SFB 823

Some comments on goodness-of-fit tests for the parametric form of the copula based on L^2-distances

Axel Bücher, Holger Dette,

Discussion Paper

Nr. 6/2009

SFB 823

Some comments on goodness-of-fit tests for the parametric form of the copula based on L^2-distances

Axel Bücher
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
e-mail: axel.buecher@ruhr-uni-bochum.de

Holger Dette
Ruhr-Universität Bochum
Fakultät für Mathematik
44780 Bochum, Germany
email: holger.dette@ruhr-uni-bochum.de

6 July 2009

Abstract

In a recent paper Fermanian (2005) studied a goodness-of-fit test for the parametric form of a copula, which is based on an L^2-distance between a parametric and nonparametric estimate of the copula density. In the present paper we investigate the asymptotic properties of the proposed test statistic under fixed alternatives. We also study the impact of different estimates for the parameters of the finite dimensional family of copulas specified by the null hypothesis and illustrate the performance of a parametric bootstrap procedure for the approximation of the critical values.

Keywords and Phrases: goodness-of-fit tests, Copula, L^2-distance, parametric bootstrap
AMS Subject Classification: 62G; 62P

1 Introduction

Nowadays copulas are widely used by practitioners to analyze dependence structures in various applications including finance, actuarial science and hydrology [see e.g. Frees and Valdez (1998), Embrechts, McNeil and Straumann (2002), McNeil, Frey and Embrechts (2005) and Genest and Favre (2007)]. The copula of a multivariate distribution describes its dependence structure as a complement to the behaviour of its margins, and as a consequence the estimation of the distribution can be splitted into the estimation of the marginals and the copula. Several parametric families of copulas

$$\mathcal{C} = \{C_\theta \mid \theta \in \Theta\} \tag{1.1}$$

(here $\Theta \subset \mathbb{R}^q$ denotes an arbitrary subset) have been proposed in the literature in order to reflect various aspects of dependency [see e.g. the monographs of Joe (1997) or Nelsen (2006)]. Because misspecification of the copula function can have a serious impact on the statistical analysis, several authors have pointed out the importance of goodness-of-fit tests for the hypothesis of a parametric family (1.1) [see e.g. Malevergne and Sornette (2003), Cui and Sun (2004), Fermanian (2005), Scaillet (2007), Dobric and Schmid (2007), Genest and Rémillard (2008) among many others].
In a recent paper Fermanian (2005) proposed a test for the hypothesis

$$H_0 : C \in \mathcal{C} \, , \; H_1 : \; C \notin \mathcal{C} \tag{1.2}$$

where C denotes the copula of a d-dimensional distribution and $\mathcal{C}$ the parametric class defined by (1.1). The test is based on an L^2-distance between a nonparametric and parametric estimate of the copula density, and Fermanian (2005) proved asymptotic normality of the corresponding test statistic under the null hypothesis H_0 and specific assumptions on the estimates of the parameters of the copula density.
The present paper has several purposes. First we provide a more sophisticated analysis of the test proposed by Fermanian (2005) and study the asymptotic properties of the test statistic under fixed alternatives. It is shown that in this case an appropriately standardized version of the test statistic is also asymptotically normal distributed. Moreover, in contrast to the null hypothesis, it turns out that under the alternative the form of the parametric estimate has a substantial impact on the asymptotic distribution of the test statistic. In particular, we investigate two estimation methods for the parameter of the copula density, namely the common maximum likelihood and minimum L^2-distance estimation technique. For these estimates we prove asymptotic normality of the standardized test statistic under the null hypothesis and fixed alternatives with different rates of convergence in both cases. These results can be used for the construction of confidence regions for a measure of deviation, say M^2, between the parametric family of copulas and the "true" copula or for testing *precise hypotheses* of the form $H_0 : M^2 \leq \Delta$ vs. $H_1 : M^2 > \Delta$, where $\Delta \geq 0$ is a preassigned measure of accuracy [see Berger and Delampady (1987)]. These hypotheses are motivated by the observation that in practice a copula will never be exactly of a given parametric form (which would correspond to the case $\Delta = 0$), but in the best case approximately given by a parametric form (which would correspond to a small value of Δ).
Secondly it has been pointed out by several authors that under the null hypothesis the normal approximation of test statistics based on L^2-distances is not very accurate [see e.g. Härdle and Mammen (1993) or Fan and Linton (2003)]. Therefore we propose a parametric bootstrap procedure for the approximation of the critical values and investigate its performance by means of a simulation study. In particular, it is demonstrated that the bootstrap test based on the L^2-distance yields a reliable approximation of the nominal level and has similar power properties as several goodness-of-fit tests, which were investigated in a recent paper by Genest, Rémillard and Beaudoin (2008).
The remaining part of the paper is organized as follows. In Section 2 we introduce the necessary notation and define the test statistic. The asymptotic properties of the maximum likelihood and the minimum L^2-distance in the parametric family of copulas are investigated in Section 3 under

a correct and incorrect specification of the parametric family of copulas. In Section 4 we prove asymptotic normality of the test statistic under the null hypothesis and fixed alternatives with different rates of convergence in both cases. Section 5 is devoted to a small simulation study in order to investigate the finite sample properties of a parametric bootstrap procedure based on the L^2-distance. Finally, some more technical results are presented in the Appendix.

2 Testing for the form of the copula with the L^2-distance

Throughout this paper let $X_1, \ldots, X_n$ denote independent identically distributed d-dimensional random variables with joint continuous distribution function H and copula C, which has a density τ_0 supported in the cube $[0,1]^d$. We denote by $X_i = (X_{i,1}, \ldots, X_{i,d})^T$ the components of the vector X_i $(i = 1, \ldots, n)$ and by F_j the marginal distribution of the jth component $(j = 1, \ldots, d)$, which yields by Sklar's theorem [see e.g. Nelsen (2006)]

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \qquad \text{a.e.}\,. \tag{2.1}$$

We define by

$$F_{n,r}(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}\{X_{j,r} \leq x\} \tag{2.2}$$

the empirical distribution function of the r-th marginal distribution F_r $(r = 1, \ldots, d)$ and

$$Y_i := (F_1(X_{i,1}), \ldots, F_d(X_{i,d}))^{\mathrm{T}} \tag{2.3}$$

$$Y_{n,i} := (F_{n,1}(X_{i,1}), \ldots, F_{n,d}(X_{i,d}))^{\mathrm{T}}, \tag{2.4}$$

(note that the distribution function of the random variable Y_i is given by the copula C). We assume for the parametric class of copulas defined by (1.1) that the parameter space Θ is compact with non empty interior and that C_θ has a density supported in $[0,1]^d$, say $\tau(\cdot, \theta)$, which is two and three times continuously differentiable with respect to the first and second argument, respectively. In the following discussion $\hat{\theta}$ denotes an (under the null hypothesis consistent) estimate of the parameter θ, which will be specified in the following section, and we denote by $\hat{\tau}(\cdot) = \tau(\cdot, \hat{\theta})$ the corresponding parametric estimate of the copula density. Moreover, the nonparametric kernel estimate of the copula density is defined by

$$\tau_n(u) := \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{u - Y_{n,i}}{h}\right) \tag{2.5}$$

[see e.g. Charpentier, Fermanian and Scaillet (2007)] and $\omega : [0,1]^d \longrightarrow \mathbb{R}_0^+$ denotes a two times continuously differentiable weight function with compact support contained in the cube $[\varepsilon_1, 1 - \varepsilon_1]^d \subset [0,1]^d$, where $\varepsilon_1 > 0$. We further assume that $\tau(\cdot, \cdot)$ is uniformly continuous on

$[\varepsilon_0, 1-\varepsilon_0]^d \times \Theta$ for some $\varepsilon_0 \in (0, \varepsilon_1)$. For testing the parametric hypothesis (1.2) Fermanian (2005) proposed the test statistic

$$J_n = J_n(\widehat{\theta}) := \int (\tau_n - K_h * \widehat{\tau})^2(u)\, \omega(u)\, du. \tag{2.6}$$

where $*$ denotes convolution and $K_h(x) = K(x/h)/h$. Here K denotes the kernel of the density estimate (2.5), and the convolution operator is applied to the parametric estimate of the copula density in order to reduce the bias [see e.g. Bickel and Rosenblatt (1973) or Härdle and Mammen (1993)]. Because the asymptotic properties of the test statistic, especially under a fixed alternative, depend sensitively on the specific choice of the parametric estimate, we study in the following section two estimation methods for the parameter θ of the family of copula densities.

3 Estimation of the parameter of the copula density

Throughout this paper let τ_0 denote the "true" density of the copula C and consider the Kulback Leibler distance

$$D_{KL}(\tau_0, \tau(\cdot, \theta)) = \int \log(\tau_0(u))\tau_0(u)\, du - \int \log(\tau(u, \theta))\tau_0(u)\, du, \tag{3.1}$$

We define $L(\theta) = \int \log \tau(u, \theta)\tau_0(u)\, du$ and

$$\theta^*_{ML} = \underset{\theta \in \Theta}{\operatorname{argmax}}\, L(\theta) \tag{3.2}$$

as the parameter corresponding to the best approximation of the copula density τ_0 by the parametric class $\{\tau(\cdot, \theta) \mid \theta \in \Theta\}$ with respect to the Kulback Leibler distance and assume that θ^*_{ML} is attained at an unique interior point of Θ. The maximum likelihood estimate of the parameter θ is defined as

$$\widehat{\theta}_{ML} := \underset{\theta \in \Theta}{\operatorname{argmax}}\ L_n(\theta). \tag{3.3}$$

where $L_n(\theta) := \frac{1}{n} \sum_{i=1}^n \log \tau(Y_{n,i}, \theta)$ denotes the likelihood function [see Genest, Ghoudi and Rivest (1995)]. We assume that $\hat{\theta}_{ML}$ is also attained at an interior point of the parameter space Θ and that the parametric class of copula densities satisfies the following assumptions of regularity:

(a) $\mathbb{E}\left[\|\partial_\theta \log \tau(Y_i, \theta^*_{ML})\| + \left\|\partial^2_{y\theta} \log \tau(Y_i, \theta^*_{ML})\right\| + \left\|\partial^3_{yy\theta} \log \tau(Y_i, \theta^*_{ML})\right\|\right] < \infty$

(b) There exist constants α, $\beta > 0$, such that for any point Y^*_{ni} with $\|Y^*_{ni} - Y_i\| \le \|Y_{ni} - Y_i\|$

$$\left\|\partial^3_{yy\theta} \log(Y^*_{ni}, \theta^*_{ML})\right\| \le \alpha \left\|\partial^3_{yy\theta} \log \tau(Y_i, \theta^*_{ML})\right\| + \beta \left\|\partial^3_{yy\theta} \log \tau(Y_{ni}, \theta^*_{ML})\right\|$$

(c) For all $u \in (0,1)^d$ we have with the notation $r(t) := t(1-t)$

$$\left\|\partial^3_{yy\theta} \log \tau(u, \theta^*_{ML})\right\| \le const\, r(u_1)^{a_1} \dots r(u_d)^{a_d},$$

where $a_k = \frac{\delta - 1}{p_k}$ mit $\frac{1}{p_1} + \cdots + \frac{1}{p_d} = 1$ and $\delta > 0$.

(d) For all $u \in (0,1)^d$

$$\left\|\partial^2_{\theta\theta} \log \tau(u, \theta^*_{ML})\right\| \le const\, r(u_1)^{b_1} \dots r(u_d)^{b_d},$$

where $b_k = \frac{\zeta - 1}{q_k}$ and $\frac{1}{q_1} + \dots + \frac{1}{q_d} = 1$, $\zeta > 0$.

(e) For all $u \in (0,1)^d$ we have in a neighbourhood $V(\theta^*_{ML})$ of the point θ^*_{ML}

$$\sup_{\theta \in V(\theta^*_{ML})} \left\|\partial^3_{\theta\theta\theta} \log \tau(u, \theta)\right\| \le const\, r(u_1)^{c_1} \dots r(u_d)^{c_d},$$

where $c_k = \frac{\eta - 1}{p'_k}$ and $\frac{1}{p'_1} + \dots + \frac{1}{p'_d} = 1$, $\eta > 0$.

Fermanian (2005) states that these assumptions are satisfied for most of the commonly used copula families [see also Hu (1998), who proved some of these assumptions for Clayton-, Frank- and Gauß-copulas]. Our first result establishes a stochastic expansion for the maximum likelihood estimate, from which asymptotic normality can be derived.

Theorem 3.1. *If the assumptions stated in Section 2 and 3 are satisfied and the maximum likelihood estimate $\hat{\theta}_{ML}$ is consistent, i.e. $\hat{\theta}_{ML} \xrightarrow{P} \theta^*_{ML}$, then*

$$\sqrt{n}(\widehat{\theta}_{ML} - \theta^*_{ML}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} D(Y_i) + o_P(1) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)\,, \tag{3.4}$$

where

$$\begin{aligned} D(Y_i) &= A^{-1}(\partial_\theta \log \tau(Y_i, \theta^*_{ML}) + h(Y_i)), \\ h(Y_i) &= \int \partial^2_{y\theta} \log \tau(u, \theta^*_{ML})(\mathbb{1}(Y_i \le u) - u)\, \tau_0(u)\, du, \\ A &= -\mathbb{E}\left[\partial^2_\theta \log \tau(Y_i, \theta^*_{ML})\right], \\ \Sigma &= \mathrm{Var}(D(Y_i)). \end{aligned}$$

Proof. The proof follows by similar arguments as presented in part D of the paper by Fermanian (2005). Because this author only considers the null hypothesis and some of the arguments in this reference seem to be incorrect (see our Remark 3.2), we present here the main steps. Because $\hat{\theta}_{ML}$ is an interior point of the parameter space, we obtain by means of Taylor expansions

$$\begin{aligned} 0 \;=\; &\partial_\theta\, L_n(\widehat{\theta}_{ML}) = S_0 + S_1 + S_2 \\ &+ \partial^2_{\theta\theta}\, L_n(\theta^*_{ML})\, (\widehat{\theta}_{ML} - \theta^*_{ML}) + \frac{1}{2}(\widehat{\theta}_{ML} - \theta^*_{ML})^T \partial^3_{\theta\theta\theta} L_n(\tilde{\theta})\, (\widehat{\theta}_{ML} - \theta^*_{ML})\,, \end{aligned} \tag{3.5}$$

where the terms S_0, S_1, S_2 are defined by

$$S_0 \;=\; \frac{1}{n} \sum_{i=1}^{n} \partial_\theta\, \log \tau(Y_i, \theta^*_{ML}), \tag{3.6}$$

$$S_1 \;=\; \frac{1}{n} \sum_{i=1}^{n} \partial^2_{y\theta}\, \log \tau(Y_i, \theta^*_{ML})(Y_{n,i} - Y_i), \tag{3.7}$$

$$S_2 \;=\; \frac{1}{2n} \sum_{i=1}^{n} (Y_{n,i} - Y_i)^T\, \partial^3_{yy\theta}\, \log \tau(Y^*_{n,i}, \theta^*_{ML})\, (Y_{n,i} - Y_i), \tag{3.8}$$

and $\tilde{\theta}$ and Y_{ni}^* satisfy $\|\tilde{\theta} - \theta_{ML}^*\| \le \|\hat{\theta}_{ML} - \hat{\theta}_{ML}^*\|$ and $\left\|Y_{n,i}^* - Y_i\right\| \le \|Y_{n,i} - Y_i\|$ $(i = 1, \ldots, n)$, respectively. Note that $E[S_0] = 0$ and therefore S_0 is a sum of centered, independent and identically distributed random variables. Using a Hoeffding approximation it follows for the second term

$$S_1 = \frac{1}{n}\sum_{i=1}^{n} h(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right), \tag{3.9}$$

where the random variable $h(Y_i)$ is defined in Theorem 3.1. Finally, we have (using the assumptions (a) - (c)) that $S_2 = O_P(\frac{\log_2 n}{n}) = o_P(\frac{1}{\sqrt{n}})$. Combining these estimates we obtain

$$\partial_\theta\, L_n(\theta_{ML}^*) = S_0 + S_1 + S_2 = \frac{1}{n}\sum_{i=1}^{n} A \cdot D(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right), \tag{3.10}$$

where we have again used the notation of Theorem 3.1. Note that it follows from Proposition A.1 in Genest, Ghoudi and Rivest (1995) that $\partial_\theta^2\theta L_n(\theta_{ML}^*) \xrightarrow{P} -A$, which implies (observing that $\partial_{\theta\theta\theta}^3 L_n(\tilde{\theta})$ is bounded in probability, because of assumption (e) and $\tilde{\theta} \xrightarrow{P} \theta_{ML}^*$)

$$-\partial_\theta L_n(\theta_{ML}^*) = (-A + o_P(1))\,(\widehat{\theta}_{ML} - \theta_{ML}^*) \tag{3.11}$$

The assertion of the Theorem now follows from (3.5) and (3.10) by a standard argument. □

Remark 3.2.

(a) Note that Theorem 3.1 requires the consistency of the maximum likelihood estimate $\hat{\theta}_{ML} \xrightarrow{P} \theta_{ML}^*$, where θ_{ML}^* is the best approximation of the "true" copula density τ_0 by the parametric class $\{\tau(\cdot, \theta) | \theta \in \Theta\}$ with respect to the Kulback Leibler distance. If the parametric model has been correctly specified, this was proved in Theorem 4.2.1 of Hu (1998) and similar arguments could be used to establish consistency of $\hat{\theta}_{ML}$ if the model has been misspecified.

(b) It should be pointed out here that Fermanian (2005) considered estimates of the parameter θ satisfying

$$\hat{\theta} - \theta_0 = \frac{1}{n} A^{-1}(\theta_0) \sum_{i=1}^{n} B(\theta_0, Y_i) + o_p(n^{-1/2}(\log n)^{-1/2})$$

where θ_0 is the "true" parameter of the copula (this means that the model has been correctly specified). However, we were not able to find estimates in the literature satisfying this assumption (in particular the proof presented in Appendix D of Fermanian (2005) seems to be not correct).

In the remaining part of this section we derive a similar expansion for an estimate minimizing an L^2-distance between the "true" copula density and the parametric class of densities specified by the null hypothesis. To be precise we define

$$\theta_{L^2}^* := \underset{\theta \in \Theta}{\operatorname{argmin}} \int \big(\tau(u, \theta) - \tau_0(u)\big)^2 \omega(u)\, du \tag{3.12}$$

as the parameter corresponding to the best L^2-approximation of τ_0 by the parametric class $\{\tau(\cdot,\theta)|\theta \in \Theta\}$ and we assume that it is attained at an unique interior point of the parameter space Θ. If the model is correctly specified and θ_0 the true parameter corresponding to the density $\tau_0(\cdot) = \tau(\cdot,\theta_0)$, then $\theta^*_{L^2} = \theta_0$. The empirical analogue of (3.12) is given by

$$\widehat{\theta}_{L^2} = \underset{\theta\in\Theta}{\operatorname{argmin}} \int \big(\tau(u,\theta) - \tau_n(u)\big)^2 \omega(u)\, du \ , \tag{3.13}$$

where τ_n denotes the kernel estimate of the copula defined by (2.5). The following result provides a stochastic expansion for the difference $\hat{\theta}_{L^2} - \theta^*_{L^2}$.

Theorem 3.3. *If the L^2-estimate is consistent, i.e. $\hat{\theta}_{L^2} \xrightarrow{P} \theta^*_{L^2}$, and the conditions $h \longrightarrow 0$, $nh^d \longrightarrow \infty$,*

$$\frac{\log_2^2 n}{nh^{4+\frac{d}{2}}} \longrightarrow 0 \tag{3.14}$$

$$\frac{\log(h^{-d})}{nh^d} \rightarrow 0 \tag{3.15}$$

$$\frac{\log(h^{-d})}{\log_2 n} \rightarrow \infty \tag{3.16}$$

$$n^{1-\alpha}h^d \longrightarrow \infty \tag{3.17}$$

are satisfied for some $\alpha \in (0, \frac{3}{4})$, then it follows that

$$\widehat{\theta}_{L^2} - \theta^*_{L^2} + B \ = \frac{1}{n}\sum_{i=1}^n D_n(Y_i) \ + \ o_P(\frac{1}{\sqrt{n}}), \tag{3.18}$$

where the bias is given by

$$B := \int (K_h * \tau_0(u) - \tau_0(u)) C^{-1}\, \partial_\theta \tau(u,\theta^*_{L^2})\, \omega(u)\, du = O(h^2). \tag{3.19}$$

and

$$\delta(u) \ := \ \tau(u,\theta^*_{L^2}) - \tau_0(u) \tag{3.20}$$

$$C \ = \ \int \{\partial_\theta \tau(u,\theta^*_{L^2})\partial_\theta^T \tau(u,\theta^*_{L^2}) + \delta(u)\, \partial^2_{\theta\theta}\tau(u,\theta^*_{L^2})\}\, \omega(u)\, du. \tag{3.21}$$

$$D_n(Y_i) \ := \ C^{-1}\left[\int (K_h(u-Y_i) - \mathbb{E}\,[K_h(u-Y_i)])\, \partial_\theta \tau(u,\theta^*_{L^2})\, \omega(u)\, du + r_n(Y_i)\right] \tag{3.22}$$

$$r_n(Y_i) \ := \ \mathbb{E}\,[h_n(Y_k,Y_i)\,|\,Y_i] \tag{3.23}$$

$$h_n(Y_k,Y_i) \ := \ \frac{-1}{h}\int (dK)_h(u-Y_k)(\mathbb{1}(Y_i \le Y_k) - Y_k)\, \partial_\theta \tau(u,\theta^*_{L^2})\, \omega(u)\, du. \tag{3.24}$$

In particular, we have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n D_n(Y_i) \xrightarrow{\mathcal{D}} \mathcal{N}_q(0,\Sigma), \tag{3.25}$$

where the asymptotic variance is given by

$$
\begin{aligned}
(3.26) \qquad \Sigma \;=\; & C^{-1}\,\mathbb{E}\Big[\partial_\theta \tau(Y_i,\theta^*_{L^2})\,\partial_\theta^T \tau(Y_i,\theta^*_{L^2})\,\omega^2(Y_i) \\
& -\mathbb{E}\left[\partial_\theta\,\tau(Y_i,\theta^*_{L^2})\,\omega(Y_i)\right]\,\mathbb{E}\left[\partial_\theta^T\,\tau(Y_i,\theta^*_{L^2})\,\omega(Y_i)\right] \\
& -2\sum_{r=1}^{d}\int \partial_\theta \tau(Y_i,\theta^*_{L^2})\,\omega(Y_i)\,(\tau_0\,\partial_\theta^T \tau(\cdot,\theta^*_{L^2})\,\omega)(v_1,\ldots,Y_{ir}\ldots,v_d)\,d\underline{v}_{-r} \\
& +\sum_{r,s=1}^{d}\int (\tau_0\,\partial_\theta(\cdot,\theta^*_{L^2})\,\omega)(u_1,\ldots,Y_{ir},\ldots,u_d)(\tau_0\,\partial_\theta^T(\cdot,\theta^*_{L^2})\,\omega) \\
& (v_1,\ldots,Y_{is},\ldots,v_d)\,d\underline{u}_{-r}\,d\underline{v}_{-s}\Big]C^{-1}
\end{aligned}
$$

and the symbol $d\underline{u}_{-r}$ *means integration with respect to the* $(d-1)$*-dimensional variable* $\underline{u}_{-r} = (u_1,\ldots,u_{r-1},u_{r+1},\ldots,u_d)$.

Proof. With the notation

$$
\begin{aligned}
Q_n(\theta) &= -\int \big(\tau(u,\theta)-\tau_n(u)\big)^2\omega(u)\,du, \\
Q(\theta) &= -\int \big(\tau(u,\theta)-\tau_0(u)\big)^2\omega(u)\,du, \\
\psi_n(\theta) &= \partial_\theta Q_n(\theta) \\
\psi(\theta) &= \partial_\theta\, Q(\theta)
\end{aligned}
$$

we obtain for same $\tilde\theta$ with $\|\tilde\theta-\theta^*_{L^2}\| \le \|\hat\theta_{L^2}-\theta^*_L\|$ the expansion

$$
\begin{aligned}
0 \;=\; & \partial_\theta\, Q_n(\widehat\theta_{L^2}) = \partial_\theta\, Q_n(\theta^*_{L^2}) \\
(3.27) \qquad & +\partial_\theta^2 Q_n(\theta^*_{L^2})\,(\widehat\theta_{L^2}-\theta^*_{L^2}) + \frac{1}{2}(\widehat\theta_{L^2}-\theta^*_{L^2})^T\partial_\theta^3\,Q_n(\tilde\theta)\,(\widehat\theta_{L^2}-\theta^*_{L^2})\,.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(3.28) \qquad \partial_\theta\, Q_n(\theta^*_{L^2}) &= \frac{2}{n}\sum_{i=1}^{n}\int \big(K_h(u-Y_{n,i})-\tau(u,\theta^*_{L^2})\big)\partial_\theta\tau(u,\theta^*_{L^2})\,\omega(u)\,du \\
&= 2\Big(A_{n1}+A_{n2}+A_{n3}+A_{n4}\Big)\,,
\end{aligned}
$$

where the random variables A_{nj} $(j = 1, \dots, 4)$ are defined by

$$
\begin{aligned}
A_{n1} &= \frac{1}{n}\sum_{i=1}^{n}\int (K_h(u - Y_i) - (K_h * \tau_0)(u))\partial_\theta\,\tau(u, \theta^*_{L^2})\,\omega(u)\,du, \\
A_{n2} &= \frac{1}{n}\sum_{i=1}^{n}\int \beta_{ni}(u)\partial_\theta\,\tau(u, \theta^*_{L^2})\,\omega(u)\,du, \\
A_{n3} &= \frac{1}{n}\sum_{i=1}^{n}\int \gamma^*_{ni}(u)\partial_\theta\,\tau(u, \theta^*_{L^2})\,\omega(u)\,du, \\
A_{n4} &= \int (K_h * \tau_0(u) - \tau(u, \theta^*_{L^2}))\partial_\theta\,\tau(u, \theta^*_{L^2})\,\omega(u)\,du,
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_i(u) &:= K_h(u - Y_i), \\
\beta_{ni}(u) &:= \frac{-1}{h}(dK)_h(u - Y_i)(Y_{n,i} - Y_i), \\
\gamma^*_{ni}(u) :&= \frac{1}{2\,h^2}(d^2K)_h(u - Y^*_{n,i})(Y_{n,i} - Y_i)^{(2)}.
\end{aligned}
\tag{3.29}
$$

Here d^jK denotes the jth derivative of the kernel K (note that $K_h(u - Y_{ni}) = \alpha_i(u) + \beta_{ni}(u) + \gamma^*_{ni}(u)$). Observing the estimate $\|Y_{n,i} - Y_i\|_\infty = O_P\big((\frac{\log_2 n}{n})^{\frac{1}{2}}\big)$ it follows that $A_{n3} = O_p(\frac{\log_2 n}{nh^2}) = o_p(\frac{1}{\sqrt{n}})$. For the term A_{n2} we obtain by a straightforward but tedious calculation (see Appendix A)

$$
A_{n2} = \frac{1}{n}\sum_{i=1}^{n} r_n(Y_i) + o_p(\frac{1}{\sqrt{n}})\,, \tag{3.30}
$$

where we used the notation in (3.23). Finally, $A_{n4} = O(h^2)$ by a standard calculation, which yields

$$
\partial_\theta\,Q_n(\theta^*_{L^2}) - 2\int (K_h * \tau_0(u) - \tau_0(u))\,\partial_\theta\,\tau(u, \theta^*_{L^2})\,\omega(u)\,du = \frac{2}{n}\sum_{i=1}^{n} C\,D_n(Y_i) \;+\; o_P(\frac{1}{\sqrt{n}}), \tag{3.31}
$$

where the matrix C is defined in (3.21). Under the assumptions of Theorem 3.3 it can be shown that the kernel estimate of the copula density converges uniformly in probability to the "true" density τ_0 on $[\varepsilon, 1 - \varepsilon]^d$ (for any $\varepsilon > 0$), and it follows

$$
\begin{aligned}
\partial^2_\theta\,Q_n(\theta^*_{L^2}) &= 2\int \Big(\big(\tau_n(u) - \tau(u, \theta^*_{L^2})\big)\partial^2_{\theta\theta}\,\tau(u, \theta^*_{L^2}) - \partial_\theta\,\tau(u, \theta^*_{L^2})\partial^T_\theta\,\tau(u, \theta^*_{L^2})\Big)\omega(u)\,du \\
&\xrightarrow{P} -2\,C.
\end{aligned}
\tag{3.32}
$$

Observing (3.27) this gives

$$
\partial_\theta\,Q_n(\theta^*_{L^2}) \;=\; 2\,\big(C + o_P(1)\big)(\widehat{\theta}_{L^2} - \theta^*_{L^2}). \tag{3.33}
$$

and the assertion (3.18) follows because the matrix C is positive definite (note that $\theta^*_{L^2}$ minimizes the function $Q(\theta)$ and $\theta^*_{L^2}$ is an interior point of Θ).
Note that the dominating term of the right hand side of (3.18) is a sum of centered i.i.d. random variables and the asymptotic normality now follows from the central limit theorem for triangular arrays and the Cramér-Wold device. For the calculation of the asymptotic covariance we note that

$$\begin{aligned}\Sigma_n = \mathbb{E}\left[D_n(Y_i)D_n(Y_i)^T\right] &= C^{-1}\,\mathbb{E}\left[C\,D_n(Y_i)(C\,D_n(Y_i))^T\right]C^{-1} \\ &= C^{-1}\{E_D^{(1)} + 2E_D^{(2)} + E_D^{(3)}\}C^{-1}\,,\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}E_D^{(1)} &= \mathbb{E}\Big[\int (K_h(u-Y_i) - E\,K_h(u-Y_i))\,\partial_\theta\,\tau(u,\theta^*_{L^2}) \\ &\qquad (K_h(v-Y_i) - E\,K_h(v-Y_i))\,\partial_\theta^T\,\tau(v,\theta^*_{L^2})\,\omega(u)\,\omega(v)\,du\,dv\Big], \\ 2\,E_D^{(2)} &= \mathbb{E}\Big[\int (K_h(u-Y_i) - E\,K_h(u-Y_i))\,\partial_\theta\,\tau(u,\theta^*_{L^2})\omega(u)\,du\,r_n^T(Y_i)\Big], \\ E_D^{(3)} &= \mathbb{E}\left[r_n(Y_i)\,r_n^T(Y_i)\right].\end{aligned}$$

The assertion of the theorem now follows by a straightforward but tedious evaluation of the expressions $E_D^{(j)}$ $(j = 1,2,3)$ observing that

$$r_n(Y_i) = -\sum_{i=1}^{d}\int (\tau_0\nabla\tau(\cdot,\theta^*_{L^2})\,\omega)(u_1,\ldots,u_{r-1},Y_{ir},u_{r+1},\ldots,u_d)\,d\underline{u}_{-r}, + O(h)$$

□

Remark 3.4. If the condition

$$\|\partial_\theta Q(\theta)\mid_{\theta=\theta^*_{L^2}}\| < \inf\{\|\partial_\theta Q(\theta)\| \mid \theta\in\Theta;\|\theta-\theta^*_{L^2}\|\geq\varepsilon\}$$

is satisfied for all $\varepsilon > 0$, then it has been shown by Bücher (2008) (using Theorem 5.9 in van der Vaart (1998)) that under the bandwidth conditions stated in Theorem 3.3 the estimate $\hat\theta_{L^2}$ is consistent, that is $\hat\theta_{L^2} \xrightarrow{P} \theta^*_{L^2}$.

4 Weak convergence of the statistic $J_n(\hat\theta)$

In this section we study the asymptotic properties of the goodness-of-fit test for the parametric form of the copula which is based on the L^2-distance J_n with the estimator $\hat\theta_{\mathrm{ML}}$ or $\hat\theta_{\mathrm{L}^2}$. We begin with a statement of the asymptotic properties of the statistic $J_n(\hat\theta)$ under the null hypothesis of a correct specification of the copula family.

Theorem 4.1. *If the assumptions stated in Section 2 and 3 are satisfied and*

$$nh^d \to \infty, \frac{\log_2^2 n}{nh^{4+\frac{d}{2}}} \to 0$$

, *then*

$$nh^{\frac{d}{2}}\Big(J_n(\hat\theta_{ML}) - \frac{1}{nh^d}\int K^2(t)\,\tau_0(u-ht)\,\omega(u)\,dt\,du + \frac{1}{nh}\int \tau_0^2\omega\sum_{r=1}^{d}\int K_r^2\Big) \xrightarrow{\mathcal{D}_{\mathcal{H}_0}} \mathcal{N}(0,2\sigma^2), \tag{4.1}$$

where the asymptotic variance is given by

$$\sigma^2 = \int \tau_0^2\omega(u)\,du \int\Big(\int K(u)K(u+v)\,du\Big)^2 dv. \tag{4.2}$$

If additionally $nh^{4+d}\to 0$, $nh^{6+\frac{d}{2}}\to 0$, *then*

$$\begin{aligned} nh^{\frac{d}{2}}\Big(J_n(\hat\theta_{L^2}) &- \frac{1}{nh^{\frac{d}{2}}}\int K^2(t)\tau_0(u-ht)\,\omega(u)\,dt\,du + \frac{1}{nh}\int \tau_0^2\omega\sum_{r=1}^{d}\int K_r^2 \\ &- \int\big(\partial_\theta^T\tau(u,\theta_0)\,B\big)^2\omega(u)\,du\Big) \xrightarrow{\mathcal{D}_{\mathcal{H}_0}} \mathcal{N}(0,2\sigma^2), \end{aligned}$$

where σ^2 *is given as above and*

$$\begin{aligned} B &= \int (K_h*\tau_0(u)-\tau_0(u))C^{-1}\,\partial_\theta\,\tau(u,\theta_0)\,\omega(u)\,du = O(h^2), \\ C &= \int \partial_\theta\tau(u,\theta_0)\,\partial_\theta^T\tau(u,\theta_0)\,\omega(u)\,du. \end{aligned} \tag{4.3}$$

The proof of this result follows by similar arguments as given by Fermanian (2005), where some modifications are necessary, because the estimators $\hat\theta_{ML}$ and $\hat\theta_{L^2}$ do not satisfy the assumptions (3.1) of this paper. The details are omitted for the sake of brevity.
We now concentrate on the corresponding results under fixed alternatives.

Theorem 4.2. *If the assumptions of Theorem 4.1 are satisfied and the null hypothesis is not valid, that is* $\tau_0(\cdot)\neq\tau(\cdot,\theta)$ *for all* $\theta\in\Theta$, *then*

$$\sqrt{n}\Big(J_n(\hat\theta_{ML}) - b_1\Big) \xrightarrow{\mathcal{D}_{\mathcal{H}_1}} \mathcal{N}(0,\sigma^2_{\mathcal{H}_1}).$$

where the bias is given by

$$b_1 = \int (K_h*(\tau_0-\tau^*))^2(u)\,\omega(u)\,du.$$

The asymptotic variance is given by

$$\sigma^2_{\mathcal{H}_1} = 4\cdot\{\sigma_{11}+\sigma_{22}+\sigma_{33}+2\sigma_{12}-2\sigma_{13}-2\sigma_{23}\}$$

where

$$
\begin{aligned}
\sigma_{11} &:= \operatorname{Var}((\tau_0-\tau^*)\,\omega(Y_i)),\\
\sigma_{12} &:= -\,\mathbb{E}\Big[(\tau_0-\tau^*)(Y_i)\,\omega(Y_i)\sum_{r=1}^{d}\int(\tau_0-\tau^*)\tau_0\,\omega(u_1,\dots,Y_{ir},\dots,u_d)\,d\underline{u}_{-r}\Big],\\
\sigma_{13} &:= \beta_{ML}^T\,A^{-1}\mathbb{E}\Big[(\tau_0-\tau^*)\,\omega(Y_i)\Big(\partial_\theta\log\tau(Y_i,\theta^*_{ML})\\
&\qquad\qquad +\partial^2_{y\theta}\log\tau(Y_j,\theta^*_{ML})(\mathbb{1}(Y_i\le Y_j)-Y_j)\Big)\Big],\\
\sigma_{22} &:= \mathbb{E}\Big[\sum_{r,s=1}^{d}\int(\tau_0-\tau^*)\tau_0\,\omega(u_1,\dots,Y_{ir},\dots,u_d)\\
&\qquad\qquad (\tau_0-\tau^*)\tau_0\,\omega(v_1,\dots,Y_{is},\dots,v_d)\,d\underline{u}_{-r}\,d\underline{v}_{-s}\Big],\\
\sigma_{23} &:= \beta_{ML}^T\,A^{-1}\mathbb{E}\Big[\sum_{r=1^d}\int(\tau_0-\tau^*)\tau_0\,\omega(v_1,\cdots,Y_{ir},\cdots,v_d)\,d\underline{v}_{-r}\\
&\qquad\qquad \Big(\partial_\theta\log\tau(Y_i,\theta^*_{ML})+\partial^2_{y\theta}\log\tau(Y_j,\theta^*_{ML})(\mathbb{1}(Y_i\le Y_j)-Y_j)\Big)\Big],\\
\sigma_{33} &:= \beta_{ML}^T\,\operatorname{Var}(D(Y_i))\,\beta_{ML}
\end{aligned}
$$

and we have used the notation $\tau^*(u)=\tau(u,\theta^*_{ML})$,

$$\beta_{ML}=\int(\tau_0-\tau^*)\,\omega(u)\,\partial_\theta\,\tau(u,\theta^*_{ML})\,du.$$

Similarly, we have

$$\sqrt{n}\Big(J_n(\hat\theta_{L^2})-b_1-2\,b_2\Big)\xrightarrow{\mathcal{D}_{\mathcal{H}_1}}\mathcal{N}(0,\sigma^2_{\mathcal{H}_1}),$$

where

$$
\begin{aligned}
b_1 &= \int(K_h*(\tau_0-\tau^*))^2(u)\,\omega(u)\,du,\\
b_2 &= \int K_h(u-t)(\tau_0-\tau^*)(t)K_h(u-s)\,\partial_\theta^T\tau(s,\theta^*_{L^2})\,B\,\omega(u)\,du\,ds\,dt\;=O(h^3),
\end{aligned}
$$

$$\sigma^2_{\mathcal{H}_1}=4(\sigma_{11}+\sigma_{22}+2\sigma_{12}),$$

with

$$\begin{aligned}
\sigma_{11} &:= \mathrm{Var}((\tau_0 - \tau^*)\,\omega(Y_i)),\\
\sigma_{12} &:= -\,\mathbb{E}\Big[\sum_{r=1}^{d}\int (\tau_0-\tau^*)\tau_0\,\omega(u_1,\ldots,Y_{ir},\ldots,u_d)(\tau_0-\tau^*)(Y_i)\,\omega(Y_i)\,d\underline{u}_{-r}\Big],\\
\sigma_{22} &:= \mathbb{E}\Big[\sum_{r,s=1}^{d}\int (\tau_0-\tau^*)\tau_0\,\omega(u_1,\ldots,Y_{ir},\ldots,u_d)\\
&\qquad (\tau_0-\tau^*)\tau_0\,\omega(v_1,\ldots,Y_{is},\ldots,v_d)\,d\underline{u}_{-r}\,d\underline{v}_{-s}\Big].
\end{aligned}$$

Remark 4.3. Note that the choice $\omega(\cdot) = \tau(\cdot, \theta^*_{ML})^{-1}$ yields

$$\begin{aligned}
\beta_{ML} &= \int \tau_0(u)\frac{\partial_\theta\ \tau(u,\theta^*_{ML})}{\tau(u,\theta^*_{ML})}du - \int \partial_\theta\ \tau(u,\theta^*_{ML})du\\
&= \int \tau_0(u)\partial_\theta \log\tau(u,\theta)\ |_{\theta=\theta^*_{ML}}\ du
\end{aligned}$$

and the term β_{ML} in the first part of Theorem 4.2 vanishes. By a careful inspection of the proof of Theorem 4.2 it can be shown that for the choice $\omega(\cdot) = \tau(\cdot, \hat\theta_{ML})^{-1}$ the asymptotic variance of the statistic $\sqrt{n}(J_n(\hat\theta_{ML}) - b_1)$ simplifies substantially and is given by

$$\sigma^2_{\mathcal{H}_1} = 4(\sigma_{11} + 2\sigma_{12} + \sigma_{22})\ .$$

Proof of Theorem 4.2. We restrict ourselves to a proof of the first part, the corresponding result for the estimator $\hat\theta_{L^2}$ is derived similarly [see Bücher (2008)]. We use the decomposition

$$J_n(\hat\theta_{ML}) = W_{n1} + W_{n2} + W_{n3} + W_{n4} + W_{n5} + W_{n6}, \tag{4.4}$$

where the quantities W_{nj} are given by

$$\begin{aligned}
W_{n1} &= \int (\tau_n - K_h * \tau_0)^2(u)\,\omega(u)\,du,\\
W_{n2} &= \int (K_h * (\tau_0 - \tau^*))^2(u)\,\omega(u)\,du,\\
W_{n3} &= \int (K_h * (\tau^* - \widehat{\tau}))^2(u)\,\omega(u)\,du,\\
W_{n4} &= 2\int (\tau_n - K_h * \tau_0)(u)(K_h * (\tau_0 - \tau^*))(u)\,\omega(u)\,du,\\
W_{n5} &= 2\int (\tau_n - K_h * \tau_0)(u)(K_h * (\tau^* - \widehat{\tau}))(u)\,\omega(u)\,du,\\
W_{n6} &= 2\int (K_h * (\tau_0 - \tau^*))(u)(K_h * (\tau^* - \widehat{\tau}))(u)\,\omega(u)\,du,
\end{aligned}$$

$\tau^*(\cdot) = \tau(\cdot, \theta^*_{ML})$ denotes the best approximation of the "true" copula density τ_0 by the parametric family with respect to the Kulback Leibler distance and $\hat{\tau}(\theta) = \tau(\cdot, \hat{\theta}_{ML})$ its corresponding estimate. From the proof of Theorem 4.1 [see Fermanian (2005)] we obtain

$$W_{n1} = o_p(n^{-1/2}), \;\; W_{n3} = o_p(n^{-1/2}),$$

and the assertion of the theorem follows, if the weak convergence

$$\sqrt{n}\; (W_{n4} + W_{n5} + W_{n6}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{\mathcal{H}_1}) \tag{4.5}$$

can be established. In order to prove this result we investigate the term W_{n4}, W_{n5} and W_{n6} separately. Recalling the notation in (3.29) we obtain the decomposition

$$W_{n4} \;=\; W^{(1)}_{n4} + W^{(2)}_{n4} + W^{(3)}_{n4}\,,$$

where

$$W^{(1)}_{n4} \;=\; \frac{2}{n}\sum_{i=1}^{n} Z_i = \frac{2}{n}\sum_{i=1}^{n} (K_h * g_h)(Y_i) - \mathbb{E}\,[(K_h * g_h)(Y_i)], \tag{4.6}$$

$$W^{(2)}_{n4} \;=\; \frac{2}{n}\sum_{i=1}^{n} \int \beta_{ni} g_h(u)\, du, \tag{4.7}$$

$$W^{(3)}_{n4} \;=\; \frac{2}{n}\sum_{i=1}^{n} \int \gamma^*_{ni} g_h(u)\, du, \tag{4.8}$$

$g_h(u) = (K_h * (\tau_0 - \tau^*))(u)\omega(u)$ and equation (4.6) defines the random variables Z_i in an obvious manner. Obviously, the term $W^{(1)}_{n4}$ is a sum of i.i.d. random variables, and a straightforward but tedious calculation shows that $W^{(3)}_{n4} = o_p(n^{-1/2})$ (for this estimate we use the conditions on the bandwidth and the estimate $\|Y_{n,i} - Y_i\|^2 = O_p(\frac{\log_2 n}{n})$). For the remaining term we use a further decomposition

$$W^{(2)}_{n4} = \frac{2}{n}\sum_{i=1}^{n} \int \beta_{ni} g_h\, du = W^{(2,1)}_{n4} + W^{(2,2)}_{n4}, \tag{4.9}$$

where

$$W^{(2,1)}_{n4} \;=\; \frac{-2}{n^2 h}\sum_{i=1}^{n} \int (dK)_h(u - Y_i)(1 - Y_i) g_h(u)\, du \;=\; o_P\left(\frac{1}{\sqrt{n}}\right),$$

$$W^{(2,2)}_{n4} \;=\; \frac{2}{n^2}\sum_{k\neq i} \frac{-1}{h} \int (dK)_h(u - Y_i)(\mathbb{1}(Y_k \leq Y_i) - Y_i) g_h(u)\, du.$$

The second term can be approximated by

$$W^{(2,2)}_{n4} \;=\; \frac{2}{n}\sum_{i=1}^{n} s_n(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right), \tag{4.10}$$

where

$$s_n(Y_i) = \mathbb{E}\left[k_n(Y_k, Y_i) \,|\, Y_i\right] \tag{4.11}$$

and

$$k_n(Y_k, Y_i) := \frac{-1}{h} \int (dK)_h(u - Y_k)(\mathbb{1}(Y_i \leq Y_k) - Y_k) g_h(u)\, du.$$

[see Appendix B.] A standard calculation now shows that

$$s_n(Y_i) = -\sum_{r=1}^{d} \mathbb{E}\left[(\tau_0 - \tau^*)\tau_0\, \omega(Y_i)|Y_{ir}\right] + O(h),$$

which gives

$$W_{n4} = \frac{2}{n} \sum (Z_i + s_n(Y_i)) + o_p(n^{-1/2}), \tag{4.12}$$

where Z_i and $s_n(Y_i)$ are defined by (4.6) and (4.11), respectively. An application of the Cauchy-Schwarz inequality shows that

$$|W_{n5}| \;\leq\; 2\Big(W_{n1}\, W_{n3}\Big)^{\frac{1}{2}} = o_P\left(\frac{1}{\sqrt{n}}\right), \tag{4.13}$$

and for the remaining term W_{n6} we have

$$W_{n6} = -\frac{2}{n} \sum_{i=1}^{n} B(Y_i) \;+\; o_P\left(\frac{1}{\sqrt{n}}\right), \tag{4.14}$$

where $B(Y_i) := \beta_{ML}^T\, D(Y_i)$ as shown in Appendix C. Therefore we obtain from (4.5) that

$$\sqrt{n}\; (J_n(\hat{\theta}_{ML}) - b_1) = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} (Z_i + s_n(Y_i) - B(Y_i)) + o_p(1),$$

and the assertion of the theorem now follows from the central limit theorem and a straightforward but tedious calculation of $\mathrm{Var}(Z_i + s_n(Y_i) - B(Y_i))$. □

5 Simulation Study

In this section we study the performance of a parametric bootstrap goodness-of-fit test based on the statistic $J_n(\widehat{\theta}_{ML})$. From Theorem 4.1 we get, using the notation $\widehat{\tau}(\cdot) = \tau(\cdot, \widehat{\theta}_{ML})$, that

$$T_n := nh^{\frac{d}{2}} \frac{J_n - \frac{1}{nh^d} \int K^2(t)\widehat{\tau}(u - ht)\, \omega(u)\, dt\, du + \frac{1}{nh} \int \widehat{\tau}^2\omega \sum_{r=1}^{d} \int K_r^2}{\sqrt{2} \int \widehat{\tau}^2\omega(u)\, du \int \left(\int K(u)K(u+v)\, du\right)^2 dv}$$

converges weakly to the standard normal distribution. With $u_{1-\alpha}$ denoting the $(1-\alpha)$-quantile of the $\mathcal{N}(0,1)$-distribution we therefore obtain, by rejecting $\mathcal{H}_0$ for $T_n > u_{1-\alpha}$, an asymptotic level-α test. Since this normal approximation does not provide sufficiently exact critical values for small sample sizes [see e.g. Härdle and Mammen (1993) or Fan and Linton (2003)], we propose a parametric bootstrap procedure in order to approximate the critical values. For this purpose we proceed as follows.

In a first step compute the ML-estimate $\widehat{\theta}_{ML}$ and simulate for each $b = 1, \ldots, B$ with $B \in \mathbb{N}$ independent identically distributed random vectors $Y_1^{b*}, \ldots, Y_n^{b*}$ with distribution function $C_{\widehat{\theta}_{ML}}$. In a second step we calculate for every of the B samples the statistic

$$T_n^{b*} = T_n^{b*}(Y_1^{b*}, \ldots, Y_n^{b*})$$

and denote by

$$H_{n,b}^*(t) = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}\{T_n^{b*} \leq t\}$$

the empirical distribution function of $T_n^{1*}, \ldots, T_n^{B*}$. We determine the $(1-\alpha)$-quantile of this distribution and use it as a critical value for the goodness-of-fit test statistic $T_n = T_n(X_1, \ldots, X_n)$.

For our simulation study we use a similar setting as in Genest et. al. (2008). We choose a sample size of $n = 150$ and check the performance of the parametric bootstrap-procedure for two copula families, namely the Gauss- and the Clayton-Copula. The true copula is chosen from the Clayton-, Gauss-, Frank- or Gumbel-Family with parameter determined by the Kendall's-τ-coefficient taking values in $\{0.25, 0.5, 0.75\}$. We use $B = 100$ Bootstrap-Replications and make 100 replications of the whole procedure in order to estimate the power of the test. Considering the parameters emerging in the definition of T_n we choose

$$\begin{aligned} \omega &= \mathbb{1}_{[0.25, 0.975]} \\ h &= 0.7\, n^{-1/6} \\ K(u,v) &= (15/16)^2\, (1-u^2)^2 (1-v^2)^2 \mathbb{1}_{[-1,1]^2}(u,v) \end{aligned}$$

The results are presented in Table 1. From this table it can be seen that the level of the test is well approximated in almost all cases, but there appears an effect of underestimation for larger τ-coefficients.

Regarding the power of the test our results are quite comparable to other simulation studies considering the goodness-of-fit testing for copula families, see e.g. Genest et al. (2008). With stronger dependence, measured by the τ-coefficient, the test performs substantially better than in case of weak dependence. In comparison to the tests studied in Genest et al. (2008) it is remarkable that our bootstrap test outperforms all tests within that paper in case of the true copula family being Frank. In the other cases a comparison is more difficult. For example, if the copula under the null hypothesis is Gauss but the true copula is Gumbel, the L^2-test proposed in this paper yields similar results as most of the tests investigated by Genest et. al. (2008), but there exist also more powerful tests. On the other hand, if the true copula is Clayton, the bootstrap version of

the test proposed by Fermanian (2005) yields a power comparable to the best tests investigated by Genest et. al. (2008). A similar observation can be made if the copula under the null hypothesis is Clayton. In this case the L^2-test always yields a similar power as the best test considered by Genest. et.al. (2008).
For further conclusions and interpretations of the results, especially in comparison to other tests, we refer to the extensive simulation study in the paper of Genest et al. (2008).

6 Appendix

6.1 Proof of identity (3.30)

Using the notation

$$h^*_{nm}(Y_i, Y_k) := \frac{-1}{h} \int (dK)_h(u - Y_i)(\mathbb{1}(Y_k \le Y_i) - Y_i)\, \partial_{\theta_m} \tau(u, \theta^*_{L^2})\, \omega(u)\, du \tag{6.1}$$

(where ∂_{θ_m} denotes the derivative with respect to the m-th component of the vector θ, for $m = 1, \dots, q$) we obtain the decomposition

$$A_{n2m} = A^{(1)}_{n2m} + A^{(2)}_{n2m} \tag{6.2}$$

for the m-th component of the vector A_{n2}, where the random variables $A^{(i)}_{n2m}$ $(i = 1, 2)$ are defined by

$$A^{(1)}_{n2m} = \frac{-1}{n^2 h} \sum_{i=1}^{n} \int (dK)_h(u - Y_i)(1 - Y_i) \partial_{\theta_m} \tau(u, \theta^*_{L^2})\, \omega(u)\, du, \tag{6.3}$$

$$A^{(2)}_{n2m} = \frac{1}{n^2} \sum_{k \neq i} h^*_{nm}(Y_i, Y_k). \tag{6.4}$$

A straightforward standard calculation yields the estimate $A^{(1)}_{n2m} = o_P(n^{-\frac{1}{2}})$. The second term $A^{(2)}_{n2m}$ can be identified as a non-degenerate U-statistic

$$A^{(2)}_{n2m} = \frac{1}{n^2} \sum_{i<j} \tilde{h}_{nm} = \frac{n-1}{2n} \binom{n}{2}^{-1} U_n, \tag{6.5}$$

where $\tilde{h}_{nm}(Y_i, Y_k) = h^*_{nm}(Y_i, Y_k) + h^*_{nm}(Y_k, Y_i)$ denotes the symmetrized kernel of U_n. A straightforward but tedious calculation yields the estimate $\mathbb{E}\left[\tilde{h}^2_{nm}(Y_i, Y_k)\right] = O(h^{-2}) = o(n)$, so that the assumptions of Lemma 3.1 in Zheng (1996) are fulfilled. This result gives

$$\tilde{A}^{(2)}_{n2m} - A^{(2)}_{n2m} = o_P(n^{-\frac{1}{2}}), \tag{6.6}$$

where $\tilde{A}^{(2)}_{n2m}$ denotes the orthogonal projection $\tilde{A}^{(2)}_{n2m} = \frac{1}{n} \sum_{i=1}^{n} \tilde{r}_{nm}(Y_i)$ and $\tilde{r}_{nm}(Y_i) := \mathbb{E}\left[\tilde{h}_{nm}(Y_i, Y_k)|Y_i\right]$. Finally, observing that $\mathbb{E}\left[h^*_{nm}(Y_i, Y_k)|Y_i = y_i\right] = 0$, it follows that $r_{nm}(Y_i) = \tilde{r}_{nm}(Y_i)$ which yields the assertion in equation (3.30). □

Copula under $\mathcal{H}_0$	True Copula	τ-Coeff	0.15	0.1	0.05
Clayton	Clayton	0.25	0.14	0.1	0.08
		0.5	0.12	0.08	0.03
		0.75	0.08	0.05	0.01
	Frank	0.25	0.81	0.74	0.63
		0.5	1	1	0.98
		0.75	1	1	1
	Gauss	0.25	0.63	0.54	0.41
		0.5	0.96	0.95	0.91
		0.75	1	1	0.98
	Gumbel	0.25	0.89	0.83	0.73
		0.5	0.99	0.99	0.98
		0.75	1	1	1
Gauss	Clayton	0.25	0.4	0.31	0.21
		0.5	0.89	0.84	0.77
		0.75	0.99	0.99	0.97
	Frank	0.25	0.5	0.4	0.25
		0.5	0.9	0.83	0.73
		0.75	1	1	1
	Gauss	0.25	0.15	0.07	0.02
		0.5	0.12	0.1	0.04
		0.75	0.15	0.04	0.01
	Gumbel	0.25	0.2	0.18	0.14
		0.5	0.41	0.31	0.17
		0.75	0.37	0.26	0.16

Table 1: *Simulated rejection probabilities of the L^2-type test for various null hypotheses and alternatives. The sample size is $n = 150$ and $B = 100$ parametric Bootstrap-replications have been performed. The last three columns show the percentage of rejection of $\mathcal{H}_0$ at level $\alpha \in \{0.15, 0.1, 0.05\}$.*

6.2 Proof of identity (4.10)

The proof of equation (4.10) follows along the same lines as the one given in appendix A via application of Lemma 3.1 in Zheng (1996). Using the notation $\tilde{k}_n(Y_i, Y_k) := k_n^*(Y_i, Y_k) + k_n^*(Y_k, Y_i)$ for the symmetrizied kernel we obtain the following identification of $W_{n4}^{(2,2)}$ as a non-degenerate U-statistic:

$$W_{n4}^{(2,2)} = \frac{2}{n^2} \sum_{i<k} \tilde{k}_n(Y_i, Y_k) = \frac{n-1}{n} \binom{n}{2}^{-1} U_n. \tag{6.7}$$

A straightforward calculation shows $\mathbb{E}[\tilde{k}_n^2(Y_i, Y_k)] = O(h^{-2}) = o(n)$, and an application of Lemma 3.1 in Zheng (1996) yields

$$\widehat{W}_{n4}^{(2,2)} - W_{n4}^{(2,2)} = o_P(n^{-\frac{1}{2}}), \tag{6.8}$$

where $\widehat{W}_{n4}^{(2,2)}$ denotes the orthogonal projection $\widehat{W}_{n4}^{(2,2)} = \frac{2}{n} \sum_{i=1}^n \tilde{s}_n(Y_i)$ with $\tilde{s}_n(Y_i) = \mathbb{E}[\tilde{k}_n(Y_i, Y_k)|Y_i]$. Observing that $\mathbb{E}\left[k_n^*(Y_i, Y_k|Y_i = y_i\right] = 0$ we obtain $\tilde{s}_n(Y_i) = s_n(Y_i)$ and the assertion follows. □

6.3 Proof of identity (4.14)

By means of a Taylor expansion we obtain the decomposition

$$W_{n6} = W_{n6}^{(1)} + W_{n6}^{(2)}, \tag{6.9}$$

where the random variables $W_{n6}^{(i)}$ are defined by

$$\begin{aligned} W_{n6}^{(1)} &= -2 \int K_h(u-t)(\tau_0 - \tau^*)(t) K_h(u-s) \partial_\theta^T \tau(s, \theta_{ML}^*)(\widehat{\theta}_{ML} - \theta_{ML}^*)\, \omega(u)\, dt\, ds\, du, \\ W_{n6}^{(2)} &= - \int K_h(u-t)(\tau_0 - \tau^*)(t) K_h(u-s)(\widehat{\theta}_{ML} - \theta_{ML}^*)^T \partial_{\theta\theta}^2 \tau(s, \tilde{\theta}) \\ &\qquad (\widehat{\theta}_{ML} - \theta_{ML}^*)\, \omega(u)\, dt\, ds\, du, \end{aligned}$$

for some $\tilde{\theta}$ with $\|\tilde{\theta} - \theta_{ML}^*\| \leq \|\widehat{\theta}_{ML} - \theta_{ML}^*\|$. A straightforward calculation yields

$$W_{n6}^{(2)} = O_P\left(\left\|\widehat{\theta}_{ML} - \theta_{ML}^*\right\|^2\right) = O_P(n^{-1}) = o_P(n^{-\frac{1}{2}}).$$

Using identity (3.4) from Theorem 3.1 we obtain

$$W_{n6}^{(1)} = -\frac{2}{n} \sum_{i=1}^n \int K_h(u-t)(\tau_0 - \tau^*)(t) K_h(u-s) \partial_\theta^T \tau(s, \theta_{ML}^*)\, \omega(u)\, du\, dt\, ds\, D(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right).$$

Finally, considering expansions of $(\tau_0 - \tau^*)$ and ω, the dominating sum can be estimated by $\frac{-2}{n} \sum_{i=1}^n \beta_{ML}^T\, D(Y_i) + o_P(n^{-\frac{1}{2}})$, which yields the assertion in equation (4.14). □

Acknowledgments. The authors would also like to thank M. Stein, who typed parts of this paper with considerable technical expertise. Financial support by the Deutsche Forschungsgemeinschaft (SFB 823, Statistik nichtlinearer dynamischer Prozesse) is gratefully acknowledged. The work of H. Dette was supported in part by a NIH grant award IR01GM072876:01A1.

References

Berger, J.O., Delampady, M. (1987). Testing precise hypotheses. Statist. Sci. 2, 317-335.

Bickel, P. and Rosenblatt, M. (1973). On some global measures of the deviations of density function estimators. Ann. Statist. 1, 1071-1095.

Bücher, A. (2008). Validation von Copula-Modellen. Diplomarbeit, Ruhr-Universität Bochum, Germany.

Charpentier, A., Fermanian, J.-D. and Scaillet, O. (2007). The estimation of copulas: theory and practice. In Copulas: From theory to application in finance, edited by Rank, J., published by Risk Publications, London.

Cui, S. and Sun, Y. (2004). Checking for the gamma frailty distribution under the marginal proportional hazards frailty model. Statistica Sinica 14, 249-267.

Dobrić, J. and Schmid, F. (2007). A goodness of fit test for copulas based on Rosenblatt's transformation. Computational Statistics & Data Analysis 51, 4633-4642.

Embrechts, P., McNeil, A.J. and Straumann, D. (2002). Correlation and dependence in risk management: properties and pitfalls . In Risk management: value at risk and beyond, edited by Dempster M., published by Cambridge University Press, Cambridge.

Fan, Y. and Linton, O. (2003). Some higher order theory for a consistent nonparametric model specification test. Journal of Statistical Planning and Inference 109, no. 1-2, 125-154.

Fermanian, J.-D. (2005). Goodness-of-fit tests for copulas. Journal of Multivariate Analysis, 95, 119- 152.

Frees, E.W. and Valdez, E.A. (1998). Understanding relationships using copulas. North American Actuarial Journal 2, 1-25.

Genest, C. and Favre, A.-C. (2007). Everything you always wanted to know about copula modeling but were afraid to ask. Journal of Hydrologic Engineering 12, 347-368.

Genest C., Ghoudi, K. and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. Biometrika 82, 543-552.

Genest, C. and Rémillard, B. (2008). Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. Annales de l'Institut Henri Poincaré. Probabilités et Statistiques 44 (in press).

Genest, C., Rémillard, B. and Beaudoin, D. (2008). Goodness-of-fit tests for copulas: A review and a power study. Insurance: Mathematics and Economics, 42, in press.

Härdle, W. and Mammen, E. (1993). Testing parametric versus nonparametric regression, Ann. Statist. 21, 1926-1947.

Hu, H.L. (1998). Large sample theory of pseudo-maximum likelihood estimates in semiparametric models. PhD Thesis, University of Washington, Department of Statistics, Seattle.

Joe, H. (1997). Multivariate Models and Dependence Concepts. Chapman & Hall, London.

Klaassen, C.A.J. and Wellner, J.A. (1997). Efficient estimation in the bivariate normal copula model: normal margins are least favourable. Bernoulli 3, 55-77.

Malevergne, Y. and Sornette, D. (2003). Testing the Gaussian copula hypothesis for financial assets dependences. Quantitative Finance 3, 231-250.

McNeil, A.J., Frey, R. and Embrechts, P. (2005). Quantitative Risk Management. Princeton University Press, Princeton, N.J.

Nelson, R.B. (2006). An Introduction to Copulas, second ed. Springer, New York.

Scaillet, O. (2007). Kernel based goodness-of-fit tests for copulas with fixed smoothing parameters. Journal of Multivariate Analysis 98, 533-543.

Van der Vaart, A.W. (1998). Asymptotic Statistics. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge.

Zheng, J.X. (1996). A consistent test of functional form via nonparametric estimation techniques. Journal of Econometrics 75, 263-289.