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Preprint 2008-24 Dezember 2008

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Symmetry of functions and exchangeability of random variables

Karl Friedrich Siburg · **Pavel A. Stoimenov**

Abstract We present a new approach for measuring the degree of exchangeability of two continuous, identically distributed random variables or, equivalently, the degree of symmetry of their corresponding copula.

While the opposite of exchangeability does not exist in probability theory, the contrary of symmetry is quite obvious from an analytical point of view. Therefore, leaving the framework of probability theory, we introduce a natural measure of symmetry for bivariate functions in an arbitrary normed function space. Restricted to the set of copulas this yields a general concept for measures of (non-)exchangeability of random variables. The fact that copulas are never antisymmetric leads to the notion of maximal degree of antisymmetry of copulas.

We illustrate our approach by various norms on function spaces, most notably the Sobolev norm for copulas.

Keywords copula · exchangeability · symmetry · Sobolev space

Mathematics Subject Classification (2000) 60B10 · 60E05 · 62H05

1 Introduction

The concept of exchangeability has been studied extensively and has found important applications in many areas of statistics; see the survey in Galambos (1982). Recently, attention has been devoted to the ways in which identically distributed random variables fail to be exchangeable (Durante et al, 2008; Durante, 2008; Nelsen, 2007; Klement and Mesiar, 2006). Recall that two random variables X and Y are called exchangeable if (X,Y) and (Y,X) have the same distribution, i.e. if their joint distribution function $F_{X,Y}$ is symmetric:

$$F_{X,Y}(x,y) = F_{X,Y}(y,x).$$

Exchangeable random variables are necessarily identically distributed.

An alternative approach to characterise exchangeability is provided by the theory of copulas. In particular, Sklar's theorem (Sklar, 1959) states that for real-valued random variables X and Y with joint distribution function $F_{X,Y}$ and univariate margins F_X and F_Y, respectively, there exists a copula $C_{X,Y}$ such that

$$F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y)). \tag{1.1}$$

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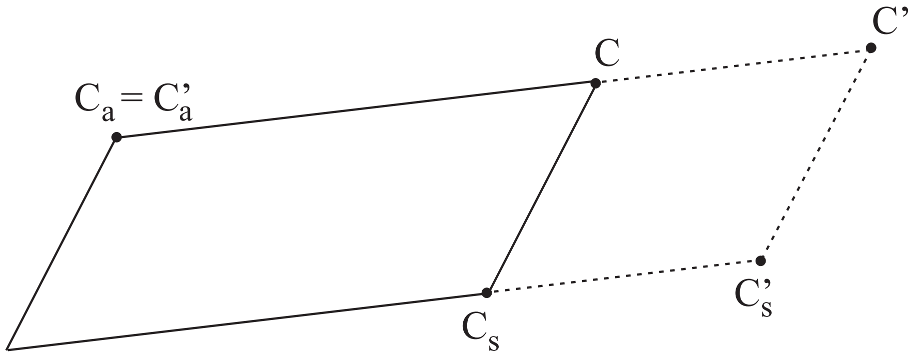


Fig. 1.1 The decomposition of copulas into symmetric and antisymmetric part

Moreover, if F_X and F_Y are continuous the function $C_{X,Y}$ is unique and will be referred to as the copula of (X,Y). It follows that, for continuous identically distributed random variables X and Y, exchangeability is equivalent to the symmetry of their copula $C = C_{X,Y}$:

$$C = C^\top$$

where $f^\top(x,y) = f(y,x)$ is the transpose of a function f.

In order to measure the degree of non-exchangeability of random variables, respectively, the degree of non-symmetry of copulas, one usually equips the set $\mathfrak{C}$ of copulas with the L^p-norm $\|\cdot\|_p$ where $p \in [1,\infty]$ and defines the quantity

$$\mu_p(C) = \|C - C^\top\|_p$$

as a measure of non-exchangeability of X and Y (Klement and Mesiar, 2006; Nelsen, 2007; Durante et al, 2008). In other words, $\mu_p(C)$ measures the norm of the antisymmetric part of C which, for a general function f, is defined by

$$f_a = \frac{(f - f^\top)}{2}.$$

However, it completely ignores the symmetric part

$$f_s = \frac{(f + f^\top)}{2}$$

which also carries essential information and, hence, must be taken into account when dealing with symmetry properties of functions. For instance, consider the decomposition of two copulas C and C' as given in Figure 1.1. Although they both possess the same antisymmetric part, it is obvious that C' is more symmetric than C.

Therefore, we suggest a more general approach to the investigation of symmetry properties of copulas. Leaving the class $\mathfrak{C}$, we consider arbitrary functions $f : I^2 \to \mathbb{R}$ and decompose them into their symmetric and antisymmetric parts. Then, a natural measure of symmetry is given by

$$\delta(f) = \frac{\|f_s\|^2 - \|f_a\|^2}{\|f\|^2}$$

where $\|\cdot\|$ is any norm on some function space V containing the set $\mathfrak{C}$ of copulas. The normalization by $\|f\|^2$ is enforced by the fact that any measure of symmetry should be scale invariant; as a consequence, we always have $-1 \le \delta(f) \le 1$. Note that this is in contrast to the normalization by $\max_{C\in\mathfrak{C}} \mu_p(C)$ proposed in Durante et al (2008) which assumes the compactness of $\mathfrak{C}$ and may be impossible to compute. Finally, the restriction of δ to $\mathfrak{C}$ yields a measure of symmetry for copulas, respectively, a measure of exchangeability for random variables. This approach has various advantages:

(i) Decomposing a copula C into its symmetric and antisymmetric function parts yields much more information than just a single number.
(ii) In contrast to μ_p, the measure δ incorporates the symmetric as well as the antisymmetric part of a copula. Consequently, it is able to tell whether, and even how much, a copula is more symmetric than antisymmetric.
(iii) The definition of δ is not restricted to L^p-norms; it is even well defined for norms with respect to which the set of copulas is not compact.
(iv) Finally, the measure δ is scale invariant by construction, regardless of the chosen norm.

We point out that, although the opposite of exchangeability does not exist in probability theory, the opposite of symmetry is well defined. Moreover, since copulas are never antisymmetric, we have $\delta(C) > -1$ for all $C \in \mathfrak{C}$, and we call the number

$$\alpha = \inf_{C \in \mathfrak{C}} \delta(C)$$

the maximal degree of antisymmetry of copulas (with respect to the given norm $\|\cdot\|$). Any copula C with $\delta(C) = \alpha$ will be called maximally antisymmetric. Note that, in general, it is not clear whether maximally antisymmetric copulas exist.

There is one distinguished norm that has turned out to be perfectly fit for the investigation of probabilistic properties of copulas. This norm is induced by a modified Sobolev scalar product on the linear span of $\mathfrak{C}$, and its resulting geometry reflects probabilistic features of the underlying random variables in a surprising and very precise way; see Siburg and Stoimenov (2007, 2008).

The paper is organized as follows. Section 2 recalls the concepts of symmetry and antisymmetry for functions and introduces a natural measure for the degree of symmetry of a given function. Section 3 applies this to the set of copulas and introduces the notion of maximally antisymmetric copulas. In addition, several examples of normed function spaces containing the set of copulas are presented. In Section 4, we investigate the particular case of the Sobolev norm and give a characterization of maximally antisymmetric copulas in geometric, as well as probabilistic, terms.

2 Measures of symmetry for functions—the analytical viewpoint

This section discusses the symmetry properties of an arbitrary function $f : I^2 \to \mathbb{R}$ where $I = [0,1]$ is the closed unit interval. In this general context, it is obvious what the opposite of symmetric means.

Definition 2.1 For any given function $f : I^2 \to \mathbb{R}$ the transposed function $f^\top : I^2 \to \mathbb{R}$ is defined by

$$f^\top(x,y) = f(y,x).$$

A function $f : I^2 \to \mathbb{R}$ is symmetric if $f = f^\top$, and antisymmetric if $f = -f^\top$. For a given $f : I^2 \to \mathbb{R}$ we call

$$f_s = \frac{f + f^\top}{2} \quad \text{and} \quad f_a = \frac{f - f^\top}{2}$$

its symmetric, respectively antisymmetric, part.

With these definitions we have the following decomposition:

$$f = f_s + f_a = (f - f_s) + (f - f_a). \tag{2.1}$$

Moreover, a function f is symmetric if, and only if, $f = f_s$, and antisymmetric if, and only if, $f = f_a$.

Let $(V, \|\cdot\|)$ be any normed vector space of functions $f : I^2 \to \mathbb{R}$. In view of the decomposition (2.1), there are two "points of reference" for the investigation of symmetry properties of a given function $f \in V$. In order to measure the degree of antisymmetry and symmetry of f, one would like to construct a functional δ which compares $\|f_s\|$ and $\|f_a\|$, and which takes its minimal, respectively maximal, value precisely for the antisymmetric, respectively symmetric, functions. Moreover, symmetry properties are scale invariant, i.e., the measure δ should satisfy the invariance condition $\delta(\lambda f) = \delta(f)$ for every $\lambda \neq 0$.

From this analytical viewpoint, the following measure of symmetry on $(V, \|\cdot\|)$ is natural. We point out that the zero function is symmetric as well as antisymmetric, so we cannot define the measure of symmetry for $f = 0$.

Definition 2.2 Let $(V, \|\cdot\|)$ be a normed vector space of functions $f : I^2 \to \mathbb{R}$. Then the measure of symmetry of $(V, \|\cdot\|)$ is defined as the functional $\delta : V \setminus \{0\} \to \mathbb{R}$ with

$$\delta(f) = \frac{\|f_s\|^2 - \|f_a\|^2}{\|f\|^2}$$

where $f = f_s + f_a$ is the decomposition of f into its symmetric and antisymmetric part.

Note that, in view of (2.1), we can also write

$$\delta(f) = \frac{\|f - f_a\|^2 - \|f - f_s\|^2}{\|f\|^2} = \frac{\|f + f^\top\|^2 - \|f - f^\top\|^2}{4\|f\|^2}. \tag{2.2}$$

If the norm $\|\cdot\|$ on V is induced by a scalar product $\langle\cdot,\cdot\rangle$ then the measure of symmetry δ can be represented as

$$\delta(f) = \frac{\langle f, f_s - f_a\rangle}{\|f\|^2} = \frac{\langle f, f^\top\rangle}{\|f\|^2}. \tag{2.3}$$

If, in addition, the norm is invariant under transposition, i.e., $\|f^\top\| = \|f\|$, then

$$\delta(f) = \Big\langle \frac{f}{\|f\|}, \frac{f^\top}{\|f^\top\|} \Big\rangle \tag{2.4}$$

measures the (cosine of the) angle between the unit vectors pointing in the direction of f and $f^\top$, respectively. Norms that come from a scalar product and are invariant under transposition, are characterized by the following result.

Proposition 2.3 *Let $\|\cdot\|$ be a norm on V satisfying $\|f^\top\| = \|f\|$ for all $f \in V$. Then $\|\cdot\|$ is induced by a scalar product if, and only if,*

$$\|f\|^2 = \|f_s\|^2 + \|f_a\|^2$$

for all $f \in V$.

Proof It is well known that $\|\cdot\|$ is induced by a scalar product if, and only if, the "parallelogram law"

$$\|g+h\|^2 + \|g-h\|^2 = 2(\|g\|^2 + \|h\|^2)$$

holds for all $g, h \in V$; see, e.g., Hewitt and Stromberg (1975, (16.6)). Now apply this to $g = f_s$ and $h = f_a$, and observe that $f_s - f_a = f^\top$. □

Let us return to the general situation where $(V, \|\cdot\|)$ is any normed vector space of functions $f : I^2 \to \mathbb{R}$.

Theorem 2.4 *The measure of symmetry $\delta : (V \setminus \{0\}, \|\cdot\|) \to (\mathbb{R}, |\cdot|)$ satisfies the following properties:*

(i) $\delta(V \setminus \{0\}) = [-1, 1]$.
(ii) $\delta(f) = -1$ *if and only if f is antisymmetric.*
(iii) $\delta(f) = 1$ *if and only if f is symmetric.*
(iv) δ *is continuous.*
(v) $\delta(\lambda f) = \delta(f)$ *for every* $\lambda \neq 0$.

Proof Properties (ii)–(v) follow immediately from the definition. The inclusion $\delta(V \setminus \{0\}) \subset [-1, 1]$ in (i) is obvious as well. As for the reverse inclusion, note that δ is a continuous mapping on the connected set $V \setminus \{0\}$ that attains the values -1 and 1. Thus, the Intermediate Value Theorem implies $[-1, 1] \subset \delta(V \setminus \{0\})$, proving (i). □

In addition to the transposition $f \mapsto f^\top$, there is another operation on functions from I^2 to $\mathbb{R}$ given by

$$\widehat{f}(x,y) = x + y - 1 + f(1-x, 1-y) \tag{2.5}$$

whose motivation stems from probabilistic considerations; see Section 3.

Proposition 2.5 *Both operations $f \mapsto f^\top$ and $f \mapsto \widehat{f}$ are involutions, i.e.*

$$(f^\top)^\top = f \quad \text{and} \quad \widehat{\widehat{f}} = f$$

for all $f \in V$.

Proof This is verified by straightforward calculations. □

Next we investigate under which conditions the measure of symmetry δ is invariant under these operations. For the following result, let $i : I^2 \to I^2$ be the involution

$$i(x,y) = (1-x, 1-y).$$

Theorem 2.6 *Let δ be the measure of symmetry of $(V, \|\cdot\|)$. Then the following holds:*

(i) $\delta(f^\top) = \delta(f)$ *if, and only if,* $\|f^\top\| = \|f\|$.
(ii) $\delta(\widehat{f}) = \delta(f)$ *if* $\|\widehat{f}\| = \|f \circ i\| = \|f\|$.

Proof The first assertion follows from

$$(f^\top)_s = f_s \quad \text{and} \quad (f^\top)_a = -f_a. \tag{2.6}$$

Moreover, it is an easy calculation to show that

$$(\widehat{f})_s = \widehat{f_s} \quad \text{and} \quad (\widehat{f})_a = f_a \circ i, \tag{2.7}$$

which proves the second part. □

The following examples show that the invariance conditions in Theorem 2.6 are quite strong and usually not satisfied.

Example 2.7 (i) Let

$$(V, \|\cdot\|) = (L^p(I^2, \mathbb{R}), \|\cdot\|_{L^p})$$

be the usual L^p-space with $p \in [1, \infty]$. Then it is easy to check that $\|\cdot\| = \|\cdot\|_{L^p}$ satisfies the conditions $\|f^\top\| = \|f \circ i\| = \|f\|$. On the other hand, it is not true that $\|\widehat{f}\| = \|f\|$ for every $f \in L^p(I^2, \mathbb{R})$.

(ii) More generally, consider any involution $\iota : I^2 \to I^2$, and let

$$(V, \|\cdot\|) = (L^p(I^2, \mathbb{R}), \|\cdot\|_w)$$

be equipped with a weighted L^p-norm for some weight function $w > 0$ that is not invariant under ι:

$$\|f\|_w^p = \int_{I^2} w|f|^p \, d\lambda$$

where $w \circ \iota \neq w$ and λ denotes the Lebesgue measure. Then there is always an element $f \in V$ with $\|f \circ \iota\|_w \neq \|f\|_w$.

3 Measures of symmetry for copulas—the probabilistic viewpoint

Recall from the introduction that one can associate with each pair (X,Y) of continuous random variables a unique function $C_{X,Y} : I^2 \to I$, called the copula of (X,Y), such that (1.1) holds. In the following, we collect some basic properties of copulas and refer to Nelsen (2006) for proofs and more details. As for the general definition, a copula is a function $C : I^2 \to I$ satisfying the following conditions:

(i) $C(x,0) = C(0,y) = 0$ for all $x,y \in I$.
(ii) $C(x,1) = x$ and $C(1,y) = y$ for all $x,y \in I$.
(iii) $C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geq 0$ for all rectangles $[x_1,x_2] \times [y_1,y_2] \subset I^2$.

The set of all copulas will be denoted by $\mathfrak{C}$.

If $C_{X,Y}$ is the copula of (X,Y), it is easy to show that the following identities hold, the second of which motivates our earlier definition in (2.5):

$$C_{Y,X} = (C_{X,Y})^\top$$
$$C_{g(X),g(Y)} = \begin{cases} C_{X,Y} & \text{if } g \text{ is strictly increasing} \\ \widehat{C_{X,Y}} & \text{if } g \text{ is strictly decreasing} \end{cases}$$

for any strictly monotone transformation $g : \mathbb{R} \to \mathbb{R}$. In particular, the function $\widehat{C_{X,Y}}$ is also a copula, called the survival copula of (X,Y).

Definition 3.1 Let $(V, \|\cdot\|)$ be any normed vector space of functions $f : I^2 \to \mathbb{R}$ such that $\mathfrak{C} \subset V$, and let δ be its measure of symmetry. Then the restriction of δ to $\mathfrak{C}$ is called the corresponding measure of symmetry for copulas:

$$\delta(C) = \frac{\|C_s\|^2 - \|C_a\|^2}{\|C\|^2} = \frac{\|C + C^\top\|^2 - \|C - C^\top\|^2}{4\|C\|^2}. \tag{3.1}$$

Viewed as a functional $(X,Y) \mapsto \delta(C_{X,Y})$ of pairs of continuous random variables, we also call δ the corresponding measure of exchangeability for random variables.

Note that, if C is a copula, then its transposed function $C^\top$ is also a copula. Therefore, since $\mathfrak{C}$ is a convex set, the symmetric part C_s of a copula is a copula itself. The antisymmetric part C_a, however, is never a copula because $C_a = 0$ on ∂I^2.

The properties of a general measure of symmetry as given in Theorem 2.4 translate into the following properties of the corresponding measure of symmetry for copulas.

Theorem 3.2 *The measure of symmetry for copulas* $\delta : (\mathfrak{C}, \|\cdot\|) \to (\mathbb{R}, |\cdot|)$ *satisfies the following properties:*

(i) $\delta(\mathfrak{C}) \subset (-1,1]$.
(ii) $\delta(C) = 1$ *if and only if* C *is symmetric.*
(iii) δ *is continuous.*

Proof Since copulas can never be antisymmetric, it follows that $\delta(C) > -1$ for all $C \in \mathfrak{C}$. The remaining assertions are contained in Theorem 2.4. □

This result prompts the following definition.

Definition 3.3 The number

$$\alpha = \inf_{C \in \mathfrak{C}} \delta(C)$$

is called the maximal degree of antisymmetry of copulas with respect to $(V, \|\cdot\|)$. A copula C is called maximally antisymmetric if $\delta(C) = \alpha$.

Note that, in general, $\mathfrak{C}$ need not be compact with respect to $\|\cdot\|$ (Darsow and Olsen, 1995). Consequently, it may be not clear whether there are maximally antisymmetric copulas at all.

We conclude this section with examples of vector spaces $(V, \|\cdot\|)$ containing the set $\mathfrak{C}$.

Example 3.4 It is well known that copulas are Lipschitz continuous L^p-functions for any $p \in [1,\infty]$, so that we may choose

$$(V, \|\cdot\|) = (L^p(I^2,\mathbb{R}), \|\cdot\|_{L^p})$$

with $p \in [1,\infty]$. The classical situation deals with the case $p = \infty$ where

$$\|\cdot\| = \|\cdot\|_\infty = \|\cdot\|_{L^\infty}$$

on $\mathfrak{C}$ is the maximum norm (Durante et al, 2008; Klement and Mesiar, 2006; Nelsen, 2007). For this particular choice, we immediately see that $\|C\|_\infty = 1$ and $\|C + C^\top\|_\infty = 2$ for every $C \in \mathfrak{C}$, so that we have

$$\delta_\infty(C) = 1 - \left(\frac{\|C - C^\top\|_\infty}{2}\right)^2. \tag{3.2}$$

Therefore, the measure of symmetry δ_∞ is actually a function of the distance $\|C - C^\top\|_\infty$ alone. Consequently, for the maximum norm, the measure of symmetry $\delta_\infty(C)$ for copulas and the quantity $\|C - C^\top\|_\infty$ essentially share the same properties.

As for the maximal degree of antisymmetry, Nelsen (2007) showed that $\|C - C^\top\|_\infty \le 1/3$ as well as $\|C_{2/3} - C_{2/3}^\top\|_\infty = 1/3$ where C_θ is the copula given by

$$C_\theta = \min(x, y, \max(x - \theta, 0) + \max(y - (1 - \theta), 0)) \tag{3.3}$$

for $0 \le \theta \le 1$. C_θ is the singular copula whose support consists of two line segments, one from $(0, 1-\theta)$ to $(\theta, 1)$, and the other from $(\theta, 0)$ to $(1, 1-\theta)$. Hence, in view of (3.2), we have

$$\alpha_\infty = \frac{35}{36} \tag{3.4}$$

and $C_{2/3}$ is a maximally antisymmetric copula. Actually, Nelsen gave a complete description of maximally antisymmetric copulas in this setting, and proved that all maximally antisymmetric copula agree on the diagonal in I^2; compare Theorem 4.7.

Related results were obtained by Klement and Mesiar (2006). Finally, Durante et al (2008) computed that $\|C_\theta - C_\theta^\top\|_\infty = 1 - \theta$ for every $\theta \in [2/3, 1]$.

Example 3.5 In addition to L^p-spaces, there is another natural class of function spaces containing the set of copulas. Since $C^{0,1}(I^2,\mathbb{R}) = W^{1,\infty}(I^2,\mathbb{R})$ (Evans, 1998) we have $\mathfrak{C} \subset W^{1,p}(I^2,\mathbb{R})$ for $p \in [1,\infty]$ where $W^{1,p}(I^2,\mathbb{R})$ is the standard Sobolev space; compare also Darsow and Olsen (1995). Therefore, we may consider

$$(V, \|\cdot\|) = (W^{1,p}(I^2,\mathbb{R}), \|\cdot\|_{W^{1,p}})$$

with $p \in [1,\infty]$.

Among these spaces, the Sobolev space $W^{1,2}(I^2,\mathbb{R})$ is a Hilbert space with respect to the standard Sobolev scalar product

$$\langle f, g\rangle_{W^{1,2}} = \int_{I^2} fg\, d\lambda + \int_{I^2} \nabla f \cdot \nabla g\, d\lambda$$

with λ being the Lebesgue measure. On the linear span of $\mathfrak{C}$, which will be denoted by span($\mathfrak{C}$), there is also the (modified) Sobolev scalar product defined by

$$\langle f, g\rangle_0 = \int_{I^2} \nabla f \cdot \nabla g\, d\lambda \tag{3.5}$$

with induced norm

$$\|f\|_0 = \|\nabla f\|_{L^2} = \left(\int_{I^2} |\nabla f|^2\, d\lambda\right)^{1/2}. \tag{3.6}$$

Actually, in view of Poincaré's inequality, the norms $\|\cdot\|_0$ and $\|\cdot\|_{W^{1,2}}$ are equivalent on span($\mathfrak{C}$). In contrast to the standard Sobolev norm, the modified Sobolev norm $\|\cdot\|_0$ has distinguished geometric and probabilistic properties. We are not going to dwell on this here, but refer the reader to Siburg and Stoimenov (2007, 2008) where the geometry of $\mathfrak{C}$ with respect to $\langle\ ,\ \rangle_0$, as well as its probabilistic interpretations, is described in detail. Thus, another natural choice for $(V,\|\cdot\|)$ is given by

$$(V,\|\cdot\|) = (\mathrm{span}(\mathfrak{C}),\|\cdot\|_0).$$

The corresponding measure of symmetry for copulas, δ_0, is discussed in the next section.

4 The Sobolev measure of symmetry for copulas

This section considers the case where the norm $\|\cdot\|_0$ on span($\mathfrak{C}$) is the modified Sobolev norm given by (3.6), coming from the scalar product (3.5) for copulas introduced in **?**. Let δ_0 be the measure of symmetry with respect to $\|\cdot\|_0$. Note that the norm $\|\cdot\|_0$ satisfies $\|C^\top\|_0 = \|C\|_0$ for all $C \in \mathfrak{C}$. Hence, in view of (2.4), the measure δ_0 can be written as

$$\delta_0(C) = \frac{\langle C, C^\top\rangle_0}{\|C\|_0^2} = \left\langle \frac{C}{\|C\|_0}, \frac{C^\top}{\|C^\top\|_0} \right\rangle_0 \tag{4.1}$$

which has a distinctive geometric interpretation.

In the following, we denote by

$$C^-(x,y) = \max(x+y-1,0) \quad , \quad C^+(x,y) = \min(x,y)$$

the lower and upper Fréchet-Hoeffding bound, respectively, and by

$$P(x,y) = xy$$

the product copula modelling stochastic independence. C^- and C^+ are the minimal and maximal copula, respectively, since we have

$$C^-(x,y) \le C(x,y) \le C^+(x,y) \tag{4.2}$$

for all $(x,y) \in I^2$ and any $C \in \mathfrak{C}$. We refer to Nelsen (2006) for proofs and more details.

The set $\mathfrak{C}$ carries an algebraic structure discovered by Darsow et al (1992). More precisely, they introduced a product for copulas by setting

$$(A*B)(x,y) = \int_0^1 \partial_2 A(x,t)\partial_1 B(t,y)\,dt\,,$$

where ∂_i denotes the partial derivative with respect to the i-th variable. Darsow et al. proved that $A*B \in \mathfrak{C}$ whenever $A,B \in \mathfrak{C}$. Furthermore,

$$P*C = C*P = P \quad \text{and} \quad C^+ * C = C * C^+ = C \tag{4.3}$$

for all $C \in \mathfrak{C}$; thus, P and C^+ are the null, respectively unit, element for the $*$-product.

It turns out that the Sobolev scalar product $\langle\cdot,\cdot\rangle_0$ allows an algebraic interpretation. In fact, it is an elementary calculation to show that the Sobolev scalar product is related to the $*$-product via the following representation formula which relates the geometric and the algebraic structures on the set of copulas.

Lemma 4.1 (Siburg and Stoimenov (2008)) *We have*

$$\langle A,B\rangle_0 = \int_0^1 (A^\top * B + A * B^\top)(t,t)\,dt.$$

Furthermore, we need the following facts about the scalar product of two copulas.

Lemma 4.2 **(Siburg and Stoimenov (2008))** *For any A,B in $\mathfrak{C}$, we have*

$$\frac{1}{2} \leq \langle A,B\rangle_0 \leq 1$$

where both bounds are sharp. Moreover, $\langle A,B\rangle_0 = 1$ if, and only if, $A = B$ and $\|A\|_0 = \|B\|_0 = 1$.

Recall from Definition 3.3 that the maximal degree of antisymmetry of copulas is given by $\alpha = \inf_{C\in\mathfrak{C}} \delta(C)$. For the special case of the Sobolev norm $\|\cdot\|_0$ we obtain the following result.

Theorem 4.3 *The maximal degree of antisymmetry with respect to $\|\cdot\|_0$ is*

$$\alpha_0 = \frac{1}{2}.$$

Proof In view of (4.1) and Lemma 4.2 we have

$$\alpha_0 = \inf_{C\in\mathfrak{C}} \delta_0(C) \geq \frac{1}{2}. \tag{4.4}$$

In order to show that this estimate is sharp we provide an example of a copula C that satisfies $\delta_0(C) = 1/2$. For this, consider the copula C_θ from (3.3) with $0 \leq \theta \leq 1$. In order to compute $\delta_0(C_\theta)$ we need to calculate the scalar product $\langle C_\theta, C_\theta^\top\rangle_0$ and the norm $\|C_\theta\|_0$. To do so, we have depicted in Figure 4.1 the gradients ∇C_θ and $\nabla C_\theta^\top$, as well as their Euclidean scalar product $\nabla C_\theta \cdot \nabla C_\theta^\top$. Integrating $|\nabla C_\theta|^2$ and $\nabla C_\theta \cdot \nabla C_\theta^\top$ yields $\|C_\theta\|_0^2 = 1$ as well as

$$\langle C_\theta, C_\theta^\top\rangle_0 = \theta + \theta(1-\theta) + (1-3\theta)^2 = 8\theta^2 - 4\theta + 1. \tag{4.5}$$

Therefore, $\delta_0(C_\theta) = \langle C_\theta, C_\theta^\top\rangle_0 = 8\theta^2 - 4\theta + 1$, and this function takes its minimal value $1/2$ precisely at the point $\theta = 1/4$. Hence the copula $C_{1/4}$ satisfies

$$\delta_0(C_{1/4}) = \frac{1}{2},$$

which completes the proof of the theorem. □

As mentioned before, the norm $\|\cdot\|_0$ establishes a strong link between the geometry and probabilistic aspects of copulas (Siburg and Stoimenov, 2008). In particular, this norm detects the so-called mutual complete dependence of random variables.

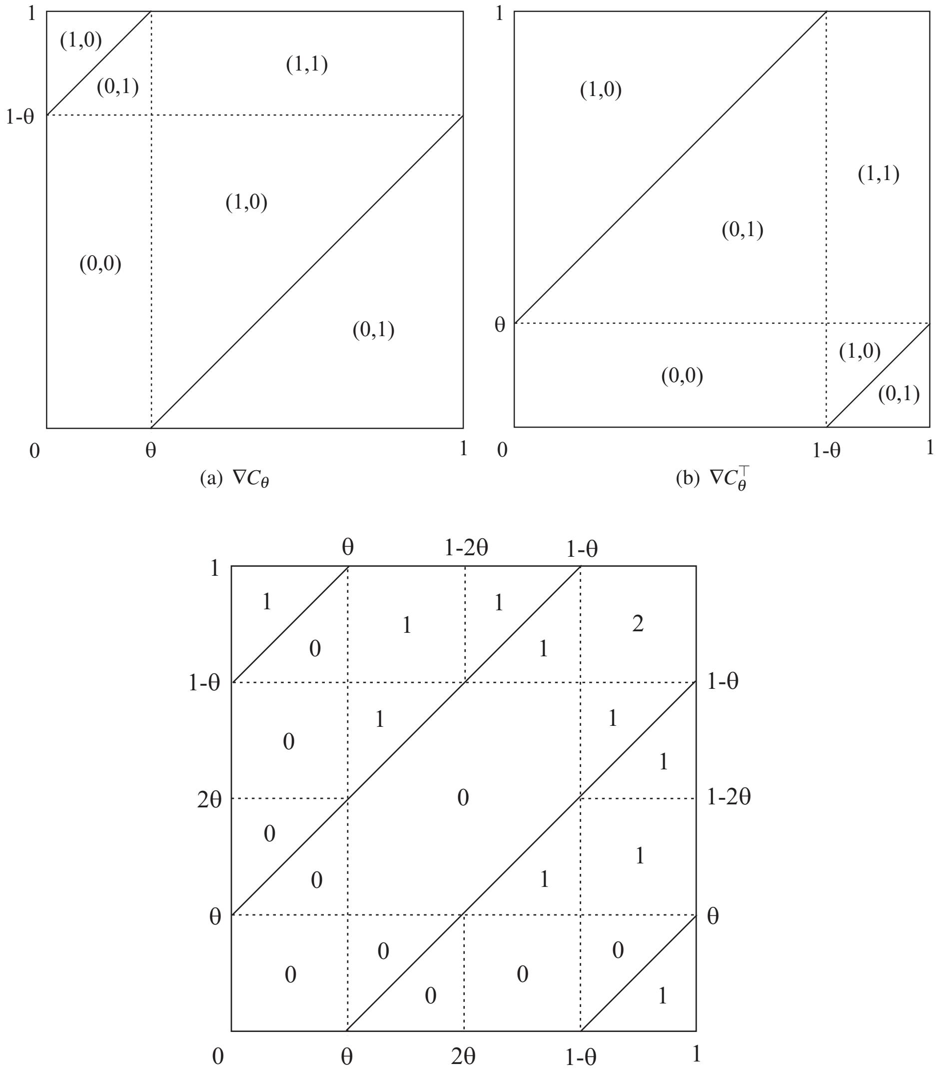
Definition 4.4 Two random variables X and Y are called mutually completely dependent if there are Borel measurable functions g,h such that $Y = g(X)$ a.e. and $X = h(Y)$ a.e.

The next result provides a complete characterization of maximally antisymmetric copulas in terms of geometric, as well as probabilistic, properties. This is another manifestation of the exceptional role of the norm $\|\cdot\|_0$.

Theorem 4.5 *Let $C = C_{X,Y}$ be the copula of two random variables X and Y. Then the following assertions are equivalent:*

(i) *C is maximally antisymmetric with respect to $\|\cdot\|_0$.*
(ii) *$\|C\|_0 = 1$, and $\langle C, C^\top\rangle_0 = 1/2$.*
(iii) *C and $C^\top$ realize the diameter of $\mathfrak{C}$, i.e., $\|C - C^\top\|_0 = diam(\mathfrak{C})$.*
(iv) *X and Y are mutually completely dependent, and $\langle C, C^\top\rangle_0 = 1/2$.*

Proof It is clear from (4.1) and Lemma 4.2 that $(i) \Leftrightarrow (ii)$. It follows from Lemma 4.2 that $(ii) \Leftrightarrow (iii)$; see Siburg and Stoimenov (2008, Cor. 15). Finally, $(ii) \Leftrightarrow (iv)$ follows from the fact that $\|C\|_0 = 1$ is equivalent to X and Y being mutually completely dependent (Siburg and Stoimenov, 2007, Thm. 9). □



(a) ∇C_θ

(b) $\nabla C_\theta^\top$

(c) $\nabla C_\theta \cdot \nabla C_\theta^\top$

Fig. 4.1 Calculating the scalar product $\langle C_\theta, C_\theta^\top \rangle_0$ in (4.5)

Note that Theorem 4.5 implies the following probabilistic consequence of maximal non-exchangeability with respect to the Sobolev norm $\|\cdot\|_0$.

Corollary 4.6 *Maximally non-exchangeable random variables with respect to* $\|\cdot\|_0$ *are mutually completely dependent.*

Finally, we will derive a result analogous to that of Nelsen who showed that all maximally antisymmetric copulas with respect to the maximum norm $\|\cdot\|_\infty$ coincide on the diagonal; see Nelsen (2007, Thm. 3.1).

Theorem 4.7 *If* $\langle C, C^\top \rangle_0 = 1/2$ *then*

$$(C * C)(t,t) = C^-(t,t)$$

for all $t \in I$.

Proof If $\langle C, C^\top \rangle_0 = 1/2$ then the representation formula (4.1) implies

$$\frac{1}{2} = \langle C, C^\top \rangle_0 = \int_0^1 (C^\top * C^\top + C * C)(t,t)\,dt = 2\int_0^1 (C * C)(t,t)\,dt$$

so that $C * C$ and C^- have the same mean value along the diagonal:

$$\int_0^1 (C * C)(t,t)\,dt = \frac{1}{4} = \int_0^1 C^-(t,t)\,dt.$$

But the lower Fréchet-Hoeffding bound satisfies $C \geq C^-$ for any copula C, in particular $C * C \geq C^-$, so that $C * C$ and C^- have to agree on the diagonal. □

As a consequence, we obtain the following analog to Nelsen's result.

Corollary 4.8 *If C is maximally antisymmetric with respect to $\|\cdot\|_0$ then $C * C$ agrees with C^- along the diagonal.*

Proof This follows immediately from Theorem 4.5 and Theorem 4.7. □

We end this paper with some remarks concerning possible sample versions for the above measures δ_∞ and δ_0. Whereas $\delta_\infty(C)$ can easily be estimated from a sample by using the empirical copula, the estimation of $\delta_0(C)$ is a much more difficult task since it involves the estimation of the partial derivatives of C. This requires more advanced techniques using kernel density estimators and will be pursued in future investigations.

References

Darsow WF, Olsen ET (1995) Norms for copulas. Internat J Math Math Sci 18(3):417–436
Darsow WF, Nguyen B, Olsen ET (1992) Copulas and Markov processes. Illinois J Math 36(4):600–642
Durante F (2008) Construction of non-exchangeable bivariate distribution functions. Statist Papers
Durante F, Klement EP, Sempi C, Úbeda-Flores M (2008) Measures of non-exchangeability for bivariate random vectors. Statist Papers
Evans LC (1998) Partial differential equations. American Mathematical Society, Providence, RI
Galambos J (1982) Exchangeability. In: Kotz S, Johnson NL (eds) Encyclopedia of Statistical Sciences, John Wiley & Sons, New York, pp 573–577
Hewitt E, Stromberg K (1975) Real and abstract analysis. Springer-Verlag, New York
Klement EP, Mesiar R (2006) How non-symmetric can a copula be? Comment Math Univ Carolin 47(1):141–148
Nelsen RB (2006) An introduction to copulas, 2nd edn. Springer, New York
Nelsen RB (2007) Extremes of nonexchangeability. Statist Papers 48(2):329–336
Siburg KF, Stoimenov PA (2007) A measure of mutual complete dependence. Technical report University of Dortmund
Siburg KF, Stoimenov PA (2008) A scalar product for copulas. J Math Anal Appl 344(1):429–439
Sklar A (1959) Fonctions de répartition à n dimensions et leurs marges. Publ Inst Statist Univ Paris 8:229–231

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