SFB 823

Discussion Paper

A fluctuation test for constant correlation

Dominik Wied, Matthias Arnold

Nr. 3/2009

SFB 823

A fluctuation test for constant correlation

Dominik Wied* and Matthias Arnold

Institut für Wirtschafts- und Sozialstatistik, TU Dortmund, Dortmund, Germany

July 1, 2009

Abstract

We propose a new test for constant correlation. It bases on successively estimated correlations and compares these with the estimated correlation of the whole data set. In contrast to existing tests for this problem, our test does not require that possible change points are known or that there is normality in the data. To derive the asymptotic null distribution, we develop a generalized delta-method on function spaces. Here, the considered random function is not multiplied by a scalar, but by another function. To achieve this, we generalize the concept of Hadamard differentiability. We show analytically that the test has non-trivial power against local alternatives. A simulation study confirms our analytical findings.

Keywords. Structural break, Delta-method, Hadamard differentiability, Brownian motion, Diversification meltdown.

JEL numbers: C02, C12

*corresponding author: Dominik Wied, Institut für Wirtschafts- und Sozialstatistik, Fakultät Statistik, TU Dortmund, 44221 Dortmund, Germany, email: dominik.wied@tu-dortmund.de, Fon: +49/231/7555735, Fax: +49/231/7555284. The financial support by the Deutsche Forschungsgemeinschaft (SFB 823, Statistik nichtlinearer dynamischer Prozesse) is gratefully acknowledged.

1 Introduction

This paper presents a new fluctuation test for the null hypothesis of constant correlation in a bivariate sample. Many existing tests for this problem look for a change point in the correlation which is assumed to be known. The time series is separated into several parts and one assumes that the correlation is constant in these parts. Hence, one obtains different χ^2-tests, see Tang (1995), Jennrich (1970) and Pearson und Wilks (1933) for example.

Fischer (2007) uses another approach. He tests whether the correlations change according to special trigonometric functions. This approach is more flexible but the author does not calculate the (exact or asymptotic) distribution of his test statistics.

In this paper, we propose a fluctuation test for constant correlation with a test statistic whose asymptotic distribution we can calculate exactly. The basic idea is to reject the null hypothesis if the empirical correlations fluctuate too much. A comparable approach was used in Ploberger et al. (1989) or Sen (1980), albeit for the parameters in the linear regression model. Our assumptions are weaker than the assumptions proposed by Fischer (2007), Tang (1995), Jennrich (1970) or Pearson und Wilks (1933). We do not need that possible change points are known, we allow for m-dependence in the data and the variances need not be constant. Additionally, the test is non-parametric, especially the normality assumption is not needed.

The asymptotic null distribution is the distribution of the maximum of the absolute value of a one-dimensional Brownian bridge. To derive it, we extend the concept of the delta-method on function spaces to the case where the considered random function is not multiplied by a scalar but by another function. For this, we develop a generalization of Hadamard differentiability. To our knowledge, these generalizations have not been proposed in the literature before.

One possible field of application is econometrics. The question here is whether one can confirm the often discussed diversification meltdown - the fact that correlations are higher in bear markets than in bull markets - empirically. This question is very relevant for the portfolio theory basing on Markowitz (1952). Since the normality assumption is not needed, we can apply the test e.g. on data from a t-distribution which is very popular to model stock returns.

There is a broad literature concerning this problem. Longin und Solnik (1995) find empirical evidence for the diversification meltdown examining stock indices from seven countries for the period 1960 to 1990. Ragea (2003) gets a different result. He looks at daily returns for the period 1999 to 2002. Despite hectic on the markets (e.g. because of september 11th) the correlations seem to be constant. King et al. (1994) point out that changes in correlation between markets are driven mostly by movements in unobservable variables.

The next section presents the test statistic and its asymptotic null distribution. The third section analyzes the local power. The fourth section presents a simulation study and in the fifth section, we apply the test to stock returns. All proofs are deferred to the appendix.

2 The test statistic and its asymptotic null distribution

Let $(X_i, Y_i)', i \in \{1, \dots, T\}$, be bivariate random vectors with finite $(4+\alpha^*)$th moments for an $\alpha^* > 0$. We want to test whether the correlation between X_i and Y_i,

$$\rho_i = \frac{Cov(X_i, Y_i)}{\sqrt{Var(X_i)}\sqrt{Var(Y_i)}},$$

is constant for all i:

$$H_0 : \rho_i = \rho_0 \forall\ i \in \{1, \dots, T\} \text{ vs. } H_1 : \exists i \in \{1, \dots, T-1\} \text{ with } \rho_i \neq \rho_{i+1}$$

for a constant $\rho_0 \in (-1, 1)$. Let

$$\tau(z) = [2 + z(T-2)], z \in [0,1],\ \bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i,\ \bar{Y}_k = \frac{1}{k}\sum_{i=1}^{k} Y_i,$$

$$\overline{(X^2)}_k = \frac{1}{k}\sum_{i=1}^{k} X_i^2,\ \overline{(Y^2)}_k = \frac{1}{k}\sum_{i=1}^{k} Y_i^2,\ \overline{(XY)}_k = \frac{1}{k}\sum_{i=1}^{k} X_i Y_i,$$

$$\hat{\rho}_k = \frac{\sum_{i=1}^{k}(X_i - \bar{X}_k)(Y_i - \bar{Y}_k)}{\sqrt{\sum_{i=1}^{k}(X_i - \bar{X}_k)^2}\sqrt{\sum_{i=1}^{k}(Y_i - \bar{Y}_k)^2}}. \tag{1}$$

Expression (1) describes the empirical correlation coefficient, calculated from the first k observations. The test statistic is defined as

$$Q_T(X,Y) = c \max_{2 \le j \le T} \frac{j}{\sqrt{T}} |\hat{\rho}_j - \hat{\rho}_T|,$$

where c is a constant which is used to derive the asymptotic null distribution in explicit form. It is cumbersome to write down, for the case of independence it can be found in appendix A.1.

The test rejects the null hypothesis of constant correlation if the empirical correlations fluctuate too much. This fluctuation is expressed in the term $\max_{2 \le j \le T} |\hat{\rho}_j - \hat{\rho}_T|$. Because of the weighting factor $\frac{j}{\sqrt{T}}$, deviations at the beginning are tied down compared to deviations in the end. This compensates for the fact that $\hat{\rho}_j$ tends to fluctuate more at the beginning where it is calculated from fewer observations.

We impose the following assumptions:

(A1) The random variables X_i and Y_i, $i \in \{1, \ldots, T\}$, are defined on a common probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

(A2) $\mathbb{E}(X_i) = \mathbb{E}(Y_i) = 0 \; \forall \; i \in \{1, \ldots, T\}$.

(A3) Let

$$U_i := \begin{pmatrix} X_i^2 - \mathbb{E}(X_i^2) & Y_i^2 - \mathbb{E}(Y_i^2) & X_i & Y_i & X_iY_i - \mathbb{E}(X_iY_i) \end{pmatrix}'$$

and $S_j := \sum_{i=1}^{j} U_i$, then

$$\begin{aligned} \lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T} S_T S_T'\right) &= \lim_{\min(k,T)\to\infty} \mathbb{E}\left(\frac{1}{T}(S_{k+T} - S_k)(S_{k+T} - S_k)'\right) \\ &= \operatorname{plim}_{T\to\infty} \frac{1}{T} S_T S_T' =: D_1 >_L 0, \end{aligned}$$

where $>_L 0$ means that D_1 is positive definite.

(A4) The $(2+\alpha)$th moments of the components of U_i are uniformly bounded for an $\alpha > 0$, the quadratic components of U_i are uniformly integrable.

(A5) The random vectors (X_i, Y_i) and (X_{i+n}, Y_{i+n}) are independent for all i and $n > m$ for a number $m \in \mathbb{N}$, thus they are m-dependent.

(A6) Under H_0, for $f(i) \in \{\mathbb{E}(X_i^2), \mathbb{E}(Y_i^2), \mathbb{E}(X_iY_i)\}$ it holds $f(i) = c_f + d_{fi}$ with

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} d_{fi} = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{T} d_{fi}^2 = 0$$

for every $z \in [0,1]$ and

$$c_{\mathbb{E}(X_i^2)} =: \sigma_x^2,\ c_{\mathbb{E}(Y_i^2)} =: \sigma_y^2,\ c_{\mathbb{E}(X_iY_i)} =: \sigma_{xy}.$$

With this assumption,

$$\frac{1}{T}\sum_{i=1}^{T} f(i) - c_f = o(\frac{1}{\sqrt{T}}).$$

In addition, we assume that all these moments are uniformly bounded.

(A7) For $j \in \{0, \ldots, m\}$ and

$$\begin{aligned} g(i,j) \in \{&Cov(X_i^2, X_{i+j}^2), Cov(X_i^2, Y_{i+j}^2), Cov(X_i^2, X_{i+j}Y_{i+j}), \\ &Cov(Y_i^2, Y_{i+j}^2), Cov(Y_i^2, X_{i+j}Y_{i+j}), Cov(X_iY_i, X_{i+j}Y_{i+j})\} \end{aligned}$$

we have

$$\lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} g(i,j) = c_{gj}\ \forall j \in \{0, \ldots, m-1\}.$$

In addition, we assume that all these moments are uniformly bounded.

Because of assumption (A4), $\mathbb{E}(X_i^{4+\alpha^*}) < \infty,\ \mathbb{E}(Y_i^{4+\alpha^*}) < \infty\ \forall\ i \in \{1, \ldots, T\}$ for an $\alpha^* > 0$, there is no assumption on the fifth moments. Assumption (A6) does not restrict these moments to be asymptotically equal but that the moments may not fluctuate too much. The assumption can be weakened to

(A8) For a bounded function g that is not identically zero and that can be approximated by step functions,

$$\begin{aligned} \mathbb{E}(X_i^2) &= a_2 + a_2\frac{1}{\sqrt{T}}g\left(\frac{i}{T}\right) \\ \mathbb{E}(Y_i^2) &= a_3 + a_3\frac{1}{\sqrt{T}}g\left(\frac{i}{T}\right) \\ \mathbb{E}(X_iY_i) &= a_1 + a_1\frac{1}{\sqrt{T}}g\left(\frac{i}{T}\right). \end{aligned}$$

This a situation in which the variances fluctuate in a similar way; this is realistic for the modeling of stock returns. A typical example for the function g is

$$g(z) = \begin{cases} 0, & z \leq z_0, \\ \\ g_0, & z > z_0. \end{cases}$$

This assumption does not contradict the other ones except of (A6). It is violated because $Cov(X_i, Y_i) = \sigma_{xy} + d_i$ with

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} d_i = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{\tau(z)} g\left(\frac{i}{T}\right) = \int_0^z g(u)du \neq 0$$

for at least one $z \in [0, 1]$.

We rewrite the test statistic as $\sup_{0\leq z\leq 1} |K_T(z)|$ with

$$K_T(z) = c\frac{\tau(z)}{\sqrt{T}}(\hat{\rho}_{\tau(z)} - \hat{\rho}_T). \tag{2}$$

Our main result is

Theorem 2.1. *Under H_0 and assumptions (A1) - (A7) or (A1) - (A5), (A7) and (A8),*

$$\sup_{0\leq z\leq 1} |K_T(z)| \to_d \sup_{0\leq z\leq 1} |B(z)|.$$

Here, B is the one-dimensional Brownian bridge. The limit distribution is called Kolmogorov-Smirnov-(KS-)distribution and has a distribution function with an explicit functional form, see Billingsley (1968, p. 85). For the proof of theorem 2.1 which is given in appendix A.2, we consider different function spaces, either $D[\epsilon, 1]$ for $\epsilon \geq 0$, or a product space in which each component is either $D[\epsilon, 1]$ or $D^+[\epsilon, 1]$, the space of càdlàg-functions whose values are bounded away from zero. We always use the supremum norm together with the σ-field generated by the open balls, see Davidson (1994, p. 435), Gill (1989) or Pollard (1984, chapter 4).

3 Local power

In this section, we consider local alternatives of the form

$$\rho_{i,T} = \rho_0 + \frac{1}{\sqrt{T}} g\left(\frac{i}{T}\right) \quad (i \in \{1, \ldots, T\})$$

with constant variances, i.e. we introduce

(A9) For g as in (A8),

$$\mathbb{E}(X_i^2) = \sigma_x^2$$
$$\mathbb{E}(Y_i^2) = \sigma_y^2$$
$$\mathbb{E}(X_iY_i) = \sigma_{xy} + \frac{1}{\sqrt{T}} g\left(\frac{i}{T}\right).$$

All assumptions from section 2 except (A6) remain, especially assumption (A7) remains, maybe with other limits. We get

Theorem 3.1. *Under assumptions (A1) - (A5), (A7) and (A9),*

$$\sup_{z\in[0,1]} \left| c\frac{\tau(z)}{\sqrt{T}}(\hat{\rho}_{\tau(z)} - \rho_T) \right| \to_d \sup_{z\in[0,1]} |B(z) + C(z)|,$$

where $C(z)$ is a deterministic function.

The proof is in appendix A.4.

For local alternatives the supremum is now taken over the absolute value of a Brownian bridge plus a deterministic function $C(z)$. Its distribution is rather unwieldy, but we get a result for the local power for arbitrarily large g. For this, we rewrite assumption (A9) to $g(z) = Mh(z)$ for a function h and a constant M. It follows

Corollary 3.2. *Let $P_{H_1}(M)$ be the rejection probability for given M if the alternative is true. Let $\epsilon > 0$ and h arbitrary but not constant. Then there is a M_0 so that*

$$\lim_{T\to\infty} P_{H_1}(M) > 1 - \epsilon$$

for all $M > M_0$.

4 Some finite sample simulations

Next, we examine the finite sample null distribution for bivariate normal and bivariate t_5-distributions, each with constant variance $Var(X_i) = Var(Y_i) = 1$. We use a theoretical level of 0.05 and generate 5000 repetitions. The motivation for the t_5-distribution is that the t-distribution is popular to model stock returns and that we need finite $(4+\alpha^*)$th moments for the test.

The results are listed in table 1. In general, the test keeps the size; the lower $|\rho|$, the lower is the size. For the t_5-distribution, the test is rather conservative, especially for $\rho = 0$.

Table 1: Empirical level for the normal and t_5-distribution under the null hypothesis

T	$\rho = -0.9$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
	a) Normal distribution				
200	0.105	0.048	0.043	0.052	0.093
500	0.059	0.041	0.038	0.045	0.063
1000	0.052	0.048	0.043	0.050	0.052
	b) t_5-distribution				
200	0.095	0.053	0.031	0.049	0.093
500	0.062	0.040	0.034	0.044	0.059
1000	0.049	0.039	0.032	0.036	0.053

We compare the finite sample power for 5 different alternatives for bivariate t_5-distributions, each with constant variance $Var(X_i) = Var(Y_i) = 1$ and 5000 replications (table 2). Here, we assume that the covariance changes according to the principle in (A9).

(B1) $\rho_i = 0.5, i \leq \frac{T}{2}$, and $\rho_i = 0.7, i > \frac{T}{2}$,

(B2) $\rho_i = 0.5, i \leq \frac{T}{4}$, and $\rho_i = 0.7, i > \frac{T}{4}$,

(B3) $\rho_i = -0.5, i \leq \frac{T}{2}$, and $\rho_i = 0.5, i > \frac{T}{2}$,

(B4) $\rho_i = -0.5, i \leq \frac{T}{4}$, and $\rho_i = 0.5, i > \frac{T}{4}$,

(B5) $\rho_i = 0.5, i \leq \frac{T}{4}$, and $\rho_i = 0.7, \frac{T}{4} < i \leq \frac{3}{4}T$, and $\rho_i = 0.5, i > \frac{3}{4}$.

Table 2 shows that for higher T, the power increases for all alternatives.

Table 2: Empirical power of the test for 5 different alternatives and the t_5-distribution

Alternative	$T = 200$	$T = 500$	$T = 1000$
(B1)	0.310	0.565	0.815
(B2)	0.258	0.455	0.689
(B3)	0.971	0.995	0.999
(B4)	0.926	0.991	0.997
(B5)	0.113	0.197	0.382

The differences between the different alternatives are underlined with an analysis of the different functions C_1 from theorem 3.1. It holds

$$\text{(B1)}\ \ C_1(z) = \begin{cases} -\frac{1}{10}\sqrt{T}z, & z \le \frac{1}{2}, \\ \frac{1}{10}\sqrt{T}z - \frac{1}{10}\sqrt{T}, & z > \frac{1}{2}. \end{cases}$$

$$\text{(B2)}\ \ C_1(z) = \begin{cases} -\frac{3}{20}\sqrt{T}z, & z \le \frac{1}{4}, \\ \frac{1}{20}\sqrt{T}z - \frac{1}{20}\sqrt{T}, & z > \frac{1}{4}. \end{cases}$$

$$\text{(B3)}\ \ C_1(z) = \begin{cases} -\frac{1}{2}\sqrt{T}z, & z \le \frac{1}{2}, \\ \frac{1}{2}\sqrt{T}z - \frac{1}{2}\sqrt{T}, & z > \frac{1}{2}. \end{cases}$$

$$\text{(B4)}\ \ C_1(z) = \begin{cases} -\frac{3}{4}\sqrt{T}z, & z \le \frac{1}{4}, \\ \frac{1}{4}\sqrt{T}z - \frac{1}{4}\sqrt{T}, & z > \frac{1}{4}. \end{cases}$$

$$\text{(B5)}\ \ C_1(z) = \begin{cases} -\frac{1}{10}\sqrt{T}z, & z \le \frac{1}{4}, \\ \frac{1}{10}\sqrt{T}z - \frac{1}{20}\sqrt{T}, & \frac{1}{4} < z \le \frac{3}{4}, \\ -\frac{1}{10}\sqrt{T}z + \frac{1}{10}\sqrt{T}, & z > \frac{3}{4}. \end{cases}$$

The first four functions are triangle functions, the fifth one is a jagged function. The order of the absolute maximums is listed in table 3, we can see that this order corresponds to the order of the empirical power. This makes sense because a higher absolute maximum leads more likely to a rejection of the null hypothesis, see the shape in theorem 3.1. The fact that the maximums are multiples of $\sqrt{T}$ is reflected by the increase of the empirical power for higher T.

Table 3: Maximums of the power functions

Alternative	(B3)	(B4)	(B1)	(B2)	(B5)
Maximum	$\frac{1}{4}\sqrt{T}$	$\frac{3}{16}\sqrt{T}$	$\frac{1}{20}\sqrt{T}$	$\frac{3}{80}\sqrt{T}$	$\frac{1}{40}\sqrt{T}$

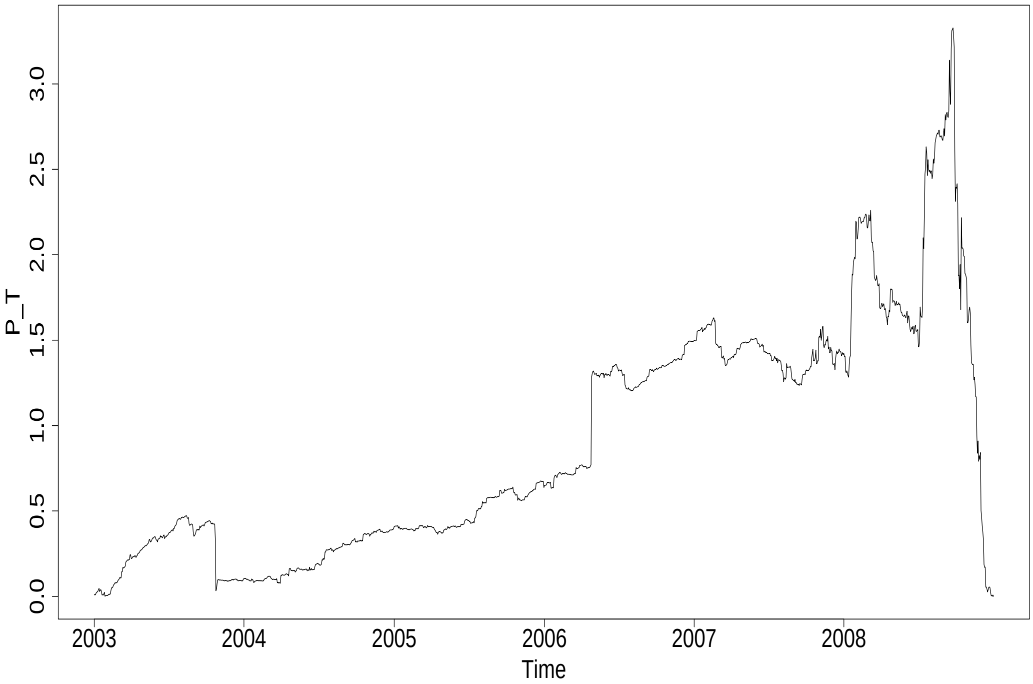
5 Application to stock returns

We apply the fluctuation test to daily returns of Bank of America and McDonalds stocks for the period 2003 to 2008. The data source is http://de.finance.yahoo.com/, the calculations were made with R, version 2.8.0, on an Intel Core 2 Duo machine. Following the diversification meltdown, the correlations should not be constant but become higher in the time of the financial crisis. We have $T = 1511$ and we suppose that all assumptions are fulfilled. To simplify the calculation, we assume the random vectors to be independent, hence we use the test statistic with the constant c_{iid}. The value of the test statistic is 3.328 and the p-value smaller than 0.001, thus the null hypothesis of constant correlation is clearly rejected. Figure 1 shows the behavior of

$$P_T(j) := c\frac{j}{\sqrt{T}}|\hat{\rho}_j - \hat{\rho}_T|$$

for all dates. The maximal value is taken on september 23rd, 2008, one week after september, 15th, the day on which Lehman Brothers announced insolvency. Since this day was the climax of the worldwide financial crisis up to now, we can conclude that in this time, the correlations changed structurally. Indeed, the estimated correlation in the time before september 23rd is 0.306 and 0.666 after it. With a test basing on known change points like Jennrich (1970), we would not have been able to detect the change point.

Figure 1: Behavior of P_T for all dates



A Appendix section

A.1 The constant from the test statistic

Lemma A.1. *If the random vectors (X_i, Y_i) are i.i.d., the standardizing factor c_{iid} is given by*

$$c_{iid} = \sqrt{\hat{F}_1 \hat{D}_{3,1} + \hat{F}_2 \hat{D}_{3,2} + \hat{F}_3 \hat{D}_{3,3}}$$

where

$$\begin{pmatrix} \hat{F}_1 & \hat{F}_2 & \hat{F}_3 \end{pmatrix} = \begin{pmatrix} \hat{D}_{3,1}\hat{E}_{11} + \hat{D}_{3,2}\hat{E}_{21} + \hat{D}_{3,3}\hat{E}_{31} \\ \hat{D}_{3,1}\hat{E}_{12} + \hat{D}_{3,2}\hat{E}_{22} + \hat{D}_{3,3}\hat{E}_{32} \\ \hat{D}_{3,1}\hat{E}_{13} + \hat{D}_{3,2}\hat{E}_{23} + \hat{D}_{3,3}\hat{E}_{33} \end{pmatrix}',$$

$$\hat{E}_{11} = \frac{1}{T}\sum_{i=1}^{T} X_i^4 - \left(\frac{1}{T}\sum_{i=1}^{T} X_i^2\right)^2,$$

$$\hat{E}_{12} = \hat{E}_{21} = \frac{1}{T}\sum_{i=1}^{T} X_i^2 Y_i^2 - \frac{1}{T}\sum_{i=1}^{T} X_i^2 \frac{1}{T}\sum_{i=1}^{T} Y_i^2,$$

$$\hat{E}_{13} = \hat{E}_{31} = \frac{1}{T}\sum_{i=1}^{T} X_i^3 Y_i - \frac{1}{T}\sum_{i=1}^{T} X_i^2 \frac{1}{T}\sum_{i=1}^{T} X_i Y_i,$$

$$\hat{E}_{22} = \frac{1}{T}\sum_{i=1}^{T} Y_i^4 - \left(\frac{1}{T}\sum_{i=1}^{T} Y_i^2\right)^2,$$

$$\hat{E}_{23} = \hat{E}_{32} = \frac{1}{T}\sum_{i=1}^{T} X_i Y_i^3 - \frac{1}{T}\sum_{i=1}^{T} Y_i^2 \frac{1}{T}\sum_{i=1}^{T} X_i Y_i,$$

$$\hat{E}_{33} = \frac{1}{T}\sum_{i=1}^{T} X_i^2 Y_i^2 - \left(\frac{1}{T}\sum_{i=1}^{T} X_i Y_i\right)^2,$$

$$\hat{D}_{3,1} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_y}\hat{\sigma}_x^{-3}, \hat{D}_{3,2} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x}\hat{\sigma}_y^{-3}, \hat{D}_{3,3} = \frac{1}{\hat{\sigma}_x\hat{\sigma}_y}$$

where

$$\hat{\sigma^2}_x = \overline{(X^2)}_T - (\bar{X}_T)^2, \hat{\sigma^2}_y = \overline{(Y^2)}_T - (\bar{Y}_T)^2, \hat{\sigma}_{xy} = \overline{(XY)}_T - \bar{X}_T\bar{Y}_T.$$

Proof. See the discussion before lemma A.4. □

A.2 Proof of theorem 2.1 with assumptions (A1) - (A7)

For the proof of theorem 2.1 with assumptions (A1) - (A7), we need several lemmas as auxiliary results. At first, we just consider the interval $[\epsilon, 1]$ for arbitrary $\epsilon > 0$. The first lemma is

Lemma A.2. *On* $D[\epsilon, 1]^5$,

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{\tau(\cdot)} \begin{pmatrix} X_i^2 - \sigma_x^2 \\ Y_i^2 - \sigma_y^2 \\ X_i \\ Y_i \\ X_iY_i - \sigma_{xy} \end{pmatrix} = \frac{\tau(\cdot)}{\sqrt{T}} \begin{pmatrix} \overline{(X^2)}_{\tau(\cdot)} & - & \sigma_x^2 \\ \overline{(Y^2)}_{\tau(\cdot)} & - & \sigma_y^2 \\ \bar{X}_{\tau(\cdot)} & & \\ \bar{Y}_{\tau(\cdot)} & & \\ \overline{(XY)}_{\tau(\cdot)} & - & \sigma_{xy} \end{pmatrix} =: U(\cdot) \to_d D_1^{\frac{1}{2}}\ W_5(\cdot),$$

where $W_5(\cdot)$ is a 5-dimensional Brownian motion and

$$D_1 = D_1' = \lim_{T\to\infty} \frac{1}{T} \sum_{j=1}^{T} \sum_{i=1}^{T}$$

$$\begin{pmatrix} Cov(X_i^2, X_j^2) & Cov(X_i^2, Y_j^2) & Cov(X_i^2, X_j) & Cov(X_i^2, Y_j) & Cov(X_i^2, X_jY_j) \\ \cdot & Cov(Y_i^2, Y_j^2) & Cov(Y_i^2, X_j) & Cov(Y_i^2, Y_j) & Cov(Y_i^2, X_jY_j) \\ \cdot & \cdot & Cov(X_i, X_j) & Cov(X_i, Y_j) & Cov(X_i, X_jY_j) \\ \cdot & \cdot & \cdot & Cov(Y_i, Y_j) & Cov(Y_i, X_jY_j) \\ \cdot & \cdot & \cdot & \cdot & Cov(X_iY_i, X_jY_j) \end{pmatrix}.$$

Proof.

$$\frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \begin{pmatrix} X_i^2 - \sigma_x^2 \\ Y_i^2 - \sigma_y^2 \\ X_i \\ Y_i \\ X_iY_i - \sigma_{xy} \end{pmatrix} = \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \begin{pmatrix} X_i^2 - \mathbb{E}(X_i^2) \\ Y_i^2 - \mathbb{E}(Y_i^2) \\ X_i \\ Y_i \\ X_iY_i - \mathbb{E}(X_iY_i) \end{pmatrix} + \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \begin{pmatrix} \mathbb{E}(X_i^2) - \sigma_x^2 \\ \mathbb{E}(Y_i^2) - \sigma_y^2 \\ 0 \\ 0 \\ \mathbb{E}(X_iY_i) - \sigma_{xy} \end{pmatrix}$$

$$=: A_1 + A_2.$$

Consider the first component of A_2:

$$A_{2,1}(z) = \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \left(\mathbb{E}(X_i^2) - \sigma_x^2\right).$$

With assumption (A6), $A_{2,1}$ converges to 0 for every fixed $z \in [\epsilon, 1]$. Because of $\tau(z) \geq T\epsilon - 2$, the convergence is uniform on $[\epsilon, 1]$, i.e. $A_{2,1}$ converges to 0 in probability in the supremum norm. Analoguely, all other components of A_2 converge to 0, hence A_2.

The sum in A_1 can be separated into one sum from $i = 1$ to $[Tz]$ called A_3 and one sum from $[Tz + 1]$ to $[Tz + (1 - z)2]$, called A_4. We show that A_4 converges in probability to the zero function in the supremum norm. For this, we first show that for fixed z, A_4 converges to 0 in probability. If

$$[Tz + (1 - z)2] < [Tz + 1],$$

A_4 is equal to 0. For

$$[Tz + (1-z)2] \geq [Tz + 1]$$

our argument builds on the Markov inequality. A_4 consists of two summands at most so that for the expectation of the first component A_{41},

$$\mathbb{E}\left(\left|\left(\frac{1}{\sqrt{T}} \sum_{i=[Tz+1]}^{[Tz+(1-z)2]} (X_i^2 - \mathbb{E}(X_i^2))\right)\right|\right) \leq \frac{1}{\sqrt{T}} 2 \sup_{i \in \mathbb{N}} \mathbb{E}(|X_i^2 - \mathbb{E}(X_i^2)|).$$

Since the second moment of the X_i is uniformly bounded (Assumption (A4)), $\sup_{i\in\mathbb{N}} \mathbb{E}(|X_i^2 - \mathbb{E}(X_i^2)|)$ is finite. Thus, the right hand side converges to 0 for $T \to \infty$. With the Markov inequality it holds for arbitrary $\epsilon > 0$

$$\mathbb{P}\left(\left|\left(\frac{1}{\sqrt{T}} \sum_{i=[Tz+1]}^{[Tz+(1-z)2]} (X_i^2 - \mathbb{E}(X_i^2))\right)\right| > \epsilon\right)$$
$$\leq \frac{1}{\epsilon} \frac{1}{\sqrt{T}} 2 \sup_{i \in \mathbb{N}} \mathbb{E}(|X_i^2 - \mathbb{E}(X_i^2)|) \to 0 \ (T \to \infty).$$

The same argument holds for the other components of A_4.

Consequently, all finite-dimensional distributions converge in probability and therefore in distribution to 0. We show the tightness of the process similarly to the method on page 138 in Billingsley (1968). At first, we show the tightness of every single component (exemplarily for the first one); with this, the tightness of the whole vector follows.

$$B := \mathbb{E}\left(|A_{41}(t) - A_{41}(t_1)|^{1+\frac{\alpha}{2}} \cdot |A_{41}(t_2) - A_{41}(t)|^{1+\frac{\alpha}{2}}\right) \leq \frac{1}{T^{1+\frac{\alpha}{2}}} 4C$$

for $\epsilon \leq t_1 \leq t \leq t_2 \leq 1$ and a constant C because of the uniform boundedness. If $[Tt_2] - [Tt_1] = 0$, then $B = 0$. If $[Tt_2] - [Tt_1] \geq 1$, we get

$$\frac{1}{T^{1+\frac{\alpha}{2}}} 4C \leq ([Tt_2] - [Tt_1])^{1+\frac{\alpha}{2}} \frac{1}{T^{1+\frac{\alpha}{2}}} 4C = 4C\left(\frac{[Tt_2] - [Tt_1]}{T}\right)^{1+\frac{\alpha}{2}}$$

and the condition of theorem 15.6 in Billingsley (1968) is fulfilled. Thus, A_4 converges as a process in distribution (and also in probability) to the zero function. On A_3, we apply the multivariate invariance principle from Philipps und Durlauf (1986, p. 475) which bases on a univariate invariance principle from McLeish (1975). With the continuous mapping theorem, CMT, see van der Vaart (1998, p. 259), the lemma follows. □

Lemma A.3. *On $D[\epsilon, 1]$,*

$$\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) \to_d D_3 D_2 D_1^{\frac{1}{2}}\ W_5(\cdot),$$

where

$$\rho_0^* = \frac{\sigma_{xy}}{\sigma_x \sigma_y},$$

$$D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$D_3 = \begin{pmatrix} -\frac{1}{2}\frac{\sigma_{xy}}{\sigma_y}\sigma_x^{-3} & -\frac{1}{2}\frac{\sigma_{xy}}{\sigma_x}\sigma_y^{-3} & \frac{1}{\sigma_x\sigma_y} \end{pmatrix}.$$

Proof. We apply the generalized delta-method that is described in appendix A.3 two times on $U(\cdot)$. At first, we have

$$f_1 : D[\epsilon, 1]^5 \to D[\epsilon, 1]^3$$

$$f_1(x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} x_1 - x_3^2 \\ x_2 - x_4^2 \\ x_5 - x_3 x_4 \end{pmatrix}$$

with the generalized Hadamard-differential for $\theta = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 \end{pmatrix}' \in D[\epsilon, 1]^5$,

$$f'_{1,\theta} : D[\epsilon, 1]^5 \to D[\epsilon, 1]^3,$$

$$f'_{1,\theta}(h) = f'_\theta(h_1, h_2, h_3, h_4, h_5) = \begin{pmatrix} 1 & 0 & -2\theta_3 & 0 & 0 \\ 0 & 1 & 0 & -2\theta_4 & 0 \\ 0 & 0 & -\theta_4 & -\theta_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix}.$$

Here, $M_T(z)$ (see appendix A.3) is

$$M_T(z) = \begin{pmatrix} \overline{(X^2)}_{\tau(z)} & \overline{(Y^2)}_{\tau(z)} & \bar{X}_{\tau(z)} & \bar{Y}_{\tau(z)} & \overline{(XY)}_{\tau(z)} \end{pmatrix}'.$$

Second, we have

$$f_2 : D^+[\epsilon,1]^2 \times D[\epsilon,1] \to D[\epsilon,1]$$
$$f_2(x_1,x_2,x_3) = \frac{x_3}{\sqrt{x_1 x_2}}$$

with the generalized Hadamard-differential for $\theta = \begin{pmatrix}\theta_1 & \theta_2 & \theta_3\end{pmatrix}' \in D^+[\epsilon,1]^2 \times D[\epsilon,1]$

$$f'_{2,\theta} : D^+[\epsilon,1]^2 \times D[\epsilon,1] \to D[\epsilon,1]$$

$$f'_{2,\theta}(h) = f'_\theta(h_1,h_2,h_3) = \begin{pmatrix}-\frac{1}{2}\frac{\theta_3}{\sqrt{\theta_2}}\theta_1^{-\frac{3}{2}} & -\frac{1}{2}\frac{\theta_3}{\sqrt{\theta_1}}\theta_2^{-\frac{3}{2}} & \frac{1}{\sqrt{\theta_1\theta_2}}\end{pmatrix} \cdot \begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}.$$

Here, $M_T(z)$ is

$$M_T(z) = \begin{pmatrix}\overline{(X^2)}_{\tau(z)} - (\bar{X}_{\tau(z)})^2 & \overline{(Y^2)}_{\tau(z)} - (\bar{Y}_{\tau(z)})^2 & \overline{(XY)}_{\tau(z)} - \bar{X}_{\tau(z)}\bar{Y}_{\tau(z)}\end{pmatrix}'.$$

□

Now, one can show that

$$(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) \to_d W_1(\cdot),$$

on $D[\epsilon,1]$, where W_1 is a one-dimensional Brownian motion. $(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}$ has to be estimated consistently. This number is a continuous composition of moments of X_i and Y_i that appear in the matrices D_3 and

$$E = \begin{pmatrix}E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{pmatrix} := D_2D_1D_2'$$
$$= \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\begin{pmatrix}Cov(X_i^2,X_j^2) & Cov(X_i^2,Y_j^2) & Cov(X_i^2,X_jY_j)\\ Cov(X_i^2,Y_j^2) & Cov(Y_i^2,Y_j^2) & Cov(Y_i^2,X_jY_j)\\ Cov(X_i^2,X_jY_j) & Cov(Y_i^2,X_jY_j) & Cov(X_iY_i,X_jY_j)\end{pmatrix}.$$

We show the estimation procedure for E_{12}.

$$E_{12} = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} Cov(X_i^2, Y_i^2) + 2\sum_{j=1}^{m} \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} Cov(X_i^2, Y_{i+j}^2).$$

Let j be fixed, w.l.o.g. 1. With assumption (A7),

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} Cov(X_i^2, Y_{i+1}^2) &= \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} (\mathbb{E}(X_i^2 Y_{i+1}^2) - \mathbb{E}(X_i^2)\mathbb{E}(Y_{i+1}^2)) \\
&= \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} \mathbb{E}(X_i^2 Y_{i+1}^2) - \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} \mathbb{E}(X_i^2)\mathbb{E}(Y_{i+1}^2) \\
&=: k_1 + k_2.
\end{aligned}$$

k_1 can be estimated consistently by $\hat{k}_1 = \frac{1}{T}\sum_{i=1}^{T-1} X_i^2 Y_{i+1}^2$, using a law of large numbers, see Davidson (1994, theorem 19.2). With assumption (A7) and the Cauchy-Schwarz-inequality for the last summand, k_2 is equal to

$$\begin{aligned}
&c_X c_Y + c_X \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} d_{Yi} + c_Y \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} d_{Xi} + \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} d_{Xi} d_{Yi} \\
&= c_X c_Y = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} \mathbb{E}(X_i^2) \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} \mathbb{E}(Y_i^2).
\end{aligned}$$

Analogue to k_1, a consistent estimator is $\hat{k}_2 = \frac{1}{T}\sum_{i=1}^{T} X_i^2 \; \frac{1}{T}\sum_{i=1}^{T-1} Y_i^2$. With the CMT, we get the estimator c (compare lemma A.1).

Now, we extend the convergence result to the interval $[0, 1]$.

Lemma A.4. *On* $D[0, 1]$,

$$W_T(\cdot) := c\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) \to_d W_1(\cdot).$$

Proof. We define the following functions:

$$W_T^{\epsilon}(z) = \begin{cases} W_T(z), & z \geq \epsilon \\ 0 & z < \epsilon \end{cases},$$

$$W^{\epsilon}(z) = \begin{cases} W_1(z), & z \geq \epsilon \\ 0 & z < \epsilon \end{cases}.$$

It holds with the previous lemmas

$$W_T^\epsilon(\cdot) \to_d W^\epsilon(\cdot)$$

for $T \to \infty$ on $D[0,1]$ and also

$$W^\epsilon(\cdot) \to_d W_1(\cdot)$$

for rational $\epsilon \to 0$ on $D[0,1]$.

The convergence of $W_T(\cdot)$ on $D[0,1]$ follows with theorem 4.2 in Billingsley (1968) if we can show that

$$\lim_{\epsilon\to 0} \limsup_{T\to\infty} \mathbb{P}(\sup_{z\in[0,1]} |W_T^\epsilon(z) - W_T(z)| \geq \eta) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \mathbb{P}(\sup_{z\in[0,\epsilon]} |W_T(z)| \geq \eta) = 0$$

for all $\eta > 0$. Now,

$$\begin{aligned}
&\sup_{z\in[0,\epsilon]} |W_T(z)| \\
&= \sup_{z\in[0,\epsilon]} \left| \frac{\frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} (X_i - \bar{X}_{\tau(z)})(Y_i - \bar{Y}_{\tau(z)}) - \frac{\rho_0}{\sqrt{T}} \sqrt{\frac{1}{\tau(z)} \sum_{i=1}^{\tau(z)} (X_i - \bar{X}_{\tau(z)})^2 \frac{1}{\tau(z)} \sum_{i=1}^{\tau(z)} (Y_i - \bar{Y}_{\tau(z)})^2}}{\sqrt{\frac{1}{\tau(z)} \sum_{i=1}^{\tau(z)} (X_i - \bar{X}_{\tau(z)})^2 \frac{1}{\tau(z)} \sum_{i=1}^{\tau(z)} (Y_i - \bar{Y}_{\tau(z)})^2}} \right| \\
&=: \sup_{z\in[0,\epsilon]} \left| \frac{D_1(z)}{D_2(z)} \right|.
\end{aligned}$$

By a strong law of large numbers, see Davidson (1994, theorem 19.5), and the CMT, D_2 goes to $\sigma_x\sigma_y$ almost surely for fixed $z > 0$ for $T \to \infty$. The same holds for $\bar{X}_T$ and $\bar{Y}_T$ with the limit 0. Let now $\delta > 0$ be arbitrary. By Egoroff's Theorem, see Davidson (1994, theorem 18.4), there is a set $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta) \geq 1 - \delta$ and a number $M(\delta) > 0$ so that $|D_1(z) - \sigma_x\sigma_y| < \delta$, $|\bar{X}_{\tau(z)}| < \delta$ and $|\bar{Y}_{\tau(z)}| < \delta$ on Ω_δ for $\tau(z) \geq M(\delta)$. Hence, for $z \geq \frac{M(\delta)}{T}$, for large enough T,

$$\sup_{z\in[\frac{M(\delta)}{T},\epsilon]} \left| \frac{1}{D_2(z)} \right| \leq \frac{1}{\sigma_x\sigma_y - \delta} < \infty.$$

Straightforward calculation yields

$$\sup_{z\in[\frac{M(\delta)}{T},\epsilon]} |D_1(z)| \leq C_1(\delta) \sup_{z\in[\frac{M(\delta)}{T},\epsilon]} D_3(z)$$

for a constant $C_1(\delta)$, where $D_3(z)$ is the sum of finitely many functions $D_3^i(z)$ with $\sup_{z\in[\frac{M(\delta)}{T},\epsilon]}|D_3^i(z)| \to_d \sup_{z\in[0,\epsilon]}|W_1(z)|$.

We have

$$\sup_{z\in[0,\frac{M(\delta)}{T}]}|W_T(z)| \leq \frac{C_2(\delta)}{\sqrt{T}}$$

for a constant $C_2(\delta)$; this goes to 0 for $T \to \infty$.

Since $W(0) = 0$ $\mathbb{P}$-almost everywhere, it holds

$$\lim_{\epsilon\to 0}\limsup_{T\to\infty}\mathbb{P}(\sup_{z\in[0,\epsilon]}|W_T(z)| \geq \eta) = 0$$

on Ω_δ. Since $\delta > 0$ was arbitrary, the lemma follows. □

Lemma A.5.

$$B_T(\cdot) := c\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_T) \to_d B(\cdot)$$

on $D[0,1]$, where $B(\cdot)$ is a one-dimensional Brownian bridge.

Proof. Define

$$W_T(\cdot) := c\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*)$$

and

$$B_T(z) = W_T(z) - \frac{\tau(z)}{T}W_T(1) =: h\left(W_T(z), \frac{\tau(z)}{T}\right).$$

Since $\frac{\tau(z)}{T}$ converges to z, the lemma follows with the CMT and the definition of the Brownian bridge. □

Applying the CMT another time proofs theorem 2.1.

A.3 Generalized Delta-method

Define

$$\mathbb{G}_1 := \mathbb{H}_1 \times \ldots \times \mathbb{H}_k(k\text{-times}, k \geq 1, \mathbb{H}_i \in \{D[\epsilon,1], D^+[\epsilon,1], \epsilon \geq 0\})$$

$$\mathbb{G}_2 := \mathbb{H}_1 \times \ldots \times \mathbb{H}_l(l\text{-times}, l \geq 1, \mathbb{H}_i \in \{D[\epsilon,1], D^+[\epsilon,1], \epsilon \geq 0\})$$

and the supremum norms corresponding to these spaces, $||\cdot||_{\mathbb{G}_1}$ and $||\cdot||_{\mathbb{G}_2}$.

Definition A.6 (Generalized Hadamard-differentiability)**.** *Let $\theta \in \mathbb{G}_1$. A function $f : \mathbb{G}_1 \to \mathbb{G}_2$ is generalized Hadamard-differentiable in θ if there exists a continuous, linear map $f'_\theta : \mathbb{G}_1 \to \mathbb{G}_2$ (the generalized Hadamard differential) so that*

$$\lim_{T\to\infty} \left\| \frac{f(\theta+r_T h_T)-f(\theta)}{r_T} - f'_\theta(h) \right\|_{\mathbb{G}_2} = 0$$

for all $r_T \in D[\epsilon, 1]$ with $r_T(z) \neq 0 \forall z \in [\epsilon, 1] \, \forall T$, $h_T, h \in \mathbb{G}_1$ with $||r_T||_{D[\epsilon,1]} \to 0$ and $||h_T - h||_{\mathbb{G}_1} \to 0$ so that $\theta + r_T h_T \in \mathbb{G}_1$ for all T.

Theorem A.7 (Generalized Delta-method)**.** *Let the assumptions of definition A.6 be fulfilled so that $f : \mathbb{G}_1 \to \mathbb{G}_2$ is generalized Hadamard-differentiable in θ. Let $M_T : \Omega \to \mathbb{G}_1$ be random functions so that*

$$r_T \cdot (M_T - \theta) \to_d M$$

as $T \to \infty$ for a sequence $r_T \in D[\epsilon, 1]$ with $||\frac{1}{r_T}||_{D[\epsilon,1]} \to 0, r_T(z) \neq 0 \ \forall z, \forall T$, and a random function M in $\mathbb{G}_1$. Then,

$$r_T \cdot (f(M_T) - f(\theta)) \to_d f'_\theta \, M$$

where f'_θ is the generalized Hadamard-differential of f at θ.

Proof. For each T, we define a function

$$g_T(h) = r_T \cdot \left(f(\theta + \frac{1}{r_T} h) - f(\theta) \right)$$

on $\mathbb{G}_T := \{h : \theta + \frac{1}{r_T} h \in \mathbb{G}_1\}$. Since f is generalized Hadamard-differentiable, it holds

$$\lim_{T\to\infty} ||g_T(h_T) - f'_\theta(h)||_{\mathbb{G}_2} = 0$$

for each sequence h_T with $||h_T - h||_{\mathbb{G}_1} \to 0$ and $h \in \mathbb{G}_1$. With the CMT, it follows

$$r_T \cdot (f(M_T) - f(\theta)) = g_T(r_T \cdot (M_T - \theta)) \to_d f'_\theta \, M.$$

∎

The main difference to the Delta-method from van der Vaart (1998, p. 297) is that here, r_T is an element from $D[\epsilon, 1]$ and not just a sequence of real numbers. Hence, we need the stronger assumption that r_T goes to 0 in the supremum norm on $D[\epsilon, 1]$. We need this in the proof of the asymptotic null distribution to separate $\hat{\rho}_{\tau(z)}$ and ρ_0^* - here, $\tau(z)$ cannot be written in the vector as a factor.

Straightforward calculation of definition A.6 gives us the Hadamard differentials used in lemma A.3. We make use of the fact that in these special cases the differentials are the same as they were for the analogue functions not applied on function spaces but on $\mathbb{R}^k$. During the calculation for f_2 used in lemma A.3, we have to ensure that an expression like $\frac{r_T}{\sqrt{\theta_1 \theta_2}}$ for functions θ_1 and θ_2 tends to 0 in the supremum norm. For this, it is necessary that the values of θ_1 and θ_2 are bounded away from 0, hence that they are in $D^+[\epsilon, 1]$. To make this clear, we distinguish between $D[\epsilon, 1]$ and $D^+[\epsilon, 1]$.

A.4 Proofs of the local power

Proof of theorem 3.1

Transferring the proof of lemma A.2, we obtain that $U(\cdot)$ converges to $D_1^{\frac{1}{2}}\ W_5(\cdot) + A$ with $A = \begin{pmatrix} 0 & 0 & 0 & 0 & \int_0^z g(u)du \end{pmatrix}'$. This lies in the fact that A_2 equals to

$$A_2 = \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{\sqrt{T}} g(\frac{i}{T}) \end{pmatrix}'.$$

The fifth component converges as a process to the deterministic function $\int_0^{\cdot} g(u)du$.

Also all other proofs can be transferred and it holds

$$\begin{aligned} \frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) &\rightarrow_d D_3 D_2 D_1^{\frac{1}{2}}\ W_5(\cdot) + D_3 D_2 A \\ &\stackrel{L}{=} (D_3 D_2 D_1 D_2' D_3')^{\frac{1}{2}} W_1(\cdot) + D_3 D_2 A \end{aligned}$$

and

$$(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)}-\rho_0^*) \rightarrow_d W_1(\cdot)+(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}D_3D_2A$$

$$\stackrel{L}{=} W_1(\cdot)+(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}\cdot\frac{\int_0^{\cdot} g(u)du}{\sigma_x\sigma_y}.$$

The constant c converges in probability to $(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}$. Thus,

$$c\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)}-\rho_T) \rightarrow_d B(\cdot)+C(\cdot),$$

where

$$C(z)=\frac{(D_3D_2D_1D_2'D_3')^{-\frac{1}{2}}}{\sigma_x\sigma_y}\left(\int_0^z g(u)du - z\int_0^1 g(u)du\right),$$

a deterministic function depending on z.

Proof of corollary 3.2

Analogously to the proof of theorem 3.1, it holds

$$\sup_{z\in[0,1]}\left|c\frac{\tau(z)}{\sqrt{T}}(\hat{\rho}_{\tau(z)}-\rho_T)\right| \rightarrow_d \sup_{z\in[0,1]}|B(z)+MC_1|$$
$$= M\sup_{z\in[0,1]}\left|\frac{B(z)}{M}+C_1\right|,$$

where $C_1 \neq 0$ for at least one z. Hence,

$$M\sup_{z\in[0,1]}\left|\frac{B(z)}{M}+C_1\right| \geq MC_2$$

for a constant C_2. Thus, the test statistic becomes arbitrarily large, especially larger than every quantile of the distribution under H_0.

It is necessary that h is not constant because the test statistic would equal to $\sup_{z\in[0,1]}|B(z)|$ otherwise. Since we integrate Mh from 0 to 1, asymptotically also late structural changes are detected if M is sufficiently large.

A.5 Proof of theorem 2.1 with assumptions (A1) - (A5), (A7) and (A8)

Analogously to A.4, we transfer the proof of lemma A.2 with

$$A_2 = \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \begin{pmatrix} a_2 \frac{1}{\sqrt{T}} g(\frac{i}{T}) & a_3 \frac{1}{\sqrt{T}} g(\frac{i}{T}) & 0 & 0 & a_1 \frac{1}{\sqrt{T}} g(\frac{i}{T}) \end{pmatrix}'.$$

Straightforward calculation yields that $C(z)$ then equals to 0.

References

P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.

J. Davidson. *Stochastic Limit Theory.* Oxford University Press, 1994.

M. Fischer. Are correlations constant over time? *Sonderforschungsbereich 649: "Okonomisches Risiko 12 (SFB 649 Papers)*, 2007.

R.D. Gill. Non- and semi-parametric maximum likelihood estimators and the von mises method - part 1. *Scandinavian Journal of Statistics*, 16:97–128, 1989.

R.I. Jennrich. An asymptotic chi-square test for the equality of two correlation matrices. *Journal of the American Statistical Association*, 65:904–912, 1970.

M. King, E. Sentana, und S. Wadhwani. Volatility and links between national stock markets. *Econometrica*, 62(4):901–933, 1994.

F. Longin und B. Solnik. Is the correlation in international equity returns constant: 1960-1990? *International Money and Finance*, 14(1):3–26, 1995.

H.M. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

D.L. McLeish. Invariance principles for dependent variables. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 32:165–178, 1975.

E.S. Pearson und S.S. Wilks. Methods of statistical analysis appropriate for k samples of two variables. *Biometrika*, 25:353–378, 1933.

P.C.B. Philipps und S.N. Durlauf. Multiple time series regression with integrated processes. *Review of Economic Studies*, 53:473–495, 1986.

W. Ploberger, W. Krämer, und K. Kontrus. A new test for structural stability in the linear regression model. *Journal of Econometrics*, 40:307–318, 1989.

D. Pollard. *Convergence of Stochastic Processes*. Series in Statistics, Springer, New York, 1984.

V. Ragea. Testing correlation stability during hectic financial markets. *Journal Financial Markets and Portfolio Management*, 17(3):289–308, 2003.

P.K. Sen. Asymptotic theory of some tests for a possible change in the regression slope occuring at an unknown time point. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 52:203–218, 1980.

G.Y.N. Tang. Intertemporal stability in international stock markets relationships: A revisit. *The Quarterly Review of Economics and Finance*, 35:579–593, 1995.

A.W. van der Vaart. *Asymptotic Statistics*. Cambridge University Press, Cambridge, 1998.