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Semi-Pullbacks and Bisimulations in Categories of Stochastic Relations

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Abstract

The problem of constructing a semi-pullback in a category is intimately connected to the problem of establishing the transitivity of bisimulations. Edalat shows that a semi-pullback can be constructed in the category of Markov processes on Polish spaces, when the underlying transition probability functions are universally measurable, and the morphisms are measure preserving continuous maps. We demonstrate that the simpler assumption of Borel measurability suffices. Markov processes are in fact a special case: we consider the category of stochastic relations over Standard Borel spaces. At the core of the present solution lies a selection argument from stochastic dynamic optimization. An example shows that (weak) pullbacks do not exist in the category of Markov processes. We show that bisimilar labelled Markov processes are characterized through a weak negation-free logic, which provides a simplification and generalization of previous results by Desharnais, Edalat and Panangaden. The construction of pullbacks provides a rather general answer to Panangaden's question regarding the transitivity of the bisimulation relation in categories of Markov processes.

Keywords: Bisimulation, semi-pullback, stochastic relations, labelled Markov processes, Hennessy-Milner logic.

1 Introduction

The existence of semi-pullbacks in a category makes sure that the bisimulation relation is transitive, provided bisimulation between objects is defined as a span of morphisms [13]. Edalat investigates this question for categories of Markov processes and shows that semi-pullbacks exist [8]. The category he focusses on has as objects universally measurable transition probability functions on Polish spaces, the morphisms are continuous, surjective, and probability preserving maps. His proof is constructive and makes essentially use of techniques of analytic spaces (which are continuous images of Polish spaces). The result implies that the semi-pullback of those transition probabilities which are measurable with respect to the Borel sets of the Polish spaces under consideration may in fact be universally measurable rather than simply Borel measurable. This then demands some unpleasant technical machinery when logically characterizing bisimulation for labelled Markov processes, cf. [3]. Quite apart from that, it is somewhat annoying that forming the semi-pullback for Borel Markov processes seems to require a change of categories. This is so because this Borel measurability comes naturally with the Borel sets of a Polish space, whereas universal measurability requires a somewhat elaborate completion process.

The distinction between measurability and universal measurability (both terms are defined in Sect. 2) may seem negligible at first. Measurability is the natural concept in measurable spaces (like continuity in topological spaces, or homomorphisms in groups), thus stochastic concepts are usually formulated in terms of it. Universal measurability requires a completion process using all (σ)-finite measures on the measure space under consideration. In a Polish space the Borel sets as the measurable sets are generated by the open (or by the closed) sets, so the generators are well known, comparable generators for the universally measurable sets are not that easy identified, let alone put to use. Thus it appears to be sensible to search for solutions for the problem of constructing semi-pullbacks for stochastic relations or labelled Markov processes first within the realm of Borel sets.

This short note shows among others that the semi-pullback of Borel Markov processes exists within the category of these processes, when the underlying space is Polish (like the real line). Edalat considers transition probability functions from one Polish space into itself, this paper considers the slightly more general notion of a stochastic relation, cf. [15, 1, 5, 4], i.e., transition sub-probability functions from one Polish space to another one. Rather than constructing the function explicitly, as Edalat does, we rely chiefly on a selection argument: we show that the problem can be formulated in terms of measurable set-valued maps for which a measurable selector exists.

The paper's contributions are as follows. First it is shown that one can in fact construct semi-pullbacks in a category of stochastic relations between Polish spaces (and, by the way, an example shows that weak pullbacks do not exist). The second contribution is the reduction of an existential argument to a selection argument, a technique borrowed from dynamic optimization. Third it is shown that the solution for characterizing bisimulations for labelled Markov processes proposed by Desharnais, Edalat and Panagaden [3] can be carried over to Standard Borel spaces with their simple Borel structure. This gives a conceptually simpler result and widens its applicability.

This note is organized as follows: Sect. 2 collects some basic facts from topology, and from measure theory. It is shown that assigning a Polish space its set of subprobability measures is an endofunctor on this category, opening the road to discuss applications through monads, cf.[9, 6]. Sect. 3 defines the category of stochastic relations, shows how to formulate the problem in terms of a set-valued function, and proves that a selector for that function exists. This implies the existence of semi-pullbacks for some related categories, too. A counterexample destroys the hope for strengthening this results to weak pullbacks. Finally, we show in Sect. 4 that the bisimulation relation is transitive for the category of stochastic relations, and that bisimilar labelled Markov processes are characterized through a weak negation free logic. Sect. 5 wraps it all up by summarizing the results and indicating areas of further work. It is clear that the topological assumptions should be weakened further.

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2 A Small Dose Measure Theory

This Section collects some basic facts from topology and measure theory for the reader's convenience and for later reference.
A *Polish space* $(X, \mathcal{T})$ is a topological space which has a countable dense subset, and which is metrizable through a complete metric, a measurable space $(X, \mathcal{A})$ is a set X with a σ-algebra $\mathcal{A}$. The *Borel sets* $\mathcal{B}(X, \mathcal{T})$ for the topology $\mathcal{T}$ is the smallest σ-algebra on X which contains $\mathcal{T}$. A *Standard Borel* space $(X, \mathcal{A})$ is a measurable space such that the σ-algebra $\mathcal{A}$ equals $\mathcal{B}(X, \mathcal{T})$ for some Polish topology $\mathcal{T}$ on X. Although the Borel sets are determined uniquely through the topology, the converse does not hold, as we will see in a short while. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a map $f : X \to Y$ is $\mathcal{A} - \mathcal{B}$*-measurable* whenever

$$f^{-1}[\mathcal{B}] \subseteq \mathcal{A}$$

holds, where

$$f^{-1}[\mathcal{B}] := \{f^{-1}[B] | B \in \mathcal{B}\}$$

is the set of inverse images

$$f^{-1}[B] := \{x \in X | f(x) \in B\}$$

of elements of $\mathcal{B}$. Note that $f^{-1}[\mathcal{B}]$ is in any case an σ-algebra. If the σ-algebras are the Borel sets of some topologies on X and Y, resp., then a measurable map is called *Borel measurable* or simply a *Borel* map. The real numbers $\mathbb{R}$ carry always the Borel structure induced by the usual topology which will not be mentioned explicitly when talking about Borel maps.
A map $f : X \to Y$ between the topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is *continuous* iff the inverse image of an open set from $\mathcal{S}$ is an open set in $\mathcal{T}$. Thus a continuous map is also measurable with respect to the Borel sets generated by the respective topologies.
When the context is clear, we will write down Polish spaces without their topologies, and the Borel sets are always understood with respect to the topology. Measurable maps with respect to the Borel sets of a Polish topology will simply be called *Borel maps.*
The following statement will be most helpful in the sequel. It states that, given a measurable map between Polish spaces, we can find a finer Polish topology on the domain, which has the same Borel sets, and which renders the map continuous, or makes a sequence of Borel sets clopen ($\equiv$ closed and open); formally:

Proposition 1 *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{S})$ *be Polish spaces.*

1. *If $f : X \to Y$ is a Borel measurable map, then there exists a Polish topology $\mathcal{T}'$ on X such that $\mathcal{T}'$ is finer than $\mathcal{T}$ (hence $\mathcal{T} \subseteq \mathcal{T}'$), $\mathcal{T}$ and $\mathcal{T}'$ have the same Borel sets, and f is $\mathcal{T}' - \mathcal{S}$ continuous.*

2. *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of Borel sets in X, then there exists a finer Polish topology $\mathcal{T}'$ on X such that every A_n is clopen with respect to $\mathcal{T}'$, and $\mathcal{T}$ and $\mathcal{T}'$ have the same Borel sets.*

Proof [18, Cor. 3.2.5, Cor. 3.2.6] □
Given two measurable spaces X and Y, a *stochastic relation* $K : X \rightsquigarrow Y$ is a Borel map from X to the set $\mathbf{S}(Y)$, the latter denoting the set of all subprobability measures on (the Borel sets) of Y. The latter set carries the *weak*-σ-algebra*. This is the smallest σ-algebra on $\mathbf{S}(Y)$ which renders all maps $\mu \mapsto \mu(B)$ measurable, where $B \subseteq Y$ is measurable. Hence $K : X \rightsquigarrow Y$ is a stochastic relation iff

1. $K(x)$ is a subprobability measure on (the Borel sets of) Y for all $x \in X$,

2. $x \mapsto K(x)(B)$ is a measurable map for each Borel set $B \subseteq Y$.

Let Y be a Polish space, then $\mathbf{S}(Y)$ is usually equipped with the topology of weak convergence. This is the smallest topology on $\mathbf{S}(Y)$ which makes the map $\mu \mapsto \int_Y f\, d\mu$ continuous for each continuous and bounded $f : Y \to \mathbb{R}$. It is well known that this topology is Polish [16, Thm. II.6.5], and that its Borel sets is just the weak*-σ-algebra [14, Thm. 17.24]. If X is a Standard Borel space, then $\mathbf{S}(X)$ is also one: select a Polish topology $\mathcal{T}$ on X which induces the measurable structure, then $\mathcal{T}$ will give rise to the Polish topology of weak convergence on $\mathbf{S}(X)$ which in turn has the weak-*-σ-algebra as its Borel sets.
A Borel map $f : X \to Y$ between the Polish spaces X and Y induces a Borel map

$$\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$$

upon setting ($\mu \in \mathbf{S}(X), B \subseteq Y$ Borel)

$$\mathbf{S}(f)(\mu)(B) := \mu(f^{-1}[B])$$

It is easy to see that a continuous map f induces a continuous map $\mathbf{S}(f)$, and we will see in a moment that $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ is onto, provided $f : X \to Y$ is. Denote by $\mathbf{P}(X)$ the subspace of all probability measures on X.
Let $\mathcal{F}(X)$ be the set of all closed and non-empty subsets of the Polish space X, and call for Polish Y a relation, i.e., a set-valued map $F : X \to \mathcal{F}(Y)$ *$\mathcal{C}$-measurable* iff the weak inverse

$$\exists F(C) := \{x \in X | F(x) \cap C \neq \emptyset\}$$

for a compact set $C \subseteq Y$ is measurable. A *selector* s for such a relation F is a single-valued map $s : X \to Y$ such that $\forall x \in X : s(x) \in F(x)$ holds. $\mathcal{C}$-measurable relations have Borel selectors:

Proposition 2 *Let X and Y be Polish spaces. Then each $\mathcal{C}$-measurable relation F has a measurable selector.*

Proof Since closed subsets of Polish spaces are complete, the assertion follows from [11, Theorem 3]. □
Postulating measurability for $\exists F(C)$ for open or for closed sets C leads to the general notion of a measurable relation. These relations are a valuable tool in such diverse fields as stochastic dynamic programming [19] and descriptive set theory [14]. Overviews are provided in [18, Chapter 5] and [10, 19].
As a first application it is shown that $\mathbf{S}$ actually constitutes an endofunctor on the category of Standard Borel spaces with surjective measurable map as morphisms. This implies that $\mathbf{S}$ is the functorial part of a monad $\langle \mathbf{S}, \eta, \mu \rangle$ very similar to the one studied by Giry, cf. [9]. The crucial part is evidently to show that $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ is a surjection whenever $f : X \to Y$ is one. This is done through a measurable selection argument using Prop. 2.

Lemma 1 $\mathbf{S}$ *is an endofunctor on the category* $\mathfrak{SB}$ *of Standard Borel spaces with surjective Borel maps as morphisms.*

Proof 1. Let X and Y be Standard Borel spaces, and endow these spaces with a Polish topology the Borel sets of which form the respective σ-algebras. Since $\mathbf{S}(X)$ is a Polish space under the topology of weak convergence, and since a Borel map $f : X \to Y$ induces a Borel map $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ with all the compositional properties a functor should have, only surjectivity of the induced map has to be shown.
2. In view of Prop. 1 it is no loss of generality to assume that f is continuous (otherwise consider a finer Polish topology with the same Borel sets rendering f continuous). Continuity and surjectivity together imply that $y \mapsto f^{-1}[\{y\}]$ has closed and non-empty values in X, and constitutes a $\mathcal{C}$-measurable relation, which has a measurable selector $g : Y \to X$ by Prop. 2, so that $f(g(y)) = y$ always holds. Let $\nu \in \mathbf{S}(Y)$, and define $\mu \in \mathbf{S}(X)$ upon setting

$$\mu(A) := \nu(g^{-1}[A])$$

for $A \subseteq X$ Borel. Since $g^{-1}\left[f^{-1}[B]\right] = B$ for $B \subseteq Y$, it is now easy to establish that $\mathbf{S}(f)(\mu) = \nu$ holds. □
Finally, the concept of universal measurability is needed. Let $\mu \in \mathbf{S}(X, \mathcal{A})$ be a subprobability on the measurable space $(X, \mathcal{A})$, then $A \subseteq X$ is called *μ-measurable* iff there exist $M_1, M_2 \in \mathcal{A}$ with $M_1 \subseteq A \subseteq M_2$ and $\mu(M_1) = \mu(M_2)$. The μ-measurable subsets of X form a σ-algebra $\mathcal{M}_\mu(\mathcal{A})$. The σ-algebra $\mathcal{U}(\mathcal{A})$ of universally measurable sets is defined by

$$\mathcal{U}(\mathcal{A}) := \bigcap \{\mathcal{M}_\mu(\mathcal{A}) | \mu \in \mathbf{S}(X, \mathcal{A})\}$$

(in fact, one considers usually all finite or σ-finite measures, but it is easy to see that these definitions lead to the same universally measurable sets). If $f : X_1 \to X_2$ is an $\mathcal{A}_1$-$\mathcal{A}_2$-measurable map between the measurable spaces $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$, then it is well known that f is also $\mathcal{U}(\mathcal{A}_1)$-$\mathcal{U}(\mathcal{A}_2)$-measurable [12, Prop. I.B.6]; the converse does not hold, and one usually cannot conclude that a map $g : X_1 \to X_2$ which is $\mathcal{U}(\mathcal{A}_1)$-$\mathcal{A}_2$-measurable is also $\mathcal{A}_1$-$\mathcal{A}_2$-measurable.

3 Semi-Pullbacks

The category $\mathfrak{SRel}$ of stochastic relations has as objects triplets $\langle X, Y, K \rangle$, where X and Y are Standard Borel spaces, and $K : X \rightsquigarrow Y$ is a stochastic relation. A morphism

$$\langle \varphi, \psi \rangle : \langle X, Y, K \rangle \to \langle X', Y', K' \rangle$$

is a pair of surjective Borel maps $\varphi : X \to X'$ and $\psi : Y \to Y'$ such that

$$K' \circ \varphi = \mathbf{S}(\psi) \circ K$$

holds, rendering the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle K'} \\ \mathbf{S}(Y) & \xrightarrow[\mathbf{S}(\psi)]{} & \mathbf{S}(Y') \end{array}$$

commutative. Thus we have for $x \in X, B' \subseteq Y'$ Borel the equality

$$K'(\varphi(x))(B') = K(x)(\psi^{-1}[B']),$$

so that morphisms are in particular measure preserving. Morphisms compose componentwise. The category of Markov processes is a subcategory of $\mathfrak{SRel}$: it has as objects pairs $\langle X, K\rangle$, where X is a Standard Borel space, and $K : X \rightsquigarrow X$ is a stochastic relation, i.e., a Borel measurable transition probability function. Morphisms are surjective and measurable measure preserving maps.

Edalat [8] investigates a similar category, called $\mathfrak{MProc}$ for easier reference: the objects are pairs $\langle X, K\rangle$ such that X is a Polish space, and K is a universally measurable transition sub-probability function. This requires that for each Borel set $A \subseteq X$ the map $x \mapsto K(x)(A)$ is $\mathcal{U}(\mathcal{B}(X))$-measurable, and that $K(x) \in \mathbf{S}(X, \mathcal{B}(X))$ for each $x \in X$. Morphisms in $\mathfrak{MProc}$ are surjective and continuous maps which are measure preserving. Note that an object $\langle X, K\rangle$ in $\mathfrak{MProc}$ has the property that for each Borel set $A \subseteq X$ and for each $r \in \mathbb{R}$ the set $\{x \in X | K(x)(A) \leq r\}$ is universally measurable; since each Borel set is measurable, this is a weaker condition than the one we will be investigating.

Assume that $\langle \varphi_i, \psi_i\rangle : \langle X_i, Y_i, K_i\rangle \to \langle X, Y, K\rangle$ $(i = 1, 2)$ are morphisms in $\mathfrak{SRel}$, then a *semi-pullback* for this pair of morphisms is an object $\langle A, B, N\rangle$ together with morphisms $\langle \alpha_i, \beta_i\rangle : \langle A, B, N\rangle \to \langle X_i, Y_i, K_i\rangle$ $(i = 1, 2)$ so that this diagram is commutative in $\mathfrak{SRel}$:

$$\begin{array}{ccc} \langle A, B, N\rangle & \xrightarrow{\langle \alpha_1, \beta_1\rangle} & \langle X_1, Y_1, K_1\rangle \\ {\scriptstyle \langle \alpha_2, \beta_2\rangle}\downarrow & & \downarrow{\scriptstyle \langle \varphi_1, \psi_1\rangle} \\ \langle X_2, Y_2, K_2\rangle & \xrightarrow[\langle \varphi_2, \psi_2\rangle]{} & \langle X, Y, K\rangle \end{array}$$

This means in particular that

$$\begin{aligned} K_1 \circ \alpha_1 &= \mathbf{S}(\beta_1) \circ N, \\ K_2 \circ \alpha_2 &= \mathbf{S}(\beta_2) \circ N \end{aligned}$$

should hold, so that a bisimulation is to be constructed (cf. Def. 1). The condition that $\langle A, B, N\rangle$ is the object underlying a semi-pullback may be formulated in terms of measurable maps as follows: N is a map from the Standard Borel space A to the Standard Borel space $\mathbf{S}(B)$ so that N is also a measurable selector for the set-valued function

$$b \mapsto \{\mu \in \mathbf{S}(B) \,|\, (K_1 \circ \alpha_1)(b) = \mathbf{S}(\beta_1)(\mu), (K_2 \circ \alpha_2)(b) = \mathbf{S}(\beta_2)(\mu)\}.$$

This translates the problem of finding the object $\langle A, B, N\rangle$ of a semi-pullback to a selection problem for set-valued maps, provided the spaces A and B together with the morphisms are identified.
It should be noted that the notion of a semi-pullback depends only on the measurable structure of the Standard Borel spaces involved. The topological structure enters only through Borel sets, and Borel measurability. From Prop. 1 we see that there are certain degrees of freedom for selecting a Polish topology that generates the Borel sets. They will be capitalized upon in the sequel.
Our goal is to establish:

Theorem 1 *$\mathfrak{SRel}$ has semi-pullbacks for each pair of morphisms*

$$\begin{array}{ccc} & & \langle X_1, Y_1, K_1\rangle \\ & & \Big\downarrow{\scriptstyle \langle \varphi_1, \psi_1\rangle} \\ \langle X_2, Y_2, K_2\rangle & \xrightarrow[\langle \varphi_2, \psi_2\rangle]{} & \langle X, Y, K\rangle \end{array}$$

with a common range.

We begin with a measure-theoretic and rather technical observation: in terms of probability theory, it states that there exists under certain conditions a common distribution for two random variables with values in a Polish space with preassigned marginal distributions. This is a cornerstone for the construction leading to the proof of Theorem 1, it shows in particular where Edalat's work enters the present discussion.

Proposition 3 *Let Z_1, Z_2, Z be Polish spaces,*

$$\zeta_i : Z_i \to Z \;\; (i = 1, 2)$$

continuous and surjective maps, define

$$S := \{\langle x_1, x_2\rangle \in Z_1 \times Z_2 | \zeta_1(x_1) = \zeta_2(x_2)\},$$

and let $\nu_1 \in \mathbf{P}(Z_1), \nu_2 \in \mathbf{P}(Z_2), \nu \in \mathbf{P}(S)$ such that

$$\forall E_i \in \zeta_i^{-1}[\mathcal{B}(Z)] : \mathbf{P}(\pi_i)(\nu)(E_i) = \nu_i(E_i) \;\; (i = 1, 2)$$

holds, where $\pi_1 : S \to Z_1, \pi_2 : S \to Z_2$ are the projections; S carries the trace of the product topology. Then there exists $\mu \in \mathbf{P}(S)$ such that

$$\forall E_i \in \mathcal{B}(Z_i) : \mathbf{P}(\pi_i)(\mu)(E_i) = \nu_i(E_i) \;\; (i = 1, 2)$$

holds.

Proof It is not difficult to see that $\zeta_i : Z_i \to Z$ are morphisms in Edalat's category of probability measures on Polish spaces. The assertion then follows from the proof of [8, Cor. 5.4]. □

In important special cases, there are other ways of establishing the Proposition, as will be discussed briefly.

Remark: 1. If $\zeta_i : Z_i \to Z$ are bijections, then the Blackwell-Mackey Theorem [18, Thm. 4.5.7] shows that $\zeta_i^{-1}\left[\mathcal{B}(Z)\right] = \mathcal{B}(Z_i)$. In this case the given measure $\nu \in \mathbf{P}(S)$ is the desired one.

2. If Z_1, Z_2, Z are not only Polish but also locally compact (like the real line $\mathbb{R}$), then a combination of the Riesz Representation Theorem and the equally famous Hahn-Banach Theorem can be used to construct the desired measure directly. This is the line of attack in [7]. Consequently, the somewhat heavy machinery of regular conditional distributions on analytic spaces need not be used (on the other hand, the Hahn-Banach Theorem relies on the Axiom of Choice which is not listed among the light weight tools either). —

The preparations for establishing that $\mathfrak{SRel}$ has semi-pullbacks are complete.

Proof of Theorem 1

1. In view of Prop. 1 we may assume that the respective σ-algebras on X_1 and X_2 are obtained from Polish topologies which render φ_1 and K_1 as well as φ_2 and K_2 continuous. These topologies are fixed for the proof. Put

$$\begin{aligned} A &:= \{\langle x_1, x_2\rangle \in X_1 \times X_2 | \varphi_1(x_1) = \varphi_2(x_2)\}, \\ B &:= \{\langle y_1, y_2\rangle \in Y_1 \times Y_2 | \psi_1(y_1) = \psi_2(y_2)\}, \end{aligned}$$

then both A and B are closed, hence Polish. $\alpha_i : A \to X_i$ and $\beta_i : B \to Y_i$ are the projections, $i = 1, 2$. The diagrams

$$\begin{array}{ccccc} X_1 & \xrightarrow{\varphi_1} & X & \xleftarrow{\varphi_2} & X_2 \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle K_2} \\ \mathbf{S}(Y_1) & \xrightarrow[\mathbf{S}(\psi_1)]{} & \mathbf{S}(Y) & \xleftarrow[\mathbf{S}(\psi_2)]{} & \mathbf{S}(Y_2) \end{array}$$

are commutative by assumption, thus we know that for $x_i \in X_i$

$$\begin{aligned} K(\varphi_1(x_1)) &= \mathbf{S}(\psi_1)(K_1(x_1)) \\ K(\varphi_2(x_2)) &= \mathbf{S}(\psi_2)(K_2(x_2)) \end{aligned}$$

both hold. The construction implies that

$$(\psi_1 \circ \beta_1)(y_1, y_2) = (\psi_2 \circ \beta_2)(y_1, y_2)$$

is true for $\langle y_1, y_2\rangle \in B$, and $\psi_1 \circ \beta_1 : B \to Y$ is surjective.

2. Fix $\langle x_1, x_2\rangle \in A$. Lemma 1 shows that $\mathbf{S}$ is an endofunctor on $\mathfrak{SB}$, in particular that the image of a surjective map under $\mathbf{S}$ is onto again, so that there exists $\mu \in \mathbf{S}(S)$ with

$$\mathbf{S}(\psi_1 \circ \beta_1)(\mu) = K(\varphi_1(x_1)),$$

consequently,

$$\mathbf{S}(\psi_i \circ \beta_i)(\mu) = \mathbf{S}(\psi_i)(K_i(x_i)) \ (i = 1, 2).$$

But this means

$$\forall E_i \in \psi_i^{-1}\left[\mathcal{B}\left(Y\right)\right] : \mathbf{S}\left(\beta_i\right)\left(\mu\right)(E_i) = K_i(x_i)(E_i)\ (i = 1, 2).$$

Put

$$\Gamma(x_1, x_2) := \{\mu \in \mathbf{S}\left(B\right) | \mathbf{S}\left(\beta_1\right)\left(\mu\right) = K_1(x_1) \wedge \mathbf{S}\left(\beta_2\right)\left(\mu\right) = K_2(x_2)\},$$

then Prop 3 shows that $\Gamma(x_1, x_2) \neq \emptyset$.
3. Since K_1 and K_2 are continuous,

$$\Gamma : A \to \mathcal{F}(\mathbf{S}\left(B\right))$$

is easily established. The set $\exists\Gamma(C)$ is closed in A for compact $C \subseteq \mathbf{S}\left(B\right)$. In fact, let $(\langle x_1^{(n)}, x_2^{(n)}\rangle)_{n\in\mathbb{N}}$ be a sequence in this set with $x_i^{(n)} \to x_i$, as $n \to \infty$ for $i = 1, 2$, thus $\langle x_1, x_2\rangle \in A$. There exists $\mu_n \in C$ such that $\mathbf{S}\left(\beta_i\right)\left(\mu_n\right) = K_i(x_i^{(n)})$. Because C is compact, there exists a converging subsequence $\mu_{s(n)}$ and $\mu \in C$ with $\mu = \lim_{n\to\infty} \mu_{s(n)}$ in the topology of weak convergence. Continuity of K_i implies that

$$\mathbf{S}\left(\beta_i\right)\left(\mu\right) = K_i(x_i),$$

consequently $\langle x_1, x_2\rangle \in \exists\Gamma(C)$, thus this set is closed, hence measurable. From Prop. 2 it can now be inferred that there exists a measurable map $N : A \to \mathbf{S}\left(B\right)$ such that

$$N(x_1, x_2) \in \Gamma(x_1, x_2)$$

holds for every $\langle x_1, x_2\rangle \in A$. Thus $N : A \rightsquigarrow B$ is a stochastic relation with

$$\begin{aligned} K_1 \circ \alpha_1 &= \mathbf{S}\left(\beta_1\right) \circ N, \\ K_2 \circ \alpha_2 &= \mathbf{S}\left(\beta_2\right) \circ N \end{aligned}$$

Hence $\langle A, B, N\rangle$ is the desired semi-pullback. □
Specializing Theorem 1, we list some categories of stochastic relations which have semi-pullbacks.

Corollary 1 *The following categories have semi-pullbacks:*

1. *Objects are Standard Borel spaces with a sub-probability measure attached, morphisms are measure preserving and surjective Borel maps (continuous maps, resp.).*
2. *Objects are Markov processes over Standard Borel spaces (Polish spaces), morphisms are measure preserving and surjective Borel maps (continuous maps, resp.).*
3. *Objects are stochastic relations over Polish spaces, morphisms $\langle\varphi, \psi\rangle$ are as in $\mathfrak{SRel}$ with ψ continuous. In the subcategory in which φ is also continuous semi-pullbacks exists, too*

Proof By specialization from Theorem 1. Whenever continuity enters the game, its proof shows that the semi-pullback has the continuity property, too. □
Hence we know that the semi-pullback $\langle X, K\rangle$ for morphisms involving Markov processes is a Markov process again (whereas Edalat's main result [8, Cor. 5.2] permits only to conclude that K is a universally measurable transition sub-probability function).

Remark: One might be tempted now and ask for pullbacks or at least for weak pullbacks in the categories involved, now that the upper left hand corner of a pullback diagram can be filled. Recall that in a category the pair

$$\langle \tau_1 : c \to a_1, \tau_2 : c \to a_2 \rangle$$

is a weak pullback for the pair

$$\rho_1 : a_1 \to b, \rho_2 : a_2 \to b$$

of morphisms iff it is a semi-pullback (so that

$$\rho_1 \circ \tau_1 = \rho_2 \circ \tau_2$$

holds), and if

$$\langle \tau_1' : c' \to a_1, \tau_2' : c' \to a_2 \rangle$$

is another semi-pullback for that pair, then there exists a morphism $\theta : c' \to c$ so that

$$\tau_i' = \tau_i \circ \theta \ (i = 1, 2)$$

holds; θ is the *factor.* It θ is unique, then the weak pullback is called a pullback.
The following example shows that even the category of Standard Borel spaces with probability measures where the morphisms are surjective and measure preserving measurable maps does not have always weak pullbacks: Let μ be the uniform distribution on $A := \{1, 2, 3\}$, put $B := \{a, b\}$ with

$$\nu(a) := \frac{2}{3}, \nu(b) := \frac{1}{3}.$$

Let $f : A \to B$ with

$$f(1) := f(2) := a, f(3) := b.$$

Then

$$f : \langle A, \mu\rangle \to \langle B, \nu\rangle$$

is a morphism. Now compute the semi-pullback $\langle P, \gamma\rangle$ for the kernel pair represented by f. Then

$$\begin{aligned} P &= \{\langle x, x'\rangle | f(x) = f(x')\} \\ &= \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}, \end{aligned}$$

and a suitable instance for γ is determined easily (e.g., $\gamma(\langle 3, 3\rangle) = \frac{1}{3}$, all other pairs in P can be assigned $\frac{1}{6}$). The identity $\iota : \langle A, \mu\rangle \to \langle A, \mu\rangle$ has the property $f \circ \iota = f \circ \iota$. If a weak pullback exists, then we know about the factor ρ that $\rho(a) = \langle a, a\rangle$ holds for all $a \in A$; since f is not injective, ρ cannot be onto. This is a contradiction.
The reason for this is evidently that a weak pullback in e.g. $\mathfrak{SRel}$ would induce a weak pullback in the category of sets with ordinary maps as morphisms, but that it cannot be guaranteed there that the factor is onto, even if the morphisms for which the pullback is computed are.
Consequently, semi-pullbacks are the best we can do in $\mathfrak{SRel}$.

4 Bisimulation

This section demonstrates that the bisimulation relation on objects of $\mathfrak{SRel}$ is transitive, and serves as an application for the result that semi-pullbacks exist in this category. A final application is provided by proving the well known result due to Desharnais, Edalat and Panagaden that bisimilarity of labelled Markov processes may be characterized through a simple negation-free modal logic; the processes are based on Standard Borel spaces with measurable — rather than universally measurable — transition sub-probability functions.
We define a bisimulation for two objects in $\mathfrak{SRel}$ through a span of morphisms in that category [13, 17]. This is similar to the notion of 1-bisimulation investigated in [5] for the comma category $\mathbb{1}_{\mathcal{M}} \downarrow \mathbf{S}$, were $\mathcal{M}$ is the category of all measurable spaces with measurable maps as morphisms.

Definition 1 *An object P in $\mathfrak{SRel}$ together with morphisms $\langle \sigma_1, \tau_1 \rangle : P \to Q_1$ and $\langle \sigma_2, \tau_2 \rangle : P \to Q_2$ is called a* bisimulation of objects Q_1 and Q_2.

Let $P := \langle X, Y, K \rangle$ and $Q_i := \langle X_i, Y_i, K_i \rangle$, then we get the familiar commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{\sigma_1} & X & \xrightarrow{\sigma_2} & X_2 \\ \downarrow{\scriptstyle K_1} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle K_2} \\ \mathbf{S}(Y_1) & \xleftarrow[\mathbf{S}(\tau_1)]{} & \mathbf{S}(Y) & \xrightarrow[\mathbf{S}(\tau_2)]{} & \mathbf{S}(Y_2) \end{array}$$

Thus we have effectively established Theorem 1 by constructing a bisimulation for the objects serving as domains for the morphisms investigated in the semi-pullback.
We apply the semi-pullback for establishing the fact that the bisimulation relation is transitive in $\mathfrak{SRel}$.

Proposition 4 *The bisimulation relation between objects in the category $\mathfrak{SRel}$ of stochastic relations is transitive. The same is true for the subcategories of Markov processes introduced in Cor. 1.*

Proof Consider as e.g. in the proof for [17, Theorem 5.4] the diagram in Fig. 1.
The lower triangles are given bisimulations, and the upper diamond with its dotted lines is the semi-pullback for the pair

$$\begin{aligned} \langle \sigma_2, \tau_2 \rangle : \langle A_1, B_1, L_1 \rangle &\to \langle X_2, Y_2, K_2 \rangle \\ \langle \sigma_3, \tau_3 \rangle : \langle A_2, B_2, L_2 \rangle &\to \langle X_2, Y_2, K_2 \rangle \end{aligned}$$

which exists by Thm. 1 for $\mathfrak{SRel}$, and by Cor. 1 for the subcategories. Then

$$\begin{aligned} \langle \sigma \circ \zeta_1, \tau \circ \xi_1 \rangle : \langle A_3, B_3, L_3 \rangle &\to \langle X_1, Y_1, K_1 \rangle \\ \langle \zeta_2 \circ \sigma_4, \xi_2 \circ \tau_4 \rangle : \langle A_3, B_3, L_3 \rangle &\to \langle X_3, Y_3, K_3 \rangle \end{aligned}$$

is the desired bisimulation. □

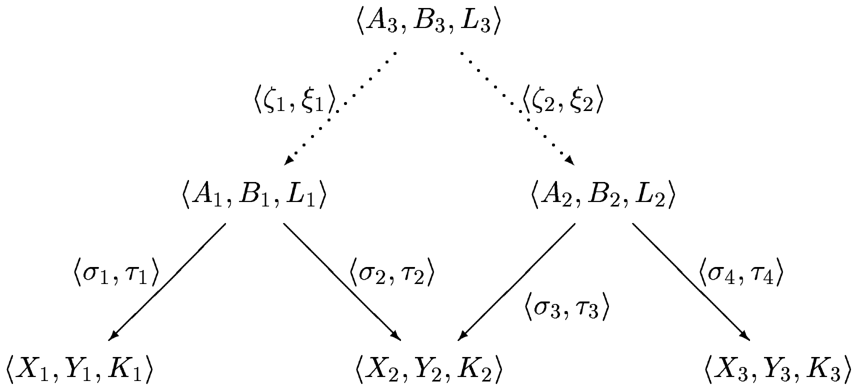

Figure 1: Transitivity of Bisimulation Diagrams

Finally it will be shown that bisimulations for labelled Markov processes can be characterized through a Hennessy-Milner logic. This follows the lines of [3], but we will capitalize on the possibility to construct semi-pullbacks in categories of Markov processes over Polish spaces with Borel (rather than universally) measurable transition sub-probabilities. Hence we can characterize bisimulation in what seems to be a much more natural way from a probabilistic point of view. Some work has to be done for keeping the argumentation within the realm of Polish spaces.
Fix a countable set L of actions.

Definition 2 *Let S be a Standard Borel space, and assume that $k_a : S \rightsquigarrow S$ is a stochastic relation for each $a \in \mathsf{L}$. Then $(S, (k_a)_{a \in \mathsf{L}})$ is called a* labelled Markov process.

S serves as a state space for the process. If the process is in state $s \in S$, and action $a \in \mathsf{L}$ is taken, then $k_a(s, B)$ is the probability for the next state to be a member of Borel set $B \subseteq S$. Before proceeding, recall that a subset $A \subseteq X$ of a Polish space X is called *analytic* iff there exists a Polish space P and a continuous map $f : P \to X$ such that $A = f[P]$ holds. If A is equipped with the trace of the Borel sets of X, viz., $\{A \cap B | B \in \mathcal{B}(X)\}$ then A together with this σ-algebra is called an *analytic space*. The definition of a labelled Markov process found in [3] resembles the one given above, but assumes that the state space is analytic; *generalized* labelled Markov processes are introduced in which the transition sub-probability is assumed to be universally measurable.
Returning to Def. 2, let $(S, (k_a)_{a \in \mathsf{L}})$ and $(S', (k'_a)_{a \in \mathsf{L}})$ be labelled Markov processes with the same set L of actions. A *morphism*

$$f : (S, (k_a)_{a \in \mathsf{L}}) \to \left(S', (k'_a)_{a \in \mathsf{L}}\right)$$

is a surjective Borel map $f : S \to S'$ such that

$$\forall a \in \mathsf{L} : k'_a \circ f = \mathbf{S}(f) \circ k_a$$

holds, so that f is probability preserving for each action. Thus we have for each action $a \in \mathsf{L}$ a morphism between the objects (S, k_a) and (S', k'_a) in the category described in Cor. 1.(2).

Corollary 2 *The category of labelled Markov processes with morphisms described above has semi-pullbacks.*

Proof Apply Cor. 1 for each action separately and collect the results. □

From now on we omit the set L of actions when writing down labelled Markov processes.

In essentially the same way bisimulations are introduced through a span of morphisms: the labelled Markov processes $(S,(k_a))$ and $(S',(k'_a))$ are called *bisimilar* iff there exists a labelled Markov process $(T,(\ell_a))$ and morphisms $(T,(\ell_a)) \to (S,(k_a))\,, (T,(\ell_a)) \to (S',(k'_a))\,.$

We follow [3] in introducing syntax and semantics of the Hennessy-Milner logic $\mathcal{L}$. The syntax is given by

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a \rangle_q \phi$$

Here $a \in \mathsf{L}$ is an action, and q is a rational number. Fix a labelled Markov process $(S,(k_a))$, then satisfaction of a state s for a formula ϕ is defined inductively. This is trivial for $\top$ and for formulas of the form $\phi_1 \wedge \phi_2$. The more complicated case is making an a-move: $s \models \langle a \rangle_q \phi$ holds iff we can find a measurable set $A \subseteq S$ such that $\forall s' \in A : s' \models \phi$ and $k_a(s,A) \geq q$ both hold. Intuitively, we can make an a-move in a state s to a state that satisfies ϕ with probability greater than q.

Denote by Φ the set of all formulas, and put $[\![\phi]\!]_S := [\![\phi]\!] := \{s \in S | s \models \phi\}$ as usual as the set of states that satisfy formula ϕ (we omit the subscript S if the context is clear). Let $(S',(k'_a))$ be another labelled Markov process, then define for $s \in S, s' \in S'$ the relation $s \approx s'$ iff s and s' satisfy all the same formulas. Formally, $s \approx s'$ holds iff $1_{[\![\phi]\!]}(s) = 1_{[\![\phi]\!]}(s')$ holds for all $\phi \in \Phi$, 1_A denoting the indicator function for the set A. Now define for labelled Markov processes

$$(S,(k_a)) \sim \left(S',(k'_a)\right) :\Leftrightarrow \begin{cases} \forall s \in S \exists s' \in S' : s \approx s' \\ \text{and} \\ \forall s' \in S' \exists s \in S : s' \approx s. \end{cases}$$

Hence relation $\sim$ indicates that two labelled Markov processes satisfy exactly the same formulas for logic $\mathcal{L}$.

We will establish for labelled Markov processes the equivalence of bisimilarity and satisfying the same formulas, and we will follow essentially the line of attack pursued in [3]. But we want to stay within the realm of Standard Borel spaces. Here comes the crucial trick. Working as in [3] with the set of equivalence classes with the final Borel structure for the quotient map for $\approx$ would bring us into the realm of analytic spaces, or, as Arveson writes: "It often happens that the quotient of a standard Borel space by a very regular equivalence relation fails to be standard" [2, p. 71]. Instead we will work with a Borel set which intersects each equivalence class in exactly one element (what is usually called a *Borel cross section*, cf. [18, p. 186]). Now this is the plan: we investigate the relation $\approx$ and show that it has a Borel cross section T. Because T is a Borel set in a Polish space, it is a Standard Borel space itself. With T comes a surjection

$$f_T : S \to T$$

which has all the necessary properties of the quotient map, so that we can construct from $(S,(k_a))$ another labelled Markov process $(T,(h_a))$ with f_T now acting as morphism. This is then applied to the case that $(S,(k_a)) \sim (S',(k'_a))$ by forming the sum of the processes and constructing from this sum through relation $\approx$ morphisms the semi-pullback of which will yield the desired bisimulation. So the plan is very similar to that in [3], but the terrain will be operated on in a slightly different manner.

Some important properties of relation $\approx$ are collected now.

Lemma 2 *Let $(S,(k_a))$ be a labelled Markov process, then*

1. $\approx\subseteq S \times S$ *is a Borel set.*
2. *There exists a Borel set $T \subseteq S$ such that T intersects each equivalence class in exactly one state.*
3. *There exists a Borel map $f_T : S \to T$ such that for all $s, s' \in S : s \approx s'$ implies $f_T(s) = f_T(s')$, and $\forall s \in S : s \approx f_T(s)$ holds.*

Proof 0. From [3, Prop. 9.1] (or by structural induction) it is seen that $[\![\phi]\!]$ is a measurable subset of S. The set

$$\mathcal{F} := \{[\![\phi]\!] \,|\phi \in \Phi\}$$

is countable, so we can find by Prop. 1 a Polish topology $\mathcal{T}$ on S having the same Borel sets as the given σ-algebra such that every $[\![\phi]\!]$ is closed. We assume throughout this proof that S is equipped with $\mathcal{T}$.
1. The equivalence class $[s]$ for $s \in S$ can be represented as

$$[s] = \bigcap\{[\![\phi]\!] \,|\phi \in \Phi \text{ with } s \models \phi\} \cap \bigcap\{S \setminus [\![\phi]\!] \,|\phi \in \Phi \text{ with } s \not\models \phi\}$$

Thus $[s]$ is a G_δ-set (i.e., a countable intersection of open sets), since in a Polish space each closed set is a G_δ. In particular, $[s]$ is a Borel set.
2. Enumerate $\mathcal{F}$ as $(F_n)_{n\in\mathbb{N}}$, then

$$S \times S \setminus \approx \;=\; \bigcup_{n\in\mathbb{N}} \left((S \setminus F_n) \times F_n\right) \cup \left(F_n \times (S \setminus F_n)\right).$$

Since each F_n is Borel, $\approx\subseteq S \times S$ is Borel, too. This establishes the first claim.
3. Thus the relation $\approx$ is a Borel subset of $S \times S$, and each equivalence class is a G_δ. Hence the second assertion follows from [18, Theorem 5.9.2], and from [18, Prop. 5.1.9] f_T is constructed with the desired properties, yielding the third assertion. □
By virtue of Prop. 1, T is a Standard Borel space with the Borel sets of S that are contained in T as the σ-algebra. The Borel map f_T that comes with T may be interpreted as a selection map, since $s \approx f_T(s)$ always holds. Hence f_T picks from each class a representative in a measurable way; this will help bypassing the cumbersome construction of the factor space in [3]. f_T is plainly surjective. We will capitalize on these properties in the sequel.

Lemma 3 *Let $(S,(k_a)), T, f_T$ as above. The σ-algebra on T is generated by*

$$\mathcal{B}_0 := \{f_T\left[[\![\phi]\!]\right] |\phi \in \Phi\},$$

and $\mathcal{B}_0$ is closed under finite intersections.

Proof 0. We need to show that $f_T\left[[\![\phi]\!]\right]$ is always a Borel set in T, that $\mathcal{B}_0$ contains the intersection of two of its members, and that $\mathcal{B}_0$ separates points. Then the assertion will follow from the Mackey's Unique Structure Theorem [2, Theorem 3.3.5].
1. Fix $\phi \in \Phi$, and put $A := f_T\left[[\![\phi]\!]\right]$, then A is the image of a Borel set by a Borel map, hence is analytic. Similarly, $B := f_T\left[S \setminus [\![\phi]\!]\right]$ is an analytic set, and $A \cup B = T$. The construction of T and f_T imply that $A \cap B = \emptyset$. Hence we have two non-empty analytic sets that partition a Polish space, and Souslin's Theorem [18, Theorem 4.4.3] implies that A is a Borel set.

2. Now let $t_1, t_2 \in T$ with $t_1 \neq t_2$. There are $s_1, s_2 \in S$ with $t_1 = f_T(s_1), t_2 = f_T(s_2)$ and $s_1 \not\approx s_2$. Consequently, there is a formula ϕ with, say, $s_1 \models \phi$ and $s_2 \not\models \phi$. But this means $t_1 \in f_T[[\![\phi]\!]], t_2 \notin f_T[[\![\phi]\!]]$.
3. We need to show that

$$f_T[[\![\phi_1 \wedge \phi_2]\!]] = f_T[[\![\phi_1]\!]] \cap f_T[[\![\phi_2]\!]]$$

holds. The non-trivial inclusion follows from the construction of T: if $t = f_T(s_1) = f_T(s_2)$ with $s_1 \models \phi_1$ and $s_2 \models \phi_2$, then $s_1 \approx s_2$, and $s_1, s_2 \in [\![\phi_1 \wedge \phi_2]\!]$. Hence $t \in f_T[[\![\phi_1 \wedge \phi_2]\!]]$. □
From these data a labelled Markov process can be constructed:

Corollary 3 *Let S, T, f_T be as above, and put for $a \in \mathsf{L}, s \in S$ and the measurable $B \subseteq T$*

$$h_a(f_T(s))(B) := k_a(s)(f_T^{-1}[B]).$$

This defines a labelled Markov process $(T, (h_a))$ such that

$$f_T : (S, (k_a)) \to (T, (h_a))$$

is a morphism.

Proof 0. For each formula ϕ, the equality

$$[\![\phi]\!] = f_T^{-1}[f_T[[\![\phi]\!]]]$$

holds. For, if $s \in f_T^{-1}[f_T[[\![\phi]\!]]]$, then $f_T(s) \models \phi$, since $s \approx f_T(s)$, $s \models \phi$ also holds, thus $s \in [\![\phi]\!]$. This settles the non-obvious inclusion.
1. It is not difficult to see that for each $\phi \in \Phi$ the equality

$$k_a(s_1)([\![\phi]\!]) = k_a(s_2)([\![\phi]\!])$$

holds, provided that $s_1 \approx s_2$ (see [3, Lemma 9.6]). Thus $h_a(t)(B)$ is well-defined for each $B \in \mathcal{B}_0$. Since $\mathcal{B}_0$ generates the Borel sets on T, and since $\mathcal{B}_0$ is closed under finite intersections, $h_a(t)$ is well-defined on the Borel sets of T.
2. For establishing measurability properties of h_a, we put

$$\mathcal{C} := \{B \subseteq T | t \mapsto h_a(t)(B) \text{ is measurable and } B \text{ is Borel}\}$$

Then $\mathcal{C}$ is a σ-algebra by the familiar properties of measurable maps. We claim that $\mathcal{B}_0 \subseteq \mathcal{C}$ holds. In fact, let $\phi \in \Phi$ be a formula, then $t \mapsto h_a(t)(f_T[[\![\phi]\!]])$ has shown to be measurable. Since the half-closed intervals $[q, +\infty[$ with q rational generate the Borel sets on $\mathbb{R}$, it suffices to show that their inverse images are Borel sets in T. Since

$$\begin{aligned}(h_a(\cdot)(f_T[[\![\phi]\!]]))^{-1}[[q, +\infty[] &= \{t \in T | h_a(t)(f_T[[\![\phi]\!]]) \geq q\} \\ &= f_T[[\![\langle a \rangle_q \phi]\!]],\end{aligned}$$

and the latter set is indeed a Borel set in T by Lemma 3, we see that $\mathcal{B}_0 \subseteq \mathcal{C}$. Consequently, $\mathcal{C}$ is just the Borel σ-algebra on T.
3. This establishes that $(T, (h_a))$ is a labelled Markov process. By construction, f_T is a morphism $(S, (k_a)) \to (T, (h_a))$. □
We can now prove that satisfying the same formulas and bisimilarity are equivalent. The proof follows the trail laid out in [3], but it makes use of the constructions developed so far.

Theorem 2 *Two labelled Markov processes are bisimilar iff they satisfy the same formulas.*

Proof 1. The "only if"- part follows from [3, Cor. 9.3], so only the "if"-part needs to be established. We proceed as in the proof of [3, Theorem 9.10] by constructing from the labelled Markov processes $(S, (k_a))$ and $(S', (k'_a))$ a diagram of the form

$$\begin{array}{ccc} & & (S', (k'_a)) \\ & & \downarrow \\ (S, (k_a)) & \longrightarrow & (T, (h_a)) \end{array}$$

From this, a semi-pullback (Cor. 2) will provide the desired bisimulation.
2. Let S_0 be the sum of the Standard Borel spaces S and S', hence S_0 is a Standard Borel space again. Put for $a \in \mathsf{L}, s \in S_0$ and the Borel set $B \subseteq S_0$

$$\ell_a(s)(B) := \begin{cases} k_a(s)(S \cap B), & s \in S \\ k'_a(s)(S' \cap B), & s \in S' \end{cases}$$

Thus $(S_0, (\ell_a))$ is a labelled Markov process.
Construct T and f_T from S_0 and from the equivalence relation $\approx$ on S_0 according to Lemma 2, and define the labelled Markov process $(T, (h_a))$ as in Lemma 3. Let

$$i : S \to S_0, i' : S' \to S_0$$

be the embeddings of S resp. S' into S_0. Then

$$f_T \circ i : S \to T, f_T \circ i' : S' \to T$$

are surjective, since $(S, (k_a)) \sim (S', (k'_a))$. Both are morphisms. □
The *bisimulation type* of a labelled Markov process is that subset of the formulas Φ that the process satisfies, so that each equivalence class of processes with respect to $\sim$ is uniquely characterized through a subset of Φ. Since Φ is countable, it turns out that there are at most $2^{\aleph_0}$ bisimulation types.

5 Conclusion

We show that one can construct a semi-pullback in the category of Markov processes over Standard Borel spaces with continuous and measure preserving maps as morphisms. This is actually a special case of a more general result which deals with stochastic relations over Standard Borel spaces in which the class of Polish spaces mentioned above serve as target spaces for transition probability functions. It is shown that in the latter category the bisimulation relation is transitive. It is finally shown that the characterization of bisimulation through satisfiability in a simple logic may be derived in this conceptually simpler context, too.
Rather than constructing the object underlying a semi-pullback explicitly, we rely on selection arguments from the theory of set-valued relations. This gives probably less technical insight into the nature of the object one looks for, but is easier to apply, and it permits drawing from the rich well of topology, in particular utilizing the weak topology on the space of all

subprobability measures. Selection arguments may be a helpful way of constructing objects; we illustrate this by showing that the map which assigns each Polish space its subprobabilities and each surjective Borel measurable map the corresponding measure transform is actually a functor which may be difficult to establish otherwise.
Further work will address the characterization of bisimilarity of stochastic relations through a suitable logic. These relations may be viewed as the many-sorted cousins of Markov processes, so that a similar characterization would be desirable.

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