitz-continuous functions $f \in C_0(\mathbb{R})$. As the space of Lipschitz continuous functions in $C_0(\mathbb{R})$ is $\|\cdot\|_{\infty}$ -dense in $C_0(\mathbb{R})$, a simple ε -argument together with the triangle inequality ensures that Eq. (4.10) holds in fact for all $f \in C_0(\mathbb{R})$. Therefore,

$$\frac{\ln(\text{ch}(S_n^{(\alpha_n, \beta)})) - n\rho}{\sqrt{n}} \quad (4.11)$$

tends in distribution to $(N(0, \sigma^2 - \rho^2))$. This in particular shows that $\ln(\text{ch}(S_n^{(\alpha_n, \beta)}))/n \rightarrow \rho^2$ and thus $e^{-2S_n^{(\alpha_n, \beta)}} \rightarrow 0$ in probability. Using

$$x - \ln 2 \leq \ln(\text{ch } x) \leq x + \ln(1 + e^{-2x}) \leq x + e^{-2x}$$

and thus

$$\ln \text{ch } S_n^{(\alpha_n, \beta)} - e^{-2S_n^{(\alpha_n, \beta)}} \leq S_n^{(\alpha_n, \beta)} \leq \ln \text{ch } S_n^{(\alpha_n, \beta)} + \ln 2,$$

we obtain the theorem from Eq. (4.11). \square

It is possible to derive variants of Theorem 4.2 for $\alpha_n, \beta_n \rightarrow \infty$ in certain coupled ways. Clearly, these variants usually do not have a geometric meaning anymore. Nevertheless we discuss the case $\beta_n \rightarrow \infty$ and $\alpha_n = \beta_n + c$ for some constant $c \geq 0$. For $c = 0$, this can be transformed by some quadratic transformation (see p.44 of [15]) into the case $\beta = -1/2$ and $\alpha_n \rightarrow \infty$ considered above. It is therefore not surprising that this transformation also appears in the following limit theorem:

4.9 Theorem. *Let $c \geq 0$ be a constant, and let $(\beta_n)_{n \in \mathbb{N}} \subset [-1/2, \infty[$ be an increasing sequence of indices with $\lim_{n \rightarrow \infty} n/\sqrt{\beta_n} = 0$. Let $\nu \in M^1([0, \infty[)$ with a finite second moment $\int_0^{\infty} x^2 d\nu(x) < \infty$. Then, $\rho := \int_0^{\infty} \ln(\text{ch}(2x)) d\nu(x) < \infty$ and $\sigma^2 := \int_0^{\infty} (\ln(\text{ch}(2x)))^2 d\nu(x) < \infty$ exist, and for the random walks $(S_n^{(\beta_n - c, \beta_n)})_{n \geq 0}$ on $[0, \infty[$,*

$$\frac{2 \cdot S_n^{(\beta_n + c, \beta_n)} - n\rho}{\sqrt{n}}$$

tends in distribution for $n \rightarrow \infty$ to $N(0, \sigma^2 - \rho^2)$.

The proof of this second CLT is based on the observation that for $\beta \rightarrow \infty$, $\alpha = \beta + c$, and $f \in C([0, \infty[)$,

$$\delta_r *_{(\alpha, \beta)} \delta_s(f) \rightarrow f\left(\frac{1}{2} \text{arch}(\text{ch}(2s)\text{ch}(2r))\right);$$

see Lemma 4.12 below. Again, this motivates the following notions.

4.10 Definition. (1) We equip the set $[0, \infty[$ with the commutative semigroup operation

$$s \bullet r := \frac{1}{2} \cdot \text{arch}(\text{ch}(2r) \cdot \text{ch}(2s)) \quad (r, s \geq 0),$$

and extend this operation again in a linear, weakly continuous way to the space of all bounded Borel measures. We also notice that the mapping $Q(x) := \ln(\text{ch}(2x))$ establishes a semigroup isomorphism from $([0, \infty[, \bullet)$ onto $([0, \infty[, +)$.

- (2) A function $f \in C([0, \infty[)$ is called arche-Lipschitz continuous with constant L , if for all $x, y \geq 0$,

$$\left| f\left(\frac{1}{2}\operatorname{arch}(e^x)\right) - f\left(\frac{1}{2}\operatorname{arch}(e^y)\right) \right| \leq L|x - y|.$$

This notion is again motivated by the fact, that for a function $f \in C_0([0, \infty[)$, which is Lipschitz in the usual sense with constant L , the function $\tilde{f}(x) := f(Q(x))$ is arche-Lipschitz with the same L .

Before starting with the central parts of the proof, we note the collect some elementary equations for hyperbolic functions:

$$\operatorname{arch}(\sqrt{x + 1/2}) = \frac{1}{2} \cdot \operatorname{arch}(2x) \quad (x \geq 0); \quad (4.12)$$

$$|\operatorname{ch} s \cdot \operatorname{ch} r + i \cdot \operatorname{sh} s \cdot \operatorname{sh} r| = \left(\frac{\operatorname{ch}(2s)\operatorname{ch}(2r)}{2} + 1/2 \right)^{1/2} \quad (s, r \geq 0); \quad (4.13)$$

$$\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + i \cdot \operatorname{sh} r \cdot \operatorname{sh} s| = \frac{1}{2} \operatorname{arch}(\operatorname{ch}(2s)\operatorname{ch}(2r)) \quad (s, r \geq 0). \quad (4.14)$$

The following results are the key for the proof of Theorem 4.9

4.11 Lemma. *Let $f \in C([0, \infty[)$ be arche-Lipschitz continuous with constant L . Then, for all $r, s \geq 0$, $t \in [0, 1]$ and $\varphi \in [0, \pi]$,*

$$\begin{aligned} & \left| f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + te^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s|) - f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + i \cdot \operatorname{sh} r \cdot \operatorname{sh} s|) \right| \\ & \leq 8L \frac{|i - te^{i\varphi}|}{\sin^2 \varphi}. \end{aligned} \quad (4.15)$$

Proof. Put

$$A := |\operatorname{ch} r \cdot \operatorname{ch} s + i \cdot \operatorname{sh} r \cdot \operatorname{sh} s| \quad \text{and} \quad B := |\operatorname{ch} r \cdot \operatorname{ch} s + te^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s|.$$

Using Eq. (4.12), Definition 4.10(2), and the mean value theorem for $f(x) := \ln(2x^2 - 1)$, we obtain

$$\begin{aligned} |f(\operatorname{arch} A) - f(\operatorname{arch} B)| &= \left| f\left(\frac{1}{2}\operatorname{arch}(2A^2 - 1)\right) - f\left(\frac{1}{2}\operatorname{arch}(2B^2 - 1)\right) \right| \\ &\leq L \cdot |\ln(2A^2 - 1) - \ln(2B^2 - 1)| \\ &\leq 4L \cdot |A - B| \cdot \frac{y}{2y^2 - 1} \leq 2L \cdot |A - B| \frac{1}{y - 1/\sqrt{2}} \end{aligned} \quad (4.16)$$

for some y between A and B . On the other hand,

$$\begin{aligned} B &\geq \operatorname{ch} r \cdot \operatorname{ch} s + t \cdot \cos \varphi \cdot \operatorname{sh} r \cdot \operatorname{sh} s \\ &= (1 + (1 + t \cdot \cos \varphi) \cdot \operatorname{sh} r \cdot \operatorname{sh} s) =: D \end{aligned}$$

and $A \geq D$, and thus $y \geq D$. Moreover, we have $|A - B| \leq |i - te^{i\varphi}| \operatorname{sh} r \cdot \operatorname{sh} s$ and

$$1 + t \cdot \cos \varphi \geq \frac{1}{2}(1 + \cos \varphi) \geq \frac{1}{4}(1 - \cos^2 \varphi) = \frac{1}{4} \sin^2 \varphi.$$

We thus conclude that

$$\begin{aligned} |f(\operatorname{arch} A) - f(\operatorname{arch} B)| &\leq 2L|i - te^{i\varphi}| \frac{\operatorname{sh} r \cdot \operatorname{sh} s}{(1 - 1/\sqrt{2}) + (1 + t \cos \varphi) \operatorname{sh} r \cdot \operatorname{sh} s} \\ &\leq 2L \frac{|i - te^{i\varphi}|}{1 + t \cos \varphi} \leq 8L \frac{|i - te^{i\varphi}|}{\sin^2 \varphi}. \end{aligned}$$

□

4.12 Lemma. *There is a constant $C = C(c)$ such that for all $r, s \in [0, \infty[$, all sufficiently large β , and all arche-Lipschitz continuous functions f on $[0, \infty[$ with constant L ,*

$$|\delta_r *_{(\beta+c, \beta)} \delta_s(f) - \delta_r \bullet \delta_s(f)| \leq CL/\sqrt{\beta}.$$

Proof. We only consider the case $c > 0$. By the definition of the Jacobi convolution, Eq. (4.14), and Lemma 4.11,

$$\begin{aligned} &|\delta_r *_{(\beta+c, \beta)} \delta_s(f) - \delta_r \bullet \delta_s(f)| \\ &\leq c_{(\beta+c, \beta)} \int_0^1 \int_0^\pi \left| f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + te^{i\varphi} \operatorname{sh} r \cdot \operatorname{sh} s|) - \right. \\ &\quad \left. - f(\operatorname{arch} |\operatorname{ch} r \cdot \operatorname{ch} s + i \cdot \operatorname{sh} r \cdot \operatorname{sh} s|) \right| \cdot (1 - t^2)^{c-1} (t \sin \varphi)^{2\beta} dt d\varphi \\ &\leq 8Lc_{(\beta+c, \beta)} \int_0^1 \int_0^\pi |i - te^{i\varphi}| (1 - t^2)^{c-1} t^{2\beta+1} \sin^{2\beta-2} \varphi dt d\varphi \\ &\leq 8Lc_{(\beta+c, \beta)} \int_0^1 \int_0^\pi (\cos \varphi + (1 - t \sin \varphi)) (1 - t^2)^{c-1} t^{2\beta+1} \sin^{2\beta-2} \varphi dt d\varphi \\ &= 8Lc_{(\beta+c, \beta)} \int_0^1 \int_0^\pi (\cos \varphi + (1 - t) + t(1 - \sin \varphi)) (1 - t^2)^{c-1} t^{2\beta+1} \sin^{2\beta-2} \varphi dt d\varphi \\ &\leq 8Lc_{(\beta+c, \beta)} \int_0^1 \int_0^\pi (\cos \varphi + (1 - t^2) + t \cos^2 \varphi) (1 - t^2)^{c-1} t^{2\beta+1} \sin^{2\beta-2} \varphi dt d\varphi. \end{aligned}$$

Using the definition of $c_{(\beta+c, \beta)}$, the beta integrals

$$\int_0^1 (1 - t^2)^{a-1} t^{b-1} dt = \frac{\Gamma(a)\Gamma(b/2)}{2\Gamma(a + b/2)}$$

and

$$\int_0^\pi |\cos \varphi|^{2a-1} \sin^{2b-1} \varphi d\varphi = \frac{\Gamma(a)\Gamma(b)}{2\Gamma(a + b)},$$

and $\Gamma(x + 1/2)/\Gamma(x) = O(\sqrt{x})$ for $x \rightarrow \infty$, the lemma follows readily. □

The proofs of Lemmas 4.6 and 4.7, Proposition 4.8, and finally of the central limit theorem 4.2 can now be transferred word by word to this setting (by using Definition 4.10 instead of 4.3) which then leads to the proof of Theorem 4.9.

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