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MEHLER SEMIGROUPS, ORNSTEIN-UHLENBECK PROCESSES AND BACKGROUND DRIVING LÉVY PROCESSES ON LOCALLY COMPACT GROUPS AND ON HYPERGROUPS

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Abstract. For finite dimensional vector spaces it is well-known that there exists a 1–1-correspondence between distributions of Ornstein-Uhlenbeck type processes (w.r.t. a fixed group of automorphisms) and (background driving) Lévy processes. An analogous result could be proved for simply connected nilpotent Lie groups. Here we extend this correspondence to a class of commutative hypergroups.

Introduction

Let $\mathbb{V}$ be a $d-$dimensional real vector space and let $(T_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms. *M-semigroups (or skew semigroups)* are continuous one-parameter families of probabilities $(\mu(t))_{t\geq 0}$ on $\mathbb{V}$ satisfying $\mu(t+s) = \mu(t)\star T_t(\mu(s))$, $\forall s, t \geq 0$. These skew or M-semigroups are distributions of (generalized) Ornstein-Uhlenbeck-processes (resp. Mehler semigroups of transition kernels) and correspond in a 1-1-manner to continuous convolution semigroups, the distributions of Lévy processes (called *background driving Lévy processes*). The correspondence is expressed by path-wise random integral representations of the involved processes. See [25] for $d = 1$, [2] or [32] and the literature mentioned there. More generally, for random integrals of additive processes see [37]. It should be mentioned that limits of M-semigroups are self-decomposable laws and vice versa. For the background of self-decomposability and random integral representations on vector spaces see e.g. the monograph [26], or [39, 28, 27], furthermore, [1, 38, 37], and the literature mentioned there. For some applications of self-decomposability see e.g., [4, 29] and the references there.

For locally compact groups $\mathbb{G}$ admitting a continuous one-parameter group $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$, Ornstein-Uhlenbeck processes (or Mehler semigroups of transition kernels) resp. M-semigroups on the one side and Lévy processes resp. continuous convolution semigroups on the other, are defined verbatim as in the vector space case. In the group case – as random integral representations are in general not available – at least for contractible simply connected nilpotent Lie groups a 1-1-correspondence between M-semigroups and continuous convolution semigroups is established via Lie-Trotter product formulas

$$(LT1) \quad \mu(t) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{n-1} T_{\frac{kt}{n}}\left(\mu_{t/n}\right) \qquad (LT2) \quad \mu_t = \lim_{n\to\infty} \mu(t/n)^n$$

which may be understood as weak versions of random integral representations. See e.g., [14], §2.14, [16], Theorem C, [15]. (For a process-approach under some technical conditions see e.g., [30].)

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The proof relies (i) on the construction of (space-time-) Lévy processes resp. continuous convolution semigroups on the *space-time building* $\Gamma := \mathbb{G} \rtimes \mathbb{R}$, (ii) on the existence of common cores for generators of continuous convolution semigroups and (iii) on Lie-Trotter formulas for addition of generators of C_0-contraction semigroups. The second property, the existence of common cores, proved independently and nearly simultaneously by J. Faraut, K. Harzallah, F. Hirsch, J.P. Roth, [12, 11, 21, 22, 23, 24, 35], is crucial. See also [13, 8, 9, 19]. (In fact, for our purpose a slight generalization of this result is needed, see Theorem 1.9 b), c) below.)

As a corollary it follows that the Bruhat test functions $\mathcal{D}(\mathbb{G})$ and – for direct and semidirect extensions $\Gamma = \mathbb{G} \rtimes \mathbb{R}$ – that the subspaces $\mathcal{D}(G) \otimes \mathcal{D}(\mathbb{R}) \subseteq \mathcal{D}(\Gamma)$ are common cores for generators of continuous convolution semigroups on $\mathbb{G}$ and Γ respectively. A key result which enables e.g. to verify $(LT1)$ and $(LT2)$. (Recall that for Lie groups $\mathcal{D}(\mathbb{G})$ is just $C_c^\infty(\mathbb{G})$.)

Recently M. Rösler [36] and M. Voit [40] investigated hypergroup structures on the cone of non-negative definite $d \times d-$matrices with a *group like* behaviour. In particular, the structure of the automorphism group is well-known, a homomorphic image of $\mathrm{GL}(\mathbb{R}^d)$. In fact, for $a \in \mathrm{GL}(\mathbb{R}^d)$ there corresponds an automorphism $\mathcal{K} \ni \kappa \mapsto T_a(\kappa) := (a\kappa^2 a^*)^{1/2} \in \mathcal{K}$. In [17] some probabilistic aspects of these hypergroup structures were investigated, especially divisibility, (semi-)stability and also self-decomposability and M-semigroups. However, the problem of existence of background driving Lévy processes and the correspondence by Lie-Trotter formulas was not investigated there. This is the main target of the present investigations.

Note that a version of the above-mentioned theorem of F. Hirsch et al. for hypergroups is proved in the thesis S. Menges [33], 5.26. There also the existence of a common core for convolution semigroups on commutative hypergroups is established ([33], 5.17, 5.22). However, for non-Abelian hypergroups there is no natural candidate for a common core as e.g., $\mathcal{D}(\mathbb{G})$ for general locally compact groups. To find such function spaces on semi-direct extensions and to show a core property which allows to prove the analogues of $(LT1)$ and $(LT2)$ is a crucial tool of this investigation.

In Section 1 we collect notations and basic facts for continuous convolution semigroups and invariant C_0-contraction semigroups, including a sketch of the afore mentioned Theorem of F. Hirsch et al. (in its slightly generalized form.) In Section 2 we apply these results to the case of locally compact groups (generalizing slightly the already published results for nilpotent Lie groups). Section 3 contains the main results: Theorem 3.1 and 3.2. The proof of the first is a consequence of the results collected in Section 2, whereas Section 4 is concerned with the proof of Theorem 3.2, the hypergroup case: For a class of hypergroups containing the afore mentioned hypergroups on matrix cones the existence of background driving Lévy processes and the correspondence via the Lie-Trotter formulas is established. The proof is quite technical and sometimes cumbersome, but I was unable to find a more elegant version.

1. Notations and basic facts

Let $\mathbb{G}$ be a locally compact group or a hypergroup. (Or a locally compact semigroup with unit e and with a *nice behaviour at* ∞: for all compact $M, N \subseteq \mathbb{G}$ the set $\{z \in \mathbb{G} : \forall x \in N \; xz \text{ or } zx \in M\}$ is relatively compact.) According to the Riesz representation theorem measures $\mu \in \mathcal{M}^b(\mathbb{G})$ are identified with continuous linear functionals on $C_0(\mathbb{G})$, the dual pairing is denoted by $\int_{\mathbb{G}} f d\mu = \langle f, \mu \rangle$.

Measures are also identified with linear operators, the convolution operators acting e.g. on $C_0(\mathbb{G})$ from right resp. left:

$$R_\mu : \quad (R_\mu f)(x) := \int f d(\varepsilon_x \star \mu) = \langle f, \varepsilon_x \star \mu \rangle$$

$$L_\mu : \quad (L_\mu f)(x) := \int f d(\mu \star \varepsilon_x) = \langle f, \mu \star \varepsilon_x \rangle$$

In particular, for $\mu = \varepsilon_{x_0}$ we use the abbreviations $R_{x_0} := R_{\varepsilon_{x_0}}$ resp. $L_{x_0} := L_{\varepsilon_{x_0}}$ for the right and left translations.

We collect some well-known *properties of convolution operators* which are tacitly used in the sequel. (See e.g., [18, 13], and for hypergroups, [5].)

Proposition 1.1. ***a)*** *R_μ and L_μ are linear operators acting on $C_0(\mathbb{G})$ with $||R_\mu||_\infty = ||L_\mu||_\infty = ||\mu||_\infty$*
b) $R_\mu L_\nu = L_\nu R_\mu$ *for all* $\mu, \nu \in \mathcal{M}^b(\mathbb{G})$
c) $R_{\mu\star\nu} = R_\mu R_\nu$ *and* $L_{\mu\star\nu} = L_\nu R_\mu$ *for all* $\mu, \nu \in \mathcal{M}^b(\mathbb{G})$
d) $\langle f, \mu \star \nu \rangle = \langle R_\mu f, \nu \rangle = \langle L_\nu f, \mu \rangle \quad \forall \mu, \nu \in \mathcal{M}^b(\mathbb{G}), f \in C_0(\mathbb{G})$
In particular, for $\nu = \varepsilon_e$ *resp.* $= \varepsilon_{x_0}$
d1) $\langle f, \mu \rangle = R_\mu f(e) = L_\mu f(e) \quad \forall \mu \in \mathcal{M}^b(\mathbb{G}), f \in C_0(\mathbb{G})$
d2) $f(x_0) = \langle f, \varepsilon_{x_0} \rangle = R_{x_0} f(e) = L_{x_0} f(e) \quad \forall f \in C_0(\mathbb{G})$
d3) $R_\mu f(x_0) = \langle f, \varepsilon_{x_0} \star \mu \rangle = \langle R_\mu f, \varepsilon_{x_0} \rangle = \langle L_{x_0} f, \mu \rangle = \langle L_{x_0} R_\mu f, \varepsilon_e \rangle$ $\forall \mu, \nu \in \mathcal{M}^b(\mathbb{G}), f \in C_0(\mathbb{G}), x_0 \in \mathbb{G}$
d4) $L_\mu f(x_0) = \langle f, \mu \star \varepsilon_{x_0} \rangle = \langle L_\mu f, \varepsilon_{x_0} \rangle = \langle R_{x_0} f, \mu \rangle = \langle R_{x_0} L_\mu f, \varepsilon_e \rangle$ $\forall \mu, \nu \in \mathcal{M}^b(\mathbb{G}), f \in C_0(\mathbb{G}), x_0 \in \mathbb{G}$

Proposition 1.2. *Let $f \in C_0(\mathbb{G})$, and let $x_0 \in \mathbb{G}$ such that $|f(x_0)| = ||f||_\infty$. Then $||f||_\infty = |R_{x_0} f(e)| = ||R_{x_0} f||_\infty$*

⟦ R_{x_0} is a contraction (Proposition 1.1 a)), hence $||R_{x_0} f||_\infty \leq ||f||_\infty$. On the other hand, according to property d2) in Proposition 1.1, $|f(x_0)|$ $= |\langle R_{x_0} f, \varepsilon_e \rangle|$, whence $||f||_\infty = |f(x_0)| = |R_{x_0} f(e)| \leq ||R_{x_0} f||_\infty$ ⟧

Let $T := R_\lambda$, $\lambda \in \mathcal{M}^b(\mathbb{G})$. T is left invariant, i.e. $TL_x = L_x T \; \forall x \in \mathbb{G}$ (see Proposition 1.1) and $\langle f, \lambda \rangle = Tf(e)$, $Tf(x) = \langle L_x f, \lambda \rangle$. This is a motivation to define

Definition 1.3. *A subspace $\mathbb{D} \subseteq C_0(\mathbb{G})$ is called left invariant if $L_x \mathbb{D} \subseteq \mathbb{D}$, $\forall x \in \mathbb{G}$, and a linear operator $U : \mathbb{D} \to C_0(\mathbb{G})$ is called left invariant if $\mathbb{D}$ is left invariant and $UL_x = L_x U \; \forall x \in \mathbb{G}$. Hence $UL_\nu = L_\nu U$ for all $\nu \in \mathcal{M}^b(\mathbb{G})$ with $L_\nu(\mathbb{D}) \subseteq \mathbb{D}$.*

In this case, we define the linear functional $A : \mathbb{D} \to \mathbb{C}$ by $\langle f, A \rangle := Tf(e)$, hence (according to 1.1. d2)) $Uf(x) = L_x Uf(e) = UL_x f(e) = \langle L_x f, A \rangle$. This motivates the notation $U = R_A$ (in analogy to Proposition 1.1. d3)).

Definition 1.4. *Let $U : \mathbb{D} \to C_0(\mathbb{G})$ be a linear operator acting on a subspace $\mathbb{D} \subseteq C_0(\mathbb{G})$. U is called dissipative if for all $f \in \mathbb{D}$, for all $x_0 \in \mathbb{G}$ such that $f(x_0) = ||f||_\infty$ it follows $\Re\left(Uf(x_0)\right) \le 0$.*

Proposition 1.5. ***a)*** *Let $(T_t)_{t\ge 0}$ be a C_0-contraction semigroup on $C_0(\mathbb{G})$ with infinitesimal generator $\left(U := \frac{d^+}{dt}|_{t=0}T_t, D(U)\right)$. Then the domain $D(U)$ is dense and U is closed and dissipative. Furthermore, $(I - U)D(U) = C_0(\mathbb{G})$.*

b) *Conversely, let U be dissipative with dense domain $\mathbb{D}$. Then $(U, \mathbb{D})$ is closable, and the closure $\left(\overline{U}, \overline{\mathbb{D}}\right)$ is closed and dissipative. Furthermore, $(I - U)(\mathbb{D})$ is dense in $(I - \overline{U})(\overline{\mathbb{D}})$.*

c) *If in addition, $(I - U)(\mathbb{D})$ is dense in $C_0(\mathbb{G})$ then $\left(\overline{U}, \overline{\mathbb{D}}\right)$ is the generator of a (uniquely determined) C_0-contraction semigroup $(T_t)_{t\ge 0}$.*

In the latter case, $\mathbb{D}$ is called 'core' for the generator of $(T_t)_{t\ge 0}$.

⟦ This characterization of generators of contraction semigroups as dissipative operators is known as *Theorem of Lumer-Phillips* ([31]). ⟧

As a consequence of the Riesz representation theorem we obtain

Proposition 1.6. *A left invariant linear operator $T = R_A$ – A defined as above in 1.3 – defined on $\mathbb{D} := C_0(\mathbb{G})$ is the convolution operator of a bounded measure $A = \lambda \in \mathcal{M}^b(\mathbb{G})$, and conversely.*

In particular, a C_0-semigroup of invariant operators on $C_0(\mathbb{G})$ is representable as $(T_t = R_{\lambda_t})_{t\ge 0}$ where $(\lambda_t)_{t\ge 0}$ is a continuous convolution semigroup in $\mathcal{M}^b(\mathbb{G})$ with $\lambda_0 = \varepsilon_e$.

We adopt the following notations: $\mathcal{M}^1(\mathbb{G})$ denotes the set of probability measures and $\mathcal{M}^{(1)}(\mathbb{G}) := \left\{\lambda \in \mathcal{M}^b(\mathbb{G}) : ||\lambda|| \le 1\right\}$.

In the sequel we shall always tacitly assume for continuous convolution semigroups that $\lambda_0 = \varepsilon_e$. Let $(\lambda_t)_{t\ge 0}$, $\lambda_0 = \varepsilon_e$, be a continuous convolution semigroup in $\mathcal{M}^b(\mathbb{G})$ with corresponding C_0-operator semigroup $(T_t = R_{\lambda_t})_{t\ge 0}$. Then the infinitesimal generator $(U, D(U))$ is a left invariant operator. If moreover, $(\lambda_t) \subseteq \mathcal{M}^{(1)}(\mathbb{G})$ then $(U, D(U))$ is (left invariant and) dissipative.

In view of Popositions 1.5 and 1.6 we have:

Proposition 1.7. *Let $(U, D(U))$ be left invariant and dissipative and assume $(I - U)D(U) = C_0(\mathbb{G})$, hence U is the generator of a C_0-contraction semigroup $(T_t)_{t\ge 0}$. Then $T_t = R_{\lambda_t}$ for some continuous convolution semigroup $(\lambda_t)_{t\ge 0} \subseteq \mathcal{M}^{(1)}(\mathbb{G})$.*

⟦ For $\alpha > 0$ the resolvent $I_\alpha := (U - \frac{1}{\alpha}I)^{-1}$ is bonded, obviously left invariant, hence a convolution operator of a bounded measure. Any T_t is representable as limit of exponentials of resolvent operators, hence is itself left invariant. ⟧

Remark 1.8. *Let $\mathbb{D}$ be a core for the generator of a semigroup of convolution operators $(R_{\lambda_t})_{t\ge 0}$. Then, by a slight abuse of language, we call $\mathbb{D}$ a core for the continuous convolution semigroup $(\lambda_t)_{t\ge 0}$.*

Now we are ready to formulate the announced result of J. Faraut, K. Harzallah, F. Hirsch and J.P. Roth ([12, 11, 21, 22, 23, 24, 35]). We restrict to the case of continuous convolution semigroups with trivial idempotents $\lambda_0 = \varepsilon_e$. As mentioned in the above cited literature, the results generalize easily to continuous convolution semigroups with non-trivial idempotents λ_0. (If $\lambda_t \geq 0$ then $\lambda_0 = \omega_K$, a Haar measure on some compact sub-(hyper)group K).

Theorem 1.9. *Let $\mathbb{D}$ be a dense linear subspace of $C_0(\mathbb{G})$.*
a) *Assume (i) $L_x\mathbb{D} \subseteq \mathbb{D}\ \forall x \in \mathbb{G}$ and (ii) $R_x\mathbb{D} \subseteq \mathbb{D}\ \forall x \in \mathbb{G}$*

Let $U : \mathbb{D} \to C_0(\mathbb{G})$ be a left invariant and dissipative linear operator. Then the closure $(\overline{U}, \overline{\mathbb{D}})$ is the generator of a left invariant contraction semigroup $(T_t = R_{\lambda_t})_{t\geq 0}$. I.e., $\mathbb{D}$ is a core for the continuous convolution semigroup $(\lambda_t) \subseteq \mathcal{M}^{(1)}(\mathbb{G})$.

b) *More generally, (ii) may be replaced by (ii') $R_x\mathbb{D} \subseteq \overline{\mathbb{D}}\ \ \forall x \in \mathbb{G}$.*
c) *Let $(U, D(U))$ be a dissipative, closed and left invariant operator. Assume $\mathbb{D} \subseteq D(U)$ to be left-invariant (i), and assume furthermore (ii") $R_x\mathbb{D} \subseteq D(U)\ \forall x \in \mathbb{G}$.*

Then $(U, D(U))$ is the generator of a left invariant contraction semigroup $(R_{\lambda_t})_{t\geq 0}$ and $\widetilde{\mathbb{D}} := \operatorname{span}\{R_x\mathbb{D} : x \in \mathbb{G}\}$ is a left- and right invariant core for $(U, D(U))$ (resp. for $(\lambda_t)_{t\geq 0}$) .

The following sketch of a **proof** follows – with different notations – the lines of the proofs in [21, 22]. See also [13]. For hypergroups a proof (of a)) is contained in the thesis [33], 5.26.

Condition (ii') is weaker than (ii), hence *b)* ⇒ *a)*.
To prove b) we first note that

1. Condition (i) implies $L_\nu\mathbb{D} \subseteq \overline{\mathbb{D}}\ \forall \nu \in \mathcal{M}^b(\mathbb{G})$. In fact, approximating ν by measures ν_n with finite supports such that $L_{\nu_n} \to L_\nu$ in the strong operator topology and observing $L_{\nu_n}\mathbb{D} \subseteq \mathbb{D}$ for all n yields $L_{\nu_n}f \to L_\nu f$ for $f \in \mathbb{D}$, and furthermore, $UL_{\nu_n}f = L_{\nu_n}Uf \to L_\nu Uf$. Hence $L_\nu f \in \overline{\mathbb{D}}$ and $\overline{U}L_\nu f = L_\nu Uf$.

Analogously, $\forall g \in \overline{\mathbb{D}}$ we obtain $L_\nu g \in \overline{\mathbb{D}}$ and $\overline{U}L_\nu g = L_\nu\overline{U}g$. (This applies in particular for $f \in \mathbb{D}, g := R_{x_0}f$.)

2. Let $\nu \in ((I - U)\mathbb{D})^\perp$. Since $(I - U)\mathbb{D}$ is dense in $(I - \overline{U})\overline{\mathbb{D}}$, we have $\nu \perp (I - \overline{U})\overline{\mathbb{D}}$.

Let $f \in \mathbb{D}$, let $x_0 \in \mathbb{G}$ such that $||L_\nu f||_\infty = |L_\nu f(x_0)| = |R_{x_0}L_\nu f(e)|$, i.e., for some c with $|c| = 1$ we have $||L_\nu f||_\infty = c \cdot L_\nu f(x_0)$. W.l.o.g. we may assume $c = 1$, else replace f by $c \cdot f$.

As $g := R_{x_0}f \in \overline{\mathbb{D}}$ by assumption (ii') we have $0 = \langle (I - \overline{U})R_{x_0}f, \nu\rangle = \langle L_\nu(I - \overline{U})g, \varepsilon_e\rangle = L_\nu g(e) - \overline{U}L_\nu g(e) = R_{x_0}L_\nu f(e) - \overline{U}R_{x_0}L_\nu f(e) = ||R_{x_0}L_\nu f||_\infty - \overline{U}R_{x_0}L_\nu f(e)$. Since $||L_\nu f||_\infty = (R_{x_0}L_\nu f)(e) = ||R_{x_0}L_\nu f||_\infty$ (cf. Proposition 1.2) and $\overline{U}$ is dissipative, we have $\Re\overline{U}R_{x_0}L_\nu f(e) \leq 0$. Therefore, $||R_{x_0}L_\nu f||_\infty = 0$. According to property a) in 1.1, $||L_\nu f||_\infty = 0$ follows. Since $\mathbb{D}$ is dense in $C_0(\mathbb{G})$ we have proved $\nu = 0$.

3. Therefore, $(I - U)\mathbb{D}$ is dense in $C_0(\mathbb{G})$.

Assertion b) (and hence a)) follows by Proposition 1.7.

To prove c), put $\widetilde{\mathbb{D}} := \operatorname{span}\{R_x\mathbb{D} : x \in \mathbb{G}\}$.
Claim: $\widetilde{\mathbb{D}}$ is a core for $(U, D(U))$. Hence $(U, D(U))$ is maximal dissipative and therefore a generator.

$\llbracket$ Obviously, $\mathbb{D} \subseteq \widetilde{\mathbb{D}} \subseteq D(U)$. Hence $\widetilde{\mathbb{D}}$ is dense, by construction left and right invariant and therefore according to a), $\widetilde{\mathbb{D}}$ is a core for the closure of the restriction $(U, \widetilde{\mathbb{D}})$. Since $(U, D(U))$ is closed, we observe $\overline{\widetilde{\mathbb{D}}} \subseteq D(U)$, hence $\left(\overline{U}, \overline{\widetilde{\mathbb{D}}}\right) = (U, D(U))$ and $(I-U)D(U) = C_0(\mathbb{G})$ since $(I-U)\widetilde{\mathbb{D}}$ is dense in $C_0(\mathbb{G})$ $\rrbracket$ □

We obtain immediately the well known result:

Corollary 1.10. *Let $\mathbb{G}$ be a locally compact group. Then the Bruhat test function space $\mathcal{D}(\mathbb{G})$ is a common core for all continuous convolution semigroups $(\lambda_t)_{t\geq 0}$ in $\mathcal{M}^{(1)}(\mathbb{G})$, in particular, for continuous convolution semigroups of probabilities.*

$\llbracket$ $\mathcal{D}(\mathbb{G})$ is dense, left- and right-invariant and – according to the Lévy-Khinchin-Hunt representation – $\mathcal{D}(\mathbb{G})$ is contained in the domain of the generator of any continuous convolution semigroup. Cf. e.g., [18], 4.4.18, 4.5.8 for continuous convolution semigroups of probabilities, see e.g. [8, 9, 12, 10, 11, 13, 42, 43] for the more general case $\mathcal{M}^{(1)}(\mathbb{G})$. $\rrbracket$

Corollary 1.11. *Let $\mathbb{G}$ be an Abelian locally compact group or an Abelian hypergroup. Then the space of 'analytic vectors' $\mathcal{A} := (L^1_c(\widehat{\mathbb{G}}))^\vee$ is a common core for all continuous convolution semigroups $(\lambda_t)_{t\geq 0}$ in $\mathcal{M}^{(1)}(\mathbb{G})$. (Here L^1_c denotes the space of functions with compact support which are integrable on the dual $\widehat{\mathbb{G}}$ w.r.t. the Haar resp. Plancherel measure, and $^\vee$ denotes the inverse Fourier transform.) Analogously, $C_c(\widehat{\mathbb{G}})^\vee$ and $L^2_c(\widehat{\mathbb{G}})^\vee$ share this property.*

$\llbracket$ $\mathcal{A}$ is dense and left- and right-invariant. Furthermore, for any $f \in \mathcal{A}$ and any continuous convolution semigroup $t \mapsto R_{\lambda_t} f = (\widehat{\lambda_t} \cdot \widehat{f})^\vee = \left(e^{t\cdot\psi} \cdot \widehat{f}\right)^\vee$ (with $\psi := \log \widehat{\lambda_1}$) is analytic. Therefore in particular, f is contained in the domain of the generator. For groups a proof is found in e.g. [7], for hypergroups see [33], 5.17, 5.22. $\rrbracket$

Remark 1.12. *For later use we note that the cores $\mathcal{D}(\mathbb{G})$ and $\mathcal{A}$ constructed above in Corollary 1.10 resp. 1.11 are invariant under automorphisms of $\mathbb{G}$.*

2. Semidirect products $\Gamma = \mathbb{G} \rtimes \mathbb{R}$: The case of locally compact groups

Throughout in this Section $\mathbb{G}, \mathbb{G}_i$ denote locally compact topological groups.

First we note a further corollary to Theorem 1.9:

Corollary 2.1. *Let $\mathbb{G}_i,\ i = 1, 2$ be locally compact groups with test function spaces $\mathcal{D}(\mathbb{G}_1), \mathcal{D}(\mathbb{G}_2)$ respectively. Then the subspace $\mathbb{D} := \mathcal{D}(\mathbb{G}_1) \otimes \mathcal{D}(\mathbb{G}_1) \subseteq \mathcal{D}(\mathbb{G}_1 \otimes \mathbb{G}_2)$ is a common core for continuous convolution semigroups in $\mathcal{M}^{(1)}(\mathbb{G}_1 \otimes \mathbb{G}_2)$.*

⟦ On the one hand, $\mathbb{D} \subseteq \mathcal{D}(\mathbb{G}_1 \otimes \mathbb{G}_2) \subseteq D(U)$ for any generator $(U, D(U))$ of a continuous convolution semigroup as mentioned in Corollary 1.10. On the other hand, $\mathbb{D}$ satisfies the conditions (i) and (ii) of Theorem 1.9 a). ⟧

Now let $\mathbb{G}$ denote a locally compact group and let $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$ be a continuous one parameter group. The semidirect product $\Gamma = \mathbb{G} \rtimes \mathbb{R}$ is the Cartesian product $\mathbb{G} \otimes \mathbb{R}$ equipped with the group operation $(x, s)(y, t) := (xT_s(y), s + t)$. Γ is a locally compact group and hence $\mathcal{D}(\Gamma)$ is a common core for continuous convolution semigroups in $\mathcal{M}^{(1)}(\Gamma)$. First we have

Proposition 2.2. *Let $\mathbb{G}$ be a Lie group. Then $\mathbb{D} := \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R}) \subseteq \mathcal{D}(\Gamma)$ is a common core for continuous convolution semigroups in $\mathcal{M}^1(\Gamma)$*

Proof: In contrast to the above mentioned Corollary 2.1 now the proof relies on the weaker assumption (ii') in Theorem 1.9 b).

1. Left invariance (i) is obvious: For $\varphi \otimes \psi \in \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$ we have

$$L_{(y,t)}(\varphi \otimes \psi)(x, s) = \varphi(yT_t(x)) \cdot \psi(s + t) =: \varphi_1(x) \cdot \psi_1(s)$$

Hence $L_{(y,t)}(\varphi \otimes \psi) \in \mathbb{D}\ \forall (y, t) \in \Gamma$

2. Condition (ii') is fulfilled:

Let $(U = R_A, D(U))$ be the generator of $(R_{\lambda_t})_{t\geq 0}$, with a continuous convolution semigroup $(\lambda_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\Gamma)$. According to Corollary 1.10 $\mathbb{D} \subseteq \mathcal{D}(\Gamma) \subseteq D(U)$. Let $(\overline{U}, \overline{\mathbb{D}})$ denote the closure of the restriction $(U, \mathbb{D})$.

We have to show for all $(y, t) \in \Gamma$ that $R_{(y,t)}\mathbb{D} \subseteq \overline{\mathbb{D}}$. In fact,

$$R_{(y,t)}(\varphi \otimes \psi)(x, s) = \varphi(xT_s(y)) \cdot \psi(s + t)$$

We fix $\varepsilon_n > 0, \delta_n > 0, s_i^{(n)} \in \mathbb{R}, i \leq i \leq N_n$. Let $\mathrm{supp}\psi \subseteq [a, b] \subseteq \bigcup_{i=1}^{N_n}[s_i^{(n)} - \delta_n, s_i^{(n)} + \delta_n]$. Choose furthermore $\gamma_i^{(n)} \in \mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}\gamma_i^{(n)} \subseteq [s_i^{(n)} - \delta_n, s_i^{(n)} + \delta_n], 0 \leq \gamma_i^{(n)} \leq 1$ and $\sum_1^{N_n} \gamma_i^{(n)} \equiv 1$ on $[a, b]$. Put $\psi_i^{(n)} := \gamma_i^{(n)} \cdot \psi$. Let $\varepsilon_n \to 0$ and choose $\gamma_i^{(n)}$ and δ_n such that

$$||(x, s) \mapsto \sum_{i=1}^{N_n} \Big(\varphi(xT_s(y)) - \varphi(xT_{s_i^{(n)}}(y))\Big) \cdot \psi_i^{(n)}(s + t)||_{C_0^{(2)}(\Gamma)} < \varepsilon_n$$

We have

$$H(x, s) := R_{(y,t)} (\varphi \otimes \psi) (x, s) =$$
$$\sum_{i=1}^{N_n} \Big(\varphi(xT_s(y)) - \varphi(xT_{s_i^{(n)}}(y))\Big)\cdot\psi_i^{(n)}(s+t)+\sum_{i=1}^{N_n} \varphi(xT_{s_i^{(n)}}(y))\cdot\psi_i^{(n)}(s+t)$$
$$=: G_n(x, s) + F_n(x, s)$$

By construction, $||G_n||_{C_0^{(2)}(\Gamma)} \to 0$, furthermore, $F_n \in \mathbb{D}, H \in \mathcal{D}(\Gamma) \subseteq D(U)$ and $F_n \to H$ in $C_0(\Gamma)$. The Lévy-Khinchin-Hunt representation (cf. e.g., [18], 4.4.18, 4.5.8, [19], resp. [8, 12, 10, 11, 13, 42, 43]) yields that the restriction of the generator $U = R_A : C_0^{(2)}(\Gamma) \to C_0(\Gamma)$ is continuous. Whence $||UG_n||_\infty \to 0$ and $UF_n \to UH$.

Therefore we have $H_n \to H$ and $UH_n \to UH$, whence $H \in \overline{\mathbb{D}}$, as asserted.

3. Now the proof follows by Theorem 1.9 b). □

In all examples we have in mind, the underlying group is a (simply connected, nilpotent) Lie group. Nevertheless it is worth to point out that this result is true for general locally compact groups $\mathbb{G}$ which admit a continuous one-parameter group of automorphisms $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$:

Theorem 2.3. *Let $\mathbb{G}$ be a locally compact group with $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$. We define as above the semidirect extension $\Gamma = \mathbb{G} \rtimes \mathbb{R}$ and put again $\mathbb{D} := \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$.*
Let $(\lambda_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\Gamma)$ be a continuous convolution semigroup with generating functional A resp. infinitesimal generator $(U = R_A, D(U))$. Then $\mathbb{D}$ is a core for $(\lambda_t)_{t\geq 0}$ resp. for $(U = R_A, D(U))$.

We sketch a **proof:**
$\mathbb{D}$ is dense in $C_0(\Gamma)$ and $\mathbb{D} \subseteq \mathcal{D}(\Gamma) \subseteq D(U)$. As before, it follows immediately that $\mathbb{D}$ is left invariant.
Claim: $R_{(y,t)}\mathbb{D} \subseteq \overline{\mathbb{D}}$. (Again $(\overline{U}, \overline{\mathbb{D}})$ denotes the closure of the restriction $(U, \mathbb{D})$.)

As in Proposition 2.2, let $\delta_n \to 0$, let $\varphi \otimes \psi \in \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$, define as in proposition 2.2, $H := R_{(y,t)}\varphi \otimes \psi$, $\psi = \sum \psi_i^{(n)}$ and decompose as before $H = F_n + G_n$.

(T_t) is a continuous one-parameter group. The connected component $\mathbb{G}_0$ is characteristic and $\mathbb{G}/\mathbb{G}_0$ is totally disconnected. Therefore, the induced automorphisms $\overline{T}_t$ act trivially on $\mathbb{G}/\mathbb{G}_0$.

Choose an open subgroup $\mathbb{G}_1 \subseteq \mathbb{G}$ such that $\mathbb{G}_1/\mathbb{G}_0$ is compact. Then, (e.g., according to [14], 3.1.22) we have $\mathbb{G}_1 = \lim\limits_{\leftarrow} \mathbb{G}_1/K^\alpha$ with compact normal T_t-invariant subgroups K^α. Hence $\Gamma_1 := \mathbb{G}_1 \rtimes \mathbb{R}$ is an open subgroup of Γ and $\Gamma_1 = \lim\limits_{\leftarrow} \Gamma_1/L^\alpha$ with $L^\alpha = K^\alpha \otimes \{0\}$.

The Lévy-Khinchin-Hunt representation for general locally compact groups (cf. e.g., [18, 19] resp. [8, 12, 10, 11, 13, 42, 43]) yields that $A = B + \eta$ where η is a bounded measure (a Poisson generator), and B is supported by Γ_1. We have $U = R_A = R_B + R_\eta$ and, as η is bounded, $||R_\eta G_n||_\infty \to 0$, and $R_\eta F_n \to R_\eta H$.
Hence w.l.o.g. we may assume that $\Gamma = \Gamma_1$ is Lie projective.

Since $\varphi \in \mathcal{D}(\mathbb{G})$ is constant on $K^\alpha-$cosets for some K^α and all functions involved are hence left $K^\alpha-$invariant, we may assume w.l.o.g. that $\mathbb{G} = \mathbb{G}_1/K^\alpha$ resp. $\Gamma = \Gamma_1/L^\alpha$. Thus the proof is reduced to the case of Lie groups, which was proved in Proposition 2.2. □

Lie-Trotter formulas. We recall Lie-Trotter product formulas for addition of generators of C_0 semigroups and its applications to continuous convolution semigroups. For the background see e.g., P.R. Chernoff [6], 1.1, and the literature mentioned there. For continuous convolution semigroups see e.g., [13].

Proposition 2.4. ***a)*** *The sum $U + V$ of generators of C_0- contraction semigroups $(U, D(U))$ and $(V, D(V))$ defines a dissipative operator on $D(U) \cap D(V)$. If $D(U) \cap D(V)$ is a core for $\overline{U + V}$ (hence for the generator of a contraction semigroup) then the involved semigroups are related by the Lie-Trotter formula:*

$$(LT) \qquad e^{t(U+V)} = \lim_{n\to\infty} \left(e^{(t/n)U} e^{(t/n)V}\right)^n$$

(Convergence in the strong operator topology.)

b) *Applying this to continuous convolution semigroups (resp. to the corresponding convolution operators) we obtain:*
Let $(\mu_t)_{t\geq 0}$, $(\nu_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\mathbb{G})$ *be continuous convolution semigroups in on a locally compact group* $\mathbb{G}$. *Let* $\mathbb{D}$ *be a common core for all continuous convolution semigroups (e.g.,* $\mathbb{D} = \mathcal{D}(\mathbb{G})$*). Then the sum of the generators is at least defined on* $\mathbb{D}$ *and its closure generates a continuous convolution semigroup* $(\lambda_t)_{t\geq 0}$. *Furthermore, the Lie-Trotter formula for continuous convolution semigroups holds true:*

$$(LT*) \qquad \lambda_t = \lim_{n\to\infty} \left(\mu_{t/n} \star \nu_{t/n}\right)^n$$

3. The main results

In the following we consider a sub-semigroup of $\mathcal{M}^1(\Gamma)$, defined as $\mathcal{M}^1_*(\Gamma) := \{\mu \otimes \varepsilon_t : \mu \in \mathcal{M}^1(\mathbb{G}), t \in \mathbb{R}\}$. (Analogously $\mathcal{M}^{(1)}_*(\Gamma)$, $\mathcal{M}^{(1)}_{*,+}(\Gamma)$, $\mathcal{M}^b_*(\Gamma)$ etc. are defined). Recall the definition of an M-semigroup in the Introduction: A continuous family $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^1(\mathbb{G})$ is a M-semigroup iff

$$\mu(s+t) = \mu(s) \star T_s(\mu(t)) \quad \text{for all } s, t \geq 0.$$

Obviously, $(\mu(t))_{t\geq 0}$ is a M-semigroup in $\mathcal{M}^1(\mathbb{G})$ iff $(\lambda_t := \mu(t) \otimes \varepsilon_t)_{t\geq 0}$ is a continuous convolution semigroup in $\mathcal{M}^1_*(\Gamma)$. Furthermore, as immediately verified, for $f \in \mathbb{D} := \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$ the generator U of (R_{λ_t}) splits as $Uf = (W+P)f$ (resp. $Wf = (U-P)f$), with $Wf = \frac{d^+}{dt}\big|_{t=0} R_{\mu(t)\otimes\varepsilon_0} f$ and $\pm Pf = \frac{d^+}{dt}\big|_{t=0} R_{\varepsilon_e \otimes \varepsilon_{\pm t}}$. W and $\pm P$ – by construction dissipative invariant operators – are extended to generators of continuous convolution semigroups $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0}$ and $\left(p^{\pm}_t := \varepsilon_e \otimes \varepsilon_{\pm t}\right)_{t\geq 0}$ respectively. (Cf. Theorem 2.3). Therefore, the steps in Section 2 yield the following result (cf. e.g., [14]), 2.14 III, [16], Theorem C. See also e.g., [15, 3] for applications:

Theorem 3.1. *Let* $\mathbb{G}$ *be a locally compact group and* $\mathbb{T} := (T_t)_{t\geq 0} \subseteq \mathrm{Aut}(\mathbb{G})$ *a fixed continuous one-parameter group. Furthermore, let* $\Gamma := \mathbb{G} \rtimes \mathbb{R}$ *denote the semidirect extension of* $\mathbb{G}$ *defined by* $\mathbb{T}$. *Then*
a) $\mathbb{D} := \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$ *is a core for any continuous convolution semigroup of probabilities in* $M^1_*(\Gamma)$.
b) *There exists a bijection* $(\mu(t))_{t\geq 0} \leftrightarrow (\mu_t)_{t\geq 0}$ *between M-semigroups and continuous convolution semigroups, i.e., between (distributions of) Ornstein-Uhlenbeck processes and (background driving) Lévy processes. The bijection is expressed by the 'forward and backward Lie-Trotter formulas'*

$$(LT1) \quad \mu(t) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{n-1} T_{kt/n}\left(\mu_{t/n}\right) \quad (LT2) \qquad \mu_t = \lim_{n\to\infty} (\mu(t/n))^n$$

For (matrix cone-) hypergroups we shall prove in analogy to the group case:

Theorem 3.2. *Let* $\mathcal{K}$ *be a matrix cone hypergroup (investigated in* [36, 40]*) with fixed continuous one parameter group* $\mathbb{T} := (T_t)_{t\geq 0} \subseteq \mathrm{Aut}(\mathcal{K})$. *Define the semidirect hypergroup-product* $\Gamma := \mathbb{G} \rtimes \mathbb{R}$ *in canonical way.*

Then the assertions a) and b) of Theorem 3.1 hold true in this situation, where $\mathcal{D}(\mathbb{G})$ *and* $\mathbb{D}$ *have to be replaced by suitable function spaces* $\mathcal{A}$ *and* $\widetilde{\mathbb{D}}$ *(defined in the proof of Theorem 4.21 and in 4.23 below) on the hypergroups* $\mathcal{K}$ *and* Γ *respectively.*

In particular, $\widetilde{\mathbb{D}}$ is again a common core for all continuous convolution semigroups in $\mathcal{M}^1_*(\Gamma)$.

The proof of Theorem 3.1, worked out in Section 2, relied mainly on the Theorem 1.9 b). In fact, Theorem 3.1, in particular a), is well-known and was used several times – at least in the case of Lie groups – without pointing out that the original version of Theorem 1.9 a) needs a straight forward generalization (i.e. condition (ii') instead of (ii)) to handle the case of semidirect products. (See e.g. [14], §2.14, [16]). We included a proof in order to show the differences to the case of hypergroups:

The proof of Theorem 3.2 is more complicated and not straight forward. In fact, the details are quite technical, but I was unable to find a better way. The proof will be carried out in Section 4, in a series of propositions, which may be interesting in their own right. Here we sketch an **outline of the proof**:

1. Assume $(\mu(t))_{t\geq 0}$ to be a M-semigroup on $\mathcal{K}$ with corresponding space-time semigroup (λ_t) in $\mathcal{M}^1_*(\Gamma)$. Then we construct a suitable core $\mathcal{E}$ for (λ_t) such that on $\mathcal{E}$ the generator U of the convolution operators (R_{λ_t}) splits $U = W + P$, W generating a continuous convolution semigroup $(\sigma_t = \mu_t \otimes \varepsilon_0)_{t\geq 0}$ concentrated on $\mathcal{K}\otimes\{0\} \cong \mathcal{K}$, and P generates the semigroup of shifts $\big(p_t^+ := \varepsilon_{(e,t)}\big)_{t\geq 0}$. (Note that the constructed core $\mathcal{E}$ still depends on (λ_t).)

2. Then the Lie-Trotter formula (LT) (Proposition 2.4 a)) applied to $U = W + P$ yields $(LT1)$. Hence $(\mu(t))_{t\geq 0} \mapsto (\mu_t)_{t\geq 0}$ is established.

3. Conversely, let (μ_t) be a continuous convolution semigroup on a matrix cone hypergroup $\mathcal{K}$. On these hypergroups there exists a subspace $\mathcal{A}$ which is a common core for all continuous convolution semigroups on $\mathcal{K}$ and is invariant under shifts and automorphisms. (Cf. 1.11, 1.12). By means of $\mathcal{A}$ we construct a subspace $\widetilde{\mathbb{D}} \subseteq C_0(\Gamma)$ which is a common core for continuous convolution semigroups in $\mathcal{M}^1_*(\Gamma)$.

4. Furthermore, let V be the generator of $(\mu_t)_{t\geq 0}$, let $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0}$ with generator W, and let P as above, then $U = W + P$ is (the restriction to $\widetilde{\mathbb{D}}$ of) the generator of a continuous convolution semigroup $(\lambda_t = \mu(t) \otimes \varepsilon_t)_{t\geq 0} \subseteq \mathcal{M}^1_*(\Gamma)$. Applying the Lie-Trotter formulas to $U = W + P$ resp. $W = U - P$ and considering the space component, i.e., the projection to $\mathcal{K}$, we obtain $(LT1)$ and $(LT2)$ respectively.

5. Together with step 1. this yields the bijection $(\mu(t))_{t\geq 0} \leftrightarrow (\mu_t)_{t\geq 0}$ as asserted.

4. Semidirect products $\Gamma = \mathcal{K} \rtimes \mathbb{R}$: The case of matrix cone hypergroups $\mathcal{K}$

As announced in Theorem 3.2 our aim is to establish a 1-1-correspondence between M-semigroups and continuous convolution semigroups on a class of hypergroups with 'group-like behaviour': Such hypergroups on the cone of non-negative definite matrices were recently investigated, cf. [36, 40], a class of hypergroups which share many features with locally compact groups. In particular, the group of automorphisms is well known, and there exist continuous one-parameter groups of automorphisms in abundance. (See e.g. [17] for an overview of some probabilistic structures on these hypergroups, in particular, the first section contains a collection of basic properties.) In the sequel we have these examples in mind, but results and proofs depend only on particular properties of $\mathcal{K}$, thus could be generalized to larger classes of hypergroups.

Definition 4.1. *Let $\mathcal{K}$ be the cone of positive semidefinite $d \times d$-matrices endowed with a hypergroup structure (investigated in [36, 40]). (We restrict for convenience to the case of real matrices.) $\mathcal{K}$ is a commutative Hermitean hypergroup, furthermore, self-dual (i.e., $\widehat{\mathcal{K}}$ is a hypergroup $\cong \mathcal{K}$), with Pontryagin and Godement property. In particular, Lévy's continuity theorem is valid. $\mathcal{K}$ is aperiodic, i.e., without idempotents except the unit e. The unit of the hypergroup $\mathcal{K}$ is the zero-matrix, denoted by e.*

Automorphisms of $\mathcal{K}$ are obtained in the following way: $\mathcal{K}$ is considered as subset of the $d \times (d-1)/2-$dimensional vector space $\mathbb{H} := \mathcal{K} - \mathcal{K}$ of (real) Hermitean matrices. For $a \in \mathrm{GL}(\mathbb{R}^d)$ put $T_a : \mathbb{H} \ni \kappa \mapsto ((a\kappa)(a\kappa)^*)^{1/2} = (a\kappa^2 a^*)^{1/2} \in \mathcal{K}$. The restriction to $\mathcal{K}$ defines an hypergroup automorphism of $\mathcal{K}$. Let $(T_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group in $\mathrm{Aut}(\mathcal{K})$. Then there exists a continuous one-parameter group $(a_t = \exp(tQ))_{t\in\mathbb{R}} \subseteq \mathrm{GL}(\mathbb{R}^d)$ such that $T_t = T_{a_t}$ $\forall t \in \mathbb{R}$. And conversely, $(T_{a_t}) \subseteq \mathrm{Aut}(\mathcal{K})$ for any one-parameter group (a_t). In the following we fix $T_t := T_{a_t}$ with $a_t = \exp t \cdot Q$, $t \in \mathbb{R}$.

Let $\mathbb{V} := \mathbb{H} \otimes \mathbb{R}$, the Cartesian product, containing $\Gamma := \mathcal{K} \otimes \mathbb{R}$ as a subset. Γ, endowed with a convolution structure $\varepsilon_{(x,s)} * \varepsilon_{(y,t)} := \left(\varepsilon_x \star \varepsilon_{T_s(y)}\right) \otimes \varepsilon_{s+t}$ for $(x,s),(y,t) \in \Gamma$ and with involution defined by $(x,s)^- = (T_{-s}(x)^-, -s)$ is a (non commutative) hypergroup. (The axioms are easily verified. Note that in our case, $\mathcal{K}$ is Hermitean, hence in particular, $T_{-s}(x)^- = T_{-s}(x)$.) Therefore, the notation $\Gamma =: \mathcal{K} \rtimes \mathbb{R}$ is justified.

Probabilities on $\mathcal{K}$ resp. on Γ act by convolution on $C_0(\mathcal{K})$ and $C_0(\Gamma)$ respectively. We denote the left and right convolution operators as follows: Let $f \in C_0(\mathcal{K}), g \in C_0(\Gamma)$, $z \in \mathcal{K}, (z,r) \in \Gamma$.

$$\begin{aligned}
\overset{\bullet}{R}_z f(x) &= f(x \star z) = \int_{\mathcal{K}} f(y) d\left(\varepsilon_x \star \varepsilon_z\right)(y) \\
\overset{\bullet}{L}_z f(x) &= f(z \star x) = \int_{\mathcal{K}} f(y) d\left(\varepsilon_z \star \varepsilon_x\right)(y) \\
R_{(z,r)} g(x,s) &= f\left((x,s) * (z,r)\right) = \int_{\Gamma} g(y,u) d\left(\varepsilon_{(x,s)} * \varepsilon_{(z,r)}\right)(y,u) \\
L_{(z,r)} g(x,s) &= f\left((z,r) * (x,s)\right) = \int_{\Gamma} g(y,u) d\left(\varepsilon_{(z,r)} * \varepsilon_{(x,s)}\right)(y,u)
\end{aligned}$$

In an analogous way we define for measures λ on Γ resp. μ on $\mathcal{K}$ the left resp. right convolution operators $\overset{\bullet}{R}_\mu$, $\overset{\bullet}{L}_\mu$, R_λ, L_λ on $\mathcal{K}$ resp. Γ.

Definition 4.2. *In the following we restrict again our considerations to measures on the 'space-time building' Γ of the particular form $\lambda = \mu \otimes \varepsilon_u \in \mathcal{M}^1_*(\Gamma) := \{\mu \otimes \varepsilon_u : \mu \in \mathcal{M}^1(\mathcal{K}), u \in \mathbb{R}\}$. In that case we have*

$$R_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} g\left(x \star T_s(y), s+u\right) d\mu(y)$$

$$L_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} g\left(y \star T_u(x), s+u\right) d\mu(y)$$

Note that for $g = \varphi \otimes \psi$ we obtain (with $\psi_u : s \mapsto \psi(s+u)$):

$$R_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} \varphi\left(x \star T_s(y)\right) d\mu(y) \cdot \psi_u(s) = \left(\overset{\bullet}{R}_{T_s(\mu)}\, \varphi\right)(x) \cdot \psi_u(s)$$

$$L_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} \varphi\left(y \star T_u(x)\right) d\mu(y) \cdot \psi_u(s) = \left(\overset{\bullet}{L}_{\mu}\, \varphi\right)(T_u(x)) \cdot \psi_u(s)$$

The involution on Γ induces involutions on on spaces of functions and measures:

Let $g \in \mathbb{C}^b(\Gamma)$. Then $\widetilde{g}(x,s) := g\left((x,s)^-\right) = g\left(T_{-s}(x), -s\right)$

Let $\lambda \in \mathcal{M}^b(\Gamma)$. Then $\int_\Gamma f d\widetilde{\lambda} := \int_\Gamma \widetilde{f} d\lambda$

In particular, for $\lambda = \mu \otimes \varepsilon_u$ we obtain $\widetilde{\lambda} = T_{-u}(\mu) \otimes \varepsilon_{-u}$.

We recall the notations of left invariant operators and subspaces introduced in Section 2; we have to distinguish between invariant operators on $\mathcal{K}$ and on the non-commutative hypergroup Γ.

Proposition 4.3. ***a)*** *For $\lambda, \mu \in \mathcal{M}^b(\Gamma)$ we have $\widetilde{(\lambda * \mu)} = \widetilde{\mu} * \widetilde{\lambda}$*

b) *For $\lambda \in \mathcal{M}^b(\Gamma), f \in C_0(\Gamma)$ we have $\widetilde{(R_\lambda f)} = L_{\widetilde{\lambda}} \widetilde{f}$*

The existence of background driving Lévy processes: the mapping $(\mu(t))_{t\geq 0} \mapsto (\mu_t)_{t\geq 0}$.

The hypergroup $\mathcal{K}$ is embedded into a vector space $\mathbb{H}$, hence inherits a differentiable structure. Note that the action of T_t on $\mathcal{K}$ resp. $\mathbb{H}$ is smooth: $t \mapsto \left(T_{\exp tQ}(\kappa)\right)^2 = \exp tQ\ \kappa^2 \exp tQ^* =: \kappa(t)^2$ is an entire function, and $\mathcal{K} \ni x \mapsto x^{1/2} \in \mathcal{K}$ is holomorphic on $\mathcal{K}_0 := \mathcal{K} \cap \mathrm{GL}(\mathbb{R}^d)$. If the kernel $N(\kappa) \neq \{0\}$ then $N(\kappa(t)) = \exp(-tQ^*)N(\kappa)$ and $N(\kappa(t))^\perp = \exp(-tQ)N(\kappa)^\perp$, hence the projections onto these subspaces depend analytically on t. Whence the assertion easily follows.

We define particular differential operators:

Definition 4.4. *For $f \in C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$ (i.e. with continuous derivatives in $C_0(\mathbb{H}\otimes\mathbb{R})$) and $(x,s) \in \mathbb{H}\otimes\mathbb{R}$ we put*

$$\begin{aligned} Xf(x,s) &:= \frac{d^+}{dt}\Big|_{t=0} f(T_t(x), s+t) = \lim_{t\searrow 0} \frac{1}{t}\left(f(T_t(x), s+t) - f(x,s)\right) \\ Pf(x,s) &:= \frac{d^+}{dt}\Big|_{t=0} f(x, s+t) = \lim_{t\searrow 0} \frac{1}{t}\left(f(x, s+t) - f(x,s)\right) \\ Sf(x,s) &:= \frac{d^+}{dt}\Big|_{t=0} f(T_t(x), s) = \lim_{t\searrow 0} \frac{1}{t}\left(f(T_t(x), s) - f(x,s)\right) \end{aligned}$$

For the restriction to $(x,s) \in \Gamma$ we obtain:

Proposition 4.5. *Let $\lambda \in \mathcal{M}^b(\Gamma), f \in C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$, $(x,s) \in \Gamma$*

a) $Xf(x,s) = \lim_{t\searrow 0} L_{\frac{1}{t}\left(\varepsilon_{(e,t)} - \varepsilon_{(e,0)}\right)} f(x,s)$

b) $Pf(x,s) = \lim_{t\searrow 0} R_{\frac{1}{t}\left(\varepsilon_{(e,t)} - \varepsilon_{(e,0)}\right)} f(x,s)$

Hence

c) $R_\lambda X f(x,s) = X R_\lambda f(x,s)$ ***d)*** $L_\lambda P f(x,s) = P L_\lambda f(x,s)$

e) $\sup_{(x,s)\in\Gamma} |X R_\lambda f(x,s)| \leq ||\lambda|| \sup_{(x,s)\in\Gamma} |Xf(x,s)|$

f) $\sup_{(x,s)\in\Gamma} |S R_\lambda f(x,s)| \leq ||\lambda|| \sup_{(x,s)\in\Gamma} |Sf((x,s)|$.

⟦ a)–e) are obvious, only f) needs a proof:
It is sufficient to prove the assertion for $\lambda = \varepsilon_{(y,u)}$. A simple calculation shows $SR_{(y,u)}f(x,s) = R_{(T_s(y),u)}Sf(x,s)$. Whence

$$\sup_{(x,s)\in\Gamma} |SR_{(y,u)}f(x,s)| = \sup_{(x,s)\in\Gamma} |R_{(T_s(y),u)}Sf(x,s)| \leq$$

$$\sup_{(y',u)\in\Gamma}\ \sup_{(x,s)\in\Gamma} |R_{(y',u)}Sf(x,s)| \leq \sup_{(y',u)\in\Gamma} ||R_{(y',u)}Sf||_{C_0(\Gamma)} \leq ||Sf||_{C_0(\Gamma)}.$$ ⟧

Proposition 4.6. *Let $f \in C_0^{(1)}(\mathbb{H}\otimes\mathbb{R}), (x,s) \in \mathbb{H}\otimes\mathbb{R}$.*

$$Xf(x,s) = \lim_{t\searrow 0}\frac{1}{t}\left(f\left(T_t(x),s\right) - f(x,s)\right) + \lim_{t\searrow 0}\frac{1}{t}\left(f(x,s+t) - f(x,s)\right)$$

$$=: Sf(x,s) + Pf(x,s)$$

⟦ $Xf(x,s) =$

$$\lim_{t\searrow 0}\left[\tfrac{1}{t}\left(f\left(T_t(x),s+t\right) - f(x,s+t)\right) + \tfrac{1}{t}\left(f(x,s+t) - f(x,s)\right)\right]$$

The second terms converge to $Pf(x,s)$, hence also the first terms are convergent, to $S'f(x,s)$ say. Now

$$S'f(x,s) = \lim_{t\searrow 0}\left[\tfrac{1}{t}\left(f\left(T_t(x),s+t\right) - f(T_t(x),s)\right)\right.$$

$$\left.+\tfrac{1}{t}\left(f\left(T_t(x),s\right) - f(x,s)\right) - \tfrac{1}{t}\left(f(x),s+t) - f(x,s)\right)\right]$$

The first and third terms converge to $Pf(x,s)$ and $-Pf(x,s)$ respectively, hence $S'f = Sf$ as asserted. ⟧

The differential operators X and P are related by

Proposition 4.7. $\left(X\widetilde{f}\right)(x,s) = -\left(\widetilde{Pf}\right)(x,s)$

$$\begin{aligned}
\left[X\widetilde{f}(x,s)\right. &= \lim_{t\searrow 0}\frac{1}{t}\left(\widetilde{f}\left(T_t(x),s+t\right) - \widetilde{f}(x,s)\right) \\
&= \lim_{t\searrow 0}\frac{1}{t}\left(f\left(T_{-s-t}T_t(x),-s-t\right) - f\left(T_{-s}(x),-s\right)\right) \\
&= \lim_{t\searrow 0}\frac{1}{t}\left(f\left(T_s(x),-s-t\right) - f\left(T_{-s}(x),-s\right)\right) \\
&= -\left(Pf\right)\left(T_{-s}(x),-s\right) = -\left(\widetilde{Pf}\right)(x,s) \quad \Big]
\end{aligned}$$

Definition 4.8. *We introduce semi-norms on $C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$:*

$$||f||_{(0)} := \sup_{(x,s)\in\Gamma}|f(x,s)|, \quad ||f||_{(1)} := \sup_{(x,s)\in\Gamma}|Xf(x,s)| = ||Xf||_{(0)}$$

and $||f||_{(2)} := ||Sf||_{(0)}$. Finally we put $|||f||| := \sum_{j=0}^{2}||f||_{(j)}$.

$\mathcal{B}$ denotes the completion of $C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$ w.r.t. $|||\cdot|||$. (Since functions coinciding on Γ are identified, the Banach space $\mathcal{B}$ may be considered as subspace of $C_0(\Gamma)$.)

Proposition 4.9. ***a)*** *$\mathcal{B}$ is dense in $C_0(\Gamma)$ w.r.t. $||\cdot||_\infty (= ||\cdot||_{(0)})$.*
b) *For all $f \in \mathcal{B}$ there exist $Xf,\ Pf,\ Sf \in C_0(\Gamma)$.*
c) *For all $\lambda \in \mathcal{M}^b(\Gamma)$, for all $f \in \mathcal{B}$ we have $|||R_\lambda f||| \leq ||\lambda|| \cdot |||f|||$.*
d) *In particular, for a continuous convolution semigroup $(\lambda_t)_{t\geq 0}$ in $\mathcal{M}^{(1)}(\Gamma)$ the operators $(R_{\lambda_t})_{t\geq 0}$ may be considered as C_0-contraction semigroup on $C_0(\Gamma)$ as well as on $\mathcal{B}$.*

$[\![$ a), b) are obvious, c) and d) are immediate consequences of Proposition 4.5 e) and f). $]\!]$

Definition 4.10. *In the following let* $(\lambda_t = \mu(t) \otimes \varepsilon_t)_{t\geq 0}$ *be a continuous convolution semigroup in* $\mathcal{M}^1_*(\Gamma)$ *with* $\lambda_0 = \varepsilon_{(e,0)}$. *Let* $(U, D(U))$ *resp.* $\left(\overset{*}{U}, D(\overset{*}{U})\right)$ *denote the infinitesimal generators of the* C_0- *contraction semigroups* $(R_{\lambda_t})_{t\geq 0}$ *on* $C_0(\Gamma)$ *and on* $\mathcal{B}$ *respectively.*

Proposition 4.11. $D(\overset{*}{U})$ *is dense in* $D(U)$ *and in* $C_0(\Gamma)$*, furthermore,* $D(\overset{*}{U})$ *is a core for* $(U, D(U))$.

$[\![$ In fact, by construction $D(\overset{*}{U}) \subseteq D(U)$ and $D(\overset{*}{U})$ is dense in $\mathcal{B}$ w.r.t. $|||\cdot|||$. Hence also dense in $C_0(\Gamma)$ w.r.t. $||\cdot||_{(0)}$. Furthermore, $(I- \overset{*}{U})D(\overset{*}{U}) = \mathcal{B}$, hence $(I-U)D(\overset{*}{U})$ is dense in $C_0(\Gamma)$. Whence the assertion. $]\!]$

Remark 4.12. $D(U)$ *is left invariant since* U *is left invariant. But the* $|||\cdot|||-$*defining operators* X *and* S *are not left invariant. Hence we can not conclude that* $D(\overset{*}{U})$ *is left invariant. That is the reason why we have to use more complicated constructions in the sequel*

Proposition 4.13. *There exists a core* $\mathcal{E}$ *for* $(R_{\lambda_t})_{t\geq 0}$ *(resp.* $(U, D(U))$*) such that* $\mathcal{E} \subseteq D(U) \cap D(P)$

Proof: Let $f \in D(U)$, $\psi \in \mathcal{D}(\mathbb{R})$. Put $g = g_{f,\psi} : (x,s) \mapsto f(x,s)\cdot\psi(s)$.

Let $\mathcal{E}_0 := \text{span}\{g_{f,\psi} : f \in D(U), \psi \in \mathcal{D}(\mathbb{R})\}$.

1. $\mathcal{E}_0 \subseteq D(U)$.

In fact, we prove for $g := g_{f,\psi} : Ug(x,s) = Uf(x,s)\cdot\psi(s) + f(x,s)\cdot\psi'(s)$:

$$[\![\quad \begin{aligned} & \frac{1}{t}\int_{\mathcal{K}} g(x\star T_s(y), s+t) - g(x,s)d\mu(t)(y) \\ = & \frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y), s+t)\cdot\psi(s+t) - f(x,s)\cdot\psi(s)d\mu(t)(y) \\ = & \left[\frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y), s+t) - f(x,s)d\mu(t)(y)\right]\cdot\psi(s+t) \\ + & \int_{\mathcal{K}} f(x,s)d\mu(t)(y)\cdot\left[\frac{1}{t}\left(\psi(s+t)-\psi(s)\right)\right] \\ \overset{t\to 0}{\longrightarrow} & \; Uf(x,s)\cdot\psi(s) + \psi'(s)\cdot f(x,s). \end{aligned} \quad]\!]$$

Convergence is uniform in (x,s) since ψ and ψ' have compact support and $Uf \in C_0(\Gamma)$.

2. $\mathcal{E}_0$ is dense in $C_0(\Gamma)$. In fact, let L_n be compact intervals, $L_n \nearrow \mathbb{R}$, e.g., $L_n = [-k_n, k_n]$ with $k_n \nearrow \infty$. Let $\psi_n \in \mathcal{D}(\mathbb{R}), 1_{L_n} \leq \psi_n \leq 1_{L_{n+1}}$. Then $f(x,s)\cdot\psi_n(s) \to f(x,s)$ uniformly in $(x,s) \in \Gamma$ (since $f \in C_0(\Gamma)$).

3. $(I-U)\mathcal{E}_0$ is dense in $C_0(\Gamma)$.

$[\![$ We show: $\forall \varepsilon > 0\ \forall\ f \in D(U)$ there exists a $g \in \mathcal{E}_0$ such that $||(I-U)f - (I-U)g||_\infty = ||(f-g) - (Uf - Ug)||_\infty < \varepsilon$. (Note that $(I-U)D(U) = C_0(\mathcal{K})$.)

Let $f \in D(U)$, choose L_n, ψ_n as above, and assume in addition that $||\psi_n{}'||_\infty \to 0$. Put $g_n(x,s) := f(x,s)\cdot\psi_n(s)$. Then $(I-U)g_n(x,s) = g_n(x,s) - Uf(x,s)\cdot\psi_n(s) - f(x,s)\cdot\psi_n'(s)$, therefore, $|(I-U)f(x,s) -$

$(I-U)g_n(x,s)| \leq ||f-g_n||_{(0)}+|Uf(x,s)|\cdot|1-\psi_n(s)|+||f||_{(0)}\cdot||\psi_n'||_\infty \to 0$. Convergence is again uniform in (x,s) since $Uf \in C_0(\Gamma)$. $]\!]$

4. The above steps remain true if $\mathcal{E}_0$ is replaced by

$$\mathcal{E} := \text{span}\left\{g_{f,\psi} : f \in D(\overset{*}{U}), \psi \in \mathcal{D}(\mathbb{R})\right\}.$$

(According to 4.11, $D(\overset{*}{U})$ is a core for $(U, D(U))$.)

5. In that case we have in addition $\mathcal{E} \subseteq D(P)$ (and $P\mathcal{E} \subseteq C_0(\Gamma)$).

$[\![$ Since $D(\overset{*}{U}) \subseteq \mathcal{B} \subseteq D(P)$ (cf. Proposition 4.9) and $Pg_{f,\psi}(x,s) = Pf(x,s)\cdot\psi(s) + f(x,s)\cdot\psi'(s)$.) $]\!]$ □

Note that in contrast to $\mathcal{E}_0$ the core $\mathcal{E}$ is not left invariant but the core $D(U)\cap D(P)$ is:

Proposition 4.14. *$D(U)\cap D(P)$ is a core for $(U,D(U))$ (since the core $\mathcal{E}$ is contained in $D(U)\cap D(P)$ according to 4.13). Furthermore, $D(U)\cap D(P)$ is obviously left invariant, since U and P are left invariant.*

For $f \in D(U)\cap D(P)$ we have:

$$Uf = Wf + Pf,$$

where $Wf(x,s) = \lim_{t\searrow 0} \frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y),s) - f(x,s)d\mu(t)(y)$
$= \lim_{t\searrow 0}\frac{1}{t}\left(R_{\mu(t)\otimes\varepsilon_0} - I\right) f(x,s)$

$[\![$ $Uf(x) = \lim_{t\searrow 0}\frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y), s+t) - f(x,s)d\mu(t)(y) =$
$= \lim_{t\searrow 0}\frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y),s+t) - f(x,s+t)d\mu(t)(y) +$
$\lim_{t\searrow 0}\frac{1}{t}\int_{\mathcal{K}} f(x,s+t) - f(x,s)d\mu(t)(y) =: Wf(x,s) + Pf(x,s)$

Furthermore,

$$\begin{aligned} Wf(x,s) = \lim_{t\searrow 0}&\left[\frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y),s) - f(x,s)d\mu(t)(y)\right.\\ &+\frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y),s+t) - f(x\star T_s(y),s)d\mu(t)(y)\\ &\left.-\frac{1}{t}\int_{\mathcal{K}} f(x,s+t) - f(x,s)d\mu(t)(y)\right] \end{aligned}$$

The second and the third terms converge to $Pf(x,s)$ and $-Pf(x,s)$ respectively, whence

$$Wf(x,s) = \lim_{t\searrow 0}\frac{1}{t}\int_{\mathcal{K}} f(x\star T_s(y),s) - f(x,s)d\mu(t)(y)$$

follows. $]\!]$

Definition 4.15. $\Lambda := \left\{\mathcal{K}\ni x \mapsto f(x,0) =: \overset{\bullet}{f}(x) : f \in D(U)\cap D(P)\right\}$

Proposition 4.16. *Λ is $||\cdot||-$dense in $C_0(\mathcal{K})$, left invariant (and also right invariant, as $\mathcal{K}$ is Abelian).*

$[\![$ $D(U)\cap D(P)$ is a left invariant subspace of $C_0(\Gamma)$. In other words, $L_{(y,u)}(D(U)\cap D(P)) \subseteq (D(U)\cap D(P))\ \forall (y,u)\in\Gamma$. Considering $u=0$ we obtain $\overset{\bullet}{L}_y(\Lambda)\subseteq\Lambda\ \forall y\in\mathcal{K}$. $]\!]$

Now we are ready to prove the *existence of a background driving Lévy process:*

Proposition 4.17. *As introduced afore, we put* $\overset{\bullet}{f}$ *for the restriction of* f *to* $\{(y,0) : y \in \mathcal{K}\} \equiv \mathcal{K}$. *With this notation we have:*

$\Lambda \ni \overset{\bullet}{f} \mapsto Wf(\cdot, 0) =: V\,\overset{\bullet}{f}$

is a left invariant operator $\Lambda \to C_0(\mathcal{K})$. V *is dissipative (by construction) and has a unique extension to the generator of a semigroup of convolution operators* $(R_{\mu_t})_{t\geq 0}$ *for a continuous convolution semigroup* $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}_+(\mathcal{K})$. *In particular,* Λ *is a core for* $(\mu_t)_{t\geq 0}$.

⟦ Λ is dense in $C_0(\mathcal{K})$ and left invariant. Since $\mathcal{K}$ is Abelian, Λ is (trivially) right invariant. By construction, V is dissipative, whence according to Theorem 1.9 a) the existence of $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\mathcal{K})$ follows.

Furthermore, according to Proposition 4.14, $V = \lim_{t\searrow 0} V_t$ where $V_t = \frac{1}{t}\left(R_{\mu(t)} - I\right)$ and $\mu(t) \in \mathcal{M}^1(\mathcal{K})$. Hence $R_{\mu_s} = \lim_{t\searrow 0} \exp s \cdot V_t$, thus $\mu_s = \lim_{t\searrow 0} \exp s\frac{1}{t}\left(\mu(t) - \varepsilon_e\right) \geq 0$ for all $s \geq 0$. ⟧

Proposition 4.18. *Let* $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}_+(\mathcal{K})$, W *and* V *be defined as in Proposition 4.17. Let* $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0} \subseteq \mathcal{M}^1_*(\Gamma)$ *denote the corresponding continuous convolution semigroup, concentrated on* $\mathcal{K}\otimes\{0\} \cong \mathcal{K}$. *Put furthermore* $\left(p_t^{\pm} := \varepsilon_{(e,\pm t)}\right)_{t\in\mathbb{R}_+}$.

Then W *and* $\pm P$ *are the generators of* $(R_{\sigma_t})_{t\geq 0}$ *and* $\left(R_{p_t^{\pm}}\right)_{t\geq 0}$ *respectively.*

⟦ For all $(x,s) \in \Gamma$ we have:

$$\begin{aligned}
Wf(x,s) &= \lim_{t\searrow 0} \frac{1}{t}\int_{\mathcal{K}} f\left(x \star T_s(y), s\right) - f(x,s)d\mu(t)(y) \\
&= \lim_{t\searrow 0} \frac{1}{t}\int_{\mathcal{K}} \left(L_{(e,s)}f\right)\left(T_{-s}(x)\star y, 0\right) - \left(L_{(x,s)}f\right)\left(T_{-s}(x), 0\right) d(\mu(t)(y) \\
&= V\,\overset{\bullet}{g}\left(T_{-s}(x)\right) \quad (\text{with } \ g := L_{(e,s)}f) \\
&= \frac{d^+}{dt}\Big|_{t=0} \overset{\bullet}{R}_{\mu_t}\overset{\bullet}{g}\left(T_{-s}(x)\right) = \frac{d^+}{dt}|_{t=0} R_{\sigma_t} g\left(T_{-s}(x), 0\right) \\
&= \frac{d^+}{dt}|_{t=0} R_{\sigma_t} f\,(x,s)
\end{aligned}$$

(cf. Proposition 4.17.) ⟧

In view of Proposition 4.14, application of the Lie-Trotter formula (LT) (Proposition 2.4 a)) to the decomposition $U = W + P$ yields

Proposition 4.19. *With the notations introduced above we obtain:*

$$(LT1) \qquad \mu(t) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{n-1} T_{kt/n}\left(\mu_{t/n}\right)$$

⟦ The Lie-Trotter formula $(LT*)$ yields $\lambda_t = \lim_{n\to\infty}\left(\sigma_{t/n} \star p^+_{t/n}\right)^n$. Considering the projection to the $\mathcal{K}-$component yields $(LT1)$. ⟧

In 4.17 we have proved $\mu_t \in \mathcal{M}_+^{(1)}(\mathcal{K})$. Now we are ready to prove

Proposition 4.20. *$\mu_t \in \mathcal{M}^1(\mathcal{K})$ for all $t \geq 0$.*

$[\![$ Assume $||\mu_t|| < 1$ for some $t > 0$. Then, as μ_t are positive, $||\mu_t|| = e^{-ct}$ for some $c > 0$. Therefore, in $(LT1)$ the right hand side has norm $\leq e^{-ct}$. A contradiction to the assumption $\mu(t) \in \mathcal{M}^1(\mathcal{K})$. $]\!]$

We have proved that for any M-semigroup $(\mu(t))_{t\geq 0}$ there exists a continuous convolution semigroup $(\mu_t)_{t\geq 0}$, the background driving Lévy process, such that $(LT1)$ holds true. In fact, the following results prove uniqueness of $(\mu_t)_{t\geq 0}$ and bijectivity of the mapping $(\mu_t)_{t\geq 0} \mapsto (\mu(t))_{t\geq 0}$. Bijectivity is proved by the inverse Lie-Trotter formula $(LT2)$.

The existence of M-semigroups: The mapping $(\mu(t))_{t\geq 0} \mapsto (\mu_t)_{t\geq 0}$

First we show

Theorem 4.21. *Let $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup in $\mathcal{M}^1(\mathcal{K})$. Then there exists a M-semigroup $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^1(\mathcal{K})$ such that (LT1) and (LT2) hold:*

$$(LT1) \quad \mu(t) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{n-1} T_{kt/n}\left(\mu_{t/n}\right) \qquad (LT2) \quad \mu_t = \lim_{n\to\infty} \left(\mu(t/n)\right)^n$$

Proof: At the first glance it seems obvious to consider as before

$$W = \frac{d^+}{dt}\Big|_{t=0} R_{\mu_t\otimes\varepsilon_0} =: \frac{d^+}{dt}|_{t=0} R_{\sigma_t}$$

and to apply the Lie-Trotter formula to the representation $U = W + P$ resp. $W = U - P$. But a priori there is no 'natural' common domain for U, W, P. Therefore we have to find a slightly different approach. This will be done in the subsequent steps, formulated as propositions.

Let $(\mu_t)_{t\geq 0} \in \mathcal{M}^1(\mathcal{K})$ be given, define $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0} \subseteq \mathcal{M}_*^1(\Gamma)$, put for $t > 0$, $W_t := \frac{1}{t}\left(R_{\sigma_t} - I\right)$, $V_t := \frac{1}{t}\left(\overset{\bullet}{R}_{\mu_t} - I\right)$ (acting on $C_0(\Gamma)$ and $C_0(\mathcal{K})$ respectively). Furthermore, let $(W, D(W))$ and $(V, D(V))$ be the generators of the corresponding contraction semigroups (R_{σ_t}) and $\left(\overset{\bullet}{R}_{\mu_t}\right)$.

Let $\mathcal{A} \subseteq D(V)$ denote a core for $(\mu_t)_{t\geq 0}$ with the following properties:
(1) $\mathcal{A}$ is left invariant (and right invariant, as $\mathcal{K}$ is Abelian).
(2) $T_s(\mathcal{A}) \subseteq \mathcal{A}$ for all automorphisms T_s.

[Such cores exist for $\mathcal{K}$, e.g., $\mathcal{A} = \left(L_c^1(\widehat{\mathcal{K}})\right)^\vee$, the space of analytic vectors, as mentioned in 1.11, 1.12.]

Define $\mathbb{D} := \mathcal{A} \otimes \mathcal{D}(\mathbb{R}) \subseteq C_0(\Gamma)$. Then we have:

(i) $\mathbb{D} \subseteq D(W)$

$[\![$ Let $f := \varphi \otimes \psi \in \mathbb{D}$. Then

$W_t f(x,s) = \frac{1}{t}\left(\int_{\mathcal{K}} \varphi\left(x \star T_s(y)\right) - \varphi(x) d\mu_t(y)\right) \cdot \psi(s)$

$= \left(V_t\gamma\right)\left(T_{-s}(x)\right) \cdot \psi(s) \xrightarrow{t\to 0} \left(V\gamma\right)\left(T_{-s}(x)\right) \cdot \psi(s)$

(with $\gamma := \varphi \circ T_s \in \mathcal{A}$.) We define: $Wf(x,s) := \lim_{t\searrow 0} W_t f(x,s)$ $]\!]$

(ii) $\mathbb{D}$ is left invariant and dense in $C_0(\Gamma)$

⟦ Obviously $\mathbb{D}$ is dense in $C_0(\Gamma)$. To prove invariance we consider $L_{(y,t)}(\varphi \otimes \psi)(x,s) = (\varphi \otimes \psi)(y \star T_t(x), s+t) =$

$(\varphi \circ T_t)(T_{-t}(y) \star x) \cdot \psi(s+t) = \overset{\bullet}{L}_{(T_{-t}(y))}(\varphi \circ T_t)(x) \cdot \psi(s+t) =: g(x) \cdot \xi(s)$

with $g = \overset{\bullet}{L}_{(T_{-t}(y))}(\varphi \circ T_t) \in \mathcal{A}$ and $\xi \in \mathcal{D}(\mathbb{R})$. ⟧ □

Proposition 4.22. *Let $f \in \mathbb{D}$, $(z,u) \in \Gamma$. Then $R_{(z,u)} f \in D(W)$.*

⟦ In fact, by definition

$$\begin{aligned}
&W_t R_{(z,u)}(\varphi \otimes \psi)(x,s) = \\
&= \frac{1}{t} \int (\varphi(x \star T_s(z) \star T_{s+u}(y)) - \varphi(x \star T_s(z)) \, d\mu_t(y) \cdot \psi(s+u) \\
&= V_t(\varphi \circ T_{s+u}) \big(T_{-(s+u)}(x) \star T_u(z)\big) \cdot \psi(s+u) \\
&=: \overset{\bullet}{R}_z ((V_t \varphi_{s,u}) \circ T_{-u})(T_{-s}(x)) \cdot \psi_u(s) \quad (\text{with } \varphi_{s,u} := \varphi \circ T_{s+u}) \\
&\xrightarrow{t \to 0} \overset{\bullet}{R}_z ((V \varphi_{s,u}) \circ T_{-u})(T_{-s}(x)) \cdot \psi_u(s) \\
&= V(\varphi \circ T_{s+u}) \big(T_{-(s+u)}(x) \star T_u(z)\big) \cdot \psi(s+u) \\
&=: W\big(R_{(z,u)}(\varphi \otimes \psi)\big)(x,s)
\end{aligned}$$

Convergence is again uniform on Γ. ⟧

Definition 4.23. *Let* $\widetilde{\mathbb{D}} := \operatorname{span}\left\{R_{(z,u)} f : (z,u) \in \Gamma, f \in \mathbb{D}\right\}$

Proposition 4.24. *$\widetilde{\mathbb{D}}$ is dense in $C_0(\Gamma)$ and left and right invariant.*

Furthermore, $\widetilde{\mathbb{D}} \subseteq D(W) \cap D(P)$.

W and $\pm P$ are, as limits of convolution operators, left invariant and by construction dissipative. Hence U shares this property.

Therefore, according to Theorem 1.9 c), $\widetilde{\mathbb{D}}$ is a core for P, W and $U := W + P$. (Note that $W = U - P$.)

⟦ Only $\widetilde{\mathbb{D}} \subseteq D(P)$ needs a proof:

$$P R_{(z,u)}(\varphi \otimes \psi)(x,s) = \lim_{t \searrow 0} R_{\frac{1}{t}\left(\varepsilon_{(e,t)} - \varepsilon_{(e,0)}\right)} R_{(z,u)}(\varphi \otimes \psi)(x,s) =$$

$$\lim_{t \searrow 0} (\varphi(x \star T_s(z))) \cdot \frac{1}{t}(\psi(s+u+t) - \psi(s+u)) = \varphi(x \star T_s(z)) \cdot \psi'(s+u)$$

Convergence is uniform since ψ and ψ' have compact supports. Whence the assertion. ⟧

Proposition 4.25. *The afore announced Lie-Trotter formulas $(LT1)$ and $(LT2)$ (cf. 4.21) hold.*

⟦ Applying the Lie-Trotter formula (LT) (cf. Proposition 2.4 a)) to $U = W + P$ resp. $W = U - P$ yields $\lambda_t = \lim_{n \to \infty} \left(\sigma_{t/n} * p^{+}_{t/n}\right)^n$ resp. $\sigma_t = \lim_{n \to \infty} \left(\lambda_{t/n} * p^{-}_{t/n}\right)^n$, $t \geq 0$. Projecting to the space component $\mathcal{K}$ yields $(LT1)$ resp. $(LT2)$ ⟧

We have proved, that $(\mu(t))_{t \geq 0}, (\lambda_t)_{t \geq 0} \subseteq \mathcal{M}_+(\mathcal{K})$ and have norm $||\mu(t)||, ||\lambda_t|| \leq 1$. Comparing the norms in $(LT1)$ and $(LT2)$ yields again that $\mu(t)$ and hence λ_t are probabilities.

The proof of Theorem 3.2 is complete. □

Remark 4.26. *In the particular situation with given continuous convolution semigroup $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}^1(\mathcal{K})$ it is possible to find an alternative proof for the existence of a corresponding M-semigroup $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^1(\mathcal{K})$ satisfying (LT1):*

The alternative proof avoids space-time semigroups and relies heavily on the fact that $\mathcal{K}$ is Abelian (this was used also before, to find an example $\mathcal{A}$ of a suitable function space with prescribed properties) and on the validity of Lévy's continuity theorem.

$[\![$ Let $\widehat{\mu}_t = e^{tL}$ with strongly negative definite $-L : \widehat{\mathcal{K}} \,(\equiv \mathcal{K}) \to \mathbb{R}$. (For definitions and properties of positive and negative definite functions on hypergroups see e.g. [5, 41, 20]). L is a continuous function and $\mathbb{R} \ni s \mapsto T_s \in \mathrm{Aut}(\mathcal{K})$ is continuous. Define

$$M(t) := \int_0^t L \circ T_s^* ds = \lim_{n\to\infty} \tfrac{t}{n} \sum_{k=0}^{n-1} L \circ T_{kt/n}^* =: \lim_{n\to\infty} M_n(t)$$

(where T_s^* denotes the dual automorphism acting on $\widehat{\mathcal{K}} \,(\cong \mathcal{K})$).

Obviously, $M_n(\cdot)$ are continuous and $-M_n(\cdot)$ are strongly negative definite functions with corresponding continuous convolution semigroups $\left(\rho_t^{(n)} := \bigstar_{k=0}^{n-1} \mu_{k,t}^{(n)}\right)_{t\geq 0}$, where $\mu_{k,t}^{(n)} := T_{kt/n}(\mu_{t/n})$, $\widehat{\rho_t^{(n)}} = e^{M_n(t)}$. Moreover, $e^{M_n(t)} \stackrel{n\to\infty}{\longrightarrow} e^{M(t)}$ (for all $t \geq 0$), and the limit is continuous at e. Therefore, according to Lévy's continuity theorem for hypergroups, there exist probabilities $\mu(t) \in \mathcal{M}^1(\mathcal{K})$ with $\widehat{\mu(t)} = e^{M(t)}$ and, since by construction, $t \mapsto M(t)$ is continuous, $t \mapsto \mu(t)$ is weakly continuous. Furthermore, by construction, $\mu(t) = \lim_{n\to\infty} \bigstar_{k=0}^{n-1} T_{kt/n}(\mu_{t/n})$. I.e., $(LT1)$ holds. And in addition, $\forall s,t \geq 0,\ M(s+t) = M(s) + M(t) \circ T_s^*$. Hence $(\mu(t))_{t\geq 0}$ is a M-semigroup. $]\!]$

Note that by construction, $t \mapsto M(t) = \int_0^t L \circ T_s^* ds$ is differentiable in $t = 0$ with $\frac{d^+}{dt}\,|_{t=0} M(t) = L$. (*)

On the other hand, if (*) is assumed for strongly negative definite functions $-M(t), t \geq 0$, and $-L : \widehat{\mathcal{K}} \to \mathbb{R}$ is continuous and strongly negative definite, then there exist $\mu(t) \in \mathcal{M}^1(\mathcal{K})$ and a continuous convolution semigroup $(\mu_t)_{t\geq 0}$, such that $(LT2)$ holds.

$[\![$ In fact, $\widehat{\mu(t/n)}^{\,n} = e^{t\cdot \frac{n}{t}\cdot M(\frac{t}{n})} \to e^{t\cdot L}$. Lévy's continuity theorem proves the assertion $(LT2)$: $\mu(t/n)^n \to \mu_t$. $]\!]$

As Fourier transforms on $\widehat{\mathcal{K}}$ are real valued, it is easily shown that $(LT2)$ is equivalent to the differentiability condition (*).

Hence we obtain:

Remark 4.27. *The proof of $(LT1)$ and $(LT2)$ in Proposition 4.25 shows in view of Theorem 3.2 that for any M-semigroup on $\mathcal{K}$ with Fourier transform $\widehat{\mu(t)} = e^{M(t)}$, $t \geq 0$, the logarithms $M(t)$ are differentiable at $t = 0$ and $\frac{d^+}{dt}\,|_{t=0} M(t) = L$, the logarithm of the Fourier transform of the background driving Lévy process.*

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