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CONTINUOUS CONVOLUTION HEMIGROUPS INTEGRATING A SUB-MULTIPLICATIVE FUNCTION

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ABSTRACT. In [19, 20] E. Siebert obtained the following remarkable result: A Lévy process on a completely metrizable topological group $\mathbb{G}$, resp. a continuous convolution semigroup of probabilities satisfies a moment condition $\int f d\mu_t < \infty$ for some sub-multiplicative f if and only if the jump measure of the process resp. the Lévy measure η of the continuous convolution semigroup satisfies $\int_{\complement U} f d\eta < \infty$ for some neighbourhood U of the unit e. Here we generalize this result to *additive* processes on (second countable) locally compact groups resp. to convolution *hemi*groups $(\mu_{s,t})_{s \leq t}$.

INTRODUCTION

A probability ν on a normed vector space $(\mathbb{V}, || \cdot ||)$ possesses a $k-$th moment, if $\int ||x||^k d\nu < \infty$, equivalently, if $f : x \to 1 + ||x||^k$ is $\nu-$integrable. f is continuous, sub-multiplicative, symmetric and satisfies $f(0) = 1$. Hence moment conditions are integrability conditions for (particular) sub-multiplicative functions.

For investigations in limit theorems on more general structures, in particular on locally compact groups, investigations of integrability of sub-multiplicative functions provide interesting tools. In [19], Theorem 1, [20], Theorem 5, E. Siebert obtained characterizations of integrability of such f for continuous convolution semigroups resp. for Lévy processes, in terms of the behaviour of the Lévy measures, resp. the jump-measures of the processes: [19] is based on analytical methods whereas in [20] the emphasis is laid on the behaviour of the processes. In fact, a partial key result, [20], Theorem 4, is proved for (general) additive processes resp. for convolution hemigroups. Whereas the afore mentioned characterization of integrability of sub-multiplicative f (relying on [20], Theorem 5,) is proved there only for continuous convolution semigroups resp. for Lévy processes.

For particular *hemi*groups and particular f ('logarithmic moments') appearing in investigations of self-decomposability resp. of (generalized) Ornstein-Uhlenbeck processes on homogeneous groups, Siebert's results were already generalized: For homogeneous groups see e.g. [3, 5], for vector spaces see e.g., [12]. (For 'logarithmic moments' consider the sub-multiplicative functions $f : x \mapsto 1 + \log(1 + ||x||) \approx \log^+(||x||)$.)

Hemigroups resp. additive processes turned out to be essential for investigations in various applications. The background for hemigroups

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on locally compact groups is found e.g., in [21], [7], [8], [9] and the references mentioned there; see also [1] for further applications.

The afore mentioned result ([20], Theorem 5, resp. [19], Theorem 1) relies on a splitting of the underlying Lévy measure of the continuous convolution semigroup $(\mu_t)_{t\geq 0}$ (resp. the jump-measure of the underlying process) into a part with bounded support and a bounded measure. Hence we obtain two continuous convolution semigroups $\left(\mu_t^{(i)}\right)_{t\geq 0}$, $i = 1, 2$: For the first any f is integrable, the second one is a Poisson semigroup, and the underlying continuous convolution semigroup $(\mu_t)_{t\geq 0}$ is represented by a perturbation series in terms of $\left(\mu_t^{(i)}\right)_{t\geq 0}$, $i = 1, 2$. This technique allows to reduce the investigations to the Poisson part, and we obtain ([19, 20]): f is integrable w.r.t. the underlying continuous convolution semigroup iff f is integrable w.r.t. the bounded part of the Lévy measure.

Here, in Theorem 4.3, we generalize Siebert's results to (Lipschitz-continuous) convolution *hemigroups* on locally compact groups. We start in Section 1 with perturbation series for *operator hemigroups* (also called generalized semigroups or evolution families) to provide the tools for the next sections. Then, following (and generalizing) the proofs in [20] resp. [19], we obtain a version of Siebert's characterization in the general situation. At the first glance, a slightly weaker version, since an additional technical condition (4.4) is needed. This condition is however always satisfied for continuous convolution semigroups.

1. Perturbation series representations for hemigroups of operators

Definition 1.1. *Let $\mathbb{B}$ be a separable Banach space, and $\mathcal{B}(\mathbb{B})$ the Banach space of bounded operators. A family $\{U_{t,t+s}\}_{0\leq s\leq s+t\leq T} \subseteq \mathcal{B}(\mathbb{B})$, $(T \leq \infty)$ is called continuous hemigroup of operators if $(s,t) \mapsto U_{t,t+s}$ is continuous w.r.t. the strong operator topology, $U_{s,s} = I$ for all s, and $U_{s,r}U_{r,t} = U_{s,t}$ for all $s \leq r \leq t$, and finally $||U_{t,t+s}|| \leq M\mathrm{e}^{\beta s}$ for all $t, s \geq 0$, for some $M \geq 1$ and $\beta \geq 0$.*

To simplify notations, here we shall throughout restrict to the case $M = 1$ and frequently also $\beta = 0$, i.e., we restrict to contractions.

Hemigroups of operators were investigated under different notations, e.g., *evolution families* or *evolution operators* ([14, 15, 6, 10]) or *semi-groupes generaliseés* ([16]), etc. In view of the applications to distributions of additive processes we prefer the expression *operator hemigroups* (cf. [8]) in analogy to the standard notations in probability theory.

Theorem 1.2. ***a)*** *Let $\{U_{s,t}\}_{0\leq s\leq t}$ be a continuous hemigroup of contractions. Let $\mathbb{R} \ni t \mapsto C(t) \in \mathcal{B}(\mathbb{B})$ be a measurable mapping, uniformly bounded, $||C(t)|| \leq \beta$ for all $t \geq 0$. Then*

$$V_{t,t+s} := \sum_{k\geq 0} V_{t,t+s}^{(k)} \quad \textit{with}$$

$$V_{t,t+s}^{(0)} := U_{t,t+s}, \quad V_{t,t+s}^{(k+1)} := \int_0^s V_{t,t+u}^{(0)} C(t+u) V_{t+u,t+s}^{(k)} du$$

defines a continuous hemigroup satisfying a growth condition $||V_{t,t+s}|| \leq \mathrm{e}^{\beta s}$ for all $t, s \geq 0$.

b) *If* $s \mapsto U_{t,t+s}$ *is a.e. differentiable with* $\frac{\partial^+}{\partial s} U_{t,t+s}|_{s=0}(x) =: A(t)(x)$ *for* $x \in D(A(t))$*, and if* $\mathbb{D} := \bigcap_{t\geq 0} D(A(t))$ *is dense, then for all* $x \in \mathbb{D}$ $s \mapsto V_{t,t+s}(x)$ *is differentiable a.e. with* $\frac{\partial^+}{\partial s} V_{t,t+s}(x)|_{s=0} = A(t)x + C(t)x$*, resp. in integrated form:* $V_{t,t+s}(x) = \int_0^s V_{t,t+u}\left(A(u)+C(u)\right)(x)du$

c) *In particular, let* $C(t) = c(t)(S(t) - I)$ *with contractions* $S(\cdot)$*,* $0 \leq c(\cdot) \leq \beta$*, where* $t \mapsto c(t)$ *and* $t \mapsto S(t)$ *are measurable. Then we obtain representations*

$$
\begin{aligned}
V_{t,t+s} &= \mathrm{e}^{-\beta s} \sum_{k\geq 0} W^{(k)}_{t,t+s}, \quad \textit{with} \quad ||W^{(k)}_{t,t+s}|| \leq \frac{\beta^k s^k}{k!} && (1.1)\\
W^{(0)}_{t,t+s} &:= U_{t,t+s}, \quad W^{(k+1)}_{t,t+s} := \int_0^s W^{(0)}_{t,t+u} \widetilde{C}(t+u) W^{(k)}_{t+u,t+s} du, \quad \textit{where} \\
\widetilde{C}(\tau) &= C(\tau) + \beta\cdot I = c(\tau)S(\tau) + (\beta - c(\tau))\cdot I \\
&\quad \textit{alternatively,} \\
V_{t,t+s} &= \mathrm{e}^{-\beta s} \sum_{k\geq 0} \frac{s^k\beta^k}{k!} \widetilde{W}^{(k)}_{t,t+s} && (1.2)\\
\textit{with} &\quad ||\widetilde{W}^{(0)}_{t,t+s}|| \leq 1, \ \widetilde{W}^{(k)}_{t,t+s} := \frac{k!}{s^k\beta^k} W^{(k)}_{t,t+s}
\end{aligned}
$$

Proof. Consider the Banach space of measurable functions $L^1(\mathbb{R}_+, \mathbb{B}) = \left\{ f : \mathbb{R}_+ \to \mathbb{B} : ||f||_* := \int_{\mathbb{R}_+} ||f(t)||dt < \infty \right\}$

$$
\begin{aligned}
\text{Then} \quad \mathcal{P}_s &: \quad \left(\mathcal{P}_s f\right)(t) := U_{t,t+s}\left(f(t+s)\right), && (1.3)\\
\text{and} \quad \mathcal{Q}_s &: \quad \left(\mathcal{Q}_s f\right)(t) := \mathrm{e}^{s\cdot C(t)}\left(f(t)\right), \ \forall t, s \geq 0, && (1.4)
\end{aligned}
$$

define continuous one-parameter *semi*-groups of '*space-time*' operators on $L^1(\mathbb{R}_+, \mathbb{B})$, where $(\mathcal{P}_s)_{s\geq 0}$ are contractions and $|||\mathcal{Q}_s||| \leq \mathrm{e}^{s\cdot\beta}$, $s \geq 0$, $|||\cdot|||$ denoting the operator norm on $\widetilde{\mathbb{B}} := (L^1(\mathbb{R}_+, \mathbb{B}), ||\cdot||_*)$. See e.g., [16], II.7, [8], 8.6, 8.7 for the space-time semigroup (1.3), with $\widetilde{\mathbb{B}} := C_0(\mathbb{R}_+, \mathbb{B})$. Here, to ensure $\mathcal{Q}_s\widetilde{\mathbb{B}} \subseteq \widetilde{\mathbb{B}}$ in (1.4), we had to use $\widetilde{\mathbb{B}} := L^1(\mathbb{R}_+, \mathbb{B})$.

Let $\mathbb{T}$ and $\mathbb{S}$ denote the generators of $(\mathcal{P}_s)_{s\geq 0}$ and $(\mathcal{Q}_s)_{s\geq 0}$ respectively. In particular, $\mathbb{S} : \ (\mathbb{S}f)(t) := C(t)(f(t))$, $t \geq 0$, is a bounded operator. Let $(\mathcal{R}_s)_{s\geq 0}$ denote the semigroup generated by $\mathbb{T} + \mathbb{S}$. (The addition of generators is well defined since $\mathbb{S}$ is bounded.)

According to T. Kato [13], IX, §2, Theorem 2.1, (2.4), (2.5), resp. [11], (13.2.4)–(13.2.6), or [16], II.3, $(\mathcal{R}_s)_{s\geq 0}$ is representable by a norm-convergent *perturbation series* in $\mathcal{B}(\widetilde{\mathbb{B}})$:

$\mathcal{R}_s = \sum_{k\geq 0} \mathfrak{V}^{(k)}_s$ where $\mathfrak{V}^{(0)}_s = \mathcal{P}_s$ and $\mathfrak{V}^{(k+1)}_s = \int_0^s \mathcal{P}_u \mathbb{S} \mathfrak{V}^{(k)}_{s-u} du.$

(Obviously, we have $\mathfrak{V}^{(k+1)}_s = \int_0^s \mathcal{P}_{s-u} \mathbb{S} \mathfrak{V}^{(k)}_u du$, cf. e.g., [13], [11].)

Let $f \in \widetilde{\mathbb{B}}, \ k \geq 0, \ t, s \geq 0, \ 0 \leq u \leq s.$

Claim: $\forall t, s \geq 0, k \in \mathbb{Z}_+$ there exist operators $V^{(k)}_{t,t+s} \in \mathcal{B}(\mathbb{B})$ such that

$$\left(\mathfrak{V}^{(k)}_s f\right)(t) = V^{(k)}_{t,t+s}\left(f(t+s)\right) \quad \lambda^1 - \text{a.e.} \tag{1.5}$$

$[\![$ $k = 0$: $\left(\mathfrak{V}^{(0)}_s f\right)(t) = \left(\mathcal{P}_s f\right)(t) = U_{t,t+s}\left(f(t+s)\right)$, hence the assertion with $V^{(0)}_{t,t+s} = U_{t,t+s}$.

$k+1>0:$ Assuume that (1.5) is proved for $k' \leq k$. Then

$$\left(\mathfrak{V}_r^{(k+1)} f\right)(w) = \int_0^r \left(\mathfrak{V}_u^{(0)} \mathbb{S} \mathfrak{V}_{r-u}^{(k)} f\right)(w) du$$
$$= \int_0^r U_{w,w+u}\left(h_k(w+u)\right) du =: (*),$$

where $h_k(w') := C(w')(g_k(w'))$, $g_k(w') := V_{w',w'+r-u}^{(k)}\left(f(w'+r-u)\right)$.

For $w' := w+u$ we obtain therefore

$(*) = \int_0^r U_{w,w+u} C(w+u) V_{w+u,w+r}^{(k)}\left(f(w+r)\right) du$. Inserting $r = s, w = t$ this yields

$$\left(\mathfrak{V}_s^{(k+1)} f\right)(t) =$$
$$\int_0^s U_{t,t+u} C(t+u) V_{t+u,t+s}^{(k)}\left(f(t+s)\right) du =: V_{t,t+s}^{(k+1)}\left(f(t+s)\right)$$

Put $f = \varphi \otimes x$, $x \in \mathbb{B}, \varphi \in L^1(\mathbb{R}_+)$, i.e., $f : t \mapsto \varphi(t)x$, where $0 \leq \varphi \leq 1$, and $\varphi \equiv 1$ on $[a,b]$. Then for $s, t, s+t \in [a,b]$ we obtain:

$V_{t,t+s}^{(k+1)}\left((\varphi \otimes x)(s+t)\right) = V_{t,t+s}^{(k+1)}(x) = \int_0^s U_{t,t+u} C(t+u) V_{t+u,t+s}^{(k)}(x) du$, as asserted. $\rrbracket$

Note that (1.5) holds true for λ^1−a almost all t. But considering the particular $f := \varphi \otimes x$ as above, continuity of $(t, r+s) \mapsto U_{t,t+s}(x)$ $(\forall x)$ yields that $(t, t+s) \mapsto V_{t,t+s}^{(k)}(x)$ is continuous ($\forall x$ and $\forall k$.) Hence for $f = \psi \otimes x$, $\psi \in L^1 \cap C_0(\mathbb{G})$, (1.5) is valid for all $t \geq 0$.

Note that $V_{t,t+u}^{(0)} = U_{t,t+u}$, $V_{t',t'+s'}^{(1)} = \int_0^{s'} U_{t',t'+u_1} C(t'+u_1) U_{t'+u_1,t'+s'} du_1$, hence, inserting $t' = t+u$, $s' = s-u$

$$V_{t,t+s}^{(2)} =$$
$$\int_0^s \int_0^{s-u} U_{t,t+u} C(t+u) U_{t+u,t+u+u_1} C(t+u+u_1) U_{t+u+u_1,t+s} du_1 du$$
$$\cdots$$

whence by induction

$$V_{t,t+s}^{(k+1)} = \int_0^s \int_0^{w_0} \cdots \int_0^{w_k} U_{t,t+v_0} C(t+v_0) \cdots$$
$$\cdots U_{t+v_k} C(t+v_{k+1}) U_{t+v_{k+1},t+s} du_{k+1} \cdots du_1 du \tag{1.6}$$

where $v_0 := u, v_i := u + \sum_1^i u_j, w_i := s - v_i$.

Whence immediately $|||V_{t,t+s}^{(k)}||| \leq \frac{s^k \beta^k}{k!}$ follows, hence $||V_{t,t+s}|| \leq e^{\beta s}$.

Finally, the relations $\mathcal{R}_s\left(\varphi \otimes x\right)(t) = \left(\sum_k V_{t,t+s}^{(k)}(x)\right) \cdot \varphi(t+s) =: V_{t,t+s}(x) \cdot \varphi(t+s)$ and $\mathcal{R}_s \mathcal{R}_{s'} = \mathcal{R}_{s+s'}$ yield the hemigroup property $V_{t,t+s} V_{t+s,t+s+s'} = V_{t,t+s+s'}$. (Here, φ, s, s', t are suitably chosen as above.)

b) **Claim:** Let $x \in \mathbb{D}$ then

$\frac{d^+}{ds} V_{t,t+s}(x)|_{s=0} = \sum_k \frac{d^+}{ds} V_{t,t+s}^{(k)}(x)|_{s=0} = A(t)(x) + C(t)(x)$

$\llbracket$ $k = 0:$ By assumption, $\frac{d^+}{ds} V_{t,t+s}^{(0)}(x)|_{s=0} = \frac{d^+}{ds} U_{t,t+s}(x)|_{s=0} = A(t)(x)$ for $x \in D(A(t))$.

Furthermore, for $f \in D(\mathbb{T})$ we have $\frac{d^+}{ds} \mathcal{R}_s f|_{s=0} = \mathbb{T} f + \mathbb{S} f$.

If $x \in \mathbb{D}$ and $\varphi \in C^1 \cap L^1(\mathbb{R}_+)$ then $f := \varphi \otimes x \in D(\mathbb{T})$, and $(\mathbb{T} f)(t) = \frac{d^+}{ds}\left(U_{t,t+s}(x) \cdot \varphi(t+s)\right)|_{s=0} = A(t)(x) \cdot \varphi(t) + x \cdot \varphi'(t)$. On the other hand, $\mathbb{S}(\varphi \otimes x)(t) = C(t)(x) \cdot \varphi(t)$. Moreover, $\frac{d^+}{ds} e^{s\mathbb{S}}|_{s=0} = \mathbb{S}$

is bounded, hence we obtain for λ^1-almost all t

$$\begin{aligned}\frac{d^+}{ds}\left(V_{t,t+s}(x)\varphi(t+s)\right)|_{s=0} &= \frac{d^+}{ds}\mathcal{R}_s\,|_{s=0}(\varphi\otimes x)(t)\\ &= \frac{d^+}{ds}\left((U_{t,t+s}(x)\cdot\varphi(t+s)\right)|_{s=0} + C(t)(x)\cdot\varphi(t)\\ &= x\cdot\varphi'(t) + \left(A(t)+C(t)\right)(x)\cdot\varphi(t)\end{aligned}$$

Whence the assertion follows if we choose φ and $t, t+s$ suitable as before.]

c) Proof of the special case:
Put $\mathbb{S} =: \widetilde{\mathbb{S}} - \beta I$, i.e. define $\widetilde{C}(t) := c(t)S(t) + (\beta - c(t))\cdot I$ and $\widetilde{\mathbb{S}} : t \mapsto \widetilde{C}(t)\,(f(t))$. Denote by $(\mathcal{R})_{s\geq 0}$ the semigroup generated by $\mathbb{T}+\widetilde{\mathbb{S}}$ and represent $\widetilde{\mathcal{R}}_s$ by a perturbation series. In view of $\mathcal{R}_s = \widetilde{\mathcal{R}}_s\cdot \mathrm{e}^{-s\cdot\beta}$, the assertion follows. □

2. Continuous hemigroups of probabilities and perturbation series

In the following let $\mathbb{G}$ denote a locally compact topological group. $\mathbb{G}$ is assumed to be second countable. By $\mathcal{M}^1(\mathbb{G})$ we denote the convolution semigroup of probabilities, $\star$ denotes convolution. We use the abbreviation $\langle\nu, f\rangle = \int_{\mathbb{G}} f d\nu$.

In the sequel we apply the results of Section 1 to operators defined by convolution hemigroups on a locally compact group. (Cf. Definition 2.1 below). There, $\mathbb{B} := C_0(\mathbb{G})$ and $\mu \in \mathcal{M}^b(\mathbb{G})$ is identified with the convolution operator $R_\mu : R_\mu f(x) := \int_{\mathbb{G}} f(xy)d\mu(y)$, $f \in C_0(\mathbb{G})$.

Definition 2.1. ***a)*** *A continuous convolution semigroup is a one-parameter family of probabilities $(\mu_s)_{s\geq 0}$ depending continuously on s, and fulfilling $\mu_{s+t} = \mu_s \star \mu_t$ for all $s,\ t \geq 0$. Throughout we assume $\mu_0 = \varepsilon_0$.*
b) *(Cf.* [21, 7, 8]*.) A **convolution hemigroup** is a two-parameter family of probabilities $(\mu_{t,t+s})_{0\leq t\leq t+s\leq T}$, depending continuously on the parameters $(t, t+s)$ and fulfilling $\mu_{t,t+s}\star\mu_{t+s,t+s+s'} = \mu_{t,t+s+s'}$ for all $0 \leq t \leq t+s \leq t+s+s' \leq T$, for some $0 < T \leq \infty$.*

If $(\mu_{t,t+s})_{0\leq t\leq t+s\leq T}$ is a convolution hemigroup of probabilities then the convolution operators $\left(U_{t,t+s} := R_{\mu_{t,t+s}}\right)_{0\leq t\leq t+s\leq T}$ form a continuous hemigroup of contractions on the Banach space $\mathbb{B} := C_0(\mathbb{G})$.

We will frequently make use of the following well-known observation:

Lemma 2.2. *Let $(\mu_{t,t+s})_{0\leq t\leq t+s}$ be a separately continuous hemigroup, i.e., $t \mapsto \mu_{s,t}$ and $s \mapsto \mu_{s,t}$ are continuous, and $\mu_{t,t} = \varepsilon_e$ for all t. Then $\forall T < \infty$, for all sequences $0 \leq t_n \leq t_n+s_n \leq T$ with $s_n \to 0$ we obtain: $\mu_{t_n,t_n+s_n} \to \varepsilon_e$.*

Consequently, for all neighbourhoods U of e and all $s_n \to 0$ we obtain: $\sup_{0\leq t\leq T} \mu_{t,t+s_n}(\complement U) \to 0$.

Proof. For all subsequences $(n') \subseteq \mathbb{N}$ there exists a converging subsequence $(n'') \subseteq (n')$, i.e., $t_n \overset{(n'')}{\longrightarrow} t_0 \in [0,T]$. Hence $\forall r > t_0$ we have $r \geq t_n + s_n$ for sufficiently large $n \geq n(r)$ and by continuity, $\mu_{t_n,t_n+s_n}\star\mu_{t_n+s_n,r} \to \mu_{t_0,r}$ along (n''), and also $\mu_{t_n+s_n,r} \to \mu_{t_0,r}$. Whence by the shift-compactness theorem ([17], III, Theorem 2.1, 2.2, [7], Theorem 1.21) we obtain that $\{\mu_{t_n,t_n+s_n}\}$ is relatively compact and all accumulation points ν satisfy $\nu \star \mu_{t_0,r} = \mu_{t_0,r}$. Hence, considering $r = r_n \searrow t_0$, it follows $\nu \star \varepsilon_e = \varepsilon_e$, whence $\nu = \varepsilon_e$.

Hence we have shown: For all subsequences $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ such that $\mu_{t_n,t_n+s_n} \to \varepsilon_e$ along (n''). Whence the assertion follows. □

Corollary 2.3. *For a hemigroup $(\mu_{s,s+t})$ as above we obtain: For all functions $\varphi \in C^b(\mathbb{G})_+$ for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for $0 \le t \le t+s \le T,\ s \le \delta$ it follows $\langle \mu_{t,t+s}, \varphi\rangle \ge \varphi(e) - \varepsilon$.*

Let $(\mu_t)_{t\ge 0}$ be a continuous convolution semigroup with corresponding C_0-contraction semigroup (R_{μ_t}) acting on $C_0(\mathbb{G})$. The infinitesimal generator is defined as $N := \frac{d^+}{dt} R_{\mu_t}\,|_{t=0}$. Then $D(N) \supseteq \mathcal{D}(\mathbb{G})$, the Schwartz-Bruhat space and moreover, $\mathcal{D}(\mathbb{G})$ is a core for N. The *generating functional* is defined as $\langle A, f\rangle := Nf(e) = \frac{d^+}{dt}\langle \mu_t, f\rangle\,|_{t=0}$ for $f \in \mathcal{D}(\mathbb{G})$. In fact, A is canonically extended to $\mathcal{E}(\mathbb{G}) := \{f \in C^b(\mathbb{G}) : f\cdot\varphi \in \mathcal{D}(\mathbb{G})\ \forall \varphi \in \mathcal{D}(\mathbb{G})\}$. (For details see e.g., [7], IV, 4.1–4.5).) As a consequence of E. Siebert's characterization of generating functionals ([18], Satz 5, [7], 4.4.18, 4.5.8) we obtain for Lipschitz-continuous hemigroups $(\mu_{t,t+s})$ that $\frac{d^+}{ds}\langle \mu_{t,t+s}, f\rangle\,|_{s=0} =: \langle A(t), f\rangle$ exists λ^1- a.e. and defines a family of generating functionals $(A(t))_{0\le t\le T}$. (For details see e.g., [21], Theorem 4.3, Corollary 4.5., [8, 9].)

$(\mu_{s,s+t})$ is a priori defined for $0 \le t \le t+s \le T$ (for some $T \le \infty$). If the hemigroup is (a.e.) differentiable with generating functionals $A(t) = \frac{\partial^+}{\partial s}\mu_{t,t+s}|_{s=0}$ and if $T < \infty$ we continue the hemigroup beyond time T defining $A(T+t) := A(t), 0 \le t \le T$, etc.

Next we apply the results of Section 1 to convolution hemigroups. Tacitly we identify measures with convolution operators on $\mathbb{B} := C_0(\mathbb{G})$ and we identify the generating functionals of continuous convolution semigroups with generators of the corresponding C_0-contraction semigroups.

We note the following corollaries to Theorem 1.2:

Corollary 2.4. *Let $(\mu_{t,t+s})$ be a Lipschitz-continuous hemigroup in $\mathcal{M}^1(\mathbb{G})$ with a family of generating functionals $A(t) = \frac{\partial^+}{\partial s}\mu_{t,t+s}|_{s=0}$, for λ^1-almost all t. (For details the reader is referred e.g., to [20], [21], [8].) Let, for $t \ge 0$, $\gamma(t) := c(t)\cdot(\rho(t) - \varepsilon_e)$ be Poisson generators, where $\rho(t) \in \mathcal{M}^1(\mathbb{G})$ and $0 \le c(t) \le \beta$. Furthermore, $t \mapsto c(t)$ and $t \mapsto \rho(t) \in \mathcal{M}^1(\mathbb{G})$ are assumed to be measurable.*

Then there exists an a.e. differentiable hemigroup $(\nu_{t,t+s})$ with generating functionals $\frac{\partial^+}{\partial s}\nu_{t,t+s}|_{s=0} = A(t) + \gamma(t)$, for a.a. $t \ge 0$.

$\nu_{t,t+s}$ admits a representation by perturbation series :

$$\nu_{t,t+s} = \mathrm{e}^{-\beta\cdot s}\sum_{k\ge 0}\nu^{(k)}_{t,t+s}$$

where $\nu^{(0)}_{t,t+s} = \mu_{t,t+s}$, $\nu^{(k+1)}_{t,t+s} = \int_0^s \mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}du$, and $\sigma(r) := c(r)\rho(r) + (\beta - c(r))\varepsilon_e \in \mathcal{M}^b_+(\mathbb{G})$.

Furthermore, $\nu^{(k)}_{t,t+s} \in \mathcal{M}^b_+(\mathbb{G})$ with $||\nu^{(k)}_{t,t+s}|| \le \frac{\beta^k\cdot s^k}{k!}$ for $k \ge 0$.

Proof. Immediate consequence of Theorem 1.2 c), since $||\sigma(r)|| = \beta$ and $||\mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}|| = \beta\cdot||\nu^{(k)}_{t+u,t+s}||$, for all $0 \le t \le t+u \le t+s, k \in \mathbb{Z}_+$. □

In particular we are interested in the following *special case*:

Corollary 2.5. *Let* $(\nu_{t,t+s})$ *be a Lipschitz-continuous hemigroup in* $\mathcal{M}^1(\mathbb{G})$ *with a family of generating functionals* $A(t) = \frac{\partial^+}{\partial s}\nu_{t,t+s}|_{s=0}$, *for* λ^1-*almost all* t. *Let* U *be an open neighbourhood of* e *in* $\mathbb{G}$ *such that the Lévy measures satisfy*

$$\eta_{A(t)}(\complement U) =: c(t) \le \beta < \infty \quad \text{for all} \quad t \tag{2.1}$$

$t \mapsto A(t)$, *hence* $t \mapsto c(t)$ *are measurable. Put* $\gamma(t) := c(t)\,(\rho(t) - \varepsilon_e)$ *with* $\rho(t) := \frac{1}{c(t)}\eta_{A(t)}|_{\complement U} \in \mathcal{M}^1(\mathbb{G})$ *and put* $\overline{A}(t) := A(t) - \gamma(t)$. *Let finally* $(\mu_{t,t+s})$ *be the hemigroup generated by* $\left(\overline{A}(t)\right)$, $t \ge 0$.

Then $(\nu_{t,t+s})$ *admits a series representation*

$$\nu_{t,t+s} = \mathrm{e}^{-\beta s}\sum_{k\ge 0}\nu_{t,t+s}^{(k)}$$

with summands $\nu_{t,t+s}^{(k)}$ *sharing the properties described in Corollary 2.4*

⟦ Put $\gamma(t) := \eta_{A(t)}|_{\complement U} - \eta_{A(t)}(\complement U)\cdot\varepsilon_e = c(t)\,(\rho(t) - \varepsilon_e)$, hence $\sigma(t) = \eta_{A(t)}|_{\complement U} + \left(\beta - \eta_{A(t)}(\complement U)\right)\cdot\varepsilon_e$ and apply Corollary 2.4. ⟧

3. Sub-multiplicative and sub-additive functions

First we collect some properties of sub-multiplicative and sub-additive functions. At first we note the nearly obvious

Lemma 3.1. *Let* $f : \mathbb{G} \to \mathbb{R}_+$ *be sub-multiplicative and* $g : \mathbb{G} \to \mathbb{R}_+$ *sub-additive. Then*

a) *If* $f \ne 0$ *then* $f(e) \ge 1$. *If* $f \ne 0$ *and symmetric, i.e.,* $f(x^{-1}) = f(x)\ \forall x$ *then* $f \ge 1$. *In fact, Proposition 3.3 below shows that* $f \ge f(e)$.

b) $k := f + 1$ *and* $h := g + 1$ *are sub-multiplicative and* ≥ 1.

c) $h := \mathrm{e}^g$ *is sub-multiplicative and* ≥ 1.

d) *If* $f \ge 1$ *then* $h := \log f$ *is sub-additive and* ≥ 0. *Hence according to b),* $\log(g+1)+1$ *is sub-multiplicative and* ≥ 1.

e) *If* $f \ge 1$ *then* $\widetilde{f} : x \mapsto f(x^{-1})$ *is sub-multiplicative and* ≥ 1. *Furthermore,* $h := \max\left(f, \widetilde{f}\right)$ *is sub-multiplicative,* ≥ 1 *and symmetric.*

f) *If* $f \ge 1$ *then* $1/f$ *is super-multiplicative and* $0 < 1/f \le 1$.

To avoid complicated notations we restrict in the following Section to *continuous symmetric sub-multiplicative functions with* $f(e) = 1$. In view of the results mentioned above, and in view of applications we have in mind there is no serious loss of generality.

Lemma 3.2. *Let* g *be sub-additive, symmetric and* ≥ 0. *Then* $g(xy) \ge |g(x) - g(y)|$ *for all* $x, y \in \mathbb{G}$.

⟦ $g(x) = g\left((xy)y^{-1}\right) \le g(xy) + g(y)$ and on the other hand, we have $g(y) = g\left(x^{-1}(xy)\right) \le g(x) + g(xy)$. Whence the assertion. ⟧

Proposition 3.3. *Let* $f : \mathbb{G} \to [1, \infty)$ *be sub-multiplicative and symmetric. Then we have:*

$f(xy) \ge \frac{f(x)}{f(y)}\cdot 1_{\{f(x)\ge f(y)\}} + \frac{f(y)}{f(x)}\cdot 1_{\{f(y)>f(x)\}}$

Whence in particular, $f(xy) \ge \max\left\{\frac{f(x)}{f(y)}, \frac{f(y)}{f(x)}, 1\right\}$

⟦ Applying Lemma 3.2 to $g := \log f$ yields:
$f(xy) = \mathrm{e}^{g(xy)} \geq \mathrm{e}^{|g(x)-g(y)|} = \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geq f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}}$ ⟧

Proposition 3.4. *Let $f : \mathbb{G} \to [1, \infty)$ be measurable, symmetric and sub-multiplicative. Let $\mu, \nu, \lambda \in \mathcal{M}^b_+(\mathbb{G})$. Then we have:*
a) $\langle \mu \star \nu, f\rangle \leq \langle \mu, f\rangle \cdot \langle \nu, f\rangle$
b) $\langle \mu \star \nu, f\rangle \geq \max\{\langle \mu, f\rangle \cdot \langle \nu, 1/f\rangle \ , \ \langle \mu, 1/f\rangle \cdot \langle \nu, f\rangle\}$
Hence
c) $\langle \mu \star \nu \star \lambda, f\rangle \geq$
$\max\left\{\langle \mu, f\rangle \cdot \langle \nu, \frac{1}{f}\rangle \cdot \langle \lambda, \frac{1}{f}\rangle, \ \langle \mu, \frac{1}{f}\rangle \cdot \langle \nu, f\rangle \cdot \langle \lambda, \frac{1}{f}\rangle, \ \langle \mu, \frac{1}{f}\rangle \cdot \langle \nu, \frac{1}{f}\rangle \cdot \langle \lambda, f\rangle\right\}$

Proof. a) is obvious.

$$
\begin{aligned}
&b) \quad \langle \mu \star \nu, f\rangle = \int\int f(xy) d\mu(x) d\nu(y) \\
&\overset{Prop.3.3}{\geq} \int\int \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geq f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}} d\nu(y) d\mu(x) \\
&= \int f(x) \int \frac{1}{f(y)} \left(1_{\{f(x)\geq f(y)\}} + \frac{f(y)^2}{f(x)^2} \cdot 1_{\{f(y)>f(x)\}}\right) d\nu(y) d\mu(x) \\
&\geq \int f(x) \int \frac{1}{f(y)} \left(1_{\{f(x)\geq f(y)\}} + 1_{\{f(y)>f(x)\}}\right) d\nu(y) d\mu(x) \\
&= \langle \mu, f\rangle \cdot \langle \nu, 1/f\rangle
\end{aligned}
$$

The other assertions are now obvious. □

Proposition 3.5. *Let f be continuous, symmetric, sub-multiplicative, let $\mu_n, \mu \in \mathcal{M}^b_+(\mathbb{G})$ with $\mu_n \to \mu$ weakly. Then*
$\langle \mu, f\rangle \leq \liminf \langle \mu_n, f\rangle$

⟦ For all $N > 0$ we have $\langle \mu_n, f \wedge N\rangle \to \langle \mu, f \wedge N\rangle$ by assumption, hence
$\langle \mu, f\rangle = \sup_N \langle \mu, f \wedge N\rangle = \sup_N \lim_n \langle \mu_n, f \wedge N\rangle \leq \liminf_n \langle \mu_n, f\rangle$ ⟧

Proposition 3.6. *Let $f : \mathbb{G} \to [1, \infty)$ be continuous, sub-multiplicative, symmetric with $f(e) = 1$. Let $(\mu_{t,t+s})_{0\leq t\leq t+s}$ be a continuous hemigroup with $\langle \mu_{t_0,t_0+s_0}, f\rangle < \infty$. Then* $\sup\limits_{t_0\leq t\leq t+s\leq t_0+s_0} \langle \mu_{t,t+s}, f\rangle < \infty$.

Proof. Let $\alpha \in (0, 1)$. Then there exist a $\delta = \delta(\alpha) > 0$ such that for $0 < u - v < \delta$ we have $\langle \mu_{u,v}, 1/f\rangle > \alpha$ (cf. Lemma 2.2, Corollary 2.3). Furthermore, according to Lemma 3.4 we have $\langle \mu_{t_0,t_0+s_0}, f\rangle \geq \langle \mu_{t_0,t_0+v}, 1/f\rangle\langle \mu_{t_0+v,t_0+u}, f\rangle\langle \mu_{t_0+u,t_0+s_0}, 1/f\rangle$. Consequently, choose t_1, s_1 such that $t_0 \leq t_1 \leq t_1 + s_1 \leq t_0 + s_0 < \delta$, $t_1 - t_0 < \delta$ and $t_0+s_0-t_1-s_1 < \delta$, then $\langle \mu_{t_1,t_0+s_0}, f\rangle \leq \langle \mu_{t_0,t_0+s_0}, f\rangle\cdot\alpha^{-1}$, $\langle \mu_{t_0,t_1+s_1}, f\rangle \leq \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-1}$, and $\langle \mu_{t_1,t_1+s_1}, f\rangle \leq \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-2}$.

Let $[t_*, t_* + s_*] \subseteq [t_0, t_0 + s_0]$ be a sub-interval of length $s_* < \delta$. Then there exist $t_0 < \cdots < t_i < t_{i+1} < \ldots t_{N+1} := t_0 + s_0$ such that $t_{i+1} - t_i < \delta$ $\forall i$ and $t_* = t_{i_0}, t_* + s_* = t_{i_0+1}$ for some i_0. Therefore, repeating the above consideration $N-$times, we obtain $\langle \mu_{t_*,t_*+s_*}, f\rangle \leq \langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-2N}$.

Hence for any sub-interval $[t, t+s] \subseteq [t_0, t_0+s_0]$, decomposing $[t, t+s]$ in at most N sub-intervals of lengths $\leq \delta$ we obtain $\langle \mu_{t,t+s}, f\rangle \leq \left(\langle \mu_{t_0,t_0+s_0}, f\rangle \cdot \alpha^{-2N}\right)^N$. (Note that $N \approx [s_0/\delta] + 1$ can be chosen independently from the particular decomposition.) □

4. Moments of Lipschitz-continuous hemigroups and their Lévy-measures

The following key-result is proved in [20], Theorem 4:

Proposition 4.1. *Let $(\mu_{t,t+s})_{t,s\geq 0}$ be a Lipschitz continuous hemigroup with generating functionals $(A(t))$, resp. $B(s,t) := \int_s^t A(\tau)d\tau$ and Lévy measures $\eta_{A(\tau)}$ and $\eta_{B(s,t)} = \int_s^t \eta_{A(\tau)}d\tau$ respectively.*

Assume that there exists a neighbourhood U of e such that

$$\eta_{A(\tau)}\left(\complement U\right) = 0 \ \forall \tau, \ \text{hence} \quad \eta_{B(s,t)}\left(\complement U\right) = 0, \ \forall \ s < t \tag{4.1}$$

Then for any continuous sub-multiplicative function $f : \mathbb{G} \to [1, \infty)$, for all $0 < T < \infty$ we have:

$$\sup_{0\leq t\leq t+s\leq T} \langle \mu_{t,t+s}, f\rangle < \infty \tag{4.2}$$

In fact, more is shown there: Let $\alpha > 0, r \in (0, \alpha)$. Then $\exists t > 0$: $\sup\limits_{0\leq s\leq t} \langle \mu_{r,r+s}, f\rangle \leq \int \sup\limits_{0\leq s\leq t} f\left(X_r^{-1}X_{r+s}\right) dP \leq \beta(t)$.

There $\beta(t) \searrow 1$ (with $t \searrow 0$) and $(X_r^{-1}X_{r+s})$ denote the increments of an additive process with distributions $(\mu_{r,r+s})_{r,r+s\geq 0}$.

Hence, if $f(e) = 1$, then $\sup\left(\langle \mu_{r,r+s}, f\rangle - 1\right) \to 0$. This proves in particular the assertion (4.2) if $[0, T]$ is covered by a finite number of small intervals.

Lemma 4.2. *Let $(\nu_{t,t+s})$ be represented by a perturbation series as in Corollaries 2.4, 2.5: $\nu_{t,t+s} = \mathrm{e}^{-\beta\cdot s}\sum_{k\geq 0}\nu_{t,t+s}^{(k)}$, where $\nu_{t,t+s}^{(0)} = \mu_{t,t+s}$, $\nu_{t,t+s}^{(k+1)} = \int_0^s \mu_{t,t+u} \star \sigma(t+u) \star \nu_{t+u,t+s}^{(k)}du$.*

Then for continuous symmetric sub-multiplicative functions $f \geq 1$ with $f(e) = 1$ we have:

a) $\langle \nu_{t,t+s}, f\rangle = \mathrm{e}^{-\beta s}\sum_{k\geq 0}\langle \nu_{t,t+s}^{(k)}, f\rangle$, $\langle \nu_{t,t+s}^{(0)}, f\rangle = \langle \mu_{t,t+s}, f\rangle$ *and*

$$\langle \nu_{t,t+s}^{(k+1)}, f\rangle \leq$$

$$\int_0^{s_0} \cdots \int_0^{s_k} \prod_{i=0}^{k+1} \langle \mu_{t_i,t_{i+1}}, f\rangle \cdot \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle du_{k+1}\cdots du_0 \tag{4.3}$$

where $t_0 = t$, $t_i := t_i + u_i$, $t_{k+1} := t+s$, $s_0 := s, s_i := s - \sum_1^i u_j$.

b) $\langle \nu_{t,t+s}, f\rangle \geq \langle \mu_{t,t+s}, f\rangle \cdot \mathrm{e}^{-\beta s}$.

c) $\langle \nu_{t,t+s}, f\rangle \geq C \cdot D \cdot \mathrm{e}^{-\beta s}\int_0^s \langle \sigma(t+u), f\rangle du$
with positive constants $C = C(t, t+s), D = D(t, t+s) \in (0, 1]$.

d) *Furthermore, we observe*
$\int_0^s \langle \sigma(t+u), f\rangle du = \int_0^s c(t+u)\langle \rho(t+u), f\rangle du + \int_0^s (\beta - c(t+u))\, du \cdot \varepsilon_e$
with $\int_0^s (\beta - c(t+u))\, du \leq \beta \cdot s$.

⟦ a) and b) follow immediately by 2.4, 2.5 (in view of (1.6)) and by Proposition 3.4.

Analogously, c) follows applying 3.4 to

$$\langle \nu_{t,t+s}, f\rangle \geq \mathrm{e}^{-\beta s}\int_0^s \langle \mu_{t,t+u} \star \sigma(t+u) \star \mu_{t+u,t+s}, f\rangle du$$

defining $C := \inf\limits_{0\leq u\leq s}\langle \mu_{t,t+u}, 1/f\rangle$ and $D := \inf\limits_{0\leq u\leq s}\langle \mu_{t+u,t+s}, 1/f\rangle$. (Recall that $f(e) = 1$.) ⟧

Now we have the means to formulate the main result:

Theorem 4.3. *Let $(\nu_{t,t+s})$ be a Lipschitz-continuous hemigroup with generating functionals $A(\tau)$ and $B(s,t) = \int_s^t A(\tau)d\tau$ respectively. Assume as in Corollary 2.5 (2.1)*

$$c(\tau) := \eta_{A(\tau)}\left(\complement U\right) \leq \beta,\ 0 \leq \tau \leq T \tag{4.4}$$

for some neighbourhood U of the unit e. Let as before, $f : \mathbb{G} \to [1,\infty)$ be continuous, sub-multiplicative and symmetric with $f(e) = 1$. Then the following assertions are equivalent:

(i) $\langle \nu_{t,t+s}, f\rangle < \infty$ *for all* $0 \leq t \leq t+s \leq T$

(ii) $\langle \nu_{0,T}, f\rangle < \infty$

(iii) $\int_0^T \langle \sigma(\tau), f\rangle d\tau < \infty$ *(with the notations introduced in 2.5).*

(iv) $\langle \eta_{B(0,T)}, f1_{\complement U}\rangle = \int_0^T \int_{\complement U} f d\eta_{A(\tau)} d\tau < \infty$

(v) *For all* $s \in (0,T)$ $\sup_{0\leq t\leq t+s\leq T} \langle \eta_{B(t,t+s)}, f1_{\complement U}\rangle < \infty$

Proof. We use the notations introduced above, in particular in 2.5.
"$(i) \Leftrightarrow (ii)$" cf. Lemma 3.6.
"$(iii) \Leftrightarrow (iv)$" Note that $\sigma(\tau) \geq 0$, $\beta T \geq \int_0^T \beta - \eta_{A(r)}(\complement U)dr \geq 0$ and and $\langle \eta_{B(0,T)}, f1_{\complement U}\rangle = \int_0^T \langle \sigma(\tau), f\rangle d\tau - \int_0^T \beta - \eta_{A(r)}(\complement U)dr$ (cf. Lemma 4.2 d)). Whence the assertion follows.
"$(iv) \Leftrightarrow (v)$" is obvious, since the integrands are non-negative.
"$(ii) \Rightarrow (iii)$" follows by Lemma 4.2 c). (Note that $C \cdot D > 0$).
"$(iii) \Rightarrow (ii)$" According to Lemma 4.2 a) we have to show that $\mathrm{e}^{-\beta T}\sum_k \langle \nu_{0,T}^{(k)}, f\rangle < \infty$. For $k = 0$ we have $\langle \nu_{0,T}^{(0)}, f\rangle = \langle \mu_{0,T}, f\rangle \leq \sup_{t\leq t+s\leq T}\langle \mu_{t,t+s}, f\rangle =: M_0 < \infty$ (cf. Proposition 4.1.) Note that $1 \leq M_0 \leq M_0^2$ and by assumption (iii), $\int_0^T \langle \sigma(\tau), f\rangle d\tau < \infty$. $t \mapsto \Gamma(t) := \int_0^t \langle \sigma(v), f\rangle dv$ is increasing, bounded on $[0,T]$ and absolutely continuous w.r.t. $\lambda^1|_{[0,T]}$. Hence for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\forall\ s < \delta(\varepsilon), \forall t$ we have $\Gamma(t,t+s) := \Gamma(t+s) - \Gamma(t) < \varepsilon$. Furthermore, for all $k \in \mathbb{Z}_+0$, $d > 0$ we have in view of (4.3):

$$\begin{aligned}
&\langle \nu_{t,t+s}^{(k+1)}, f\rangle \overset{3.4c),\ (4.3)}{\leq} \\
&\int_0^{s_0} \cdots \int_0^{s_k} \prod_{i=0}^{k+1} \langle \mu_{t_i,t_{i+1}}, f\rangle \cdot \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle du_{k+1}\cdots du_0 \leq \\
&M_0^{k+2} \cdot \int_0^{s_0} \cdots \int_0^{s_k} \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle du_{k+1}\cdots du_0 \leq \\
&M_0 \cdot (M_0 \cdot d)^{k+1}
\end{aligned} \tag{4.5}$$

(with the notations introduced in (4.3)), if $s < \delta(d)$, hence $s_i < \delta(d)$.

To prove the last estimate of (4.5) note that $\int_0^{s_k} \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle du_{k+1} = \prod_{i=0}^{k-1} \langle \sigma(t_{i+1}), f\rangle \cdot \int_0^{s_k} \langle \sigma(t_k + u_{k+1}), f\rangle du_{k+1} \leq \prod_{i=0}^{k-1} \langle \sigma(t_{i+1}), f\rangle \cdot d$, etc.

Let $0 < c < 1$, choose $0 < d < c/M_0$. (Note that M_0 only depends on T.) We begin with $0 = t_0$. Put $t_{i+1} := t_i + s_i$ and choose $s_i < \delta(d)$, hence $\Gamma(t_i, t_{i+1}) < d$. Then according to (4.5) we observe $\langle \nu_{t_0,t_1}, f\rangle \leq \mathrm{e}^{-\beta s_1}\sum_k \langle \nu_{t_0,t_0+s_1}^{(k)}, f\rangle \leq \mathrm{e}^{-\beta s_1}\left(1-c\right)^{-1} \cdot M_0$.

Now replace t_0 by $t_0 + s =: t_1$, $s < \delta(d)$ etc. After N repetitions, $N \approx T/\delta(d)$, the interval $[0,T]$ is covered and we obtain in view of Proposition 3.4: $\langle \nu_{0,T}, f\rangle \leq \prod_1^N \langle \nu_{t_i,t_{i+1}}, f\rangle \leq \mathrm{e}^{-\beta T}\left(1-c\right)^{-N} \cdot M_0^N$. □

5. Appendix

Remarks 5.1. ***a)*** *The constant M_0 in (4.5) depends on the the length of the chosen interval: Put $M_0 = M_0(s)$ if the behaviour of $\nu_{t,t+u}$ is considered in the interval $0 \leq u \leq s$.*

If $(\nu_{t,t+s})$ is time-homogeneous, i.e. if $(\nu_s := \nu_{t,t+s})_{s\geq 0}$ (and also $(\mu_s := \mu_{t,t+s})_{s\geq 0}$) are continuous convolution semigroups, then we have with $M_0(s) = \sup_{u\leq s}\langle \mu_u, f\rangle$:

$$\langle \nu_{t,t+s}, f\rangle = \langle \nu_s, f\rangle \leq M_0(s)\mathrm{e}^{-s\beta}\mathrm{e}^{M_0(s)\langle\sigma,f\rangle} \tag{5.1}$$

where $A(t) \equiv A$, $\sigma(t) \equiv \sigma = \eta_A|_{\complement U}$, $\beta := \sigma(\mathbb{G}) = \eta_A(\complement U) \equiv c(t)$.

With different notations the upper bound (5.1) is found in [20]*, proof of Theorem 5. In fact, in the time-homogeneous case sharper estimates are available:*

$$\langle \nu_s, f\rangle = \langle \nu_{t,t+s}, f\rangle = \mathrm{e}^{-\beta s}\sum_k \langle \nu^{(k)}_{t,t+s}, f\rangle \text{ with } \langle \nu^{(0)}_{t,t+s}, f\rangle = \langle \mu_{t,t+s}, f\rangle$$

$$\langle \nu^{(k+1)}_{t,t+s}, f\rangle \leq \int_0^s \langle \mu_u, f\rangle\langle\sigma, f\rangle\langle \nu^{(k)}_{t+u,t+s}, f\rangle du$$

$$\leq M_0(s)\langle\sigma, f\rangle \int_0^s \langle \nu^{(k)}_{t+u,t+s}, f\rangle du \leq \cdots \leq \frac{M_0(s)}{(k+1)!}\left(M_0(s)\langle\sigma, f\rangle\right)^{k+1}$$

Whence (5.1) follows.

b) *E. Siebert's results in* [19, 20] *for the time-homogeneous case are proved for general continuous convolution semigroups, and in that case the restrictive condition (2.1) is trivially fulfilled (for any $T > 0$). It is natural to conjecture that the assertions of Theorem 4.3 hold true also without condition (2.1) resp (4.4). But up to now no proof is available.*

c) *Throughout, in order to avoid problems with measurability and in view of* [20]*, Theorem 4, we assumed $\mathbb{G}$ to be second contable. In fact, this is not a serious restriction:*

Firstly, w.l.o.g. we may assume $\mathbb{G}$ to be $\sigma-$compact, since the group generated by the supports $\bigcup_{0\leq t<t+s\leq T} \operatorname{supp}(\nu_{t,t+s})$ is $\sigma-$compact.

As well known (cf. e.g., [2]*, page 101, exerc. 11) a $\sigma-$compact group is representable as projective limit of second countable groups $\mathbb{G} = \lim_{\leftarrow} \mathbb{G}/K$, $K \in \mathfrak{K}$, a set of compact normal subgroups with $\bigcap_{K\in\mathfrak{K}} = \{e\}$.*

Let f be as above, then $W := \{f = 1\}$ is a closed subgroup and f is $K-$invariant for any compact subgroup $K \subseteq W$. Moreover, $g := \log f$ is uniformly continuous by Lemma 3.2. Let e.g., $\psi : x \mapsto x^2/(1 + x^2)$, then $h := \psi \circ g$ is uniformly continuous and bounded. Hence h is K_0-invariant for some $K_0 \in \mathfrak{K}$ (cf. the above reference [2]*). Therefore, h, hence also f is K_0-invariant. Thus integrability of f w.r.t. $(\nu_{t,t+s})$ can be reduced to the case of second countable groups.*

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