Extrinsically Immersed Symplectic Symmetric Spaces

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Abstract

Let (V, Ω) be a symplectic vector space and let $\phi : M \to V$ be a symplectic immersion. We show that $\phi(M) \subset V$ is (locally) an extrinsic symplectic symmetric space (e.s.s.s.) in the sense of [CGRS] if and only if the second fundamental form of ϕ is parallel.

Furthermore, we show that any symmetric space which admits an immersion as an e.s.s.s. also admits a *full* such immersion, i.e., such that $\phi(M)$ is not contained in a proper affine subspace of V, and this immersion is unique up to affine equivalence.

Moreover, we show that any extrinsic symplectic immersion of M factors through to the full one by a symplectic reduction of the ambient space. In particular, this shows that the full immersion is characterized by having an ambient space V of minimal dimension.

1 Introduction

Ever since their introduction by É. Cartan ([C]), symmetric spaces have been studied intensely from various viewpoints. In [F2], D. Ferus introduced the notion of an *extrinsic symmetric space* which is a Riemannian symmetric space admitting an embedding into an Euclidean vector space such that the geodesic reflection at each $p \in M$ is the restriction of the reflection in the normal plane of M in p. In fact, Ferus gave a classification of all Riemannian extrinsic symmetric spaces, showing that most Riemannian symmetric spaces admit such an embedding. Combining this with his results from [F1], he showed that a submanifold of Euclidean space is locally extrinsically symmetric if and only if its shape operator is parallel.

Evidently, the concept of extrinsic symmetric spaces may be generalized to other classes as well. See [Ka, Ki] for results on extrinsic pseudo-Riemannian symmetric spaces and [GS] for extrinsic CR-symmetric spaces. In both cases, many examples of such embeddings are given, but a classification appears to be out of reach.

In [CGRS], *extrinsic symplectic symmetric spaces* (e.s.s.s.) were considered for the first time, and their basic algebraic and geometric properties were described. However, in contrast to the aforementioned cases, only few types of e.s.s.s. are known, all of which have a 3-step nilpotent transvection group.

In this article, we further investigate e.s.s.s. In fact, we slightly generalize this notion to *extrinsic symplectic immersions* by which we mean an immersion $\phi : M \to V$ of a symmetric space M into the symplectic vector space (V, Ω) such that $\phi^*(\Omega)$ is non-degenerate and such that for all $p \in M$

$$\phi \circ s_p = \sigma_{T_p} \circ \phi,$$

where $s_p : M \to M$ is the symmetric reflection at p, and $\sigma_{T_p} : V \to V$ is the reflection in the affine normal space of $T_p := d\phi(T_p M)$. If ϕ is an embedding, then $\phi(M) \subset V$ is an e.s.s.s. While we do not know of any example of an extrinsic symplectic immersion which is *not* an embedding, we nevertheless use this notion as it allows us to state our results more elegantly. As a first result, we obtain in analogy to the aforementioned result from [F1]:

Theorem A *Let (V, Ω) be a symplectic vector space and let $\phi : M \to V$ be a symplectic immersion, i.e., such that $\phi^*(\Omega)$ is non-degenerate.*

Then the shape operator of ϕ is parallel if and only if there is an extrinsic symplectic immersion $\hat{\phi} : \hat{M} \to V$ of a symmetric space $\hat{M}$ and a local diffeomorphism $\varepsilon : M \to \hat{M}$ such that $\phi = \hat{\phi} \circ \varepsilon$.

In contrast to the Riemannian case, e.s.s.s. which are intrinsically equivalent do not need to be extrinsically equivalent. For instance, a symplectic vector space, regarded as a flat symmetric space, can be embedded non-affinely into a higher-dimensional vector space as an e.s.s.s. ([CGRS]). It is precisely this non-uniqueness which we want to illuminate.

An extrinsic symplectic immersion $\phi : M \to V$ is called *full* if there is no proper affine subspace of V containing $\phi(M)$.

Theorem B *Let M be a symmetric space admitting an extrinsic symplectic immersion. Then M also admits a full extrinsic symplectic immersion which is unique up to affine equivalence.*

We point out that Theorem B reveals a feature of e.s.s.s. which is not valid for pseudo-Riemannian extrinsic symmetric spaces. Indeed, there are full extrinsic pseudo-Riemannian symmetric spaces which are intrinsically equivalent but with ambient spaces of different dimensions; see e.g. examples 7.3 and 7.4 in [Ka].

Consider an extrinsic symplectic immersion $\phi : M \to V$ on the symplectic space (V, Ω), and let $V' \subset V$ be the smallest affine subspace containing $\phi(M)$. Assuming that $0 \in \phi(M)$ we may assume that V' is a linear. Let (V_0, Ω_0) be the symplectic reduction of V', i.e., $V_0 := V'/N$ where $N \subset V'$ is the null space of $\Omega|_{V'}$, and $\pi^*(\Omega_0) = \Omega|_{V'}$ for the canonical projection $\pi : V' \to V_0$.

Theorem C *Let $\phi : M \to V$ be an extrinsic symplectic immersion with $0 \in \phi(M)$, let $V' \subset V$, $V_0 = V'/N$ and $\pi : V' \to V_0$ be as in the preceding paragraph. Then $\phi_0 := \pi \circ \phi : M \to V_0$ is the full extrinsic symplectic immersion.*

As an immediate consequence of Theorem C we get

Corollary D *An extrinsic symplectic immersion $\phi : M \to V$ is full if and only if the ambient space V is of smallest possible dimension.*

This paper is structured as follows. After setting up our notation in section 2, we define in section 3 the notion of an extrinsic symplectic immersion and discuss their relation to extrinsic symplectic morphisms of Lie algebras. In section 4, we prove Theorem A and finally, section 5 is devoted to the proofs of the remaining statements.

2 Preliminaries

2.1 Symmetric spaces

We recall some basic facts on symmetric spaces which can be found e.g. in [H].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $\sigma : G \to G$ be an involution, i.e., a Lie group homomorphism satisfying $\sigma^2 = Id$. Then $Fix(\sigma) := \{g \in G \mid \sigma(g) = g\}$ is a closed subgroup of G with Lie algebra $\mathfrak{k} = \{x \in \mathfrak{g} \mid d\sigma(x) = x\}$. Let $K \subset G$ be a closed σ-invariant subgroup with Lie algebra $\mathfrak{k}$ (i.e., the identity component of K coincides with that of $Fix(\sigma)$). Then the coset space $M := G/K$ is a manifold. Since $\sigma : G \to G$ is an involution, so is $d\sigma : \mathfrak{g} \to \mathfrak{g}$, hence we may decompose

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{1}$$

where $d\sigma|_{\mathfrak{k}} = Id$ and $d\sigma|_{\mathfrak{p}} = -Id$, and hence we get for the Lie brackets

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{2}$$

Conversely, given a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ satisfying (2), then the map $d\sigma|_{\mathfrak{k}} = Id$ and $d\sigma|_{\mathfrak{p}} = -Id$ is a Lie algebra homomorphism, hence it is the differential of an involuton $\sigma : G \to G$, if G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and hence gives rise to a symmetric space $M = G/K$ as described above.

By definition of K, σ induces an involution $s_0 : M \to M$ by $s_0(gK) := \sigma(g)K$. For $p = gK \in M$, we define the *symmetry at p s_p* : $M \to M$ by $s_p := L_g \circ s_0 \circ L_{g^{-1}}$, where $L_g : M \to M$ denotes the canonical left action by $g \in G$. Then one easily verifies the well-definedness of s_p as well as the following properties:

1. $s_p^2 = Id_M$, and p is an isolated fixed point of s_p for all $p \in M$.

2. $s_p \circ s_q \circ s_p = s_{s_p(q)}$ for all $p, q \in M$.

It is known that the group $\hat{G} := \langle s_p \mid p \in M \rangle \subset \mathit{Diff}(M)$ can be presented by $\{s_p \mid p \in M\}$ subject to these relations. Moreover, the identity component of $\hat{G}$ is known to be a Lie group, called the *transvection group of M*, and its Lie algebra is $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p} \lhd \mathfrak{g}$.

A manifold with smooth maps s_p for all $p \in M$ satisfying these relations is called a *symmetric space*. Thus, we have described above how a connected Lie group G with an involution $\sigma : G \to G$ induces the structure of a symmetric space on $M := G/K$ for a closed σ-invariant subgroup $K \subset G$ with Lie algebra $\mathfrak{k}$.

This process can be reverted. Namely, if M is a symmetric space, let G be the (connected component of the) transvection group, and for a fixed $p_0 \in M$ define the involution

$$\sigma := Ad_{s_{p_0}} : G \to G,$$

which is a well defined involution as $\sigma(s_p) = s_{s_{p_0}(p)} \in \hat{G}$ for all p. Since p_0 is an isolated fixed point of s_{p_0}, it follows that $K := Stab(p_0) \subset G$ is indeed a σ-invariant Lie subgroup with Lie algebra $\mathfrak{k}$ from (1) and hence fits the above description. Observe that if we assume $M = G/K$ to be simply connected, then K is connected and hence $K = Fix(\sigma)_0$.

2.2 The curvature of symplectic symmetric spaces

A symmetric pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called *symplectic*, if there is an $ad_{\mathfrak{k}}$-invariant symplectic form ω on $\mathfrak{p}$. Likewise, a symplectic space M is called *symplectic*, if there is a symplectic form on M which is invariant under all symmetries and hence under the transvection group.

It is elementary to show that the correspondence between simply connected symplectic symmetric spaces and symplectic symmetric pairs described in section 2.1 is a bijection. Our task shall be to investigate the curvature of a symplectic symmetric space and to give a criterion when such a symplectic symmetric space admits an extrinsic symplectic immersion.

If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a symplectic symmetric pair, then $(ad_{\mathfrak{k}})|_{\mathfrak{p}}$ yields a homomorphism $ad : \mathfrak{k} \to \mathfrak{sp}(\mathfrak{p}, \omega)$, and its *curvature* is defined as the map (cf. [H, IV Thm. 4.2])

$$R : \Lambda^2 \mathfrak{p} \longrightarrow \mathfrak{sp}(\mathfrak{p}, \omega), \qquad R(x, y) := -ad_{[x,y]}|_{\mathfrak{p}}.$$

The Jacobi identity of $\mathfrak{g}$ is equivalent to the $ad_{\mathfrak{k}}$-invariance of R and R satisfying the *first Bianchi identity*, i.e., R being contained in the *space of formal curvatures of* $\mathfrak{sp}(\mathfrak{p}, \omega)$

$$K(\mathfrak{sp}(\mathfrak{p}, \omega)) = \{R \in \Lambda^2 \mathfrak{p}^* \otimes \mathfrak{sp}(\mathfrak{p}, \omega) \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \text{ for all } x, y, z \in \mathfrak{p}\}.$$

Thus, there is a one-to-one correspondence between transvective symplectic symmetric pairs and elements $R \in K(\mathfrak{sp}(\mathfrak{p}, \omega))$ such that $ad_k R = 0$ for all $k = R(x, y) \in \mathfrak{sp}(\mathfrak{p}, \omega)$.

As a representation space of $\mathfrak{sp}(\mathfrak{p}, \omega)$, $K(\mathfrak{sp}(\mathfrak{p}, \omega))$ is isomorphic to $\Lambda^2(\mathfrak{sp}(\mathfrak{p}, \omega))$ with an explicit isomorphism given by ([CGRS, Lemma 1.8])

$$R_{A_1 \wedge A_2}(x, y) := \frac{1}{2}((A_1 x) \circ (A_2 y) - (A_1 y) \circ (A_2 x)) \in S^2(\mathfrak{p}) \cong \mathfrak{sp}(\mathfrak{p}, \omega), \qquad (3)$$

where the identification $S^2(\mathfrak{p}) \cong \mathfrak{sp}(\mathfrak{p}, \omega)$ is given by

$$(x \circ y) \cdot z := \omega(x, z)y + \omega(y, z)x.$$

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Observe that our isomorphism differs from the one given in ([CGRS, Lemma 1.8]) by a factor $\frac{1}{2}$, but this will be convenient later on in Lemma 5.2.

Thus, we can find an element

$$\mathcal{R} = \sum_{i,j} a_{ij} A_i \wedge A_j \in \Lambda^2 \mathfrak{sp}(\mathfrak{p}, \omega), \tag{4}$$

where (a_{ij}) is a non-degenerate skew-symmetric matrix and $A_i \in \mathfrak{sp}(\mathfrak{p}, \omega)$ such that

$$
\begin{aligned}
-ad_{[x,y]} \;&=\; \mathcal{R}(x,y) := \sum_{i,j} a_{ij} R_{A_i \wedge A_j}(x,y) \\
&=\; \frac{1}{2} \sum_{i,j} a_{ij}((A_i x) \circ (A_j y) - (A_j y) \circ (A_i x)) \\
&=\; \sum_{i,j} a_{ij}(A_i x) \circ (A_j y)
\end{aligned}
$$

for all $x, y \in \mathfrak{p}$. We may write (4) also as

$$\mathcal{R} = \sum_i A_i \wedge A^i, \text{ where } A^i := \sum_j a_{ij} A_j.$$

The $(ad_{\mathfrak{k}})|_{\mathfrak{p}}$-equivariance of $\mathcal{R}$ is equivalent to

$$0 = ad_k \mathcal{R} = 2 \sum_{i,j} a_{ij}[ad_k, A_i] \wedge A_j = 2 \sum_i [ad_k, A_i] \wedge A^i$$

for all $k \in \mathfrak{k}$, which implies

Proposition 2.1 *Let $\mathfrak{k} \subset \mathfrak{sp}(\mathfrak{p}, \omega)$ and let $\mathcal{R} \in \Lambda^2 \mathfrak{sp}(\mathfrak{p}, \omega) \cong K(\mathfrak{sp}(\mathfrak{p}, \omega))$ be given as in (4). Then $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$ becomes a Lie algebra and hence a symmetric pair whose curvature is represented by $\mathcal{R}$ if and only if there are $\kappa_{ij} = \kappa_{ji} \in \mathfrak{k}^*$ such that*

$$[ad_k, A_i] = \sum_j \kappa_{ij}(k) A^j \text{ for } \kappa_{ij} = \kappa_{ji} \in \mathfrak{k}^*. \tag{5}$$

3   Extrinsic symplectic immersions

In this section, we shall generalize the notion of an *extrinsic symplectic symmetric space* from [CGRS] to extrinsic symplectic immersions. We begin by introducing some notation.

We define $ASp(V, \Omega) = Sp(V, \Omega) \ltimes V$ to be the group of affine symplectic transformations of V and let

$$Lin : ASp(V, \Omega) \longrightarrow Sp(V, \Omega)$$

be the canonical projection $ASp(V) \to ASp(V, \Omega)/V \cong Sp(V, \Omega)$ which is thus a homomorphism.

Definition 3.1 *Let (V, Ω) be a symplectic vector space, and let (W, p) be a pair of a subspace $W \subset V$ such that $\Omega|_W$ is non-degenerate, and a point $p \in V$. The reflection in (W, p) is the involution $\sigma_{(W,p)} \in ASp(V, \Omega)$ given by*

$$\sigma_{(W,p)}(p) = p, \ \ Lin(\sigma_{W,p})|_W = -Id_W, \text{ and } Lin(\sigma_{W,p})|_{W^\perp} = Id_{W^\perp}.$$

Here, $^\perp$ refers to the orthogonal complement w.r.t. Ω.

Note that the definition of the reflections implies that for all $g \in ASp(V, \Omega)$ we have

$$Ad_g(\sigma_{(W,p)}) = \sigma_{(Lin(g) \cdot W, g \cdot p)}. \tag{6}$$

Definition 3.2 *Let M be a symmetric space and (V, Ω) a symplectic vector space. An extrinsic symplectic immersion of M is an immersion $\phi : M \to V$ such that for all $p \in M$ we have*

1. *ϕ is a symplectic immersion, i.e., $\phi^*(\Omega)$ is non-degenerate or, equivalently, the restriction of Ω to $T_p := d\phi_p(T_p M)$ is non-degenerate,*

2. *$\phi \circ s_p = \sigma_{(T_p, \phi(p))} \circ \phi$.*

We define $N_p := (T_p)^\perp$ to be the Ω-orthogonal complement, so that we have the decompositions $V = T_p \oplus N_p$ for all $p \in M$.

Thus, there is a well defined homomorphism

$$\imath_\phi : G \longrightarrow ASp(V, \Omega), \ \imath_\phi(s_p) := \sigma_{(T_p, \phi(p))}, \tag{7}$$

where G is the transvection group of M, since this definition respects the relations for s_p. Then evidently, we have the relation

$$\imath_\phi(g) \cdot \phi(p) = \phi(g \cdot p) \text{ for all } g \in G \text{ and } p \in M. \tag{8}$$

It follows that

$$
\begin{aligned}
Lin(\imath_\phi(g))(T_p) &= d(\imath_\phi(g) \circ \phi)(T_p M) \\
&= d(\phi \circ L_g)(T_p M) \text{ by (8)} \\
&= d\phi(T_{g \cdot p} M) = T_{g \cdot p},
\end{aligned}
$$

where $L_g : M \to M$ denotes the action of $g \in G$ on M. That is, (8) implies

$$Lin(\imath_\phi(g))(T_p) = T_{g \cdot p} \text{ and } Lin(\imath_\phi(g))(N_p) = N_{g \cdot p} \text{ for all } g \in G \text{ and } p \in M. \tag{9}$$

Let us now write $M = G/K$ which induces the Lie algebra decomposition (1), and suppose that $\phi : M \to (V, \Omega)$ is an extrinsic symplectic immersion with $\phi(p_0) = 0$ for $p_0 := eK \in M$. Then by (8) the diagram

$$\tag{10}$$

$$
\begin{array}{ccc}
G & \xrightarrow{\imath_\phi} & ASp(V, \Omega) \\
\downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle \pi_0} \\
M & \xrightarrow{\phi} & V
\end{array}
$$

commutes, where $pr : G \to M = G/K$ and $\pi_0 : ASp(V, \Omega) \to ASp(V, \Omega)/Sp(V, \Omega) \cong V$ are the canonical projections, i.e., $\pi_0(g) = g \cdot 0$ for all $g \in ASp(V, \Omega)$. Moreover, we consider the Ω-orthogonal decomposition

$$V = W_1 \oplus W_2, \tag{11}$$

where $W_1 := T_{p_0} = d\phi(T_{p_0}M)$ is a non-degenerate subspace by hypothesis. Let $\sigma_0 := \sigma_{(W_1, 0)}$, so that $\sigma_0 = \imath_\phi(s_{p_0})$ and hence by (7)

$$\imath_\phi \circ \sigma = \imath_\phi \circ Ad_{s_{p_0}} = Ad_{\sigma_0} \circ \imath_\phi, \tag{12}$$

whose derivative reads

$$d\imath_\phi \circ d\sigma = Ad_{\sigma_0} \circ d\imath_\phi, \tag{13}$$

where $d\sigma : \mathfrak{g} \to \mathfrak{g}$ is the involution with eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$. If we decompose $d\imath_\phi = \Lambda + \tau : \mathfrak{g} \to \mathfrak{asp}(V, \Omega) = \mathfrak{sp}(V, \Omega) \oplus V$, then (13) is equivalent to

$$\Lambda \circ d\sigma = Ad_{\sigma_0} \circ \Lambda \quad \text{and} \quad \tau \circ d\sigma = \sigma_0 \circ \tau. \tag{14}$$

Since $Ad_{\sigma_0} : Sp(V, \Omega) \to Sp(V, \Omega)$ is an involution with differential $Ad_{\sigma_0} : \mathfrak{sp}(V, \Omega) \to \mathfrak{sp}(V, \Omega)$, we get the corresponding symmetric space decomposition

$$\mathfrak{sp}(V, \Omega) = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}, \quad \text{where} \quad \begin{aligned} \tilde{\mathfrak{k}} &= \{x \in \mathfrak{sp}(V, \Omega) \mid xW_i \subset W_i\} \text{ and} \\ \tilde{\mathfrak{p}} &= \{x \in \mathfrak{sp}(V, \Omega) \mid xW_i \subset W_{i+1}\}, \text{ taking } i \bmod 2. \end{aligned} \tag{15}$$

Definition 3.3 *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a symmetric pair and (V, Ω) a symplectic vector space. We call a Lie algebra homomorphism*

$$d\imath : \mathfrak{g} \longrightarrow \mathfrak{asp}(V, \Omega), d\imath(x) := \Lambda(x) + \tau(x) \in \mathfrak{sp}(V, \Omega) \oplus V$$

an extrinsic symplectic morphism *of $\mathfrak{g}$ if there is a symplectic orthogonal decomposition $V = W_1 \oplus W_2$ such that*

1. *$\mathfrak{k} = \ker(\tau)$ and $\tau : \mathfrak{p} \to W_1$ is a linear isomorphism,*

2. *We have $\Lambda \circ d\sigma = Ad_{\sigma_0} \circ \Lambda$ for the involution $d\sigma : \mathfrak{g} \to \mathfrak{g}$ and $Ad_{\sigma_0} : \mathfrak{sp}(V, \Omega) \to \mathfrak{sp}(V, \Omega)$, or, equivalently, with the splitting from (15)*

$$\Lambda(\mathfrak{k}) \subset \tilde{\mathfrak{k}} \text{ and } \Lambda(\mathfrak{p}) \subset \tilde{\mathfrak{p}}.$$

Remark. The reader who is familiar with [CGRS] will note that $d\imath = \Lambda + \tau$ is an extrinsic symplectic morphism of $\mathfrak{g}$ if and only if Λ satisfies the conditions in [CGRS, Lemma 1.5] when identifying $\mathfrak{p} \cong W_1$ by the isomorphism τ.

We now wish to describe how the notions of the extrinsic symplectic immersion of a symmetric space M and extrinsic symplectic morphisms of a symmetric pair are related.

Theorem 3.4 *(cf. [CGRS, Lemmas 1.5 and 1.7])*

1. *Let $M = G/K$ be a symmetric space and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding symmetric pair. Let $\phi : M \to (V, \Omega)$ be an extrinsic symplectic immersion of M with $\phi(eK) = 0$, and let $\imath_\phi : G \to ASp(V, \Omega)$ be the Lie group homomorphism from (7).*

   Then $d\imath_\phi : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ is an extrinsic symplectic morphism of the symmetric pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

2. *Conversely, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a symmetric pair and let $d\imath : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ be an extrinsic symplectic morphism.*

   Then there is a symmetric space $M = G/K$ corresponding to this symmetric pair and an injective extrinsic symmetric immersion $\phi : M \to (V, \Omega)$ such that $d\imath = d\imath_\phi$.

Proof. Let $\phi : M \to V$ be an extrinsic symplectic immersion of M with $\phi(p_0) = 0$ where $p_0 = eK \in G/K = M$, let $W_1 := T_{p_0} = d\phi(T_{p_0} M)$ and consider the splitting (11).

Since (10) commutes, it follows that $\tau = d(\pi_0 \circ \imath_\phi)_e = d(\phi \circ pr)_e$, and since ϕ is an immersion, we have $\ker(\tau) = \ker(dpr_e) = \mathfrak{k}$. Moreover, since $pr : G \to M$ is a submersion, it follows that $Im(\tau) = d(\pi_0 \circ \imath_\phi)(\mathfrak{g}) = d\phi(T_{p_0} M) = W_1$, showing the first property in definition 3.3. The second follows immediately from (14).

To show the second statement, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and G be the simply connected Lie group with Lie algebra $\mathfrak{g}$, let $\sigma : G \to G$ be the involution with differential $d\sigma$ and $\imath : G \to ASp(V, \Omega)$ the homomorphism with differential $d\imath : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$. By Definition 3.3, it follows that (14) holds as $\tau(\mathfrak{k}) = 0$ and $\tau(\mathfrak{p}) = W_1$. Since this is equivalent to (13), integration yields that $\imath$ satisfies (12).

We let $K := \imath^{-1}(Sp(V, \Omega)) \subset G$ which is closed and has $\mathfrak{k}$ as its Lie algebra since $x \in \mathfrak{k}$ iff $d(\pi_0 \circ \imath)(x) = 0$. Moreover, for $k \in K$ we have by (12) $\imath(\sigma(k)) = Ad_{\sigma_0}\imath(k) \in Sp(V, \Omega)$, since Ad_{σ_0} leaves $Sp(V, \Omega)$ invariant. Therefore, $K \subset G$ is a closed σ-invariant subgroup with Lie algebra $\mathfrak{k}$, hence $M := G/K$ is a symmetric space corresponding to the symmetric pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

There is an induced map $\phi : M \to ASp(V, \Omega)/Sp(V, \Omega) = V$ for which (10) commutes with $\imath_\phi := \imath$, and ϕ is an injective immersion. The commutativity of (10) implies that $\phi(gK) = (\phi \circ pr)(g) = \pi_0(\imath_\phi(g)) = \imath_\phi(g) \cdot 0 = \imath_\phi(g) \cdot \phi(eK)$ for all $g \in G$, so that (8) holds, which was shown to imply (9). Therefore, $T_{gK} = Lin_{\imath(g)}(T_{eK}) = Lin_{\imath(g)}(W_1)$, and since $\Omega|_{W_1}$ is non-degenerate, $\Omega|_{T_p}$ is non-degenerate for all $p \in M$, i.e., $\phi : M \to V$ is a symplectic immersion.

For $g, h \in G$ and $p := gK$ and $q := hK \in M = G/K$ we have

$$
\begin{aligned}
\phi(s_p(q)) &= \phi(g\sigma(g^{-1}h)K) = (\imath_\phi(g\sigma(g^{-1}h)))(0) \text{ by (10)}\\
&= \imath_\phi(g) \cdot \imath_\phi(\sigma(g^{-1}h))(0) = \imath_\phi(g) \cdot Ad_{\sigma_0}(\imath_\phi(g^{-1}h))(0) \text{ by (12)}\\
&= \imath_\phi(g) \cdot \sigma_0 \cdot \imath_\phi(g)^{-1} \cdot \imath_\phi(h)(0) \text{ as } \sigma_0(0) = 0\\
&= Ad_{\imath_\phi(g)}(\sigma_{(T_{eK},0)}) \cdot \phi(q) \text{ by (10)}\\
&= \sigma_{(Lin(\imath_\phi(g)) \cdot T_{eK}, \imath_\phi(g)(0))} \cdot \phi(q) \text{ by (6)}\\
&= \sigma_{(T_p, \phi(p))} \cdot \phi(q) \text{ by (9)},
\end{aligned}
$$

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and hence,

$$\phi \circ s_p = \sigma_{(T_p, \phi(p))} \circ \phi$$

for all $p \in M$ which is the second property for ϕ being a symplectic immersion of M. ∎

Note that if $\iota : (V_1, \Omega_1) \to (V_2, \Omega_2)$ is a symplectic isomorphism and $\phi : M \to V_1$ is an extrinsic symplectic immersion with associated homomorphism $\imath_\phi : G \to ASp(V, \Omega)$, then $\imath_{\iota \circ \phi} = Ad_\iota \circ \imath_\phi$. Thus, from Theorem 3.4 we immediately obtain

Corollary 3.5 *Let $M = G/K$ be a simply connected symmetric space with corresponding Lie algebra decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then there is a one-to-one correspondence between affine equivalence classes of extrinsic symplectic immersions $\phi : M \to (V, \Omega)$ and conjugacy classes of extrinsic symplectic morphisms $d\imath : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$.*

If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a symmetric pair admitting an extrinsic symplectic morphism $d\imath = \Lambda + \tau :$ $\mathfrak{g} \to \mathfrak{asp}(V, \Omega)$, then evidently, $\tau^*(\Omega)$ is an $ad_\mathfrak{k}$-invariant symplectic form on $\mathfrak{p}$. Thus, for the splitting of the Lie algebra $\mathfrak{g}$ from (1) we can describe the curvature $R : \Lambda^2 \mathfrak{p} \to \mathfrak{k}$ as in (4). In this language, the results from [CGRS] yield the following.

Proposition 3.6 *([CGRS]) Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a symplectic symmetric pair with the $ad_\mathfrak{k}$-invariant symplectic form $\omega \in \Lambda^2 \mathfrak{p}^*$, let $\mathcal{R} = \sum_{i,j} a_{ij} A_i \wedge A_j \in \Lambda^2 \mathfrak{sp}(\mathfrak{p}, \omega)$ be its curvature and let $\kappa_{ij} = \kappa_{ji} \in \mathfrak{k}^*$ be as in (5). Then $\mathfrak{g}$ admits an extrinsic symplectic morphism if and only if*

$$\kappa_{ij}(R(x, y)) = \omega((A_i A_j + A_j A_i)x, y) \text{ for all } x, y \in \mathfrak{p} \text{ and all } i, j$$

4 Parallelity of the second fundamental form

Let $\phi : M \to (V, \Omega)$ be a symplectic immersion of a manifold M into a symplectic vector space, i.e., $\phi^*(\Omega)$ is non-degenerate. Since $T_p := d\phi(T_p M) \subset V$ is a non-degenerate subspace, we obtain the Ω-orthogonal splitting $V = T_p \oplus N_p$ for all $p \in M$. Thus, we have the pullback vector bundles

$$\mathcal{T} := \{(p, v) \in M \times V \mid v \in T_p\} \text{ and } \mathcal{N} := \{(p, v) \in M \times V \mid v \in N_p\}.$$

Evidently, $\mathcal{T} = TM$ is the tangent bundle of M as ϕ is an immersion, and $\mathcal{N} \to M$ is called the *normal bundle of ϕ*. For the direct sum of these vector bundles, we have

$$M \times V = \mathcal{T} \oplus \mathcal{N}. \tag{16}$$

The canonical flat connection $\overset{\circ}{\nabla}$ on V induces connections on TM and $\mathcal{N}$ by the definitions

$$\nabla^\mathcal{T}_X Y := (\overset{\circ}{\nabla}_X Y)_\mathcal{T} \text{ and } \nabla^\mathcal{N}_X \xi := (\overset{\circ}{\nabla}_X \xi)_\mathcal{N},$$

where the subscripts refer to the components of the vector bundle splitting (16). Moreover, the *second fundamental form* and the *shape operator of ϕ* are defined as

$$\alpha(X, Y) := (\overset{\circ}{\nabla}_X Y)_\mathcal{N} \text{ and } A_\xi X := (\overset{\circ}{\nabla}_X \xi)_\mathcal{T}, \tag{17}$$

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which are related by the identity

$$\Omega(\alpha(X,Y),\xi) = \Omega(A_\xi X, Y). \tag{18}$$

Since α is symmetric, it follows that $A_\xi \in \mathfrak{sp}(T_p, \Omega)$.

Proof of Theorem A. If $\phi : M \to V$ is an extrinsic symplectic immersion, then $s_p : M \to M$ can be extended to bundle maps $s_p : TM \to TM$ and $s_p : \mathcal{N} \to \mathcal{N}$ in a canonical way, and these maps preserve the connections $\nabla^{\mathcal{T}}$ and $\nabla^{\mathcal{N}}$. Therefore, for all $X, Y, Z \in T_p$ we have

$$s_p((\nabla_X \alpha)(Y, Z)) = (\nabla_{s_p(X)} \alpha)(s_p Y, s_p Z) = (\nabla_{-X} \alpha)(-Y, -Z) = -(\nabla_X \alpha)(Y, Z).$$

On the other hand, $s_p((\nabla_X \alpha)(Y, Z)) = (\nabla_X \alpha)(Y, Z)$ as $(\nabla_X \alpha)(Y, Z) \in N_p$, so that $(\nabla_X \alpha)(Y, Z) = 0$ and hence, α is parallel. This evidently also holds if $M \to \hat{M} \to V$ is the composition of a local diffeomorphism and an extrinsic symplectic immersion.

Conversely, let $\phi : M \to (V, \Omega)$ be a symplectic immersion such that α is parallel. Fix $p_0 \in M$ and assume w.l.o.g. that $\phi(p_0) = 0$. We consider the Ω-orthogonal decomposition $V = W_1 \oplus W_2$ with $W_1 := d\phi_{p_0}(T_{p_0}M)$. As in [F1], we define the map

$$\Lambda : W_1 \longrightarrow \mathfrak{sp}(V, \Omega), \quad \Lambda(x)(y + \xi) = A_\xi x + \alpha(x, y),$$

where $x, y \in W_1$ and $\xi \in W_2$. By (18) and the symmetry of α, we have indeed $\Lambda(x) \in \mathfrak{sp}(V, \Omega)$, and in fact, $\Lambda(x) \in \tilde{\mathfrak{p}}$ with the splitting $\mathfrak{sp}(V, \Omega) = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ from (15). Now we can apply verbatim the proof in [F1] of the corresponding statement for Riemannian immersions to the present case to see that

1. $\mathfrak{g} := \{R(x, y) \mid x, y \in W_1\} \oplus \{\Lambda(x) \mid x \in W_1\} =: \mathfrak{k} \oplus \mathfrak{p} \subset \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}} \subset \mathfrak{sp}(V, \Omega)$ is a Lie subalgebra and hence a symmetric pair,

2. The map $d\imath : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ given by $d\imath(k) := k$ for $k \in \mathfrak{k}$ and $d\imath(x) := \Lambda(x) + x$ for $x \in \mathfrak{p}$ is an extrinsic symplectic morphism.

By Theorem 3.4, there is an injective extrinsic symplectic immersion

$$\hat{\phi} : \hat{M} \longrightarrow V$$

of a symmetric space $\hat{M}$ with $d\imath_{\hat{\phi}} = d\imath$, and $\hat{\phi}(\hat{p}_0) = 0 = \phi(p_0)$ and $d\hat{\phi}(T_{p_0}\hat{M}) = W_1 = d\phi(T_{p_0}M)$ for some $\hat{p}_0 \in \hat{M}$. Therefore, the second fundamental form of ϕ and $\hat{\phi}$ at $0 \in V$ coincide, and since both have parallel second fundamental form, it follows that their second fundamental forms coincide and hence, $\phi(M) \subset \hat{\phi}(\hat{M})$. As $\hat{\phi}$ is injective, it follows that $\phi = \hat{\phi} \circ \varepsilon$ for some map $\varepsilon : M \to \hat{M}$, and clearly, ε is a local diffeomorphism. ∎

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5 Full extrinsic symplectic immersions

Suppose that $d\imath = \Lambda + \tau : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ is an extrinsic symplectic morphism, and let $V = W_1 \oplus W_2$ be the Ω-orthogonal decomposition with $W_1 = \tau(\mathfrak{p})$, and we let $\omega := \tau^*(\Omega)$ be the symplectic form on $\mathfrak{p}$ induced by this morphism. We define the map

$$\mathcal{A} : W_2 \longrightarrow \mathfrak{sp}(\mathfrak{p}, \omega), \qquad \tau(\mathcal{A}(\xi)(x)) := \Lambda(x)\xi \in W_1. \tag{19}$$

Indeed, $\mathcal{A}(\xi) \in \mathfrak{sp}(\mathfrak{p}, \omega)$ as

$$
\begin{aligned}
\omega(\mathcal{A}(\xi)x, y) - \omega(\mathcal{A}(\xi)y, x) &= (\tau^*\Omega)(\tau^{-1}\Lambda(x)\xi, y) - (\tau^*\Omega)(\tau^{-1}\Lambda(y)\xi, x) \\
&= \Omega(\Lambda(x)\xi, \tau(y)) - \Omega(\Lambda(y)\xi, \tau(x)) \\
&= \Omega(\Lambda(x)\tau(y) - \Lambda(y)\tau(x), \xi) = 0,
\end{aligned}
$$

since $\mathfrak{sp}(V, \Omega) \ni [\Lambda(x) + \tau(x), \Lambda(y) + \tau(y)] = [\Lambda(x), \Lambda(y)] + (\Lambda(x)\tau(y) - \Lambda(y)\tau(x))$, so that $\Lambda(x)\tau(y) - \Lambda(y)\tau(x) = 0$ for all $x, y \in \mathfrak{p}$.

Lemma 5.1 *The map $\mathcal{A} : W_2 \to \mathfrak{sp}(\mathfrak{p}, \omega)$ from (19) is $\mathfrak{k}$-equivariant; i.e., for all $k \in \mathfrak{k}$ and $\xi \in W_2$, the following identity holds:*

$$\mathcal{A}(\Lambda(k)\xi) = [k, \mathcal{A}(\xi)].$$

Proof. This follows from the following calculation for $x \in \mathfrak{p}$, $k \in \mathfrak{k}$ and $\xi \in W_2$:

$$
\begin{aligned}
\tau(\mathcal{A}(\Lambda(k)\xi)(x)) &= \Lambda(x)\Lambda(k)\xi = \Lambda(k)\Lambda(x)\xi - [\Lambda(k), \Lambda(x)]\xi \\
&= \Lambda(k)\tau(\mathcal{A}(\xi)x) - \Lambda(k\ x)\xi = \tau(k\ \mathcal{A}(\xi)x) - \tau(\mathcal{A}(\xi)k\ x) \\
&= \tau([k, \mathcal{A}(\xi)](x)).
\end{aligned}
$$
■

Lemma 5.2 (cf. [CGRS, (1.26)]) *Let $d\imath : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ be an extrinsic symplectic morphism, let $V = W_1 \oplus W_2$ and $\mathcal{A} : W_2 \to \mathfrak{sp}(\mathfrak{p}, \omega)$ be as above. Let $\Lambda^2\mathcal{A} : \Lambda^2 W_2 \to \Lambda^2\mathfrak{sp}(\mathfrak{p}, \omega)$ be the canonical extension. Then the curvature $\mathcal{R}$ of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is represented by*

$$\mathcal{R} = (\Lambda^2\mathcal{A})(\Omega|_{W_2})^{-1} \in \Lambda^2(\mathfrak{sp}(\mathfrak{p}, \omega))$$

with the identification from (3) and where $(\Omega|_{W_2})^{-1} \in \Lambda^2 W_2$ is the inverse of $\Omega|_{W_2} \in \Lambda^2 W_2^*$. That is, if $\{e_i\}$ is a basis of W_2 such that $\Omega_{ij} := \Omega(e_i, e_j)$ has the inverse matrix Ω^{ij}, and if $A_i := \mathcal{A}(e_i)$, then

$$\mathcal{R} = \sum_{i,j} \Omega^{ij} R_{A_i \wedge A_j}.$$

Once again, observe that in our convention the identification $\Lambda^2(\mathfrak{sp}(V, \Omega) \cong K(\mathfrak{sp}(V, \omega)$ differs from that in ([CGRS]) by a factor $\frac{1}{2}$.

Definition 5.3 *Let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *be a symplectic symmetric pair, and let* $\mathcal{R} \in \Lambda^2(\mathfrak{sp}(\mathfrak{p}, \omega))$ *be the element representing the curvature of this space. Then*

$$\mathrm{srk}(\mathfrak{g}, \mathfrak{k}) := \mathrm{rk}(\mathcal{R})$$

is called the symplectic curvature rank*. Here, we consider the rank of* $\mathcal{R}$ *regarded as a 2-form.*

Note that because of Lemma 5.2 we have $\dim W_2 \geq \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$, and hence $\dim V \geq \dim \mathfrak{p} + \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$ for any extrinsic symplectic morphism.

Let $V' \subset V$ be a linear subspace. We define the Lie group

$$Stab(V') := \{g \in ASp(V, \Omega) \mid g \cdot V' \subset V'\},$$

whose Lie algebra is given as

$$\mathfrak{stab}(V') := \{A + v \in \mathfrak{asp}(V, \Omega) = \mathfrak{sp}(V, \Omega) \oplus V \mid v \in V' \text{ and } A \cdot V' \subset V'\}.$$

We call an extrinsic symplectic immersion $\phi : M \to V$ *full* if there is no proper affine subspace $V' \subsetneq V$ such that $\phi(M) \subset V'$. Likewise, we call an extrinsic symplectic morphism $d\imath : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ *full*, if there is no proper linear subspace $V' \subsetneq V$ such that $d\imath(\mathfrak{g}) \subset \mathfrak{stab}(V') \subset \mathfrak{asp}(V, \Omega)$. Evidently, ϕ is full if and only if $d\imath_\phi$ is.

Proposition 5.4 *Let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *be a symmetric pair,* $V = W_1 \oplus W_2$ *a symplectic vector space, and let* $d\imath = \Lambda + \tau : \mathfrak{g} \to \mathfrak{asp}(V, \Omega)$ *be an extrinsic symplectic morphism. Consider the map* $\mathcal{A} : W_2 \to \mathfrak{sp}(\mathfrak{p}, \omega)$ *from (19). Then the following are equivalent.*

1. *$d\imath$ is full.*

2. *$\dim(W_2) = \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$, i.e., $\dim V = \dim \mathfrak{p} + \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$.*

3. *$\mathcal{A}$ is injective.*

4. *$W_2 = span\{\Lambda(x)\tau(y) \mid x, y \in \mathfrak{p}\}$.*

Proof. Since $\Lambda^2 \mathcal{A}(\Omega|_{W_2})^{-1} = \mathcal{R}$ by Lemma 5.2, it follows that $\mathrm{rk}(\mathcal{A}) \geq \mathrm{rk}(\mathcal{R}) = \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$. Thus, if $\dim W_2 = \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$, then $\mathcal{A}$ must be injective.

Conversely, if $\mathcal{A}$ is injective, then $\mathrm{rk}(\Lambda^2 \mathcal{A}(\Omega|_{W_2})^{-1}) = \dim W_2$ since $\Omega|_{W_2}$ is non-degenerate. Thus, $\dim W_2 = \mathrm{rk}(\mathcal{R}) = \mathrm{srk}(\mathfrak{g}, \mathfrak{k})$ by Lemma 5.2, which shows that the second and third statement are equivalent.

Let $W_2' := span\{\Lambda(x)\tau(y) \mid x, y \in \mathfrak{p}\} \subset W_2$. Then $\xi \in (W_2')^\perp$ if and only if $0 = \Omega(\xi, \Lambda(x)\tau(y)) = -\Omega(\Lambda(x)\xi, \tau(y)) = -\Omega(\tau(\mathcal{A}(\xi)x), \tau(y)) = -\omega(\mathcal{A}(\xi)x, y)$ for all $x, y \in \mathfrak{p}$, using the definition of $\mathcal{A}$ in (19), which implies that $(W_2')^\perp = \ker(\mathcal{A})$, and this shows the equivalence of the third and fourth statement.

Let $V' \subset V$ be such that $d\imath(\mathfrak{g}) \subset \mathfrak{stab}(V') \subset \mathfrak{asp}(V, \Omega)$. Since $\tau(\mathfrak{p}) = W_1$, it follows that $W_1 \subset V'$. Thus, $[d\imath(x), \tau(y)] = \Lambda(x)\tau(y) \in V'$ for all $x, y \in \mathfrak{p}$, i.e., $W_1 \oplus W_2' \subset V'$. If

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$W_2' = W_2$, then we must have $V' = V$ which shows that the fourth statement implies the first.

On the other hand, $W_1 \oplus W_2'$ is invariant under $\Lambda(x)$ for all $x \in \mathfrak{p}$ by definition, and W_2' is invariant under $\Lambda(k)$ for $k \in \mathfrak{k}$, since

$$\Lambda(k)(\Lambda(x)\tau(y)) = [\Lambda(k), \Lambda(x)]\tau(y) + \Lambda(x)\Lambda(k)\tau(y) = \Lambda(k\ x)\tau(y) + \Lambda(x)\tau(k\ y) \in W_2'.$$

Thus, $d\imath(\mathfrak{g}) \subset \mathfrak{stab}(W_1 \oplus W_2')$, so that the first statement implies the fourth. $\blacksquare$

Proposition 5.5 *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a symmetric pair with two full extrinsic symplectic morphisms $d\imath_i = \Lambda_i + \tau_i : \mathfrak{g} \to \mathfrak{asp}(V_i, \Omega_i)$ such that $\tau_1^*(\Omega_1) = \tau_2^*(\Omega_2)$. Then $d\imath_1$ and $d\imath_2$ are affinely equivalent, i.e., there is a linear symplectic isomorphism $\iota : (V_1, \Omega_1) \to (V_2, \Omega_2)$ such that $d\imath_2 = Ad_\iota \circ d\imath_1$.*

Proof. Consider the Ω_i-orthogonal decompositions $V_i = W_1^i \oplus W_2^i$ where $\tau_i : \mathfrak{p} \to W_1^i$ is an isomorphism for $i = 1, 2$. Then $\mathcal{A}_i : W_2^i \to \mathfrak{sp}(\mathfrak{p}, \omega)$ is injective for $i = 1, 2$ by Proposition 5.4 and moreover, the images $\mathcal{A}_i(W_2^i) \subset \mathfrak{sp}(\mathfrak{p}, \omega)$ coincide by Lemma 5.2. Thus, we may define the linear isomorphisms

$$\iota_1 := \tau_2 \circ \tau_1^{-1} : W_1^1 \longrightarrow W_1^2, \text{ and } \iota_2 : \mathcal{A}_2^{-1} \circ \mathcal{A}_1 : W_2^1 \longrightarrow W_2^2.$$

By our hypothesis, $\tau_i^*(\Omega_i) = \omega$. Moreover, by Lemma 5.2 $(\mathcal{A}_i^{-1})^*(\Omega_i^{-1}) = \mathcal{R} \in \Lambda^2(\mathfrak{sp}(V, \omega)$. Thus, $(\iota_i)^*(\Omega_2|_{W_i^2}) = \Omega_1|_{W_i^1}$ for $i = 1, 2$, i.e.,

$$\iota := \iota_1 \oplus \iota_2 : (V_1, \Omega_1) \longrightarrow (V_2, \Omega_2)$$

is a symplectic isomorphism. Since $\tau_2 = \iota_1 \circ \tau_1$ and $\mathcal{A}_1 = \mathcal{A}_2 \circ \iota_2$, we have for $x, y \in \mathfrak{p}$ and $\xi \in W_2^2$

$$
\begin{aligned}
\Omega_2(\iota\Lambda_1(x)\iota^{-1}\xi, \tau_2(y)) &= (\iota^*\Omega_2)(\Lambda_1(x)\iota_2^{-1}\xi, \iota_1^{-1}\tau_2(y)) \\
&= \Omega_1(\tau_1(\mathcal{A}_1(\iota_2^{-1}(\xi))x), \tau_1(y)) \\
&= (\tau_1^*\Omega_1)(\mathcal{A}_2(\xi)x, y) = (\tau_2^*\Omega_2)(\mathcal{A}_2(\xi)x, y) \\
&= \Omega_2(\tau_2(\mathcal{A}_2(\xi)x), \tau_2(y)) \\
&= \Omega_2(\Lambda_2(x)\xi, \tau_2(y)),
\end{aligned}
$$

and since $\iota\Lambda_1(x)\iota^{-1}(W_i^2) \subset W_{i+1}^2$ and $\Lambda_2(x)(W_i^2) \subset W_{i+1}^2$ taking indices mod 2, it follows that $\iota\Lambda_1(x)\iota^{-1} = \Lambda_2(x)$ for all $x \in \mathfrak{p}$. Similarly, we have for $k \in \mathfrak{k}$ and $x, y \in \mathfrak{p}$

$$
\begin{aligned}
\Omega_2(\iota\Lambda_1(k)\iota^{-1}\tau_2(x), \tau_2(y)) &= (\iota^*\Omega_2)(\Lambda_1(k)\iota_1^{-1}\tau_2(x), \iota_1^{-1}\tau_2(y)) \\
&= \Omega_1(\Lambda_1(k)\tau_1(x), \tau_1(y)) \\
&= \Omega_1(\tau_1(k\ x), \tau_1(y)) \\
&= (\tau_1^*\Omega_1)(k\ x, y) = (\tau_2^*\Omega_2)(k\ x, y) \\
&= \Omega_2(\tau_2(k\ x), \tau_2(y)) \\
&= \Omega_2(\Lambda_2(k)\tau_2(x), \tau_2(y)),
\end{aligned}
$$

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and since $\iota\Lambda_1(k)\iota^{-1}(W_1^2) \subset W_1^2$ and $\Lambda_2(k)(W_1^2) \subset W_1^2$, it follows that $\iota\Lambda_1(k)\iota^{-1}|_{W_1^2} = \Lambda_2|_{W_1^2}$. Finally, for $k \in \mathfrak{k}$ and $\xi \in W_2^2$ we have by Lemma 5.1

$$
\begin{aligned}
\iota\Lambda_1(k)\iota^{-1}\xi &= \iota_2\Lambda_1(k)\iota_2^{-1}\xi \\
&= \mathcal{A}_2^{-1}\mathcal{A}_1(\Lambda_1(k)(\iota_2^{-1}(\xi))) \\
&= \mathcal{A}_2^{-1}[k, \mathcal{A}_1(\iota_2^{-1}(\xi))] \\
&= \mathcal{A}_2^{-1}[k, \mathcal{A}_2(\xi)] \\
&= \mathcal{A}_2^{-1}\mathcal{A}_2(\Lambda_2(k)(\xi)) = \Lambda_2(k)(\xi),
\end{aligned}
$$

so that $\iota\Lambda_1(k)\iota^{-1}|_{W_2^2} = \Lambda_2|_{W_2^2}$, which completes showing that

$$\Lambda_2 = Ad_\iota \circ \Lambda_1 \text{ and } \tau_2 = \iota \circ \tau_1,$$

and therefore, $d\iota_2 = Ad_\iota \circ d\iota_1$. ∎

Proof of Theorem C. Let $\phi : M \to V$ be an extrinsic symplectic immersion with $0 \in \phi(M)$, let $V' \subset V$ be the smallest subspace containing $\phi(M)$, and let $N \subset V'$ be the nullspace of $\Omega|_{V'}$. Then $V_0 := V'/N$ has an induced symplectic structure Ω_0 such that $\pi^*(\Omega_0) = \Omega|_{V'}$, where $\pi : V' \to V_0$ is the canonical projection.

Since $T_p \subset V'$ for all $p \in M$ and $\Omega|_{T_p}$ is non-degenerate, it follows that $N \subset T_p^\perp$ and hence, $N \cap T_p = 0$ for all $p \in M$. Thus, $\phi_0 := \pi \circ \phi : M \to V_0$ is an immersion, and $\Omega_0|_{T_p^0}$ is non-degenerate, where $T_p^0 = d\phi_0(T_pM) = \pi(T_p)$. Moreover, for the reflections we have

$$\pi \circ \sigma_{(T_p,\phi(p))} = \sigma_{(T_p^0,\phi_0(p))} \circ \pi$$

for all p, whence $\sigma_{(T_p^0,\phi_0(p))} \circ \phi_0 = \sigma_{(T_p^0,\phi_0(p))} \circ \pi \circ \phi = \pi \circ \sigma_{(T_p,\phi(p))} \circ \phi = \pi \circ \phi \circ s_p = \phi_0 \circ s_p$, using that ϕ is an extrinsic symplectic immersion. Thus, $\phi_0 : M \to V_0$ is an extrinsic symplectic immersion as well.

If $\phi_0(M) \subset V_0' \subset V_0$, then $\phi(M) \subset \pi^{-1}(V_0') \subset V'$. Since we chose $V' \subset V$ to be the smallest subspace containing $\phi(M)$, it follows that $V_0' = V_0$, and this implies that $\phi_0 : M \to V_0$ is full. ∎

Theorem B from the introduction now follows immediately from combining Theorem C with Proposition 5.5, and Corollary D is an immediate consequence of Theorem C.

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