Universität Dortmund
Fachbereich Informatik
Lehrstuhl für Software-Technologie

MEMO Nr. 134

Tracing Relations Probabilistically

Ernst-Erich Doberkat

März 2003

Internes Memorandum des
Lehrstuhls für Software-Technologie
Prof. Dr. Ernst-Erich Doberkat
Fachbereich Informatik
Universität Dortmund
Baroper Straße 301

D-44227 Dortmund

ISSN 0933-7725

Tracing Relations Probabilistically

Ernst-Erich Doberkat
Chair for Software Technology
University of Dortmund
doberkat@acm.org

February 27, 2003

Abstract

We investigate similarities between non-deterministic and probabilistic ways of describing a system in terms of computation trees. We first show that the construction of traces for both kinds of relations follow the same principles of construction (which could be described in terms of monads, but this does not happen here). Finally representations of measurable trees in terms of probabilistic relations are given.
Keywords: Probabilistic relations, specification techniques (nondeterministic, stochastic), representation theory.

1 Introduction

This paper investigates the relationship between trees and probabilistic relations. Trees arise in a natural way when the behavior of a system is specified through relations. Given a family $(R_n)_{n\in\mathbb{N}}$ of non-deterministic relations $R_n \subseteq X \times X$ over a state space X, R_n describes the behavior of the system at step n, so that the current state x_n may be followed by any state x_{n+1} with $\langle x_n, x_{n+1}\rangle \in R_n$. Rolling out the R_n yields a computation tree, the tree of traces that describe all possible paths through the system. If, on the other hand, a system is described through a sequence $(K_n)_{n\in\mathbb{N}}$ of probabilistic relations K_n, then computing the traces for K_n gives the probabilistic analogue of a computation tree. We are interested in the relationship between the non-deterministic and the probabilistic relations, seeing a probabilistic relation as a refinement of a non-deterministic one: whereas a non-deterministic relation R specifies for a state x through the set $R(x) := \{y|\langle x, y\rangle \in R\}$ all possible subsequent states, a probabilistic relation K attaches a weight $K(x)(dy)$ to each next state y (to be more precise: it describes for a set A the probability $K(x)(A)$ that y is a member of A). Introducing a relation $\models$ between R and K which says that $R \models K$ iff $K(x)$ assigns positive probability to exactly the members of $R(x)$, we see that $R \models K$ indicates K being a probabilistic refinement of R.
The problem discussed in this paper is, then, whether it is possible to find a probabilistic refinement for a given computation tree T. Thus we investigate the problem of finding for T a sequence $(K_n)_{n\in\mathbb{N}}$ of probabilistic relations such that after a computation history $w \in T$ and a given state x at time n the set $\{y|wxy \in T\}$ of all possible next states for this computation is exactly the set of states that are assigned positive probability by $K_n(x)$. If this is answered in the positive, then not only single steps in a non-deterministically specified computation can be refined stochastically but also whole traces arising from those specifications have a probabilistic refinement. This sheds further light on the relationship between stochastic and non-deterministic relations (compare [4, 6]).
In fact, it can be shown that under some not too restrictive conditions a computation tree has a probabilistic representation. The restrictions are topological in nature: we first show (Proposition 5) that a probabilistic representation can be established provided the set of all possible offsprings at any given time is compact. This condition is relaxed to the assumption that the state space is σ-compact, using a topological characterization of the body of a tree over the natural numbers (Proposition 6).

Overview We introduce in the next section computation trees and define their probabilistic counterparts. It is shown that a computation tree is spawned by the traces of a sequence of non-deterministic relations. This also works the other way around: each computation tree T exhibits a certain lack of memory in the actions it describes, thus it generates a sequence of relations for which T is just the corresponding tree. The probabilistic analogue is also studied: we show under which conditions the probabilistic counterparts of computation trees are spawned by probabilistic relations; it turns out that memoryless relations between X and the set X^∞ of all X-sequences characterize the situation completely. Section 3 introduces measurable trees as those class of trees for which a characterization is possible. It gives the mentioned representations, first for the compact, then for the σ-compact case. It turns out that the latter case is interesting in its own right because it requires studying trees over the natural numbers. The paper closes with a generalization of the following convexity result:

suppose that $R_i \models K_i$ for $i = 1, 2$, and put for $0 \leq p \leq 1$ the convex combination

$$K_1 \oplus_p K_2 : x \mapsto p \cdot K_1 + (1 - p) \cdot K_2,$$

then

$$R_1 \cup R_2 \models K_1 \oplus_p K_2,$$

indicating that a non-deterministic specification leaves much room for probabilistic refinements.

Related Work Non-deterministic relations have been studied extensively since the times of Ernst Schröder, [2] gives a good overview. Probabilistic relations have been introduced by Panangaden [14] drawing an analogy to non-deterministic ones through a Kleisli construction for two popular monads. These relations have been studied in the context of bisimulation and labelled Markov processes [3, 7, 5] and probabilistic testing [18]. The relationship between nondeterminism and probabilities is studied from different angles [15, 4, 6].

Acknowledgement The author is grateful to Georgios Lajios for his critical comments.

2 Computation Trees

Denote for a set V by V^* as usual the free semigroup on V with ϵ as the empty word; V^∞ is the set of all infinite sequences based on V. If $v \in V^*$, $w \in V^* \cup V^\infty$, then $v \preceq w$ iff v is an initial piece of w, in particular $\sigma |_k \preceq \sigma$ for all $\sigma \in V^\infty, k \in \mathbb{N}$, where $(s_n)_{n\in\mathbb{N}} |_k := s_0 \dots s_k$ is the prefix of $(s_n)_{n\in\mathbb{N}}$ of length $k + 1$. Denote for $M \subseteq V^*$ all words of length n by $\pi_n(M)$.
A *tree* T *on* V is a subset of V^* which is closed under the prefix operation, thus $w \in T$ and $v \preceq w$ together imply $v \in T$. The *body* $[T]$ *of* T [17] is the set of all sequences on V each finite prefix is in T, thus

$$[T] := \{\sigma \in V^\infty | \forall k \in \mathbb{N} : \sigma |_k \in T\}.$$

Clearly a finite tree like a binary search tree or a heap has an empty body.
Suppose we specify the n^{th} step in a process through relation $R_n \subseteq V \times V$. Execution spawns a tree by rendering explicit the different possibilities opening up for exploitation. Put $\mathcal{R} := (R_n)_{n\in\mathbb{N}}$ and

$$\mathsf{Tree}\,(\mathcal{R}) := \{v \in V^* | v_0 \in \mathsf{dom}(R_0), v_j \in R_{j-1}(v_{j-1}) \text{ for } 1 \leq j \leq |v|\} \cup \{\epsilon\},$$

then $\mathsf{Tree}\,(\mathcal{R})$ is a tree with body

$$[\mathsf{Tree}\,(\mathcal{R})] = \{\alpha \in V^\infty | \alpha_0 \in \mathsf{dom}(R_0) \;\&\; \forall j \geq 1 : \alpha_j \in R_{j-1}(\alpha_{j-1})\}.$$

This is the computation tree associated with $\mathcal{R}$; $\mathsf{Tree}\,(\mathcal{R})$ collects all finite, $[\mathsf{Tree}\,(\mathcal{R})]$ all infinite traces.
In fact, each tree T spawns a sequence of relations: Define

$$R_0^T := V^2 \cap T,$$

and inductively for $k \geq 1$

$$\begin{aligned} \langle x_k, x_{k+1}\rangle \in R_k^T \quad \Leftrightarrow \quad & \exists \langle x_0, x_1\rangle \in R_0^T \exists x_2 \in R_1^T(x_1) \dots \exists x_{k-1} \in R_{k-2}^T(x_{k-2}) : \\ & x_k \in R_{k-1}^T(x_{k-1}) \wedge x_0 x_1 \dots x_k x_{k+1} \in T. \end{aligned}$$

Example 1 *Let $T := \{\epsilon\} \cup H_{12}$, where H_k is the tree underlying a heap of size k, the nodes being the binary representations of the corresponding numbers. Then*

$$\begin{aligned} R_0^T &= \{\langle 1,0\rangle, \langle 1,1\rangle\}, \\ R_1^T &= \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}, \\ R_2^T &= R_1^T. \end{aligned}$$

We see that

$$\mathsf{Tree}\left(\left(R_n^T\right)_{n\in\mathbb{N}}\right) = \{\epsilon\} \cup H_{15}$$

holds, to that T is not generated from the relations R^T.

The trees T which may be represented through $\mathsf{Tree}\left(\left(R_n^T\right)_{n\in\mathbb{N}}\right)$ are of interest, they turn out to be memoryless in the sense that the behavior described by the tree at time $k+1$ depends directly only on the behavior at time k, once the initial input is provided.
Let for the sets X, Y be A as subset of some set X, and $f : X \to 2^Y$ be a set-valued map. Then define

$$A \otimes f := \{\langle a, b\rangle | a \in A, b \in f(a)\}$$

as the product of A and f. It is clear that each subset $M \subseteq X \times Y$ can be represented as a product $M = \pi_X[M] \otimes f_M$ with $f_M(a) := \{b \in Y | \langle a, b\rangle \in M\}$.
Using this product, we define memoryless trees through the following observation: $\pi_{n+1}(T)$ can be decomposed as a product

$$\pi_{n+1}(T) = \pi_n(T) \otimes J_n$$

with $J_n : X^n \to 2^X$. Thus the next letter x_n in a word $x_0 \dots x_n \in T$ is under this decomposition an element of $J_n(x_0, \dots, x_{n-1})$, and the tree being memoryless means that the latter set depends on x_{n-1} only. Thus J_n is induced by a map $f_n : X \to 2^X$ in the sense that

$$J_n(x_0, \dots, x_{n-1}) = f_n(x_{n-1})$$

holds for all $\langle x_0, \dots, x_{n-1}\rangle \in X^n$.

Definition 1 *A tree T over V is called* memoryless *iff for each $n \in \mathbb{N}$ with $n \geq 2$ the set*

$$\pi_{n+1}(T)$$

can be written as

$$\pi_n(T) \otimes J_n,$$

where $J_n : X^n \to 2^X$ is induced by a map $X \to 2^X$.

This means that only the length of the history and the initial input determines the behavior of a memoryless tree. Heaps, for example, are not always memoryless:

Example 2 *Let T be the tree according to Example 1, then using the notation of the decomposition above*

$$\begin{aligned} J_1(1) &= \{0,1\} \\ J_2(1,0) = J_2(11) &= \{0,1\} \\ J_3(1,0,0) = J_3(101) &= \{0,1\} \\ J_3(1,1,0) &= \{0\} \\ J_3(1,1,1) &= \emptyset. \end{aligned}$$

Clearly, $\{\epsilon\} \cup H_k$ is memoryless iff $k = 2^t - 1$ for some t.

It is not difficult to see that the tree $\mathsf{Tree}\left((R_n)_{n\in\mathbb{N}}\right)$ is memoryless, and that for a memoryless tree T with associated maps $J_n : X \to 2^X$ the equality

$$R^T_{n-1} = \{\langle x, y\rangle | x \in \mathsf{dom}(J_n), y \in J_n(x)\}$$

for all $n \geq 2$ holds.

Proposition 1 *Let T be a tree over V. Then the following conditions are equivalent:*

1. *The sequence $\mathcal{R} = \left(R^T_n\right)_{n\in\mathbb{N}}$ of relations $R^T_n \subseteq V \times V$ defined through T has the property that $T = \mathsf{Tree}\left(\mathcal{R}\right)$ holds.*
2. *T is memoryless.*

Proof Both implications are established through inductions on the length of words. ■
In the remainder of the paper we will not distinguish relations from the associated set valued maps. We will see soon (Prop. 2) that a similar notion will be helpful to characterize the probabilistic analogue of trees.
Turning to the stochastic side of the game, we denote for a measurable space X by $\mathbf{P}(X)$ the set of a probability measures on (the σ-algebra of) X. We usually omit mentioning the σ-algebra underlying a measurable space and talk about its members as *measurable subsets*, or as *Borel subsets*, if X is a metric space, see below. $\mathbf{P}(X)$ is endowed with the $*-\sigma$-algebra, i.e. the smallest σ-algebra that makes for each measurable subset A of X the evaluation map $\mu \mapsto \mu(A)$ measurable.
Define for the measurable map $f : X \to Y$ and for $\mu \in \mathbf{P}(X)$ the *image* of μ under f by

$$\mathbf{P}(f)(\mu)(B) := \mu(f^{-1}[B]),$$

then $\mathbf{P}(f)(\mu) \in \mathbf{P}(Y)$. It is easily established that $\mathbf{P}(f) : \mathbf{P}(X) \to \mathbf{P}(Y)$ is measurable. This makes $\mathbf{P}$ a functor on the category of measurable spaces with measurable maps as morphisms; in fact, it is the functorial part of monad investigated by Giry [9].
A *probabilistic relation $K : X \rightsquigarrow Y$ between the measurable spaces X and Y* [1, 14, 6, 4] is a measurable map $K : X \to \mathbf{P}(Y)$, consequently it has these properties:

1. for all $x \in X$, $K(x)$ is a probability measure on Y,
2. for all $B \subseteq Y$ which are measurable, $x \mapsto K(x)(B)$ is a measurable map on X, where measurability of real functions always refers to the Borel sets in $\mathbb{R}$.

Sometimes a probabilistic relation K is called a *Markov relation.* When general probabilistic relations are used to model computations, Markov relations model terminating computations (so that for a non-Markovian relation K, for which $K(x)$ is a subprobability measure, the difference $1 - K(x)(Y)$ may be interpreted as the amount of nontermination on input x, since this is the probability for "no state at all" [13]).
Probabilistic relations may be composed similar to set theoretic ones: let $K : X \rightsquigarrow Y$ and $L : Y \rightsquigarrow Z$ be probabilistic relations, then define for $x \in X$ and the measurable subset $C \subseteq Z$ the *(ordinary) product of* K *and* L by

$$(K \odot L)(x)(C) := \int_Y L(y)(C)\ K(x)(dy),$$

thus

$$K \odot L : X \rightsquigarrow Z.$$

This is just the Kleisli product in Giry's monad, giving further support for the analogy we are exploring: The composition of relations is the Kleisli product in the well known monad the functorial part of which is the powerset functor [12, Ex. VI.2.1]
If $\mu \in \mathbf{P}(X), K : X \rightsquigarrow Y$, define for the measurable subset $A \subseteq X \times Y$

$$(\mu \otimes K)(A) := \int_X K(x)(A_x)\ \mu(dx)$$

with

$$A_x := \{y \in Y | \langle x, y\rangle \in A\}.$$

Consequently, $\mu \otimes K \in \mathbf{P}(X \times Y)$.
A Polish space X is a completely metrizable separable topological space; as usual, we take the Borel sets as the σ-algebra on a Polish space. If X is Polish, so are [16]

- $\mathbf{P}(X)$ under the topology of weak convergence; the $* - \sigma$-algebra coincides with the Borel sets,
- X^n under the product topology for $n \in \mathbb{N}$, and X^* under the topological sum of $(X^n)_{n\in\mathbb{N}}$,
- X^∞ under the topological product; the Borel sets are the σ-algebra generated by sets of the form $\prod_{n\in\mathbb{N}} A_n$, where all $A_n \subseteq X$ are Borel sets, and all but a finite number equal X.

The topology of weak convergence on $\mathbf{P}(X)$ is the smallest topology for which the evaluation maps $\mu \mapsto \int_X f\ d\mu$ are continuous for every bounded and continuous function $f : X \to \mathbb{R}$. If Q is another topological space, then by the celebrated Portmanteau Theorem [16, Theorem II.6.1] a map $g : Q \to \mathbf{P}(X)$ is continuous for this topology iff the set $\{q \in Q | g(q)(U) > 0\}$ is open for each open subset U of X.
An important example of a Polish space is furnished by the *Baire space* $\mathcal{N} := \mathbb{N}^\infty$, where the natural numbers $\mathbb{N}$ have the discrete topology, so that each subset of $\mathbb{N}$ is open; $\mathcal{N}$ carries the product topology. This space is interesting since it is the prototypical Polish space in the following sense [17, Theorem 2.6.9]: Every Polish space is a one-to-one and continuous image of a closed subset of $\mathcal{N}$. A base for the topology on $\mathcal{N}$ consists of sets of the form $A \times \prod_{k>n} \mathbb{N}$, where $A \subseteq \mathbb{N}^n, n \in \mathbb{N}$. Closed subsets of $\mathcal{N}$ may be characterized in terms of trees:

Lemma 1 *A set $D \subseteq \mathcal{N}$ is closed iff $D = [T]$ for some tree T over $\mathbb{N}$. The body $[T]$ of a tree T over $\mathbb{N}$ is a Polish space.*

Proof The first assertion follows from [17, Prop. 2.2.13]. Since closed subsets of Polish spaces are Polish again in their relative topology, the second part is established. ■

One immediate consequence of working in a Polish space is that *disintegration of measures* is possible: Suppose $\mu \in \mathbf{P}(X_1 \times X_2)$ is a probability measure on the product of the Polish spaces X_1 and X_2. Then there exists a probability μ_1 on X_1 and a probabilistic relation $K : X_1 \rightsquigarrow X_2$ such that $\mu = \mu_1 \otimes K$ holds.

Fix for the rest of the paper X as a Polish space. The product X^n is always equipped with the product topology, the free monoid X^* has always the topological sum, and $\mathbf{P}(X)$ always the topology of weak convergence as the respective topologies.

Now suppose that a sequence $\mathcal{K} := (K_n)_{n\in\mathbb{N}}$ of probabilistic relations $K_n : X \rightsquigarrow X$ is given. Define inductively a sequence $K_0^n : X \rightsquigarrow X^{n+1}$ by setting $K_0^0 := K_0$, and for $x \in X, A \subseteq X^{n+2}$ measurable

$$K_0^{n+1}(x)(A) := \int_{X^{n+1}} K_{n+1}(x_n)(\{x_{n+1} | \langle x_0, \dots, x_{n+1}\rangle \in A\})\ K_0^n(x)(d\langle x_0, \dots x_n\rangle).$$

Let K_n specify probabilistically the n^{th} state transition of a system, then $K_0^n(x)(A)$ gives the probability that the sequence $\langle x_1, \dots x_n\rangle$ is an element of A, provided the system was initially in state x.

It is not difficult to see that the sequence $(K_0^n)_{n\in\mathbb{N}}$ forms a projective system: for the measurable subset $A \subseteq X^{n+1}$ and for $x \in X$ the equality

$$K_0^{n+1}(x)(A \times X) = K_0^n(x)(A)$$

holds for each $n \in \mathbb{N}$. This is the exact probabilistic counterpart to the property that a tree is closed with respect to prefixes.

Denote the resp. projections $(x_n)_{n\geq 0} \mapsto \langle x_0, \dots, x_n\rangle$ by $proj_{n+1}$. Standard arguments [16, V.3] show that there exists a uniquely determined probabilistic relation

$$\mathcal{K}_0^\infty : X \rightsquigarrow X^\infty$$

such that for all $x \in X$ the equality

$$\mathbf{P}(proj_{n+1})(\mathcal{K}_0^\infty(x)) = K_0^n(x)$$

holds. Thus $\mathcal{K}_0^\infty(x)(A)$ is the probability that the infinite sequence σ of states the system is running through is an element of A, provided the system starts in x. Averaging out the starting state x through an initial probability $\mu \in \mathbf{P}(X)$, i.e. forming

$$\mathsf{Tree}(\mathcal{K})_\mu(A) := \int_X \mathcal{K}_0^\infty(x)(A)\ \mu(dx)$$

yields a probability measure on X^∞. This is the probabilistic analogue to the body $[\mathcal{R}]$ of the tree formed from the sequence $\mathcal{R}$ of non-deterministic relations.

We have shown how to reverse this construction in the non-deterministic case by showing that each tree T yields a sequence of relations $\mathcal{R}$ with $T = \mathsf{Tree}(\mathcal{R})$. Investigating similarities between non-deterministic relations and their probabilistic counterparts, the question arises

whether this kind of reversal is also possible for the probabilistic case. To be more specific: Under which conditions does there exist for a probability measure $\nu^\infty \in \mathbf{P}(X^\infty)$, a sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of probabilistic relations $K_n : X \rightsquigarrow X$ and an initial probability $\mu \in \mathbf{P}(X)$ such that the representation $\nu^\infty = \mathsf{Tree}(\mathcal{K})_\mu$ holds?
Since in this case $\mu = \mathbf{P}(proj_0)(\nu^\infty)$ must hold, the question is reduced to conditions under which we can construct for a probabilistic relation $L : X \rightsquigarrow X^\infty$ a sequence $\mathcal{K}$ of probabilistic relations $X \rightsquigarrow X$ such that $L = \mathcal{K}_0^\infty$.

Definition 2 *A transition probability $L : X \rightsquigarrow X^\infty$ is called* memoryless *iff the projection*

$$\mathbf{P}(proj_{n+1})(L(x))$$

can be written for each $n \in \mathbb{N}, x \in X$ as a disintegration

$$\mathbf{P}(proj_n)(L(x)) \otimes J_n$$

with $J_n : X \rightsquigarrow X$, where J_n is independent of x.

The reader may wish to compare the definition of a memoryless probabilistic relation to that of a memoryless tree in Def. 1. Similarly, a comparison of Prop. 1 for the set valued case with Prop. 2 addressing the probabilistic case may be illuminating. Fix x, and interpret $\mu := L(x)$ in Definition 2 as the joint distribution of a stochastic process $(\zeta_i)_{i\geq 0}$ with $\zeta_i : \Omega \to X$ over the probability space $(\Omega, \mathcal{A}, \mathsf{P})$. Since X is Polish, there exists for each $n \in \mathbb{N}$ a regular conditional distribution of ζ_n conditional to $\langle \zeta_0, \ldots, \zeta_{n-1}\rangle$ (cf. [16, Theorem V.8.1]), hence a probabilistic relation $J_n : X^n \rightsquigarrow X$ such that for the Borel sets $B_1 \subseteq X^n, B_2 \subseteq X$

$$\begin{aligned}\mu_n(B_1 \times B_2) &= \mathsf{P}(\langle \zeta_0, \ldots, \zeta_{n-1}\rangle \in B_1, \zeta_n \in B_2)\\ &= \int_{B_1} J_n(x_0, \ldots x_{n-1})(B_2)\ \mu_{n-1}(d\langle x_0, \ldots, x_{n-1}\rangle).\end{aligned}$$

Consequently, a memoryless distribution corresponds to a Markov process, since in this case $J_n(x_0, \ldots, x_{n-1})$ only depends on the last state x_{n-1} and not on the whole history $x_0, \ldots, x_{n-1}$. Taking things a little further, consider a sequence $L_n : X \rightsquigarrow X$ of probabilistic relations. Define for $x \in X$ the probabilistic relation $L_\infty : X \rightsquigarrow X^\infty$ upon setting

$$L_\infty(x) := \bigotimes_{n\in\mathbb{N}} L_n(x),$$

then L_∞ is evidently memoryless. This corresponds to an independent stochastic process. Memoryless transition probabilities characterize those relations that arise through refinements.

Proposition 2 *Let $L : X \rightsquigarrow X^\infty$: be a probabilistic relation. Then the following conditions are equivalent:*

1. *There exists a sequence $\mathcal{K} = (K_n)_{n\in\mathbb{N}}$ of probabilistic relations $K_n : X \rightsquigarrow X$ such that $L = \mathcal{K}_0^\infty$ holds.*
2. *L is memoryless.*

Proof 1. "(1) ⇒ (2):" The probabilistic relation $\mathcal{K}_0^\infty$ is plainly memoryless. The construction shows that

$$\begin{aligned}\mathbf{P}\left(proj_{n+1}\right)\left(\mathcal{K}_0^\infty(x)\right) &= K_0^n(x) \\ &= K_0^{n-1}(x) \oplus K_n\end{aligned}$$

holds.
2. "(2) ⇒ (1):" Define inductively $K_0(x) := \mathbf{P}\left(proj_1\right)(L(x))$, and let K_{n+1} be determined through

$$\mathbf{P}\left(proj_{n+2}\right)(L(x)) = \mathbf{P}\left(proj_{n+1}\right)(L(x)) \otimes K_{n+1}.$$

From the definition of K_0^n above we see through an inductive argument that

$$K_0^n(x) = \mathbf{P}\left(proj_{n+1}\right)(L(x))$$

holds, because the equality

$$K_0^{n+1}(x) = K_0^n(x) \otimes K_{n+1}$$

is inferred from the defining equation. Since the extension of the projective system $(K_0^n)_{n\in\mathbb{N}}$ to a probabilistic relation $X \rightsquigarrow X^\infty$ is unique, the assertion follows. ■

Thus there are in fact striking similarities between non-deterministic relations and their probabilistic counterparts, when it comes to specify reactive, i.e., long running behavior. Both generate memoryless trees, and from these trees the single step behavior can be recovered. This is always true for the non-deterministic case (since we consider here only possibilities without attaching any constraints), it is possible in the probabilistic case under the condition that probabilities on product spaces can be suitably decomposed.

The next section will deal with a transfer between non-deterministic and probabilistic relations: Given is a tree, can we generate it probabilistically?

3 Representing Measurable Trees

A stochastic representation K of a non-deterministic relation R should have the following properties: we have $K(x)(R(x)) = 1$ for each x, indicating that a state transition in state x is guaranteed to lead to a state in $R(x)$, and we want the latter set to be exactly the set of all target states. This latter condition on exact fitting of $R(x)$ is a bit cumbersome to formulate if the space X is not finite or countable. But the topological structure on X comes in helpful now. We want $R(x)$ to be the smallest set of all states for which each open neighborhood U has positive probability $K(x)(U)$.

This is captured through the support of a probability: Given $\mu \in \mathbf{P}(X)$, define $\mathsf{supp}(\mu)$ as the smallest closed subset $F \subseteq X$ such that $\mu(F) = 1$, thus

$$\mathsf{supp}(\mu) := \bigcap\{F \subseteq X | F \text{ is closed and } \mu(F) = 1\}.$$

It can be shown that $\mu(\mathsf{supp}(\mu)) = 1$, and $x \in \mathsf{supp}(\mu)$ iff $\mu(U) > 0$ for each neighborhood U of x. So this is exactly what we want.

Definition 3 *Let Y be a measurable space, and Z be a Polish space, $R \subseteq Y \times Z$ a non-deterministic relation, and $K : Y \rightsquigarrow Z$ a probabilistic one. Then*

$$R \models K \Leftrightarrow \forall y \in Y : R(y) = \mathsf{supp}(K(y)).$$

We say that K represents *R.*

These are some elementary properties of the representation:

Example 3 *Let $f : X \to X$ be a measurable map, and put*

$$\Delta_f(x) := \delta_{f(x)},$$

δ_x denoting the Dirac measure on point x. Then

$$Graph(f) \models \Delta_f.$$

Example 4 *Assume $R_i \models K_i$ for $i = 1, 2$, where $R_i \subseteq X \times Y$, then*

$$R_1 \cup R_2 \models K_1 \oplus_p K_2.$$

Here the union is taken element wise, $0 \leq p \leq 1$, and

$$(K_1 \oplus_p K_2)(x)(A) := p \cdot k_1(x)(A) + (1 - p) \cdot K_2(x)(A)$$

is the convex combination of K_1 and K_2.

This will be generalized considerably in Proposition 7.
Measurable relations will provide a link between non-deterministic and stochastic systems, as we will see. Let us fix some notations first.
Assume that Y is a measurable, and that Z is a Polish space. A relation $R \subseteq Y \times Z$ induces as above a set-valued map through

$$Y \ni y \mapsto R(y) := \{z \in Z | \langle y, z\rangle \in R\} \in 2^Z.$$

If $R(y)$ always takes closed and non-empty values, and if the (weak) inverse

$$(\exists R)(G) := \{y \in Y | R(y) \cap G \neq \emptyset\}$$

is a measurable set, whenever $G \subseteq Z$ is open, then R is called a *measurable relation on* $Y \times Z$. Since Z is Polish, R is a measurable relation iff the strong inverse

$$(\forall R)(F) := \{y \in Y | R(y) \subseteq F\}$$

is measurable, whenever $F \subseteq Z$ is closed [10, Theorem 3.5]. It is well known that a measurable relation R constitutes a measurable subset of $Y \times Z$.
It is immediate that the support yields a measurable relation for a probabilistic relation $K : Y \rightsquigarrow Z$: put

$$R_K := \{\langle y, z\rangle \in Y \times Z | z \in \mathsf{supp}(K(y))\},$$

then

$$(\forall R_K)(F) = \{y \in Y | K(y)(F) = 1\}$$

is true for the closed set $F \subseteq Z$, and

$$(\exists R_K)(G) = \{y \in Y | K(y)(G) > 0\}$$

holds for the open set $G \subseteq Z$. Both sets are measurable. In fact, if $y \mapsto K(y)$ is weakly continuous, then $\exists R_K(G)$ is open for G open. It is also plain that $R \models K$ implies that R has to be a measurable relation. Given a set-valued relation R, a probabilistic relation K that satisfies R can be found. For this, R has to take closed values, and a measurability condition is imposed; from [6] we get:

Proposition 3 *Let $R \subseteq Y \times Z$ be a measurable relation for Z Polish. If Z is σ-compact, or if $R(y)$ assumes compact values for each $y \in Y$, then there exists a probabilistic relation $K : Y \rightsquigarrow Z$ with $R \models K$.*

Compactness plays an important role in the sequel, so we state the representation only for this case, leaving aside a more general formulation.
We are interested in trees. The notion of a measurable tree is introduced as an analogue to measurable relations.

Definition 4 *The tree $T \subseteq X^*$ is called a* measurable tree *iff the following conditions are satisfied:*

1. *T is memoryless,*
2. *$[T] \neq \emptyset$,*
3. *$T^\bullet := \{\langle v, x\rangle \in X^* \times X | vx \in T\}$ constitutes a measurable relation on $X^* \times X$.*

The last condition implies that $T^\bullet(v)$ is a closed subset of X for all $v \in T$. The condition $[T] \neq \emptyset$ makes sure that $\forall v \in T : T^\bullet(v) \neq \emptyset$, so that the tree continues to grow, hence T has the proper range for a measurable relation. Since $T^\bullet$ constitutes a measurable relation the graph of which is just T, it follows that T is a measurable subset of X^*. The first condition constraints our attention to memoryless trees; this is not too restrictive because a tree that is represented through a stochastic relation is memoryless in view of Prop. 1.
The easy way to represent a measurable tree through a probabilistic relation would be to capitalize on Prop. 3: if T is a measurable tree such that either $T^\bullet(v)$ is always compact, or if X is σ-compact, then there exists a probabilistic relation $K^\bullet : X^* \rightsquigarrow X$ such that $T^\bullet \models K^\bullet$. This solution, however, is less than satisfactory: Given $v \in X^*$ and the measurable subset $A \subseteq X$, the probability $K^\bullet(v)(A)$ depends directly on the entire history v. Having a look at the construction of the relations $(S_n)_{n\in\mathbb{N}}$ based on the tree, we see, however, that the tree has a kind of a Markov property: the state x_{n+1} at time $n+1$ depends *directly* only on some state x_n with $\langle x_n, x_{n+1}\rangle \in S_{n+1}$ and only indirectly on the entire history. This observation is not reflected in the representation above, hence we will need some refinement of the arguments above.
We start with a simple observation: If all relations R_n are measurable relations, then $\mathsf{Tree}\,(\mathcal{R})$ is a measurable tree:

Lemma 2 *Construct $\mathsf{Tree}\,(\mathcal{R})$ from the sequence $\mathcal{R} = (R_n)_{n\in\mathbb{N}}$ as above, then this tree has the following properties:*

1. *$\mathsf{Tree}\,(\mathcal{R}) \subseteq X^*$ is a Borel set, provided each $R_n \subseteq X \times X$ is,*
2. *$\mathsf{Tree}\,(\mathcal{R}) = \{\langle v, x\rangle | v \in \mathsf{Tree}\,(\mathcal{R}), x \in X \text{ so that } vx \in \mathsf{Tree}\,(\mathcal{R})\}$ is a measurable relation on $X^* \times X$, provided each R_n is.*

Proof 1. Define inductively

$$\begin{aligned} B_0 &:= X, \\ B_1 &:= R_1, \\ B_{k+1} &:= B_k \times X \cap X^k \times R_{k+1}. \end{aligned}$$

Then each B_k is a Borel subset of X^{k+1} under assumption 1. Since $\mathsf{Tree}\,(\mathcal{R}) = \bigcup_{k\geq 0} B_k$, the assertion follows.
2. It is easy to see that

$$\begin{aligned}(\forall \mathsf{Tree}\,(\mathcal{R}))(F) &= \bigcup_{n\geq 1} \left(B_{n-1} \cap X^{n-1} \times (\forall R_{n-1})(F)\right), \\ (\exists \mathsf{Tree}\,(\mathcal{R}))(G) &= \bigcup_{n\geq 1} \left(B_{n-1} \cap X^{n-1} \times (\exists R_{n-1})(G)\right),\end{aligned}$$

holds. This implies the second part of the assertion. ∎
Since a probabilistic relation generates a measurable relation, this has as an easy consequence:

Proposition 4 *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of probabilistic relations $K_n : X \rightsquigarrow X$. Then $\mathsf{Tree}\left((\mathsf{supp}(K_n))_{n\in\mathbb{N}}\right)$ constitutes a measurable tree.*

We can show now that under a compactness condition a measurable tree may be generated from some probabilistic refinement. Recall that the set $\mathcal{K}(X)$ of all compact non-void subsets of X is a Polish space when endowed with the Vietoris topology, and that measurability of a compact-valued relation $R \subseteq X \times X$ is equivalent to measurability of the map $R : X \to \mathcal{K}(X)$.

Proposition 5 *Let T be a measurable tree on X, and assume that $T \cap X^k$ is compact for each $k \geq 0$. Then there exists a sequence $(K_n)_{n\geq 0}$ of probabilistic relations $K_n : X \rightsquigarrow X$ such that*

$$T = \mathsf{Tree}\left((\mathsf{supp}(K_n))_{n\in\mathbb{N}}\right)$$

holds.

The proof of this statement makes substantial use of some non trivial properties of Borel sets in Polish spaces.
Proof 1. Define the sequence $(R_k^T)_{k\geq 0}$ of relations for T as above, then there exists for each $k \geq 1$ a measurable subset $D_k \subseteq X$ such that

$$R_k^T : D_k \to \mathcal{K}(X)$$

is a measurable map. This will be shown now. Fix $k \geq 0$, and let $(x_n)_{n\geq 0} \subseteq R_{k+1}^T(x')$ be a sequence, thus we can find $v_n \in T \cap X^k$ with $v_n x' x_n \in T \cap X^{k+2}$. Since the latter set is compact, we can find a convergent subsequence $(v_{n_\ell} x' x_{n_\ell})_{\ell\geq 0}$ and $vx'x \in T$ with $v_{n_\ell} x' x_{n_\ell} \to vx'x$, as $\ell \to \infty$. Consequently, $R_{k+1}^T(x')$ is closed, and sequentially compact, hence compact, since X is Polish. Thus $R_{k+1}^T(x) \in \mathcal{K}(X)$, provided the former set is not empty. The domain D_k of R_k^T is

$$\begin{aligned}D_k &= \pi_X\left[\{\langle v, x\rangle \in T \times X | T(vx) \neq \emptyset\}\right] \\ &= \pi_X\left[\{\langle v, x\rangle \in T \times X | T(vx) \cap X \neq \emptyset\}\right].\end{aligned}$$

If we can show that $(\forall R_{k+1}^T)(F)$ is Borel in X whenever $F \subseteq X$ is closed, then measurability of D_k will follow (among others).
2. In fact, if $F \subseteq X$ is closed, then the compactness assumption for T implies that

$$\{\langle v, x\rangle \in T \times X | T(vx) \cap F \neq \emptyset\}$$

is closed, consequently,

$$H^{(F)} := \{\langle v, x\rangle \in T \times X | T(vx) \subseteq F\}$$

is a G_δ set, since F is one. Hence $H^{(F)}$ is Borel. Because the section $H^{(F)}_x$ is compact for each $x \in X$, the Novikov Theorem [17, Th. 5.7.1] implies now that

$$\pi_X \left[H^{(F)}\right] = (\forall R^T_{k+1})(F)$$

is measurable.
3. The map $R^T_{k+1} : D_k \to \mathcal{K}(X)$ is measurable for each $k \geq 0$. Because R^T_{k+1} takes compact and nonempty values in a Polish space we can find by Prop. 3 a probabilistic relation $K_{k+1} : X \rightsquigarrow X$ such that $R^T_{k+1} \models \mathsf{supp}(K_{k+1})$. Hence

$$T = \mathsf{Tree}\left((\mathsf{supp}(K_n))_{n\in\mathbb{N}}\right)$$

is established. ■
This result makes the rather strong assumption that each slice $T \cap X^n$ of the tree at height n is compact. A little additional work will show that this may be relaxed to σ-compactness. For this fix a measurable tree T over a σ-compact Polish space X (so X may be represented as

$$X = \bigcup_{n\in\mathbb{N}} X_n,$$

where each X_n is compact) such that $T \subseteq X^*$ is closed. Define for $\alpha = n_0 \dots n_{k-1} \in \mathbb{N}^k$ the compact set

$$X_\alpha := X_{n_0} \times \cdots \times X_{n_{k-1}},$$

and put

$$S := \{\alpha \in \mathbb{N}^* | T \cap X_\alpha \neq \emptyset\}.$$

Clearly, S is a tree over $\mathbb{N}$. Now let $\sigma \in [S]$, and set

$$T_\sigma := \{v \in X^* | v \in T \cap X_{\sigma|_{|v|}}\}.$$

From the construction it is clear that

$$T = \overline{\bigcup_{\sigma\in[S]} T_\sigma}$$

holds, the bar denoting topological closure.
Since $T_\sigma \cap X^n$ is compact for each $n \in \mathbb{N}$, $T \subseteq X^*$ is closed, and T is a measurable tree over X, the condition of Prop. 5 is satisfied. Thus there exist probabilistic relations $(K_{n,\sigma})_{n\in\mathbb{N}}$ such that

$$T_\sigma = |(\mathsf{supp}(K_{n,\sigma}))_{n\in\mathbb{N}}|$$

is true.
A representation of T will be obtained by pasting the relations $(K_{n,\sigma})_{n\in\mathbb{N}}$ along their index σ. Since $[S]$ may be uncountable, we have probably more than countably many of these families of probabilistic relations, so gluing cannot be done through simply summing up all members. The observation that the body $[S]$ of tree S is a Polish space will come in helpful

now: we construct a probability measures on the set of indices and integrate the $(K_{n,\sigma})$ with this measure.
The following Lemma helps with the construction. Call a probability measure on a Polish space *thick* iff it assigns positive probability to each non-empty open set. Construct for example on the real line the probability measure

$$A \mapsto \int_A f(x)\ dx$$

with a strictly increasing and continuous density $f : \mathbb{R} \to \mathbb{R}_+$, then this constitutes a thick measure. But it can be said more:

Lemma 3 *Let P be a Polish space.*

1. *There exists a thick probability measure for P.*
2. *Assume that Q is a Polish space, and that $\phi : P \to \mathbf{P}(Q)$ is continuous, where $\mathbf{P}(Q)$ is endowed with the topology of weak convergence. Define*

$$\mu^\bullet(A) := \int_P \phi(p)(A)\ \mu_\ell(dp).$$

Then

$$\mathsf{supp}(\mu^\bullet) = \overline{\bigcup_{p\in P} \mathsf{supp}(\phi(p))}$$

holds.

Proof 1. Let $(r_n)_{n\in\mathbb{N}}$ be a countable dense sequence for P, and define

$$\mu_\ell := \sum_{n\in\mathbb{N}} 2^{-(n+1)} \cdot \delta_{r_n},$$

the infinite sum taken as the weak limit of the partial sums. This limit exists because for each bounded and measurable map $f : P \to \mathbb{R}$ we know that

$$\sum_{n\in\mathbb{N}} 2^{-(n+1)} \cdot |f(r_n)| \leq \sup_{p\in P} |f(p)| < \infty$$

holds, thus we may take limits, as $n \to \infty$:

$$\begin{aligned} \int_P f\ d\left(\sum_{k=0}^{n} 2^{-(k+1)} \cdot \delta_{r_k}\right) &= \sum_{k=0}^{n} 2^{-(k+1)} \cdot f(r_k) \\ &\to \sum_{k=0}^{\infty} 2^{-(k+1)} \cdot f(r_k) \\ &= \int_P f\ d\mu_\ell \end{aligned}$$

Then $\mu_\ell \in \mathbf{P}(P)$, and the construction shows that μ_ℓ has the desired properties.

2. Continuity implies that $\phi : P \rightsquigarrow Q$ is in particular a probabilistic relation, so that the integral defining $\mu^\bullet$ exists. The familiar properties of the integral show that $\mu^\bullet$ is σ-additive, thus $\mu^\bullet \in \mathbf{P}(Q)$. Since

$$\begin{aligned}\mu^\bullet\left(\overline{\bigcup_{p\in P} \mathsf{supp}(\phi(p))}\right) &\geq \int_P \mu_p\left(\mathsf{supp}(\phi(p))\right)\ \mu_\ell(dp)\\ &= 1,\end{aligned}$$

and since the properties of μ_ℓ make sure that $\mu^\bullet(U) = 0$ implies that $\phi(p)(U) = 0$ holds for all $p \in P$ for a non-empty open set U, the desired equality follows. ■

Note that the continuity condition imposed above for $\phi : P \to \mathbf{P}(Q)$ is satisfied whenever the set $\{p \in P | \phi(p)(U) > 0\}$ is open for an open $U \subseteq Q$. It turns out that

$$[S] \ni \sigma \mapsto K_{n,\sigma} \in \mathbf{P}(X)$$

has this property for fixed $n \in \mathbb{N}$, since it is a continuous map, when $[S]$ has the topology inherited from the Baire space $\mathcal{N}$, and $\mathbf{P}(X)$ carries the weak topology.

Because the body of tree S is a Polish space by Lemma 1, we obtain the generalization of Prop. 5.

Proposition 6 *Let T be a measurable tree on the σ-compact Polish space X such that $T \subseteq X^*$ is closed. Then there exists a sequence $(K_n)_{n\geq 0}$ of probabilistic relations $K_n : X \rightsquigarrow X$ such that*

$$T = \mathsf{Tree}\left((\mathsf{supp}(K_n))_{n\in\mathbb{N}}\right)$$

holds.

The proof takes a thick probability on $[S]$ and pastes the $(K_{n,\sigma})$ along σ, making heavy use of the construction in Lemma 3, because $\sigma \mapsto K_{n,\sigma}(x)$ is always continuous. This is so since $T_\sigma \cap X^n$ depends only on $\sigma\mid_n$, hence only on a finite number of components of σ.

Proof 1. We use the notations from above. Let $\mu_\ell \in \mathbf{P}([S])$ be a thick probability. Assume that $\emptyset \neq U \subseteq X$ is an open set, then

$$W := \{\sigma \in [S] | K_{n,\sigma}(x)(U) > 0\}$$

can be written for fixed $n \in \mathbb{N}, x \in X$ as

$$\left(A(x) \times \prod_{k\geq n} \mathbb{N}\right) \cap [S]$$

for a suitable subset $A(x) \subseteq \mathbb{N}^n$. Thus W is open in $[S]$, hence $\sigma \mapsto K_{n,\sigma}$ is weakly continuous.
2. Define for $A \subseteq X$ measurable

$$K_n(x)(A) := \int_{[S]} K_{n,\sigma}(x)(A)\ \mu_\ell(d\sigma),$$

as in Lemma 3. Standard arguments again show that for each $n \in \mathbb{N}$ the map $x \mapsto K_n(x)(A)$ is measurable. Consequently, the construction yields the desired representation. ■

The proof relies on the existence of a thick probability on the Polish space $[S]$. In fact, such a probability may be obtained from the following construction, which offers an alternative to Lemma 3: Let $\psi : S \to [0,1]$ be a map with $\psi(\epsilon) = 1$ and $\psi(v) = \sum_{vi \in S} \psi(vi)$ for all $v \in S$, then there exists a unique $\mu \in \mathbf{P}([S])$ with

$$\forall v \in S : \psi(v) = \mu\left(\{\sigma \in [S] | \sigma\ |_{|v|} = v\}\right)$$

by [11, Ex. 17.17]. If ψ is strictly positive, then the associated measure μ is clearly thick. In the theory of convex cones, the integral over a probability measure is often interpreted as the generalization of a convex combination, cf. [8, Chapter II]. We state as a generalization of Example 4 and as a reformulation of Lemma 3 the following continuous version. It relates convex combinations of probabilistic and set theoretic relations:

Proposition 7 *Let P be a Polish space, and assume that for a family $(R_p)_{p \in P}$ of relations $R_p \subseteq X \times X$ indexed by P the stochastic representation $R_p \models K_p$ holds. Let $\mu_\ell \in \mathbf{P}(P)$ be a thick probability measure on P. Assume that the family of probabilistic relations $K_p : X \rightsquigarrow X$ has the additional property that $p \mapsto K_p(x)$ is weakly continuous for each $x \in X$. Then*

$$\overline{\bigcup_{p \in P} R_p} \models \lambda x \lambda A. \int_P K_p(x)(A)\ \mu_\ell(dp)$$

holds.

4 Conclusion

We have shown that there are some interesting similarities between non-deterministic and probabilistic ways of describing a system in terms of computation trees. We first show that the construction of traces for both kinds of relations exhibit the same principles of construction (which could be described in terms of monads, but this does not happen here). Then we give under some topological conditions representations of measurable trees in terms of probabilistic relations.

References

[1] S. Abramsky, R. Blute, and P. Panangaden. Nuclear and trace ideal in tensored *-categories. *Journal of Pure and Applied Algebra*, 143(1 – 3):3 – 47, 1999.

[2] C. Brink, W. Kahl, and G. Schmidt, editors. *Relational Methods in Computer Science.* Advances in Computing Science. Springer-Verlag, Wien, New York, 1997.

[3] J. Desharnais, A. Edalat, and P. Panangaden. Bisimulation of labelled Markov-processes. Technical report, School of Computer Science, McGill University, Montreal, 1998.

[4] E.-E. Doberkat. The demonic product of probabilistic relations. In Mogens Nielsen and Uffe Engberg, editors, *Proc. Foundations of Software Science and Computation Structures*, volume 2303 of *Lecture Notes in Computer Science*, pages 113 – 127, Berlin, 2002. Springer-Verlag.

[5] E.-E. Doberkat. Semi-pullbacks and bisimulations in categories of stochastic relations. Technical Report 130, Chair for Software Technology, University of Dortmund, November 2002.

[6] E.-E. Doberkat. The converse of a probabilistic relation. In A. Gordon, editor, *Proc. Foundations of Software Science and Computation Structures'03*, Lecture Notes in Computer Science, Berlin, 2003. Springer-Verlag. (in print).

[7] A. Edalat. Semi-pullbacks and bisimulation in categories of Markov processes. *Math. Struct. in Comp. Science*, 9(5):523 – 543, 1999.

[8] B. Fuchssteiner and W. Lusky. *Convex Cones*, volume 56 of *North-Holland Mathematical Studies.* North-Holland Publishing Company, Amsterdam, 1981.

[9] M. Giry. A categorical approach to probability theory. In *Categorical Aspects of Topology and Analysis*, volume 915 of *Lecture Notes in Mathematics*, pages 68 – 85, Berlin, 1981. Springer-Verlag.

[10] C. J. Himmelberg. Measurable relations. *Fund. Math.*, 87:53 – 72, 1975.

[11] A. S. Kechris. *Classical Descriptive Set Theory.* Number 156 in Graduate Texts in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[12] S. Mac Lane. *Categories for the Working Mathematician.* Number 5 in Graduate Texts in Mathematics. Springer-Verlag, Berlin, 2 edition, 1997.

[13] C. Morgan, A. McIver, and K. Seidel. Probabilistic predicate transformers. *ACM Trans. Prog. Lang. Syst.*, 18(3):325 – 353, May 1996.

[14] P. Panangaden. Probabilistic relations. In C. Baier, M. Huth, M. Kwiatkowska, and M. Ryan, editors, *Proc. PROBMIV*, pages 59 – 74, 1998. Also available from the School of Computer Science, McGill University, Montreal.

[15] P. Panangaden. Does combining nondeterminism and probability make sense? *Bulletin of the EATCS*, (75):182 – 189, Oct. 2001.

[16] K. R. Parthasarathy. *Probability Measures on Metric Spaces.* Academic Press, New York, 1967.

[17] S. M. Srivastava. *A Course on Borel Sets.* Number 180 in Graduate Texts in Mathematics. Springer-Verlag, Berlin, 1998.

[18] F. van Breugel, S. Shalit, and J. Worrell. Testing labelled Markov processes. In *Proc. ICALP'2002*, volume 2380 of *Lecture Notes in Computer Science*, pages 537–548, Berlin, 2002. Springer-Verlag.

/99/ T. Bühren, M. Cakir, E. Can, A. Dombrowski, G. Geist, V. Gruhn, M. Gürgrn, S. Handschumacher, M. Heller, C. Lüer, D. Peters, G. Vollmer, U. Wellen, J. von Werne
Endbericht der Projektgruppe eCCo (PG 315)
Electronic Commerce in der Versicherungsbranche
Beispielhafte Unterstützung verteilter Geschäftsprozesse
Februar 1999

/100/ A. Fronk, J. Pleumann,
Der DoDL-Compiler
August 1999

/101/ K. Alfert, E.-E. Doberkat, C. Kopka
Towards Constructing a Flexible Multimedia Environment for Teaching the History of Art
September 1999

/102/ E.-E. Doberkat
An Note on a Categorial Semantics for ER-Models
November 1999

/103/ Christoph Begall, Matthias Dorka, Adil Kassabi, Wilhelm Leibel, Sebastian Linz, Sascha Lüdecke, Andreas Schröder, Jens Schröder, Sebastian Schütte, Thomas Sparenberg, Christian Stücke, Martin Uebing, Klaus Alfert, Alexander Fronk, Ernst-Erich Doberkat
Abschlußbericht der Projektgruppe PG-HEU (326)
Oktober 1999

/104/ Corina Kopka
Ein Vorgehensmodell für die Entwicklung multimedialer Lernsysteme
März 2000

/105/ Stefan Austen, Wahid Bashirazad, Matthais Book, Traugott Dittmann, Bernhard Flechtker, Hassan Ghane, Stefan Göbel, Chris Haase, Christian Leifkes, Martin Mocker, Stefan Puls, Carsten Seidel, Volker Gruhn, Lothar Schöpe, Ursula Wellen
Zwischenbericht der Projektgruppe IPSI
April 2000

/106/ Ernst-Erich Doberkat
Die Hofzwerge — Ein kurzes Tutorium zur objektorientierten Modellierung
September 2000

/107/ Leonid Abelev, Carsten Brockmann, Pedro Calado, Michael Damatow, Michael Heinrichs, Oliver Kowalke, Daniel Link, Holger Lümkemann, Thorsten Niedzwetzki, Martin Otten, Michael Rittinghaus, Gerrit Rothmaier
Volker Gruhn, Ursula Wellen
Zwischenbericht der Projektgruppe Palermo
November 2000

/108/ Stefan Austen, Wahid Bashirazad, Matthais Book, Traugott Dittmann, Bernhard Flechtker, Hassan Ghane, Stefan Göbel, Chris Haase, Christian Leifkes, Martin Mocker, Stefan Puls, Carsten Seidel, Volker Gruhn, Lothar Schöpe, Ursula Wellen
Endbericht der Projektgruppe IPSI
Februar 2001

/109/ Leonid Abelev, Carsten Brockmann, Pedro Calado, Michael Damatow, Michael Heinrichs, Oliver Kowalke, Daniel Link, Holger Lümkemann, Thorsten Niedzwetzki, Martin Otten, Michael Rittinghaus, Gerrit Rothmaier
Volker Gruhn, Ursula Wellen
Zwischenbericht der Projektgruppe Palermo
Februar 2001

/110/ Eugenio G. Omodeo, Ernst-Erich Doberkat
Algebraic semantics of ER-models from the standpoint of map calculus.
Part I: Static view
März 2001

/111/ Ernst-Erich Doberkat
An Architecture for a System of Mobile Agents
März 2001

/112/ Corina Kopka, Ursula Wellen
Development of a Software Production Process Model for Multimedia CAL Systems by Applying Process Landscaping
April 2001

/113/ Ernst-Erich Doberkat
The Converse of a Probabilistic Relation
Oktober 2002

/114/ Ernst-Erich Doberkat, Eugenio G. Omodeo
Algebraic semantics of ER-models in the context of the calculus of relations.
Part II: Dynamic view
Juli 2001

/115/ Volker Gruhn, Lothar Schöpe (Eds.)
Unterstützung von verteilten Softwareentwicklungsprozessen durch integrierte Planungs-, Workflow- und Groupware-Ansätze
September 2001

/116/ Ernst-Erich Doberkat
The Demonic Product of Probabilistic Relations
September 2001

/117/ Klaus Alfert, Alexander Fronk, Frank Engelen
Experiences in 3-Dimensional Visualization of Java Class Relations
September 2001

/118/ Ernst-Erich Doberkat
The Hierarchical Refinement of Probabilistic Relations
November 2001

/119/ Markus Alvermann, Martin Ernst, Tamara Flatt, Urs Helmig, Thorsten Langer, Ingo Röpling,
Clemens Schäfer, Nikolai Schreier, Olga Shtern
Ursula Wellen, Dirk Peters, Volker Gruhn
Project Group Chairware Intermediate Report
November 2001

/120/ Volker Gruhn, Ursula Wellen
Autonomies in a Software Process Landscape
Januar 2002

/121/ Ernst-Erich Doberkat, Gregor Engels (Hrsg.)
Ergebnisbericht des Jahres 2001
des Projektes "MuSofT – Multimedia in der SoftwareTechnik"
Februrar 2002

/122/ Ernst-Erich Doberkat, Gregor Engels, Jan Hendrik Hausmann, Mark Lohmann, Christof Veltmann
Anforderungen an eine eLearning-Plattform — Innovation und Integration —
April 2002

/123/ Ernst-Erich Doberkat
Pipes and Filters: Modelling a Software Architecture Through Relations
Juni 2002

/124/ Volker Gruhn, Lothar Schöpe
Integration von Legacy-Systemen mit Eletronic Commerce Anwendungen
Juni 2002

/125/ Ernst-Erich Doberkat
A Remark on A. Edalat's Paper *Semi-Pullbacks and Bisimulations in Categories of Markov-Processes*
Juli 2002

/126/ Alexander Fronk
Towards the algebraic analysis of hyperlink structures
August 2002

/127/ Markus Alvermann, Martin Ernst, Tamara Flatt, Urs Helmig, Thorsten Langer
Ingo Röpling, Clemens Schäfer, Nikolai Schreier, Olga Shtern
Ursula Wellen, Dirk Peters, Volker Gruhn
Project Group Chairware Final Report
August 2002

/128/ Timo Albert, Zahir Amiri, Dino Hasanbegovic, Narcisse Kemogne Kamdem,
Christian Kotthoff, Dennis Müller, Matthias Niggemeier, Andre Pavlenko, Stefan Pinschke,
Alireza Salemi, Bastian Schlich, Alexander Schmitz
Volker Gruhn, Lothar Schöpe, Ursula Wellen
Zwischenbericht der Projektgruppe Com42Bill (PG 411)
September 2002

/129/ Alexander Fronk
An Approach to Algebraic Semantics of Object-Oriented Languages
Oktober 2002

/130/ Ernst-Erich Doberkat
Semi-Pullbacks and Bisimulations in Categories of Stochastic Relations
November 2002

/131 Yalda Ariana, Oliver Effner, Marcel Gleis, Martin Krzysiak,
Jens Lauert, Thomas Louis, Carsten Röttgers, Kai Schwaighofer,
Martin Testrot, Uwe Ulrich, Xingguang Yuan
Prof. Dr. Volker Gruhn, Sami Beydeda
Endbericht der PG nightshift:
Dokumentation der verteilten Geschäftsprozesse im FBI und Umsetzung von Teilen dieser Prozesse im Rahmen eines FBI-Intranets basierend auf WAP- und Java-Technologie
Februar 2003

/132/ Ernst-Erich Doberkat, Eugenio G. Omodeo
ER Modelling from First Relational Principles
Februar 2003

/133/ Klaus Alfert, Ernst-Erich Doberkat, Gregor Engels (Hrsg.)
Ergebnisbericht des Jahres 2002 des Projektes "MuSofT – Multimedia in der SoftwareTechnik"
März 2003

/134/ Ernst-Erich Doberkat
Tracing Relations Probabilistically
März 2003