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Abstract

Let $\nu \in M^1([0,\infty[)$ be a fixed probability measure. For each dimension $p \in \mathbb{N}$, let $(X_n^p)_{n\geq 1}$ be i.i.d. $\mathbb{R}^p$-valued radial random variables with radial distribution ν. We derive two central limit theorems for $\|X_1^p + \ldots + X_n^p\|_2$ for $n,p \to \infty$ with normal limits. The first CLT for $n >> p$ follows from known estimates of convergence in the CLT on $\mathbb{R}^p$, while the second CLT for $n << p$ will be a consequence of asymptotic properties of Bessel convolutions.

Both limit theorems are considered also for $U(p)$-invariant random walks on the space of $p \times q$ matrices instead of $\mathbb{R}^p$ for $p \to \infty$ and fixed dimension q.

KEYWORDS: Radial random walks, central limit theorems, random matrices, large dimensions, matrix cones, Bessel convolution, Bessel functions of matrix argument.

Math. Subject Classification: 60F05, 60B10, 60B12, 33C70, 43A62.

1 Two central limit theorems

This paper has its origin in the following problem: Let $\nu \in M^1([0,\infty[)$ be a fixed probability measure. Then for each dimension $p \in \mathbb{N}$ there is a unique radial probability measure $\nu_p \in M^1(\mathbb{R}^p)$ with ν as its radial part, i.e., ν is the image of ν_p under the norm mapping $\varphi_p(x) := \|x\|_2$. For each $p \in \mathbb{N}$ consider i.i.d. $\mathbb{R}^p$-valued random variables X_k^p, $k \in \mathbb{N}$, with law ν_p as well as the associated radial random walks

$$\big(S_n^p := \sum_{k=1}^n X_k^p\big)_{n\geq 0}$$

on $\mathbb{R}^p$. The aim is to find limit theorems for the $[0,\infty[$-valued random variables $\|S_n^p\|_2$ for $n,p \to \infty$. In [V1] and [RV] we proved that for all sequences $p_n \to \infty$,

$$\|S_n^{p_n}\|_2^2/n \to \sigma^2 := \sigma^2(\nu) := \int_0^\infty x^2 \, d\nu(x)$$

under the condition $\sigma^2 < \infty$. Moreover, in [RV] an associated strong law and a large deviation principle were derived under the condition that p_n grows fast enough. In this paper we present two associated central limit theorems (CLTs) under disjoint growth conditions for p_n.

The first CLT holds for $p_n << n$ and is an obvious consequence of Berry-Esseen estimates on $\mathbb{R}^p$ with explicit constants depending on the dimensions p, which are due to Bentkus and Götze [B], [BG2] (for a survey about this topic we also recommend [BGPR]):

1.1 Theorem. *Assume that $\nu \in M^1([0, \infty[)$ with $\nu \neq \delta_0$ admits a finite third moment $m_3(\nu) := \int_0^\infty x^3 \, d\nu(x) < \infty$, and that $\lim_{n\to\infty} n/p_n^3 = \infty$. Then*

$$\frac{\sqrt{p_n}}{n\sigma^2\sqrt{2}}(\|S_n^{p_n}\|_2^2 - n\sigma^2)$$

tends in distribution for $n \to \infty$ to the standard normal distribution $N(0,1)$.

Proof. The radial measure ν_p on $\mathbb{R}^p$ has a covariance matrix Σ^2 which is invariant under all conjugations w.r.t orthogonal transformations. This yields that $\Sigma^2 = c_p I_p$ with I_p as identity matrix and some constant c_p. As $\sigma^2 = E(\|X_1^p\|_2^2) = pc_p$, we actually have $\Sigma^2 = (\sigma^2/p)I_p$. Theorem 2 of Bentkus [B] implies after normalization that the distribution function $F_{n,p}$ of $\frac{p}{n\sigma^2}\|S_n^p\|_2^2$ and the distribution function F_p of the χ_p^2-distribution with p degrees of freedom satisfy

$$\|F_{n,p} - F_p\|_\infty \leq C \cdot \frac{p^{3/2}}{\sqrt{n}} \tag{1.1}$$

for $n, p \in \mathbb{N}$ with a universal $C = C(\nu)$. Therefore, for $p = p_n$ as in the theorem, we have uniform convergence of distribution functions. Moreover, the classical CLT shows that for χ_p^2-distributed random variables X_p (with $E(X_p) = p$ and $Var(X_p) = 2p$), the random variables $\frac{X_p - p}{\sqrt{2p}}$ tend to the standard normal distribution $N(0,1)$ for $p \to \infty$. A combination of both results readily implies the theorem. $\square$

1.2 Remark. The main result of [BG2] suggests that for sufficiently large dimensions p and $\nu \in M^1([0, \infty[)$ with finite fourth moment $m_4(\nu) := \int_0^\infty x^4 \, d\nu(x) < \infty$,

$$\|F_{n,p} - F_p\|_\infty \leq C \cdot \frac{p^2}{n} \qquad (n, p \in \mathbb{N}) \tag{1.2}$$

holds (the dependence of the constants is not clearly noted in [BG2] and difficult to verify). If (1.2) is true, then Theorem 1.1 holds under the weaker condition $\lim_{n\to\infty} n/p_n^2 = \infty$. We also remark that the results of [BG1] indicate that the method of the proof of Theorem 1.1 above cannot go much beyond this condition.

In this paper, we derive the following complementary CLT for $p_n >> n$:

1.3 Theorem. *Assume that $\nu \in M^1([0, \infty[)$ admits a finite fourth moment $m_4(\nu) := \int_0^\infty x^4 \, d\nu(x) < \infty$, and that $\lim_{n\to\infty} n^2/p_n \to 0$. Then*

$$\frac{\|S_n^{p_n}\|_2^2 - n\sigma^2}{\sqrt{n}}$$

tends in distribution for $n \to \infty$ to the normal distribution $N(0, m_4(\nu) - \sigma^4)$ on $\mathbb{R}$.

1.4 Remark. (1) We do not know precise conditions on the dimensions p_n under which the CLTs 1.1 and 1.3 hold (possibly under sufficiently strong moment conditions for ν).

In order to obtain a reasonable guess, notice that the CLTs imply, after suitable normalizations, convergence of moments of $\|S_n^{p_n}\|_2^2 - n\sigma^2$ to moments of normal distributions. We therefore computed the second moment

$$E\big((\|S_n^{p_n}\|_2^2 - n\sigma^2)^2\big) = n(m_4(\nu) - \sigma^4) + 2\frac{n(n+1)}{p_n}\sigma^4. \tag{1.3}$$

This leads to the **conjecture** that the assertion of Theorem 1.1 holds if and only if $n/p_n \to \infty$ holds, and the assertion of Theorem 1.3 precisely for $n/p_n \to 0$ where the limit conditions are necessary by Eq. (1.3). We also verified that these conditions also work for the third and fourth moments. Unfortunately, the formulas become too complicated to detect a general limit pattern for arbitary moments at present in order to derive convergence of all moments. The case $p_n = cn$ for some constant c might be also an interestig field of research.

(2) A comparison of Theorems 1.1 and 1.3 has the following possible implication to statistics: Assume that $\nu \in M^1([0, \infty[)$ is known and that the random variable $\|S_n^{p_n}\|_2$ can be observed with a known time parameter n, but an unknown dimension p which has to be estimated. Then p can be recovered in a reasonable way for $n >> p^3$ while this is not the case for $n << \sqrt{p}$.

In this paper we are going to derive two generalizations of the preceding CLTs:

The first extension concerns a matrix-valued version: For fixed dimensions $p, q \in \mathbb{N}$ let $M_{p,q} = M_{p,q}(\mathbb{F})$ be the space of $p \times q$-matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. This is a Euclidean vector space of real dimension dpq with scalar product $\langle x, y \rangle = \mathfrak{R}tr(x^*y)$ where $x^* := \overline{x}^t$, $\mathfrak{R}t := \frac{1}{2}(t + \overline{t})$ is the real part of $t \in \mathbb{F}$, and tr is the trace in $M_q := M_{q,q}$. A measure on $M_{p,q}$ is called radial if it is invariant under the action of the unitary group $U_p = U_p(\mathbb{F})$ by left multiplication, $U_p \times M_{p,q} \to M_{p,q}$, $(u, x) \mapsto ux$. This action is orthogonal w.r.t. the scalar product above, and, by uniqueness of the polar decomposition, two matrices $x, y \in M_{p,q}$ belong to the same U_p-orbit if and only if $x^*x = y^*y$. Thus the space $M_{p,q}^{U_p}$ of U_p-orbits in $M_{p,q}$ is naturally parameterized by the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$-matrices over $\mathbb{F}$. We identify $M_{p,q}^{U_p}$ with Π_q via $U_q x \simeq (x^*x)^{1/2}$, i.e., the canonical projection $M_{p,q} \to M_{p,q}^{U_p}$ will be realized as the mapping

$$\varphi_p : M_{p,q} \to \Pi_q, \quad x \mapsto (x^*x)^{1/2}.$$

The square root is used here in order to ensure for $q = 1$ and $\mathbb{F} = \mathbb{R}$ that the setting above with $\Pi_1 = [0, \infty[$ and $\varphi_p(x) = \|x\|$ appears. By taking images of measures, φ_p induces a Banach space isomorphism between the space $M_b^{U_q}(M_{p,q})$ of all bounded radial Borel measures on $M_{p,q}$ and the space $M_b(\Pi_q)$ of bounded Borel measures on the cone Π_q. In particular, for each $\nu \in M^1(\Pi_q)$ there is a unique radial probability measure $\nu_p \in M^1(M_{p,q})$ with $\varphi_p(\nu_p) = \nu$.

As in the case $q = 1$, we now consider for each $p \in \mathbb{N}$ i.i.d. $M_{p,q}$-valued random variables X_k^p, $k \in \mathbb{N}$, with law ν_p and the associated radial random walks $\big(S_n^p := \sum_{k=1}^n X_k^p\big)_{n \geq 0}$.

1.5 Definition. We say that $\nu \in M^1(\Pi_q)$ admits a k-th moment ($k \in \mathbb{N}$) if

$$m_k(\nu) := \int_{\Pi_q} \|s\|^k \, d\nu(s) < \infty$$

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where $\|s\| = (tr s^2)^{1/2}$ is the Hilbert-Schmidt norm. If the second moment exists, the second moment of ν is defined as the matrix-valued integral

$$\sigma^2 := \sigma^2(\nu) := \int_{\Pi_q} s^2 \, d\nu(s) \in \Pi_q.$$

With these notions, the following generalizations of Theorems 1.1 and 1.3 hold:

1.6 Theorem. *Assume that $m_4(\nu) < \infty$. Moreover, let $\lim_{n\to\infty} n/p_n^4 = \infty$. Then*

$$\frac{\sqrt{p_n}}{n}(\varphi_{p_n}(S_n^{p_n})^2 - n\sigma^2)$$

tends in distribution to some normal distribution $N(0, T^2)$ on the vector space H_q of hermitian $q \times q$-matrices over $\mathbb{F}$ (with a covariance matrix $T^2 = T^2(\sigma^2)$ described in the proof below for $\mathbb{F} = \mathbb{R}$).

Proof. We regard $M_{p,q} = M_{p,q}(\mathbb{F})$ as $\mathbb{F}^p \otimes \mathbb{F}^q$. The radial measure ν_p on $M_{p,q}$ has a covariance matrix Σ_p^2 which is invariant under all conjugations w.r.t. U_p, i.e., we have $\Sigma_p^2 = I_p \otimes T_p$ for some $T_p \in \Pi_q$. As $\sigma^2 = E((X_k^p)^* X_k^p) = pT_p$, we have $\Sigma_p^2 = \frac{1}{p} \cdot I_p \otimes \sigma^2$.

Moreover, Theorem 1 of Bentkus [B] implies after normalization that there is an universal constant $C > 0$ such that

$$\left|P(\frac{\sqrt{p}}{\sqrt{n}}S_n^p \in K) - N(0, \Sigma_1^2)(K)\right| \leq C \cdot \frac{p^2}{\sqrt{n}}$$

for all convex sets $K \subset M_{p,q}$ and all n, p. Therefore, for $p = p_n$ as in the theorem,

$$\left|P(\frac{\sqrt{p_n}}{\sqrt{n}}S_n^{p_n} \in K) - N(0, \Sigma_1^2)(K)\right| \to 0 \qquad \text{for} \quad n \to \infty$$

uniformly in all convex sets $K \subset M_{p,q}$. Using the projections $\varphi_p : M_{p,q} \to \Pi_q$, we obtain

$$\left|P(\frac{p_n}{n} \cdot \varphi_{p_n}(S_n^{p_n})^2 \in L) - W_{p_n}(L)\right| \to 0 \qquad \text{for} \quad n \to \infty \qquad (1.4)$$

uniformly in all convex sets $L \subset \Pi_q$ where the measures $W_{p_n} := \varphi_{p_n}^2(N(0, \Sigma_1^2))$ are certain Wishart distributions on Π_q with p_n degrees of freedom. By definition, the measures W_{p_n} appear as the distribution of a p_n-fold sum of iid W_1-distributed random variables on the vector space H_q with expectation σ^2 and some covariance matrix $T^2 = T^2(\sigma^2)$ described below. A combination of Eq. (1.4) and the classical CLT on H_q then readily implies the theorem.

We finally compute T^2 for $\mathbb{F} = \mathbb{R}$. Let $X = (X_1, \ldots, X_q)$ be a $\mathbb{R}^q$-valued, standard normal distributed random variable, and $\sigma \in \Pi_q$ the positive semidefinite root of σ^2. Then $Y := X\sigma$ is $\mathbb{R}^q$-valued with distribution $N(0, \sigma^2)$, and the H_q-valued random variable $Y^*Y = X^*\sigma^2 X$ has the distribution W_1. We notice that $E(X_i^4) = 3$, $E(X_i^2 X_j^2) = 1$ for $i \neq j$, and that $E(X_i X_j X_k X_l) = 0$ whenever at least one index appears only once in $\{i, j, k, l\}$. This implies that

$$(T^2)_{(i,j),(k,l)} = Kov(Y_iY_j, Y_kY_l) = E(Y_iY_jY_kY_l) - E(Y_iY_j)E(Y_kY_l)$$

$$= \sum_{a,b,c,d=1}^{q} \sigma_{a,i}\sigma_{b,j}\sigma_{c,k}\sigma_{d,l}E(X_aX_bX_cX_d) \quad - \quad \Big(\sum_{a=1}^{q}\sigma_{a,i}\sigma_{a,j}\Big)\Big(\sum_{a=1}^{q}\sigma_{a,k}\sigma_{a,l}\Big)$$

$$= \sum_{a,b=1}^{q} \big(\sigma_{a,i}\sigma_{a,k}\sigma_{b,j}\sigma_{b,l} + \sigma_{a,i}\sigma_{a,l}\sigma_{b,j}\sigma_{b,k}\big)$$

$$= (\sigma^2)_{i,k}(\sigma^2)_{j,l} + (\sigma^2)_{i,l}(\sigma^2)_{j,k}. \qquad (1.5)$$

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The computation for $\mathbb{F} = \mathbb{C}, \mathbb{H}$ is similar. $\qquad\square$

Notice that the proof is analog to that of Theorem 1.1. The slightly stronger condition is required as here certain convex sets in $M_{p,q}$ instead of balls are used, where only weaker convergence results form [B] are available. As for $q = 1$, we expect that Theorem 1.6 remains true under slightly weaker conditions than $n/p_n^4 \to \infty$.

1.7 Theorem. *Assume that $m_4(\nu) < \infty$ and $\lim_{n\to\infty} n^2/p_n = 0$. Then*

$$\frac{1}{\sqrt{n}}(\varphi_{p_n}(S_n^{p_n})^2 - n\sigma^2)$$

tends in distribution to the normal distribution $N(0, \Sigma^2)$ on H_q where Σ^2 is the covariance matrix of $\varphi_{p_n}(X_1^{p_n})$ (which is independent of p_n).

Notice that for $q = 1$, Theorem 1.7 completely agrees with 1.3. Theorem 1.7 will appear in Section 3 below as a special case of the even more general CLT 3.8.

We next turn to this generalization: Consider again the Banach space isomorphism $\varphi_p : M_b^{U_q}(M_{p,q}) \to M_b(\Pi_q)$. The usual group convolution on $M_{p,q}$ induces a Banach-$*$-algebra-structure on $M_b(\Pi_q)$ such that this becomes a probability-preserving Banach-$*$-algebra isomorphism. The space Π_q together with this new convolution becomes a so-called commutative orbit hypergroup; see [J],[BH], and [R]. Moreover, for $p \geq 2q$, Eq. (3.5) and Corollary 3.2 of [R] show that the convolution of point measures on Π_q induced from $M_{p,q}$ is given by

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f\left(\sqrt{r^2 + s^2 + svr + rv^*s}\right) \Delta(I - vv^*)^{\mu-\rho} \, dv \qquad (1.6)$$

with $\mu := pd/2$, $\rho := d\left(q - \frac{1}{2}\right) + 1$,

$$D_q := \{v \in M_q : v^*v < I\}$$

(where $v^*v < I$ means that $I - v^*v$ is positive definite), and with the normalization constant

$$\kappa_\mu := \int_{D_q} \Delta(I - v^*v)^{\mu-\rho} dv. \qquad (1.7)$$

The convolution on $M_b(\Pi_q)$ is just given by bilinear, weakly continuous extension.

It was observed in [R] that Eq. (1.6) defines a commutative hypergroup $(\Pi_q, *_\mu)$ for all indices $\mu \in \mathbb{R}$ with $\mu > \rho - 1$, where $0 \in \Pi_q$ is the identity and the involution is the identity mapping. These hypergroups are closely related with a product formula for Bessel functions J_μ on the cone Π_q and are therefore called Bessel hypergroups. For details we refer to [FK], [H], and in particular [R]. For general indices μ, these Bessel hypergroups $(\Pi_q, *_\mu)$ have no group interpretation as in the cases $\mu = pd/2$ with integral p, but nevertheless the notion of random walks on these hypergroups is still meaningful. For $q = 1$, such structures and associated random walks were investigated by Kingman [K] and many others; see [BH].

1.8 Definition. Fix $\mu > \rho - 1$ and a probability measure $\nu \in M^1(\Pi_q)$. A Bessel random walk $(S_n^\mu)_{n\geq 0}$ on Π_q of index μ and with law ν is a time-homogeneous Markov chain on Π_q with $S_0^\mu = 0$ and transition probability

$$P(S_{n+1}^\mu \in A | S_n^\mu = x) = (\delta_x *_\mu \nu)(A)$$

for $x \in \Pi_q$ and Borel sets $A \subset \Pi_q$.

This notion has its origin in the following well-known fact for the orbit cases $\mu = pd/2$, $p \in \mathbb{N}$: If for a given $\nu \in M^1(\pi_q)$ we consider the associated radial random walk $(S_n^p)_{n \geq 0}$ on $M_{p,q}$ as above, then $(\varphi_p(S_n^p))_{n \geq 0}$ is a random walk on Π_q of index $\mu = pd/2$ with law ν.

We shall derive Theorem 1.7 in Section 3 in this more general setting for $\mu \in \mathbb{R}$, $\mu \geq 2q$, as the proof is precisely the same as in the group case. The proof will rely on facts on these Bessel convolutions which we recapitulate in the next section

We finally mention that it seems reasonable that at least for $q = 1$, Theorem 1.1 may be also generalized to Bessel random walks with arbitrary indices $\mu \in \mathbb{R}$ with $\mu \to \infty$. A possible approach might work via explicit Berry-Esseen-type estimates for Hankel transforms similar as in [PV] with a careful investigation of the dependence of constants there on the dimension parameter.

2   Bessel convolutions on matrix cones

In this section we collect some known facts mainly from [R] and [V2].

Let $\mathbb{F}$ be one of the real division algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. Denote the usual conjugation in $\mathbb{F}$ by $t \mapsto \overline{t}$, the real part of $t \in \mathbb{F}$ by $\Re t = \frac{1}{2}(t + \overline{t})$, and by $|t| = (t\overline{t})^{1/2}$ its norm.

For $p, q \in \mathbb{N}$ we denote by $M_{p,q}$ the vector space of all $p \times q$-matrices over $\mathbb{F}$ and put $M_q := M_q(\mathbb{F}) := M_{q,q}(\mathbb{F})$ for abbreviation. Let further $H_q = \{x \in M_q : x = x^*\}$ the space of Hermitian $q \times q$-matrices. All these spaces are real Euclidean vector spaces with scalar product $\langle x, y \rangle := \Re tr(x^*y)$ and norm $\|x\| = \langle x, x \rangle^{1/2}$. Here $x^* := \overline{x}^t$ and tr denotes the trace. Let further

$$\Pi_q := \{x^2 : \ x \in H_q\} = \{x^*x : x \in H_q\}$$

be the cone of all positive semidefinite matrices in H_q. Bessel functions J_μ on these matrix cones with a parameter $\mu > 0$ (and suppressed parameters $\mathbb{F}$ and q) were studied from different points of view by numerous people; we here only mention [H], [FK], [R], and [RV] which are relevant here. As we do not need details, we do not recapitulate the complicated definition here and refer to these references. We only mention that for $q = 1$, and $\mathbb{F} = \mathbb{R}$, we have $\Pi_q = [0, \infty[$, and the Bessel function $\mathcal{J}_\mu$ satisfies

$$J_\mu\left(\frac{x^2}{4}\right) = j_{\mu-1}(x)$$

where $j_\kappa(z) = {}_0F_1(\kappa + 1; -z^2/4)$ is the usual modified Bessel function in one variable.

Hypergroups are convolution structures which generalize locally compact groups insofar as the convolution product of two point measures is in general not a point measure again, but just a probability measure on the underlying space. More precisely, a hypergroup $(X, *)$ is a locally compact Hausdorff space X together with a convolution $*$ on the space $M_b(X)$ of regular bounded Borel measures on X, such that $(M_b(X), *)$ becomes a Banach algebra, and $*$ is weakly continuous, probability preserving and preserves compact supports of measures. Moreover, one requires an identity $e \in X$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for $x \in X$, as well as a continuous involution $x \mapsto \bar{x}$ on X such that for all $x, y \in X$, $e \in supp(\delta_x * \delta_y)$ is equivalent to $x = \bar{y}$, and $\delta_{\bar{x}} * \delta_{\bar{y}} = (\delta_y * \delta_x)^-$. Here for $\mu \in M_b(X)$, the measure μ^- is given by $\mu^-(A) = \mu(A^-)$ for Borel sets $A \subset X$. A hypergroup $(X, *)$ is called commutative if and only if so is the convolution $*$. Thus for a commutative hypergroup $(X, *)$, $M_b(X)$ is a commutative Banach-$*$-algebra with identity δ_e. Due to weak continuity, the convolution of measures on a hypergroup is uniquely determined by convolution products of point measures.

On a commutative hypergroup $(X, *)$ there exists a (up to a multiplicative factor) unique Haar measure ω, i.e. ω is a positive Radon measure on X satisfying

$$\int_X \delta_x * \delta_y(f) d\omega(y) = \int_X f(y) d\omega(y) \quad \text{for all } x \in X, \ f \in C_c(X).$$

The decisive object for harmonic analysis on a commutative hypergroup is its dual space

$$\widehat{X} := \{\varphi \in C_b(X) : \varphi \neq 0, \ \varphi(\overline{x}) = \overline{\varphi(x)}, \ \delta_x * \delta_y(\varphi) = \varphi(x)\varphi(y) \text{ for all } x, y \in X\}.$$

Its elements are called characters. As for LCA groups, the dual of a commutative hypergroup is a locally compact Hausdorff space with the topology of locally uniform convergence and can be identified with the symmetric spectrum of the convolution algebra $L^1(X, \omega)$. For more details on hypergroups we refer to [J] and [BH].

The following theorem contains some of the main results of [R].

2.1 Theorem. *Let $\mu \in \mathbb{R}$ with $\mu > \rho - 1$. Then*

(a) *The assignment*

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f\left(\sqrt{r^2 + s^2 + svr + rv^*s}\right) \Delta(I - vv^*)^{\mu-\rho} dv, \quad f \in C(\Pi_q) \ (2.1)$$

with κ_μ as in (1.7), defines a commutative hypergroup structure on Π_q with neutral element $0 \in \Pi_q$ and the identity mapping as involution. The support of $\delta_r *_\mu \delta_s$ satisfies

$$supp(\delta_r *_\mu \delta_s) \subseteq \{t \in \Pi_q : \|t\| \leq \|r\| + \|s\|\}.$$

(b) *A Haar measure of $(\Pi_q, *_\mu)$ is given by*

$$\omega_\mu(f) = \frac{\pi^{q\mu}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta(r)^\gamma dr \qquad \text{with} \qquad \gamma = \mu - \frac{d}{2}(q-1) - 1.$$

(c) *The dual space of $\Pi_{q,\mu}$ is given by $\widehat{\Pi_{q,\mu}} = \{\varphi_s : s \in \Pi_q\}$ with*

$$\varphi_s(r) := \mathcal{J}_\mu(\tfrac{1}{4}rs^2r) = \varphi_r(s).$$

The hypergroup $\Pi_{q,\mu}$ is self-dual via the homeomorphism $s \mapsto \varphi_s$. Under this identification of $\widehat{\Pi_{q,\mu}}$ with $\Pi_{q,\mu}$, the Plancherel measure on $\Pi_{q,\mu}$ is $(2\pi)^{-2\mu q}\omega_\mu$.

The most important informal observation at this point is that the convolution (2.1) converges for $\mu \to \infty$ to the semigroup convolution

$$(\delta_r \bullet \delta_s)(f) := f(\sqrt{r^2 + s^2}), \qquad r, s \in \Pi_q$$

associated with the semigroup operation $r \bullet s := \sqrt{r^2 + s^2}$ on Π_q. We next shall make this convergence more precise, as this is the main ingredient for the proof of Theorem 1.7.

3 The central limit theorem for $\mu_n >> n$

We here derive a generalization of Theorem 1.7 for general parameters μ. We begin with some unusual notion which is needed below:

3.1 Definition. A function $f : \Pi_q \to \mathbb{C}$ is called root-Lipschitz continuous with constant L, if for all $x, y \in \Pi_q$,
$$|f(\sqrt{x}) - f(\sqrt{y})| \le L\|x - y\|.$$

3.2 Lemma. *There is a constant $C = C(q)$ such that for all $r, s \in \Pi_q$, all $\mu \ge 2\rho$, and all root-Lipschitz continuous functions f on Π_q with constant L,*
$$|\delta_r *_\mu \delta_s(f) - \delta_r \bullet \delta_s(f)| \le CL\|r\| \cdot \|s\|/\sqrt{\mu}.$$

Proof. We first recapitulate from Lemma 3.1 of [RV] that for $\mu > \rho$ and $v \in \sqrt{\mu} \cdot D_q \subset M_q$,
$$0 \le e^{-\langle v,v \rangle} - \Delta(I - \frac{1}{\mu}vv^*)^\mu \le \frac{1}{\mu}tr((vv^*)^2) \cdot e^{-\langle v,v \rangle}. \tag{3.1}$$

As all norms on Π_q are equivalent, we conclude from the first inequality that for $r, s > 0$ and suitable constants C_i,

$$
\begin{aligned}
&|\delta_r *_\mu \delta_s(f) - \delta_r \bullet \delta_s(f)| \\
&\le \frac{1}{\kappa_\mu} \int_{D_q} \left| f\left(\sqrt{r^2 + s^2 + svr + rv^*s}\right) - f\left(\sqrt{r^2 + s^2}\right) \right| \cdot \Delta(I - vv^*)^{\mu-\rho}\, dv \\
&\le \frac{L}{\kappa_\mu} \int_{D_q} \|svr + rv^*s\| \cdot \Delta(I - vv^*)^{\mu-\rho}\, dv \\
&\le \frac{C_1 L\|r\| \cdot \|s\|}{\kappa_\mu} \int_{D_q} \|v\| \cdot \Delta(I - vv^*)^{\mu-\rho}\, dv \\
&= \frac{C_1 L\|r\| \cdot \|s\|}{\kappa_\mu(\mu - \rho)^{(dq^2+1)/2}} \int_{\sqrt{\mu-\rho} \cdot D_q} \|v\| \cdot \Delta(I - \frac{1}{\mu-\rho}vv^*)^{\mu-\rho}\, dv \\
&\le \frac{C_1 L\|r\| \cdot \|s\|}{\kappa_\mu(\mu - \rho)^{(dq^2+1)/2}} \int_{M_q} \|v\| \cdot e^{-\langle v,v \rangle}\, dv \\
&\le C_2 \frac{L\|r\| \cdot \|s\|}{\kappa_\mu(\mu - \rho)^{(dq^2+1)/2}}
\end{aligned}
\tag{3.2}
$$

Moreover, the second inequality in (3.1) yields that for sufficiently large μ,

$$
\begin{aligned}
\kappa_\mu &= \int_{D_q} \Delta(I - v^*v)^{\mu-\rho}\, dv \\
&= \frac{1}{(\mu-\rho)^{dq^2/2}} \int_{\sqrt{\mu-\rho} \cdot D_q} \Delta(I - \frac{1}{\mu-\rho}vv^*)^{\mu-\rho}\, dv \\
&\ge \frac{1}{(\mu-\rho)^{dq^2/2}} \int_{\sqrt{\mu-\rho} \cdot D_q} \left(1 - \frac{tr((vv^*)^2)}{\mu - \rho}\right) e^{-\langle v,v \rangle}\, dv \\
&\ge C_3(\mu - \rho)^{-dq^2/2}
\end{aligned}
$$

and thus $\kappa_\mu \ge C_4(\mu - \rho)^{-dq^2/2}$ for all $\mu \ge 2\rho$ and some constant C_3. The lemma is now a consequence of Eq. (3.2). $\square$

3.3 Lemma. *There is a constant $C = C(q)$ such that for all root-Lipschitz continuous functions f on Π_q with constant L, all $\nu_1, \nu_2 \in M^1(\Pi_q)$ with $m_1(\nu_i) < \infty$, and all $\mu \geq 2\rho$,*

$$|\nu_1 *_\mu \nu_2(f) - \nu_1 \bullet \nu_2(f)| \leq CL \cdot m_1(\nu_1) \cdot m_1(\nu_2)/\sqrt{\mu}.$$

Proof. By Lemma 3.2,

$$|\nu_1 *_\mu \nu_2(f) - \nu_1 \bullet \nu_2(f)|$$
$$\leq \int_{\Pi_q} \int_{\Pi_q} |\delta_r *_\mu \delta_s(f) - \delta_r \bullet \delta_s(f)| \, d\nu_1(r) \, d\nu_2(s)$$
$$\leq \frac{CL}{\sqrt{\mu}} \int_{\Pi_q} \int_{\Pi_q} \|r\| \cdot \|s\| \, d\nu_1(r) \, d\nu_2(s) \quad = \quad CL\frac{m_1(\nu_1)m_1(\nu_2)}{\sqrt{\mu}}.$$

$\square$

3.4 Lemma. *There is a constant $C = C(q)$ such that for all root-Lipschitz continuous functions f on Π_q with constant L, all $\nu_1, \nu_2, \nu_3 \in M^1(\Pi_q)$ with $m_1(\nu_i) < \infty$, and all $\mu \geq 2\rho$,*

$$|(\nu_1 *_\mu \nu_2) \bullet \nu_3(f) - (\nu_1 \bullet \nu_2) \bullet \nu_3(f)| \leq CL \cdot m_1(\nu_1) \cdot m_1(\nu_2)/\sqrt{\mu}.$$

Proof. For $y \in \Pi_q$, consider the function $f_y(x) := f(x \bullet y)$ on Π_q. By the definition of $\bullet$ and our definition of root-Lipschitz continuity, these f_y are also root-Lipschitz continuous with the same constant L. Therefore, by Lemma 3.3,

$$|(\nu_1 *_\mu \nu_2) \bullet \nu_3(f) - (\nu_1 \bullet \nu_2) \bullet \nu_3(f)|$$
$$\leq \int_{\Pi_q} \left| \int_{\Pi_q} f_y(x) \, d(\nu_1 *_\mu \nu_2)(x) - \int_{\Pi_q} f_y(x) \, d(\nu_1 \bullet \nu_2)(x) \right| d\nu_3(y)$$
$$\leq CL \cdot m_1(\nu_1) \cdot m_1(\nu_2)/\sqrt{\mu}.$$

$\square$

For $\nu \in M^1(\Pi_q)$ and $n \in \mathbb{N}$, we denote the n-fold convolution powers of ν w.r.t. the convolutions $*_\mu$ and $\bullet$ by $\nu^{(n, *_\mu)}$ and $\nu^{(n, \bullet)}$ respectively.

3.5 Lemma. *For all $\nu \in M^1(\Pi_q)$ with $m_2(\nu) < \infty$, and all $n \in \mathbb{N}$, $\mu > \rho - 1$,*

$$m_1(\nu^{(n, *_\mu)}) \leq \sqrt{n \cdot m_2(\nu)}.$$

Proof. By the definition of the convolution $*_\mu$, the function m_2 satisfies

$$\delta_r *_\mu \delta_s(m_2) = m_2(r) + m_2(s) + \frac{1}{\kappa_\mu} \int_{D_q} tr(rvs + sv^*r) \cdot \Delta(I - vv^*)^{\mu-\rho} \, dv$$

for $r, s \in \Pi_q$ where the symmetry of the integrand and the substitution $v \mapsto -v$ immediately yield that the integral is equal to 0. Therefore,

$$\delta_r *_\mu \delta_s(m_2) = m_2(r) + m_2(s).$$

Integration yields that for all $\nu_1, \nu_2 \in M^1(\Pi_q)$ with $m_2(\nu_i) < \infty$,

$$m_2(\nu_1 *_\mu \nu_2) = m_2(\nu_1) + m_2(\nu_2).$$

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In particular we obtain by induction that for $\nu \in M^1(\Pi_q)$ with $m_2(\nu) < \infty$ and $n \in \mathbb{N}$,

$$m_2(\nu^{(n,*_\mu)}) = n \cdot m_2(\nu).$$

Therefore, by the Cauchy-Schwarz inequality,

$$m_1(\nu^{(n,*_\mu)}) = \int_{\Pi_q} \|x\| \, d\nu^{(n,*_\mu)}(x) \leq \left(\int_{\Pi_q} \|x\|^2 \, d\nu^{(n,*_\mu)}(x) \right)^{1/2} = \sqrt{n \cdot m_2(\nu)}.$$

$\square$

3.6 Remark. The preceding lemma has the following weaker variant under a weaker moment condition: For all $\nu_1, \nu_2 \in M^1(\Pi_q)$ and all $\mu > \rho - 1$,

$$m_1(\nu_1 *_\mu \nu_2) \leq m_1(\nu_1) + m_1(\nu_2).$$

For the proof, observe that the convolution $*_\mu$ has the property that for $r, s \in \Pi_q$,

$$supp(\delta_r *_\mu \delta_s) \subset \{t \in \Pi_q : \quad \|t\| \leq \|r\| + \|s\|\};$$

see for instance Theorem 3.10 of [R]. Therefore,

$$m_1(\nu_1 *_\mu \nu_2) = \int_{\Pi_q} \int_{\Pi_q} \left(\int_{\Pi_q} \|r\| \, d(\delta_x *_\mu \delta_y)(r) \right) d\nu_1(x) \, d\nu_2(y)$$

$$\leq \int_{\Pi_q} \int_{\Pi_q} (\|x\| + \|y\|) \, d\nu_1(x) \, d\nu_2(y)$$

$$= m_1(\nu_1) + m_1(\nu_2)$$

as claimed.

3.7 Proposition. *Let $\nu \in M^1(\Pi_q)$ with $m_2(\nu) < \infty$. Then there is a constant $C = C(q, \nu)$ such that for all root-Lipschitz continuous functions f on Π_q with constant L, and all $n \geq 2$, $\mu \geq 2\rho$,*

$$\left| \nu^{(n,*_\mu)}(f) - \nu^{(n,\bullet)}(f) \right| \leq CL \frac{n^{3/2}}{\sqrt{\mu}}.$$

Proof. We first observe that

$$\left| \nu^{(n,*_\mu)}(f) - \nu^{(n,\bullet)}(f) \right|$$

$$\leq \left| \nu^{(n,*_\mu)}(f) - \nu^{(n-1,*_\mu)} \bullet \nu(f) \right| + \left| (\nu^{(n-2,*_\mu)} * \nu) \bullet \nu(f) - \nu^{(n-2,*_\mu)} \bullet \nu \bullet \nu(f) \right|$$

$$+ \ldots + \left| (\nu * \nu) \bullet \nu^{(n-2,\bullet)}(f) - \nu^{(n,\bullet)}(f) \right|.$$

Moreover, by Lemmas 3.4 and 3.5, we have for $k = 2, \ldots, n$ that

$$\left| \nu^{(k,*_\mu)} \bullet \nu^{(n-k,\bullet)}(f) - \nu^{(k-1,*_\mu)} \bullet \nu \bullet \nu^{(n-k,\bullet)}(f) \right|$$

$$\leq m_1(\nu^{(k-1,*_\mu)}) \cdot m_1(\nu) \cdot \frac{C_1 L}{\sqrt{\mu}}$$

$$\leq C_1 L \frac{\sqrt{k-1}}{\sqrt{\mu}} m_1(\nu) \cdot \sqrt{m_2(\nu)}$$

with a suitable constant $C_1 > 0$. Combining this with the preceding inequality and $\sum_{k=1}^n \sqrt{k} = O(n^{3/2})$, the proposition follows. $\square$

We now fix a probability measure $\nu \in M^1(\Pi_q)$ and consider for $\mu > \rho$, the associated random walk $(S_n^\mu)_{n \in \mathbb{N}}$ on Π_q with law ν according to Definition 1.8.

3.8 Theorem. *Let $\nu \in M^1(\Pi_q)$ with a finite fourth moment $\int_{\Pi_q} \|x\|^4 \, d\nu(x) < \infty$. Assume that $n^2/\mu_n \to 0$ for $n \to \infty$. Then*

$$\frac{(S_n^{\mu_n})^2 - n\sigma^2}{\sqrt{n}}$$

tends in distribution for $n \to \infty$ to the normal distribution $N(0, \Sigma^2)$ on the vector space H_q of Hermitian $q \times q$ matrices, where Σ^2 is the covariance matrix belonging to the image measure $Q(\nu) \in M^1(\Pi_q) \subset M^1(H_q)$ under the square mapping $Q(x) := x^2$ on Π_p.

Proof. Let $f \in C_0(H_q)$ be a Lipschitz-continuous function in the usual sense on H_q with the Lipschitz constant L. Then, for $n \in \mathbb{N}$, the functions

$$f_n(x) := f\Big(\frac{x^2 - n\sigma^2}{\sqrt{n}}\Big)$$

on Π_q are root-Lipschitz with constants $n^{-1/2}L$. Therefore, by Proposition 3.7 and the assumptions of the theorem,

$$\Big| \int_{\Pi_q} f_n \, d\nu^{(n,*\mu_n)} - \int_{\Pi_q} f_n \, d\nu^{(n,\bullet)} \Big| \leq CL \frac{n}{\sqrt{\mu}_n} \to 0 \tag{3.3}$$

for $n \to \infty$ with

$$\int_{\Pi_q} f_n \, d\nu^{(n,*\mu_n)} = \int_{\Pi_q} f_n \, dP_{S_n^{\mu_n}} = \int_{H_q} f dP_{((S_n^{\mu_n})^2 - n\sigma^2)/\sqrt{n}}, \tag{3.4}$$

where P_X denotes the law of a random variable X. Moreover, as the square mapping $Q(x) := x^2$ is a isomorphism from the semigroup $(\Pi_q, \bullet)$ onto the semigroup $(\Pi_q, +)$, and denoting the classical convolution of measures on $\Pi_q \subset H_q$ associated with the operation $+$ by $*$, we see that

$$\int_{\Pi_q} f_n \, d\nu^{(n,\bullet)} = \int_{\Pi_q} f\Big(\frac{x^2 - n\sigma^2}{\sqrt{n}}\Big) d\nu^{(n,\bullet)}(x)$$

$$= \int_{\Pi_q} f\Big(\frac{x - n\sigma^2}{\sqrt{n}}\Big) dQ(\nu^{(n,\bullet)})(x)$$

$$= \int_{\Pi_q} f\Big(\frac{x - n\sigma^2}{\sqrt{n}}\Big) d(Q(\nu)^{(n,*)})(x) \tag{3.5}$$

where the latter tends by the classical central limit theorem on the euclidean space H_q to $\int_{H_q} f \, dN(0, \Sigma^2)$. Taking (3.3) and (3.4) into account, we obtain that

$$\int_{H_q} f dP_{((S_n^{\mu_n})^2 - n\sigma^2)/\sqrt{n}} \to \int_{H_q} f \, dN(0, \Sigma^2) \tag{3.6}$$

for $n \to \infty$. As the space of Lipschitz continuous functions in $C_0(H_q)$ is $\|.\|_\infty$-dense in $C_0(H_q)$, a simple ε-argument together with the triangle inequality ensures that Eq. (3.6) holds for all $f \in C_0(H_q)$. This completes the proof. $\qquad \square$

We briefly consider the case of point measures $\nu = \delta_x$. In this case we have $\Sigma^2 = 0$ and we do not need to apply the classical CLT above. In particular, the first part of the preceding proof leads to the following weak law:

3.9 Corollary. *Let $(a_n)_{n \geq 1} \subset]0, \infty[$ an increasing sequence with $a_n = o\big(\frac{n^{3/2}}{\mu^{1/2}}\big)$ for $n \to \infty$. If $\nu = \delta_x$ for some $x \in \Pi_q$, then*

$$\frac{(S_n^{\mu_n})^2 - nx^2}{a_n} \to 0 \qquad in \ probability.$$

In particular, for $a_n := n$ and $n/\mu_n \to 0$, we obtain $S_n^{\mu_n}/\sqrt{n} \to x$. We note that this particular result was derived under much weaker conditions on the μ_n in [RV] by different methods.

We finally note that a similar CLT is derived in [V3] for radial random walks on the hyperbolic spaces when time and dimension tend to infinity similar as above.

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