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A Remark on A. Edalat's Paper *Semi-Pullbacks and Bisimulations in Categories of Markov-Processes*

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Abstract

The problem of constructing a semi-pullback in a category is intimately connected to the problem of establishing the transitivity of bisimulations. Edalat shows that a semi-pullback can be constructed in the category of Markov processes on Polish spaces, when the underlying transition probability functions are universally measurable, and the morphisms are measure preserving continuous maps. We show that one needs not to resort to universal measurability, provided the Polish space is locally compact. This paper shows that Borel measurability is sufficient, since the corresponding category is closed under taking semi-pullbacks. This is in fact a special case: we consider the category of stochastic relations over Standard Borel spaces with locally compact target spaces, and establish the result there.
This gives a partial answer to Panangaden's question regarding the transitivity of the bisimulation relation in categories of Markov processes.

Keywords: Bisimulation, semi-pullback, stochastic relations, category of Markov processes.

1 Introduction

The existence of semi-pullbacks in a category makes sure that the bisimulation relation is transitive, provided bisimulation between objects is defined as a span of morphisms [JNW96]. Edalat investigates this question for categories of Markov processes and shows that semi-pullbacks exist [Eda99]. The category he focusses on has as objects universally measurable transition probability functions on Polish spaces, the morphisms are continuous, surjective, and probability preserving maps. His proof is constructive and makes essentially use of techniques of analytic spaces (which are continuous images of Polish spaces). The result implies that the semi-pullback of those transition probabilities which are measurable with respect to the Borel sets of the Polish spaces under consideration may in fact be universally measurable rather than simply Borel measurable. This then demands some unpleasant technical machinery when logically characterizing bisimulation for labelled Markov processes, cf. [DEP99]. Quite apart from that, it is somewhat annoying that forming the semi-pullback for Borel Markov processes seems to require a change of categories. This is so because this Borel measurability comes naturally with the Borel sets of a Polish space, whereas universal measurability requires a somewhat elaborate completion process.

This short note shows among others that the semi-pullback of Borel Markov processes exists within the category of these processes, when the underlying space is Polish and locally compact (like the real line). Edalat considers transition probability functions from one Polish space into itself, this paper considers the slightly more general notion of a stochastic relation, cf. [Pan98, ABP99, Dob02a, Dob80] i.e., transition probability functions from one Polish space to another one, where the target space is locally compact. Rather than constructing the function explicitly, as Edalat does, we rely on a selection argument: we show that the problem can be formulated in terms of measurable set-valued maps for which a measurable selector exists.

The paper's contributions are twofold. First it is shown that one can in fact construct semi-pullbacks in a category of stochastic relations between Polish spaces under a compactness assumption without using the somewhat heavy machinery of universal measurability and analytic spaces. Hence the trade-off presents itself as to work either in a general Polish space and then to construct universal measurable semi-pullbacks, or to accept the compactness assumptions and to be able to construct the semi-pullback in the category itself. The second contribution is the reduction of an existential argument to a selection argument, a technique borrowed from dynamic optimization.

This note is organized as follows: Sect. 2 collects some basic facts from topology, and from measure theory. It is shown that assigning a Polish space its set of subprobability measures is an endofunctor on this category, opening the road to discuss applications through monads, cf.[Gir81, Dob02b]. Sect. 3 defines the category of stochastic relations, shows how to formulate the problem in terms of a set-valued function, and proves that a selector for that function exists. This implies the existence of semi-pullbacks for some related categories, too. Finally, we show in Sect. 4 that the bisimulation relation is transitive for the category of stochastic relations. Sect. 5 wraps it all up by summarizing the results and indicating areas of further work. It is clear that the topological assumptions should be weakened further.

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2 Preliminaries

This Section collects some basic facts from topology and measure theory for the reader's convenience and for later reference.
A *Polish space* $(X, \mathcal{T})$ is a topological space which has a countable dense subset, and which is metrizable through a complete metric. The *Borel sets* $\mathcal{B}(X, \mathcal{T})$ for the topology $\mathcal{T}$ is the smallest σ-algebra on X which contains $\mathcal{T}$. A *Standard Borel* space $(X, \mathcal{A})$ is a measurable space such that the σ-algebra $\mathcal{A}$ equals $\mathcal{B}(X, \mathcal{T})$ for some Polish topology $\mathcal{T}$ on X. Although the Borel sets are determined uniquely through the topology, the converse does not hold, as we will see in a short while. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a map $f : X \to Y$ is $\mathcal{A} - \mathcal{B}$-*measurable* whenever

$$f^{-1}[\mathcal{B}] \subseteq \mathcal{A}$$

holds, where

$$f^{-1}[\mathcal{B}] := \{f^{-1}[B] | B \in \mathcal{B}\}$$

is the set of inverse images

$$f^{-1}[B] := \{x \in X | f(x) \in B\}$$

of elements of $\mathcal{B}$. Note that $f^{-1}[\mathcal{B}]$ is in any case an σ-algebra. If the σ-algebras are the Borel sets of some topologies on X and Y, resp., then a measurable map is called *Borel measurable* or simply a *Borel* map. The real numbers $\mathbb{R}$ carry always the Borel structure induced by the usual topology which will not be mentioned explicitly when talking about Borel maps.
A map $f : X \to Y$ between the topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is *continuous* iff the inverse image of an open set from $\mathcal{S}$ is an open set in $\mathcal{T}$. Thus a continuous map is also measurable with respect to the Borel sets generated by the respective topologies.
When the context is clear, we will write down Polish spaces without their topologies, and the Borel sets are always understood with respect to the topology. Measurable maps with respect to the Borel sets of a Polish topology will simply be called *Borel maps.*
It will turn out to be helpful to make more precise statements of the measurability of a Borel map:

Proposition 1 *Let X, Y be Polish spaces, and assume that $g : X \to Y$ is continuous and onto. If $f : X \to Y$ is Borel measurable such that f is constant on the atoms of $g^{-1}[\mathcal{B}(Y)]$, then f is $g^{-1}[\mathcal{B}(Y)] - \mathcal{B}(Z) -$ measurable.*

Proof Recall that an atom $A \in g^{-1}[\mathcal{B}(Y)]$ has the property that $\emptyset \neq A$, and each subset B of A is either empty or equals A. The atoms of $g^{-1}[\mathcal{B}(Y)]$ are just the inverse images $g^{-1}[\{y\}]$ of the points $y \in Y$, because these sets are clearly atomic in that σ-algebra, and since they form a partition of X. Now let $B \in \mathcal{B}(Y)$ be a Borel set, then by assumption $f^{-1}[B]$ is a Borel set in X which is the union of atoms of $g^{-1}[\mathcal{B}(Y)]$. Thus the assertion follows from the Blackwell-Mackey-Theorem [Sri98, Thm. 4.5.7]. □
The following surprising statement will be of use in the sequel. It states that, given a measurable map between Polish spaces, we can find a finer Polish topology on the domain, which has the same Borel sets, and which renders the map continuous; formally:

Proposition 2 *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be Polish spaces, and let $f : X \to Y$ be a Borel measurable map. Then there exists a Polish topology $\mathcal{T}'$ on X with these properties:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$ (hence $\mathcal{T} \subseteq \mathcal{T}'$), and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}')$,*

2. *f is $\mathcal{T}' - \mathcal{S}$ continuous.*

Proof [Sri98, Cor.3.2.6] □
Given two Polish spaces X and Y, a *stochastic relation* $K : X \rightsquigarrow Y$ is a Borel map from X to the set $\mathbf{S}(Y)$, the latter denoting the set of all subprobability measures on (the Borel sets) of Y. This set carries the weak topology, i.e., the smallest topology which makes the map $\mu \mapsto \int_Y f\ d\mu$ for all continuous functions $f : Y \to \mathbb{R}$ continuous. It is well known that the weak topology on $\mathbf{S}(Y)$ is a Polish space [Par67, Theorem II.6.5], and that its Borel sets are the smallest σ-algebra on $\mathbf{S}(Y)$ for which for any Borel set $B \subseteq Y$ the map $\mu \mapsto \mu(B)$ is measurable ([Kec94, Thm. 17.24]; this σ-algebra is sometimes called the *weak-*-σ-algebra* in stochastic dynamic optimization). Hence $K : X \rightsquigarrow Y$ is a stochastic relation iff

1. $K(x)$ is a subprobability measure on (the Borel sets of) Y for all $x \in X$,
2. $x \mapsto K(x)(B)$ is a measurable map for each Borel set $B \subseteq Y$.

A Borel map $f : X \to Y$ between the Polish spaces X and Y induces a Borel map

$$\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$$

upon setting ($\mu \in \mathbf{S}(X), B \subseteq Y$ Borel)

$$\mathbf{S}(f)(\mu)(B) := \mu(f^{-1}[B])$$

It is easy to see that a continuous map f induces a continuous map $\mathbf{S}(f)$, and we will see in a moment that $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ is onto, provided $f : X \to Y$ is. Denote by $\mathbf{P}(X)$ the subspace of all probability measures on X.
Let $\mathcal{F}(X)$ be the set of all closed and non-empty subsets of the Polish space X, and call for Polish Y a relation, i.e., a set-valued map $F : X \to \mathcal{F}(Y)$ *$\mathcal{C}$-measurable* iff the weak inverse

$$\exists F(C) := \{x \in X | F(x) \cap C \neq \emptyset\}$$

for a compact set $C \subseteq Y$ is measurable. A *selector* s for such a relation F is a single-valued map $s : X \to Y$ such that $\forall x \in X : s(x) \in F(x)$ holds. $\mathcal{C}$-measurable relations have Borel selectors:

Proposition 3 *Let X and Y be Polish spaces. Then each $\mathcal{C}$-measurable relation F has a measurable selector.*

Proof Since closed subsets of Polish spaces are complete, the assertion follows from [HV69, Theorem 3]. □
Postulating measurability for $\exists F(C)$ for open or for closed sets C leads to the general notion of a measurable relation. These relations are a valuable tool in such diverse fields as stochastic dynamic programming [Wag77] and descriptive set theory [Kec94]. Overviews are provided in [Sri98, Chapter 5] and [Him75, Wag77].
As a first application it is shown that $\mathbf{S}$ actually constitutes an endofunctor on the category of Standard Borel spaces with surjective measurable map as morphisms. This implies that $\mathbf{S}$ is the functorial part of a monad $\langle \mathbf{S}, \eta, \mu \rangle$ very similar to the one studied by Giry, cf. [Gir81]. The crucial part is evidently to show that $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ is a surjection whenever $f : X \to Y$ is one. This is done through a measurable selection argument using Prop. 3.

Lemma 1 $\mathbf{S}$ *is an endofunctor on the category* $\mathfrak{SB}$ *of Standard Borel spaces with surjective Borel maps as morphisms.*

Proof 1. Let X and Y be Standard Borel spaces, and endow these spaces with a Polish topology the Borel sets of which form the respective σ-algebras. Since $\mathbf{S}(X)$ is a Polish space under the topology of weak convergence, and since a Borel map $f : X \to Y$ induces a Borel map $\mathbf{S}(f) : \mathbf{S}(X) \to \mathbf{S}(Y)$ with all the compositional properties a functor should have, only surjectivity of the induced map has to be shown.
2. In view of Prop. 2 it is no loss of generality to assume that f is continuous (otherwise consider a finer Polish topology with the same Borel sets rendering f continuous). Continuity and surjectivity together imply that $y \mapsto f^{-1}[\{y\}]$ has closed and non-empty values in X, and constitutes a $\mathcal{C}$-measurable relation, which has a measurable selector $g : Y \to X$ by Prop. 3, so that $f(g(y)) = y$ always holds. Let $\nu \in \mathbf{S}(Y)$, and define $\mu \in \mathbf{S}(X)$ upon setting $\mu(A) := \nu(g^{-1}[A])$ for $A \subseteq X$ Borel. Since $g^{-1}\left[f^{-1}[B]\right] = B$ for $B \subseteq Y$, it is now easy to establish that $\mathbf{S}(f)(\mu) = \nu$ holds. $\Box$
We will need finally to observe the interplay between linear functionals and measures. Let for a topological space $\mathcal{C}(X)$ denote the linear space of continuous real-valued functions on X.

Proposition 4 *Let* $\mathcal{C}_0$ *be a linear subspace of* $\mathcal{C}(X)$*, where* X *is a locally compact Polish space, and assume that* $\mathcal{C}_0$ *contains all the constants. If*

$$\Lambda_0 : \mathcal{C}_0 \to \mathbb{R}$$

is a positive linear functional on $\mathcal{C}_0$ *with* $\Lambda_0(1) = 1$*, then there exists* $\mu \in \mathbf{P}(X)$ *such that* μ *represents* Λ_0*, i.e.,*

$$\Lambda_0(f) = \int_X f \, d\mu$$

holds for each $f \in \mathcal{C}_0$.

Proof 1. Since $\Lambda_0(1) = 1$, and since Λ_0 is positive, we know that

$$\Lambda_0(f) \leq ||f||_\infty := \sup_{x \in X} |f(x)|$$

holds for each $f \in \mathcal{C}_0$. By the ordered version of the Hahn-Banach Theorem [Jac78, Lemma IX.1.4], there exists a positive linear operator $\Lambda : \mathcal{C}(X) \to \mathbb{R}$ extending Λ_0 to all of $\mathcal{C}(X)$, such that

$$\Lambda(f) \leq ||f||_\infty$$

holds for all $0 \leq f \in \mathcal{C}(X)$. Since X is locally compact and σ-compact (by the existence of a countable base), the Riesz Representation Theorem [Els99, Satz 2.19 (b)] gives a (unique) measure μ on the Borel sets of X such that

$$\Lambda(f) = \int_X f \, d\mu.$$

Since Λ extends Λ_0, the assertion follows. $\Box$
The topological assumption is crucial for the application of the Riesz Representation Theorem, and it is this limitation which determines the generality of the existence of semi-pullbacks to be proven below.

3 Semi-Pullbacks

The category $\mathfrak{SRel}$ of stochastic relations has as objects triplets $\langle X, Y, K\rangle$, where X is a Standard Borel space, and Y is a locally compact Polish space, and $K : X \rightsquigarrow Y$ is a stochastic relation. A morphism

$$\langle\varphi, \psi\rangle : \langle X, Y, K\rangle \to \langle X', Y', K'\rangle$$

is a pair of surjective Borel maps $\varphi : X \to X'$ and $\psi : Y \to Y'$ with ψ continuous such that

$$K' \circ \varphi = \mathbf{S}(\psi) \circ K$$

holds, hence such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle K}\downarrow & & \downarrow{\scriptstyle K'} \\ \mathbf{S}(Y) & \xrightarrow[\mathbf{S}(\psi)]{} & \mathbf{S}(Y') \end{array}$$

is commutative. Thus we have for $x \in X, B' \subseteq Y'$ Borel the equality

$$K'(\varphi(x))(B') = K(x)(\psi^{-1}[B']),$$

so that morphisms are in particular measure preserving. Morphisms compose componentwise. The category $\mathfrak{MProc}$ of Markov processes is a subcategory of $\mathfrak{SRel}$: $\mathfrak{MProc}$ has as objects pairs $\langle X, K\rangle$, where X is a locally compact Polish space, and $K : X \rightsquigarrow X$ is a stochastic relation, i.e., a Borel measurable transition probability function. Thus we assume in contrast to [Eda99] that K is Borel measurable. This is a stronger notion than universal measurability, since each Borel map is universally measurable by [HJ70, Prop.I.B.6]. Morphisms in $\mathfrak{MProc}$ are surjective and continuous maps which are measure preserving.
Assume that $\langle\varphi_i, \psi_i\rangle : \langle X_i, Y_i, K_i\rangle \to \langle X, Y, K\rangle$ $(i = 1, 2)$ are morphisms in $\mathfrak{SRel}$, then a *semi-pullback* for this pair of morphisms is an object $\langle A, B, N\rangle$ together with morphisms $\langle\alpha_i, \beta_i\rangle : \langle A, B, N\rangle \to \langle X_i, Y_i, K_i\rangle$ $(i = 1, 2)$ so that this diagram is commutative in $\mathfrak{SRel}$:

$$\begin{array}{ccc} \langle A, B, N\rangle & \xrightarrow{\langle\alpha_1, \beta_1\rangle} & \langle X_1, Y_1, K_1\rangle \\ {\scriptstyle \langle\alpha_2, \beta_2\rangle}\downarrow & & \downarrow{\scriptstyle \langle\varphi_1, \psi_1\rangle} \\ \langle X_2, Y_2, K_2\rangle & \xrightarrow[\langle\varphi_2, \psi_2\rangle]{} & \langle X, Y, K\rangle \end{array}$$

This means in particular that

$$\begin{aligned} K_1 \circ \alpha_1 &= \mathbf{S}(\beta_1) \circ N, \\ K_2 \circ \alpha_2 &= \mathbf{S}(\beta_2) \circ N \end{aligned}$$

should hold, so that a bisimulation is to be constructed (cf. Def. 1). The condition that $\langle A, B, N\rangle$ is the object underlying a semi-pullback may be formulated in terms of measurable maps as follows: N is a map from the Standard Borel space A to the Standard Borel space $\mathbf{S}(B)$ so that N is also a measurable selector for the set-valued function

$$b \mapsto \{\mu \in \mathbf{S}(B) \mid (K_1 \circ \alpha_1)(b) = \mathbf{S}(\beta_1)(\mu), (K_2 \circ \alpha_2)(b) = \mathbf{S}(\beta_2)(\mu)\}.$$

This translates the problem of finding the object $\langle A, B, N\rangle$ of a semi-pullback to a selection problem for set-valued maps, provided the spaces A and B together with the morphisms are identified.
It should be noted that the notion of a semi-pullback depends only on the measurable structure of the Standard Borel spaces involved. The topological structure enters only through Borel sets, and Borel measurability. From Prop. 2 we see that there are certain degrees of freedom for selecting a Polish topology that generate the Borel sets. They will be capitalized upon in the sequel.
We want to establish:

Theorem 1 $\mathfrak{SRel}$ *has semi-pullbacks for each pair of morphisms*

$$\begin{aligned} \langle \varphi_1, \psi_1\rangle : \langle X_1, Y_1, K_1\rangle &\rightarrow \langle X, Y, K\rangle \\ \langle \varphi_2, \psi_2\rangle : \langle X_2, Y_2, K_2\rangle &\rightarrow \langle X, Y, K\rangle \end{aligned}$$

with a common range.

We begin with a measure-theoretic and rather technical observation: in terms of probability theory, it states that there under certain conditions a common distribution for two random variables with values in a Polish space exists with preassigned marginal distributions.

Proposition 5 *Let Z_1, Z_2, Z be Polish spaces which are locally compact,*

$$\zeta_i : Z_i \rightarrow Z \ \ (i = 1, 2)$$

continuous and surjective maps, define

$$S := \{\langle x_1, x_2\rangle \in Z_1 \times Z_2 | \zeta(x_1) = \zeta_2(x_2)\},$$

and let $\nu_1 \in \mathbf{P}(Z_1), \nu_2 \in \mathbf{P}(Z_2), \nu \in \mathbf{P}(S)$ such that

$$\forall E_i \in \zeta_i^{-1}[\mathcal{B}(Z)] : \mathbf{P}(\pi_i)(\nu)(E_i) = \nu_i(E_i) \ \ (i = 1, 2)$$

holds, where $\pi_1 : S \rightarrow Z_1, \pi_2 : S \rightarrow Z_2$ are the projections; S carries the trace of the product topology. Then there exists $\mu \in \mathbf{P}(S)$ such that

$$\forall E_i \in \mathcal{B}(Z_i) : \mathbf{P}(\pi_i)(\mu)(E_i) = \nu_i(E_i) \ \ (i = 1, 2)$$

holds.

Proof 0. By standard arguments from measure theory the assumption is tantamount to saying that

$$\int_{Z_i} f_i \, d\nu_i = \int_S f_i \circ \pi_i \, d\nu$$

holds for every $f_i : Z_i \to \mathbb{R}$ which is $\zeta_i^{-1}[\mathcal{B}(Z)]-$ measurable $(i = 1, 2)$. By the same token it will be sufficient to show that there exists $\mu \in \mathbf{P}(S)$ such that

$$\int_{Z_i} f_i \; d\nu_i = \int_S f_i \circ \pi_i \; d\mu$$

holds for every $f_i \in \mathcal{C}(Z_i)$.
1. If $\zeta_i : Z_i \to Z$ is injective, then each singleton of Z_i is a member of $\zeta_i^{-1}[\mathcal{B}(Z)]$. Hence the latter σ-algebra equals $\mathcal{B}(Z_i)$ by [Sri98, Cor. 4.5.10], since it is countably generated. Thus we assume that neither ζ_1 nor ζ_2 is injective: if both of them are, there will be nothing to demonstrate, if one of them is, and the other is not, the demonstration below will show how to handle this situation.
2. Since Z_1 and Z_2 are Polish, and since the maps involved are continuous, S is closed, hence Polish; in a similar way, local compactness is established for S. Define for $i = 1, 2$ the sets

$$\mathcal{D}_i := \{f_i \circ \pi_i | f_i \in \mathcal{C}(Z_i)\},$$

then $\mathcal{D}_i \subseteq \mathcal{C}(S)$. Let $g \in \mathcal{D}_1 \cap \mathcal{D}_2$, thus there exist $f_i \in \mathcal{C}(Z_i)$ such that

$$g = f_1 \circ \pi_1 = f_2 \circ \pi_2$$

holds. For $x_1 \neq x_2$ with $\zeta_1(x_1) = \zeta_1(x_2)$ we can find $y \in Z_2$ such that $\langle x_1, y\rangle \in S, \langle x_2, y\rangle \in S$, thus

$$g(x_1, y) = f_1(x_1) = f_2(y) \text{ and } g(x_2, y) = f_1(x_2) = f_2(y),$$

consequently, f_1 is constant on the atoms of $\zeta_1^{-1}[\mathcal{B}(Z)]$, hence f_1 is $\zeta_1^{-1}[\mathcal{B}(Z)]$-measurable by Prop. 1. A similar argument shows that f_2 is $\zeta_2^{-1}[\mathcal{B}(Z)]$-measurable. We have accordingly

$$\begin{aligned} \int_S g \; d\nu &= \int_S f_1 \circ \pi_1 \; d\nu \\ &= \int_{Z_1} f_1 \; d\nu_1, \\ \int_S g \; d\nu &= \int_S f_2 \circ \pi_2 \; d\nu \\ &= \int_{Z_2} f_2 \; d\nu_2. \end{aligned}$$

3. Define for $f \in \mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2$

$$\Lambda_0(f) := \begin{cases} \int_{Z_1} f_0 \; d\nu_1, & f = f_0 \circ \pi_1 \in \mathcal{D}_1 \\ \int_{Z_2} f_0 \; d\nu_2, & f = f_0 \circ \pi_2 \in \mathcal{D}_2, \end{cases}$$

then $\Lambda_0 : \mathcal{D} \to \mathbb{R}$ is well-defined because of part 2, it is monotone with $\Lambda_0(1) = 1$. Extend Λ_0 linearly to the linear space generated by $\mathcal{D}$, then Prop. 4 implies that there exists $\mu \in \mathbf{P}(S)$ representing Λ_0. In particular we have for $f_i \in \mathcal{C}(Z_i)$ $(i = 1, 2)$ that

$$\int_S f_i \circ \pi_i \; d\mu = \int_{Z_i} f_i \; d\nu_i$$

holds. But this implies the assertion. □

Now we have all tools for establishing that $\mathfrak{SRel}$ has semi-pullbacks.

Proof of Theorem 1

1. In view of Prop. 2 we may assume that the respective σ-algebras on X_1 and X_2 are obtained from Polish topologies which make φ_1 and K_1 as well as φ_2 and K_2 continuous. Put

$$\begin{aligned} A &:= \{\langle x_1, x_2\rangle \in X_1 \times X_2 | \varphi_1(x_1) = \varphi_2(x_2)\}, \\ B &:= \{\langle y_1, y_2\rangle \in Y_1 \times Y_2 | \psi_1(y_1) = \psi_2(y_2)\}, \end{aligned}$$

then both A and B are closed, hence Polish, and B is locally compact. $\alpha_i : A \to X_i$ and $\beta_i : B \to Y_i$ are the projections, $i = 1, 2$. The diagrams

$$\begin{array}{ccccc} X_1 & \xrightarrow{\varphi_1} & X & \xleftarrow{\varphi_2} & X_2 \\ \downarrow{\scriptstyle K_1} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle K_2} \\ \mathbf{S}(Y_1) & \xrightarrow[\mathbf{S}(\psi_1)]{} & \mathbf{S}(Y) & \xleftarrow[\mathbf{S}(\psi_2)]{} & \mathbf{S}(Y_2) \end{array}$$

are commutative by assumption, thus we know that for $x_i \in X_i$

$$\begin{aligned} K(\varphi_1(x_1)) &= \mathbf{S}(\psi_1)(K_1(x_1)) \\ K(\varphi_2(x_2)) &= \mathbf{S}(\psi_2)(K_2(x_2)) \end{aligned}$$

both hold. The construction implies that

$$(\psi_1 \circ \beta_1)(y_1, y_2) = (\psi_2 \circ \beta_2)(y_1, y_2)$$

is true for $\langle y_1, y_2\rangle \in B$, and $\psi_1 \circ \beta_1 : B \to Y$ is surjective.

2. Fix $\langle x_1, x_2\rangle \in A$. Lemma 1 shows that $\mathbf{S}$ is an endofunctor on $\mathfrak{SB}$, in particular that the image of a surjective map under $\mathbf{S}$ is onto again, so that there exists $\mu \in \mathbf{S}(S)$ with

$$\mathbf{S}(\psi_1 \circ \beta_1)(\mu) = K(\varphi_1(x_1)),$$

consequently,

$$\mathbf{S}(\psi_i \circ \beta_i)(\mu) = \mathbf{S}(\psi_i)(K_i(x_i))\ (i = 1, 2).$$

But this means

$$\forall E_i \in \psi_i^{-1}[\mathcal{B}(Y)] : \mathbf{S}(\beta_i)(\mu)(E_i) = K_i(x_i)(E_i)\ (i = 1, 2).$$

Put

$$\Gamma(x_1, x_2) := \{\mu \in \mathbf{S}(B) | \mathbf{S}(\beta_1)(\mu) = K_1(x_1) \wedge \mathbf{S}(\beta_2)(\mu) = K_2(x_2)\},$$

then Prop 5 shows that $\Gamma(x_1, x_2) \neq \emptyset$.

3. Since K_1 and K_2 are continuous, $\Gamma : A \to \mathcal{F}(\mathbf{S}(B))$ is easily established. The set $\exists\Gamma(C)$ is closed in A for compact $C \subseteq \mathbf{S}(B)$. In fact, let $(\langle x_1^{(n)}, x_2^{(n)}\rangle)_{n\in\mathbb{N}}$ be a sequence in this set with $x_i^{(n)} \to x_i$, as $n \to \infty$ for $i = 1, 2$, thus $\langle x_1, x_2\rangle \in A$. There exists $\mu_n \in C$ such that $\mathbf{S}(\beta_i)(\mu_n) = K_i(x_i^{(n)})$. Because C is compact, there exists a converging subsequence $\mu_{s(n)}$

and $\mu \in C$ with $\mu = \lim_{n\to\infty} \mu_{s(n)}$ in the topology of weak convergence. Continuity of K_i implies that

$$\mathbf{S}(\beta_i)(\mu) = K_i(x_i),$$

consequently $\langle x_1, x_2\rangle \in \exists\Gamma(C)$, thus this set is closed, hence measurable. From Prop. 3 it can now be inferred that there exists a measurable map $N : A \to \mathbf{S}(B)$ such that

$$N(x_1, x_2) \in \Gamma(x_1, x_2)$$

holds for every $\langle x_1, x_2\rangle \in A$. Thus $N : A \rightsquigarrow B$ is a stochastic relation with

$$\begin{aligned} K_1 \circ \alpha_1 &= \mathbf{S}(\beta_1) \circ N, \\ K_2 \circ \alpha_2 &= \mathbf{S}(\beta_2) \circ N \end{aligned}$$

Hence $\langle A, B, N\rangle$ is the desired semi-pullback. □
The proof shows that in $\mathfrak{MProc}$ the semi-pullback is in $\mathfrak{MProc}$ again, so that we have established:

Corollary 1 *$\mathfrak{MProc}$ has semi-pullbacks.* □

4 Bisimulation

This very short section demonstrates that the bisimulation relation on objects of $\mathfrak{SRel}$ is transitive, and serves as an application for the result that semi-pullbacks exist in this category. We define a bisimulation for two objects in $\mathfrak{SRel}$ through a span of morphisms in that category. This is similar to the notion of 1-bisimulation investigated in [Dob02a] for the comma category $1\!\!1_{\mathcal{M}} \downarrow \mathbf{S}$, were $\mathcal{M}$ is the category of all measurable spaces with measurable maps as morphisms.

Definition 1 *An object P in $\mathfrak{SRel}$ together with morphisms $\langle\sigma_1, \tau_1\rangle : P \to Q_1$ and $\langle\sigma_2, \tau_2\rangle : P \to Q_2$ is called a* bisimulation of objects Q_1 and Q_2.

Let $P := \langle X, Y, K\rangle$ and $Q_i := \langle X_i, Y_i, K_i\rangle$, then we get the familiar commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{\sigma_1} & X & \xrightarrow{\sigma_2} & X_2 \\ \downarrow K_1 & & \downarrow K & & \downarrow K_2 \\ \mathbf{S}(Y_1) & \xleftarrow[\mathbf{S}(\tau_1)]{} & \mathbf{S}(Y) & \xrightarrow[\mathbf{S}(\tau_2)]{} & \mathbf{S}(Y_2) \end{array}$$

Thus we have effectively established Theorem 1 by constructing a bisimulation for the objects serving as domains for the morphisms investigated in the semi-pullback.
We want to apply the semi-pullback for establishing the fact that the bisimulation relation is transitive in $\mathfrak{SRel}$.

Proposition 6 *The bisimulation relation between objects in the category $\mathfrak{SRel}$ of stochastic relations is transitive. The same is true for the subcategory $\mathfrak{MProc}$ of Markov processes.*

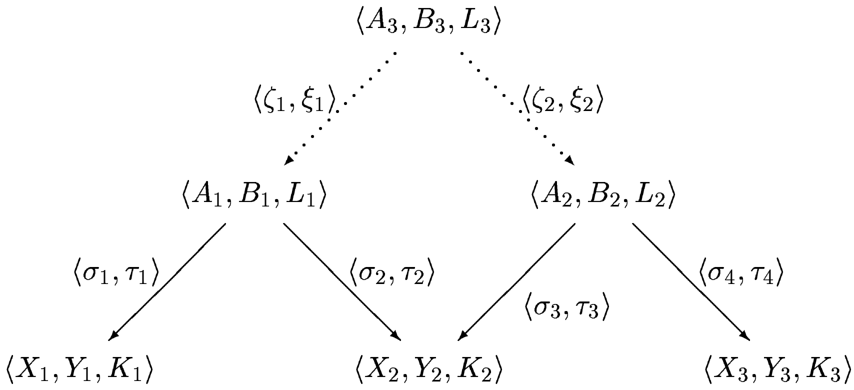

Figure 1: Transitivity of Bisimulation Diagrams

Proof Consider as e.g. in the proof for [Rut00, Theorem 5.4] the diagram in Fig. 1.
The lower triangles are given bisimulations, and the upper diamond with its dotted lines is the semi-pullback for the pair

$$\begin{aligned} \langle \sigma_2, \tau_2 \rangle : \langle A_1, B_1, L_1 \rangle &\rightarrow \langle X_2, Y_2, K_2 \rangle \\ \langle \sigma_3, \tau_3 \rangle : \langle A_2, B_2, L_2 \rangle &\rightarrow \langle X_2, Y_2, K_2 \rangle \end{aligned}$$

which exists by Thm. 1 for $\mathfrak{SRel}$, and by Cor. 1 for the subcategories. Then

$$\begin{aligned} \langle \sigma \circ \zeta_1, \tau \circ \xi_1 \rangle : \langle A_3, B_3, L_3 \rangle &\rightarrow \langle X_1, Y_1, K_1 \rangle \\ \langle \zeta_2 \circ \sigma_4, \xi_2 \circ \tau_4 \rangle : \langle A_3, B_3, L_3 \rangle &\rightarrow \langle X_3, Y_3, K_3 \rangle \end{aligned}$$

is the desired bisimulation. □

5 Conclusion

We show that one can construct a semi-pullback in the category of Markov processes with continuous and measure preserving maps as morphisms, provided the processes work on a Polish space which is locally compact (like the reals $\mathbb{R}$ in the natural topology). This is actually a special case of a more general result which deals with stochastic relations over Standard Borel spaces in which the class of Polish spaces mentioned above serve as targets for transition probability functions. It is shown that in the latter category the bisimulation relation is transitive. This renders the proofs and constructions in [DEP99] easier, and it is to be expected that the results on characterizing bisimilarity of states will hold in the category **LMP** of Markov processes rather than in **LMP*** of generalized Markov processes of that paper, provided the requirement of working in Polish spaces is strengthened to *locally compact Polish.*
Rather than constructing the object underlying a semi-pullback explicitly, we rely on selection arguments from the theory of set-valued relations. This gives probably less technical insight into the nature of the object one looks for, but is easier to apply, and it permits drawing from the rich well of topology, in particular the weak topology on the space of all subprobability measures. Selection arguments may be a helpful way of constructing objects; we illustrate this by showing that the map which assigns each Polish space its subprobabilities and each

surjective Borel measurable map the corresponding measure transform is actually a functor which may be difficult to establish otherwise.
The compactness assumptions should be removed for obtaining a more general and easier applicable result. This will be addressed. Applications of the technique of measurable selectors to stochastic relations will also be investigated further; [Dob02a] provides some examples of the interplay between set-valued and stochastic relations.

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