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Quasiperiodic Tilings - Substitution Versus Inflation

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Quasiperiodic Tilings - Substitution Versus Inflation

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Contents

1 Summary

One of the main tools for constructing quasiperiodic tilings in $\mathbb{E}^d$ is *inflation*: Given a family $\mathcal{F} := \{T_1, T_2, ..., T_\kappa\}$ consisting of a finite number of prototiles $T_1, T_2, ..., T_\kappa$ and real number $\eta > 1$. Then every prototile T_κ is replaced by a cluster *infl*(T_κ) of congruent copies of prototiles, whose union equals ηT_κ. Obviously this process can be iterated arbitrarily often and so leads to a tiling of $\mathbb{E}^d$.
If the condition

$$\bigcup{}_\nu (T_{\lambda_\nu} \in infl(T_\kappa)) = \eta T_\kappa \tag{1}$$

is canceled and even

$$\mu_d \bigcup{}_\nu (T_{\lambda_\nu} \in infl(T_\kappa)) = \eta^d \mu_d(T_\kappa) \tag{2}$$

(μ_d the Lebesgues-measure in $\mathbb{E}^d$)

is no longer required, still interesting nonperiodic tilings can be constructed. We then speak of a *substitution* and denote it by $subst(T_\kappa)$, provided

(a) it can be iterated arbitrarily often, and

(b) $subst^n(T_\kappa)$(is a cluster[1]) consisting of $\mathcal{F}$-tiles for every $n \in \mathbb{N}$.

2 Definitions

Definition 1

The *species* of all global tilings $\mathcal{T}$ of $\mathbb{E}^d$ by $\mathcal{F}$-tiles with the property, that for every cluster $\mathcal{C}$ of $\mathcal{T}$ there is an n and a prototile T_κ, such that $subst^n(T_\kappa)$ contains a cluster congruent to $\mathcal{C}$, is denoted by $\boldsymbol{S}(\mathcal{F}, subst)$.

Definition 2

Let $a_{\kappa,\lambda}$ denote the number of copies of T_λ, wich occur in $infl(T_\kappa)$, then the $\kappa * \kappa$-matrix

$$M_{subst} = \begin{pmatrix} a_{11} & \dots & a_{1\kappa} \\ \vdots & \vdots & \vdots \\ a_{\kappa 1} & \dots & a\kappa\kappa \end{pmatrix} \tag{3}$$

is called the *substition - matrix.*

Definition 3

A $\kappa * \kappa$-matrix M is called *primitive*, if $det(M) \neq 0$, and for some n all entries of M^n are strictly positive.

(c) In the sequal we require M_{subst} to be primitive.

Definition 4

a) A cluster $\mathcal{C}$ of a global tiling $\mathcal{T}$ of $\mathbb{E}^d$ is called *repetitive* with respect to $\mathcal{T}$, if there is a radius r such that every ball of radius r does contain a congruent copy of $\mathcal{C}$.
b) $\boldsymbol{S}(\mathcal{F}, subst)$ is called repetitive, if every cluster $\mathcal{C}$ of $\boldsymbol{S}$ is repetitive with respect to every tiling $\mathcal{T} \in \boldsymbol{S}$ and if the radius $r := r(\mathcal{C})$ depends on $\mathcal{C}$ only.
c) $\boldsymbol{S}$ is called *uniformly* repetitive, if there is a constant C, such that $r(\mathcal{C}) \leq C \cdot diam(\mathcal{C})$.

3 Remarks

Remark 1

Assume M_{subst} to be primitive. Then $\boldsymbol{S}{:=}\boldsymbol{S}(\mathcal{F}, subst)$ is uniformly repetitive and all members of $\boldsymbol{S}$ are locally isomorphic. (Proof by standard arguments.)

[1] I.e. a family of subsets of $\mathbb{E}^d$ without any overlap and union homeomorphic to $\mathbb{B}^d$.

Theorem 1

Given a protoset $\mathcal{F} := \{T_1, T_2, ..., T_\kappa\}$ and a substitutionmatrix M_{subst}. If M_{subst} is primitive and if there are arbitrarily large clusters in $\boldsymbol{S}(\mathcal{F}, subst)$, whose translates never overlap, then $\boldsymbol{S}(\mathcal{F}, subst)$ is aperiodic. (Proof by standard arguments.)

Theorem 2

If $\boldsymbol{S}{:=}\boldsymbol{S}(\mathcal{F}, subst)$ is repetitive but aperiodic, then no cluster $\mathcal{C}$ of $\boldsymbol{S}$ determines a global tiling $\mathcal{T}$ of $\boldsymbol{S}$.

Proof of theorem 2

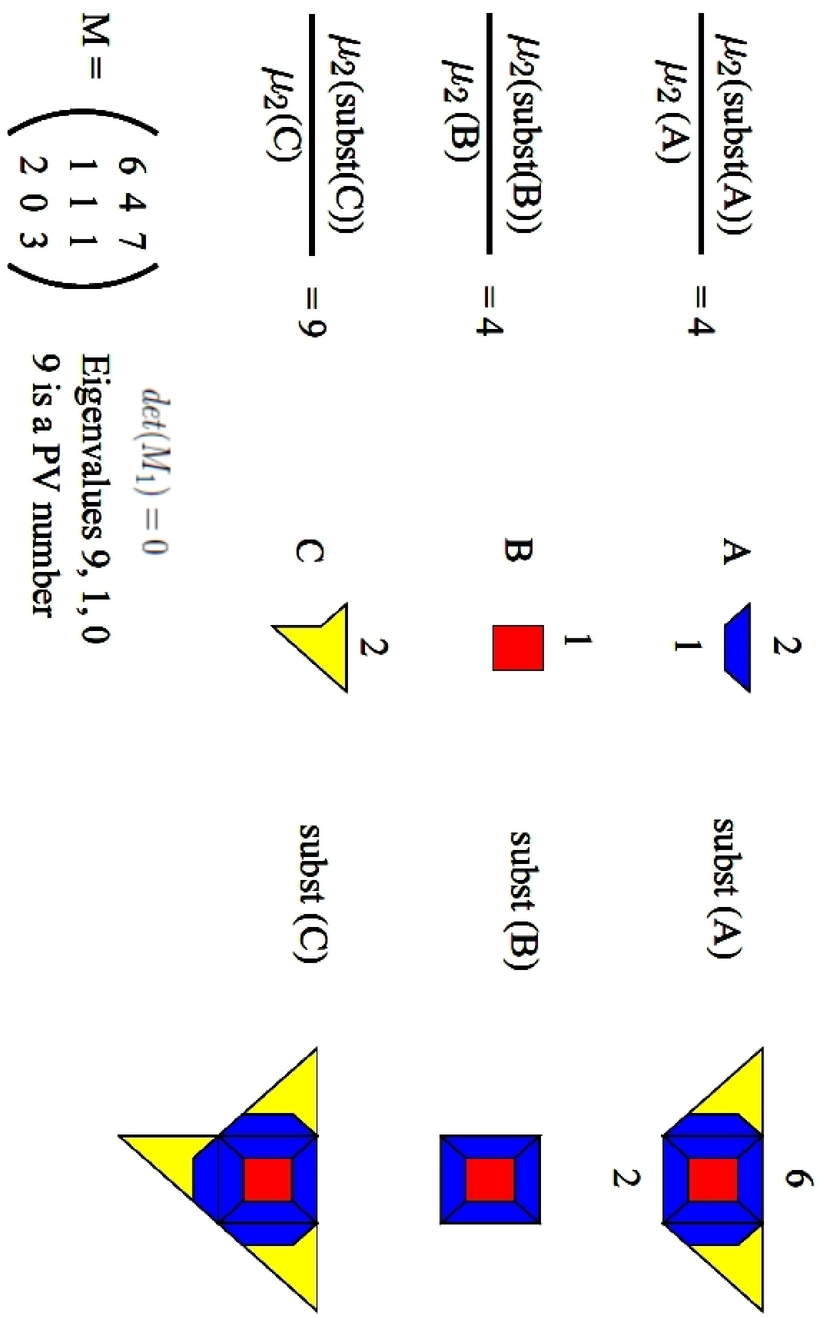
Given $\mathcal{C} \subset \mathcal{T} \in \boldsymbol{S}$. There must be an x for which $\mathcal{C} \cap (\mathcal{C} + x) = \emptyset$. If $\mathcal{C}$ would determine $\mathcal{T}$, so would do also $\mathcal{C} + x$. Hence we had $\mathcal{T} = \mathcal{T} + x$. (equal, not only congreunt!) ↯
A trivial consequence is the following Corollary.

Corollary

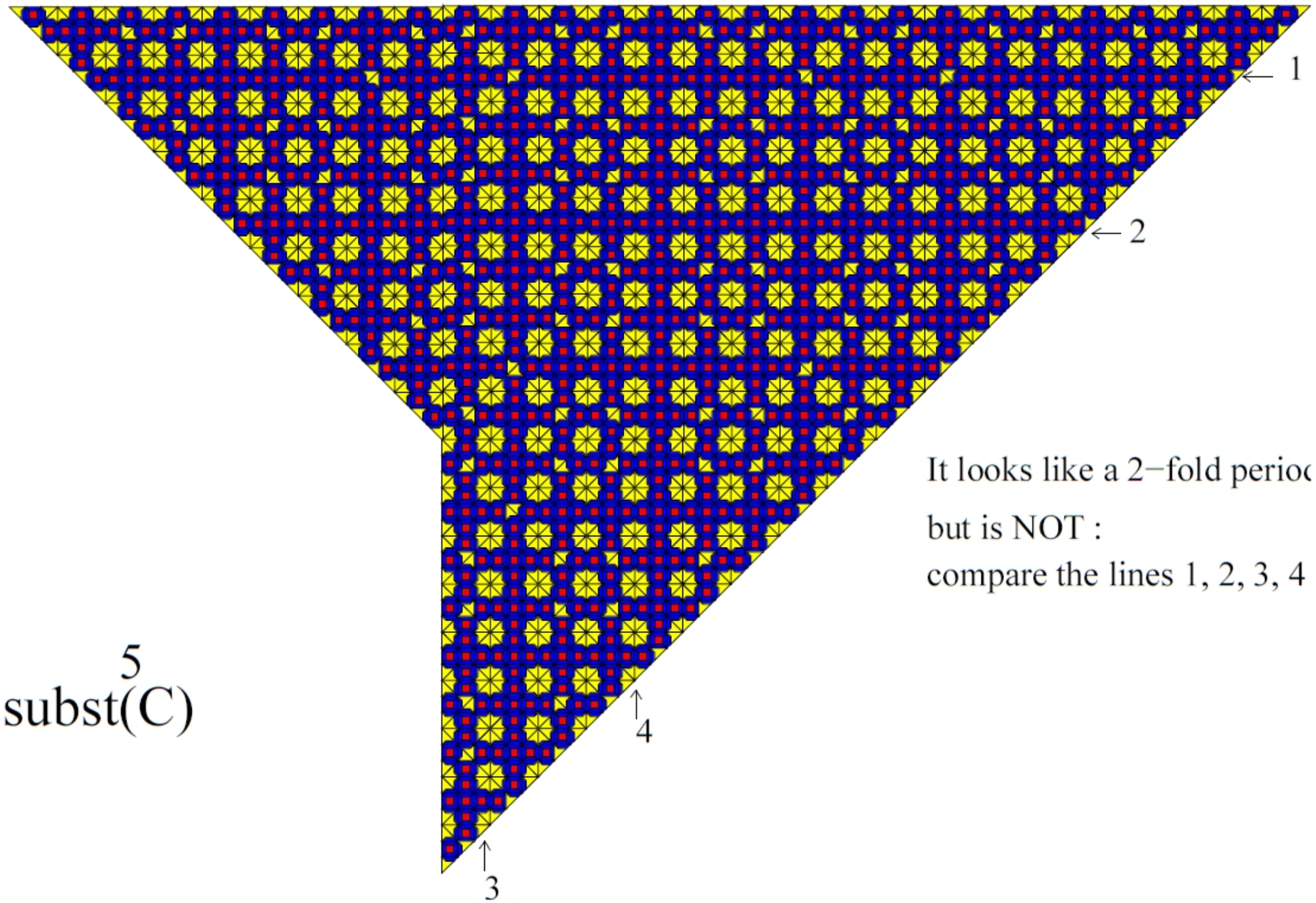
If M_{subst} is primitive and $\boldsymbol{S}{:=}\boldsymbol{S}(\mathcal{F}, subst)$ is aperiodic, then there are $c := 2^{\aleph_0}$ congruence-classes of tilings belonging to $\boldsymbol{S}$.

4 Examples

Example 1

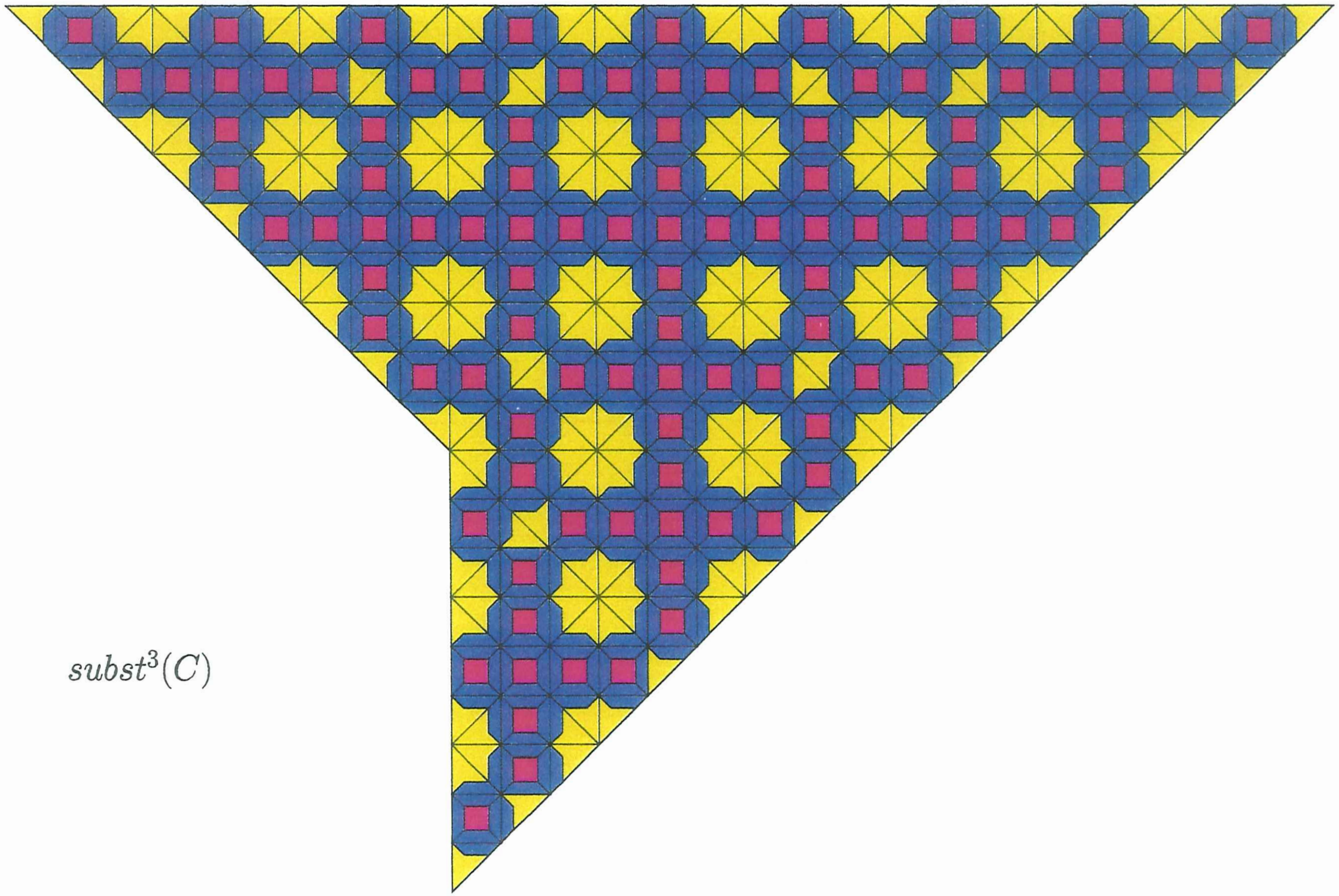


To Example 1



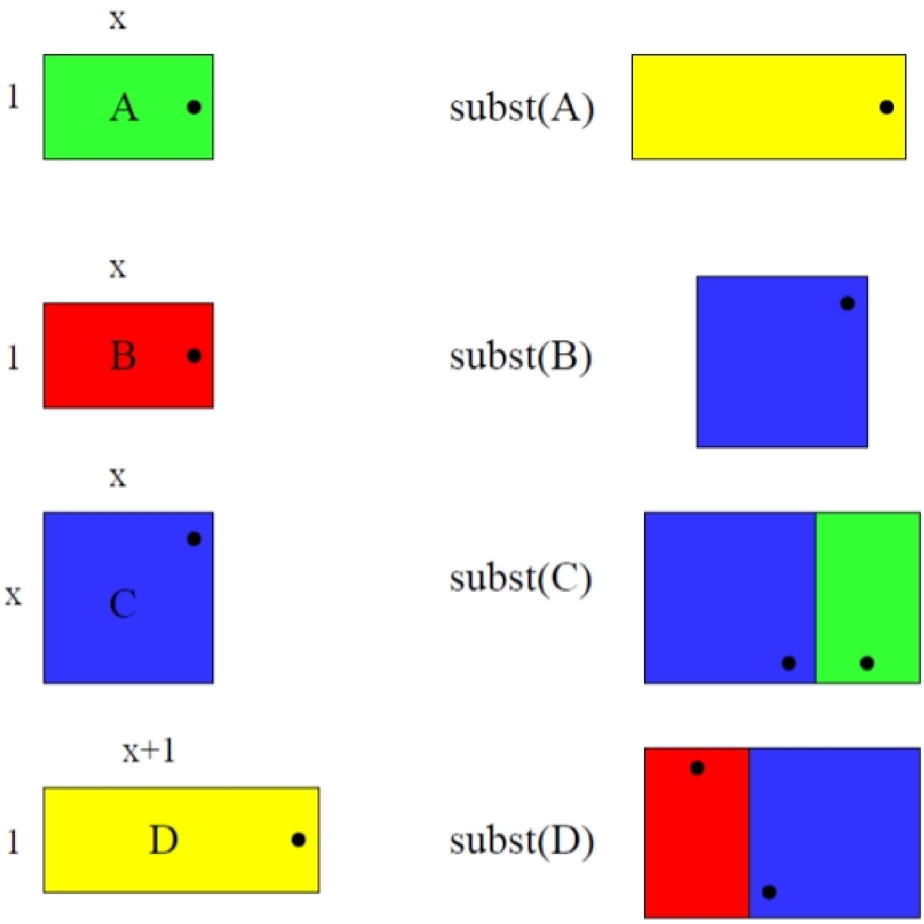
$\mathrm{subst}^5(C)$

To Example 1



$subst^3(C)$

Example 2



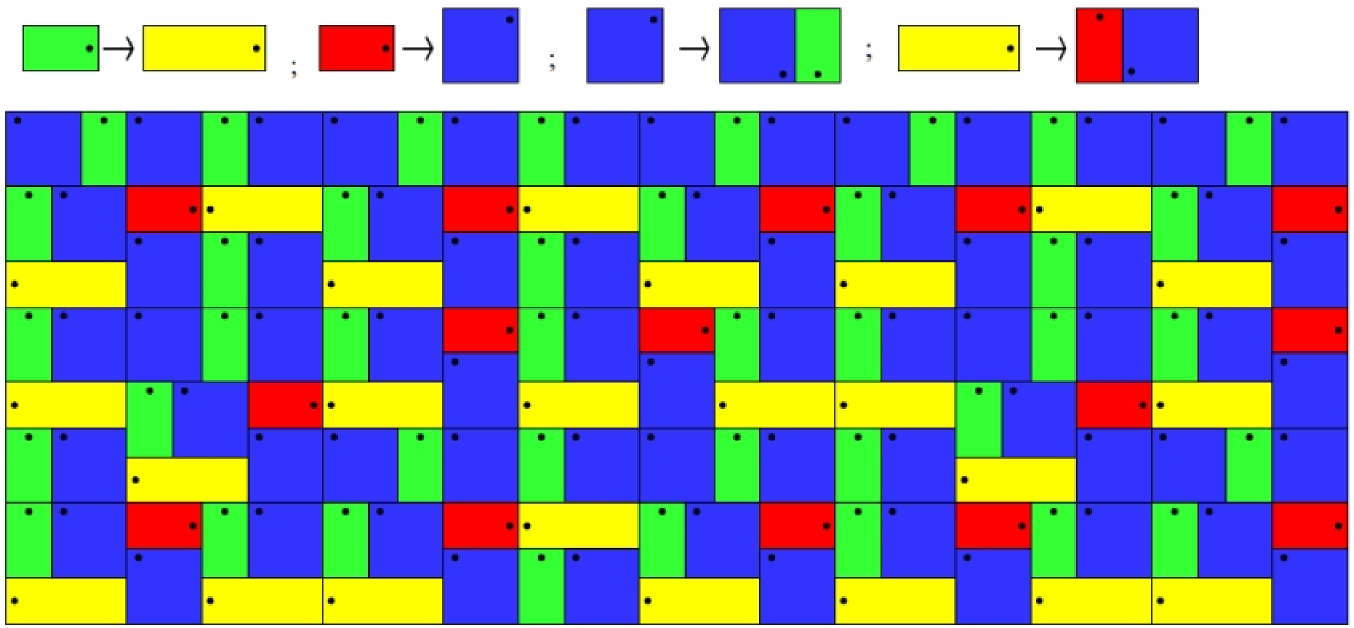
$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues:

$\tau := \frac{1+\sqrt{5}}{2}; -\tau^{-1}; +1; -1$

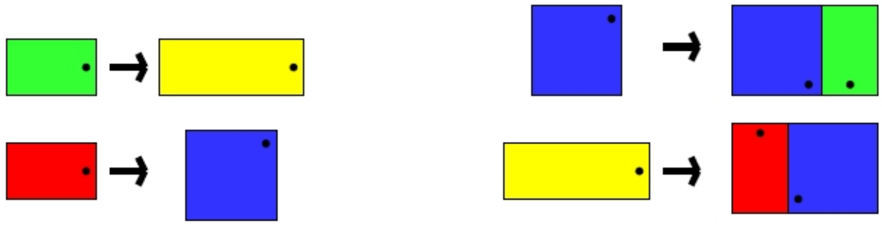
hence τ is a PV number

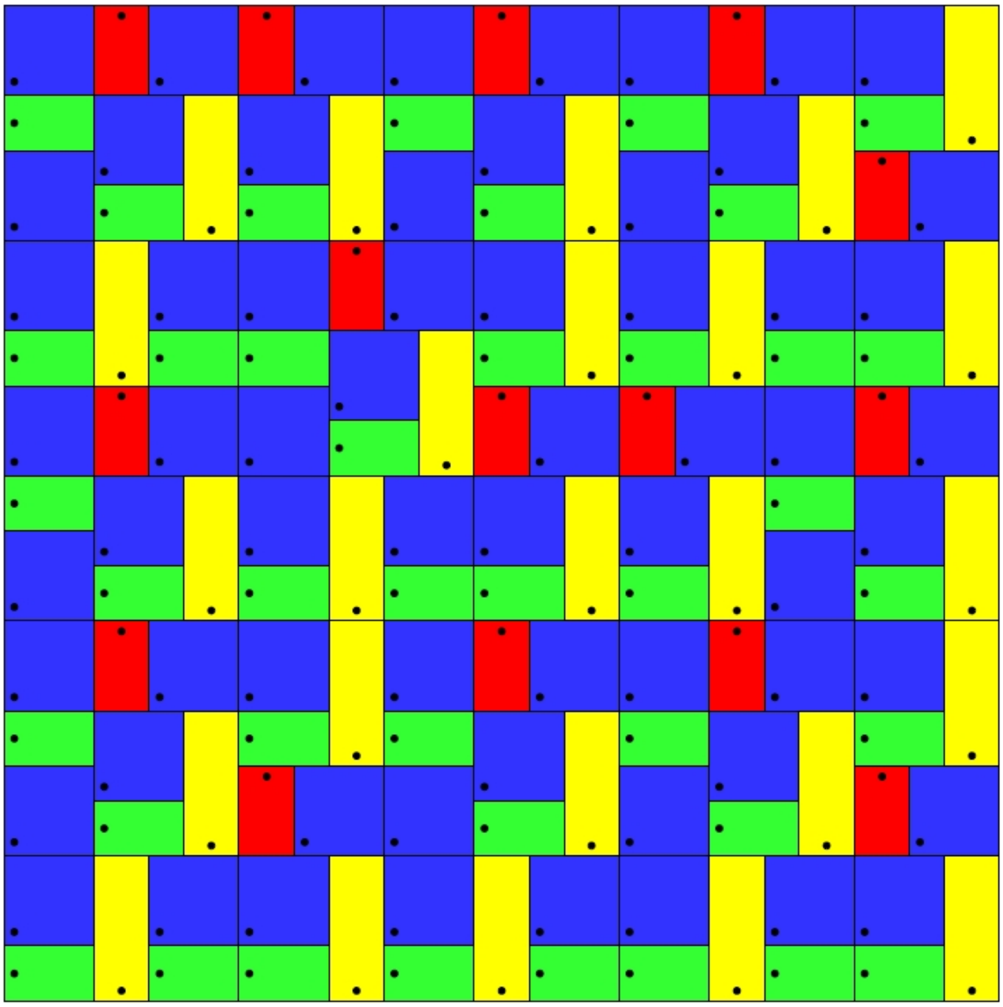
To Example 2



$\mathrm{subst}^{12}(A) \qquad x := \tau$

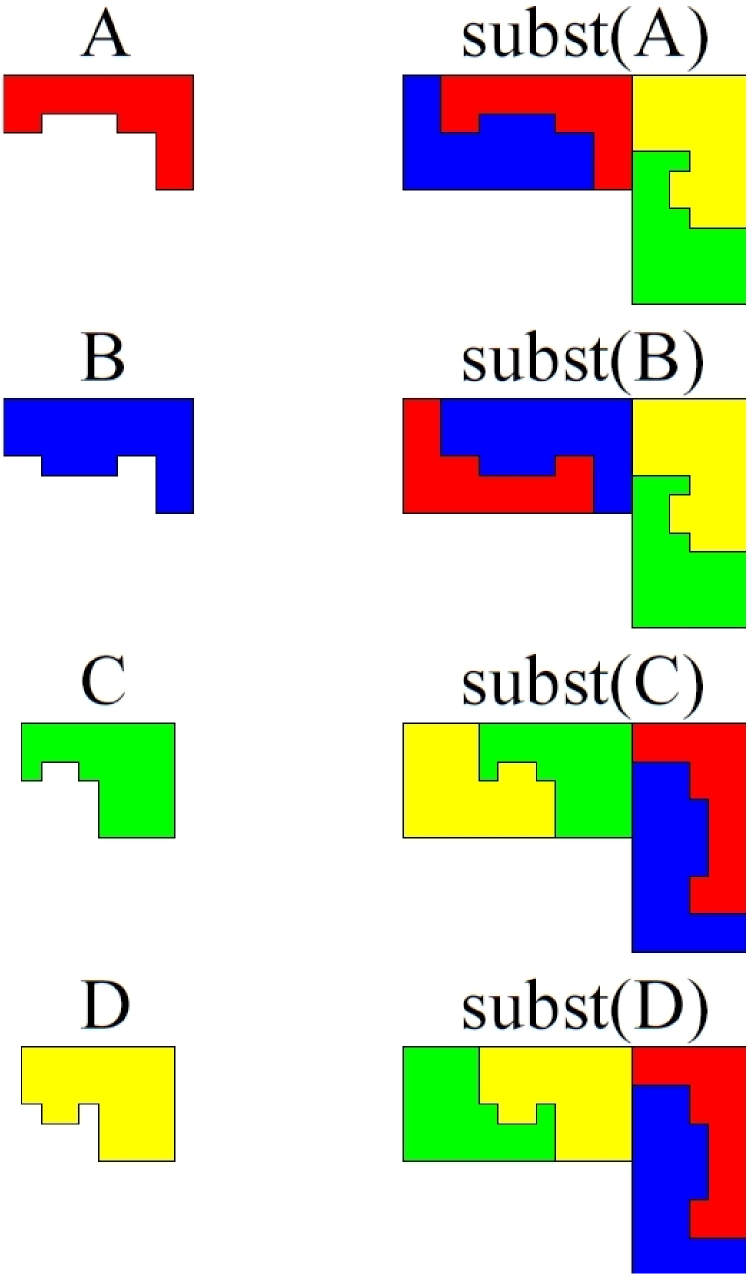
To Example 2





$\mathrm{subst}^{12}(B)$ $\quad$ $x := \tau$

Example 3



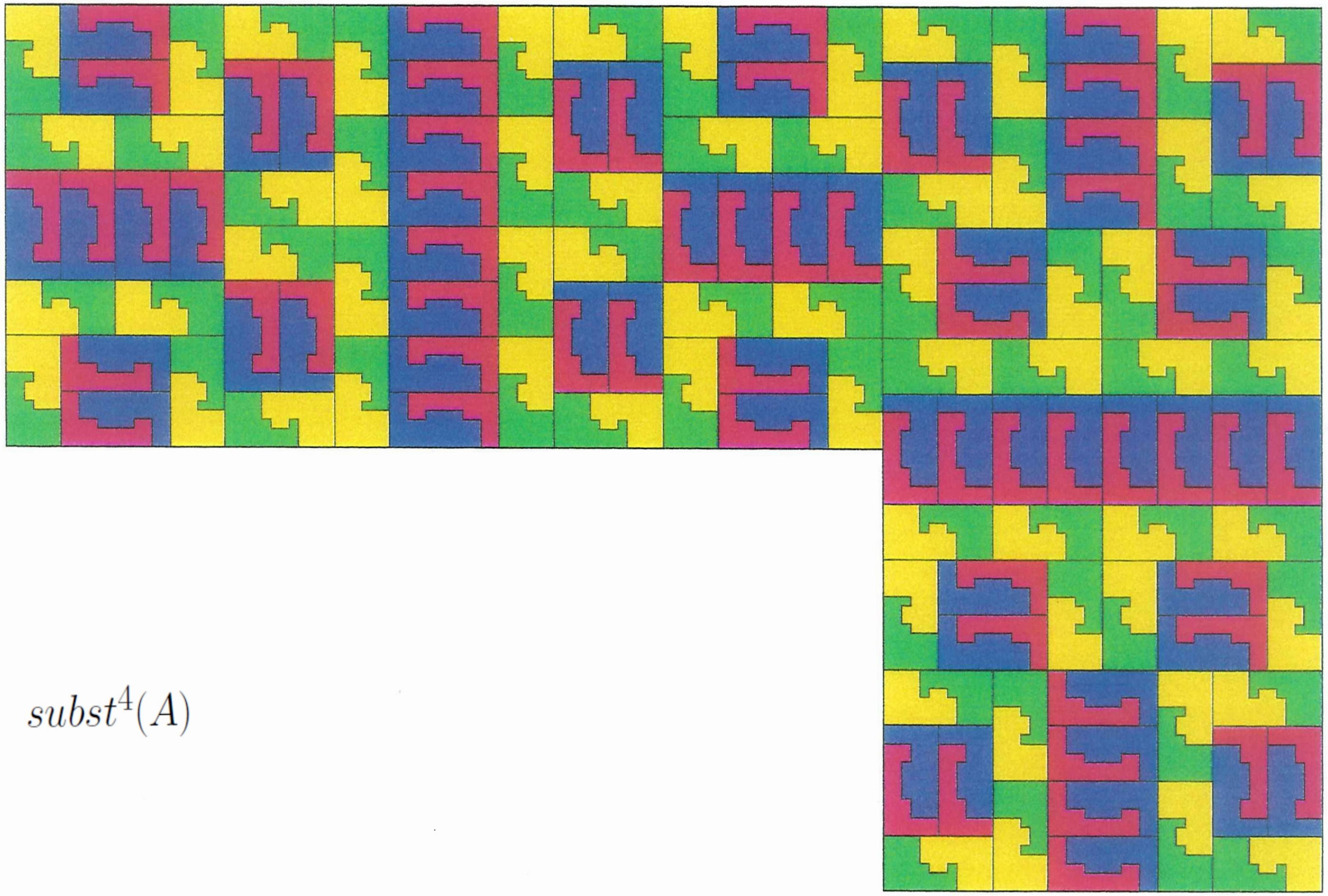
$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$det(M_3) = 0$

Eigenvalues:
$\lambda_1 = 4, \ \lambda_2 = 1, \ \lambda_3 = \lambda_4 = 0$

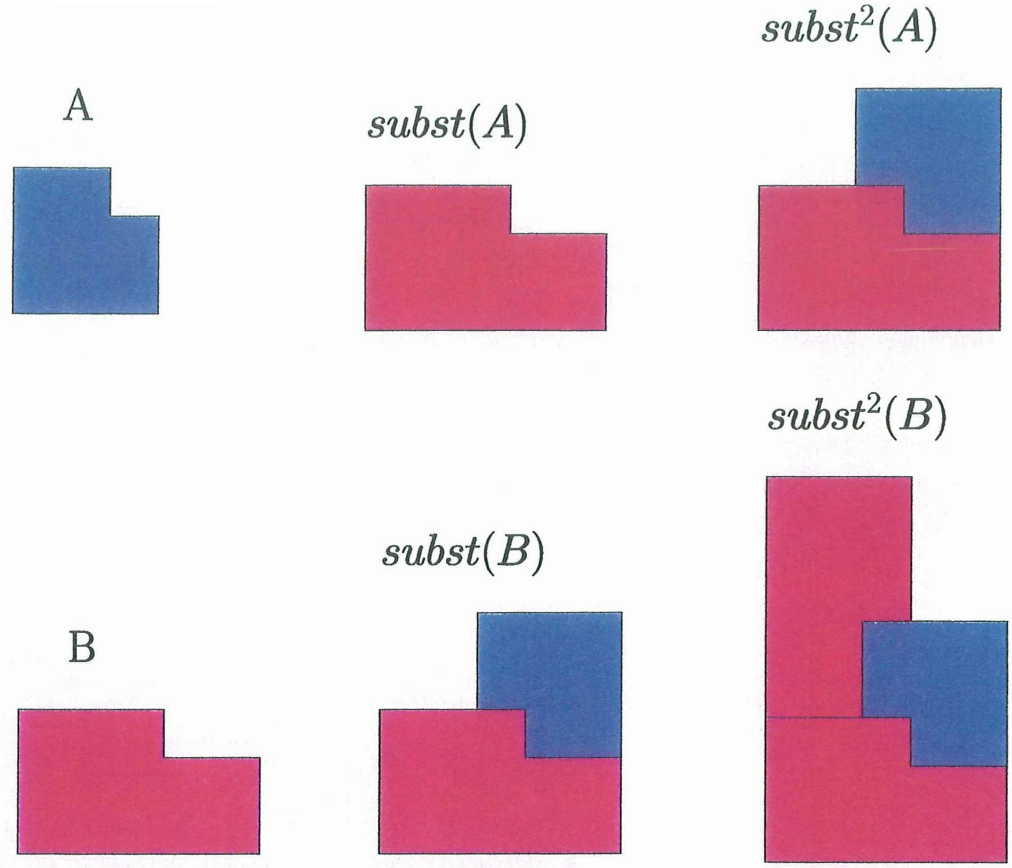
$\eta = \sqrt{\lambda_1} = 2$ is a PV-number
$S(\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}, \mathbf{subst})$ is quasiperiodic

To Example 3



$subst^4(A)$

Example 4



$$M_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

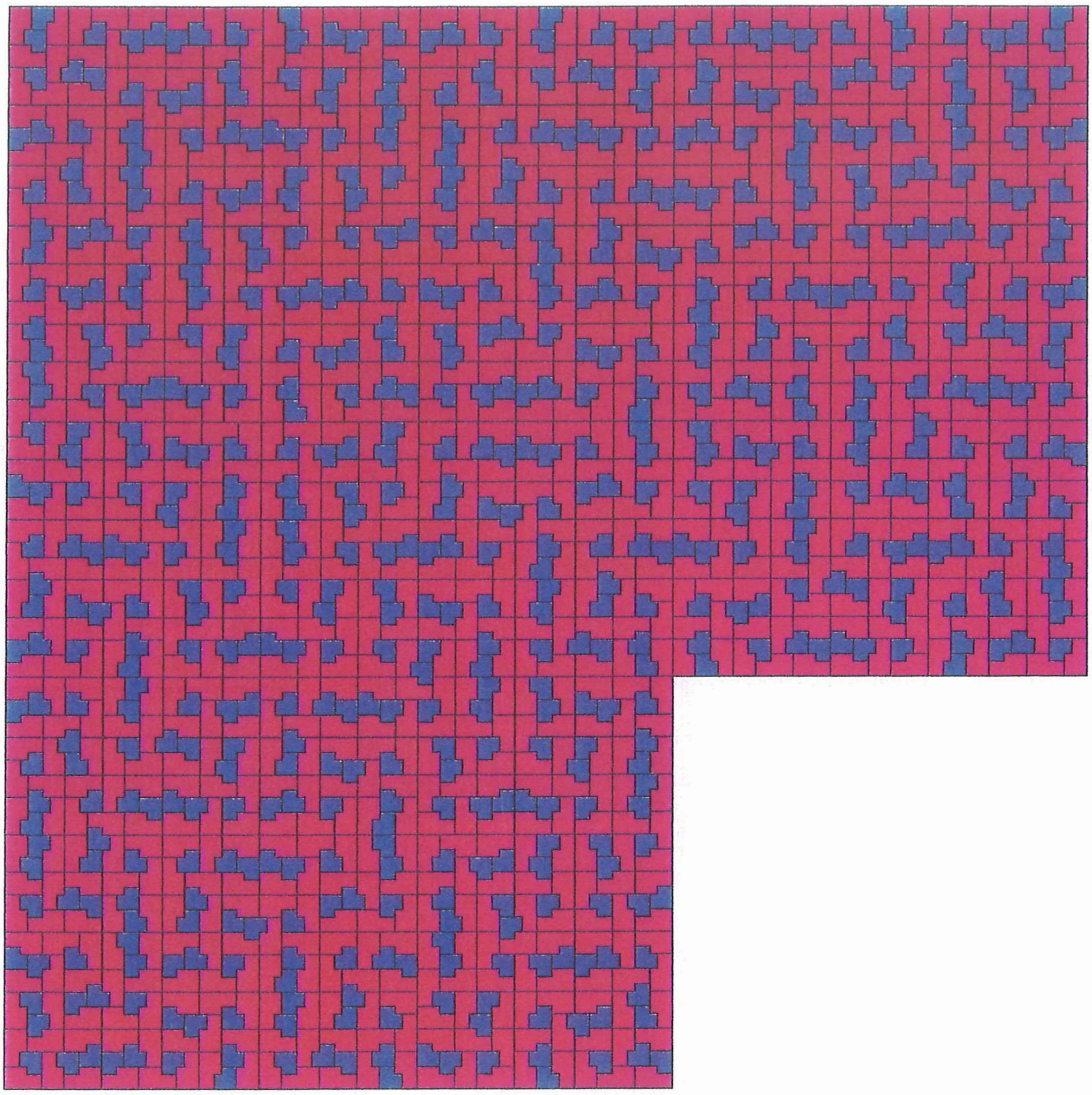
$det(M_4) = -1$

$\lambda_1 = \tau$, $\lambda_2 = -\tau^{-1}$

$\eta = \sqrt{\tau}$ is not a PV-number

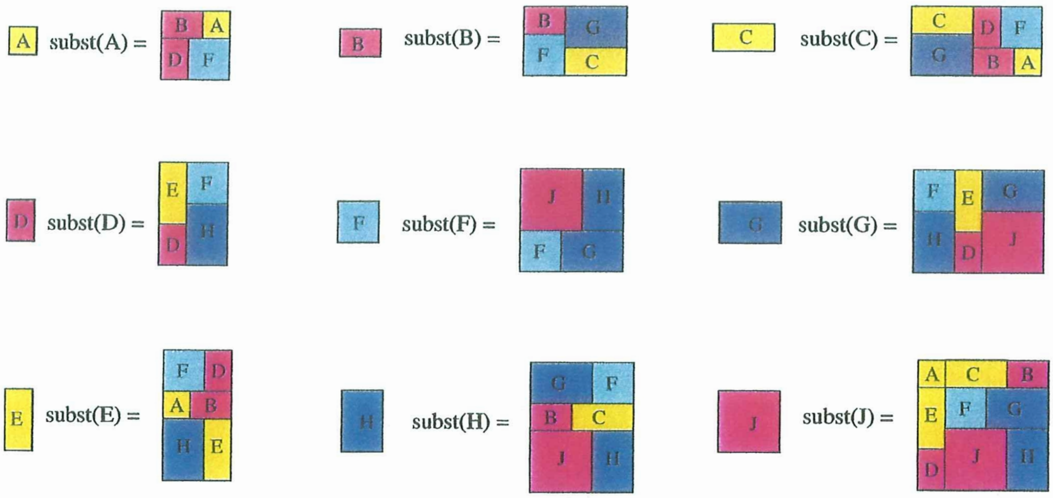
$\mathrm{S} = (\{A, B\}, subst)$ is aperiodic but not quasiperiodic

To Example 4



$subst^{16}(A) = subst^{15}(B)$

Example 5



$$M_5 := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$det(M_5) = 1$

Eigenvalues :

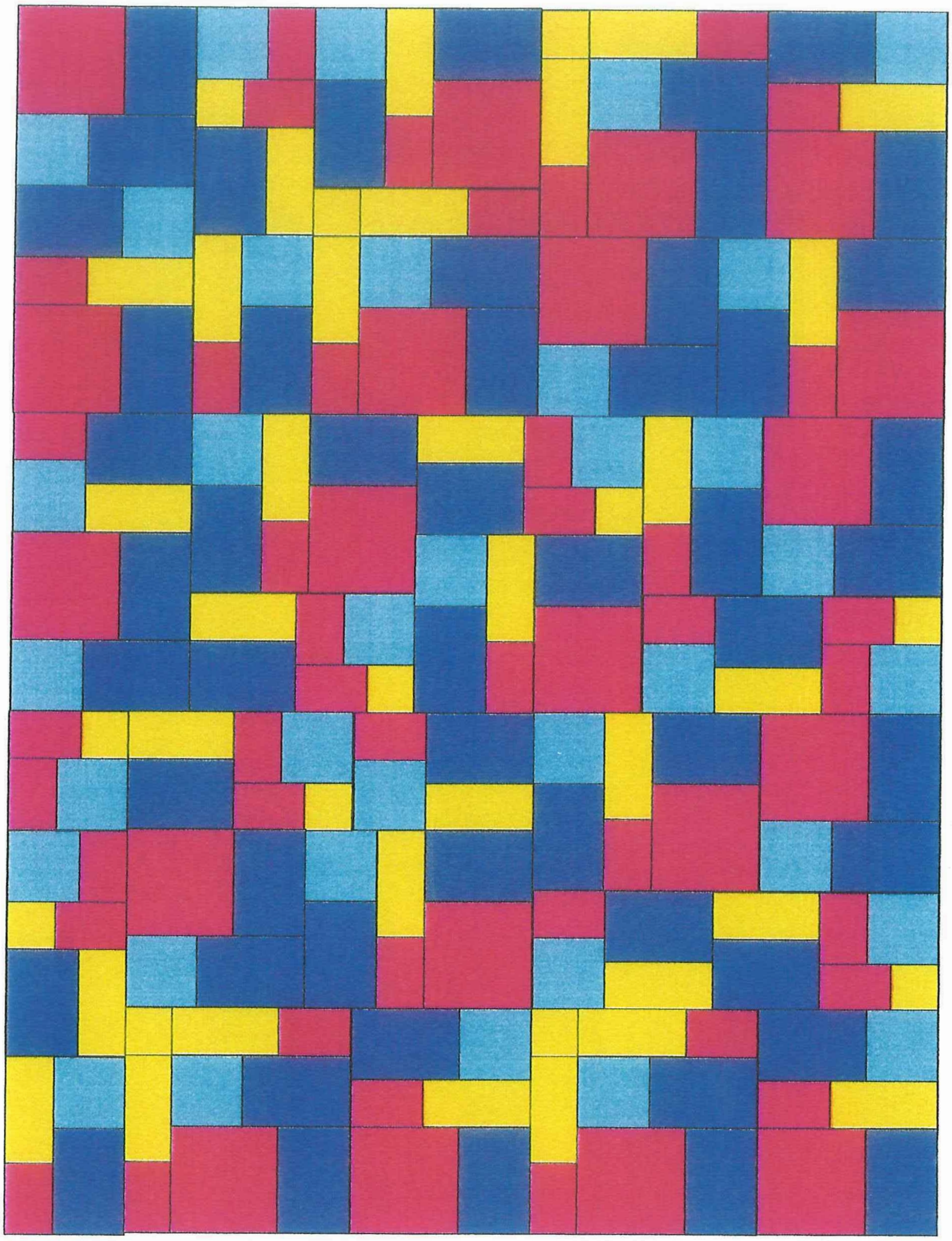
$\lambda_1 \approx 5.404313581$ $\quad \lambda_{2,3} \approx -0.202157 \pm 0.379697i$

$\lambda_4 \approx 0.430160$ $\quad \lambda_{5,6} \approx 0.784920 \mp 1.307141i$

$\lambda_7 = \lambda_4$, $\lambda_{8,9} = \lambda_{5,6}$

$\chi_M(x) = (x^3 - 5x^2 - 2x - 1)(x^3 - 2x^2 + 3x - 1)^2$

To Example 5



x:=1.5 $subst^3(H)$

M_5^2 is strictly positive and hence.

(I) M_5 is primitive.

If ξ denotes real root of $z^3 - z - 1$, then $\lambda_1 = \xi^6 = \xi^{-3}$ and $\eta = \sqrt{\lambda_1} = \xi^3 = 1 + \xi \approx 2.324718$. η is the leading Eigenvalue of the matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

It satisfies $\eta^3 - 3\eta^2 - 2\eta - 1 = 0$. This polynomial is irreducible over $\mathbf{Q}$; the algebraic conjugates of η are $\eta', \eta'' \approx 0.337641 \pm 0.562280i$, hence

(II) η is a PV-number.

(I),(II) imply

(III) If the factor $q(T) := \frac{area(subst(T))}{area(T)}$ is the same for all $T \in \{A, B, ..., J\}$, then $S(A, B, ...J, subst)$ is *quasiperiodic.*

In fact $q(A)=q(B)=q(D)=q(F)=(1+x)^2$,
$q(C)=q(E)=q(G)=q(H)=(1+x)\frac{1+x+x^2}{x^2}$ and $q(J)=(\frac{1+x+x^2}{x^2})^2$.
These values coincide precisely if $1 + x + x^2 = x^2 + x^3$, i.e. if $x^3 = 1 + x$ and $x \in \mathbb{R}$.
In this case $x = \xi$ and subst becomes an inflation with $\eta = 1 + \xi$.

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