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# Regularization schemes for degenerate Richards equations and outflow conditions

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**Abstract:** We analyze regularization schemes for the Richards equation and a time discrete numerical approximation. The original equations can be doubly degenerate, therefore they may exhibit fast and slow diffusion. Additionally, we treat outflow conditions that model an interface separating the porous medium from a free flow domain. In both situations we provide a regularization with a non-degenerate equation and standard boundary conditions, and discuss the convergence rates of the approximations.

**Keywords:** doubly degenerate equation, outflow problem, numerical approximation, variational inequalities

## 1 Introduction

The Richards equation models flow in unsaturated porous media, using the saturation  $s$  and a transformed pressure variable  $u$ . On a bounded domain  $\Omega \subset \mathbb{R}^n$  one considers the following nonlinear degenerate problem for all times  $t \in (0, T)$ .

$$\partial_t s = \operatorname{div}(\nabla u + K(s)), \quad u \in \Phi(s) \quad \text{on } \Omega, \quad (1.1)$$

$$s|_{t=0} = s_0 \text{ on } \Omega, \quad u = u_D \text{ on } \Gamma_D, \quad n \cdot (\nabla u + K(s)) = 0 \text{ on } \Gamma_N, \quad (1.2)$$

$$u \leq u_{out}, \quad n \cdot (\nabla u + K(s)) \leq 0, \quad (u - u_{out}) n \cdot \nabla u = 0 \quad \text{on } \Gamma_{out}. \quad (1.3)$$

In this set of equations,  $\Phi$  is a nonlinear monotone constitutive law,  $K$  is a vectorial coefficient that combines permeability and gravity, the boundary  $\partial\Omega$  is decomposed into an impermeable part  $\Gamma_N$ , a Dirichlet part  $\Gamma_D$ , and an outflow part  $\Gamma_{out}$ ,  $n$  is the outer normal. The value  $u_{out}$  is the transformed pressure corresponding to a vanishing capillary pressure: the fluid can flow out of the medium only if  $u$  exceeds this value. Without loss of generality we will later set  $u_{out} = 0$ . The monotone graph  $\Phi \subset \mathbb{R} \times \mathbb{R}$  can be doubly degenerate, i.e. it can have flat parts (vanishing diffusion) and infinite slope (elliptic behavior). Some physical background on the equations and, in particular, on the outflow condition, is collected below.

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We study two simplifications of the above equations. One regards standard boundary conditions, i.e.  $\Gamma_{out} = \emptyset$  such that the boundary condition (1.3) is not used. In the other simplification we study the outflow condition, but neglect gravity (setting  $K \equiv 0$ ) and assume a non-degenerate constitutive law  $\Phi$ .

In both cases, a regularization method for the above set of equations is introduced and analyzed. We derive convergence of solutions as the regularization parameter tends to zero. Furthermore, we derive convergence rates for a numerical scheme that can be used to approximate the regularized system.

**Literature.** Due to its importance in practical applications and its interesting analytical features, a vast literature is concerned with the description of flow in porous media. The Richards equation is widely used to model the flow of one phase in unsaturated porous media. If two phases (e.g. water and air) must be modelled, we are led to the closely related two-phase flow equations, see e. g. [1, 8, 9, 10, 11, 17, 26]. Instead, if the medium is partially saturated and the unsaturated phase is not modelled, we are led to the dam problem [6], where the outflow condition is of particular importance. A system with similar double degeneracy appears in a model for the evolution of bacterium species in [5]. The analytical results and methods in those models are closely related to ours.

For the analytical treatment of the doubly degenerate equations a fundamental reference is [2], where existence, regularity and uniqueness of solutions is treated. We emphasize that the uniqueness is shown only in the case that the function  $b = \Phi^{-1}$  is Lipschitz continuous. This is improved by Otto in [20] to more general single-valued functions  $b$ , but we recall at this point that  $b$  is not single valued in our case.

Existence results for the outflow boundary condition for the same nonlinear equation are provided with the formulation as variational inequalities in [1, 3], and with a regularization procedure in [25]. A homogenization of many interface conditions was performed in [10, 11, 26].

Numerical schemes for simply degenerate equations are investigated in e. g. [4, 13, 14, 16, 18, 22, 23]. The doubly degenerate case appears in [15, 21, 24]. Closest to our results are [21] and [24], treating also the doubly degenerate case with a regularization procedure. To relate our results to these works we note that in the first paper it is assumed that  $\Phi'(s) = 0$  has only one solution  $s$ , and that  $\Phi$  is convex and single valued. The problem considered in the second paper can be brought to the form of (1.1). The assumptions there allow for a nonlinearity  $\Phi$  that may be multivalued, but  $\Phi'(s) = 0$  has again only one solution. Here we allow both,  $\Phi$  can be multi-valued and vanish on intervals. In all papers mentioned above only standard boundary conditions are considered. Concerning the problem with an outflow boundary, we are not aware of any numerical approach.

**Main results on doubly degenerate equations.** In the doubly degenerate case, we restrict our analysis to  $\Gamma_{out} = \emptyset$ . To regularize the degenerate problem (1.1)–(1.2) we consider a smooth approximation  $\Phi_\delta$  of  $\Phi$  satisfying  $0 < m_\delta \leq \Phi'_\delta \leq M_\delta < \infty$  on  $\mathbb{R}$ , where  $m_\delta$  and  $M_\delta$  are  $\delta$ -depending. The regularized problem is non-degenerate parabolic, having a unique solution pair  $(s_\delta, u_\delta)$ ,

$$\begin{aligned} \partial_t s_\delta &= \operatorname{div}(\nabla u_\delta + K(s_\delta)), & u_\delta &= \Phi_\delta(s_\delta) & \text{on } \Omega_T := \Omega \times (0, T], \\ s_\delta(0) &= s_0 \text{ on } \Omega, & u_\delta &= u_D \text{ on } \Gamma_D, & n \cdot (\nabla u_\delta + K(s_\delta)) = 0 \text{ on } \Gamma_N. \end{aligned} \tag{1.4}$$

As shown e. g. in [3, 25], under natural assumptions on  $\Phi$ , a sequence  $\delta \searrow 0$  exists such that  $(s_\delta, u_\delta)$  converges weakly to a solution  $(s, u)$  of the degenerate problem (1.1). Our interest here is to compare approximate solutions. This, in particular, gives a quantitative result on the convergence  $(s_\delta, u_\delta) \rightarrow (s, u)$ , the convergence of the whole sequence, and a uniqueness result of solutions in a viscosity sense. Our main result in Theorem 2.2 is the comparison estimate

$$\sup_{t \in [0, T]} \|s_\delta - s_\varepsilon\|_{H^{-1}(\Omega)}^2 \leq C \max\{\varepsilon, \delta\}. \quad (1.5)$$

Here,  $C$  depends on the data  $s_0, u_D$ , but it is independent of  $\varepsilon$  and  $\delta$ .

**Main results on outflow boundary conditions.** The contact of the porous medium with free space across a portion  $\Gamma_{out}$  can not be described with a Dirichlet or a Neumann condition. The outflow condition takes the form of complementing inequalities, which can also be stated with a variational inequality. In this part, we restrict to non-degenerate functions  $\Phi$  and neglect gravity, therefore  $K \equiv 0$ .

For regularizing the outflow problem we use the form considered in [25]. A similar approach was employed for two-phase flow equations, see [17]. We choose a nonlinear Robin condition for the normal flux

$$-n \cdot \nabla u_\delta = F_\delta(u_\delta) \quad \text{on } \Gamma_{out}, \quad (1.6)$$

where  $F_\delta$  is an increasing function approximating the graph  $\{(u, U) \in \mathbb{R}^2 : u \cdot U = 0, u \leq 0, U \geq 0\}$ . Our main result in Theorem 3.3 is a convergence result

$$\|s_\delta - s\|_{L^2(\Omega_T)}^2 + \|u_\delta - u\|_{L^2(\Omega_T)}^2 \rightarrow 0, \quad (1.7)$$

where  $(s, u)$  is a solution of the limit problem. The convergence analysis can be made more precise. We will show that the convergence is monotone,  $u_\delta \geq u_\varepsilon$  for  $\delta \geq \varepsilon$ . If the pressure data are controlled, an error estimate is given in Proposition 3.6. Furthermore, a numerical scheme is analyzed and the error estimate

$$\|\hat{s}_\delta^h(t) - s_\delta(t)\|_{L^2(\Omega)}^2 \leq C (h^{1/2} + \delta^{-1}h^{3/4})$$

for the solution  $\hat{s}_\delta^h$  of the numerical scheme is derived in Theorem 4.1.

**Physical background.** The Richards equation models unsaturated flow in a porous medium occupying a region  $\Omega \subset \mathbb{R}^n$ . The unknown physical variables are the capillary pressure  $p$ , the saturation  $s$ , and the velocity  $v$ . Darcy's law relates forces to the velocity with the help of the permeability  $k$  which is assumed to be given by an algebraic relation  $k = k(s)$ . Similarly, the capillary pressure law imposes an algebraic relation  $p \in p_c(s)$ . With the unit vector  $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$  pointing upward, the force of gravity is  $-ge_n$ . Assuming a unit density, conservation of mass demands  $\partial_t s + \operatorname{div} v = 0$  and can be written as

$$\partial_t s = \operatorname{div} (k(s)(\nabla p + ge_n)), \quad p \in p_c(s).$$

Emphasis lies on the fact that  $p_c$  is, in general, a multi-valued function. This reflects the physical fact that in a fully saturated medium ( $s = 1$ ), the pressure inside the fluid can

increase to arbitrary values. Additionally, the permeability degenerates, namely  $k(s) = 0$  for small values of  $s$ . This again is a physical fact, reflecting that below some critical saturation  $\alpha$ , the fluid is no longer occupying connected regions in the medium. Then the permeability vanishes.

In the case of  $x$ -independent coefficient functions  $k$  and  $p_c$ , one uses the Kirchhoff-transform to simplify the problem. Choosing a function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  with  $\Phi'(s) = k(s)p'_c(s)$  such that  $\nabla[\Phi(s)] = k(s)p'_c(s)\nabla s = k\nabla p$ , setting  $K(s) = k(s)g e_n$  provides the form (1.1). The equations are complemented with initial values  $s_0$  for the saturation  $s$ , a Dirichlet condition  $u_D$  for the transformed pressure  $u = \Phi(s)$  on  $\Gamma_D \subset \partial\Omega$ , and a homogeneous Neumann condition for the velocity at an impenetrable boundary,  $n \cdot v = 0$ .

The physical *outflow condition* is

$$p \leq 0, \quad v \cdot n \geq 0, \quad p(v \cdot n) = 0 \quad \text{on } \Gamma_{out} \subset \partial\Omega.$$

It states that the capillary pressure can not exceed zero, since otherwise it is favorable for the fluid to leave the porous medium. Water can only leave the medium, but can not enter. The third condition is that water can exit only when a vanishing capillary pressure is reached. Translated into the variables  $(s, u)$  the outflow condition is as given in (1.3). We refer to [19] for more details on the modelling assumptions that lead to the outflow condition.

## 2 Standard boundary conditions

In this section we analyze the regularization procedure for the Richards equation with standard boundary conditions. We include gravity and study both the original and the regularized equation

$$\begin{aligned} \partial_t s &= \operatorname{div}(\nabla\Phi(s) + K(s)), \\ \partial_t s_\delta &= \operatorname{div}(\nabla\Phi_\delta(s_\delta) + K(s_\delta)), \end{aligned}$$

together with the boundary conditions of (1.2) and (1.4). Our aim is to verify the convergence  $s_\delta \rightarrow s$  and to determine the convergence rate. Before stating the main result, we specify the assumptions on the nonlinearities  $\Phi$ ,  $K$  and the regularization  $\Phi_\delta$ , as well as on the data.

### Assumptions on the limit problem

**The constitutive relations.** In this section two kinds of degeneracies are allowed: slow diffusion and fast diffusion. Both are encoded in  $\Phi$ , which may be multivalued in  $s = 1$ , and its derivative may vanish or blow up. To be precise, we assume that the graph  $\Phi \subset \mathbb{R}^2$  is defined by a monotonically increasing, locally Lipschitz continuous function  $\tilde{\Phi} : (-\infty, 1) \rightarrow \mathbb{R}$  with  $\lim_{s \rightarrow -\infty} \tilde{\Phi}(s) = -1$  and  $\lim_{s \rightarrow 1} \tilde{\Phi}(s) = \beta \in \mathbb{R} \cup \{\infty\}$ . The graph  $\Phi$  is determined by  $\tilde{\Phi}$  through the following relation for all  $(s, u) \in \mathbb{R}^2$ ,

$$u \in \Phi(s) : \iff (s, u) \in \Phi \iff u = \tilde{\Phi}(s) \text{ or } (s = 1 \text{ and } u \geq \beta).$$

The value  $-1$  on the left end point is fixed arbitrarily and is used only to specify the properties of  $\Phi$ . We also define  $\alpha = \inf\{s \in (-\infty, 1) : \Phi(s) > -1\}$ .

On  $K : (-\infty, 1) \rightarrow \mathbb{R}^n$  we assume  $K \equiv 0$  on  $(-\infty, \alpha]$ , and the uniform bound

$$|K'|^2 \leq C\Phi'$$

for some positive constant  $C$ . This is satisfied e.g. for quadratic functions  $k(s) \sim (s - \alpha)_+^2$  near  $s = \alpha$ ,  $K(s) = k(s)ge_n$  and a capillary pressure with derivative bounded from below such that  $\Phi'(s) \geq ck(s)$ . Furthermore, we assume that  $K'$  is bounded and extend  $K$  by  $K(1)$  for arguments  $s \geq 1$ . In particular,  $K$  is bounded on the real line.

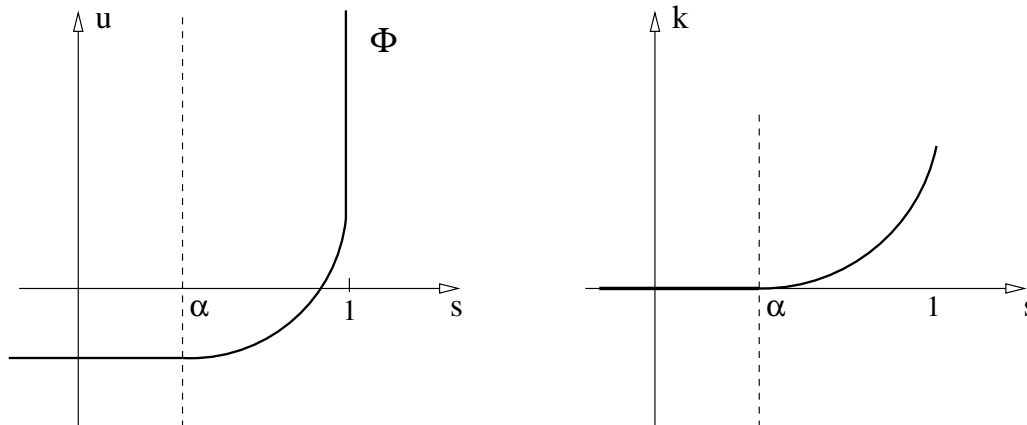


Figure 1: Typical graphs of the nonlinear coefficient functions  $\Phi$  and  $k$ , in the doubly degenerate case.

**Initial and boundary conditions.** We emphasize that for the doubly degenerate case considered in this section, only standard Dirichlet and Neumann boundary conditions are treated. The sets  $\Gamma_D$  and  $\Gamma_N$  are two relatively open subsets of  $\partial\Omega$  with empty intersection and  $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ . We assume that  $\Gamma_D$  has a positive  $n - 1$  dimensional measure and that the boundary values are given by a time-independent smooth function  $u_D$ , in particular  $u_D \in L^\infty(\Omega) \cap H^1(\Omega)$ . To have an easy proof in Lemma 2.1, we assume that the boundary admits  $C^1(\bar{\Omega})$  solutions of Poisson problems on  $\Omega$ . This is the case e.g. for smooth boundaries. The smoothness assumptions on  $\partial\Omega$  and  $u_D$  are for notational convenience and can be avoided with additional regularization steps. We finally assume that the initial and boundary conditions are compatible, in the sense that the initial saturation  $s_0 \in L^\infty(\Omega)$  and  $u_D$  of the boundary data are related by  $u_D \in \Phi(s_0)$ .

## The regularized problem

In what follows  $\delta > 0$  is a small regularization parameter, and  $\Phi_\delta : \mathbb{R} \rightarrow \mathbb{R}$  is a monotonically increasing Lipschitz approximation of  $\Phi$ . It satisfies  $\Phi_\delta(\mathbb{R}) = \mathbb{R}$  and

$$\begin{aligned} \Phi_\delta(s) &= \Phi(s) & \text{for all } s \in (\alpha + \rho(\delta), 1 - \delta), \\ \Phi'_\delta(s) &= \frac{1}{\delta} & \text{for all } s \geq 1, \quad \Phi'_\delta(s) = \delta & \text{for all } s \leq \alpha. \end{aligned}$$

The function  $\rho : [0, 1) \rightarrow [0, 1)$  appearing above is continuous and monotonically increasing with  $\rho(0) = 0$ . We furthermore assume the monotonicity in the regularization parameter,

namely  $\Phi_\varepsilon \geq \Phi_\delta$  on  $\mathbb{R}$  for all  $\varepsilon \leq \delta$ . A possible choice of  $\Phi_\delta$  is, for  $s_0 \in (\alpha, 1)$  and sufficiently small  $\delta > 0$ ,

$$\Phi_\delta(s) = \int_{s_0}^s \min \left\{ \frac{1}{\delta}, \max\{\delta, \Phi'\} \right\}. \quad (2.1)$$

In the following,  $(s, u)$  denotes a solution pair to the limit problem (1.1), whereas  $(s_\delta, u_\delta)$  solves the regularized problem (1.4). The existence of solutions  $(s, u)$  is obtained e.g. in [2]. The existence of solutions  $(s_\delta, u_\delta)$  is a consequence of standard parabolic theory. Notice that the regularized problem is a simplification of the problem considered in Section 3, where outflow boundary conditions are included. The existence proof there, which is based on time discretization, also works for the regularized problems in this section. A direct outcome of the analysis carried out here is also the existence of  $(s, u)$ , a result that has been obtained previously in the literature. However, our aim here goes beyond this existence result, to the convergence of the regularization process by obtaining estimates for the approximation error. In this sense we mention that convergence results for doubly degenerate parabolic problems are obtained in [15], based on compactness arguments. Error estimates are obtained in [24] also for doubly degenerate equations. In the present framework, the assumptions on the nonlinearities there can be translated into the Hölder continuity of the inverse of  $\Phi$ , which we do not require here.

**Lemma 2.1** (Uniform bounds). *The solution  $u_\delta$  is bounded from above,*

$$\sup_{\Omega_T} u_\delta \leq M(u_D, K, \Omega), \quad (2.2)$$

whereas for  $s_\delta$  one has the lower bound

$$\inf_{\Omega_T} s_\delta \geq s_{\min}(s_0, \alpha, \Omega). \quad (2.3)$$

Additionally,  $s_\delta$  satisfies the upper bound

$$\sup_{\Omega_T} s_\delta \leq 1 + C(u_D, K, \Omega) \delta. \quad (2.4)$$

The inequalities hold for all  $\delta > 0$ , the constants  $M$ ,  $s_{\min}$  and  $C$  depend on the indicated data, but are independent of  $\delta$ .

*Proof.* For  $k_1 = \|K\|_{L^\infty(\mathbb{R})} + 1$  and large  $\sigma > 0$ , we consider the solution  $v \in H^1(\Omega)$  of the following problem,

$$\Delta v = -1 \text{ in } \Omega, \quad \partial_n v = k_1 \text{ on } \Gamma_N, \quad v = \|u_D\|_{L^\infty(\Omega)} + \sigma \text{ on } \Gamma_D.$$

Since  $v$  is superharmonic, it is bounded from below by its values on the boundary, hence, in particular, positive. Furthermore, it is bounded uniformly from above by  $M = M_0(u_D, K, \Omega) + \sigma$ . We claim the following comparison principle: the solution  $u_\delta$  remains for all times below  $v$ , i.e.  $u_\delta(x, t) \leq v(x)$  for all  $x \in \Omega$  and all  $t \geq 0$ .

We show the claim for  $C^1(\bar{\Omega}) \cap C^2(\Omega)$ -solutions  $u_\delta(\cdot, t)$  and  $v$  and note that the result can be generalized to less regular solutions with an approximation argument. We consider the first time instance  $t \geq 0$  such that, in some  $x \in \Omega$ , the solutions touch, i.e.  $u_\delta(x, t) =$



$v(x)$ . Because of  $u_\delta(t = 0) = \Phi_\delta(s_0) \leq u_D < v$ , the time instance satisfies  $t > 0$ . Furthermore,  $x$  can not be on  $\Gamma_D$  because of  $v > u_D = u_\delta$  on  $\Gamma_D$ . On  $\Gamma_N$  holds  $\partial_n u_\delta = -K(s_\delta) \cdot n < k_1 = \partial_n v$ , hence  $x$  can not lie on the Neumann boundary. Finally, for an inner point  $x \in \Omega$ , we have, in this point,

$$\partial_t s_\delta = \Delta u_\delta + \nabla \cdot [K(\Phi_\delta^{-1}(u_\delta))] \leq -1 + K'(\Phi_\delta^{-1}(v)) \cdot \nabla(\Phi_\delta^{-1}(v)), \quad (2.5)$$

where we used that in a maximum of  $u_\delta - v$  we have  $\Delta u_\delta \leq \Delta v = -1$ , and the same gradients,  $\nabla u_\delta = \nabla v$ . We now exploit  $v \geq \sigma$  and  $(\Phi_\delta^{-1})'(\zeta) = 1/\Phi_\delta'(\Phi_\delta^{-1}(\zeta)) \rightarrow 0$  for  $\zeta \rightarrow \infty$ , independent of  $\delta$ . The boundedness of  $K'$  and  $\nabla v$  (independent of  $\sigma$ ) implies that, choosing  $\sigma$  large enough, the right hand side of (2.5) is negative for all  $\delta$ . This contradicts the minimality of  $t$  in the choice of the point  $(x, t)$ . Therefore  $u_\delta$  remains for all times below  $v$ .

The proof for the lower bounds is similar and uses the fact that  $K(s) = 0$  for  $s \leq \alpha$ . The upper bound for  $s_\delta$  is a consequence of (2.2) and  $\Phi_\delta' \geq \delta^{-1}$  on  $[1, \infty)$ .  $\square$

**Theorem 2.2** (Doubly degenerate equations). *Let  $T > 0$  define a time interval,  $\varepsilon, \delta > 0$  regularization parameters, and  $s_\delta$  and  $s_\varepsilon$  solutions of the regularized problems (1.4) with the nonlinearities  $\Phi_\delta$  and  $\Phi_\varepsilon$ , respectively. Then there holds*

$$\sup_{t \in [0, T]} \|s_\delta - s_\varepsilon\|_{H^{-1}(\Omega)}^2 \leq C \max\{\varepsilon, \delta\}. \quad (2.6)$$

Here,  $C$  depends on the data  $s_0, u_D$ , but it is independent of  $\varepsilon$  and  $\delta$ .

*Proof.* Without loss of generality we assume  $\varepsilon < \delta$ . We consider the two approximate solutions  $s_\varepsilon, s_\delta$  solving

$$\begin{aligned} \partial_t s_\delta &= \nabla \cdot (\nabla[\Phi_\delta(s_\delta)] + K(s_\delta)), \\ \partial_t s_\varepsilon &= \nabla \cdot (\nabla[\Phi_\varepsilon(s_\varepsilon)] + K(s_\varepsilon)). \end{aligned}$$

Subtracting these equations, we find for  $\sigma = s_\delta - s_\varepsilon$  the equation

$$\partial_t \sigma = \Delta[\Phi_\delta(s_\delta) - \Phi_\varepsilon(s_\varepsilon)] + \nabla \cdot [K(s_\delta) - K(s_\varepsilon)]. \quad (2.7)$$

From this equation we will derive an estimate for  $\sigma$ . We emphasize that  $s_\delta$  and  $s_\varepsilon$  are sufficiently regular for the subsequent operations since they are solutions of non-degenerate problems.

*Step 1. Test function and error evolution.* We multiply the equation with the test-function  $G = (-\Delta)^{-1}\sigma$ . To be precise, we treat the time  $t \in (0, T)$  as a parameter and introduce the function  $G : \Omega_T \rightarrow \mathbb{R}$  as the solution of the elliptic problem

$$\begin{aligned} -\Delta_x G(\cdot, t) &= \sigma(\cdot, t) \text{ on } \Omega, \\ G(\cdot, t) &= 0 \text{ on } \Gamma_D, \quad \partial_n G(\cdot, t) = 0 \text{ on } \Gamma_N. \end{aligned}$$

Multiplying (2.7) with  $G = G(\sigma)$ , for the time derivative we have

$$\partial_t \frac{1}{2} \|\nabla G\|_{L^2}^2 = \int_\Omega \nabla G \cdot \nabla \partial_t G = \int_\Omega G \partial_t (-\Delta G) = \int_\Omega G \partial_t \sigma.$$

The right hand side in (2.7) yields

$$\begin{aligned}
& - \int_{\Omega} \nabla \cdot (\nabla[\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] + [K(s_{\delta}) - K(s_{\varepsilon})]) G \\
& = \int_{\Omega} \nabla[\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] \nabla G + \int_{\Omega} [K(s_{\delta}) - K(s_{\varepsilon})] \nabla G \\
& = - \int_{\Omega} [\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] \Delta G + \int_{\Omega} [K(s_{\delta}) - K(s_{\varepsilon})] \nabla G \\
& = \int_{\Omega} [\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) + \int_{\Omega} [K(s_{\delta}) - K(s_{\varepsilon})] \nabla G.
\end{aligned}$$

The two boundary integrals over the Dirichlet boundary vanish, since both  $G$  and  $\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})$  satisfy a homogeneous Dirichlet condition on  $\Gamma_D$ . Similarly, since  $n \cdot (\nabla u_{\delta} + K(s_{\delta}))$  vanishes on  $\Gamma_N$ , integrals over the Neumann boundary are vanishing as well. Then (2.7) yields an evolution equation for the error,

$$\partial_t \frac{1}{2} \|\nabla G\|_{L^2(\Omega)}^2 + \int_{\Omega} [\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) + \int_{\Omega} [K(s_{\delta}) - K(s_{\varepsilon})] \cdot \nabla G = 0. \quad (2.8)$$

*Step 2. Error estimate.* The estimate for  $\sigma$  in  $H^{-1}(\Omega)$  is a consequence of an estimate for  $G(\cdot, t)$  in  $H^1(\Omega)$ . To conclude the latter, we exploit the monotonicity of  $\Phi_{\delta}$ , and start by considering for every fixed  $t$ , the set of points where  $s_{\delta}$  is smaller than  $s_{\varepsilon}$ , that is  $\Omega_{\delta\varepsilon} := \{x \in \Omega : s_{\delta} < s_{\varepsilon}\}$ . Recalling the  $\varepsilon$ -monotonicity of the approximate nonlinearities,  $\Phi_{\delta} \leq \Phi_{\varepsilon}$ , on this set we calculate

$$\begin{aligned}
& \int_{\Omega_{\delta\varepsilon}} [\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) \\
& = \int_{\Omega_{\delta\varepsilon}} [\Phi_{\delta}(s_{\delta}) - \Phi_{\delta}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) + \int_{\Omega_{\delta\varepsilon}} [\Phi_{\delta}(s_{\varepsilon}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) \\
& \geq \int_{\Omega_{\delta\varepsilon}} [\Phi_{\delta}(s_{\delta}) - \Phi_{\delta}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}).
\end{aligned}$$

Since  $\Phi_{\delta}$  is monotone, this is a positive term. On the set  $\Omega_{\varepsilon\delta} := \{x \in \Omega : s_{\varepsilon} \leq s_{\delta}\}$  we can perform a similar calculation. We distinguish between the set of intermediate and of large values of  $s_{\varepsilon}$ ,

$$\begin{aligned}
\Omega_{\varepsilon\delta 1} & := \{x \in \Omega_{\varepsilon\delta} : s_{\varepsilon} \leq 1 - \delta\} = \{x \in \Omega : s_{\varepsilon} \leq 1 - \delta \text{ and } s_{\varepsilon} \leq s_{\delta}\} \\
\Omega_{\varepsilon\delta 2} & := \{x \in \Omega_{\varepsilon\delta} : s_{\varepsilon} > 1 - \delta\} = \{x \in \Omega : s_{\delta} \geq s_{\varepsilon} > 1 - \delta\}.
\end{aligned}$$

Decomposing the domain and inserting 0 we find

$$\begin{aligned}
& \int_{\Omega_{\varepsilon\delta}} [\Phi_{\delta}(s_{\delta}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) \\
& = \int_{\Omega_{\varepsilon\delta}} [\Phi_{\delta}(s_{\delta}) - \Phi_{\delta}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) + \int_{\Omega_{\varepsilon\delta 1}} [\Phi_{\delta}(s_{\varepsilon}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon}) \\
& \quad + \int_{\Omega_{\varepsilon\delta 2}} [\Phi_{\delta}(s_{\varepsilon}) - \Phi_{\varepsilon}(s_{\varepsilon})] (s_{\delta} - s_{\varepsilon})
\end{aligned}$$

$$\begin{aligned}
 &\geq \int_{\Omega_{\varepsilon\delta}} [\Phi_\delta(s_\delta) - \Phi_\delta(s_\varepsilon)](s_\delta - s_\varepsilon) - C\delta \int_{\Omega_{\varepsilon\delta 1}} |s_\delta - s_\varepsilon| - C\delta \\
 &\geq \int_{\Omega_{\varepsilon\delta}} [\Phi_\delta(s_\delta) - \Phi_\delta(s_\varepsilon)](s_\delta - s_\varepsilon) - C\delta.
 \end{aligned}$$

In the first inequality, on  $\Omega_{\varepsilon\delta 1}$  where  $s_\varepsilon \leq 1 - \delta$ , we use the uniform approximation  $|\Phi_\delta(\cdot) - \Phi_\varepsilon(\cdot)| \leq C\delta$  on  $[s_{min}, 1 - \delta]$ . On  $\Omega_{\varepsilon\delta 2}$  we use  $|s_\delta - s_\varepsilon| \leq C\delta$  which follows from  $s_\delta \leq 1 + C\delta$  of Lemma 2.1, and  $\Phi_\delta(s_\varepsilon) \leq \Phi_\varepsilon(s_\varepsilon) \leq C$ . In the second inequality we use the uniform bounds for the saturation and adapted the constant  $C$ .

We can therefore rewrite the error equation (2.8) as

$$\begin{aligned}
 &\partial_t \frac{1}{2} \|\nabla G(\sigma)\|_{L^2(\Omega)}^2 + \int_{\Omega} [\Phi_\delta(s_\delta) - \Phi_\delta(s_\varepsilon)](s_\delta - s_\varepsilon) \\
 &\leq - \int_{\Omega} [K(s_\delta) - K(s_\varepsilon)] \cdot \nabla G + C\delta.
 \end{aligned} \tag{2.9}$$

In a given point  $x \in \Omega$  we compare the  $K$ -difference with the  $\Phi$ -difference.

$$\begin{aligned}
 K(s_\delta) - K(s_\varepsilon) &= \int_0^1 K'(s_\varepsilon + \xi(s_\delta - s_\varepsilon)) d\xi (s_\delta - s_\varepsilon) \\
 \Phi_\delta(s_\delta) - \Phi_\delta(s_\varepsilon) &= \int_0^1 \Phi'_\delta(s_\varepsilon + \xi(s_\delta - s_\varepsilon)) d\xi (s_\delta - s_\varepsilon)
 \end{aligned}$$

Since  $|K'|^2 \leq C\Phi'_\delta$  on  $\mathbb{R}$  we find

$$\begin{aligned}
 \left| \int_0^1 K'(s_\varepsilon + \xi(s_\delta - s_\varepsilon)) d\xi \right|^2 &\leq \int_0^1 |K'(s_\varepsilon + \xi(s_\delta - s_\varepsilon))|^2 d\xi \\
 &\leq C \int_0^1 \Phi'_\delta(s_\varepsilon + \xi(s_\delta - s_\varepsilon)) d\xi.
 \end{aligned}$$

This gives

$$\begin{aligned}
 |K(s_\delta) - K(s_\varepsilon)|^2 &\leq C \int_0^1 \Phi'_\delta(s_\varepsilon + \xi(s_\delta - s_\varepsilon)) d\xi |s_\delta - s_\varepsilon|^2 \\
 &= C[\Phi_\delta(s_\delta) - \Phi_\delta(s_\varepsilon)](s_\delta - s_\varepsilon).
 \end{aligned}$$

Hence, for any  $\eta > 0$ , the right hand side of (2.9) is estimated by

$$\begin{aligned}
 \left| \int_{\Omega} [K(s_\delta) - K(s_\varepsilon)] \cdot \nabla G \right| &\leq \left( \int_{\Omega} |K(s_\delta) - K(s_\varepsilon)|^2 \right)^{1/2} \|\nabla G\|_{L^2(\Omega)} \\
 &\leq C\eta \int_{\Omega} [\Phi_\delta(s_\delta) - \Phi_\delta(s_\varepsilon)](s_\delta - s_\varepsilon) + \frac{1}{\eta} \|\nabla G\|_{L^2(\Omega)}^2.
 \end{aligned}$$

Taking  $\eta = 1/C$ , (2.9) gives

$$\partial_t \|\nabla G\|_{L^2(\Omega)}^2 \leq C \|\nabla G\|_{L^2(\Omega)}^2 + C\delta.$$

Applying the inequalities of Gronwall and Poincaré, since  $\sigma$  vanishes at  $t = 0$  we find

$$\sup_{t \in [0, T]} \|G(\cdot, t)\|_{H^1(\Omega)}^2 \leq C\delta.$$

This implies the desired estimate for  $\sigma$  in  $L^\infty(0, T; H^{-1}(\Omega))$ .  $\square$

In the following Corollary we use the term *viscosity solution* for every  $L^2(\Omega_T)$ -weak limit  $(s, u)$  of solutions  $(s_\delta, u_\delta)$  to the regularized system.

**Corollary 2.3.** *There exists a unique viscosity solution  $(s, u)$  to the doubly degenerate system. It is a weak solution of the limit problem and satisfies, in particular,  $u \in \Phi(s)$  almost everywhere. The regularized solutions converge strongly to the viscosity solution and satisfy, with  $C$  independent of  $\delta$ ,*

$$\sup_{t \in [0, T]} \|s_\delta - s\|_{H^{-1}(\Omega)}^2 \leq C\delta. \quad (2.10)$$

for all  $\delta > 0$ .

*Proof.* Starting from the family  $\{(s_\varepsilon, u_\varepsilon) : \varepsilon > 0\}$  of solutions to the regularized problem, exploiting the uniform bounds, we can select a weakly convergent subsequence with a limit  $(s, u)$ . Estimate (2.6) of Theorem 2.2 implies the uniqueness of the limit. Results of [3, 25] show that  $(s, u)$  solves the constitutive relation  $u \in \Phi(s)$  and the degenerate problem in the form of a variational inequality or in the distributional sense.

Sending  $\varepsilon \searrow 0$  in (2.6), by lower semi-continuity of norms, we obtain the estimate (2.10). In particular, this provides the strong convergence of an arbitrary sequence of regularized solutions  $s_\delta$  to the unique viscosity solution  $s$ .  $\square$

### 3 The outflow boundary condition

In this section outflow boundary conditions are studied. Here only a non-degenerate constitutive law is considered, and gravity terms are neglected. We start by introducing a weak solution concept with variational inequalities. The concept is strong enough to guarantee the uniqueness of solutions. Our main interest is the analysis of a regularization approach for the outflow problem, which is similar to [25]. The existence of solutions to these regularized problems is achieved in Subsection 3.2, by means of a time discretization method. The strong convergence to solutions of the original outflow problem is shown in Subsection 3.3. Subsection 3.4 is devoted to the convergence analysis of the approximate solutions.

**Assumptions on the data.** For the open bounded subset  $\Omega \subset \mathbb{R}^n$  we assume here that the boundary is decomposed into three parts,  $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N \cup \bar{\Gamma}_{out}$  with three disjoint relatively open parts  $\Gamma_i$ . We additionally demand  $\bar{\Gamma}_D \cap \bar{\Gamma}_{out} = \emptyset$ . We impose an impermeability condition of  $\Gamma_N$ . The Dirichlet data on  $\Gamma_D$  are given by a smooth time-independent function  $u_D : \Omega \rightarrow \mathbb{R}$ , which is identified with  $u_D : \Omega_T \rightarrow \mathbb{R}$ . In particular,  $u_D \in H^1(\Omega) \cap L^\infty(\Omega)$ . The initial values are given by a function  $s_0 \in L^\infty(\Omega)$ . As in Section 2, we assume mild regularity and compatibility conditions by demanding  $u_D = \Phi(s_0)$  with  $u_D|_{\Gamma_{out}} \leq 0$ . This assumption makes sense since we imposed that the outflow boundary is not adjacent to the Dirichlet boundary,  $\bar{\Gamma}_D \cap \bar{\Gamma}_{out} = \emptyset$ .

On  $\Phi$  we assume from now on the non-degeneracy,

$$\Phi : \mathbb{R} \rightarrow \mathbb{R} \text{ monotone and } C^2 \text{ with } \Phi' \geq c_0 > 0 \text{ on } \mathbb{R}. \quad (3.1)$$

We denote the inverse by  $\Psi = \Phi^{-1}$  and use also its primitive  $\hat{\Psi} : \mathbb{R} \rightarrow \mathbb{R}$ .

### 3.1 Solution concept for the outflow condition

**Definition 1** (Variational solution of the limit problem). *A pair  $(s, u) \in L^2(\Omega_T) \times L^2(\Omega_T)$  with  $u = \Phi(s)$  almost everywhere is called a variational solution of the outflow problem, if  $\partial_t s \in L^2(\Omega_T)$  with  $s(0) = s_0$ ,  $\nabla u \in L^2(\Omega_T)$ ,  $u = u_D$  on  $\Gamma_D$ ,  $u \leq 0$  on  $\Gamma_{out}$ , and*

$$\int_{\Omega_T} \partial_t s H(\zeta - u) + \nabla u \cdot \nabla [H(\zeta - u)] \geq 0 \quad (3.2)$$

for all  $\zeta \in L^2(0, T; H^1(\Omega))$  with  $\zeta = u_D$  on  $\Gamma_D$  and  $\zeta \leq 0$  on  $\Gamma_{out}$ , and all  $H : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C_b^1$ , monotonically increasing with  $H(0) = 0$ .

Interpretation: In the interior,  $H(\zeta - u)$  is an arbitrary test-function, hence the bulk equation and the initial condition are satisfied in the weak sense. On the outflow boundary the condition  $u \leq 0$  is imposed explicitly. Assuming regularity of the functions and  $(\zeta - u)|_{t=0} = 0$ , (3.2) reads

$$\int_0^T \int_{\Gamma_{out}} \partial_n u H(\zeta - u) \geq 0.$$

Since  $H(\zeta - u)$  can be an arbitrary negative function, we find  $\partial_n u \leq 0$ . Furthermore, if  $u < 0$  on a part of  $\Gamma_{out}$ ,  $H(\zeta - u)$  can have both signs there, whence  $\partial_n u = 0$  in this case.

The use of the general function  $H$  is justified by the subsequent observation.

**Lemma 3.1** (Uniqueness). *The outflow problem admits at most one variational solution, in the sense of Definition 1.*

*Proof.* Let  $(s_1, u_1)$  and  $(s_2, u_2)$  be two solutions. Given  $\eta > 0$ , we consider  $H_\eta : \mathbb{R} \rightarrow \mathbb{R}$  as a smooth approximations of the sign function. It satisfies

$$H_\eta(\xi) = 1 \text{ for } \xi \geq \eta, \quad H_\eta(\xi) = -1 \text{ for } \xi \leq -\eta, \quad \text{while } H_\eta(\xi) = -H_\eta(-\xi).$$

Using  $\zeta = u_2 \chi_{(0,t)} + u_1 \chi_{(t,T)}$  in the variational inequality of  $u_1$  and  $\zeta = u_1 \chi_{(0,t)} + u_2 \chi_{(t,T)}$  in the variational inequality of  $u_2$ , adding the inequalities gives

$$\begin{aligned} 0 &\leq \int_{\Omega_t} \partial_t (s_1 - s_2) H_\eta(u_2 - u_1) - \nabla(u_2 - u_1) \cdot \nabla [H_\eta(u_2 - u_1)] \\ &\leq \int_{\Omega_t} \partial_t (s_1 - s_2) H_\eta(u_2 - u_1) \rightarrow \int_{\Omega_t} \partial_t (s_1 - s_2) \text{sign}(u_2 - u_1) \\ &= - \int_{\Omega_t} \partial_t (s_2 - s_1) \text{sign}(s_2 - s_1) = - \int_{\Omega} |s_2 - s_1|(t), \end{aligned}$$

as  $\eta \searrow 0$ . This limit makes sense since  $\partial_t s_i \in L^2(Q_T)$  ( $i = 1, 2$ ), whereas  $H_\eta(u_2 - u_1)$  converges in  $L^2$  to  $\text{sign}(u_2 - u_1) = \text{sign}(s_2 - s_1)$  by the dominated convergence theorem. Since  $t \in (0, T)$  was arbitrary, this shows the uniqueness.  $\square$

### 3.2 Regularized outflow problem

In order to regularize the outflow condition (1.3), we impose that a pressure above the exit pressure results in a large outflow. We use

$$\partial_n u_\delta = -F_\delta(u_\delta) \quad \text{on } \Gamma_{out}, \quad (3.3)$$

where we assume that  $F_\delta$  is monotone, has at most linear growth, is continuous and satisfies

$$F_\delta(\xi) = 0 \quad \text{for all } \xi \leq 0, \quad \text{and} \quad F_\delta(\xi) \geq \frac{\xi}{\delta} \quad \text{for all } \xi > 0. \quad (3.4)$$

We furthermore assume  $F_\varepsilon \geq F_\delta$  whenever  $\varepsilon < \delta$ . The primitive of  $F_\delta$  is denoted by  $\mathcal{F}_\delta : \mathbb{R} \rightarrow \bar{\mathbb{R}}$  with  $\mathcal{F}_\delta(0) = 0$ . Notice that the non-degeneracy assumption makes no regularization of  $\Phi$  necessary.

**Lemma 3.2** (Existence for the regularized problem). *Let  $u_D \in H^1(\Omega)$ ,  $s_0 \in L^\infty(\Omega)$  be as above and  $T > 0$ . Then problem (1.1) with the regularized outflow condition (3.3) possesses on  $(0, T)$  a weak solution  $(s_\delta, u_\delta)$ . More precisely, there exists  $u_\delta \in L^2(0, T; H^1(\Omega)) \cap L^\infty(\Omega_T)$  and  $s_\delta \in L^\infty(\Omega_T)$  with  $\partial_t s_\delta \in L^2(\Omega_T)$  such that the constitutive law  $u_\delta = \Phi(s_\delta)$  holds almost everywhere in  $\Omega_T$ ,  $u_\delta = u_D$  on  $\Gamma_D$ ,  $s_\delta(t=0) = s_0$ , and*

$$\int_{\Omega_T} \{\partial_t s_\delta \varphi + \nabla u_\delta \nabla \varphi\} + \int_{(0, T) \times \Gamma_{out}} F_\delta(u_\delta) \varphi = 0 \quad (3.5)$$

for all  $\varphi \in L^2(0, T; H^1(\Omega))$  with  $\varphi = 0$  on  $\Gamma_D$ . The solution satisfies the uniform estimates

$$\|s_\delta\|_{L^\infty(\Omega_T)} + \|u_\delta\|_{L^\infty(\Omega_T)} \leq C, \quad (3.6)$$

$$\int_0^T \int_\Omega \Phi'(s_\delta) |\partial_t s_\delta|^2 + \int_\Omega |\nabla u_\delta|^2(T) + \int_{\Gamma_{out}} \mathcal{F}_\delta(u_\delta)(T) \leq C, \quad (3.7)$$

with  $C$  independent of  $\delta > 0$ .

*Proof. Step 1. Time discretization.* Given  $N \in \mathbb{N}$ , we divide the time interval by taking  $h = T/N$  and  $t_k = kh$  for  $k = 0, \dots, N$ . We seek for  $s_k$  and  $u_k$  approximating  $s_\delta(t_k)$  and  $u_\delta(t_k)$  respectively. For readability, the subscript  $\delta$  is omitted in the time discrete approximations. These solve the following elliptic problems

$$\frac{s^k - s^{k-1}}{h} - \Delta u^k = 0 \quad \text{and} \quad u^k = \Phi(s^k) \quad \text{in } \Omega, \quad (3.8)$$

$$-\partial_n u^k = F_\delta(u^k) \quad \text{on } \Gamma_{out}, \quad (3.9)$$

$$u^k = u_D \quad \text{on } \Gamma_D, \quad \partial_n u^k = 0 \quad \text{on } \Gamma_N. \quad (3.10)$$

At every time step, the solution  $s^k$  of this problem can be constructed by minimizing

$$I(u) := \frac{1}{h} \int_\Omega \hat{\Psi}(u) + \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_{\Gamma_{out}} \mathcal{F}_\delta(u) - \frac{1}{h} \int_\Omega s_\delta^{k-1} u. \quad (3.11)$$

in the affine subset  $\{u = u_D \text{ on } \Gamma_D\} \subset H^1(\Omega)$ , and by setting then  $s = \Psi(u)$ . The functional is coercive, hence the minimization problem admits a solution.

*Step 2. A priori estimates.* The maximum principle for (3.8) provides essential upper bounds for  $s^k$  that are uniform in  $h$  and  $\delta$  and depend only on the uniform bounds of  $s_0$  and  $u_D$ . Since  $\Phi$  is non-degenerate, these bounds also give the uniform boundedness of  $u^k$ . For the lower bound we use the same argument, recalling that  $F_\delta(\xi) = 0$  for  $\xi \leq 0$ . Since the uniform bounds transfer to weak limits, (3.6) is shown.

*Formal derivation of the energy estimate.* Regarding the estimate for the time derivative, it suffices to multiply the equation  $\partial_t s_\delta = \Delta u_\delta$  with  $\partial_t u_\delta = \partial_t[\Phi(s_\delta)]$  and to integrate the result over  $\Omega$ . This gives

$$\begin{aligned} \int_{\Omega} \Phi'(s_\delta) |\partial_t s_\delta|^2 + \int_{\Omega} \nabla u_\delta \cdot \partial_t \nabla u_\delta &= \int_{\Gamma_{out}} \partial_n[\Phi(s_\delta)] \partial_t[\Phi(s_\delta)] \\ &= - \int_{\Gamma_{out}} F_\delta(\Phi(s_\delta)) \partial_t[\Phi(s_\delta)] = - \int_{\Gamma_{out}} \partial_t[\mathcal{F}_\delta(\Phi(s_\delta))]. \end{aligned}$$

An integration over  $(0, T)$  yields the desired estimate.

*Rigorous derivation of the energy estimate.* We repeat the above testing procedure on the time-discrete solution. Multiplying (3.8) by  $(u^k - u^{k-1})$  leads to

$$\int_{\Omega} \frac{s^k - s^{k-1}}{h} (u^k - u^{k-1}) + \int_{\Omega} \nabla u^k \nabla (u^k - u^{k-1}) + \int_{\Gamma_{out}} F_\delta(u^k) (u^k - u^{k-1}) = 0. \quad (3.12)$$

For first term we notice that  $u^k - u^{k-1} = \Phi'(\hat{s}^k)(s^k - s^{k-1})$  for an appropriate function of intermediate values  $\hat{s}^k$ . Concerning the second term we recall that  $2\nabla u^k \nabla (u^k - u^{k-1}) = |\nabla u^k|^2 - |\nabla u^{k-1}|^2 + |\nabla(u^k - u^{k-1})|^2 \geq |\nabla u^k|^2 - |\nabla u^{k-1}|^2$ . Concerning the third term, we use that monotone functions  $g : \mathbb{R} \rightarrow \mathbb{R}$  with primitive  $G$  satisfy

$$g(x)(x - y) \geq \int_y^x g(t) dt = G(x) - G(y).$$

Taking  $g = F_\delta$  gives  $F_\delta(u^k)(u^k - u^{k-1}) \geq \mathcal{F}_\delta(u^k) - \mathcal{F}_\delta(u^{k-1})$ . Using the above into (3.12) and summing over  $k = 1, 2, \dots, K$  we find the discrete estimate.

$$\begin{aligned} &\left( \sum_{k=1}^K \int_{\Omega} h \Phi'(\hat{s}^k) \left| \frac{s^k - s^{k-1}}{h} \right|^2 \right) + \left( \sum_{k=1}^K \frac{1}{2} \int_{\Omega} |\nabla(u^k - u^{k-1})|^2 \right) \\ &+ \frac{1}{2} \int_{\Omega} |\nabla u^K|^2 + \int_{\Gamma_{out}} \mathcal{F}_\delta(u^K) \leq \frac{1}{2} \int_{\Omega} |\nabla u_D|^2, \end{aligned} \quad (3.13)$$

where we have used that  $\mathcal{F}_\delta(u_D)$  vanishes by the assumption  $u_D \leq 0$  on  $\Gamma_{out}$ .

*Step 3. The limit  $h \rightarrow 0$ .*

*Weak limits.* From the discrete solution  $(s^k)_k$  we construct the piecewise affine interpolant  $\hat{s}^h$ , from  $(u^k)_k$  we construct the piecewise constant interpolant  $\bar{u}^h$  with  $\bar{u}^h(t) = u^k$  for all  $t \in (t_{k-1}, t_k)$ . With these functions in  $L^2(\Omega_T)$ , the Euler scheme of (3.8) and (3.9) reads

$$\begin{aligned} \partial_t \hat{s}^h - \Delta \bar{u}^h &= 0 \quad \text{in } \Omega_T, \\ \partial_n \bar{u}^h &= -F_\delta(\bar{u}^h) \quad \text{on } \Gamma_{out}. \end{aligned}$$

A weak form of the above is obtained by taking  $\varphi \in L^2(0, T; H^1(\Omega))$  with  $\varphi = 0$  on  $\Gamma_D$  as a test function, leading to

$$\int_{\Omega_T} \{ \partial_t \hat{s}^h \varphi + \nabla \bar{u}^h \nabla \varphi \} + \int_{(0, T) \times \Gamma_{out}} F_\delta(\bar{u}^h) \varphi = 0. \quad (3.14)$$

By the non-degeneracy  $\Phi' \geq c_0$ , the estimate (3.13) provides

$$\|\partial_t \hat{s}^h\|_{L^2(\Omega_T)} + \|\nabla \bar{u}^h\|_{L^2(\Omega_T)} \leq C, \quad (3.15)$$

with  $C$  independent of  $h$ . This allows to choose weakly convergent subsequences with  $\partial_t \hat{s}^h \rightharpoonup \partial_t s$  and  $\nabla \bar{u}^h \rightharpoonup \nabla u$ , both in  $L^2(\Omega_T)$ . For obtaining weak solutions to (3.14) we notice first that the weak convergence of  $\hat{s}^h$  and  $\bar{u}^h$  are sufficient to pass to the limit in the bulk integral of (3.14).

*Strong limits.* To take the limit in the boundary integral and to conclude  $u = \Phi(s)$  a strong convergence is needed. This is provided by (3.15): since  $u^h$  and  $s^h$  are related by a non-degenerate function, bounds can be found for all spatial and temporal derivatives. To see this we observe that  $\hat{s}^h(t) = \lambda \bar{s}^h(t) + (1-\lambda) \bar{s}^h(t-h)$  for an appropriate  $\lambda(t) \in [0, 1]$ , thus  $\nabla \hat{s}^h(t) = \lambda \Psi'(\bar{u}^h(t)) \nabla \bar{u}^h(t) + (1-\lambda) \Psi'(\bar{u}^h(t)) \nabla \bar{u}^h(t-h)$ . The non-degeneracy  $\Psi' \leq 1/c_0$  and (3.15) give  $\|\nabla \hat{s}^h\|_{L^2(\Omega_T)} \leq C$ . Since now all derivatives are bounded uniformly,  $\hat{s}^h$  is bounded in  $H^1(\Omega_T)$ . Upon choice of a subsequence, we find the strong convergence in  $\Omega_T$  and the strong convergence of the boundary values,

$$\|\hat{s}^h - s\|_{L^2(\Omega_T)} + \|(\hat{s}^h - s)|_{\Gamma_{out}}\|_{L^2(\Gamma_{outT})} \rightarrow 0.$$

We now exploit a general principle that relates the piecewise linear and the piecewise constant interpolation (see e.g. [17] for a proof of the corresponding lemma): if one interpolation converges strongly in  $L^2(\Omega_T)$ , then the other interpolation also converges strongly. We may therefore consider as an additional quantity the piecewise constant interpolation for the saturation values,  $\bar{s}^h$ . The general principle providing the strong  $L^2(\Omega_T)$ -convergence  $\hat{s}^h \rightarrow s$  also implies the strong convergence  $\bar{s}^h \rightarrow s$ . Since  $\Phi$  is Lipschitz continuous, this gives the convergence of  $\bar{u}^h = \Phi(\bar{s}^h)$ , and we finally have

$$\|\bar{u}^h - u\|_{L^2(\Omega_T)} + \|(\bar{u}^h - u)|_{\Gamma_{out}}\|_{L^2(\Gamma_{outT})} \rightarrow 0.$$

The strong convergence of  $\bar{s}^h$  allows to take the limit in the relation  $\bar{u}^h = \Phi(\bar{s}^h)$  and to conclude  $u = \Phi(s)$  almost everywhere. Furthermore, for any fixed  $\delta > 0$ ,  $F_\delta$  has at most linear growth such that also  $F_\delta(\bar{u}^h|_{\Gamma_{out}}) \rightarrow F_\delta(u|_{\Gamma_{out}})$  in  $L^2(\Gamma_{out})$ . We can therefore take the limit in all terms of (3.14) and conclude that  $(s, u)$  is a weak solution of the regularized outflow problem.  $\square$

### 3.3 Existence of solutions to the outflow problem

**Theorem 3.3** (Existence of solutions for the limit problem). *There exists a pair  $(s, u)$  that solves the outflow problem in the sense of Definition 1. For any sequence  $\delta \rightarrow 0$  there hold the weak- $*$  convergences*

$$\begin{aligned} u_\delta &\rightharpoonup u \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty(\Omega_T), \\ s_\delta &\rightharpoonup s \in H^1(0, T; L^2(\Omega)) \cap L^\infty(\Omega_T), \end{aligned} \quad (3.16)$$

and the strong convergence

$$u_\delta \rightarrow u, \quad s_\delta \rightarrow s \quad \text{in } L^2(\Omega_T). \quad (3.17)$$



*Proof.* The a priori estimates (3.6) and (3.7) allow selecting weak-\* convergent subsequences  $u_\delta \rightharpoonup u$  and  $s_\delta \rightharpoonup s$  as claimed in (3.16). Furthermore, by the non-degenerate relation  $u_\delta = \Phi(s_\delta)$ , both convergences are actually weak in  $H^1(\Omega_T)$ . Therefore, we find the strong convergence of the sequences in  $L^2(\Omega_T)$  and thus (3.17). The theorem is shown once that we verify that the limit solves the outflow problem. In particular, the uniqueness of the limit problem, shown in Lemma 3.1, provides the convergence of the whole sequence.

We note already here that the nonlinear relation  $u = \Phi(s)$  is an immediate consequence of  $u_\delta = \Phi(s_\delta)$  and the strong convergence of both sequences.

*Step 1. Variational inequality in the  $\delta$ -problem.* Let now  $H$  be a smooth function and  $\zeta$  be a test-function as in Definition 1. We insert  $H(\zeta - u_\delta)$  as a test-function into (3.5) to calculate

$$\begin{aligned} & \int_{\Omega_T} \partial_t s_\delta H(\zeta - u_\delta) + \nabla u_\delta \nabla H(\zeta - u_\delta) \\ &= - \int_{\Omega_T} \{s_\delta \partial_t H(\zeta - u_\delta) - \nabla u_\delta \nabla H(\zeta - u_\delta)\} - \int_{\Omega} s_0 H(\zeta - u_\delta) \\ &= - \int_{(0,T) \times \Gamma_{out}} F_\delta(u_\delta) H(\zeta - u_\delta) \geq 0. \end{aligned}$$

The last inequality follows by distinguishing two cases. For points  $(x, t)$  with  $u_\delta \leq 0$  we find  $F_\delta(u_\delta) = 0$  and the integrand vanishes. If, instead,  $u_\delta > 0$  we have  $F_\delta(u_\delta) > 0$  and  $\zeta - u_\delta < \zeta \leq 0$ , hence also  $H(\zeta - u_\delta) \leq 0$  and the boundary integral is non-positive.

*Step 2. The limit  $\delta \rightarrow 0$  in the variational inequality.* The a priori estimates provide a uniform bound on  $\nabla u_\delta$  in  $L^2(\Omega_T)$ . On the other hand, by the non-degeneracy, we have a uniform bound for  $\partial_t u_\delta = \Phi'(s_\delta) \partial_t s_\delta$  in  $L^2(\Omega_T)$ . This allows to conclude, for a subsequence,  $u_\delta \rightarrow u$  for  $\delta \rightarrow 0$  strongly in  $L^2(\Omega_T)$ . The (at most) linear growth of  $H$  and the weak convergence of  $\partial_t s_\delta$  allows to perform the limit in the first term,

$$\int_{\Omega_T} \partial_t s_\delta H(\zeta - u_\delta) \rightarrow \int_{\Omega_T} \partial_t s H(\zeta - u) \text{ for } \delta \rightarrow 0.$$

The other term is more involved. Since  $\nabla u_\delta$  converges only weakly, the product  $\nabla u_\delta \nabla H(\zeta - u_\delta)$  will, in general, not converge to  $\nabla u \nabla H(\zeta - u)$ . On the other hand, oscillations of  $u_\delta$  produce a negative contribution and we can therefore expect

$$\limsup_{\delta \rightarrow 0} \int_{\Omega_T} \nabla u_\delta \nabla [H(\zeta - u_\delta)] \leq \int_{\Omega_T} \nabla u \nabla [H(\zeta - u)]. \quad (3.18)$$

Once we have shown (3.18), the variational inequality is verified for the weak limit  $(s, u)$ .

*Step 3. Relation (3.18).* We set  $v_\delta := u_\delta - u$  and  $\Psi(x, z) := H(\zeta(x) - u(x) - z)$  such that  $H(\zeta - u_\delta) = \Psi(x, v_\delta(x))$ . Exploiting  $\partial_z \Psi \leq 0$  we find the sign condition

$$\begin{aligned} & \limsup_{\delta} \int_{\Omega_T} \nabla v_\delta \nabla \Psi(v_\delta) = \limsup_{\delta} \int_{\Omega_T} \nabla v_\delta \partial_z \Psi \nabla v_\delta + \nabla v_\delta \nabla_x \Psi(x, v_\delta) \\ &= \limsup_{\delta} \int_{\Omega_T} \nabla v_\delta \partial_z \Psi \nabla v_\delta + \nabla v_\delta H'(\zeta - u_\delta) \nabla_x (\zeta - u) \\ &\leq \limsup_{\delta} \int_{\Omega_T} \nabla v_\delta H'(\zeta - u_\delta) \nabla_x (\zeta - u). \end{aligned}$$

We claim that the right hand side vanishes. such that

$$\limsup_{\delta} \int_{\Omega_T} \nabla(u_{\delta} - u) \nabla H(\zeta - u_{\delta}) \leq 0.$$

To this end we recall that  $\nabla v_{\delta} \rightharpoonup 0$  in  $L^2(\Omega_T)$ . The strong convergence of  $u_{\delta}$  implies the pointwise convergence for a subsequence and, in turn, the pointwise convergence  $H'(\zeta - u_{\delta}) \nabla_x(\zeta - u) \rightarrow H'(\zeta - u) \nabla_x(\zeta - u)$ . All functions of the sequence are bounded by a multiple of the  $L^2(\Omega_T)$ -function  $\nabla_x(\zeta - u)$ , hence the Lebesgue convergence theorem provides the strong convergence in  $L^2(\Omega_T)$ . Uniqueness of the limit provides the convergence of the whole sequence. The product with  $\nabla v_{\delta}$  vanishes in the limit.

Using the above sign condition and inserting  $v_{\delta}$  provides

$$\begin{aligned} 0 &\geq \limsup_{\delta} \int_{\Omega_T} \nabla v_{\delta} \nabla[H(\zeta - u_{\delta})] \\ &= \limsup_{\delta} \int_{\Omega_T} \nabla u_{\delta} \nabla[H(\zeta - u_{\delta})] - \lim_{\delta} \int_{\Omega_T} \nabla u \nabla[H(\zeta - u_{\delta})] \\ &= \limsup_{\delta} \int_{\Omega_T} \nabla u_{\delta} \nabla[H(\zeta - u_{\delta})] - \int_{\Omega_T} \nabla u \nabla[H(\zeta - u)]. \end{aligned}$$

The last convergence follows from  $\nabla[H(\zeta - u_{\delta})] \rightharpoonup \nabla H(\zeta - u)$  in  $L^2(\Omega_T)$ . This, in turn, is a consequence of the boundedness of  $\nabla H(\zeta - u_{\delta}) = H'(\zeta - u_{\delta}) \nabla(\zeta - u_{\delta})$  in  $L^2(\Omega_T)$  and the strong convergence  $H(\zeta - u_{\delta}) \rightarrow H(\zeta - u)$ . With this, we have verified (3.18).  $\square$

### 3.4 Monotone convergence of solution sequences

**Lemma 3.4.** *Let  $\Phi$  be non-degenerate, data  $s_0$  and  $u_D$  as in Lemma 3.2. For a sequence  $\delta \rightarrow 0$  let  $s_{\delta}$  and  $u_{\delta} = \Phi(s_{\delta})$  be solutions to  $\partial_t s_{\delta} = \Delta u_{\delta}$  with regularized outflow condition (3.3) as constructed in Lemma 3.2. Then there holds the monotonicity*

$$s_{\varepsilon} \leq s_{\delta} \text{ whenever } \varepsilon \leq \delta. \quad (3.19)$$

*Proof.* We fix two solutions  $s_{\varepsilon}$  and  $s_{\delta}$  for two parameters  $\varepsilon, \delta$ , and assume in the following  $\varepsilon \leq \delta$ . The evolution equation for  $\sigma = s_{\delta} - s_{\varepsilon}$  reads

$$\partial_t \sigma = \Delta[\Phi(s_{\delta}) - \Phi(s_{\varepsilon})]. \quad (3.20)$$

The stated monotonicity is equivalent to  $\sigma \geq 0$ , or to a vanishing negative part,  $(\sigma)_{-} = 0$ . We use a monotone approximation  $H_{\eta} \in C^1(\mathbb{R}, [-1, 0])$  of a shifted Heaviside function, demanding

$$H_{\eta}(\xi) = \begin{cases} -1 & \text{for } \xi < -\eta, \\ 0 & \text{for } \xi \geq 0. \end{cases}$$

Multiplying the evolution equation (3.20) with  $H_{\eta}(u_{\delta} - u_{\varepsilon})$  and integrating over  $\Omega$  gives

$$\begin{aligned} \int_{\Omega} H_{\eta}(u_{\delta} - u_{\varepsilon}) \partial_t \sigma + \int_{\Omega} |\nabla(u_{\delta} - u_{\varepsilon})|^2 H'_{\eta}(u_{\delta} - u_{\varepsilon}) \\ + \int_{\partial\Omega} [F_{\delta}(u_{\delta}) - F_{\varepsilon}(u_{\varepsilon})] H_{\eta}(u_{\delta} - u_{\varepsilon}) = 0. \end{aligned}$$

We take the limit  $\eta \rightarrow 0$  and use the limiting function  $H : \mathbb{R} \rightarrow [-1, 0]$  with  $H(\xi) = -1$  for  $\xi < 0$  and  $H(\xi) = 0$  for  $\xi \geq 0$ . Since  $\partial_t \sigma$  and  $F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon)$  are functions and the pointwise convergence of the uniformly bounded sequence  $H_\eta(u_\delta - u_\varepsilon) \rightarrow H(u_\delta - u_\varepsilon)$ , because the second integral is positive, we find

$$\int_{\Omega} H(u_\delta - u_\varepsilon) \partial_t \sigma + \int_{\partial\Omega} [F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon)] H(u_\delta - u_\varepsilon) \leq 0.$$

Observing that the limiting function  $H$  satisfies  $H(u_\delta - u_\varepsilon) = H(s_\delta - s_\varepsilon) = H(\sigma)$ , for the primitive  $\hat{H}(\xi) = |(\xi)_-|$  of  $H$  one has

$$\partial_t \int_{\Omega} \hat{H}(s_\delta - s_\varepsilon) + \int_{\partial\Omega} [F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon)] H(u_\delta - u_\varepsilon) \leq 0.$$

In the boundary integral we distinguish between two sets. For the points  $x$  with  $u_\delta \geq u_\varepsilon$ , the boundary integral vanishes because  $H$  vanishes. On the set with  $u_\delta < u_\varepsilon$  holds  $F_\delta(u_\delta) \leq F_\delta(u_\varepsilon) \leq F_\varepsilon(u_\varepsilon)$  and therefore  $F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon) \leq 0$ . Furthermore,  $H(u_\delta - u_\varepsilon) \leq 0$ . Therefore the boundary integral is always non-negative. An integration over  $(0, t)$  yields

$$\int_{\Omega} |(\sigma)_-|(t) = \int_{\Omega} |(s_\delta - s_\varepsilon)_-|(t) = \int_{\Omega} \hat{H}(s_\delta - s_\varepsilon)(t) \leq 0,$$

and the claim  $(\sigma)_- = 0$  follows.  $\square$

**Remark 3.5.** Using the a priori estimate (3.7), as well as  $\mathcal{F}_\delta(\xi) \geq \xi^2/(2\delta)$ , we have for the pressure on the outflow boundary

$$\int_{\Gamma_{out}} (u_\delta)_+^2 \leq C\delta,$$

for all time instances  $t \in (0, T)$ . However, this is unfortunately not an  $L^\infty$ -bound.

**Proposition 3.6** (Comparison of solutions for regularized outflow condition). *Let  $0 < \varepsilon < \delta$  be regularization parameters, and let  $s_\varepsilon \leq s_\delta$  be solutions of the regularized outflow problem as in Lemma 3.2 to fixed data  $T, s_0, u_D$ . Then there exists a  $C > 0$  independent of  $\varepsilon$  and  $\delta$  such that*

$$\sup_{t \in [0, T]} \|s_\delta - s_\varepsilon\|_{L^2(\Omega)}^2 \leq C \|(u_\delta)_+\|_{L^\infty(\Gamma_{out})}^2. \quad (3.21)$$

*Proof.* We consider two solutions,  $s_\delta$  and  $s_\varepsilon$ , with  $\varepsilon < \delta$ . By Lemma 3.4 we have  $\sigma = s_\delta - s_\varepsilon \geq 0$ . The starting point for our analysis is again the evolution equation

$$\partial_t \sigma = \Delta[u_\delta - u_\varepsilon]. \quad (3.22)$$

*Step 1. The test-function.* To define the test-function in (3.22) we introduce the coefficient function

$$a(x, t) := \begin{cases} \frac{\Phi(s_\delta(x, t)) - \Phi(s_\varepsilon(x, t))}{(s_\delta - s_\varepsilon)(x, t)} & \text{if } \sigma(x, t) > 0, \\ \Phi'(s_\varepsilon(x, t)) & \text{else,} \end{cases}$$

and define the dual problem:

$$\begin{aligned}\partial_t G + a\Delta G &= 0 \text{ on } \Omega, \\ G(T) &= u_\delta(\cdot, T) - u_\varepsilon(\cdot, T) \\ G &= 0 \text{ on } \Gamma_D \cup \{x \in \Gamma_{out} : u_\varepsilon(x) > 0\}, \\ \partial_n G &= 0 \text{ on } \Gamma_N \cup \{x \in \Gamma_{out} : u_\varepsilon(x) \leq 0\}.\end{aligned}$$

Note that this problem is backward-in-time. Its weak solution  $G = G(\cdot, t)$  satisfies the Dirichlet condition in the strong sense on the indicated measurable set, whereas the Neumann condition is encoded in the weak formulation.

*Step 2. Properties of  $a$ .* By monotonicity of  $\Phi$ , the coefficient  $a$  is always positive and bounded from below by  $c_0$ . Furthermore, the function  $a$  is bounded since  $\Phi'$  is bounded on bounded intervals and since  $s_\delta \in L^\infty$ . For the differentiability of  $a$  we denote by  $\partial_j$  an arbitrary derivative, which may be  $\partial_t$  or  $\partial_{x_j}$  for some  $1 \leq j \leq n$ , and calculate in the case  $s_\delta \neq s_\varepsilon$

$$\begin{aligned}\partial_j a &= \frac{\Phi'(s_\delta) \partial_j s_\delta - \Phi'(s_\varepsilon) \partial_j s_\varepsilon}{s_\delta - s_\varepsilon} - \frac{\Phi(s_\delta) - \Phi(s_\varepsilon)}{s_\delta - s_\varepsilon} \frac{\partial_j s_\delta - \partial_j s_\varepsilon}{s_\delta - s_\varepsilon} \\ &= \frac{\Phi'(s_\delta) - \Phi'(s_\varepsilon)}{s_\delta - s_\varepsilon} \partial_j s_\delta + (\Phi'(s_\varepsilon) - \Phi'(\xi)) \frac{\partial_j s_\delta - \partial_j s_\varepsilon}{s_\delta - s_\varepsilon}.\end{aligned}$$

In the above, for each  $(x, t)$ , the mean value theorem guarantees the existence of a  $\xi$  between  $s_\delta(x, t)$  and  $s_\varepsilon(x, t)$  such that

$$\frac{\Phi(s_\delta) - \Phi(s_\varepsilon)}{s_\delta - s_\varepsilon} = \Phi'(\xi).$$

The function  $\Phi$  is twice continuously differentiable, hence the fractions

$$\frac{\Phi'(s_\delta) - \Phi'(s_\varepsilon)}{s_\delta - s_\varepsilon} \text{ and } \frac{\Phi'(s_\varepsilon) - \Phi'(\xi)}{s_\delta - s_\varepsilon}$$

are uniformly bounded. Since  $\partial_j s_\delta$  and  $\partial_j s_\varepsilon$  are  $L^2$ , this provides similar bounds for  $\partial_j a$ . In this sense,  $a$  inherits the differentiability properties of  $s$ , in particular any  $H^1$ -estimate.

Observe that the conclusion still holds in subsets of  $\Omega_T$  where  $a = \Phi'(s_\varepsilon)$ , since in this case  $\partial_j a = \Phi''(s_\varepsilon) \partial_j s_\varepsilon$ .

*Step 3. Boundedness properties of  $G$ .* The maximum principle implies that  $G$  has the sign of its initial values  $(u_\delta - u_\varepsilon)(T)$ , i.e.  $G \geq 0$ . Furthermore,  $G$  is uniformly bounded since  $u_\delta$  and  $u_\varepsilon$  are uniformly bounded.

We furthermore obtain an energy estimate for  $G$ . Testing in the dual problem with  $G$  provides

$$\partial_t \int_\Omega \frac{1}{2} |G|^2 - \int_\Omega a |\nabla G|^2 = \int_\Omega G \nabla a \nabla G.$$

Integrating over  $(0, T)$ , since the ‘‘initial values’’ at  $t = T$  are uniformly bounded, yields bounds for  $G$  in  $L^2(0, T; H^1(\Omega))$ , uniformly in the regularization parameters. Here we use that the coefficient  $a$  has the lower bound  $c_0 > 0$ , and that  $G$  satisfies uniform bounds by the maximum principle. Furthermore, as seen in Step 2, the boundedness

$\nabla s_\delta, \nabla s_\varepsilon \in L^2(\Omega_T)$  implies a uniform bound for  $\nabla a$  in the same space. This allows absorbing the right hand side, providing uniform  $L^2(\Omega_T)$  bounds for  $\nabla G$ .

We finally derive an estimate for the total flux. Given the function  $G$ , define its total flux through the boundary as

$$J_{\varepsilon,\delta} := - \int_{(0,T) \times \Gamma_{out}} \partial_n G = \int_{(0,T) \times \Gamma_{out}} |\partial_n G|.$$

For this one has

$$J_{\varepsilon,\delta} = - \int_{\Omega_T} \Delta G = \int_{\Omega_T} \frac{1}{a} \partial_t G = \int_{\Omega} \frac{1}{a} G \Big|_{t=0}^T + \int_{\Omega_T} \frac{1}{a^2} \partial_t a G.$$

In the above,  $1/a \leq 1/c_0$ , whereas  $G$  is uniformly bounded by the maximum principle. Following from Step 2,  $\partial_t a$  satisfies the integral bounds of  $s_\delta, s_\varepsilon$ , providing the uniform boundedness of  $\partial_t a \in L^2(\Omega_T)$ . Since  $G \in L^\infty$ , we have

$$J_{\varepsilon,\delta} \leq C \text{ independent of } \varepsilon, \delta.$$

*Step 4. Estimate for  $\sigma$ .* So far, all estimates regarded the boundedness of functions by a (possibly large) constant; we now want to improve our estimate for differences. To do so, we multiply (3.22) with  $G$  and obtain

$$\begin{aligned} \int_{\Omega_T} \partial_t \sigma G &= \int_{\Omega} \sigma(T) G(T) - \int_{\Omega_T} \sigma \partial_t G \\ &= \int_{\Omega} (s_\delta - s_\varepsilon)(T) \cdot (u_\delta - u_\varepsilon)(T) + \int_{\Omega_T} \sigma a \Delta G. \end{aligned}$$

Recalling the lower bound  $\Phi' \geq c_0$ , the first term on the right gives

$$\int_{\Omega} (u_\delta - u_\varepsilon)(T) \cdot (s_\delta - s_\varepsilon)(T) \geq c_0 \|(s_\delta - s_\varepsilon)(T)\|_{L^2(\Omega)}^2.$$

Since  $\sigma a = (s_\delta - s_\varepsilon)a = u_\delta - u_\varepsilon$ , the boundary conditions imply

$$\begin{aligned} \int_{\Omega_T} \sigma a \Delta G &= \int_{\Omega_T} (u_\delta - u_\varepsilon) \Delta G = \int_{\Omega_T} \Delta(u_\delta - u_\varepsilon) G \\ &+ \int_{(0,T) \times \Gamma_{out}} (F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon)) G + \int_{(0,T) \times \Gamma_{out}} (u_\delta - u_\varepsilon) \partial_n G. \end{aligned}$$

Using the evolution equation  $\partial_t \sigma = \Delta(u_\delta - u_\varepsilon)$  we find

$$\begin{aligned} c_0 \|(s_\delta - s_\varepsilon)(T)\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} (s_\delta - s_\varepsilon)(T) \cdot (u_\delta - u_\varepsilon)(T) \\ &= \int_{\Omega_T} \partial_t \sigma G - \int_{\Omega_T} \sigma a \Delta G \\ &= - \int_{(0,T) \times \Gamma_{out}} (F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon)) G - \int_{(0,T) \times \Gamma_{out}} (u_\delta - u_\varepsilon) \partial_n G. \end{aligned}$$

Due to the boundary condition on  $G$ , the first boundary integral is nonzero only if  $u_\varepsilon \leq 0$ . Then we have  $F_\varepsilon(u_\varepsilon) = 0$  and hence

$$\int_{(0,T) \times \Gamma_{out}} (F_\delta(u_\delta) - F_\varepsilon(u_\varepsilon)) G = \int_{(0,T) \times \Gamma_{out}} F_\delta(u_\delta) G \geq 0.$$

Similarly, the second boundary integral is nonzero only if  $u_\varepsilon > 0$ , when  $0 < u_\varepsilon \leq u_\delta$  by the ordering result. By the bound for  $J_{\varepsilon,\delta}$  in Step 3, this gives

$$\begin{aligned} - \int_{(0,T) \times \Gamma_{out} \cap \{u_\varepsilon > 0\}} (u_\delta - u_\varepsilon) \partial_n G &\leq \|(u_\delta)_+\|_{L^\infty(\Gamma_{out})}^2 \int_{(0,T) \times \Gamma_{out}} |\partial_n G| \\ &= \|(u_\delta)_+\|_{L^\infty(\Gamma_{out})}^2 J_{\varepsilon,\delta} \leq C \|(u_\delta)_+\|_{L^\infty(\Gamma_{out})}^2, \end{aligned}$$

and the proof is concluded.  $\square$

## 4 Numerical scheme

In Sections 2 and 3 we have studied regularization schemes and studied the convergence  $s_\delta \rightarrow s$ . By Theorem 2.2, in the doubly degenerate case with standard boundary conditions one has

$$\|s_\delta - s\|_{L^\infty(0,T;H^{-1}(\Omega))}^2 \leq C\delta. \quad (4.1)$$

Theorem 3.3 refers to the non-degenerate case with outflow boundary conditions. In this case, as  $\delta \rightarrow 0$ ,

$$u_\delta \rightarrow u, \quad s_\delta \rightarrow s \text{ strongly in } L^2(\Omega_T). \quad (4.2)$$

We mention that Remark 3.5 together with Proposition 3.6 suggests the rate

$$\|s_\delta - s\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C\delta$$

in this case. We emphasize that this estimate is not proven. We require an  $L^\infty$ -norm in Proposition 3.6, but have only an  $L^2$ -norm in Remark 3.5.

The solutions have been constructed by a time discretization and the corresponding approximate solutions  $\hat{s}_\delta^h$  and  $\bar{u}_\delta^h$ . Our interest in this section is to analyze the error  $\hat{s}_\delta^h - s_\delta$ , where  $h > 0$  is the small time step size.

**Results for standard boundary conditions.** With slightly different implicit discretization schemes in the doubly degenerate case, convergence for  $h \rightarrow 0$  was shown in [15, 18, 24]. We recall that these results regard the standard boundary conditions and that in those contributions  $\Phi$  is not allowed to be fully degenerate: a vanishing slope on an interval and, at the same time, multi-valued.

**Analysis for outflow boundary conditions.** We briefly recall the scheme introduced in (3.8)–(3.10) for  $s_\delta^k \approx s_\delta(t_k)$  and  $u_\delta^k = \Phi(s_\delta^k) \approx u_\delta(t_k)$ . Denoting by  $(\cdot, \cdot)$  the scalar product of  $L^2(\Omega)$  and by  $(\cdot, \cdot)_{\Gamma_{out}}$  the scalar product of  $L^2(\Gamma_{out})$ . With the  $\hat{f}^h$  and  $\bar{f}^h$  denoting the piecewise linear and the piecewise constant interpolations of  $L^2(\Omega)$ -functions  $f^k$ , (3.14) reads

$$(\partial_t \hat{s}_\delta^h, \varphi) + (\nabla \bar{u}_\delta^h, \nabla \varphi) + (F_\delta(\bar{u}_\delta^h), \varphi)_{\Gamma_{out}} = 0 \quad (4.3)$$

for all  $\varphi \in H^1(\Omega)$  with  $\varphi = 0$  on  $\Gamma_D$ , and for almost all  $t \in (0, T]$ .

**Theorem 4.1** (Discretization error). *Let  $\Phi$  be non-degenerate and let the data  $s_0, u_D$ , and  $T > 0$  be given as in Lemma 3.2. Denote by  $(s_\delta, u_\delta)$  the solution of the regularized outflow problem and by  $\hat{s}_{\delta,h}$  the solution of the discretization scheme, both defined in Lemma 3.2. Then*

$$\|\hat{s}_\delta^h(t) - s_\delta(t)\|_{L^2(\Omega)}^2 \leq C (h^{1/2} + \delta^{-1}h^{3/4}), \quad (4.4)$$

for a.e.  $t \in (0, T]$ , with a constant  $C$  independent of  $\delta$  and  $h$ .

*Proof. Step 1. Preliminaries.* Existence and uniqueness for the family  $(s_\delta^k, u_\delta^k)$  together with the convergences  $\hat{s}_\delta^h \rightarrow s_\delta$  and  $\hat{u}_\delta^h \rightarrow u_\delta$  in  $L^2(\Omega_T)$  for  $h \rightarrow 0$  was shown in Lemma 3.2. Our aim here is to quantify the error introduced by a finite  $h > 0$ .

We recall that the function  $F_\delta$  is continuous with a Lipschitz constant  $C_L^\delta$  of order  $1/\delta$ . We have introduced before the primitive  $\mathcal{F}_\delta$  of this function. The following estimate was obtained in (3.13)

$$\sum_{k=1}^N \frac{1}{h} \|s_\delta^k - s_\delta^{k-1}\|^2 + \sum_{k=1}^N \int_{\Omega} |\nabla(u_\delta^k - u_\delta^{k-1})|^2 + \max_k \left\{ \|\nabla u_\delta^k\|^2 + \int_{\Gamma_{out}} \mathcal{F}_\delta(u_\delta^k) \right\} \leq C, \quad (4.5)$$

with  $C > 0$  independent of  $\delta$  and  $h$ .

*Step 2. The test-function  $G$ .* We now follow the ideas of the proof of Proposition 3.6. From now on we omit the superscript  $h$  for interpolations such as  $\hat{s}_\delta^h$  and  $\hat{s}_\delta$ . We define the two coefficient functions

$$\mu := \int_0^1 \Phi'(zs_\delta + (1-z)\hat{s}_\delta) dz \quad \text{and} \quad \zeta := \int_0^1 F'_\delta(zu_\delta + (1-z)\hat{u}_\delta) dz. \quad (4.6)$$

We note that  $c_0 \leq \mu \leq C$  and  $\zeta \geq 0$  is bounded by  $C/\delta$ . Both depend on  $x$  and  $t$ . We may write the coefficients also as

$$\mu = \begin{cases} \frac{u_\delta - \hat{u}_\delta}{s_\delta - \hat{s}_\delta}, & \text{if } s_\delta \neq \hat{s}_\delta, \\ \Phi'(s_\delta), & \text{if } s_\delta = \hat{s}_\delta, \end{cases} \quad \text{and} \quad \zeta = \begin{cases} \frac{F_\delta(u_\delta) - F_\delta(\hat{u}_\delta)}{u_\delta - \hat{u}_\delta}, & \text{if } u_\delta \neq \hat{u}_\delta, \\ F'_\delta(u_\delta), & \text{if } u_\delta = \hat{u}_\delta. \end{cases}$$

Arguing as in Step 2 of the proof of Proposition 3.6, the function  $\nabla\mu$  is bounded in  $L^2(\Omega_T)$  by the  $L^2$ -norms of  $\nabla s_\delta$  and  $\nabla \hat{s}_\delta$ . In particular, we find a uniform bound for  $\mu$  in  $H^1(\Omega)$ , independent of  $\delta$  and  $h$ .

For an arbitrary time instance  $\tilde{t} > 0$  we define  $G$  as the solution of the following backward problem.

$$\begin{aligned} \partial_t G + \mu \Delta G &= 0 \quad \text{on } \Omega \times (0, \tilde{t}), \\ G(\tilde{t}) &= s_\delta(\tilde{t}) - \hat{s}_\delta(\tilde{t}) \\ G &= 0 \quad \text{on } \Gamma_D, \\ \partial_n G &= 0 \quad \text{on } \Gamma_N, \\ -\partial_n G &= \zeta G \quad \text{on } \Gamma_{out}. \end{aligned}$$

The function  $G$  satisfies a uniform  $L^\infty$ -bound by the maximum principle. Testing the evolution equation with  $G$  provides

$$\partial_t \frac{1}{2} \int_{\Omega} |G|^2 = \int_{\Omega} \mu |\nabla G|^2 + G \nabla G \cdot \nabla \mu - \int_{\Gamma_{out}} \mu \partial_n G G.$$

The uniform bound for  $\nabla\mu \in L^2(\Omega_T)$ , the sign of the boundary integral, and the strict positivity of  $\mu$  allow to conclude

$$\|G\|_{L^2(0,\tilde{t};H^1(\Omega))} \leq C, \quad (4.7)$$

with  $C$  independent of  $\delta$  and  $h$ .

*Step 3. The testing procedure.* We subtract the equation for the time discretization  $(\hat{s}_\delta, \bar{u}_\delta)$  from the one for  $(s_\delta, u_\delta)$  and obtain

$$(\partial_t s_\delta - \partial_t \hat{s}_\delta, \varphi) + (\nabla u_\delta - \nabla \bar{u}_\delta, \nabla \varphi) + (F_\delta(u_\delta) - F_\delta(\bar{u}_\delta), \varphi)_{\Gamma_{out}} = 0. \quad (4.8)$$

Taking in the above  $\varphi = G$ , integrating in time over  $(0, \tilde{t})$ , using the definitions of  $\mu, \zeta$ , and  $G$ , as well as  $s_\delta(t=0) = s_0 = \hat{s}_\delta(t=0)$ , we obtain

$$\begin{aligned} \int_0^{\tilde{t}} (\partial_t (s_\delta - \hat{s}_\delta), G) dt &= (s_\delta - \hat{s}_\delta, G)|_{t=\tilde{t}} - \int_0^{\tilde{t}} (s_\delta - \hat{s}_\delta, \partial_t G) dt \\ &= \|(s_\delta - \hat{s}_\delta)(\tilde{t})\|_{L^2(\Omega)}^2 + \int_0^{\tilde{t}} (s_\delta - \hat{s}_\delta, \mu \Delta G) dt \\ &= \|(s_\delta - \hat{s}_\delta)(\tilde{t})\|_{L^2(\Omega)}^2 + \int_0^{\tilde{t}} (u_\delta - \hat{u}_\delta, \Delta G) dt \\ &= \|(s_\delta - \hat{s}_\delta)(\tilde{t})\|_{L^2(\Omega)}^2 - \int_0^{\tilde{t}} (\nabla u_\delta - \nabla \hat{u}_\delta, \nabla G) dt + \int_0^{\tilde{t}} \int_{\Gamma_{out}} (u_\delta - \hat{u}_\delta) \partial_n G dt \\ &= \|(s_\delta - \hat{s}_\delta)(\tilde{t})\|_{L^2(\Omega)}^2 - \int_0^{\tilde{t}} (\nabla u_\delta - \nabla \bar{u}_\delta, \nabla G) dt - \int_0^{\tilde{t}} \int_{\Gamma_{out}} (F_\delta(u_\delta) - F_\delta(\bar{u}_\delta)) G dt \\ &\quad - \int_0^{\tilde{t}} (\nabla \bar{u}_\delta - \nabla \hat{u}_\delta, \nabla G) dt - \int_0^{\tilde{t}} \int_{\Gamma_{out}} (F_\delta(\bar{u}_\delta) - F_\delta(\hat{u}_\delta)) G dt. \end{aligned}$$

Using (4.8) with  $\varphi = G$ , we see that three of the integrals cancel each other. The remaining terms provide the estimate

$$\|(s_\delta - \hat{s}_\delta)(\tilde{t})\|_{L^2(\Omega)}^2 \leq \left| \int_0^{\tilde{t}} (\nabla \bar{u}_\delta - \nabla \hat{u}_\delta, \nabla G) dt \right| + \left| \int_0^{\tilde{t}} \int_{\Gamma_{out}} (F_\delta(\bar{u}_\delta) - F_\delta(\hat{u}_\delta)) G dt \right|.$$

The uniform bounds for  $G \in L^\infty$  and  $G \in L^2(0, \tilde{t}; H^1(\Omega))$  of (4.7), and the  $C_L^\delta$ -Lipschitz continuity of  $F_\delta$  provide

$$\|(s_\delta - \hat{s}_\delta)(\tilde{t})\|_{L^2(\Omega)}^2 \leq C \left( \int_0^{\tilde{t}} \int_\Omega |\nabla \bar{u}_\delta - \nabla \hat{u}_\delta|^2 \right)^{1/2} + C_L^\delta \left( \int_0^{\tilde{t}} \int_{\Gamma_{out}} |\bar{u}_\delta - \hat{u}_\delta|^2 \right)^{1/2}. \quad (4.9)$$

At this point we use of estimate (4.5) and, in particular, that

$$\sum_{k=1}^N \|s_\delta^k - s_\delta^{k-1}\|^2 \leq Ch, \quad \text{and} \quad \sum_{k=1}^N \int_\Omega |\nabla(u^k - u^{k-1})|^2 \leq C.$$



These can be used in estimating the boundary integrals. By the trace estimates, for any  $w \in H^1(\Omega)$  one has

$$\int_{\Gamma_{out}} |w|^2 \leq C_1 \int_{\Omega} |w|^2 + C_2 \left( \int_{\Omega} |w|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla w|^2 \right)^{\frac{1}{2}}$$

where the two constants  $C_{1,2} > 0$  depend on  $\Omega$  and  $\Gamma_{out}$ , but not on  $w$ . Using the inequality  $2|ab| \leq \frac{1}{\mu}a^2 + \mu b^2$  for all reals  $a, b$  and  $\mu > 0$ , one has

$$\int_{\Gamma_{out}} |w|^2 \leq Ch^{-\frac{1}{2}} \int_{\Omega} |w|^2 + h^{\frac{1}{2}} \int_{\Omega} |\nabla w|^2,$$

for some  $C > 0$ . Using the above estimates gives

$$\sum_{k=1}^N h \int_{\Gamma_{out}} |u_{\delta}^k - u_{\delta}^{k-1}|^2 \leq Ch^{\frac{3}{2}},$$

providing estimates for the last term in (4.9). In this way we get

$$\|(s_{\delta} - \hat{s}_{\delta})(\tilde{t})\|_{L^2(\Omega)}^2 \leq C (h^{1/2} + C_L^{\delta} h^{3/4}).$$

Recalling  $C_L^{\delta} = O(1/\delta)$ , this implies the error estimates stated in the theorem.  $\square$

**Remark 4.2.** *Theorem 4.1 estimates the  $L^2$  error for the Euler implicit discretization of the regularized outflow problem. By Theorem 3.3, along any sequence  $\delta \searrow 0$  the regularized solution  $s_{\delta}$  converges strongly in  $L^2$  to  $s$ , the solution of the outflow problem, in the sense of Definition 1. Passing simultaneously  $\delta$  and  $h$  to zero, the sequence  $\hat{s}_{\delta}^h$  converges to  $s$ . Following Remark 3.5, we expect an error estimate of the form*

$$\|(s - \hat{s}_{\delta}^h)\|_{L^{\infty}(0,T;L^2(\Omega))}^2 \leq C (\delta + h^{1/2} + \delta^{-1}h^{3/4}).$$

*Based on this, one can chose  $\delta$  in such a way that the error due to the regularization is in balance with the time discretization error. Specifically, this means that  $\delta$  and  $h^{3/4}/\delta$  are of the same order, yielding  $\delta = O(h^{3/8})$  and a similar approximation error. Other choices are also possible, but they lead to a lower order in the error estimate.*

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