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Cloaking of small objects by anomalous localized resonance

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Abstract: We investigate operators $\mathcal{L}^\eta u = \nabla \cdot (a^\eta \nabla u)$ and solutions u^η of $\mathcal{L}^\eta u^\eta = 0$ to various boundary conditions. The coefficients a^η are assumed to have a real part with changing sign and a small, non-negative imaginary part. We investigate a ring geometry with radii 1 and R in two space dimensions and use Fourier expansions in polar coordinates to analyze the qualitative behavior of solutions when boundary conditions on a small inclusion $B_\varepsilon(x_0)$ are imposed. Our result is that u^η depends qualitatively on the position of the inclusion. If $|x_0|$ is larger than the cloaking radius $R^* := R^{3/2}$, then u^η behaves as if no ring were present. If, instead, $|x_0|$ is smaller than R^*, then the small inclusion is invisible in the limit $\eta \to 0$.

1 Introduction

Cloaking is a name for an invisibility effect produced by a suitable device. The essential feature of such a *cloaking device* is that it is itself invisible, and that any object inside the device or in its vicinity is also nearly invisible. In order to quantify the term *invisibility*, a process of measurement is defined. This can be done in the framework of geometrical optics [15] or of partial differential equations, Maxwell equations [8], wave and Helmholtz equations, or electrostatic equations in the context of impedance tomography [12]. In such a setting, the cloaking device and the cloaked object can be described by coefficients in the equation. The cloaking device is usually given by coefficients with extreme values, large or small, or by negative coefficients in some part of the domain. In this sense, many ideas for cloaking can be considered as realizable only since the invention of negative index metamaterials [22], for which mathematical justifications can be found for instance in [7, 3, 13, 4].

At least three different approaches to cloaking are currently under investigation. In an approach based on complex analysis [15] one constructs an optical index field which has the property that no light rays enter a certain region, but all light rays are straight lines at infinity. This can be achieved by a rather explicit construction of holomorphic mappings. This approach seems to be merely applicable in the two dimensional case.

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In other approaches, one considers a specific differential equation on a subset $\Omega \subset \mathbb{R}^n$ for $n = 2$ or $n = 3$, and studies variable coefficients in Ω. A cloak is then defined by a fixed choice of coefficients. In the change of variables method [12] the construction exploits that, in a reference domain with constant coefficients, a small subset with changed coefficients has only a small effect in measurements. After a change of variables, the small subset is a large subset in the physical domain and the new coefficients have extreme values. The result is that any object that is placed in the large subset in the physical domain, is almost invisible.

A third approach is connected with the concept of *anomalous localized resonance.* The setting looks similar to the one used in the change of variables method in that a differential equation for measurements is chosen and a variable coefficient on a physical domain defines the cloak. However, here the invisibility issue is directly linked to the fact the device has an index with changes of sign. The cloaking devices are mostly given by simple geometric objects. In [16], a ring $B_R(0) \setminus B_1(0)$ is considered such that the coefficient has the opposite sign in the ring and in the outside. It is observed that a second object placed in the vicinity of $B_R(0)$, e.g. given as a further perturbation of the coefficients, is nearly invisible for measurements.

Different measurement processes can be considered. A standard possibility is to define a large set Q that contains all the structures, and to analyze the effect of different boundary data as in electrical impedance tomography (EIT), cp. [14]. In this spirit, one may define that a measurement consists in providing the Dirichlet-to-Neumann map on ∂Q for the given coefficients in Q. Even though this map does not determine the coefficients (see, e.g. [10]), one may define cloaking as the fact that certain perturbations of the coefficients do not lead to relevant changes of the Dirichlet-to-Neumann map [9].

The effect of anomalous localized resonance. The phenomenon relies on the fact that an elliptic partial differential equation with a sign-changing coefficient can exhibit localization effects [16, 17, 20]. One observes that there exist functions which are localized around the given structure and nevertheless homogeneous solutions except for a small error. Numerical tests [5] confirm this effect which is very much in contrast to standard elliptic equation without a sign-change, where the maximum principle inhibits solutions to have maxima inside the medium. Our interpretation of the cloaking process is the following. The exterior measurement results in an answer of the small object. But the information that is sent out by the small object results only in a resonance of the cloaking device and is not radiated to outer boundaries or to infinity.

One would like to study the problem $\nabla \cdot (a \nabla u) = 0$ with index $a = 1$ outside the ring $a = -1$ inside the ring. This problem can not be treated by standard methods, for elaborate approaches we refer to [1, 2, 6]. Another approach is to consider the case of a positive dissipation coefficient η and to study the limit $\eta \to 0$, which might be considered also a low viscosity limit [16]. We follow this approach and consider an coefficient which is perturbed in the complex plane as given in (1.1). For finite $\eta > 0$ this regularized problem can be solved by the Lax-Milgram theorem. Our aim is to characterize a^η-harmonic functions u^η for various boundary conditions in the limit $\eta \to 0$.

Our main theorems describe the invisibility effect in this geometry, related to the cloaking radius $R^* = R^{3/2}$. In our first theorem, we study a small dipole radiator at a position $x_0 \in \mathbb{R}^2$, modelled by a boundary condition on $\partial B_\varepsilon(x_0)$. To these boundary data, we construct the bounded a^η-harmonic extension v^η. Our aim is to analyze the qualitative behavior of v^η. Our result is that the behavior of v^η is very different depending on whether x_0 is a point inside the ball $B_{R^*}(0)$ or not. If x_0 is outside the cloaking zone, the solution v^η looks like a dipole field (without the ring). Instead, if x_0 is inside the cloaking zone, the solution v^η is concentrated in the vicinity of the cloak and the far-field vanishes. To make this statement precise we adopt ideas of [5] and introduce the number $\mathcal{M}_q^\eta$ as a measure for the field strength at a distance q.

Theorem 2, presented in Section 5, regards a passive inclusion $B_\varepsilon(x_0)$ with coefficient $a = \infty$. In this case, we study the Dirichlet-to-Neumann map on the large ball $B_q(0)$. The invisibility effect is made precise with a number $\mathcal{N}_q^\eta(f)$. This number measures, for Dirichlet data f on $\partial B_q(0)$, how much the solution for the perturbed medium differs from the solution of the uniform medium. It turns out that the small inclusion is visible in the case $|x_0| > R^*$, but invisible in the opposite case.

It is crucial in our results that the radius $\varepsilon = \varepsilon(\eta)$ vanishes together with the dissipation coefficient η, and, moreover, we need the two convergences with controlled rates. This is not only essential for our proofs, but it reflects also a physical fact. The localized resonance appears with an angular wave number of order $|\log(\eta)|$, and the inclusion must be small compared to the corresponding wavelength at distance R^*.

1.1 Geometry and main result

We consider the two-dimensional case, $x \in \mathbb{R}^2$, and the ring $\Sigma := B_R(0) \setminus B_1(0)$ with inner radius 1 and outer radius $R > 1$. The coefficients have a positive real part outside Σ and a negative real part in Σ. With a fixed direction $i_0 \in \mathbb{C}$, $|i_0| = 1$ and $\operatorname{Im} i_0 > 0$ (e.g. $i_0 = i$) we set

$$a^\eta(x) := \begin{cases} -1 + i_0\eta & \text{for } x \in \Sigma \\ +1 & \text{for } x \in \mathbb{R}^2 \setminus \Sigma. \end{cases} \tag{1.1}$$

Here, $\eta > 0$ is a small quantity that measures losses inside the medium and we will study the limit $\eta \to 0$. The geometry is sketched in Figure 1.

We study in two-dimensions the family of operators $\mathcal{L}^\eta u := \nabla \cdot (a^\eta \nabla u)$ and are interested in $\mathcal{L}^\eta$-harmonic functions $u : U \subset \mathbb{R}^2 \to \mathbb{C}$, i.e. in solutions of

$$\mathcal{L}^\eta u = \nabla \cdot (a^\eta \nabla u) = 0. \tag{1.2}$$

We ask whether the coefficient a^η can produce a cloaking phenomenon. To this end we consider an object positioned at $x_0 \in \mathbb{R}^2$ outside the negative index ring and specialize to the case that the object is a small ball $B_\varepsilon(x_0)$. We will state and prove a mathematical theorem that makes precise the observations of [16, 17, 20] on cloaking by localized resonance. We show that the cloaking radius $R^* = R^{3/2} > R$ satisfies, in appropriate limits, the following: If x_0 is not contained in the ball $B_{R^*}(0)$, then a measurement of the whole assembly yields results as if no ring were present. If,

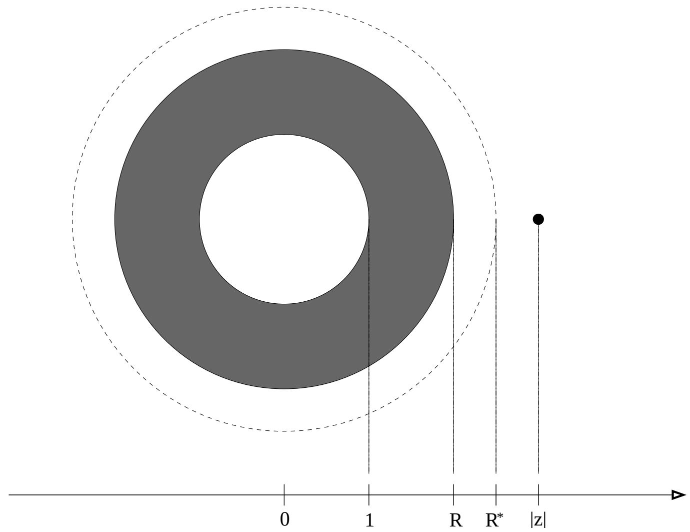

Figure 1: *Sketch of the geometry. In the black ring the permeability has a negative real part and a small imaginary part.*

instead, x_0 is contained in $B_{R^*}(0)$, then a measurement of the whole assembly does neither detect the ring nor the inclusion $B_\varepsilon(x_0)$.

Theorem 1 (Cloaking of a radiator). *Let $\mathcal{L}^\eta$ as in* (1.1), (1.2) *and $R^* := R^{3/2} > R > 1$. Let $x_0 \in \mathbb{R}^2$ be a point with $|x_0| > R$ and let $q > R^2$ be an observation radius. We study $\mathcal{L}^\eta$-harmonic bounded functions v^η on $\mathbb{R}^2 \setminus \bar{B}_\varepsilon(x_0)$ which satisfy the boundary condition $v^\eta(x_0 + \varepsilon e^{i\vartheta}) = \varepsilon^{-1} e^{i\vartheta}$ on $\partial B_\varepsilon(x_0)$. As a measure for the visibility of the dipole inclusion we use*

$$\mathcal{M}_q^\eta := \left(\int_{\partial B_q(0)} |\partial_n v^\eta|^2 \right)^{1/2}$$

and denote by $\mathcal{M}_q^* > 0$ the corresponding number for solutions of $\Delta v^* = 0$ to the same boundary conditions. We consider sequences $\eta \searrow 0$ and $\varepsilon \searrow 0$ that satisfy (3.12)*. Then the following holds.*

1. *if $|x_0| > R^*$, then $\mathcal{M}_q^\eta \to \mathcal{M}_q^*$ for $\eta \to 0$*
2. *if $|x_0| < R^*$, then $\mathcal{M}_q^\eta \to 0$ for $\eta \to 0$.*

1.2 Methods and Discussion

We have chosen to use the setting of [16] and [5] with an elementary static equation. The measurement procedure follows [5] and consists in evaluating the strength of the field on a large sphere; we evaluate either the norm of Neumann boundary values in $\mathcal{M}_q^\eta$ or the difference between the perturbed and the neutral Neumann boundary values in $\mathcal{N}_q^\eta$. Our findings on the invisibility of small objects $B_\varepsilon(x_0)$ in Theorems 1 and 2 match the numerical results of [5]. With reference to Figure 2 of that work, where (δ, r_0) is used instead of (η, ε), we must emphasize that in our results η and ε

must tend simultaneously to zero. In that sense, the "conjectured limit as $\delta \to 0$" [5] cannot be justified by our work.

We emphasize that in our model (with index 1 outside the cloak) it is essential that the coefficient takes exactly the value -1 in the cloak. Indeed, with a value $-\kappa$ in the cloak, $\kappa \neq 1$, the effect of anomalous resonance disappears as observed in Subsection 2.1. This is consistent with the results in [1, 2, 6] (see also the references therein), where well posedness of a diffraction problem is shown under the condition that $|\kappa - 1|$ is large enough.

While the setting of the problem and the measurement quantities are certainly well suited and quite general, our methods are very restrictive. We exploit here the special spherical geometry in two dimensions in order to expand solutions in terms of spherical harmonics. The proofs of our theorems are based on interaction coefficients that allow to expand spherical harmonics to center 0 into spherical harmonics to center x_0. The calculation of such interaction coefficients is based on the use of complex variables and holomorphic functions. It is desirable to extend the results to more general geometries and to understand better the underlying spectral problem in appropriate function spaces. To our knowledge, this is still an open problem.

We mention that in the time dependent problem with parabolic scaling, a negative diffusion index has a quite different effect than in elliptic or in hyperbolic problems. In the two standard approaches to backward diffusion, the region with a negative index remains immobile in the evolution, see [11].

Outline of this paper. Section 2 is devoted to the study of $\mathcal{L}^\eta$-harmonic functions on $\mathbb{R}^2$. These functions can be calculated explicitely in terms of their Fourier expansion. In Section 3 we investigate the interaction of the negative index ring with the small spherical inclusion with the help of interaction coefficients $G_{l,k}$ and $H_{k,l}$ which allow to expand every $\mathcal{L}^\eta$-harmonic function into spherical harmonics on $\partial B_\varepsilon(x_0)$. We calculate precise estimates for these coefficients and construct solutions to various boundary value problems with the help of iteration maps. Section 4 is devoted to the proof of Theorem 1, the two parts are shown in Propositions 4.1 and 4.2. In Section 5, Theorem 2 is shown by similar methods.

2 Properties of the negative-index ring

2.1 Homogeneous solutions

Following Milton [17], we study solutions of (piece-wise) elliptic problems in a radial geometry in $\mathbb{R}^2$, the points are $x \in \mathbb{R}^2$. With the angle variable θ we write $x = (r\cos(\theta), r\sin(\theta))$. We want to develop solutions $u : \mathbb{R}^2 \to \mathbb{C}$ in spherical harmonics. For fixed $k \in \mathbb{N}_*$ solutions u^η of $\mathcal{L}^\eta u^\eta = 0$ can be found with the ansatz

$$u^\eta(x) = U(r)e^{ik\theta}, \tag{2.1}$$

for $U : (0, \infty) \to \mathbb{C}$. The functions $r^{\pm k}e^{ik\theta}$ together with their complex conjugates are harmonic. Demanding the boundedness in $x = 0$ and normalizing to a unit monomial around 0 we make the following ansatz with complex numbers $a, b, \alpha, \beta \in \mathbb{C}$,

$$U_k(r) = \begin{cases} r^k & \text{for } r \le 1, \\ ar^k + br^{-k} & \text{for } 1 < r \le R, \\ \alpha r^k + \beta r^{-k} & \text{for } R < r. \end{cases} \tag{2.2}$$

It is elementary to determine relations between the numbers a, b, α, β, such that $\mathcal{L}^\eta u^\eta = 0$. For the following we denote the constant value of a^η on Σ by $A := A^\eta := -1 + i_0\eta \in \mathbb{C}$. The continuity condition for u^η and $a^\eta \partial_r u^\eta$ in $r = 1$ and $r = R$ reduces to

$$\begin{aligned} &1 = a + b, \qquad 1 = Aa - Ab \\ &aR^k + bR^{-k} = \alpha R^k + \beta R^{-k} \\ &aAR^k - bAR^{-k} = \alpha R^k - \beta R^{-k}. \end{aligned}$$

These relations allow to express the coefficients of the fundamental solution U_k explicitely,

$$\begin{aligned} a &= \frac{A+1}{2A} \qquad b = \frac{A-1}{2A} \\ \alpha &= \frac{1}{4} R^{-k} \left[\frac{(1+A)^2}{A} R^k - \frac{(1-A)^2}{A} R^{-k} \right] \\ \beta &= \frac{1}{4} R^k \left[\frac{1-A^2}{A} R^k - \frac{1-A^2}{A} R^{-k} \right] \end{aligned}$$

We emphasize that these coefficients depend on k and on η. The most important number in our analysis will be the following ratio which we call the *localization index.*

$$P_k^\eta := P(\eta, k; R, i_0) := \frac{\beta}{\alpha} \in \mathbb{C}, \tag{2.3}$$

where β and α are such that u^η from (2.1), (2.2) solves $\mathcal{L}^\eta u^\eta = 0$. We have derived the solution u^η with the angular dependence $e^{ik\theta}$ for $k > 0$. A solution for $k < 0$ is given by the same formulas with k replaced by $|k|$ (in particular in (2.2)), and we have $P_{-k}^\eta = P_k^\eta$ for all $k \in \mathbb{Z}$. The localization index satisfies

$$P_k^\eta = R^{2|k|} \frac{(1 - A_\eta^2)(R^{|k|} - R^{-|k|})}{(1 + A_\eta)^2 R^{|k|} - (1 - A_\eta)^2 R^{-|k|}} \approx R^{2|k|} \frac{2 i_0 \eta}{i_0^2 \eta^2 - 4R^{-2|k|}}. \tag{2.4}$$

With this expression for P_k^η we can already see why the cloaking radius $R^* = R^{3/2}$ will become important. To find out whether αr^k or βr^{-k} is dominant in (2.2), we consider the number $P_k^\eta / r^{2|k|}$,

$$\frac{P_k^\eta}{r^{2|k|}} = \left(\frac{R^*}{r} \right)^{2|k|} \frac{(2 i_0 \eta - i_0^2 \eta^2)(1 - R^{-2|k|})}{i_0^2 \eta^2 R^{|k|} - (2 - i_0 \eta)^2 R^{-|k|}}. \tag{2.5}$$

The limiting behavior for $\eta \to 0$ is

$$\max_k \frac{|P_k^\eta|}{r^{2|k|}} \to 0 \quad \text{if } r > R^*, \tag{2.6}$$

$$\max_k \frac{|P_k^\eta|}{r^{2|k|}} \to \infty \quad \text{if } r < R^*. \tag{2.7}$$

To see (2.6) it suffices to realize that the function $\eta/(\eta^2 R^k + R^{-k})$ is maximal (in $\eta > 0$) for $\eta = R^{-k}$ and to insert this value. For (2.7) we choose any sequence $k_n \to \infty$ and insert $\eta_n = R^{-k_n}$.

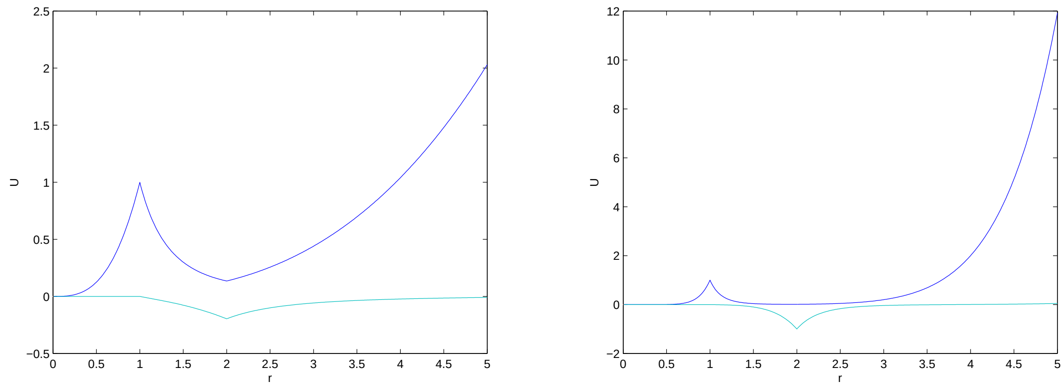

Figure 2: Real and imaginary part of U_k (dark and bright line) for $R = 2$ and cloaking radius $R^* \approx 2.8$. Left: $A = -1 + 0.05i$, $k = 3$. In this case, $P_k^\eta \approx 2 - 10^2 i$. Right: $A = -1 + i2^{-7}$, $k = 8$, such that $P_k^\eta \approx 0.25 - 8.4 \cdot 10^6 i$.

Anomalous localized resonance is related to the fact that P_k^η can become very large as described by (2.7), which provides the dominance of the β-term over the α-term for $r < R^*$. Loosely speaking, the $\mathcal{L}^\eta$-harmonic function "looks almost like" r^{-k}. It therefore has similarity to a localized homogeneous solution.

We can see from (2.4) that the denominator can become small in the limit $\eta \to 0$ only if $A_\eta \to -1$. This reflects the fact that for other limiting values than -1 no cloaking effect appears. This is consistent with [1], where it is shown that $\mathcal{L}^0$ on a bounded open subset with Dirichlet boundary conditions has a compact resolvent provided that the coefficient is not -1.

2.2 The Dirichlet-to-Neumann maps

The above localization index P_k^η can also be translated into a Dirichlet-to-Neumann operator. We consider, for arbitrary fixed $r > R$, the boundary $\Gamma := \partial B_r(0)$ and the map

$$N^{r,\eta} : H^{1/2}(\Gamma, \mathbb{C}) \to H^{-1/2}(\Gamma, \mathbb{C}), \quad u^\eta \mapsto \partial_n u^\eta|_\Gamma , \tag{2.8}$$

where u^η solves $\mathcal{L}^\eta u^\eta = 0$ in $B_r(0)$ and $n(x) = x/|x|$ is the exterior normal to $B_r(0)$. We emphasize that in this section only the ring Σ is studied and that the geometry is radially symmetric.

Since $(e^{ik\theta})_{k\in\mathbb{Z}}$ is a basis of $L^2(\Gamma, \mathbb{C})$ and $H^{\pm 1/2}(\Gamma, \mathbb{C})$, it suffices to calculate the Dirichlet-to-Neumann map for these basis functions. Since the solution keeps the θ-dependence of the boundary values, we can describe $N^{r,\eta}$ with its Fourier components

$$N^{r,\eta} = (N_k^{r,\eta})_{k\in\mathbb{Z}}, \quad N^{r,\eta}(e^{ik\theta}) = N_k^{r,\eta} \cdot e^{ik\theta}.$$

The solution to boundary values $e^{ik\theta}$ on $\partial B_r(0)$ is given by u^η as in (2.1), (2.2), and it has, in a neighborhood of $\partial B_r(0)$, the form $u^\eta = c\,(\alpha r^{|k|} + \beta r^{-|k|})e^{ik\theta}$. Therefore

$$N_k^{r,\eta} = \left.\frac{\partial_r u^\eta}{u^\eta}\right|_{\partial B_r(0)} = \frac{|k|}{r}\frac{1 - P_k^\eta r^{-2|k|}}{1 + P_k^\eta r^{-2|k|}}\,. \tag{2.9}$$

We will compare the operator $N^{r,\eta}$ of the ring geometry with the corresponding operator $N^{r,*}$ for vacuum. Since the solution without the ring is $r^{|k|}\,e^{ik\theta}$, the corresponding coefficient is

$$N_k^{r,*} = \frac{|k|}{r}. \tag{2.10}$$

For negative $k \in \mathbb{Z}$ the harmonic solutions are given by the same formulas with k replaced by $|k|$, hence $N_{-k}^{r,\eta} = N_k^{r,\eta}$ just as $N_{-k}^{r,*} = N_k^{r,*}$, and formulas (2.9) and (2.10) remain valid. From (2.9) we see once more why the number $P_k^\eta r^{-2|k|}$ is of importance. Indeed, relations (2.6) and (2.7) imply the following result.

Lemma 2.1. *For the critical radius $R^* := R^{3/2}$, the Dirichlet-to-Neumann maps have the the following properties for $\eta \to 0$. For every $r > R$ and fixed $k \in \mathbb{Z}$ holds*

$$N_k^{r,\eta} \to N_k^{r,*}. \tag{2.11}$$

In the case $r > R^*$ the convergence is uniform in k and, moreover, there holds

$$\|N^{r,\eta} - N^{r,*}\|_{\mathcal{L}(H^{1/2},H^{-1/2})} \to 0. \tag{2.12}$$

In the case $R < r < R^*$, for $\eta \to 0$, there exists $k_\eta \to \infty$ with

$$\frac{N_{k_\eta}^{r,\eta}}{N_{k_\eta}^{r,*}} \to -1. \tag{2.13}$$

Proof. The explicit formula for P_k^η in (2.4) implies immediately that $P_k^\eta \to 0$ for fixed k and $\eta \to 0$. This implies relation (2.11). Relation (2.12) is a consequence of (2.6). The uniform convergence of $N_k^{r,\eta}$ follows because (2.6) holds with an exponential rate in k. Finally, (2.13) follows from (2.7). $\square$

We read (2.12) as an invisibility result for the ring. The response of the ring to measurements from the boundary $\partial B_r(0)$ is identical to that of the empty ball $B_r(0)$, at least if the observation radius is large enough, $r > R^*$. The sign change of (2.13) will be the basis for the cloaking effect. The boundary condition $\partial_n u = -N^{r,*}u$ corresponds to solutions that are singular in 0 and bounded at infinity.

Corrector coefficients. As an abbreviation we furthermore introduce the differences

$$J_k^\eta := N_k^{r,*} - N_k^{r,\eta} = \frac{|k|}{r}\frac{2P_k^\eta r^{-2|k|}}{1 + P_k^\eta r^{-2|k|}}, \tag{2.14}$$

$$I_k^\eta := N_k^{r,*} + N_k^{r,\eta} = \frac{|k|}{r}\frac{2}{1 + P_k^\eta r^{-2|k|}}, \tag{2.15}$$

where the second equality in each line is valid for $k > 0$. The number $J_k^\eta \in \mathbb{C}$ is a measure of how different the properties of the ring are to the properties of vacuum. Instead, a small index I_k^η relates to an almost inverted sign and indicates a cloaking effect. In the calculations below there will appear the ratio between the two values, the relevant number turns out to be once more

$$\frac{J_k^\eta}{I_k^\eta} = \frac{P_k^\eta}{r^{2|k|}}\,. \tag{2.16}$$

The two coefficients will J_k^η and I_k^η will appear later on as follows. When considering a generic function $W_k(\rho,\theta) = (\rho/r)^{|k|} e^{ik\theta}$ for $k \in \mathbb{Z}$, which is harmonic on $B_r(0)$, it satisfies on $\partial B_r(0)$ the condition

$$(\partial_\rho - N^{r,\eta})W_k = (N_k^{r,*} - N_k^{r,\eta})W_k = J_k^\eta W_k.$$

Similarly, when dealing with a the function $U_k(\rho,\theta) = (\rho/r)^{-|k|} e^{ik\theta}$ for $k \in \mathbb{Z}$, which is harmonic on $\mathbb{R}^2 \setminus B_r(0)$, we can exploit that it satisfies on $\partial B_r(0)$

$$(\partial_\rho - N^{r,\eta})U_k = (-N_k^{r,*} - N_k^{r,\eta})U_k = -I_k^\eta U_k.$$

3 Interaction of the ring with an inclusion

We are interested in the effect of a small object in the vicinity of the resonant ring. We fix a point $x_0 \in \mathbb{R}^2$ with $|x_0| > R$. Furthermore, let $\varepsilon > 0$ be a radius with $\varepsilon < |x_0| - R$. We are interested in the following boundary value problem for v^η.

$$\nabla \cdot (a^\eta \nabla v^\eta) = 0 \text{ in } \mathbb{R}^2 \setminus B_\varepsilon(x_0), \tag{3.1}$$

$$v^\eta(x) = g\left(\frac{x - x_0}{\varepsilon}\right) \text{ for } x \in \partial B_\varepsilon(x_0), \tag{3.2}$$

$$v^\eta \text{ bounded}, \tag{3.3}$$

where the boundary data $g : \partial B_1(0) \to \mathbb{C}$ are given. We recall that, since a^η is not real, the solution v^η will also be complex valued.

With the aim of analyzing the interaction of the two disks $B_R(0)$ and $B_\varepsilon(x_0)$ we use the complex notation additionally for the independent variables. We identify $\mathbb{C} = \mathbb{R}^2$ by identifying a point $x = (x_1, x_2) \in \mathbb{R}^2$ with the complex number $z = x_1 + ix_2$. When we think of this identification we will write z and z_0 instead of x and x_0. We will assume that the imaginary part of the point z_0 vanishes in order to simplify the calculations below. This assumption means that $x_0 = (|x_0|, 0) \in \mathbb{R} \times \{0\}$, which is no restriction of generality since it can be acchieved by the choice of coordinates.

General boundary data g of (3.2) are expressed by its Fourier series. To do so, we denote the angle of $x - x_0$ with the real axes by $\vartheta \in [0, 2\pi)$ such that $z - z_0 = |z - z_0| e^{i\vartheta}$. We expand g with coefficients $g_l \in \mathbb{C}$, $l \in \mathbb{Z}$, as

$$g(e^{i\vartheta}) = \sum_{l\in\mathbb{Z}} g_l e^{il\vartheta} \text{ with squared norm } \|g\|^2 = \sum_{l\in\mathbb{Z}} |g_l|^2. \tag{3.4}$$

We are particularly interested in the boundary values $g = g_D$ corresponding to the coefficients $g_1 = 1$ and $g_l = 0$ for $l \neq 1$, i.e.

$$g_D(e^{i\vartheta}) = e^{i\vartheta}\,,\ \text{such that } v^\eta(z) = \frac{z - z_0}{\varepsilon} \text{ for } z \in \partial B_\varepsilon(z_0).$$

We will refer to these special data as *dipole boundary values.*

We use the abbreviations $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}_* = \{1, 2, ...\}$, and $\mathbb{N}_{**} = \{2, 3, ...\}$.

3.1 Interaction coefficients

Our aim is to derive a relation between the harmonic functions $U_k : \mathbb{C} \setminus B_R(0) \to \mathbb{C}$ and $V_l : \mathbb{C} \setminus B_\varepsilon(z_0) \to \mathbb{C}$, $k, l > 0$,

$$U_k(z) = \left(\frac{z}{R}\right)^{-k}, \qquad V_l(z) = \left(\frac{z - z_0}{\varepsilon}\right)^{-l}.$$

We develop U_k in terms of $\bar{V}_l = e^{il\vartheta}$ on $\partial B_\varepsilon(z_0)$, and we develop $\bar{V}_l$ in terms of $U_k = e^{-ik\theta}$ on $\partial B_R(0)$. We can define the corresponding operators that map boundary conditions on one sphere to the values of the harmonic extension on another sphere, $G(\bar{V}_l|_{\partial B_\varepsilon(z_0)}) := \bar{V}_l|_{\partial B_R(0)}$ and $H(U_k|_{\partial B_R(0)}) := U_k|_{\partial B_\varepsilon(z_0)}$. In Fourier coordinates with the two basis $e_l = e^{il\vartheta}$ and $e_k = e^{-ik\theta}$ the operators read

$$G : L^2(\partial B_\varepsilon(z_0)) \to L^2(\partial B_R(0)), \quad G(e^{il\vartheta}) = \sum_{k\in\mathbb{Z}} G_{l,k} e^{-ik\theta},$$
$$H : L^2(\partial B_R(0)) \to L^2(\partial B_\varepsilon(z_0)), \quad H(e^{-ik\theta}) = \sum_{l\in\mathbb{Z}} H_{k,l} e^{il\vartheta}.$$

We can identify the operators also with the corresponding operators on spaces of sequences, $l^2(\mathbb{Z}) \ni e_l \mapsto G(e_l) = (G_{l,k})_k \in l^2(\mathbb{Z})$ and $l^2(\mathbb{Z}) \ni e_k \mapsto H(e_k) = (H_{k,l})_l \in l^2(\mathbb{Z})$.

Our aim is to evaluate the coefficients $G_{l,k}$ and $H_{k,l}$ explicitely for $k, l \geq 0$. We start our calculation with a Taylor-expansion around $z_0 \in \mathbb{C}$ (or the Neumann series)

$$\frac{1}{z} = \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{z_0}\right)^{j+1} (z - z_0)^j.$$

Differentiating $k - 1$-times in $\mathbb{C}$ gives

$$(-1)^{k-1}(k-1)! \left(\frac{1}{z}\right)^k = \sum_{j=k-1}^{\infty} (-1)^j \left(\frac{1}{z_0}\right)^{j+1} \frac{j!}{(j-k+1)!} (z - z_0)^{j-k+1}.$$

Setting $j = k - 1 + l$ we find

$$\left(\frac{1}{z}\right)^k = \sum_{l=0}^{\infty} (-1)^l \left(\frac{1}{z_0}\right)^{k+l} \frac{(k+l-1)!}{l!(k-1)!} (z - z_0)^l,$$

which we may also write as

$$\left(\frac{R}{z}\right)^k = \sum_{l=0}^{\infty}(-1)^l \left(\frac{R}{z_0}\right)^k \left(\frac{\varepsilon}{z_0}\right)^l \binom{k+l-1}{l} \left(\frac{z-z_0}{\varepsilon}\right)^l . \tag{3.5}$$

We note that on $\partial B_\varepsilon(z_0)$ holds $((z-z_0)/\varepsilon)^l = e^{il\vartheta} = \bar{V}_l$. Therefore (3.5) yields

$$U_k = \sum_{l=0}^{\infty} H_{k,l}\bar{V}_l \text{ on } \partial B_\varepsilon(z_0), \qquad H_{k,l} = (-1)^l \binom{k+l-1}{l} \left(\frac{R}{z_0}\right)^k \left(\frac{\varepsilon}{z_0}\right)^l , \tag{3.6}$$

and, similarly,

$$\bar{V}_l = \sum_{k=0}^{\infty} G_{l,k} U_k \text{ on } \partial B_R(0), \qquad G_{l,k} = (-1)^l \binom{k+l-1}{k} \left(\frac{R}{z_0}\right)^k \left(\frac{\varepsilon}{z_0}\right)^l , \tag{3.7}$$

for $k, l \geq 0$. Here, the coefficients $G_{l,k}$ can be easily obtained from (3.5). We use the formula with $z = z_0 - z'$, $\varepsilon = R'$ and $R = \varepsilon'$. In the prime coordinates, the function V_k appears on the left and the functions U_l appear on the right. We exchange k and l and take the complex conjugate, exploiting the assumption $z_0 \in \mathbb{R}$. We find that $G_{l,k} = (-1)^{k+l} H_{k,l}$. Furthermore, again for $k, l > 0$, the above representation provides $G_{l,-k} = G_{-l,k} = H_{-k,l} = H_{k,-l} = 0$, the complex conjugate of the representation yields $H_{-k,-l} = H_{k,l}$ and $G_{l,k} = G_{-l,-k}$.

Our analysis of boundary value problems is based on careful estimates for the interaction coefficients $G_{l,k}$ and $H_{k,l}$. An immediate estimate for diagonal sums is given by the binomial formula,

$$\sum_{l=1}^{n} |H_{n-l+1,l}| = \frac{R}{z_0} \sum_{l=1}^{n} \binom{n}{l} \left(\frac{R}{z_0}\right)^{n-l} \left(\frac{\varepsilon}{z_0}\right)^l \leq \frac{R}{z_0} \left(\frac{R+\varepsilon}{z_0}\right)^n .$$

In particular, for $R + \varepsilon < z_0$, all coefficients are bounded in absolute value by 1, and summable over both indices.

3.2 Approximate resonant dipole radiator

We will always assume $z_0 \in \mathbb{R}$ and abbreviate $x_0 = (s, 0) \in \mathbb{R}^2$ or $z_0 = s \in \mathbb{C}$. Furthermore, we assume $R + \varepsilon < s$, which means that $\bar{B}_\varepsilon(z_0) \cap B_R(0) = \emptyset$. The aim of this section is the construction of a solution to (3.1)–(3.3). We will not succeed immediately to construct a solution to dipole boundary data $g = g_D$ along $\partial B_\varepsilon(x_0)$, but we will satisfy this condition in an approximate way. To be precise, we use the basis $\bar{V}_l \lfloor \partial B_\varepsilon(x_0) = e^{il\vartheta}$ of $L^2(\partial B_\varepsilon(x_0))$ and demand that the projection of the boundary values onto $\mathbb{C}e^{i\vartheta}$ should be $e^{i\vartheta} = \bar{V}_1$ on $\partial B_\varepsilon(z_0)$. We construct our solution with the building blocks U_k, $k \in \mathbb{N}_*$, and $\bar{V}_1$.

Definition 3.1. *We introduce the number $\lambda_d \in \mathbb{C}$ given by*

$$\lambda_d := \left[1 + \sum_{k=0}^{\infty} \frac{P_k^\eta}{s^{2k}} k \left(\frac{\varepsilon}{s}\right)^2\right]^{-1} , \tag{3.8}$$

and set coefficients $a_k \in \mathbb{C}$ to be

$$a_k := -\lambda_d \frac{P_k^\eta}{R^k s^k} \frac{\varepsilon}{s} \qquad \forall k \in \mathbb{N}. \tag{3.9}$$

With these numbers, we define the special solution $w^\eta : \mathbb{R}^2 \setminus (\bar{B}_R(0) \cup \bar{B}_\varepsilon(x_0)) \to \mathbb{C}$,

$$w^\eta(x) := \lambda_d \bar{V}_1(x) + \sum_{k=0}^{\infty} a_k\, U_k(x). \tag{3.10}$$

With the same letter w^η we refer to the $\mathcal{L}^\eta$-harmonic extension $w^\eta : \mathbb{R}^2 \setminus \bar{B}_\varepsilon(x_0) \to \mathbb{C}$. This extension exists by the subsequent lemma.

Lemma 3.2 (Approximate dipole field)**.** *The function w^η of Definition 3.1 can be extended to a solution $w^\eta : \mathbb{R}^2 \setminus \bar{B}_\varepsilon(x_0) \to \mathbb{C}$ of problem* (3.1)–(3.3).

The function w^η satisfies on $\partial B_\varepsilon(x_0)$ the boundary condition $w^\eta = \sum_{l=0}^\infty b_l \bar{V}_l$ with $b_1 = 1$. The coefficient b_l for $l \neq 1$ is given by

$$b_l = -\lambda_d \sum_{k=0}^{\infty} \frac{P_k^\eta}{R^k s^k} \frac{\varepsilon}{s} H_{k,l}\,. \tag{3.11}$$

Remark. Without the ring $B_R(0)\setminus B_1(0)$ of negative index, the solution with dipole boundary data $\bar{V}_1$ is $w^\eta = \bar{V}_1$. Therefore, if the ring introduces only a small perturbation of solutions, one might expect $\lambda_d \approx 1$, which is indeed the case for $s = |x_0| > R^*$. On the other hand, for $s = |x_0| < R^*$, expression (3.8) my define a small coefficient $\lambda_d \approx 0$. In this case, despite the dipole boundary values $\bar{V}_1$, the solution does not behave like the dipole solution $\bar{V}_1$.

Proof. Step 1. The coefficients a_k. The function w^η can be extended to an $\mathcal{L}^\eta$-harmonic function if and only if it satisfies the boundary condition $(\partial_r - N^{R,\eta})w^\eta = 0$ on $\partial B_R(0)$. This is the case if and only if

$$\begin{aligned} 0 &= (N^{R,*} - N^{R,\eta})\lambda_d \bar{V}_1 + (-N^{R,*} - N^{R,\eta}) \sum_{k=0}^{\infty} a_k\, U_k \\ &= (N^{R,*} - N^{R,\eta}) \sum_{k=0}^{\infty} \lambda_d G_{1,k} U_k - (N^{R,*} + N^{R,\eta}) \sum_{k=0}^{\infty} a_k\, U_k \\ &= \sum_{k=0}^{\infty} \lambda_d J_k^\eta G_{1,k} U_k - \sum_{k=0}^{\infty} I_k^\eta a_k\, U_k. \end{aligned}$$

Inserting $J_k^\eta / I_k^\eta = P_k^\eta R^{-2k}$ of (2.16) and $G_{1,k} = -(\varepsilon/s)(R/s)^k$, by comparing coefficients, we find

$$-\lambda_d P_k^\eta R^{-2k} (\varepsilon/s)(R/s)^k = a_k$$

and thus (3.9).

Step 2. The coefficient λ_d. We now demand that the dipole moment on $\partial B_\varepsilon(x_0)$ is one. Using $H_{k,1} = -k(\varepsilon/s)(R/s)^k$ we find

$$1 = \lambda_d + \sum_{k=0}^{\infty} a_k\, H_{k,1} = \lambda_d \left[1 + \sum_{k=0}^{\infty} \frac{P_k^\eta}{R^k s^k} k \left(\frac{\varepsilon}{s}\right)^2 \left(\frac{R}{s}\right)^k \right],$$

and hence (3.8).

Step 3. Relation (3.11). The boundary values w^η have, on the boundary $\partial B_\varepsilon(x_0)$, the expansion

$$\begin{aligned} w^\eta &= \lambda_d \bar V_1 + \sum_{k=0}^{\infty} a_k U_k = \lambda_d \bar V_1 - \lambda_d \sum_{k=1}^{\infty} \frac{P_k^\eta}{R^k s^k} \frac{\varepsilon}{s} U_k \\ &= \lambda_d \bar V_1 - \lambda_d \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{P_k^\eta}{R^k s^k} \frac{\varepsilon}{s} H_{k,l} \bar V_l\,. \end{aligned}$$

Comparing the coefficients provides relation (3.11). □

3.3 Solutions for general boundary values

We have not yet solved problem (3.1)–(3.3) for general boundary data g. In the main result of this subsection, Lemma 3.4, we will solve the problem for boundary data which have a vanishing dipole moment. Subsequently, we will use this result to correct the boundary data of the special solution w^η of Definition 3.1 to dipole data g_D.

We will construct bounded solution sequences under an assumption on the size of ε. We fix a sequence $\eta = \eta_n \to 0$ with the corresponding critical coefficients $k = k_n$ and a sequence $\varepsilon = \varepsilon_n \to 0$. We have to assume that ε_n and η_n vanish in an appropriate relation to each other. From now on, we will always assume the following.

Assumption 3.3. *We assume that the geometry is fixed by numbers $1 < R < s < R^*$ and that sequences $\eta = \eta_n \searrow 0$ and $\varepsilon = \varepsilon_n \searrow 0$ are given. We assume that there holds for some number $\delta > 0$ and the sequence $k = k_n = [-\log(\eta_n)/\log(R)]$*

$$k\varepsilon^2 \left(\frac{R^3}{s^2}\right)^k \to \infty, \qquad k^2\varepsilon^3 \left(\frac{R^3}{s^2} + \delta\right)^k \to 0. \tag{3.12}$$

The first convergence of (3.12) requires that ε is not too small and guarantees, in particular, the convergence $\lambda_d \to 0$. Instead, the second property requires that ε is small enough. It implies that the boundary values of the special solution w^η are close to $\bar V_1$; this follows from expression (3.11) where the factors $(R/s)^k$ and ε^2 are contained in the coefficient $H_{k,l}$ for $l \geq 2$.

In order to show that our results can be applied in concrete situations, we present an explicit case. We consider $b > 1$, $\delta > 0$ and $\mathbb{N} \ni n \to \infty$, and the sequences

$$\eta = \eta_n = R^{-n}, \qquad \varepsilon = \varepsilon_n = b^{-n}, \qquad k_n = n, \qquad \text{with}$$
$$b \text{ such that } b^2 < (R^3/s^2) < (R^3/s^2 + \delta) < b^3.$$

Regarding the above smallness assumptions on ε we calculate in this setting

$$k_n\varepsilon_n^2\left(\frac{R^3}{s^2}\right)^{k_n} \sim nb^{-2n}\left(\frac{R^3}{s^2}\right)^n \to \infty,$$
$$k_n^2\varepsilon_n^3\left(\frac{R^3}{s^2}+\delta\right)^{k_n} \sim n^2b^{-3n}\left(\frac{R^3}{s^2}+\delta\right)^n \to 0.$$

We can now state and prove the existence result with uniform bounds for the small-inclusion boundary value problem.

Lemma 3.4. *Let Assumption 3.3 hold. Let boundary data $g \in L^2(S^1)$ be given by the expansion $g = \sum_{l=2}^{\infty} g_l e^{il\vartheta}$ with complex numbers g_l, in particular with a vanishing dipole moment. Then there exists a solution v^η of* (3.1)–(3.3)*. More precisely, the solution can be expressed as*

$$v^\eta = V_f^\eta + U_f^\eta + \lambda^\eta w^\eta + \mu^\eta. \tag{3.13}$$

Here, w^η is the special solution of Definition 3.1 and λ^η, μ^η are complex numbers with $|\lambda^\eta| \le C_0\|g\|$. The functions V_f^η and U_f^η are given by coefficients $(f_l^\eta)_l \in l^2(\mathbb{N}_{**}, \mathbb{C})$ with $\|(f_l^\eta)_l\| \le C_0\|g\|$ as

$$V_f^\eta = \sum_{l=2}^{\infty} f_l^\eta \bar{V}_l, \quad U_f^\eta(x) = \sum_{k=0}^{\infty}\left[\sum_{l=2}^{\infty} f_l^\eta \frac{P_k^\eta}{R^{2k}} G_{l,k}\right] U_k(x)\,. \tag{3.14}$$

The number C_0 is independent of η and g.

The coefficients b_l of (3.11) *provide an expansion of the special function w^η of Lemma 3.2. They satisfy $b_1 = 1$ and*

$$\lim_{\eta\to 0}\sum_{l=2}^{\infty}|b_l^\eta|^2 = 0. \tag{3.15}$$

Proof. In the proof, we suppress the superscript η in f_l^η, V_f^η, and U_f^η. The result is shown with an iteration map. Banach's fixed point theorem will provide the existence result together with the structure properties and the estimates. The idea for the iteration is as follows. We start from a guess for the coefficients f_l. A first approximation of the solution is the function V_f. But this function does not satisfy the boundary condition on $\partial B_R(0)$ imposed by the negative index ring. We introduce the function U_f, harmonic outside $\bar{B}_R(0)$ such that $V_f + U_f$ satisfy the boundary condition on $\partial B_R(0)$. In order to treat the error on $\partial B_\varepsilon(x_0)$, we subtract an appropriate constant and an appropriate multiple of w^η. The remainder, $g - U_f - \lambda w^\eta - \mu$ is taken as a new guess for f.

Step 1. Construction of the iteration map. We investigate an iteration on the space

$$X = \left\{ f \in L^2(\partial B_\varepsilon(x_0)) : f = \sum_{l=2}^{\infty} f_l e^{il\vartheta}, f_l \in \mathbb{C} \right\},$$
$$T : X \to X, \quad f \mapsto f^{\text{new}}. \tag{3.16}$$

In order to define the iteration we start from $f \in X$ with Fourier multipliers $f_l \in \mathbb{C}$,

$$f(x) = \sum_{l=2}^{\infty} f_l e^{il\vartheta} = \sum_{l=2}^{\infty} f_l \bar{V}_l(x),$$

and consider the harmonic extension on $\mathbb{R}^2 \setminus B_\varepsilon(x_0)$,

$$V_f(x) := \sum_{l=2}^{\infty} f_l \bar{V}_l(x).$$

The field V_f does not respect the boundary condition $\partial_r - N^{R,\eta} = 0$ along $\partial B_R(0)$. We correct this by a function

$$U_f(x) = \sum_{k=0}^{\infty} c_k U_k(x),$$

which is possible if $c_k \in \mathbb{C}$ is chosen such that, on $\partial B_R(0)$,

$$\begin{aligned}
0 \overset{!}{=} (\partial_r - N^{R,\eta})(V_f + U_f) &= (N^{R,*} - N^{R,\eta}) \sum_{l=2}^{\infty} f_l \bar{V}_l + (-N^{R,*} - N^{R,\eta}) \sum_{k=0}^{\infty} c_k U_k \\
&= \sum_{k=0}^{\infty} \sum_{l=2}^{\infty} f_l (N^{R,*} - N^{R,\eta}) G_{l,k} U_k + \sum_{k=0}^{\infty} c_k (-N^{R,*} - N^{R,\eta}) U_k \\
&= \sum_{k=0}^{\infty} \sum_{l=2}^{\infty} f_l J_k^\eta G_{l,k} U_k - \sum_{k=0}^{\infty} c_k I_k^\eta U_k .
\end{aligned}$$

This can be achieved with the choice

$$c_k = \sum_{l=2}^{\infty} f_l J_k^\eta G_{l,k} / I_k^\eta = \sum_{l=2}^{\infty} f_l \frac{P_k^\eta}{R^{2k}} G_{l,k}$$

for all $k \geq 0$. We now set $\tilde{f} := g - U_f$, given by

$$\tilde{f} = \sum_{j=0}^{\infty} \left[g_j - \sum_{l\geq 2, k\geq 0} f_l G_{l,k} \frac{P_k^\eta}{R^{2k}} H_{k,j} \right] \bar{V}_j \,, \tag{3.17}$$

where we set $g_0 = g_1 = 0$. Since constant function and dipole contribution ($\bar{V}_0$ and $\bar{V}_1$) appear in $\tilde{f}$, we do not necessarily have $\tilde{f} \in X$. We post-process this by projecting onto the subspace X. With appropriate $\lambda^\eta, \mu^\eta \in \mathbb{C}$ we finally set

$$f^{\text{new}} := T(f) := \tilde{f} - \lambda^\eta w^\eta - \mu^\eta, \tag{3.18}$$

where w^η the special approximate dipole solution of Definition 3.1. The factor λ^η is chosen to generate a vanishing dipole moment of f^{new}, the factor μ in order to have a vanishing average. We must choose

$$\lambda^\eta = - \sum_{l\geq 2, k\geq 0} f_l G_{l,k} \frac{P_k^\eta}{R^{2k}} H_{k,1}. \tag{3.19}$$

Let us assume that $f \in X$ is a fixed point of T, i.e.

$$f = T(f) = \tilde{f} - \lambda^\eta w^\eta - \mu^\eta = g - U_f - \lambda^\eta w^\eta - \mu^\eta. \tag{3.20}$$

This implies that the function $v^\eta = V_f + U_f + \lambda^\eta w^\eta + \mu^\eta$ of (3.13) satisfies the boundary condition $v^\eta = g$ on $\partial B_\varepsilon(x_0)$ and the boundary condition on $\partial B_R(0)$, since both $V_f + U_f$ and w^η do. The explicit formula of (3.14) follows from the expression for c_k. The estimates for $(f_l)_l, \lambda^\eta, \mu^\eta$ follow from the fact that the fixed point is found in a ball with radius comparable to the norm of g.

Step 2. Reducing the proof to three estimates. Our aim is to treat $T : X \to X$ with a Banach fixed point argument. Interpreting G, H, I and J as operators, in light of (3.17) we must analyze the operator

$$H \circ (J \circ I^{-1}) \circ G : l^2(\mathbb{N}_{**}) \to l^2(\mathbb{N}), \tag{3.21}$$

and show that its norm is less than 1. Regarding this operator we note that H and G are bounded, but $J \circ I^{-1}$ behaves like $P_k^\eta / R^{2k} \sim R^k$ in the critical k. On the other hand, we have an estimate of the type $|H_{k,.}|, |G_{.,k}| \le ((R+\varepsilon)/s)^k$ together with some ε-factors. Assumption (3.12) will be sufficient to find a small bound for the operator.

We claim that the following relations hold in the limit $n \to \infty$.

$$\left(\sum_k \left(\frac{R}{s}\right)^k \frac{|P_k^\eta|}{R^{2k}} \frac{\varepsilon}{s} |H_{k,l}| \right)_{l \in \mathbb{N}_{**}} \quad \text{is small in } l^2(\mathbb{N}_{**}), \tag{3.22}$$

$$\left(\sum_k |G_{l,k}| \frac{|P_k^\eta|}{R^{2k}} |H_{k,1}| \right)_{l \in \mathbb{N}_{**}} \quad \text{is small in } l^2(\mathbb{N}_{**}), \tag{3.23}$$

$$\left(\sum_k |G_{l,k}| \frac{|P_k^\eta|}{R^{2k}} |H_{k,j}| \right)_{j,l \in \mathbb{N}_{**}} \quad \text{is small in } l^2(\mathbb{N}_{**}) \to l^2(\mathbb{N}_{**}). \tag{3.24}$$

The contractivity of $T : X \to X$, defined in (3.17) and (3.18), is shown once we have verified the following: smallness of the operator $H \circ J \circ I^{-1} \circ G$, boundedness of the special function w^η as in (3.15), and the smallness $\|\lambda^\eta\| \le c\|f\|_{l^2}$ for a small factor c. The smallness of the operator is a consequences of (3.24). The required boundedness of w^η is a consequence of (3.22) by (3.11) and the boundedness of λ_d. The smallness of λ^η is a consequence of (3.23) by relation (3.19).

Relation (3.15) was part of the above argument and a consequence of (3.22). For the proof of the lemma, it remains to verify (3.22)–(3.24).

An observation on uniform boundedness. We claim that for some numbers $C \in \mathbb{R}$ and $\delta_0 > 0$ there holds

$$\left(\frac{j}{K}\right)^K \delta^j \le C \quad \text{and} \quad \frac{(2j)^K}{K!} \delta^{j+K} \le C, \tag{3.25}$$

independent of $0 < \delta \le \delta_0$ and $j, K \in \mathbb{N}_*$.

Verification of (3.25). For the first expression, the boundedness is obtained by regarding $j, K \in [1, \infty)$ and to find the maximal value of the corresponding function.

We differentiate with respect to j and set the derivative to 0,

$$0 = \frac{\partial}{\partial j}\left(\frac{j^K}{K^K}\delta^j\right) = \frac{j^{K-1}\delta^j}{K^K}\left(K + j\log(\delta)\right).$$

For $\delta < 1/e$ we find the maximum for $j \le K$ and the value of the function is $(j/K)^K\delta^j \le 1$.

The boundedness of the second expression follows from Stirlings formula which implies $K! \ge cK^Ke^{-K}$ for some constant $c > 0$. Imposing smallness of δ and using the boundedness of the first expression gives

$$\frac{(2j)^K}{K!}\delta^{j+K} \le \frac{1}{c}2^K\frac{j^K\delta^j}{K^K}e^K\delta^K \le \frac{C}{c}(2e)^K\delta^K,$$

which is bounded for $\delta < 1/(2e)$. Claim (3.25) is shown.

Step 3. Conditions (3.22) *and* (3.23). We verify (3.23) with a direct calculation.

$$\begin{aligned}
A_j &:= \sum_{k\in\mathbb{N}} |G_{j,k}||P_k^\eta|R^{-2k}|H_{k,1}| \\
&= \sum_{k\in\mathbb{N}} \binom{j+k-1}{k}\left(\frac{R}{s}\right)^k\left(\frac{\varepsilon}{s}\right)^j\frac{|P_k^\eta|}{R^{2k}}\,k\left(\frac{R}{s}\right)^k\frac{\varepsilon}{s} \\
&= \sum_{k\in\mathbb{N}} \frac{|P_k^\eta|}{s^{2k}}\left(\frac{\varepsilon}{s}\right)^{j+1} j\binom{j+k-1}{j}.
\end{aligned}$$

We denote by K the critical index k, setting $K = \lfloor -\log(\eta)/\log(R)\rfloor$, the largest integer below the expression in brackets. Accordingly, we decompose the sum over k into two parts and write $A_j = A_j^{\le K} + A_j^{>K}$. In the calculation of $A_j^{\le K}$ we will use $|P_k^\eta| \le C\eta R^{4k}$, while for $A_j^{>K}$ we use $|P_k^\eta| \le CR^{2k}/\eta$.

Distinguishing additionally two cases for j we calculate first for $2 \le j \le 2K$, using $\eta \le R^{-K}$,

$$\begin{aligned}
A_j^{\le K} &\le C\sum_{k\le K}\eta\left(\frac{R^4}{s^2}\right)^k\left(\frac{\varepsilon}{s}\right)^{j+1} j\binom{k+j-1}{j} \\
&\le \sum_{k\le K}\eta\left(\frac{R^4}{s^2}\right)^k\left(\frac{\varepsilon}{s}\right)^{j+1}(3K)^j \le C(R,s)\,\eta\left(\frac{R^4}{s^2}\right)^K\varepsilon^3K^2\left(\frac{3K\varepsilon}{s}\right)^{j-2} \\
&\le C(R,s)\left[\varepsilon^3K^2\left(\frac{R^3}{s^2}\right)^K\right](1/2)^{j-2}.
\end{aligned}$$

Assumption (3.12) yields, for large n, the smallness of the squared bracket and $3K\varepsilon/s \le 1/2$, which makes, for $j \ge 2$, the above a sequence with small $l^2(\mathbb{N}_{**})$-norm.

For $j > 2K$ we modify the estimate of the binomial coefficient. We exploit (3.25) of step 2 in the inequality marked by (2) below. We assume now $j > 2K$ and $K \ge 4$

such that we can use $j+1 > \frac{1}{2}(j+2K+2) \geq 3+\frac{1}{2}(j+K)$ and hence $j+K < 2(j-2)$ in inequality marked by (1),

$$\begin{aligned}
A_j^{\leq K} &\leq C \sum_{k\leq K} \eta \left(\frac{R^4}{s^2}\right)^k \left(\frac{\varepsilon}{s}\right)^{j+1} j \binom{k+j-1}{k-1} \\
&\leq C \sum_{k\leq K} R^{-K} \left(\frac{R^4}{s^2}\right)^k \left(\frac{\varepsilon}{s}\right)^{j+1} \frac{K\,(K+j)^K}{K!} \\
&\leq C(R,s)\, R^{-K} \left(\frac{R^4}{s^2}\right)^K \left(\frac{\varepsilon}{s}\right)^3 K \frac{(2j)^K}{K!} \left(\sqrt{\varepsilon/s}\right)^{2(j-2)} \\
&\overset{(1)}{\leq} C(R,s) \left(\frac{R^3}{s^2}\right)^K \varepsilon^3 K (1/2)^j \left((\varepsilon/s)^{1/4}\right)^{j+K} \frac{(2j)^K}{K!} \\
&\overset{(2)}{\leq} C(R,s) \left[\varepsilon^3 K^2 \left(\frac{R^3}{s^2}\right)^K\right] (1/2)^{j-2}
\end{aligned}$$

for all n large enough such that $(\varepsilon/s)^{1/4} \leq \min\{1/2, \delta_0\}$ with $\delta_0 > 0$ as for (3.25). We obtain the same estimate for $A_j^{\leq K}$ as for the smaller indices j.

For $A_j^{>K}$ we exploit the estimate $1/\eta \leq CR^K$ and calculate for arbitrary $j \geq 2$, using the binomial expansion of powers and C depending on R and s,

$$\begin{aligned}
A_j^{>K} &\leq C \sum_{k>K} \frac{1}{\eta} \left(\frac{R}{s}\right)^{2k} \left(\frac{\varepsilon}{s}\right)^{j+1} j \binom{j+k-1}{j} \\
&= C \sum_{k>K} \frac{1}{\eta} \left(\frac{\varepsilon}{s}\right)^3 \left(\frac{R}{s}\right)^{2k} \left(\frac{\varepsilon}{s}\right)^{j-2} \frac{k(k+1)}{j-1} \binom{j+k-1}{j-2} \\
&\leq C\varepsilon^3 \sum_{k>K} \frac{k(k+1)}{\eta} \left[(R/s)^2 + (\varepsilon/s)\right]^{k+j-1} \\
&\leq C\varepsilon^3 K^2 \frac{1}{\eta} \left[(R/s)^2 + (\varepsilon/s)\right]^{K+j-1} \\
&\leq C\varepsilon^3 K^2 R^K \left[(R/s)^2 + (\varepsilon/s)\right]^{K+j} \\
&\leq C\varepsilon^3 K^2 \left[(R^3/s^2) + (R\varepsilon/s)\right]^K \left[(R^2/s^2) + (\varepsilon/s)\right]^j
\end{aligned}$$

which is, by assumption (3.12), of the desired type: a small multiple of a geometric series.

Concerning (3.22) we only have to compare with the previous calculation. With A_l as defined in the beginning of Step 3 we find

$$\sum_k \left(\frac{R}{s}\right)^k \frac{|P_k^\eta|}{R^{2k}} \frac{\varepsilon}{s} |H_{k,l}| = \sum_k \frac{|P_k^\eta|}{s^{2k}} \left(\frac{\varepsilon}{s}\right)^{l+1} \binom{k+l-1}{l} \leq A_l.$$

In particular, summability and smallness follows from that of A_j.

Step 4. Condition (3.24). It remains to check whether

$$(A_{jm})_{j,m\geq 2} = \left(\sum_k |G_{j,k}||P_k^\eta| R^{-2k} |H_{k,m}|\right)_{j,m\geq 2} \tag{3.26}$$

defines a small map $l^2(\mathbb{N}_{**}) \to l^2(\mathbb{N}_{**})$. The coefficients can be estimated by

$$\begin{aligned} A_{jm} &= \sum_{k\in\mathbb{N}} |G_{j,k}||P_k^\eta| R^{-2k} |H_{k,m}| \\ &= \sum_{k\in\mathbb{N}} \binom{j+k-1}{k} \left(\frac{R}{s}\right)^k \left(\frac{\varepsilon}{s}\right)^j \frac{|P_k^\eta|}{R^{2k}} \binom{m+k-1}{m} \left(\frac{R}{s}\right)^k \left(\frac{\varepsilon}{s}\right)^m \\ &= \sum_{k\in\mathbb{N}} \frac{|P_k^\eta|}{s^{2k}} \left(\frac{\varepsilon}{s}\right)^{j+m} \binom{j+k-1}{j} \binom{m+k-1}{k}. \end{aligned}$$

We will verify that, for some $\Theta \in (0,1)$ and arbitrarily small $q > 0$ there holds, for all η sufficiently small,

$$A_{jm} \le q\Theta^{j+m}. \tag{3.27}$$

This provides the smallness of the map A_{jm} in l^p-spaces.

We follow the lines of Step 3 and start with the $k \le K = \lfloor -\log(\eta)/\log(R) \rfloor$. For $j, m \le 2K$ we estimate

$$\begin{aligned} A_{jm}^{\le K} &= \sum_{k\le K} \frac{|P_k^\eta|}{s^{2k}} \left(\frac{\varepsilon}{s}\right)^{j+m} \binom{j+k-1}{j} \binom{m+k-1}{k} \\ &\le C\eta \frac{R^{4K}}{s^{2K}} \left(\frac{\varepsilon}{s}\right)^{j+m} (3K)^j (3K)^{m-1} \\ &\le C \left(\frac{R^3}{s^2}\right)^K K^2 \varepsilon^3 \left(\frac{3K\varepsilon}{s}\right)^{j+m-3}. \end{aligned}$$

Using (3.12) and $\varepsilon K \to 0$, we find the estimate (3.27). For $j, m > 2K$ and $K > 3$ we calculate

$$\begin{aligned} A_{jm}^{\le K} &\le \eta \frac{R^{4K}}{s^{2K}} \left(\frac{\varepsilon}{s}\right)^{j+m} K^2 \frac{(2j)^K}{K!} \frac{(2m)^K}{K!} \\ &\le C \frac{R^{3K}}{s^{2K}} K^2 \varepsilon^3 ((\varepsilon/s)^{1/4})^{j+m-3} \quad \frac{(2j)^K}{K!} ((\varepsilon/s)^{1/4})^{j+K} \quad \frac{(2m)^K}{K!} ((\varepsilon/s)^{1/4})^{m+K}. \end{aligned}$$

Boundedness (3.25) of Step 2 yields an estimate of the form (3.27) for large n. The remaining cases $j \le 2K < m$ and $m \le 2K < j$ are treated with analogous calculations, using for the smaller index the first estimate and for the larger index the second.

We finally consider $k > K$ and calculate

$$\begin{aligned} A_{jm}^{>K} &= \sum_{k>K} \frac{|P_k^\eta|}{s^{2k}} \left(\frac{\varepsilon}{s}\right)^{j+m} \binom{j+k-1}{j} \binom{m+k-1}{k} \\ &\le C \frac{1}{\eta} \sum_{k>K} \frac{R^{2k}}{s^{2k}} \varepsilon^3 \left(\frac{\varepsilon}{s}\right)^{j+m-3} \frac{k}{j} \binom{j+k-1}{j-1} \frac{k+1}{m-1} \binom{m+k-1}{m-2} \\ &\le C R^K \varepsilon^3 \sum_{k>K} k^2 \left[\frac{R}{s} + \frac{\varepsilon}{s}\right]^{j+k-1} \left[\frac{R}{s} + \frac{\varepsilon}{s}\right]^{m+k-1} \\ &\le C \varepsilon^3 K^2 R^K \left[\frac{R}{s} + \frac{\varepsilon}{s}\right]^{2K} (R/s + \varepsilon/s)^{j+m}. \end{aligned}$$

This is of the desired form (3.27) and provides the smallness of the operator A. ∎

Remark 3.5. *We are interested in solutions to dipole boundary data g_D on the inclusion $B_\varepsilon(x_0)$. An approximate solution to these boundary data is the special function w^η. In order to solve exactly, we correct the error with an application of Lemma 3.4 to $g = w^\eta - e^{i\vartheta} - b_0$. By* (3.15)*, these boundary data vanish in the limit $\eta \to 0$. Therefore, the corresponding solutions, given by coefficients f_l^η and λ^η, also vanish in the limit $\eta \to 0$. We conclude that the solution to exact dipole data reads*

$$v^\eta = w^\eta + V_f^\eta + U_f^\eta + \lambda^\eta w^\eta + \mu^\eta \tag{3.28}$$

with $f_l^\eta \to 0$ in $l^2(\mathbb{N}_{**})$ and $\lambda^\eta \to 0$ in $\mathbb{C}$ for $\eta \to 0$.

4 Quantitative analysis of the dipole radiator

With the last section we have shown that dipole boundary data on $\partial B_\varepsilon(x_0)$ admit $\mathcal{L}^\eta$-harmonic extensions. We keep the above situation and consider $x_0 \in \mathbb{R}^2$ with $s = |x_0| > R$, and a radius $\varepsilon > 0$ with $R + \varepsilon < s$. We study the following boundary value problem for v^η.

$$\begin{aligned} \nabla \cdot (a^\eta \nabla v^\eta) &= 0 \text{ in } \mathbb{R}^2 \setminus B_\varepsilon(x_0), & (4.1)\\ v^\eta &= \frac{e^{i\vartheta}}{\varepsilon} \text{ for } x = x_0 + \varepsilon e^{i\vartheta} \in \partial B_\varepsilon(x_0), & (4.2)\\ v^\eta &\text{ bounded.} & (4.3) \end{aligned}$$

Under assumption (3.12) this system has a solution by Lemma 3.4 and Remark 3.5. The only difference to Remark 3.5 is that we multiplied the solution by ε^{-1} in order to be in the situation of Theorem 1. The scaling is chosen such that the limit of the subsequent number $\mathcal{M}_q^\eta$ can be non-trivial.

As a measure for the strength of the far-field we use, for fixed $q > |x_0|$, the number

$$\mathcal{M}_q^\eta := \left(\int_{\partial B_q(0)} |\partial_n v^\eta|^2 \right)^{1/2}. \tag{4.4}$$

To have a comparison, we first calculate this quantity in the situation without resonant ring. We denote the solution for $a \equiv 1$ by v^*, i.e.

$$\begin{aligned} \Delta v^* &= 0 \text{ in } \mathbb{R}^2 \setminus B_\varepsilon(x_0),\\ v^* &= \frac{e^{i\vartheta}}{\varepsilon} \text{ for } x = x_0 + \varepsilon e^{i\vartheta} \in \partial B_\varepsilon(x_0),\\ v^* &\text{ bounded.} \end{aligned}$$

The function v^* can be calculated explicitely as a dipole field with $\xi = e_1 + i\, e_2$,

$$v^*(x) = \xi \cdot \nabla \log(|x - x_0|) = \frac{\xi \cdot (x - x_0)}{|x - x_0|^2}. \tag{4.5}$$

This function is clearly bounded at infinity, harmonic, and satisfies the right boundary conditions. With fixed $q > R^*$ we therefore find, independent of $\varepsilon > 0$,

$$\mathcal{M}_q^* := \left(\int_{\partial B_q(0)} |\partial_n v^*|^2 \right)^{1/2} = \left(\int_{\partial B_q(0)} \left| \partial_n \frac{\xi \cdot (x - x_0)}{|x - x_0|^2} \right|^2 \right)^{1/2} > 0\,.$$

The following proposition is our first result on cloaking and proves the statement on cloaking of Theorem 1.

Proposition 4.1 (Radiation from inside the cloaking radius). *Let a^η and $\mathcal{L}^\eta$ with $1 < R < R^*$ be as above, $x_0 \in B_{R^*}(0) \setminus \bar{B}_R(0)$ such that $s = |x_0| < R^*$, and $\varepsilon > 0$ with $R < s - \varepsilon < s + \varepsilon < R^*$. Let v^η solve (4.1)–(4.3) and $\mathcal{M}_q^\eta$ be as in (4.4) with $q > R^2$. Let $\eta \searrow 0$ and $\varepsilon = \varepsilon(\eta) \searrow 0$ satisfy (3.12). Then*

$$\lim_{\eta \to 0} \mathcal{M}_q^\eta = 0.$$

Proof. Lemma 3.4 and Remark 3.5 provide a solution $\tilde{v}^\eta$ of $\mathcal{L}^\eta \tilde{v}^\eta = 0$ with $\tilde{v}^\eta = e^{i\vartheta}$ on $\partial B_\varepsilon(x_0)$. The function of the proposition is then given as $v^\eta = \varepsilon^{-1}\tilde{v}^\eta$. By Remark 3.5 we have to analyze, with λ_d of (3.8) and a_k of (3.9),

$$\begin{aligned} v^\eta = \frac{1}{\varepsilon}\tilde{v}^\eta &= \frac{1}{\varepsilon}\left(w^\eta + V_f^\eta + U_f^\eta + \lambda^\eta w^\eta + \mu^\eta\right) \\ &= \frac{1}{\varepsilon}\lambda_d \bar{V}_1 + \frac{1}{\varepsilon}\sum_k a_k U_k + \frac{1}{\varepsilon}V_f^\eta + \frac{1}{\varepsilon}U_f^\eta + \frac{1}{\varepsilon}\lambda^\eta w^\eta + \frac{1}{\varepsilon}\mu^\eta \\ &=: F_1^\eta + F_2^\eta + F_3^\eta + F_4^\eta + F_5^\eta + F_6^\eta. \end{aligned}$$

We will show that the functions F_j^η, $j = 1, 2, ..., 5$, satisfy

$$\left\|F_j^\eta\right\|_{L^2(\partial B_q(0))} \to 0, \tag{4.6}$$

using that the above functions are given by small coefficients $(f_l)_l \in l^2(\mathbb{N}_{**})$, $f_l = f_l^\eta$, and $\lambda^\eta \in \mathbb{C}$. The constant function F_6^η does not contribute to $\mathcal{M}_q^\eta$, hence (4.6) finishes the proof of the proposition.

(a) The function $F_1^\eta = \frac{1}{\varepsilon}\lambda_d \bar{V}_1$. We only have to recall that $V_1(z) = \varepsilon/(z - z_0)$. This function and its derivatives are of order ε on $\partial B_q(0)$ and claim (4.6) for $j = 1$ follows, since $\lambda_d \to 0$ by assumption (3.12).

(b) The function $F_2^\eta = \frac{1}{\varepsilon}\sum_k a_k U_k$. The explicit formula for a_k in (3.9) provides

$$\frac{1}{\varepsilon}|a_k| = |\lambda_d| \frac{|P_k^\eta|}{R^k s^k}\frac{1}{s} \le |\lambda_d| \frac{R^{2k}}{s^k}.$$

Together with the estimate of the derivative of U_k,

$$\|\partial_n U_k\|_{L^2(\partial B_q(0))} \le Ck\left(\frac{R}{q}\right)^k, \tag{4.7}$$

exploiting $q > R^2$ and $\lambda_d \to 0$, we find

$$\|F_2^\eta\|^2_{L^2(\partial B_q(0))} \le C\sum_k |\lambda_d|^2 \frac{R^{4k}}{s^{2k}} k^2 \frac{R^{2k}}{q^{2k}} \le C|\lambda_d|^2,$$

and thus (4.6) for $j = 2$.

(c) The function $F_3^\eta = \frac{1}{\varepsilon}V_f^\eta = \frac{1}{\varepsilon}\sum_{l=2} f_l \bar{V}_l$. As in the analysis of $\bar{V}_1$ in (a) it suffices to note that

$$\|\partial_n V_l\|_{L^2(\partial B_q(0))} \le C\varepsilon^l.$$

(d) The function $F_4^\eta = \frac{1}{\varepsilon} U_f^\eta$. We here analyze the function

$$\frac{1}{\varepsilon} U_f = \frac{1}{\varepsilon} \sum_k c_k U_k \text{ with } c_k = \sum_{l\geq 2} f_l \frac{P_k^\eta}{R^{2k}} G_{l,k}$$

for $l^2(\mathbb{N})$-small coefficients $(f_l)_l$. We start by recalling that (4.7) provides us some smallness of U_k at radius q. It allows to calculate

$$\begin{aligned} \left\| \frac{1}{\varepsilon} \partial_n U_f \right\|^2_{L^2(\partial B_q(0)} &= \frac{1}{\varepsilon^2} \sum_k |c_k|^2 \|\partial_n U_k\|^2_{L^2(\partial B_q(0))} \\ \leq \frac{C}{\varepsilon^2} \sum_k \left(\sum_{l\geq 2} |f_l|\, R^k |G_{l,k}| \right)^2 k^2 \left(\frac{R}{q}\right)^{2k} &= C \sum_k \left(\sum_{l\geq 2} |f_l| \frac{|G_{l,k}|}{\varepsilon} \right)^2 k^2 \left(\frac{R^2}{q}\right)^{2k}. \end{aligned}$$

We exploit the direct estimate

$$\frac{1}{\varepsilon} |G_{l,k}| \leq \frac{1}{s} \left(\frac{R}{s} + \frac{\varepsilon}{s} \right)^{k+l-1}$$

from the binomial formula, which shows the summability in both indices. It implies

$$\left\| \frac{1}{\varepsilon} \partial_n U_f \right\|^2_{L^2(\partial B_q(0)} \leq C \|(f_l)_l\|^2_{l^2} \to 0,$$

which concludes the calculation for U_f.

(e) The function $F_5^\eta = \frac{1}{\varepsilon} \lambda^\eta w^\eta$. The function w^η/ε was analyzed in (a) and (b), the factor λ^η is small.

We have shown (4.6). With this, the proof of the proposition is complete. □

Proposition 4.2 (Radiation from outside the cloaking radius)**.** *Let $\mathcal{L}^\eta$ with $1 < R < R^*$ be as above, $x_0 \in \mathbb{R}^2 \setminus \bar{B}_{R^*}(0)$, and $\varepsilon_0 > 0$ with $R^* < |x_0| - \varepsilon_0 < |x_0| + \varepsilon_0 < q$. Then, for every sufficiently small $\eta > 0$ there exists a solution v^η of (4.1)–(4.3). Let v^* be the limiting solution given by (4.5) and $\eta = \eta_n \searrow 0$ and $\varepsilon_0 \geq \varepsilon = \varepsilon(\eta_n) \to \varepsilon_1$ be arbitrary convergent sequences. Then, with $\mathcal{M}_q^\eta$ defined by (4.4), there holds*

> *$v^\eta \to v^*$ uniformly on compact subsets of $\mathbb{R}^2 \setminus (\bar{B}_R(0) \cup \bar{B}_{\varepsilon_1}(x_0))$ and*
> *$\mathcal{M}_q^\eta \to \mathcal{M}_q^*$ for $\eta \to 0$.*

Proof. We fix a radius r with $R^* < r < |x_0| - \varepsilon$. The statements will be derived as consequences of the convergence $N^{r,\eta} - N^{r,*} \to 0$ as expressed in (2.12). We exploit that a function with boundary condition on $\partial B_r(0)$ given by $N^{r,\eta}$ can be extended to an $\mathcal{L}^\eta$-harmonic function.

Step 1. Existence of v^η. We consider the problem on a bounded domain $B_\rho(0) \setminus \bar{B}_r(0)$ with a large radius ρ. The solution v^η for fixed $\eta > 0$ and fixed inclusion radius $\varepsilon = \varepsilon(\eta)$ can be constructed as follows. We consider $Q := B_\rho(0) \setminus (\bar{B}_\varepsilon(x_0) \cup \bar{B}_r(0))$, and emphasize that Q depends on $\varepsilon(\eta)$. We search for $v_\rho^\eta \in X_d$ such that

$$\int_Q \nabla v_\rho^\eta \cdot \nabla \bar{\varphi} + \int_{\partial B_r(0)} N^{r,\eta} v_\rho^\eta \, \bar{\varphi} = 0 \quad \forall \varphi \in X$$

for the (affine) linear spaces

$$X_d = \left\{ f \in H^1(Q) : f = 0 \text{ on } \partial B_\rho(0),\ f(x_0 + \varepsilon e^{i\vartheta}) = \frac{1}{\varepsilon} e^{i\vartheta} \text{ on } \partial B_\varepsilon(x_0) \right\},$$
$$X = \left\{ f \in H^1(Q) : f = 0 \text{ on } \partial B_\rho(0),\ f = 0 \text{ on } \partial B_\varepsilon(x_0) \right\}.$$

The bilinear form satisfies, by Poincaré's inequality,

$$\begin{aligned}
&\int_Q \nabla v_\rho^\eta \cdot \nabla \bar{v}_\rho^\eta + \int_{\partial B_r(0)} N^{r,\eta} v_\rho^\eta \, \bar{v}_\rho^\eta \\
&\quad = \int_Q \nabla v_\rho^\eta \cdot \nabla \bar{v}_\rho^\eta + \int_{\partial B_r(0)} N^{r,*} v_\rho^\eta \, \bar{v}_\rho^\eta + \int_{\partial B_r(0)} (N^{r,\eta} - N^{r,*}) v_\rho^\eta \, \bar{v}_\rho^\eta \\
&\quad \geq c(Q) \|v_\rho^\eta\|_{H^1(Q)}^2 - h(\eta) \|v_\rho^\eta\|_{H^1(Q)}^2
\end{aligned}$$

with $h(\eta) \searrow 0$ for $\eta \to 0$, where $c(Q)$ depends on ρ, but is independent of η despite the η-dependence of Q. Hence the form is coercive for small $\eta > 0$. The Lax-Milgram Theorem provides a solution v_ρ^η for every $\rho > r$.

We now want to perform the limit $\rho \to \infty$. To this end we have to perform Poincaré estimates on fixed bounded domains and consider for $j \in \mathbb{N}$ the restrictions $Q_j := (Q \cap B_j(0)) \setminus B_{\varepsilon_1 + 1/j}(x_0)$ and always assume that η is sufficiently small to have $Q_j \subset Q$. We subtract suitable constants from v_ρ^η in order to have a vanishing average on $\partial B_r(0)$. With this modification, the operator $N^{r,*}$ is strictly positive and allows a Poincaré estimate with $c_j > 0$,

$$c_j \|v_\rho^\eta\|_{H^1(Q_j)}^2 \leq \int_Q \nabla v_\rho^\eta \cdot \nabla \bar{v}_\rho^\eta + \int_{\partial B_r(0)} N^{r,*} v_\rho^\eta \, \bar{v}_\rho^\eta.$$

This implies for the solutions an estimate of this norm, depending on j, but independent of $\rho > \rho_0$. By a diagonal (sub-)sequence argument with $\rho \to \infty$ and $j \to \infty$, we find a solution v^η for the problem on the unbounded domain.

Step 2. Comparison with v^*. We define $v_{\eta,\rho}^* \in X_d$ as the solution of

$$\int_Q \nabla v_{\eta,\rho}^* \cdot \nabla \bar{\varphi} + \int_{\partial B_r(0)} N^{r,*} v_{\eta,\rho}^* \, \bar{\varphi} = 0 \quad \forall \varphi \in X$$

and will exploit that the solution is an approximation of the singular limit solution v^*. Taking the difference of the two variational problems and choosing $\varphi = v_\rho^\eta - v_{\eta,\rho}^* \in X$ we find

$$\begin{aligned}
&\int_Q |\nabla (v_\rho^\eta - v_{\eta,\rho}^*)|^2 + \int_{\partial B_r(0)} N^{r,*} (v_\rho^\eta - v_{\eta,\rho}^*)(\bar{v}_\rho^\eta - \bar{v}_{\eta,\rho}^*) \\
&\quad = - \int_{\partial B_r(0)} (N^{r,\eta} - N^{r,*}) v_\rho^\eta \, (\bar{v}_\rho^\eta - \bar{v}_{\eta,\rho}^*) \leq h(\eta) \|v_\rho^\eta\|_{H^1(Q_j)} \|v_\rho^\eta - v_{\eta,\rho}^*\|_{H^1(Q_j)}
\end{aligned}$$

with $h(\eta) \searrow 0$ for $\eta \to 0$. We conclude $v_\rho^\eta - v_{\eta,\rho}^* \to 0$ in $H^1(Q_j)$ with bounds that depend on j but are independent of ρ. We therefore conclude also $v^\eta - v^* \to 0$ in $H^1_{loc}(\mathbb{R}^2 \setminus (\bar{B}_r(0) \cup \bar{B}_{\varepsilon_1}(x_0)))$. Since both functions v^η and v^* are harmonic outside the ring, locally, the strong L^2 convergence implies uniform convergence of the functions and of the derivatives on $\partial B_q(0)$. This concludes the proof. $\square$

5 Invisibility of an inclusion

In this section we transfer our results to a different experimental set-up. We are interested in the possibility to detect an inclusion with a static measurement. We assume now that the inclusion is passive: it is given by a variation of the coefficient a in the ball $B_\varepsilon(x_0)$. Similar to a tomography measurement, we prescibe a potential u^η on the boundary ∂B_q. The solution to the partial differential equation u^η is evaluated on the boundary, more precisely, the normal derivative $\partial_n u^\eta$ is evaluated on ∂B_q. In short, we want to examine how much the inclusion in $B_\varepsilon(x_0)$ changes the Dirichlet-to-Neumann map of ∂B_q.

5.1 Model for the static measurement

The geometric set-up is as before, with radii $R, \varepsilon, q > 0$ satisfying $\varepsilon \to 0$ and $q > R^2$ and with a point $x_0 \in \mathbb{R}^2$ with $R < s = |x_0| < q$. In contrast to the last section the inclusion is not radiating; the non-trivial solution u^η is a result of nonvanishing boundary data $f \in L^2(\partial B_q(0))$. We decide to study

$$\begin{aligned} \nabla \cdot (a^\eta \nabla u^\eta) &= 0 \text{ in } B_q(0) \setminus B_\varepsilon(x_0), \\ u^\eta &= \alpha \text{ on } \partial B_\varepsilon(x_0), \\ u^\eta &= f \text{ on } \partial B_q(0), \end{aligned}$$

where $\alpha \in \mathbb{R}$ is selected such that

$$\int_{\partial B_\varepsilon(x_0)} \partial_n u^\eta = 0.$$

Physical background. The above equations model an inclusion with high conductivity. Indeed, when we want to investigate a large but finite conductivity $\kappa > 1$, we study the equations

$$\begin{aligned} \nabla \cdot (a^\eta_\kappa \nabla u^\eta_\kappa) &= 0 \text{ in } B_q(0), \\ u^\eta_\kappa &= f \text{ on } \partial B_q(0), \end{aligned}$$

where we set $a^\eta_\kappa = \kappa$ in $B_\varepsilon(x_0)$ and $a^\eta_\kappa = a^\eta$ on $B_q(0) \setminus B_\varepsilon(x_0)$. It is straightforward to verify that, in the limit $\kappa \to \infty$, the solutions u^η_κ converge weakly to solutions u^η in $H^1(B_q(0) \setminus B_\varepsilon(x_0))$. This follows by the boundedness of u^η_κ in this space, the uniqueness of solutions u^η, and the uniform estimate for $\int_{B_\varepsilon(x_0)} \kappa |\nabla u^\eta_\kappa|^2$. The flux condition that determines α is satisfied for all harmonic functions on $B_\varepsilon(x_0)$ and hence for u^η_κ. This carries over to the limit function. We emphasize that, in these arguments, the small positive numbers ε and η are kept fixed.

We investigate the limit problem of $\kappa = \infty$ in order to have less parameters.

5.2 Cloaking effect

Following in spirit the work of Bruno and Lintner, we study the following number as a measure for a cloaking effect. We denote by u^* the harmonic function with boundary values f. The function u^* is the comparison solution to $a \equiv 1$, i.e. the solution when no ring and no inclusion is present. As a measure of how much the true solution differs (in our measurement) from the comparision solution, we introduce

$$\mathcal{N}_q^\eta(f) := \left(\int_{\partial B_q(0)} |\partial_n u^\eta - \partial_n u^*|^2 \right)^{1/2} .$$

We can interpret a small value of $\mathcal{N}_q^\eta(f)$ as cloaking, since in this case the measurement with data f produces a result that is very close to a measurement without ring and without inclusion.

Theorem 2 (Invisibility of a small conductor)**.** *Let the geometry be as in Theorem 1 with numbers $q > |x_0| > R$, $q > R^2$, and let the sequences $\varepsilon = \varepsilon_n$ and $\eta = \eta_n$ satisfy $\eta_n/\varepsilon_n^2 \to 0$. Then, for $f \in H^{1/2}(\partial B_q(0))$, the following holds.*

1. *In the case $|x_0| > R^*$ we find*
$$\frac{\mathcal{N}_q^\eta(f)}{\varepsilon^2} \to c(f) \text{ for } \eta \to 0,$$
with $c(f) > 0$ for all functions f with $\nabla u^*(x_0) \neq 0$.

2. *In the case $|x_0| < R^*$ we find, if ε_n has the scaling properties* (3.12),
$$\frac{\mathcal{N}_q^\eta(f)}{\varepsilon^2} \to 0 \text{ for } \eta \to 0.$$

Remark concerning the assumption $\eta_n/\varepsilon_n^2 \to 0$. We recall our example of sequences η_n and ε_n satisfying (3.12). We considered $\eta_n = R^{-n}$ and $\varepsilon_n = b^{-n}$ with $b^2 < R^3/s^2$. Since $R^3/s^2 < R$ we then demand on b that $b^2 < R$ and hence $\eta_n/\varepsilon_n^2 \to 0$ is satisfied.

Proof. Step 1: A second comparison function. We already introduced the reference function u^* without ring and without inclusion, solving $\Delta u^* = 0$ on $B_q(0)$ with $u^* = f$ on $\partial B_q(0)$. Additionally, we use the function $u_\circ^\eta$ with the ring, but without the inclusion, solving

$$\begin{aligned} \mathcal{L}^\eta u_\circ^\eta &= 0 \text{ in } B_q(0), \\ u_\circ^\eta &= f \text{ on } \partial B_q(0). \end{aligned}$$

In order to conclude the theorem from results on $u^\eta - u_\circ^\eta$, we claim on $u_\circ^\eta$ that

$$\frac{1}{\varepsilon^4} \int_{\partial B_q(0)} |\partial_n u_\circ^\eta - \partial_n u^*|^2 \to 0. \tag{5.1}$$

Relation (5.1) is a consequence of our analysis of Dirichlet-to-Neumann maps and of the assumption $\eta_n/\varepsilon_n^2 \to 0$. Indeed, we can calculate with the Neumann operators $N^{q,\eta}$ and $N^{q,*}$, and expanding f as $f = \sum_k f_k e^{ik\theta}$,

$$\begin{aligned} \frac{1}{\varepsilon^4} \int_{\partial B_q(0)} |\partial_n u_\circ^\eta - \partial_n u^*|^2 &= \frac{1}{\varepsilon^4} \int_{\partial B_q(0)} |N^{q,\eta}(f) - N^{q,*}(f)|^2 \\ &= C \frac{1}{\varepsilon^4} \sum_k |N_k^{q,\eta} - N_k^{q,*}|^2 \, |f_k|^2 = C \frac{1}{\varepsilon^4} \sum_k |J_k^\eta|^2 \, |f_k|^2, \end{aligned}$$

where J_k^η of (2.14) is evaluated with radius $r = q > R^*$. Using (2.4) and (2.6) for P_k^η shows

$$\frac{1}{\varepsilon^2}|J_k^\eta| \le C\frac{1}{\varepsilon^2}|k|\,|P_k^\eta|\,q^{-2|k|} \le C\frac{1}{\varepsilon^2}|k|\,R^{4|k|}\,\eta q^{-2|k|} \le C\frac{\eta}{\varepsilon^2}\left(|k|(R^4/q^2)^{|k|}\right) \le C\frac{\eta}{\varepsilon^2}\,.$$

This quotient vanishes in the limit $n \to \infty$, and we conclude (5.1).

Step 2: A conductor outside the cloaking radius. In this step we assume $|x_0| > R^*$. Our aim is to show that the function $w^\eta := u^\eta - u_\circ^\eta$ gives a non-trivial contribution to $\mathcal{N}_q^\eta/\varepsilon^2$, even in the limit $n \to \infty$.

We exploit that the function w^η is $\mathcal{L}^\eta$-harmonic and satisfies homogeneous boundary conditions on $\partial B_q(0)$. It solves

$$\begin{aligned}\nabla\cdot(a^\eta\nabla w^\eta) &= 0 \text{ in } B_q(0)\setminus B_\varepsilon(x_0),\\ w^\eta &= g \text{ on } \partial B_\varepsilon(x_0),\\ w^\eta &= 0 \text{ on } \partial B_q(0),\end{aligned}$$

with

$$\begin{aligned}g &= (u^\eta - u_\circ^\eta)|_{\partial B_\varepsilon(x_0)} = \alpha - u_\circ^\eta\\ &= (\alpha - u_\circ^\eta(x_0)) - \nabla u_\circ^\eta(x_0)\cdot(x-x_0) - \frac{1}{2}D^2u_\circ^\eta(x_0)(x-x_0)\cdot(x-x_0) - \dots\end{aligned}$$

The functions $u_\circ^\eta$ converge uniformly on compact sets outside $\bar{B}_R(0)$ to u^* as a consequence of (2.6); we refer to the above calculation and note that now the factor $1/\varepsilon^2$ is not included. As a consequence, in the generic case that $\nabla u^*(x_0) \neq 0$, also $\nabla u_\circ^\eta(x_0) \to \nabla u^*(x_0) \neq 0$. We investigate this case in the following.

The potential w^η coincides approximately with a dipole field. Regarding the scaling we note that w^η/ε^2 can be compared with (a complex multiple of) the solution v^η of (4.2). Similar to the proof of Proposition 4.2 one can show that $\mathcal{N}_q^\eta/\varepsilon^2$ has a non-trivial limit. This is done by comparing the field w^η/ε^2 with standard non-trivial dipole field as v^* of (4.5). We omit the details of this calculation which is analogous to the proof of Proposition 4.2, but without the limit $q \to \infty$.

Step 3: A conductor inside the cloaking radius. In this case we decompose w^η as $w^\eta = w_1^\eta + w_2^\eta + w_3^\eta + w_4^\eta$. We define the functions w_j^η as the $\mathcal{L}^\eta$-harmonic functions with the following boundary conditions: on $\partial B_\varepsilon(x_0)$, the Dirichlet conditions $w_1^\eta = \alpha - u_\circ^\eta(x_0)$, $w_2^\eta = -\nabla u_\circ^\eta(x_0)\cdot(x-x_0)$, $w_3^\eta = g - w_1^\eta - w_2^\eta$, and $w_4^\eta = 0$. We impose that the function w_j^η is bounded on $\mathbb{R}^2\setminus B_\varepsilon(x_0)$ for $j = 1,2,3$. The function w_4^η corrects the exterior boundary condition, $w_4^\eta = -w_1^\eta - w_2^\eta - w_3^\eta$ on $\partial B_q(0)$. With this choice, the function $w_1^\eta + w_2^\eta + w_3^\eta + w_4^\eta$ satisfies the same equations as w^η. By uniqueness, the decomposition of $w^\eta = w_1^\eta + w_2^\eta + w_3^\eta + w_4^\eta$ is satisfied. To complete the proof, it remains to study the single solutions w_j^η.

(a) The function w_1^η is constant and does not contribute to the Neumann data on $\partial B_q(0)$.

(b) The function $\frac{1}{\varepsilon^2}w_2^\eta$ was studied in Proposition 4.1 with the result that $\frac{1}{\varepsilon^2}w_2^\eta$ tends to 0 uniformly on compact subsets. By compactness, the η-dependence of the factor $\nabla u_\circ^\eta(x_0)$ poses no problem in the application of Proposition 4.1.

(c) The function $\frac{1}{\varepsilon^2} w_3^\eta$ has bounded Dirichlet data on $B_\varepsilon(x_0)$ with vanishing dipole moment. Such $\mathcal{L}^\eta$-harmonic functions were studied in Lemma 3.4. The solution is given in (3.13), (3.14) with l^2-bounded coefficients f_l. The contributions of V_f and U_f are small on the large radius $q > R^2$. This is shown with the same calculation as in parts (a-c) in the proof of Proposition 4.1.

(d) By points (a)-(c), the function $\frac{1}{\varepsilon^2} w_4^\eta$ has small Dirichlet values on $\partial B_q(0)$. Since it is $\mathcal{L}^\eta$-harmonic and $q > R^*$ is outside the cloaking radius, we conclude that $\frac{1}{\varepsilon^2} w_4^\eta \to 0$ as the other three functions. □

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