An Note on a Categorical Semantics for ER-Models

Ernst-Erich Doberkat
Chair for Software-Technology
University of Dortmund
doberkat@acm.org

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Abstract

We have a look at the semantics of Entity-Relationship models, a popular device for modelling data, but lacking a stringent semantics. In an earlier paper we have shown how to generate an algebraic specification for an extended ER-model; in this paper we extend the algebraic view of a model through a categorial interpretation.

Inheritance induces a tree structure for an ER-model. This is decorated with objects from a suitable category, and we ask for a unifying view of this model. Our approach suggests using colimits as the semantics. It is shown that under very mild conditions the colimit exists, and that this colimit is an adjoint to the diagonal. Finally we show how to integrate binary relations into this approach by studying two general conditions on the morphism associated to a relation.

1 Introduction

Entity-Relationship modelling is a rather popular approach to data modelling, as witnessed by the literature on data base systems [Ull88] and on software engineering [GJM91]. This popularity is partly due to the visual representability of the models constructed, and to the relative ease with which these models may be implemented. Furthermore, ER models come in many different variants, hence different kinds of problems may be subjected to ER modeling, indicating the wide applicability of this method, nay, of this family of methods.
This kind of data modeling has, however, quite a notable drawback from the conceptual point of view. Its modelling facilities are almost seductive, but the formal foundations for its semantics are not that readily described. In fact, a set theoretic semantics is easily conceived, but it is not difficult to see that set theory due to its implied semantics is not always appropriate. An alternative for giving some meaning to an ER model is provided through an algebraic way of life: [CLWW94], [Het93] and [GH91, Hoh93] formulate an algebraic view, and [Dob97] gives methods for generating an algebraic specification for an ER model.
These approaches are mainly characterized by providing some axiomatic description for the operations involved in a suitable variant of the model, and by establishing subsequently the models provided by these descriptions as the (class of) semantics for the ER model. This is a mathematically sound way providing a solid foundation for this powerful method. It enables constructing concise models when ER modeling is combined with other approaches that focus on modelling the functionality of an application. In [Dob97, Sec. 5] it is shown with an example how data modeling with ER may be combined with functional modeling using a Petri nets, essentially attributing the edges of the net with enabling terms from an algebraic specification.
This note carries the algebraic approach developed in [Dob97] a bit further. Given an algebraic specification, we may consider models for it. These models form a category, hence the specification determines objects of a category (which is usally unrelated to the category of sets). Since each entity and each relation in determined by a specification of its own, we see a collection of models, hence a collection of objects in our model category. In addition, some entities are related to each other (e.g., by IsA links). This translates to the edges of a graph, each node of which is decorated with an algebraic specification. Translating the edges to morphisms between the corresponding objects, hence one ends up with a functor from a diagram to the category under consideration.
It is this functor which is of interest here.

Carried away? It is easy to get carried away by the algebraic arguments, so the reason for why it is helpful to have such an algebraic discussion should be elaborated. First, categories abstract away all the implementation details and permit focussing on the structure of the objects under consideration and their relationships. In fact, the details of an object in a category is not accessible at all. An object may be internally as rich as Croesus (or Bill Gates) — this is of no concern, as long as this richness is not expressible on the outside. This, in turn, is determined by the set of morphisms associated with the object. Hence the approaches (ER, categories) are in fact rather similar by focussing on external relations. They may be oriented towards preserving structures (like homomorphisms familiar from group theory, or like signature morphisms from algebraic specifications), but they may also be considered as models for channels (like in the COMMUNITY approach, see [WF98]). This indicates that a categorical approach may encompass also many other views to modelling. The full generality has, however, its price: further work is required for modelling those aspects which are not determined by the structure alone, in particular questions on capturing attributes and their dependence. This will require a judicious restriction to the kind of category that is used for formulating the model.

Organization This paper is organized as follows: we first recall from [Dob97] the necessary properties of ER models and focus on the forest induced by the model. Then we have a look at the semantics of a diagram. It is formulated in terms of colimits, and we investigate the question under which conditions such a colimit exists. It turns out that an answer for the special case of trees is rather immediate, and that constructing an adjunction helps in establishing the general model for forests (supporting once more MacLane's *adjoints-are-everywhere*-hypothesis put forward in [Mac98, p. 97]). This result implies that the semantics is compostional.
The constructions address entities only, and we discuss in the final section modelling the interplay between relations and entities.

Further work Attributes are not considered in this note, they are rather a subject of further work. The solution we present is completely general as far as entities are concerned, but could probably be fine tuned when it comes to different kinds of relations. This is also delegated to the drawer labelled *Further Work.*

2 ER-Models

An entity-relationship model [Ull88, 2.4] consists of entities, relationships on these entities and attributes both on entities and relations. Only binary relations will be considered for the sake of simplicity. Entities may be related by the IsA relation: E_1 IsA E_2 indicates that each instance of E_1 is also an instance of E_2, hence shares all the attributes defined on the latter entity. Multiple inheritance is not permitted here (i.e, no entity may be related to more than one other entity via an IsA -relation). If entities are represented by their *extension*, then relations are subsets of the Cartesian product. If R relates the entities E_1 and E_2, the entities in E_1 (in E_2) are said to be in the *domain* (in the *co-domain*) of R. In the graphical representation the order of the factors for the product is not immediate, hence we number the corners of the diamond counterclockwise starting in the northern corner, identifying domain and co-domain uniquely. Attributes are mentioned for the sake of completeness; they are usually represented as maps; as usual, an attribute is a *key* for an entity iff it uniquely determines each instance. A relation R is $N:1$ iff $b_1 = b_2$ is true whenever both aRb_1 and aRb_2 hold (i.e. whenever R is a partial map), i.e. iff for each instance a in the domain of R the set $\{b : aRb\}$ contains at most one element. In a similar way $1:N$ relations are characterized: R is an $1 : N$ relation iff its inverse R^{-1} is $N : 1$. A relation is said to be $N : M$ iff there are no restrictions concerning the domain or the co-domain of the pairs participating in the relation. That a relation is $N:1$ is indicated in the graphical representation by labelling the edge leading to the domain with an $*$, and a 1 as a label for the co-domain.
Fig. 1 displays an example for modelling a simple graphical user interface.
The entities are `window`, `button`, `textfield`, `menu entry`, moreover `trigger` and `text(fixed)`, both of which are related to `menu entry` via the IsA relation, and `output window` and `icon`,

for which IsA `window` holds. The relations are `sequence`, which is an $N : M$ relation between `windows`, `residesIn`, a $1 : N$ relation between `window` and `button`, `contains` relates `textfield` and `window` as an $N{:}1$ relation, `inMenu` is an $N{:}1$ relation between `trigger` and `window`, and finally `invocation` relates `trigger` and `menu entry` $1{:}N$. Attributes are e.g. `window layout` defined on entity `window` or `button position` defined on relation `residesIn`. As usual, key attributes are underlined, and total relations or attributes carry a dot where they are total. We will, however, not deal with attributes here.

3 Preparations

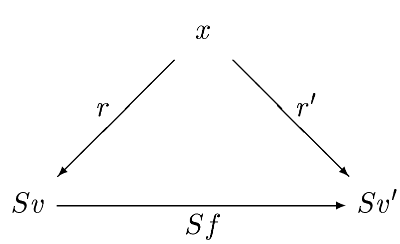
This section will relate the construction of a graph from an ER-model (see [Dob97, Sec. 3]). Furthermore it will collect some notations and results from category theory following [Mac98] for easier reference and the reader's convenience.

3.1 The Graph

Given an ER-model $\mathcal{M}$, denote by $\mathcal{E}$ and $\mathcal{R}$ the respective sets of entities and of relations. Let $N_{\mathcal{E}}$ and $N_{\mathcal{R}}$ be fresh and disjoint sets of nodes representing $\mathcal{E}$ and the domains and co-domains for the relations in $\mathcal{M}$, so that each $E \in \mathcal{E}$ is associated with a unique node $n_E \in N_{\mathcal{E}}$, similarly for $\mathcal{R}$. Construct a directed edge $n_{E_1} \to n_{E_2}$ iff E_1 IsA E_2 holds in $\mathcal{M}$.
If $r \in \mathcal{R}$ is a relation with E_1 as domain and E_2 as co-domain, generate two fresh nodes $j^{\delta(r)}$ and $j^{\gamma(r)}$ in $N_{\mathcal{E}}$ which are linked through the directed edges $j^{\delta(r)} \to n_{E_1}$ and $j^{\gamma(r)} \to n_{E_2}$ to their domain and co-domain, resp. (this reflects the fact that the domain and the co-domain of r have to be taken care of when it comes to manipulate the relation). The construction from [Dob97] additionally constructs non-directed edges $n_r \leftrightarrow n_{E_1}$ and $n_r \leftrightarrow n_{E_2}$; it also takes care of the attributes. But the simpler construction from above will suffice for the purposes of the present paper.
To illustrate things, we borrow from [Dob97] a simple ER-model for constructing a graphical user interface. This is displayed in Fig. 1. The directed graph generated from it is shown in Fig. 2.

3.2 Morphisms and all that

If $\mathcal{C}$ is a category, $\mathcal{C}(a, b)$ is the set of all morphisms $a \to b$ in $\mathcal{C}$. Suppose $\mathcal{D}$ is another category, and $S : \mathcal{C} \to \mathcal{D}$ is a functor, then $\langle v, r\rangle$ is called an *arrow from x to S* iff $r : x \to Sv$ is a morphism in $\mathcal{D}$, or, equivalenty, if $\langle v, r\rangle$ is a member of the comma category $x \downarrow S$. An inital object in this category is called *universal*; thus universal objects are unique up to isomorphisms. Thus $\langle v, r\rangle$ is a universal arrow from x to S iff for each arrow $\langle v', r'\rangle$ from x to S there exists a unique morphism $f : v \to v'$ such that $r' = Sf \circ r$ holds, thus the following diagram is commutative:



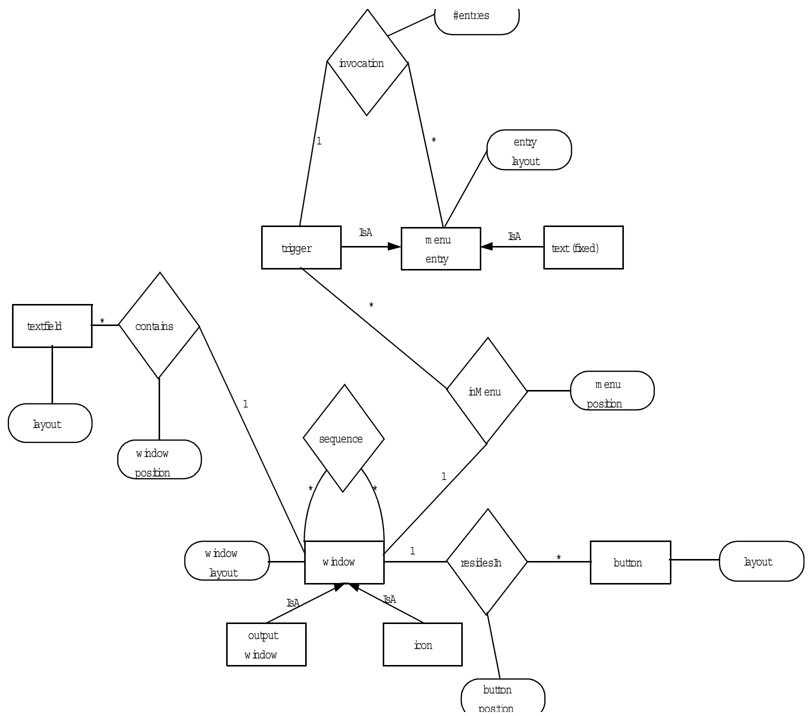


Figure 1: A simple ER-model for a GUI

Recall that a natural transformation $\tau : R \stackrel{\bullet}{\to} S$ for the functors $R, S : \mathcal{C} \to \mathcal{D}$ assigns to each object c in $\mathcal{C}$ an arrow $\tau_c : Rc \to Sc$ in $\mathcal{D}$ such that for each morphism $f : c \to c'$ the diagram

$$
\begin{array}{ccc}
Rc & \xrightarrow{\tau_c} & Sc \\
{\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle Sf} \\
Rc' & \xrightarrow[\tau_{c'}]{} & Sc'
\end{array}
$$

is commutative. Denote for two natural transformations $\tau : R \stackrel{\bullet}{\to} S$ and $\sigma : S \stackrel{\bullet}{\to} T$ their *vertical composition* by $\sigma \cdot \tau$, hence

$$(\sigma \cdot \tau)_c = \sigma_c \circ \tau_c$$

holds for each object c in $\mathcal{C}$. The vertical composition is again a natural transformation.
Trees are directed towards their root, a *forest* is a finite collection of finite trees. Each forest $\mathcal{B}$ may be considered as a category and each map $S : \mathcal{B} \to \mathcal{C}$ that assigns nodes to objects and edges to morphisms (so that each edge $e : i \to j$ in $\mathcal{B}$ yields a morphism $Se \in \mathcal{C}(S_i, S_j)$) may be considered as a functor: we can define a unique morphism $Sp \in \mathcal{C}(S_{i_0}, S_{i_k})$ for each path p from i_0 to i_k in $\mathcal{B}$ upon piecing Sp together from these edges. This is an old trick discussed at length e.g. in [Mit65, II.1], see also [Mac98, II.7].
The category $\mathcal{C}^{\mathcal{B}}$ is the category of all functors from $\mathcal{B}$ to $\mathcal{C}$, natural transformations serving as usual as morphisms between functors. These functors are called *$\mathcal{B}$-diagrams over $\mathcal{C}$*, or

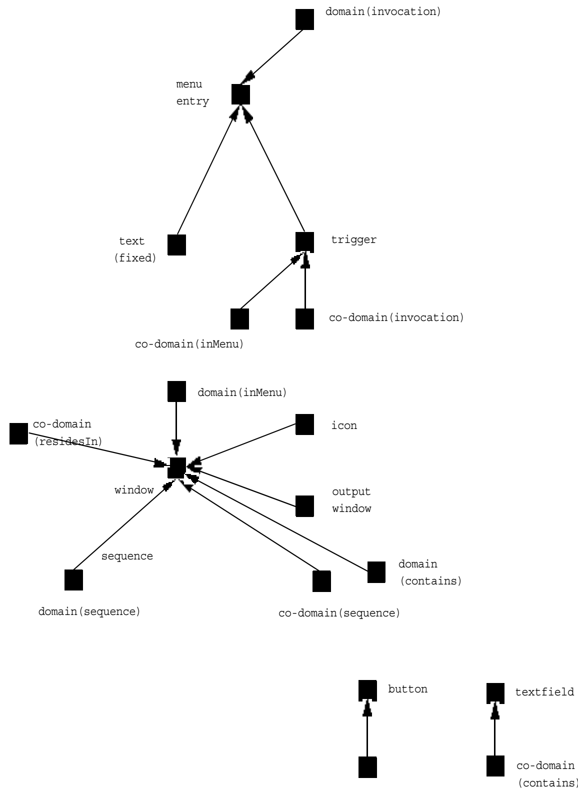
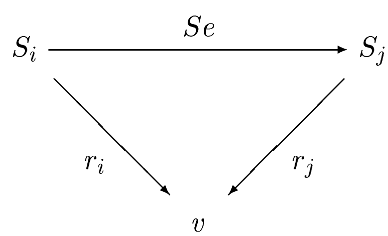


Figure 2: The graph derived from the ER-model

simply *diagrams*, if the context is clear. Denote by $\Delta_{\mathcal{B}}$ the diagonal functor from $\mathcal{C}$ to $\mathcal{C}^{\mathcal{B}}$, hence $\Delta_{\mathcal{B}}(v)$ is a constant functor on $\mathcal{B}$, mapping each object to v, and each morphism to the identity 1_v. The pair $\langle v, r\rangle$ is called a *colimit* for the diagram S iff it is a universal arrow from S to $\Delta_{\mathcal{B}}$. Thus for each node j in $\mathcal{B}$, $r_j : S_j \to v$ is a morphism in $\mathcal{C}$ such that for each edge $e : i \to j$ the diagram



commutes, and if we have another arrow $\langle v', r'\rangle$ from S to $\Delta_{\mathcal{B}}$, hence a commutative diagram for $\left(r'_j : S_j \to v'\right)_j$, then

$$r'_j = f \circ r_j$$

holds for each node j, where $f : v \to v'$ is a uniquely determined morphism in $\mathcal{C}$.
The *coproduct* $\langle v, r\rangle$ of an object a in $\mathcal{C}^n$ is a universal arrow from a to Δ_n, v being denoted

by $\coprod_{i=1,\ldots,n} a_i$, and r being identified by a collection

$$r_j : a_j \to \coprod_{i=1,\ldots,n} a_i$$

of injections. The coproduct is sometimes only identified by its object, similarly for the colimit.
Suppose that $R : \mathcal{C} \to \mathcal{D}$ and $S : \mathcal{D} \to \mathcal{C}$ are functors, and that for each pair of objects c in $\mathcal{C}$ and d in $\mathcal{D}$ there exists a bijection

$$\varphi_{c,d} : \mathcal{D}(Rc, d) \to \mathcal{C}(c, Sd)$$

that is natural in c and d. Then $\langle R, S, \varphi \rangle : \mathcal{C} \rightharpoonup \mathcal{D}$ is called an *adjunction*, and R is called the *left adjoint* for S.

4 Basic Constructions

Consider the graph $\mathcal{B}$ associated with the entities of an ER-model. This graph is actually a forest of trees, since only the nodes coming from entities are considered, reflecting the fact that we do not permit multiple inheritance, hence each entity inherits from at most one other entity. Suppose that we have constructed for each node an algebraic specification, as outlined at length in [Dob97, 4.2]. Interpreting the specification, and assuming that the specification is valid, this yields for each node a model living in that node.
Formally, we map each node j to a model Sj. These models are linked by an edge, whenever the corresponding nodes are related by the IsA-relation, hence there is a directed edge

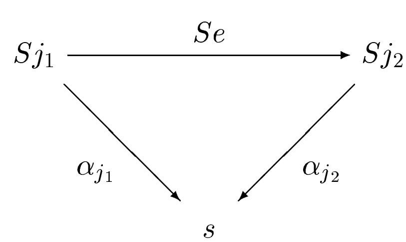
$$e : j_1 \to j_2$$

between two nodes j_1 and j_2 iff j_2 inherits from j_1. Reflecting this in the world of models (no, not Claudia Schiffer's), we establish a homomorphism

$$Se : Sj_1 \to Sj_2.$$

The models for a given specification form a category $\mathcal{C}$, so we end up with a map S associating each node in $\mathcal{B}$ an object from $\mathcal{C}$, cf. 3.2.
Now the project investigating the semantics of an ER-model may be formulated more specific: Given the functor above, associate an object — provisionally called s — and morphisms α with the diagram such that

- there is an arrow $\alpha_j : Sj \to s$ for each node in the specification tree (indicating that each model for a specification may be associated with the object through a morphism),
- the arrows are compatible, hence if $e : j_1 \to j_2$ is an edge in $\mathcal{B}$, then $\alpha_{j_1} = \alpha_{j_2} \circ Se$, hence making the diagram



  commutative,

- the object s should be as close as possible to the objects described by S: thus if there is another object s' together with morphisms $\beta_j : Sj \to s'$ having the two properties above, then β should factor uniquely through α, hence we postulate that

  $$\exists! \gamma : s \to s' \forall j : \beta_j = \gamma \circ \alpha_j$$

  holds (hence requesting information which is as precise as possible).

In other words: we are looking for a colimit for the functor S.

5 ER-Completeness

Fix a category $\mathcal{C}$ and a forest $\mathcal{B}$. The following definition is just an abbreviation.

Definition 1 *$\mathcal{C}$ is called* ER-complete *iff each $\mathcal{B}$-diagram over $\mathcal{C}$ has a colimit.*

We want to investigate these categories first.
Suppose S is a $\mathcal{B}$-diagram over $\mathcal{C}$, and let $\{w_1, \dots, w_k\}$ be the roots of the trees in $\mathcal{B}$, w_i being the root of the tree $\mathcal{T}_i$. Restrict S to this tree, obtaining a $\mathcal{T}_i$-diagram over $\mathcal{C}$. If v_i is its colimit, then $\coprod_{i=1,\dots,n} v_i$ is the colimit for S. On the other hand, suppose that the colimit for S exists, then all colimits for the trees exist, and their coproduct form the colimit. By the way, this colimit is the least upper bound of all node labels if $\mathcal{C}$ is a partially ordered set with an antisymmetric order relation.
We have demonstrated:

Observation 1 *An ER-complete category has coproducts.*

We will show now that the converse also holds. Let $\mathcal{T}$ be a tree with root w, and let S be a $\mathcal{T}$-diagram over $\mathcal{C}$. Let for each node j in $\mathcal{T}$ be π_j the unique path from j to w, and put $\tau_j^S := S\pi_j$, hence $\tau_j^S : S_j \to S_w$. Then $\langle S_w, \tau^S \rangle$ is a universal arrow from S to $\Delta_\mathcal{T}$. First, we have to show that $\langle S_w, \tau^S \rangle$ is in fact an arrow from S to $\Delta_\mathcal{T}$; this follows from the construction. Next, suppose that we have another arrow $\langle v, r \rangle$ from S to $\Delta_\mathcal{T}$, then $r_w : S_w \to v$ is a morphism in $\mathcal{C}$, and

$$r_j = r_w \circ \tau_j^S$$

holds for each node j. The latter property determines the factor for τ^S uniquely.
Thus we have established

Proposition 1 *The category $\mathcal{C}$ is ER-complete iff it has all finite coproducts.*

In fact, we have shown more: let

$$R_\mathcal{T}^0 : \mathcal{C}^\mathcal{T} \to \mathcal{C}$$

be the map that assigns to each diagram S the value S_w at the root w of tree $\mathcal{T}$. The argumentation above indicates that

$$\eta_S : S \mapsto \Delta_\mathcal{T}(R_\mathcal{T}^0 S)$$

constitutes a universal arrow from S to $\Delta_\mathcal{T} \circ R_\mathcal{T}^0$, and from MacLane's Portmanteau Theorem [Mac98, IV.1.2(ii)] we may conclude that $R_\mathcal{T}^0$ is the object function of a functor $R_\mathcal{T}$ which is the left adjoint to $\Delta_\mathcal{T}$, specifically:

Proposition 2 $\langle R_{\mathcal{T}}, \Delta_{\mathcal{T}}, \varphi\rangle : \mathcal{C}^{\mathcal{T}} \rightharpoonup \mathcal{C}$ *is an adjunction, where* φ *is defined through*

$$\varphi_{S,c} : \begin{cases} \mathcal{C}(R_{\mathcal{T}}S, c) & \to & \mathcal{C}^{\mathcal{T}}(S, \Delta_{\mathcal{T}}(c)) \\ f & \mapsto & \Delta_{\mathcal{T}}(f) \circ \eta_S \end{cases}$$

Recall that the maps φ for an adjunction are bijections. Thus the result above may be restated as follows: the meaning of each diagram is uniquely determined by the meaning which is assigned to the root.

This reflects the usual emphasis and care on modeling the root class in an inheritance hierarchy, because this class essentially determines the properties of all its descendants. Recall, moreover, that these maps are natural transformations. Consequently, homomorphic changes to the modeling of the root induce homomorphic changes to the meaning of the entire hierarchy. The latter may be considered to be a sort of continuity property: smooth changes at the root do not induce drastic changes in the object hierarchy.

Let us turn to forests. The colimit of a forest was shown to be the coproduct, the factors being determined by the constituting trees. Denote for the diagram S and the forest $\mathcal{B}$ this colimit again by $R_{\mathcal{B}}S$, then we will show that there exists a natural bijection between $\mathcal{C}(R_{\mathcal{B}}S, c)$ and $\mathcal{C}^{\mathcal{B}}(S, \Delta_{\mathcal{B}}(c))$ for each diagram S and each object c in $\mathcal{C}$, hence the adjunction generalizes to forests.

Suppose again that $\{w_1, \ldots, w_t\}$ is the collection of roots in $\mathcal{B}$, so that each node j lies in a uniquely determined tree with root $w(j)$. Denote for the diagram S the unique morphism $S_j \to S_{w(j)}$ by s_j. Then

$$R_{\mathcal{B}}S = \coprod_{i=1,\ldots,t} S_{w_i}.$$

Denote the canonical injection $S_{w_j} \to R_{\mathcal{B}}S$ by $r_{w(j)}$. Fix for the moment S and c, then a morphism $f : R_{\mathcal{B}}S \to c$ is uniquely determined by the morphisms

$$\psi_{S,c}(f)_j := f \circ r_{w(j)} \circ s_j$$

yielding for fixed S, c and f a natural transformation

$$\psi_{S,c}(f) : S \stackrel{\bullet}{\to} \Delta_{\mathcal{B}}(c).$$

It is not difficult to see that $\psi_{S,c}$ is a bijection between $\mathcal{C}(R_{\mathcal{B}}S, c)$ and $\mathcal{C}^{\mathcal{B}}(S, \Delta_{\mathcal{B}}(c))$, and it now will be shown to be natural in S and in c. Keeping c fixed, let $\tau : S' \stackrel{\bullet}{\to} S$ be a natural transformation. τ induces a unique morphism

$$\tau^{\mathcal{B}} : R_{\mathcal{B}}S' \to R_{\mathcal{B}}S$$

which makes the diagram

$$\begin{array}{ccc} S'_{w_i} & \xrightarrow{r'_{w_i}} & \coprod_{i=1,\ldots,t} S'_{w_i} \\ \downarrow{\scriptstyle \tau_{w_i}} & & \downarrow{\scriptstyle \tau^{\mathcal{B}}} \\ S_{w_i} & \xrightarrow[r_{w_i}]{} & \coprod_{i=1,\ldots,t} S_{w_i} \end{array}$$

commutative for each root w_i (the primed morphism refers to S'). Define

$$\tau_\bullet^{\mathcal{B}} : \begin{cases} \mathcal{C}(R_{\mathcal{B}}S, c) & \to & \mathcal{C}(R_{\mathcal{B}}S', c) \\ g & \mapsto & g \circ \tau^{\mathcal{B}} \end{cases}$$

and

$$\tau_* : \begin{cases} \mathcal{C}^{\mathcal{B}}(S, \Delta_{\mathcal{B}}(c)) & \to & \mathcal{C}^{\mathcal{B}}(S', \Delta_{\mathcal{B}}(c)) \\ \sigma & \mapsto & \sigma \cdot \tau \end{cases}$$

and chase a morphism $g : R_{\mathcal{B}}S \to c$ around in the diagram (note the contravariance)

$$\begin{array}{ccc} \mathcal{C}(R_{\mathcal{B}}S, c) & \xrightarrow{\psi_{S,c}} & \mathcal{C}^{\mathcal{B}}(S, \Delta_{\mathcal{B}}(c)) \\ {\scriptstyle \tau_\bullet^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle \tau_*} \\ \mathcal{C}(R_{\mathcal{B}}S', c) & \xrightarrow[\psi_{S',c}]{} & \mathcal{C}^{\mathcal{B}}(S', \Delta_{\mathcal{B}}(c)) \end{array}$$

to obtain

$$(g \circ \tau^{\mathcal{B}}) \circ r'_{w(j)} \circ s'_j = g \circ r_{w(j)} \circ s_j \circ \tau_j$$

This implies

$$\psi_{S',c} \circ \tau_\bullet^{\mathcal{B}} = \tau_* \circ \psi_{S,c},$$

hence the diagram above is commutative. Thus ψ is natural in S, and it is immediate that it is natural in c. Thus we have established another adjunction:

Proposition 3 $\langle R_{\mathcal{B}}, \Delta_{\mathcal{B}}, \psi \rangle : \mathcal{C}^{\mathcal{B}} \rightharpoonup \mathcal{C}$ *is an adjunction.*

Summarizing, we have proved

Proposition 4 *In a category with finite coproducts, the functor yielding the semantics of an ER-model is left adjoint to the diagonal.*

Although we do not need it here, it might be interesting to note that the semantic functor is colimit-preserving [Mac98, p. 119]. This implies in particular that our semantics is compostional: if the ER-model is composed as, say, the coproduct of smaller models, then the semantics behaves civilized in the sense that it composes from the semantics of the coproduct's factors, and similarly for other colimits. These aspects should be investigated further.

6 Relations

$\mathcal{R}$ is the finite set of binary relations for the ER model with diagram $\mathcal{B}$. We assume that the category $\mathcal{C}$ has finite products as well as finite coproducts, so that $S_{\mathcal{B}}$ is defined for the schema S. For each relation $r \in \mathcal{R}$ with $\delta(r)$ and $\gamma(r)$ as the domain and the codomain entities with associated nodes $n_{\delta(r)}$ and $n_{\gamma(r)}$, resp., let $j^{\delta(r)}$ and $j^{\gamma(r)}$ be the corresponding new nodes in $\mathcal{B}$. Since $\mathcal{C}$ has finite products, $(Sn_{\delta(r)}) \times (Sn_{\gamma(r)})$ exists in $\mathcal{C}$ with respective projections $\pi_{\delta(r)}$ and $\pi_{\gamma(r)}$. Because relation r may always be represented as a subobject of the Cartesian product of the domain and the codomain, we assume that there exists a monic

$$\rho^r : c_r \to (Sn_{\delta(r)}) \times (Sn_{\gamma(r)})$$

for some object c_r in $\mathcal{C}$. Hence we label the edges $(j^{\delta(r)}, n_{\delta(r)})$ and $(j^{\gamma(r)}, n_{\gamma(r)})$ with $\pi_{\delta(r)} \circ \rho^r$ and $\pi_{\gamma(r)} \circ \rho^r$, resp. The nodes are added together with the edges to the graph. This transmogrifies the graph construction outlined in [Dob97], cp. Section 4. This process yields a new diagram $\mathcal{B}^\bullet$ and a new schema $S^\bullet$ with a colimit $\Sigma := S^\bullet_{\mathcal{B}^\bullet}$.
We want to investigate the relationship between an arbitrary relation r and Σ. By construction, r helps to define the colimit.

Just a brief aside. Recall in this stage of the development that each object $\ell \in \mathcal{C}$ gives rise to a set-valued functor

$$\Lambda_\ell : \mathcal{C} \to \mathcal{S},$$

the latter denoting the category of all small sets, upon putting

$$\Lambda_\ell := \mathcal{C}(\ell, -),$$

mapping a morphism $f : a \to b$ to

$$\Lambda_\ell f : \begin{cases} \mathcal{C}(\ell, a) & \to & \mathcal{C}(\ell, b) \\ g & \mapsto & f \circ g \end{cases}$$

The famous Yoneda Lemma [Mac98, p. 61] identifies the natural transformations $\Lambda_\ell \stackrel{\bullet}{\to} \Lambda_t$ with the morphisms in $\mathcal{C}(\ell, t)$. Similarly,

$$\Gamma_\ell := \mathcal{C}(-, \ell)$$

defines a (contravariant) functor $\Gamma_\ell : \mathcal{C} \to \mathcal{S}$ mapping $f : a \to b$ to

$$\Gamma_\ell f : \begin{cases} \mathcal{C}(b, \ell) & \to & \mathcal{C}(a, \ell) \\ g & \mapsto & g \circ f \end{cases}$$

Let us return to relations. The first scenario captializes on the assumption that ρ^r is a monic. Thus the natural transformation

$$\Gamma_{c_r} \stackrel{\bullet}{\to} \Gamma_{(Sn_{\delta(r)}) \times (Sn_{\gamma(r)})}$$

implied by this morphism, again denoted by ρ^r, is a monic, too, when considered as a morphism in the category $\mathcal{S}^{\mathcal{B}}$ of all functors. In particular, ρ^r induces an injective map

$$\mathcal{C}(\Sigma, c_r) \to \mathcal{C}\left((Sn_{\delta(r)}) \times (Sn_{\gamma(r)}), c_r\right).$$

This means that each operation from Σ to the object c_r corresponds uniquely to an operation from Σ to $(Sn_{\delta(r)}) \times (Sn_{\gamma(r)})$. Since there exists the embedding as a morphism from each node to Σ, each morphism from a node to c_r gives rise to a morphism from Σ to $(Sn_{\delta(r)}) \times (Sn_{\gamma(r)})$. This models the flow of information from the colimit to the relation (and does probably not constitute an entirely surprising observation).
Next, suppose that ρ^r is a split mono, thus it has a left inverse. This is e.g. the case whenever $\mathcal{C}$ is a subcategory of $\mathcal{S}$. The Yoneda Lemma implies that the induced natural transformation

$$\Lambda_{(Sn_{\delta(r)}) \times (Sn_{\gamma(r)}} \stackrel{\bullet}{\to} \Lambda_{c_r}$$

is an epi (cp. [Mac98, Lemma IV.3]), again as a morphism in the functor category $\mathcal{S}^{\mathcal{B}}$. In particular, the induced map

$$\mathcal{C}\left((Sn_{\delta(r)}) \times (Sn_{\gamma(r)}), \Sigma\right) \to \mathcal{C}(c_r, \Sigma)$$

is an epi between sets. Epis in $\mathcal{S}$ are exactly the onto maps. Reformulating, each operation $c_r \to \Sigma$ is induced by an operation $(Sn_{\delta(r)}) \times (Sn_{\gamma(r)}) \to \Sigma$. This applies in particular to operations between c_r and Sn, where n is an arbitrary node in the forest.
Alas, this is about how far the general considerations on relations can go. More specific results require more specific assumptions, and this is — as usual — indicated as subject to further work.

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