Direct approach to L^p estimates in homogenization theory

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Abstract: We derive interior L^p-estimates for solutions of linear elliptic systems with oscillatory coefficients. The estimates are independent of ε, the small length scale of the rapid oscillations. So far, such results are based on potential theory and restricted to periodic coefficients. Our approach relies on BMO-estimates and an interpolation argument, gradients are treated with the help of finite differences. This allows to treat coefficients that depend on a fast and a slow variable. The estimates imply an L^p-corrector result for approximate solutions.

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1 Introduction

The classical (and most important) example of homogenization theory is the family of equations

$$-\nabla \cdot (A(x, x/\varepsilon) \nabla u^\varepsilon(x)) = f(x) \text{ in } \Omega, \tag{1.1}$$

for a bounded subset $\Omega \subset \mathbb{R}^n$ and $f \in H^{-1}(\Omega)$, accompanied with the boundary condition $u^\varepsilon|_{\partial\Omega} = 0$. On the coefficients A one assumes the Y-periodicity in the second variable, with $Y = (0,1)^n$ the unit cube. Under appropriate assumptions on A, it is known that the family of solutions u^ε converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to a limit function $u^0 \in H^1_0(\Omega)$, which is the solution of the homogenized problem $-\nabla\cdot(A^*(x)\nabla u^0(x)) = f(x)$. The most direct way to derive this result is with the method of two-scale convergence introduced by Allaire [1], which provides additionally a corrector result: Starting from the homogenized solution u^0 one can study the two-scale expansion of the solution in the form $\eta^\varepsilon(x) = u^0(x) + \varepsilon u^1(x, x/\varepsilon)$ and prove that $u^\varepsilon - \eta^\varepsilon \to 0$ strongly in $H^1(\Omega)$.

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Current interest in homogenization analysis stems from questions in the failure of materials. Of particular interest are norms of the strain that are sensitive to peaks. Here, one is usually concerned with L^q-norms of the gradient rather than L^2-norms, see e.g. [14].

While homogenization theory is well developed in the Hilbert space setting, much less is known for L^q-norms. We emphasize that the interest here is to have estimates that are independent of the small parameter $\varepsilon > 0$. Obviously, regularity theory could be used to find estimates for $u^\varepsilon \in H^2(\Omega)$ and embedding results yield L^q-estimates for ∇u^ε — but due to the oscillations of u^ε on the ε scale, the $H^2(\Omega)$-norms will necessarily behave like $1/\varepsilon$. A more profound obstruction to regularity in L^q-spaces is given in [6]: Without the periodicity assumption on $A(x,.)$, no ε-independent estimate $\|\nabla u^\varepsilon\|_{L^q(\Omega)}$ can hold except for $q-2$ small (in which case Meyers estimate holds).

Concerning positive results, fundamental contributions are due to Avellaneda and Lin [2], where optimal (in terms of exponents) estimates in $L^q(\Omega)$-spaces were derived for the solutions of the above equation. As in the related articles [3], [4], [5], and [7], the assumption is that $A(x,y) = A(y)$ does not depend on the slow variable, the regularity is $A \in C^\alpha(Y)$. The authors derive estimates for the singular kernels of the corresponding Greens-functions to prove the regularity result for the solutions. The strength of the result is the global character, the estimates hold up to the boundary of the domain. Results under weaker regularity assumptions on the coefficients can be deduced from an approximation method of Caffarelli and Peral [7]. Based on local energy comparison and Calderón-Zygmund type decomposition, the authors provide ε-uniform local $W^{1,p}$ bounds for elliptic equations $-\nabla \cdot (A(x/\varepsilon)\nabla u(x)) = 0$ with continuous periodic coefficients. For an analysis of the behavior near macroscopic interfaces we refer to [13].

This work follows another approach. The improvement over existing work lies in the fact that we study coefficients that may additionally depend on the slow variable, $A = A(x,y)$. The price to pay is that gradient estimates are restricted to interior estimates.

Theorem 1 provides an estimate for $\|u^\varepsilon\|_{L^q(\Omega')}$ with an optimal exponent. The method uses no potential theory but rather follows the approach sketched e.g. in [11]: the solution operator is bounded as a map $L^2 \to L^{2^*}$, and as a map $L^n \to BMO$. The L^2-result is a direct consequence of the Sobolev embedding, the BMO-result is based on a decomposition of the solution on cubes. Inhomogeneous solutions with homogeneous boundary data are treated by a testing argument, homogeneous solutions by a comparison with the solutions of the homogenized system. An interpolation argument between BMO and L^{2^*} yields the L^q-estimate. The proof of Theorem 1 is given in sections 2 and 3.

In Theorem 2 we prove an estimate for $\|\nabla u^\varepsilon\|_{L^q(\Omega')}$, again with the optimal exponent. The method is based on finite difference quotients of the solutions.

Finite differences solve an equation of the same type, if the x-differences are in accordance with the ε-periodicity of the equation. For such finite differences we can apply Theorem 1 to find L^q-estimates. We conclude with the "local lemma", Lemma 2, which states that gradients can be estimated by the ε-size finite differences. The program is carried out in section 4. This approach was already used in [17] for Lipschitz estimates in a perforated domain.

In section 5 we prove the second part of Theorem 2, which transfers the estimates for compactly supported solutions to interior estimates. The localization procedure is intricate, since the product of a solution with a smooth cut-off function behaves badly under the application of the operator with oscillatory coefficients. We circumvent this effect by using a multiplication of the solution with two-scale approximations of cut-off functions.

The a priori bound on solutions provide an improvement of the above mentioned corrector result. In Corollary 2 we show, for $f \in L^p(\Omega)$ and $\Omega' \subset \Omega$ a compact subset, that the two-scale expansions η^ε of solutions u^ε satisfy $u^\varepsilon - \eta^\varepsilon \to 0$ with the strong convergence of $W^{1,q}(\Omega')$.

Results

On a domain $\Omega \subset \mathbb{R}^n$ we study the family of operators $\mathcal{L}^\varepsilon$ acting on $v : \Omega \to \mathbb{R}^m$ as

$$\mathcal{L}^\varepsilon v(x) = -\nabla \cdot \left(A\left(x, x/\varepsilon\right) \nabla v(x)\right) \text{ in } \Omega. \tag{1.2}$$

Here, with a periodicity cell $Y = (0,1)^n$, we consider coefficients $A = A(x,y)$ that satisfy

$$A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{m^2 \times n^2} \text{ uniformly continuous and } Y\text{-periodic in } y \tag{1.3}$$

$$\text{uniform ellipticity: for } \nu > 0 \text{ holds } A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq \nu |\xi|^2 \text{ for all } \xi \in \mathbb{R}^{n \times m}. \tag{1.4}$$

For the gradient estimates of Theorem 2 further regularity conditions on the coefficient need to be imposed:

$$A \in W^{1,\rho}(\Omega, C^0(Y)) \text{ for some } \rho > n, \tag{1.5}$$

$$A \in C^{0,1}(\Omega, W^{1,n}(Y)) \cap C^{1,1}(\Omega, L^n(Y)). \tag{1.6}$$

In our strongest result, we therefore assume the regularity $A(x,.) \in C^0(Y) \cap W^{1,n}(Y)$ for almost every x. We note that this condition is neither stronger nor weaker than the Hölder continuity. Our main results are interior estimates for solutions of the boundary value problem.

Theorem 1. *Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4), $\Omega \subset \mathbb{R}^n$ bounded, and $u \in H_0^1(\Omega)$ be a weak solution of*

$$\mathcal{L}^\varepsilon u = \operatorname{div} f \text{ in } \Omega.$$

Let $\Omega' \subset \Omega$ be compactly contained, $p \in [2, n)$ and $q = np/(n-p)$. Then there holds

$$\|u\|_{L^q(\Omega')} \leq c\|f\|_{L^p(\Omega)}, \tag{1.7}$$

with a constant c depending on Ω, Ω', A, and p, but independent of f and ε. If $\Omega \subset \mathbb{R}^n$ is a domain with C^1-boundary, the above estimate holds globally, i.e. with $\Omega' = \Omega$. In the case $p = n$ and Ω' is a cube the same estimate holds with L^q replaced by BMO. In the case $p > n$ the estimate holds with $q = \infty$.

Our second main theorem lifts the orders of differentiability by one.

Theorem 2. *On $\Omega \subset \mathbb{R}^n$ we consider weak solutions $U \in H^1(\Omega)$ of*

$$\mathcal{L}^\varepsilon U = F \text{ in } \Omega.$$

Let $\Omega' \subset \Omega$ be compactly contained, $p \in [2, n)$ and $q = np/(n-p)$. Then, with a constant c depending on Ω, Ω', A, and p, but independent of F and ε, there holds

1. *If the support of U is contained in Ω' and A satisfies (1.3) – (1.5), then*

$$\|\nabla U\|_{L^q(\Omega)} \leq c\|F\|_{L^p(\Omega)}. \tag{1.8}$$

2. *If A satisfies (1.3) – (1.6) and $m = 1$, then*

$$\|\nabla U\|_{L^q(\Omega')} \leq c\left(\|F\|_{L^p(\Omega)} + \|U\|_{H^1(\Omega)}\right). \tag{1.9}$$

   The estimate holds also for systems ($m > 1$), if we have the additional Hölder regularity $A \in C^0(\Omega, C^\mu(Y))$ for some $\mu > 0$.

In the case $p > n$ the above estimates hold with $q = \infty$.

2 BMO-estimates and interpolation

Proof of Theorem 1. We realize that the Theorem holds trivially in the case $p = 2$, $q = 2^* = 2n/(n-2)$ by the continuous embedding $H^1 \to L^{2^*}$ or $H^1 \to BMO$ for $n = 2$. Proposition 1 provides a BMO-estimate on any compactly contained cube $Q \subset \Omega$. For homogeneous Dirichlet conditions, interpolation of the operators $(\mathcal{L}^\varepsilon)^{-1}\mathrm{div} : L^2(\Omega) \to H^1_0(\Omega)$ restricted to Q, i.e. $T^\varepsilon f := ((\mathcal{L}^\varepsilon)^{-1}\mathrm{div} f)\lfloor_Q$,

$$T^\varepsilon : L^2(\Omega) \to L^{2^*}(Q),$$
$$T^\varepsilon : L^n(\Omega) \to BMO(Q),$$

yields the inner L^q-estimate (1.7). For the interpolation we refer to appendix A. For the global result we can take any cube $Q \supset \Omega$ so that Ω is compactly included and apply Proposition 4 with the same interpolation argument. □

We recall that for cubes $Q_0 \subset \mathbb{R}^n$ the homogeneous space of functions of bounded mean oscillation $\mathrm{BMO}(Q_0)$ is defined by the semi-norm

$$\|u\|_{BMO(Q_0)} \equiv \sup_{Q \subset Q_0} \fint_Q |u - \fint_Q u|. \tag{2.1}$$

According to a well-known result of John and Nirenberg [12] an equivalent semi-norm is given by $\|u\|_{BMO}^2 = \sup_{Q\subset Q_0} \fint_Q |u - \fint_Q u|^2$. Observe that for a bounded measurable set $B \subset \mathbb{R}^n$ and $u \in L^2(B)$, the function $\lambda \mapsto \int_B |u-\lambda|^2\,dx$ is minimal for $\lambda = \fint_B u$. Thus there is a universal constant $c = c(n)$ such that for a function $u \in L^1_{loc}(B_R(0))$ we have

$$\|u\|^2_{BMO(Q_R(0))} \le c \sup\left\{ \fint_{B_r} |u - \fint_{B_r} u|^2 : B_r = B_r(x) \subset B_R(0) \right\}, \tag{2.2}$$

where $Q_R(0) \subset B_R(0)$ denotes the cube of sidelength R centered at the origin.

The main steps in this section are as follows. We have seen that Theorem 1 is a consequence of the BMO-estimate of Proposition 1. The Proposition follows Campanato's device based on a local decomposition of the solution, $u = v + w$, where v solves the homogeneous problem. While the w-part can handled directly, the v-part is treated seperately in Proposition 3. In that Proposition, we consider separately large and small radii. While small radii can be treated with the standard Hölder estimates of Proposition 2, large radii are treated with an homogenization argument which is carried out in Lemma 1 and Corollary 1.

Proposition 1 (BMO inner estimate)**.** *Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4). Let u be a weak solution of*

$$\mathcal{L}^\varepsilon u = \operatorname{div} f \ \textit{in}\ B_R(0).$$

Then

$$\|u\|_{BMO(Q_{R/2}(0))} \le c \left(\|f\|_{L^n(B_R(0))} + \|u\|_{H^1(B_R(0))} \right), \tag{2.3}$$

where the constant c depends on R and A, but is independent of f and ε.

Proof. Let $x \in B_{R/2}(0)$ and $B_r = B_r(x)$ with $0 < r < \min\{R/4, 1\}$. We show that there is a universal constant $C > 0$ that only depends on A so that for any $0 < \rho < r/2$

$$\fint_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left(\|f\|^2_{L^n(B_{2r})} + \|u\|^2_{H^1(B_{2r})} \right).$$

To this end we decompose $u = v + w$ where v is the weak solution of the homogeneous problem $\mathcal{L}^\varepsilon v = 0$ on B_r with $v|_{\partial B_r} = u|_{\partial B_r}$. With Poincaré's inequality

$$\int_{B_\rho} |u - \fint_{B_\rho} u|^2 \le 2 \int_{B_\rho} |v - \fint_{B_\rho} v|^2 + c\,\rho^2 \int_{B_r} |\nabla w|^2.$$

Using that $w \in H^1_0(\Omega)$ solves the equation $\mathcal{L}^\varepsilon w = \mathrm{div} f$, Young's and Hölder's inequality imply

$$\int_{B_r} |\nabla w|^2 \le c \int_{B_r} |f|^2 \le c\, r^{n-2} \|f\|^2_{L^n(B_r)}.$$

As a consequence of the proposition below, cf. (2.5), we find that

$$\int_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left(\frac{\rho}{r}\right)^n \int_{B_{r/2}} |v - \fint_{B_{r/2}} v|^2 + c\, r^n \, \|f\|^2_{L^n(B_r)}.$$

Taking into account that v is an $\mathcal{L}^\varepsilon$-minimal extension of u in B_r we find

$$\int_{B_{r/2}} |v - \fint_{B_{r/2}} v|^2 \le c\, r^2 \int_{B_r} |\nabla v|^2 \le c\, r^2 \int_{B_r} |\nabla u|^2.$$

But then Caccioppoli's inequality

$$r^2 \int_{B_r} |\nabla u|^2 \le c \int_{B_{2r}} |u - \fint_{B_{2r}} u|^2 + c\, r^2 \int_{B_{2r}} |f|^2$$

and Hölder's inequality $\|f\|^2_{L^2(B_{2r})} \le c r^{n-2} \|f\|^2_{L^n(B_{2r})}$ imply

$$\int_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left(\frac{\rho}{r}\right)^n \int_{B_{2r}} |u - \fint_{B_{2r}} u|^2 + c\, r^n \, \|f\|^2_{L^n(B_{2r})}.$$

By a standard iteration method, cf. [10] Ch.III Lemma 2.1, the factor r^n for the last term can be replaced by ρ^n. Adapting constants we find

$$\fint_{B_\rho} |u - \fint_{B_\rho} u|^2 \le C \left[\fint_{B_{2r}} |u - \fint_{B_{2r}} u|^2 + \|f\|^2_{L^n(B_{2r})} \right]$$

for any $0 < \rho < r/2$. This completes the proof. □

It remains to derive uniform bounds for the homogeneous problem. The main strategy will be to reduce everything to the following basic regularity result for elliptic systems with continuous coefficients in divergence form, that is originally due to S. Campanato [9] and C.B. Morrey Jr. [15]:

Proposition 2 (cf. [10] Ch.III Theorem 3.1)**.** *Suppose that $A \in C^0(B_R(0))$ is uniformly elliptic. If $v \in H^1(B_R(0))$ is a weak solution of $\nabla \cdot (A(x)\, \nabla v) = 0$ in $B_R(0)$, then ∇v belongs to the Morrey space $L^{2,\lambda}(B_{R/2}(0))$ for any $0 < \lambda < n$. More specifically, for any $\gamma \in [0, 1)$, the estimate*

$$\rho^2 \fint_{B_\rho(0)} |\nabla v|^2 \le C \left(\frac{\rho}{R}\right)^{2\gamma} R^2 \fint_{B_{R/2}(0)} |\nabla v|^2 \tag{2.4}$$

holds true for any $0 < \rho < R/2$ with a constant C that only depends on A and γ.

More precisely the bounds only depend on the ellipticity properties and the modulus of continuity of A. Thus, the proposition holds in a uniform fashion for equi-continuous and uniformly elliptic families $(A_\varepsilon)_{\varepsilon>0}$ of coefficient matrices. In order to apply Proposition 2 we distinguish two regimes determined by the size of ρ relative to ε. For small radii $\rho \lesssim \varepsilon$ the estimate follows from a scaling argument that provides a standard situation. In the opposite regime of large radii when $\rho \gtrsim \varepsilon$, Proposition 2 will be applied to the homogenized problem in connection with a compactness argument similar to the one in [2].

Proposition 3 (C^γ inner estimate for the homogeneous problem)**.** *Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4). Let $0 < r \le 1$ and v be a weak solution of*

$$\mathcal{L}^\varepsilon v = 0 \text{ in } B_r(0).$$

Then $v \in C^\gamma(B_{r/2}(0))$ for any $\gamma \in (0,1)$. More precisely, for any $\gamma \in [0,1)$ and $B_\rho = B_\rho(x)$

$$\fint_{B_\rho} |v - \fint_{B_\rho} v|^2 \le C(\gamma) \left(\frac{\rho}{r}\right)^{2\gamma} \fint_{B_{r/2}} |v - \fint_{B_{r/2}} v|^2 \tag{2.5}$$

holds true for any $0 < \rho < r/2$ and $x \in B_{r/2}(0)$ with some universal constant $C(\gamma)$ that only depends on γ.

Proof. After translation it is enough to prove (2.5) for $x = 0$.

1. Large radii. The estimate is based on an improvement estimate for the (squared) mean oscillation, proven in Corollary 1 below. We fix γ and consider in this first step radii $\rho > \varepsilon/\lambda$ as in Corollary 1. In view of (2.8), i.e.

$$\fint_{B_{\theta\rho}} |v - \fint_{B_{\theta\rho}} v|^2 \le \theta^{2\gamma} \fint_{B_\rho} |v - \fint_{B_\rho} v|^2$$

that holds for some $\theta \in (0, 1/2)$ and for any $0 < \varepsilon < \lambda\,\rho$, the decay estimate (2.5) follows by a k-fold iteration, with k determined by $\theta^{k+1} r < 2\rho \le \theta^k r$.

2. Small radii. In order to treat radii $0 < \rho < \varepsilon/\lambda$ we rescale by ε/λ (we can assume that $\lambda = 1$). The rescaled coefficients $y \mapsto A(\varepsilon y, y)$ are equi-continuous and uniformly elliptic as $\varepsilon \to 0$. Accordingly, weak solutions $v_\varepsilon \in H^1(B_1)$ of the rescaled equation $\nabla \cdot (A(\varepsilon y, y) \nabla v_\varepsilon(y)) = 0$ in B_1 are (uniformly) locally Hölder continuous with exponent $\gamma \in (0,1)$ and exhibit, in view of (2.4), Poincaré's and Caccioppoli's inequality, an estimate

$$\fint_{B_{\rho/\varepsilon}} |v_\varepsilon - \fint_{B_{\rho/\varepsilon}} v_\varepsilon|^2 \le C \left(\frac{\rho}{\varepsilon}\right)^{2\gamma} \fint_{B_1} |v_\varepsilon - \fint_{B_1} v_\varepsilon|^2$$

for any $0 < \rho < \varepsilon$ with a constant C that only depends on γ and A. Hence we get in the original scaling

$$\fint_{B_\rho} |v - \fint_{B_\rho} v|^2 \le C \left(\frac{\rho}{\varepsilon}\right)^{2\gamma} \fint_{B_\varepsilon} |v - \fint_{B_\varepsilon} v|^2.$$

In view of step 1, this completes the proof. ☐

Lemma 1. *Suppose that $v = v^\varepsilon$ solves $\mathcal{L}^\varepsilon v = 0$ in $B_\rho = B_\rho(0)$ where $0 < \rho \le 1$. Then for any $\theta \in (0,1)$ there is a factor $\lambda(\theta) > 0$ independent of ρ and v with the following property: For every $\gamma \in (0,1)$ and whenever $0 < \varepsilon < \lambda(\theta)\,\rho$*

$$\fint_{B_{\theta\rho}} |v - \fint_{B_{\theta\rho}} v|^2 \le C\,\theta^{2\gamma}\,\rho^2 \fint_{B_\rho} |\nabla v|^2$$

for a universal constant C that only depends on γ and A.

Proof. We first observe that inequality to be proven is scaling invariant. Rescaling x and ε by ρ we arrive at

$$\nabla \cdot (A(\rho\,x, x/\varepsilon)\nabla v) = 0 \quad \text{in} \quad B_1 = B_1(0) \quad \text{where} \quad 0 < \varepsilon < \lambda(\theta).$$

Therefore we can concentrate on the case $\rho = 1$, keeping in mind that for $0 < \rho \le 1$ the coefficients $A(\rho\,x, y)$ are uniformly elliptic and equi-continuous.
We fix $\gamma \in (0,1)$ and suppose on the contrary that for any $c > 0$ there is a $\theta \in (0,1)$, a sequence $\varepsilon_k \to 0$, and a corresponding sequence (v_k) of weak solutions of $\mathcal{L}^{\varepsilon_k} v = 0$ in B_1 so that

$$\fint_{B_\theta} |v - \fint_{B_\theta} v|^2 > c\,\theta^{2\gamma} \fint_{B_1} |\nabla v_k|^2.$$

We define a sequence of blow-up functions

$$w_k = \left(\fint_{B_1} |\nabla v_k|^2\right)^{-1/2} \left(v_k - \fint_{B_\theta} v_k\right).$$

that satisfy the equation $\mathcal{L}^{\varepsilon_k} w_k = 0$ in B_1 and with the property $\int_{B_\theta} w_k = 0$. By assumption we have

$$\fint_{B_\theta} |w_k|^2 > c\,\theta^{2\gamma}. \tag{2.6}$$

Poincaré's inequality implies an L^2 estimate for (w_k) so that for a subsequence $w_k \rightharpoonup w$ weakly in $H^1(B_1)$. Moreover, w is a weak solution of the homogenized equation $\mathcal{L}^* w = 0$. Recall that $\mathcal{L}^*$ is uniformly elliptic with continuous coefficients and that by lower semicontinuity $\fint_{B_1} |\nabla w|^2 \le 1$. Thus (2.4) and Poincaré's inequality imply

$$\fint_{B_\theta} |w|^2 \le C\,\theta^{2\gamma} \tag{2.7}$$

that holds true for some universal constant $C > 0$ that only depends on γ and A. But this is a contradiction to (2.6) since (w_k) is strongly pre-compact L^2. ☐

From the lemma we deduce the desired improvement estimate for the mean (squared) oscillation:

Corollary 1. *Suppose that $v = v^\varepsilon$ solves $\mathcal{L}^\varepsilon v = 0$ in $B_\rho = B_\rho(0)$ where $0 < \rho \le 1$. Then, for any $\gamma \in (0,1)$, there are numbers $\theta \in (0,1/2)$ and $\lambda > 0$ independent of ρ and v, so that for any $0 < \varepsilon < \lambda\,\rho$*

$$\fint_{B_{\theta\rho}} |v - \fint_{B_{\theta\rho}} v|^2 \le \theta^{2\gamma} \fint_{B_\rho} |v - \fint_{B_\rho} v|^2. \tag{2.8}$$

Proof. Let us fix $\gamma \in (0,1)$. We apply the lemma for γ replaced by $\frac{1}{2}(1+\gamma)$, $\theta \in (0,1/2)$ replaced by 2θ, and ρ replace by $\rho/2$. After adapting the constant we find with Caccioppoli's inequality

$$\fint_{B_{\theta\rho}} |v - \fint_{B_{\theta\rho}} v|^2 \le C\,\theta^{1+\gamma}\,\rho^2 \fint_{B_{\rho/2}} |\nabla v|^2 \le C\,\theta^{1-\gamma}\,\theta^{2\gamma} \fint_{B_\rho} |v - \fint_{B_\rho} v|^2$$

for any $0 < \varepsilon < \lambda\,\rho$ and some $\lambda = \lambda(\theta)$. Then the claim follows by choosing the maximal θ so that $C\,\theta^{1-\gamma} \le 1$, and the associated $\lambda > 0$. ☐

Remark 1 (Hölder estimates)**.** *In view of the Hölder estimates we got for the homogeneous problem, and the decomposition argument in Proposition 1, we infer Hölder estimates for u, and, in particular, an L^∞-estimate in case $p > n$.*

3 Global BMO-estimates

This section is devoted to the statement about global estimates in Theorem 1. Let us therefore assume that Ω is a domain of class C^1 that is compactly contained within some cube Q. After trivial extension we have $u \in H^1_0(Q)$. The goal is to extend the local estimate (2.3) to the global estimate

$$\fint_B |u - \fint_B u|^2 \le c\left(\|f\|^2_{L^n(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right) \tag{3.1}$$

for any cube $B \subset Q$ and some universal constant c that only depends on A and Ω. Observe that $\|\nabla u\|_{L^2(\Omega)} \le C(\Omega)\|f\|_{L^n(\Omega)}$, hence (3.1) implies:

Proposition 4 (BMO global estimate)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^1. Let the coefficients of the elliptic operator $\mathcal{L}^\varepsilon$ satisfy (1.3) and (1.4). Let $u \in H^1_0(\Omega;\mathbb{R}^m)$ be a weak solution of*

$$\mathcal{L}^\varepsilon u = \operatorname{div}\, f \;\; in\ \Omega.$$

Then for any cube Q that compactly contains Ω

$$\|u\|_{BMO(Q)} \le c\,\|f\|_{L^n(\Omega)}, \tag{3.2}$$

where the constant c only depends on A and Ω, but is independent of f and ε.

We only sketch the proof of the global estimate (3.1). The general device is well established, cf. e.g. [11] and the literature therein: In order to complement the local bounds we essentially have to show that they remain valid after transformation and when balls a replaced by half balls. Indeed, since $\partial\Omega$ is compact and of class C^1 there is a number $R > 0$, finitely many balls $B_R(x)$ with $x \in \partial\Omega$ and associated C^1-homeomorphisms ϕ with uniform C^1 bounds that only depend on Ω so that

$$\phi : B_R(0) \to B_R(x) \quad \text{with} \quad \phi(B_R^+(0)) = B_R(x)\cap\Omega \quad \text{and} \quad \phi(B_R^-(0)) = B_R(x)\setminus\bar{\Omega}$$

where for $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : x_n \gtrless 0\}$ the upper and lower half space, respectively, and $B_R^\pm(0) = B_R(0) \cap \mathbb{R}^n_\pm$. We let $f_\phi(y) = (\nabla\phi(y))^{-1} f(\phi(y))|\det\nabla\phi(y)|$. Then $f \mapsto f_\phi$ is an isomorphism on L^n-spaces. The transformed coefficients have the form

$$A^\varepsilon_\phi(y) = (\nabla\phi(y))^{-1} A\big(\phi(y), \phi(y)/\varepsilon\big)\,(\nabla\phi(y))^{-T}\,|\det\nabla\phi(y)|.$$

Since, for $A^\varepsilon(x) = A(x, x/\varepsilon)$, we have the identity

$$\int_{B_R^+(0)} A^\varepsilon_\phi(y)\big\langle\nabla(u\circ\phi), \nabla(\varphi\circ\phi)\big\rangle\,dy = \int_{\phi(B_R^+(0))} A^\varepsilon(x)\langle\nabla u, \nabla\varphi\rangle\,dx,$$

the transformed equation reads $-\nabla\cdot(A^\varepsilon_\phi(y)\nabla u) = \nabla\cdot f_\phi$ in $B_R^+(0)$. We need to show that the transformed coefficients $A^\varepsilon_\phi(y)$ essentially behave like $A^\varepsilon(x)$. As in case of the original coefficients, rescaling essentially preserves the modulus of continuity.

Remark 2. *Rescaling $A^\varepsilon_\phi(y) \mapsto A^\varepsilon_\phi(\varepsilon\, y)$ yields, as $\varepsilon \to 0$, a uniformly elliptic and equi-continuous family of coefficients. Accordingly, weak solutions of $-\nabla\cdot(A^\varepsilon_\phi(y)\nabla v) = 0$ in $B_R^+(0)$ exhibit uniform regularity estimates as in Proposition 2.*

Proof. It is enough to note the equi-continuity of $y \mapsto A(\phi(\varepsilon y), \phi(\varepsilon y)/\varepsilon)$ as $\varepsilon \to 0$ due to differentiability of ϕ and equi-continuity of original coefficients. □

We also need to investigate convergence properties of transformed operators $\mathcal{L}^\varepsilon_\phi v = -\nabla\cdot(A^\varepsilon_\phi(y)\nabla v)$ corresponding to $\mathcal{L}^\varepsilon$ and to establish a relation to the transformation of the homogenized operator $\mathcal{L}^*_\phi v = -\nabla\cdot(A^*_\phi(y)\nabla v)$ corresponding to $\mathcal{L}^*$. Since composition with ϕ is an isomorphism of H^1 spaces, we deduce that transformation commutes with homogenization in the following sense.

Remark 3. *Suppose that $\varepsilon_k \to 0$ and that a corresponding sequence $(v_k) \subset H^1(B_R^+(0))$ of weak solutions $\mathcal{L}^{\varepsilon_k}_\phi v_k = 0$ in $B_R^+(0)$ weakly converges $v_k \rightharpoonup v$ in $H^1(B_R^+(0))$. Then $\mathcal{L}^*_\phi u = 0$ in $B_R^+(0)$. Accordingly, Lemma 1 holds true for transformed coefficients and on half-balls.*

Proof. Let (v_k) be as above. Then there is a corresponding sequence $u_k = v_k \circ \phi^{-1} \rightharpoonup v \circ \phi^{-1} = u$ weakly in $H^1(B_R(x) \cap \Omega)$ and so that u_k solves $\nabla \cdot (A^{\varepsilon_k}(x)\nabla u_k) = 0$ in $B_R(x) \cap \Omega$. Then u solves the homogenized problem $\nabla \cdot (A^*(x)\nabla u) = 0$ in $B_R(x) \cap \Omega$. Accordingly, if $A^*_\phi(y)$ is the transform of $A^*(x)$, then v solves $\nabla \cdot (A^*_\phi(y)\nabla v) = 0$ in $B^+_R(0)$. □

Now we are in the position to discuss uniform estimates at the transformed boundary. We assume that $u \in H^1_0(\mathbb{R}^n_+)$ is trivially extended to $\mathbb{R}^n$ and for some $f \in L^n(B^+_R(0))$

$$\mathcal{L}^\varepsilon_\phi u = \mathrm{div} f \quad \text{in} \quad B^+_R(0).$$

In view of Remark 2 and 3 the arguments in the proof of Proposition 3 carry over and provide uniform estimates for the homogeneous problem. Finally, a decomposition argument as in the proof of Proposition 1 yields: For $B^+_\rho = B^+_\rho(x', 0)$ with $|x'| < R/4$ and $0 < \rho < R/4$

$$⨍_{B^+_\rho} |u - ⨍_{B^+_\rho} u|^2 \leq c \left(\|f\|^2_{L^n(B^+_R(0))} + \|\nabla u\|^2_{L^2(B^+_R(0))} \right) \tag{3.3}$$

for a constant that only depends on A and Ω but is independent of $\varepsilon \in (0, 1)$. Now suppose that for $x = (x', x_n)$ with $|x'| < R/4$ and for $0 < \rho < R/16$ the ball $B_\rho(x)$ has a non-empty intersection with the lower half space. With $B^+_{2\rho} = B^+_{2\rho}(x', 0)$ we have the rough estimate

$$⨍_{B_\rho(x)} |u - ⨍_{B_\rho(x)} u|^2 \leq c ⨍_{B^+_{2\rho}} |u|^2.$$

But in view of Poincaré's, Caccioppoli's and Hölder's inequalities we have

$$⨍_{B^+_{2\rho}} |u|^2 \leq c\,\rho^2 ⨍_{B^+_{2\rho}} |\nabla u|^2 \leq c ⨍_{B^+_{4\rho}} |u - ⨍_{B^+_{4\rho}} u|^2 + c\|f\|^2_{L^n(B^+_{4\rho})} \tag{3.4}$$

so that (3.1) follows from (3.3) and the inner estimate in Proposition 1.

4 Finite difference method

Theorem 2 provides uniform estimates for the gradient of solutions. Our approach is to consider difference quotients that are aligned with the periodicity of the problem, that is, with u evaluated at points x and $x+\varepsilon e_d$, where e_d is a coordinate vector in $\mathbb{R}^n$. Such difference quotients satisfy again an equations of the same type and we can apply Theorem 1 to find estimates. The "local lemma", Lemma 2 below, allows to transfer the estimate from the difference quotients to the gradient.

Proof of Theorem 2, item 1. We observe that Poincaré's inequality yields the estimate $\|U\|_{H^1} \leq c\|F\|_{L^2}$. Our aim is to find better integrability properties of ∇U. We first study the case $q < \infty$. The main idea of our proof is to study the discrete difference quotients of the form

$$v_d := \nabla_d^\varepsilon U(x) := \frac{U(x+\varepsilon e_d) - U(x)}{\varepsilon},$$

where $e_d \in \mathbb{R}^n$ is the d'th unit vector, $d = 1, ..., n$, and $v = (v_1, ..., v_n)$. The functions v_d are compactly supported in Ω for ε sufficiently small. They satisfy the equation

$$\mathcal{L}^\varepsilon v_d(x) = (\nabla_d^\varepsilon F)(x) + \operatorname{div}\left(\nabla_d^\varepsilon A(x) \nabla U(x+\varepsilon e_d)\right). \tag{4.1}$$

For ease of notation and without loss of generality we assume $d = n$ and write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We can write $\nabla_n^\varepsilon F = \operatorname{div}(0, ..., 0, \tilde{F}_n)$ by setting

$$\tilde{F}_n(x', x_n) = \frac{1}{\varepsilon}\int_{x_n}^{x_n+\varepsilon} F(x', \xi)\, d\xi.$$

Since $\tilde{F}_n$ is constructed as a local average of F, we can compare the L^p-norms, $\|\tilde{F}_n\|_{L^p} \leq \|F\|_{L^p}$. We now apply Theorem 1 to v_d.

$$\begin{aligned}\|v_d\|_{L^q(\Omega)} &\leq c\left(\|\tilde{F}_d\|_{L^p(\Omega)} + \|\nabla_d^\varepsilon A(.,./\varepsilon)\nabla U(.+\varepsilon e_d)\|_{L^p}\right) \\ &\leq c\left(\|F\|_{L^p(\Omega)} + \|\nabla_d^\varepsilon A\|_{L^\rho(\Omega, C^0(Y))} \cdot \|\nabla U\|_{L^{q-\delta}}\right)\end{aligned}$$

for some $\delta > 0$, since, by assumption, we have the strict inequality $1/\rho < 1/n = 1/p - 1/q$.

We next use Lemma 2 below to transfer this estimate into a result on ∇U. We start by writing the L^q-norm as a sum over local L^q-norms. We use $Q_0 = (0,1)^n$ and take the sum over all $j \in \mathbb{Z}^n$ such that $\varepsilon j \in \Omega$.

$$\int_\Omega |\nabla U|^q = \sum_j \int_{\varepsilon(j+Q_0)} |\nabla U|^q.$$

In the single cell $\varepsilon(j + Q_0)$ we use the local lemma for the rescaled functions $u(y) = U(\varepsilon j + \varepsilon y)$, $f(y) = \varepsilon^2 F(\varepsilon j + \varepsilon y)$, and $v(\varepsilon j + \varepsilon y)$. In their scaling laws, they are related to the original functions by $\nabla_y u = \varepsilon \nabla_x U$, $\mathcal{L}^1 u = f$, and $g := \nabla^1 u = \varepsilon \nabla^\varepsilon U = \varepsilon v$. Inequality (4.5) reads

$$\|\nabla_y u\|_{L^q(Q_0)} \leq c\left(\|g\|_{L^q(Q_4)} + \|f\|_{L^p(Q_5)} + \varepsilon^\alpha \|\nabla_y u\|_{L^s(Q_5)}\right),$$

which holds true for $s \geq 2$ as the exponent in the last term, Q_l are the enlarged cubes $Q_l = (-l, l)^n$. Scaling back into the original variables and taking the q'th power yields for the sinlge cell

$$\begin{aligned}
\int_{\varepsilon(j+Q_0)} |\nabla_x U|^q &= \varepsilon^n \varepsilon^{-q} \|\nabla_y u\|^q_{L^q(Q_0)} \\
&\le c\varepsilon^{n-q} \left(\varepsilon^q \varepsilon^{-n} \int_{\varepsilon(j+Q_4)} |v|^q + \varepsilon^{2q} \left(\varepsilon^{-n} \int_{\varepsilon(j+Q_5)} |F|^p \right)^{q/p} \right. \\
&\qquad\qquad \left. + \varepsilon^{\alpha q} \varepsilon^q \left(\varepsilon^{-n} \int_{\varepsilon(j+Q_5)} |\nabla U|^s \right)^{q/s} \right) \\
&= c \int_{\varepsilon(j+Q_4)} |v|^q + \left(\int_{\varepsilon(j+Q_5)} |F|^p \right)^{q/p} + \varepsilon^{n+\alpha q - nq/s} \left(\int_{\varepsilon(j+Q_5)} |\nabla U|^s \right)^{q/s} ,
\end{aligned}$$

where in the last equality we used $n + q - nq/p = 0$. Summing over all j we find for $q > s$

$$\begin{aligned}
\|\nabla U\|^q_{L^q(\Omega)} &\le c \left(\|v\|^q_{L^q(\Omega)} + \max_j \left(\int_{\varepsilon(j+Q_5)} |F|^p \right)^{(q/p)-1} \sum_j \left(\int_{\varepsilon(j+Q_5)} |F|^p \right) \right. \\
&\qquad\qquad \left. + \varepsilon^{n+\alpha q - nq/s} \|\nabla U\|^q_{L^s(\Omega)} \right) \\
&\le c \left(\|v\|^q_{L^q(\Omega)} + \|F\|^q_{L^p(\Omega)} + \varepsilon^{n+\alpha q - nq/s} \|\nabla U\|^q_{L^s(\Omega)} \right) ,
\end{aligned}$$

where ∇U is estimated like F. Inserting the v-estimate from above and exploiting $\|\nabla^\varepsilon_d A\|_{L^p(\Omega, C^0(Y))} \le c$ from (1.5), we find

$$\begin{aligned}
\|\nabla U\|_{L^q(\Omega)} &\le c \left(\|F\|_{L^p(\Omega)} + \|\nabla U\|_{L^{q-\delta}(\Omega)} + \varepsilon^{(n+\alpha q - nq/s)/q} \|\nabla U\|_{L^s(\Omega)} \right) \\
&\le c \left(\|F\|_{L^p(\Omega)} + \eta \|\nabla U\|_{L^q(\Omega)} + C_\eta \|U\|_{H^1(\Omega)} + \varepsilon^{(n+\alpha q - nq/s)/q} \|\nabla U\|_{L^s(\Omega)} \right) ,
\end{aligned}$$

where $\eta > 0$ can be chosen arbitrarily small such that we can absorb the second term of the right hand side into the left hand side. With the observation from the beginning of the proof we finally have

$$\|\nabla U\|_{L^q(\Omega)} \le c \left(\|F\|_{L^p(\Omega)} + \varepsilon^{(n+\alpha q - nq/s)/q} \|\nabla U\|_{L^s(\Omega)} \right) . \tag{4.2}$$

We note that the exponent of ε is positive for $(q/s) - 1 > 0$ small. We can therefore conclude the result by using (4.2) a finite number of times with indices s_k and $s_{k+1} = q_k = \Theta s_k$, $\Theta > 1$ fixed, starting with $s_0 = 2$. We note that we have to iterate only until $s_k > n/\alpha$, therefore the number of iterations is independent of q.

The case $q = \infty$. Only minor changes in the above arguments are necessary to treat the case $q = \infty$. Theorem 1 provided the estimate for the finite difference quotients v, which now reads

$$\|v\|_{L^\infty(\Omega)} \le c \left(\|F\|_{L^p(\Omega)} + \|\nabla U\|_{L^s} \right) .$$

In this estimate we can choose an arbitrary $s < \infty$, the constant c then depends on s. With the local lemma we may calculate

$$\sup_{\varepsilon(j+Q_0)} |\varepsilon \nabla_x U| \\ \leq c \left(\sup_{\varepsilon(j+Q_4)} |\varepsilon v| + \varepsilon^{2-(n/p)} \|F\|_{L^p(\varepsilon(j+Q_5))} + \varepsilon^{\alpha-(n/s)} \|\varepsilon \nabla_x U\|_{L^s(\varepsilon(j+Q_5))} \right).$$

Dividing by ε and taking the supremum over j we find, for $s = n/\alpha$,

$$\begin{aligned} \|\nabla U\|_{L^\infty(\Omega)} &\leq c \left(\|v\|_{L^\infty(\Omega)} + \|F\|_{L^p(\Omega)} + \varepsilon^{\alpha - \frac{n}{s}} \|\nabla U\|_{L^s(\Omega)} \right) \\ &\leq c \left(\|F\|_{L^p(\Omega)} + \|\nabla U\|_{L^s} \right). \end{aligned}$$

Together with the $L^s(\Omega)$-estimate of the first part of the proof ($q < \infty$), this provides the desired $L^\infty(\Omega)$-estimate. □

The next lemma was the key in the finite difference approach. Loosely speaking, we have: the gradient of a solution is locally as good as the finite differences and the right hand side allow.

The situation in the lemma is as follows. We consider cubes $Q_l = (-l, l)^n$, $l = 1, ..., 5$, solutions $u : Q_5 \to \mathbb{R}^m$, and investigate the gradient ∇u on the smallest cube Q_1. We assume that for $\Omega \subset \mathbb{R}^n$ the coefficients are maps $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{n^2 \times m^2}$ which are $[0,1]^n$-periodic in y, continuous, and uniformly elliptic. For some exponent $\alpha \in (0, 1)$, which provides a small factor in the final estimate, we assume that for every y the map $A(., y)$ is Hölder-continuous with exponent 2α with a y-independent upper bound. The assumptions are met by A satisfying (1.3)–(1.5).

Lemma 2 (Local lemma). *Let $K \subset \Omega \subset \mathbb{R}^n$ be compact, $\xi \in K$ a parameter, $f \in L^p(Q_5)$ a right hand side, and $\varepsilon > 0$ sufficiently small. Let the pair (p, q) satisfy either $p \in [2, n)$, $q \leq np/(n-p)$, or $p > n$, $q = \infty$. We study solutions $u : Q_5 \to \mathbb{R}^m$ of*

$$\nabla \cdot (A(\xi + \varepsilon y, y) \nabla u(y)) = f(y) \quad \forall y \in Q_5. \tag{4.3}$$

We assume to have a control on difference quotients of length 1,

$$u(y + e_d) - u(y) = g_d(y) \quad \forall y \in Q_4, \tag{4.4}$$

for $d = 1, ..., n$, $g : Q_4 \to \mathbb{R}^{m \times n}$. Then

$$\|\nabla u\|_{L^q(Q_1)} \leq c \left(\|g\|_{L^q(Q_4)} + \|f\|_{L^p(Q_5)} + \varepsilon^\alpha \|\nabla u\|_{L^2(Q_5)} \right) \tag{4.5}$$

with c depending on q and A, but independent of f, g, u, and ξ.

Proof. We first show the estimate by a contradiction argument for $p = 2$, $q = 2$, and on Q_2 instead of Q_1. To this end, let us assume that the estimate fails for some A in dimension n. We then find sequences u^k, g^k, f^k, $\xi^k \to \xi$, and ε_k of solutions of (4.3) and (4.4) such that, after rescaling and subtraction of averages,

$$g^k \to 0 \text{ in } L^2(Q_4), \quad f^k \to 0 \text{ in } L^2(Q_5), \quad \varepsilon_k^\alpha \nabla u^k \to 0 \text{ in } L^2(Q_5), \tag{4.6}$$

$$\|\nabla u^k\|_{L^2(Q_2)} = 1, \quad \int_{Q_2} u^k = 0. \tag{4.7}$$

We see that necessarily $\varepsilon_k \to 0$. We observe that each function $g_d^k : Q_4 \to \mathbb{R}^m$ is an H^1-solution of

$$\nabla \cdot (A(\xi^k + \varepsilon_k y, y)\nabla g_d^k(y)) = f^k(y + e_d) - f^k(y)$$
$$- \nabla \cdot \left(\left[A(\xi^k + \varepsilon_k y + \varepsilon_k e_d, y) - A(\xi^k + \varepsilon_k y, y) \right] \nabla u^k(y + e_d) \right).$$

The difference of the coefficients in the squared brackets is pointwise bounded by $C\varepsilon_k^{2\alpha}$. Multiplication of this equation with $g_d^k \eta$ with a cut-off function $\eta \in C_0^\infty(Q_4)$ yields

$$\|g^k\|_{H^1(Q_3)} \leq c \left(\|g^k\|_{L^2(Q_4)} + \|f^k\|_{L^2(Q_5)} + \varepsilon_k^\alpha \|\nabla u^k\|_{L^2(Q_5)} \right) \to 0.$$

This, together with (4.4) and (4.7) implies

$$\|u^k\|_{H^1(Q_3)} \leq C.$$

Choosing a subsequence we may assume for some limit function $u \in H^1(Q_3)$

$$u^k \to u \text{ strongly in } L^2(Q_3) \text{ and weakly in } H^1(Q_3),$$

and u is a weak solution of

$$\nabla \cdot (A(\xi, y)\nabla u(y)) = 0.$$

The strong convergence of u^k implies that u satisfies relation (4.4) with $g \equiv 0$ on Q_2. Hence u is a periodic solution of the homogeneous problem and must therefore be constant, thus, by (4.7), $u \equiv 0$. Finally, exploiting that u^k is a solution of (4.3), we conclude

$$\|u^k\|_{H^1(Q_2)} \leq c \left(\|u^k\|_{L^2(Q_3)} + \|f^k\|_{L^2(Q_3)} \right) \to 0,$$

which contradicts (4.7).

The general case, $p \geq 2$, $q \leq np/(n-p)$, is a consequence of the interior regularity estimates for solutions with bounded H^1-norm. □

5 Two-scale expansions

In this section we exploit the two-scale expansion of solutions and complete the proof of Theorem 2. We always assume the situation of Theorem 2, item 2, in particular the regularity assumption $A \in C^{0,1}(\Omega, W^{1,n}_{per}(Y)) \cap C^{1,1}(\Omega, L^n(Y))$ from (1.6).

We perform all calculations in the scalar case $m = 1$, the case $m > 1$ introduces only notational difficulties. Let $w_k = w_k(x, y)$ be the solutions of the cell-problems

$$\begin{aligned}&\nabla_y \cdot (A(x,y)[\nabla_y w_k(x,y) + e_k]) = 0 \text{ in } Y,\\ &w_k(x,.)\ Y\text{-periodic.}\end{aligned}$$

We first check boundedness properties of w_k. The uniform continuity of A allows to conclude, for every compact subset $\Omega' \subset \Omega$ and every $s < \infty$, the uniform bound $\|w_k(x,\cdot)\|_{W^{1,s}} \le C(\Omega')$ for all $x \in \Omega'$, i.e. $w_k \in L^\infty(\Omega'; W^{1,s}(Y))$, see e.g. [11], page 73. But even a much stronger estimate can be shown. An arbitrary x-derivative $W(x, y) = \partial_{x_l} w_k(x, y)$ satisfies

$$\nabla_y \cdot (A(x,y)[\nabla_y W(x,y)]) = -\nabla_y \cdot (\partial_{x_l} A(x,y)[\nabla_y w_k(x,y) + e_k])\,.$$

For every $x \in \Omega$, the right hand side is the divergence of a bounded function in $L^p(Y)$ for every $p < \infty$. We conclude the uniform boundedness of

$$w_k \in C^{0,1}(\Omega', W^{1,s}(Y)).$$

Second derivatives can be treated in the same way to find bounds for $w_k \in C^{1,1}(\Omega, L^s(Y))$.

With the help of the functions w_k we may, for an arbitrary smooth function η_0, construct the two-scale approximation function

$$\eta^\varepsilon(x) = \eta_0(x) + \varepsilon \sum_{k=1}^n \partial_k \eta_0(x)\, w_k(x, x/\varepsilon). \tag{5.1}$$

The function is constructed in such a way that the application of $\mathcal{L}^\varepsilon$ yields a bounded object.

$$\begin{aligned}\nabla\eta^\varepsilon(x) &= \sum_{k=1}^n \partial_k\eta_0(x)[e_k + \nabla_y w_k(x,x/\varepsilon)] + \varepsilon\sum_{k=1}^n \nabla_x(\partial_k\eta_0(x)\, w_k(x,x/\varepsilon))\\ \mathcal{L}^\varepsilon\eta^\varepsilon(x) &= -\sum_{k=1}^n \nabla_x\cdot(A(x,x/\varepsilon)\,\partial_k\eta_0(x)\,[e_k + \nabla_y w_k(x,x/\varepsilon)])\\ &\quad - \varepsilon\sum_{k=1}^n \nabla\cdot(A(x,x/\varepsilon)\cdot\nabla_x(\partial_k\eta_0(x)\, w_k(x,x/\varepsilon))).\end{aligned}$$

In the case $m > 1$ we use also scalar test-functions $\eta^\varepsilon : \Omega \to \mathbb{R}$, but they are interpreted as representing a variation in direction $\beta \in \{1, ..., m\}$, the cell solutions are $w_k^\beta : Y \to \mathbb{R}^m$, and in the last line above we then calculate $\mathcal{L}^\varepsilon(\eta^\varepsilon e_\beta) : \Omega \to \mathbb{R}^m$.

Proof of Theorem 2, item 2. We consider an H^1-solution U of $\mathcal{L}^\varepsilon U = F$ on $B_R = B_R(0) \subset \mathbb{R}^n$ for $F \in L^p(B_R)$. Our aim is to derive, for some $\Theta \in (0,1)$, an estimate

$$\|\nabla U\|_{L^q(B_{\Theta R})} \leq c \left(\|F\|_{L^p(B_R)} + \|U\|_{H^1(B_R)} \right). \tag{5.2}$$

By a covering argument, this yields the claimed estimate on arbitrary compactly contained subsets Ω'.

Let $\eta_0 \in C_0^\infty(B_{R/2})$ be a cut-off function with $\eta_0 \equiv 1$ on $B_{R/4}$. We use η^ε of equation (5.1) with support in $B_{R/2}$. The function $V^\varepsilon := U \cdot \eta^\varepsilon$ satisfies

$$\begin{aligned} \mathcal{L}^\varepsilon V^\varepsilon(x) &= -\nabla \cdot (A(x, x/\varepsilon)\nabla U(x)\eta^\varepsilon(x)) - \nabla \cdot (U(x)A(x, x/\varepsilon)\nabla\eta^\varepsilon(x)) \\ &= \eta^\varepsilon(x)\mathcal{L}^\varepsilon U(x) - 2\nabla U(x)A(x, x/\varepsilon)\nabla\eta^\varepsilon(x) + U(x)\mathcal{L}^\varepsilon\eta^\varepsilon(x). \end{aligned}$$

The regularity estimates for w_k imply uniform bounds for any $s < \infty$,

$$\nabla\eta^\varepsilon \in L^s(B_R), \qquad \mathcal{L}^\varepsilon\eta^\varepsilon \in L^n(B_R).$$

Inserting this above we find for $q = np/(n-p)$ the estimate

$$\begin{aligned} \|\mathcal{L}^\varepsilon V^\varepsilon\|_{L^p(B_R)} &\leq c \left(\|F\|_{L^p(B_R)} + \|\nabla U\|_{L^{p+\delta}(B_R)} + \|U\|_{L^q(B_R)} \right) \\ &\leq c \left(\|F\|_{L^p(B_R)} + \|U\|_{W^{1,p+\delta}(B_R)} \right) \end{aligned}$$

for some small $\delta > 0$. We can apply Theorem 2, item 1 to V^ε and find the $L^q(B_R)$-estimate for ∇V^ε. We note that in $B_{R/4}$ the gradients coincide, $\nabla V^\varepsilon = \nabla U^\varepsilon$, therefore

$$\|U\|_{W^{1,q}(B_{R/4})} \leq c \left(\|F\|_{L^p(B_R)} + \|U\|_{W^{1,p+\delta}(B_R)} \right). \tag{5.3}$$

We can iterate this estimate, starting with $p = 2$. We arrive at an arbitrary q (including $q = \infty$) after a number of iterations that depends only on n and δ. This yields (5.2).

We note that, in order to start the iteration process with $p = 2$, we need a bound for $\nabla U \in L^{2+\delta}_{loc}$. This estimate is a consequence of Meyers estimate [16] that we use in the following form: For scalar equations, A measurable and uniformly elliptic, there exists $\delta > 0$ depending on the domain and the ellipticity constant of A, such that weak solutions $U \in H^1$ of $\mathrm{div}(A\nabla u) = F$, with $F \in L^2(\Omega)$, satisfy uniform bounds for $\nabla U \in L^{2+\delta}(\Omega')$.

In the case of systems we can not start the iteration with Meyers estimate. We therefore exploit the Hölder continuity of $A(x, .)$ which implies an L^∞-bound for $\nabla_y w_k$, see e.g. [11], page 48. In this case, $\nabla\eta^\varepsilon$ is bounded in L^∞ which provides estimate (5.3) with $\delta = 0$. Again, an iteration yields (5.2). $\square$

Application to a corrector result. On a domain $\Omega \subset \mathbb{R}^n$ we study the homogenization problem

$$\mathcal{L}^\varepsilon u^\varepsilon = f \text{ in } \Omega, \qquad u^\varepsilon = 0 \text{ on } \partial\Omega$$

with coefficients $A(x,y)$ of the operator satisfying (1.3)–(1.6). We denote by $\eta_0 : \Omega \to \mathbb{R}$ the solution of the homogenized problem

$$\mathcal{L}^* \eta_0 = f \text{ in } \Omega, \qquad \eta_0 = 0 \text{ on } \partial\Omega,$$

and by η^ε from (5.1) the approximate solution to the ε-problem. For $f \in L^2$, the following corrector result holds (Allaire [1], Theorem 2.6): If

$$\eta_1(x,y) = \sum_{k=1}^n \partial_k \eta_0(x)\, w_k(x,y) \tag{5.4}$$

is such that η_1, $\nabla_x \eta_1$, and $\nabla_y \eta_1$ are admissible, then

$$u^\varepsilon - \eta^\varepsilon \to 0 \text{ strongly in } H^1(\Omega). \tag{5.5}$$

Corollary 2. *Let coefficients A satisfy (1.3)–(1.6) and let $f \in L^p(\Omega)$. For $q' < q = np/(n-p)$ and Ω' a compactly included subdomain of Ω there holds*

$$u^\varepsilon - \eta^\varepsilon \to 0 \textit{ strongly in } W^{1,q'}(\Omega'). \tag{5.6}$$

Proof. It suffices to verify the admissibility hypothesis for (5.5) and to provide uniform $L^q(\Omega')$-estimates for ∇u^ε and $\nabla \eta^\varepsilon$. Then the convergence of (5.5) implies the strong convergence in intermediate Lebesgue spaces as claimed. We note that Theorem 2, item 2, provides the uniform bound for $\nabla u^\varepsilon \in L^q(\Omega')$ with $q = np/(n-p) > p$. The boundedness of $\nabla \eta^\varepsilon \in L^s(\Omega')$ for every $s < \infty$ was already observed in the proof of Theorem 2, item 2.

It remains to analyze the regularity properties of η_0 and η_1. The homogenized operator $\mathcal{L}^*$ has Hölder-continuous coefficients $A^*(x)$, hence $\nabla_x \eta_0 \in L^q(\Omega)$. Furthermore, every x-derivative $\partial_k \eta_0$ of η_0 satisfies the equation

$$-\nabla \cdot (A^* \nabla \partial_k \eta_0) = \partial_k f + \nabla \cdot (\partial_k A^* \cdot \nabla \eta_0).$$

The right hand side is the divergence of a function in $L^p(\Omega)$ and we conclude $\eta_0 \in W^{2,p}(\Omega')$.

Regarding η_1 we have to study the cell problem. We find

$$\nabla_x \eta_1(x,y) = \sum_{k=1}^n \left(\nabla \partial_k \eta_0(x)\, w_k(x,y) + \partial_k \eta_0(x)\, \nabla_x w_k(x,y) \right),$$

hence $\nabla_x \eta_1 \in L^p(\Omega, C^0(Y))$, and

$$\nabla_y \eta_1(x,y) = \sum_{k=1}^n \partial_k \eta_0(x)\, \nabla_y w_k(x,y) \;\Rightarrow\; \nabla_y \eta_1 \in L^p(\Omega, C^0(Y)).$$

Therefore η_1, $\nabla_x \eta_1$, and $\nabla_y \eta_1$ are admissible and (5.5) holds. This concludes the proof. □

A Remarks on the interpolation argument

The interpolation argument requires an off-diagonal version of the well-known interpolation theorem of Stampacchia, cf. [18], that only requires an (L^∞, BMO) bound at the upper end-point. For the readers' convenience we briefly sketch the argument in our specific situation, mainly based on the classical Marcinkiewizc interpolation theorem along the lines of [8].
We let $Q \subset \Omega \subset \mathbb{R}^n$ and suppose that T is linear and bounded as a mapping $T : L^2(\Omega) \to L^{2^*}(Q)$ and $T : L^n(\Omega) \to \mathrm{BMO}(Q)$, respectively. We take any subdivision $\{Q_i\}_{i \in I}$ of the cube Q. Accordingly, we define

$$\mathcal{T}f(x) = \fint_{Q_i} |Tf - \fint_{Q_i} Tf| \quad \text{if} \quad x \in Q_i.$$

Then $\mathcal{T}$ is subadditive and bounded as a mapping $\mathcal{T} : L^2(\Omega) \to L^{2^*}(Q)$ and $\mathcal{T} : L^n(\Omega) \to L^\infty(Q)$, respectively. We infer from the Marcinkiewicz interpolation theorem, cf. [19] Chapter V Theorem 2.4, that $\mathcal{T} : L^p(\Omega) \to L^q(Q)$ continuously for any admissible (p, q) pair, i.e.

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{2^*} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{n} \quad \text{for some} \quad \theta \in (0,1),$$

i.e. $q = np/(n-p)$ with $2 < p < n$, with bounds that are independent of the choice of subdivisions. Thus taking the supremum over the set Δ of subdivisions of Q (and associated operators $\mathcal{T}$) we find

$$\sup_{\{Q_i\} \in \Delta} \sum_{i \in I} |Q_i| \left(\fint_{Q_i} |Tf - \fint_{Q_i} Tf| \right)^q = \sup_{\mathcal{T} \in \Delta} \|\mathcal{T}f\|^q_{L^q(Q)} \leq C \, \|f\|^q_{L^p(\Omega)}$$

with a constant C that only depends on θ and the known bounds on T. By a result of John and Nirenberg, cf. [12], the latter quantity bounds the weak L^q norm of $\tilde{T}f = Tf - \fint_Q Tf$. Thus, for any admissible pair (p, q), the operator $\tilde{T}$ is weakly bounded. Further interpolation and application of the Marcinkiewicz interpolation theorem implies in turn bounds

$$\|\tilde{T}f\|_{L^q(Q)} \leq C \, \|f\|_{L^p(\Omega)} \quad \text{thus} \quad \|Tf\|_{L^q(Q)} \leq C \left(\|f\|_{L^p(\Omega)} + \fint_Q |Tf| \right)$$

where C only depends on the previously known bounds on T and $p \in (2, n)$. Now, if we take as in our application $T = \mathcal{L}_\varepsilon^{-1}\mathrm{div} : f \mapsto u$ with end-point bounds that are independent of $\varepsilon > 0$, we get with Hölder's inequality

$$\|\tilde{T}f\|_{L^q(Q)} \leq C \, \|f\|_{L^p(\Omega)} \quad \text{thus} \quad \|\mathcal{L}_\varepsilon^{-1} f\|_{L^q} \leq C(n,p,Q) \Big(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)} \Big)$$

for $n > 2$, any $p \in (2, n)$ and $q = np/(n-p)$.

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