Existence and approximation results for shape optimization problems in rotordynamics

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Abstract

We consider a shape optimization problem in rotordynamics where the mass of a rotor is minimized subject to constraints on the natural frequencies. Our analyis is based on a class of rotors described by a Rayleigh beam model including effects of rotary inertia and gyroscopic moments. The solution of the equation of motion leads to a generalized eigenvalue problem. The governing operators are non-symmetric due to the gyroscopic terms. We prove the existence of solutions for the optimization problem by using the theory of compact operators. For the numerical treatment of the problem a finite element discretization based on a variational formulation is considered. Applying results on spectral approximation of linear operators we prove that the solution of the discretized optimization problem converges towards the solution of the continuous problem if the discretization parameter tends to zero. Finally, a priori estimates for the convergence order of the eigenvalues are presented and illustrated by a numerical example.

Keywords: gyroscopic system, generalized eigenvalue problem, existence theorem, convergence theorem, a priori estimates

1 Introduction

Typical design optimization problems in rotordynamics include the minimization of a given objective function, such as the mass of a rotor, by changing the shape of the rotor. In addition it is often the target to influence certain natural frequencies such as to increase selected ones to avoid their resonance case. This can be formulated in the following optimization problem, on which we focus in this paper.

$$\begin{array}{l} \min_r J(r) \\ \text{subject to} \\ \lambda_{\mathrm{m_i}}(r) \geq \lambda^*_{\mathrm{m_i}}, \quad i = 1, \ldots, n_c \end{array} \tag{1}$$

A continuous objective function J is minimized subject to constraints on the natural frequencies of certain modes $\lambda_{\mathrm{m_i}}$, where n_c denotes the number of constraints. These frequencies are increased above given target values $\lambda^*_{\mathrm{m_i}}$. All functions depend on a shape function r. We refer to this problem as natural frequency optimization problem. In our framework the rotational speed is fixed.

Our analysis is based on a general class of simpy supported continuous rotors which can be described by a Rayleigh beam model including the important effects of rotary inertia and gyroscopic moments (see e.g. [5] and [10]). Such systems are called gyroscopic systems. They lack some nice

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mathematical properties because the underlying operator is non-symmetric due to the influence of the gyroscopic term. This changes the corresponding theory and we can no longer assume the natural frequencies and eigenmodes to have real values but instead we have to cope with possible complex eigenvalues.

A central result of the paper is a theorem that guarantees the existence of solutions for the optimization problem (1) for the given class of rotors. In the solution process the second-order equation of motion is transformed into a first-order system and separation of variables leads to a generalized eigenvalue problem. The solution of this eigenvalue problem gives natural frequencies which are target of the optimization. Since the governing operator is non-symmetric we cannot use results from the literature (e.g. [6]) for the solution of the problem. But we are able to show that the operator of the eigenvalue problem is compact. Then the solvability of the eigenvalue problem can be shown and the spectrum is described. From these results the existence of solutions of the optimization problem can also be derived.

In a next step we introduce a suitable finite element discretization based on a variational formulation for a numerical solution of the problem. Using results on spectral approximation of linear operators (see e.g. [2, 3, 8]) it can be shown that the solution of the discretized optimization problem converges towards the solution of the continuous problem if the discretization parameter tends to zero.

Finally, we can establish a priori estimates that ensure a quadratic convergence of the eigenvalues with respect to the discretization parameter. This result is illustrated by a numerical example.

This paper is organized as follows. In Section 2 the Rayleigh beam model is introduced as the model of our choice and the equation of motion for a continuous rotor which is the basis for the forthcoming studies is given. In Section 3 the equation of motion is considered in a functional analytical framework. The solution of the equation of motion for free vibrations yields a non-symmetric generalized eigenvalue problem. The operator of the generalized eigenvalue problem is shown to be compact. The application of the Riesz-Schauder spectral theorem proves the solvability of the equation of motion. Continuity of a finite subset of eigenvalues and eigenvectors guarantees the existence of solutions for the natural frequency optimization problem. In the first section of Chapter 4 the convergence of eigenvalues and eigenvectors of a discretized problem towards those of the continuous problem is shown if the discretization parameter tends to zero. These results are used to show the convergence of the solutions of the corresponding optimization problems. In Section 5 a priori estimates for the discretization error of the considered eigenvalue problem are derived .

2 Physical model

Our rotor is a three-dimensional body which we want to describe by an one-dimensional model based on theories of lateral beam vibrations. Of course, this requires some form of approximation to the underlying physics. As mentioned above we work with a Rayleigh beam model capturing effects of rotary inertia and gyroscopic moments.

We use a static XYZ-coordinate system whose Z-axis coincides in the static position with the centerline of the shaft. We consider a shaft of length l and the spatial variable along the Z-axis is denoted by s. The motion of the rotor is described by the lateral deflections and inclinations in each point along the Z-axis (see Figure 1).

The lateral deflections in X- and Y-direction are denoted by u and v, respectively. The inclination angle θ of the tangent to the rotor deflection curve can be decomposed into two components θ_x and θ_y which are the projections of θ onto the XZ- and YZ-plane, respectively. The deflections and inclinations also depend on the time variable t. For the sake of simplicity this is not mentioned

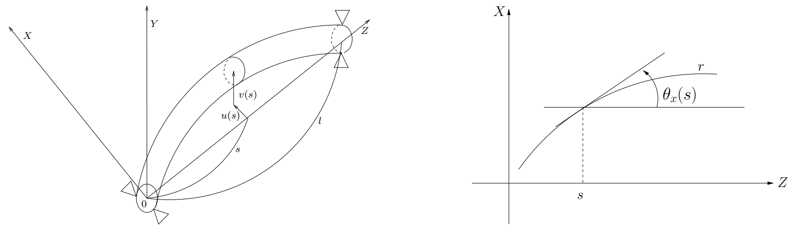

Figure 1: Model of rotor in XYZ-coordinate system (left) and inclination angle in XZ-plane (right).

explicitly in each equation. We assume the inclination angles to be small and have

$$\theta_x(s) = u'(s) \quad \text{and} \quad \theta_y(s) = v'(s), \tag{2}$$

where a prime denotes a differentiation by s. The consideration of the inclination motion leads to the gyroscopic moment in this system which is essential for our further studies. The shape of the rotor is described by a continuous function $r \in C(I)$, where $I = [0, l]$ and $r(s)$ is the radius of the shaft at position s. It is bounded from below and above by fixed functions $\underline{r}$ and $\bar{r}$ respectively and $\underline{r}, \bar{r} \in L^\infty(I)$. Using the Lagrange formalism the equation of motion can be derived. For free vibrations of a continuous rotor without damping it is given by

$$\mu \ddot{z} - \frac{1}{2}\left(I_p \ddot{z}'\right)' + i\left(I_p \omega \dot{z}'\right)' + \left(EI_a z''\right)'' = 0, \tag{3}$$

where μ denotes the mass per unit length, I_p the polar moment of inertia about the Z-axis, I_a the cross-sectional moment of inertia with respect to the Z-axis. The product EI_a is called bending rigidity where E is Young's modulus. The rotational speed is denoted by ω. As rotor support simple support at both ends is considered. In this case transverse displacements are not possible, but end rotations are permitted. The boundary conditions are written as

$$z(0,t) = 0, \quad z(l,t) = 0, \quad z''(0,t) = 0, \quad z''(l,t) = 0.$$

3 Existence theorems

In this section we prove the existence of solutions for the introduced natural frequency optimization problem. In a first step the equation of motion for the rotor is solved in Section 3.1. This is achieved by the separation of variables which leads to an eigenvalue problem and which is giving the natural frequencies and eigenmodes. The operator in the eigenvalue problem is shown to be compact and the spectrum is described by the theorem of Riesz-Schauder. This result is then used to show the existence of solutions for the optimization problem in Section 3.2.

Similar results for the non-rotating case can e.g. be found in Haslinger and Mäkinen [6] and Fichera [4]. The inclusion of the gyroscopic term, however, yields a non-symmetric system and the theory of compact operators is used. Hence our approach is an extension of existing results in the literature.

3.1 Solvability of equation of motion

Let us first consider the equation of motion for undamped free oscillations which was deduced above,

$$\mu \ddot{z} - \frac{1}{2}(I_p \ddot{z}')' + i(I_p \omega \dot{z}')' + (EI_a z'')'' = 0. \tag{4}$$

The parameters μ, I_p and I_a depend on the rotor shape function r and hence on the spatial variable s. The function r belongs to the set of admissible functions U which is given by

$$U = \{r \in C(I), \quad \underline{r} \le r \le \bar{r}, \quad |r(x) - r(y)| \le L_0|x-y|, \quad \forall x, y \in I\}, \tag{5}$$

where $I = [0, l]$ as in Section 2. The additional Lipschitz condition with constant L_0 makes U a compact subset of $C(I)$ which follows from the theorem of Arzelà-Ascoli.

Equation (4) is now transformed into an eigenvalue problem by the separation of variables and an exponential function approach for the time variable. Every solution of the eigenvalue problem leads to a solution of the original equation of motion and we just focus on these special solutions. The separation of variables is written as

$$z(s,t) = \varphi(s)\psi(t),$$

where φ and ψ only depend on one of the variables. The original equation of motion (4) becomes

$$\mu\varphi\ddot{\psi} - \frac{1}{2}\left(I_p\varphi'\right)'\ddot{\psi} + i\left(I_p\omega\varphi'\right)'\dot{\psi} + \left(EI_a\varphi''\right)''\psi = 0 \tag{6}$$

and the boundary conditions are satisfied if

$$\varphi(0) = 0,\ \varphi(l) = 0,\ \varphi''(0) = 0,\ \varphi''(l) = 0. \tag{7}$$

Equation (6) can immediately be transformed into the desired eigenvalue problem. However, before doing this, to avoid a quadratic eigenvalue problem, the second-order-equation (6) is transformed into a first-order-system by writing

$$\begin{pmatrix} \mu\varphi - \frac{1}{2}(I_p\varphi')' & 0 \\ 0 & \mu\varphi - \frac{1}{2}(I_p\varphi')' \end{pmatrix} \begin{pmatrix} \ddot{\psi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} -i(I_p\omega\varphi')' & -(EI_a\varphi'')'' \\ \mu\varphi - \frac{1}{2}(I_p\varphi')' & 0 \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \psi \end{pmatrix}.$$

Setting $\psi(t) = e^{\lambda t}$ we want to solve for $\phi = (\phi_1, \phi_2)$ the equation

$$\lambda A(\phi_1, \phi_2) = B(\phi_1, \phi_2), \tag{8}$$

where

$$A(\phi_1, \phi_2) = \begin{pmatrix} \mu\phi_1 - \frac{1}{2}(I_p\phi_1')' \\ \mu\phi_2 - \frac{1}{2}(I_p\phi_2')' \end{pmatrix} \quad \text{and} \quad B(\phi_1, \phi_2) = \begin{pmatrix} -i(I_p\omega\phi_1')' - (EI_a\phi_2'')'' \\ \mu\phi_1 - \frac{1}{2}(I_p\phi_1')' \end{pmatrix}.$$

Once (8) is solved we may set $\varphi = \phi_2$ and obtain $\phi_1 = \lambda\phi_2 = \lambda\varphi$ from the second line and the solution property for the first line.

Proposition 3.1 *Let φ satisfy the boundary conditions (7). The function $\varphi(s)e^{\lambda t}$ is a solution of equation (4) if and only if $(\lambda\varphi, \varphi)^T$ is a solution of system (8).*

Proof. We assume that $\varphi(s)e^{\lambda t}$ solves (4). Then it is obvious that $\varphi(s)e^{\lambda t}$ solves (6) where we set $\psi(t) = e^{\lambda t}$. This equation can be transformed into a first order system as shown above. If we then set $\phi_1 = \lambda\varphi$ and $\phi_2 = \varphi$ the eigenvalue problem (8) is satisfied.

For the other direction we assume that $(\lambda\varphi, \varphi)^T$ is a solution of (8). Then the first line of (8) yields that φ and $\psi(t) = e^{\lambda t}$ solve the first-order system (6). Then $z(s,t) = \varphi(s)e^{\lambda t}$ is a solution of (4). □

To prove the existence of solutions of eigenvalue problem (8) and thus for the equation of motion (4) a suitable analytical framework has to be introduced. We define the two Sobolev spaces V_1 and V_2 on $I = [0, l]$ by

$$V_1 = \{v \in H^1(I) |\ v(0) = 0,\ v(l) = 0\} = H^1_0(I),$$
$$\|v\|_{V_1} = \left(\int_I |v(s)|^2\, ds + \int_I |v'(s)|^2\, ds\right)^{1/2}$$

and

$$V_2 = \{v \in H^2(I) |\ v(0) = 0,\ v(l) = 0\},$$
$$\|v\|_{V_2} = \left(\int_I |v(s)|^2\, ds + \int_I |v'(s)|^2\, ds + \int_I |v''(s)|^2\, ds\right)^{1/2}$$

and their dual spaces are denoted by $V_1' = H^{-1}(I)$ and V_2', respectively. The conditions on the second derivatives are not imposed but turn out to be satisfied as the natural boundary conditions.

The formulas (4) and (8) are the classical formulation of the equation of motion and the eigenvalue problem. This is sufficient for the existence theorems of this chapter. For the convergence analysis of a discretized model which is done in Section 4 the weak formulation is also needed and will be introduced then.

The crucial point now is to transform the generalized eigenvalue problem given by (8) into a standard one by inverting one of the operators A or B, then to show the compactness of the resulting operator which allows the application of the Riesz-Schauder spectral theorem.

We consider the operator $M : V_1 \to V_1'$ formally written as

$$M(u) = \mu u - \frac{1}{2}(I_p u')' \tag{9}$$

and for any $v \in V_1$ defined by

$$M(u)(v) = \int \left(\mu(s)u(s)\bar{v}(s) + \frac{1}{2} I_p(s)u'(s)\bar{v}'(s)\right)\, ds,$$

and the operator $L : V_1 \times V_2 \to V_2'$ formally written as

$$L(u_1, u_2) = -i(I_p \omega u_1')' - (EI_a u_2'')''$$

and for any $v \in V_2$ defined by

$$L(u_1, u_2)(v) = i \int \omega I_p(s) u_1'(s)\bar{v}'(s)\, ds - \int EI_a(s) u_2''(s)\bar{v}''(s)\, ds.$$

Then eigenvalue problem (8) reads as

$$\lambda \underbrace{\begin{pmatrix} M(\phi_1) \\ M(\phi_2) \end{pmatrix}}_{=A(\phi_1,\phi_2)} = \underbrace{\begin{pmatrix} L(\phi_1, \phi_2) \\ M(\phi_1) \end{pmatrix}}_{=B(\phi_1,\phi_2)}, \tag{10}$$

where $A : V_1 \times V_1 \to V_1' \times V_1'$ and $B : V_1 \times V_2 \to V_2' \times V_1'$. Note that A and B map into different spaces. However, if we can find an inverse operator to B we can construct an operator $B^{-1}A$ which then is well-defined since $V_1' \subset V_2'$ and we have to solve

$$B^{-1}A\phi = \zeta\phi, \tag{11}$$

where $\zeta = 1/\lambda$.

Now the goal is to show that such an operator B^{-1} exists.
We first want to prove two lemmas giving us the invertibility of the operator M as introduced above and the operator $K : V_2 \to V_2'$ which is the second term of the operator L and is given formally by

$$K(u) = (EI_a u'')'' \tag{12}$$

and for any $v \in V_2$

$$K(u)(v) = \int EI_a(s) u''(s) \bar{v}''(s)\, ds,$$

respectively. As usual in such cases the theorem of Lax-Milgram is used.

CONVENTION: In this paper the letter C stands for a generic positive constant attaining different values at different places.

Lemma 3.1 *The operator $M : V_1 \to V_1'$ given by (9) is invertible.*

Proof. To apply the theorem of Lax-Milgram we have to show that the associated sesquilinear form $m : V_1 \times V_1 \to \mathbb{C}$ given by

$$m(u, v) = \int_I \mu(s) u(s) \bar{v}(s)\, ds + \frac{1}{2} \int_I I_p(s) u'(s) \bar{v}'(s)\, ds$$

is continuous and coercive.
The continuity and coercivity follow from the boundedness of μ and I_p. Indeed, we have

$$\begin{aligned} |m(u,v)| &= \left| \int \mu(s) u(s) \bar{v}(s)\, ds + \frac{1}{2} \int I_p(s) u'(s) \bar{v}'(s)\, ds \right| \\ &\leq C \left| \int u(s) \bar{v}(s)\, ds + \int u'(s) \bar{v}'(s)\, ds \right| = C |\langle u, v \rangle_{V_1}| \leq C \|u\|_{V_1} \|v\|_{V_1}. \end{aligned}$$

and

$$\begin{aligned} m(u,u) &= \int \mu |u(s)|^2 ds + \frac{1}{2} \int I_p |u'(s)|^2 ds \\ &\geq C \left(\int |u(s)|^2 ds + \int |u'(s)|^2 ds \right) = \|u\|_{V_1}^2. \end{aligned} \tag{13}$$

Then the application of Lax-Milgram yields

$$m(u, v) = M(u)(v)$$

and the operator $M : V_1 \to V_1^{'}$ is invertible. □

Lemma 3.2 *The operator $K : V_2 \to V_2'$ given by (12) is invertible.*

Proof. The proof is as above. The associated sesquilinear form $k : V_2 \times V_2 \to \mathbb{C}$ given by

$$k(u, v) = \int EI_a u''(s) \bar{v}''(s) ds$$

is shown to be continuous and coercive.
The continuity is obvious again due to the boundedness of I_a.

For the coercivity we have to apply the standard Poincaré inequality as well as a generalized form of it (see e.g. [1]) since we have no boundary conditions for the first derivatives. The latter inequality says, that for a nonempty convex closed cone S with apex 0,

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p} \quad \text{for all} \quad u \in S, \tag{14}$$

iff there exists a $u_0 \in S$ and a constant $C_0 < \infty$, such that for all $\xi \in \mathbb{R}^m$

$$u_0 + \xi \in S \Rightarrow |\xi| \leq C_0. \tag{15}$$

In our case the set S is chosen to be

$$S = \left\{ v \in H^1(I) | \int_0^l v = 0 \right\}$$

and having in mind that $v = u'$ and $u \in V_2$. This set is a subspace of $H^1(I)$ and hence a cone with apex 0. Then condition (15) is fulfilled with $C_0 = 0$, since for any $u_0 \in S$ and $\xi \neq 0$, we have $u_0 + \xi \notin S$. Then the estimate

$$\int |u'(s)|^2\, ds \leq C \int |u''(s)|^2\, ds \quad \forall u' \in S$$

is obtained. The coercivity follows immediately

$$\begin{aligned}
\|u\|_{V_2}^2 &= \left(\int |u(s)|^2\, ds + \int |u'(s)|^2\, ds + \int |u''(s)|^2\, ds \right) \\
&\leq C \int |u'(s)|^2\, ds + \int |u''(s)|^2\, ds \\
&\leq C \int |u''(s)|^2\, ds \\
&\leq C \int EI_a(s) u''(s) \bar{u}''(s)\, ds = C\, k(u,u).
\end{aligned} \tag{16}$$

The first inequality holds due to the standard Poincaré inequality and the second one due to the generalized Poincaré inequality. The remaining expression can again be estimated due to the boundedness of I_a leading to the last inequality. Then, by applying Lax-Milgram,

$$k(u, v) = K(u)(v)$$

and K is invertible. □

Knowing the invertibility of M and K we can construct an inverse operator to the operator B given in (10).

Lemma 3.3 *To the operator B given in (10) there exists an inverse operator $B^{-1} : V_2' \times V_1' \to V_1 \times V_2$.*

Proof. We consider an arbitrary right-hand-side $(f_1, f_2) \in V_2' \times V_1'$ and look at the system $B(\phi_1, \phi_2) = (f_1, f_2)^T$, i.e.

$$L(\phi_1, \phi_2) = -i(I_p \omega \phi_1')' - (EI_a \phi_2'')'' = f_1 \quad \in V_2', \tag{17}$$

$$M(\phi_1) = \mu\phi_1 - \frac{1}{2}(I_p \phi_1')' = f_2 \quad \in V_1'. \tag{18}$$

From Lemma 3.1 we know that M is invertible. Hence we can write

$$\phi_1 = M^{-1} f_2 \quad \in V_1.$$

Then the equation (17) can be written as

$$-(EI_a\phi_2'')'' = i(I_p\omega(M^{-1}f_2)')' + f_1 \quad \in V_2'.$$

Lemma 3.2 shows that the operator on the left-hand-side is invertible and we have

$$\phi_2 = -K^{-1}(i(I_p\omega(M^{-1}f_2)')' + f_1) \in V_2.$$

Thus we have found a preimage $(\phi_1, \phi_2) \in V_1 \times V_2$. □

The next step is to show that the operator $B^{-1}A$ is compact. The notion of compactness requires a mapping from a certain space onto itself. We notice that the space $V_1 \times V_2$ is the correct space for which the operator $B^{-1}A$ is compact.

Lemma 3.4 *The operator $B^{-1}A : V_1 \times V_2 \to V_1 \times V_2$ is compact.*

Proof. We apply the operator on an arbitrary pair $(\phi_1, \phi_2) \in V_1 \times V_2$. Note that we restrict the second component to be in V_2 whereas A also allows V_1-functions. Then we have

$$\begin{aligned} B^{-1}A(\phi_1,\phi_2) &= B^{-1}(M(\phi_1), M(\phi_2)) \\ &= (M^{-1}(M(\phi_2)), -K^{-1}(i(I_p\omega(M^{-1}(M(\phi_2)))')' + M(\phi_1))) \\ &= (\phi_2, -K^{-1}(i(I_p\omega\phi_2')' + M(\phi_1))). \end{aligned}$$

We show that the mapping is compact in each component.
For the first component we have

$$\begin{array}{ccccc} (\phi_1,\phi_2) & \overset{B^{-1}A}{\mapsto} & \phi_2 & \overset{id}{\mapsto} & \phi_2, \\ V_1 \times V_2 & \to & V_2 & \overset{cpt}{\hookrightarrow} & V_1, \end{array}$$

and for the second component

$$\begin{array}{ccccccc} (\phi_1,\phi_2) & \mapsto & i(I_p\omega\phi_2')' + M(\phi_1) & \overset{id}{\mapsto} & \dots & \overset{K^{-1}}{\mapsto} & -K^{-1}(i(I_p\omega\phi_2')' + M(\phi_1)), \\ V_1 \times V_2 & \to & V_1' & \overset{cpt}{\hookrightarrow} & V_2' & \to & V_2. \end{array}$$

All operators are linear and continuous and the composition with a compact embedding yields a compact operator. Hence, the operator $B^{-1}A : V_1 \times V_2 \to V_1 \times V_2$ is compact. □

Theorem 3.1 *Let $T := B^{-1}A$. The spectrum $\sigma(T)$ is an at most countable set with no accumulation point different from zero. Let $\sigma'(T)$ be a finite system of eigenvalues, which is separated from the rest $\sigma''(T)$ of $\sigma(T)$ by a closed Jordan curve. All eigenvalues λ of $\sigma'(T)$ depend continuously on the shape function r. The same holds for the set of corresponding eigenvectors ϕ.*

Proof. Since the operator $B^{-1}A$ is compact the spectral theorem of Riesz-Schauder can be applied and the eigenvalue problem (11) has a solution with eigenvalues ζ_i with at most one accumulation point at zero. This means that the values $\lambda_i = 1/\zeta_i$ tend to infinity. It also implies that the spectrum can be separated into two parts by a closed Jordan curve with the part inside the curve consisting of a finite number of eigenvalues and not containing zero. Then from Kato [7, IV.3.5], it follows that the eigenvalues and eigenvectors depend continuously on the closed operator $B^{-1}A$ and hence also on the shape function r. □

Remark. Our focus lies only on eigenvalues belonging to modes which are excited in the respective operating speed range. Their number is limited and can be included in a finite system of eigenvalues and can be separated from the accumulation point zero. Hence, the continuity argument of Theorem 1 holds for this case.

3.2 Solvability of optimization problem

As pointed out in Section 1, the design optimization problem we want to study for a continuous rotor is the following: Find a thickness distribution which minimizes a given continuous cost functional J subject to natural frequency constraints as given in the introduction. The rotor shape function r is bounded from below and above. More precisely, r belongs to the class of admissible functions U as in (5).

The optimization problem deals with a continuous objective function J and a constraint on the natural frequency λ_{m} of a certain mode m and is written as

$$\begin{array}{l} \min_r J(r) \\ \text{subject to} \\ \lambda_{\mathrm{m}}(r) \geq \lambda_{\mathrm{m}}^*, \\ r \in U. \end{array} \tag{19}$$

In our applications the function of the total mass of the rotor is often chosen as cost functional J. For the natural frequency λ_{m} a lower bound λ_{m}^* is given. The constraint on the natural frequencies can be put into U as defined in (5) giving the (new) class of admissible functions for this problem

$$U_c = \{r \in U |\, \lambda_{\mathrm{m}}(r) \geq \lambda_{\mathrm{m}}^*\}.$$

The theory of the last section enables us to show the existence of solutions for problem (19).

Theorem 3.2 *Let $U_c \neq \emptyset$. Then the optimization problem (19) has a solution.*

Proof. Due to Theorem 1 and the subsequent remark λ_{m} is a continuous function in r. Hence U_c is a compact subset of $C(I)$. Moreover, the objective function is assumed to be continuous in r. Since a continuous function on a compact set possesses a minimum the existence of solutions is proven. □

Remark. The shown existence and approximation results can be extended to further shape optimization problems in rotordynamics, such as the minimization of the mass subject to constraints on the critical speed and unbalance response. Details on this case can be found in [9].

4 Approximation results

4.1 Convergence of eigenvalues of discretized problem

In a first step, the convergence of the eigenvalues and eigenvectors of discretized generalized eigenvalue problems has to be shown. These functions appear in the constraints of our optimization problem. Important results in spectral approximation can be found in Babuška & Osborn [2], Chatelin [3] and Kolata [8] and can be applied to our case.

The convergence theory uses the weak formulation of the classical eigenvalue problem

$$\lambda A(\phi_1, \phi_2) = B(\phi_1, \phi_2).$$

It is obtained by multiplying the operators $A : V_1 \times V_1 \to V_1' \times V_1'$ and $B : V_1 \times V_2 \to V_2' \times V_1'$ given by,

$$A(\phi) = A(\phi_1, \phi_2) = \begin{pmatrix} \mu\phi_1 - \frac{1}{2}(I_p\phi_1')' \\ \mu\phi_2 - \frac{1}{2}(I_p\phi_2')' \end{pmatrix}$$

$$B(\phi) = B(\phi_1,\phi_2) = \begin{pmatrix} -i(I_p\omega\phi_1')' - (EI_a\phi_2'')'' \\ \mu\phi_1 - \frac{1}{2}(I_p\phi_1')' \end{pmatrix}.$$

by test functions $\eta = (\eta_1,\eta_2) \in V_2 \times V_1$ and integrating the two equations over the interval I. Considering the boundary conditions (7) we obtain

$$a(\phi,\eta) := A(\phi_1,\phi_2)(\eta_1,\eta_2) = \begin{pmatrix} \int \mu\phi_1\bar\eta_1\,ds + \frac{1}{2}\int I_p\phi_1'\bar\eta_1'\,ds \\ \int \mu\phi_2\bar\eta_2\,ds + \frac{1}{2}\int I_p\phi_2'\bar\eta_2'\,ds \end{pmatrix} \tag{20}$$

and

$$b(\phi,\eta) := B(\phi_1,\phi_2)(\eta_1,\eta_2) = \begin{pmatrix} i\int I_p\omega\phi_1'\bar\eta_1'\,ds - \int EI_a\phi_2''\bar\eta_1''\,ds \\ \int \mu\phi_1\bar\eta_2\,ds + \frac{1}{2}\int I_p\phi_1'\bar\eta_2'\,ds \end{pmatrix}. \tag{21}$$

For the further studies we use the notation $V = V_1 \times V_2$ and $\tilde V = V_2 \times V_1$ with $\|\cdot\|_V = \|\cdot\|_{\tilde V} = \|\cdot\|_{V_1} + \|\cdot\|_{V_2}$.

Remark. All lemmas and theorems formulated for the classical case then obviously hold for the weak case as well.

The approximation of the operators and the optimization problems is based on the partition of the interval $I = [0,l]$,

$$0 = a_0 < a_1 < \ldots < a_n = l,$$

where $h = \max_{i=1,\ldots,n}|a_i - a_{i-1}|$ is the discretization parameter. We assume a partition such that $h \to 0$ if $n \to \infty$. Moreover, let $P_k([a_{i-1},a_i])$ denote the space of polynomials of degree $\le k$ on the interval $[a_{i-1},a_i]$.

The rotor shape function $r \in U$, where U is given by (5), is then assumed to be approximated by piecewise constant functions r_h belonging to the set

$$\begin{aligned} U_h \;=\; \{r \in L^\infty(I)\,|\,& r^i = r|_{[a_{i-1},a_i]} \in P_0([a_{i-1},a_i]),\ i = 1,\ldots,n,\ \underline{r} \le r \le \bar r, \\ & |r^{i+1} - r^i| \le L_0 h, \quad L_0 > 0, \quad i = 1,\ldots,n-1\}. \end{aligned}$$

The spaces V_1 and V_2 defined in Section 3 are replaced by finite dimensional approximations V_1^h and V_2^h. The exact choice of these approximations depends on the degrees of freedom under consideration. Moreover, we set $V^h = V_1^h \times V_2^h$ and we assume a partition such that $\cup_{h>0}V^h$ is dense in the space $V = V_1 \times V_2$.

The discretized eigenvalue problem is now obtained by replacing the continuous rotor shape function $r \in U$ by a function $r_h \in U_h$. This is written as follows. At first, the notation of the eigenvalue problem of Section 3 is extended about a subscript r indicating the dependence on the continuous function r, i.e.

$$\lambda A_r(\phi) = B_r(\phi), \qquad \phi = (\phi_1,\phi_2) \in V_1 \times V_2$$

and

$$\lambda a_r(\phi,\eta) = b_r(\phi,\eta), \qquad \eta \in \tilde V,$$

respectively. Furthermore, we write

$$T_r = B_r^{-1}A_r, \qquad V_1 \times V_2 \to V_1 \times V_2$$

for the operator which was shown to be compact in Lemma 3.4.

The discretized subproblem for $r_h \in U_h$ is then

$$\lambda_h A_{r_h}(\phi_h) = B_{r_h}(\phi_h), \qquad \phi_h \in V^h \tag{22}$$

with eigenvalues λ_h and eigenvectors ϕ_h in V^h.

The corresponding weak formulation is

$$\lambda_h a_{r_h}(\phi_h, \eta_h) = b_{r_h}(\phi_h, \eta_h), \qquad \eta_h \in \tilde{V}_2^h. \tag{23}$$

Following Kolata [8] we now define a projection of the space $V = V_1 \times V_2$ on the space V^h. This is done by using the weak formulation.

Definition 4.1 *We define a linear operator $P_h : V \to V^h$ by*

$$b_{r_h}(P_h\phi, \eta_h) = b_r(\phi, \eta_h), \quad \forall \eta_h \in \tilde{V}_2^h.$$

Furthermore, let $T_{r_h} : V \to V^h$ be given by

$$T_{r_h} = P_h \circ T_r$$

and $T_{r_h} = B_{r_h}^{-1} A_{r_h}$.

In addition, the convergence of eigenvalues λ_h and eigenvectors ϕ_h of the discretized problem to those of the continuous problem is shown by applying the concept of strongly stable convergence (see Chatelin [3]).

Theorem 4.1 *Let $r_h \to r$ as $h \to 0$ in $L^\infty(I)$. Moreover, let S be a set bounded by a closed Jordan curve which encloses exactly one eigenvalue λ of T_r with multiplicity m. Then $\sigma(T_{r_h}) \cap S$ consists for h small enough of exactly m eigenvalues, counting their multiplicities.*

Proof. Since $\cup_{h\to 0} V^h$ is dense in V the projection operator converges pointwise towards the identity operator, $P_h \to I$. The convergence is uniform on any sequentially compact set. Since we know that T_r is compact it follows

$$\|T_r - T_{r_h}\| = \|(I - P_h)T_r\| \to 0 \quad \text{as} \quad h \to 0.$$

This implies strongly stable convergence of the sequence T_{r_h} (see Chatelin [3, Example 5.14]). For a definition of strongly stable convergence see [3, Chapter 5.2]. Let now λ be an eigenvalue of T_r with multiplicity m. Then the strongly stable convergence property of T_{r_h} guarantees that $\sigma(T_{r_h}) \cap S$ consists for h small enough of exactly m eigenvalues, counting their multiplicities [3, Proposition 5.6]. $\square$

The convergence of eigenvectors then also follows immediately by [3, Theorem 5.10].

Theorem 4.2 *Let T_{r_h} be an approximation of T_r, converging strongly stable in S. Then for any sequence of eigenvalues λ_h converging to λ and for any sequence of associated eigenvectors ϕ_h there exists a subsequence converging to an eigenvector ϕ associated with λ.*

4.2 Convergence of solutions of optimization problem

Having established the convergence of the constraint functions in Section 4.1 we now want to show the convergence of the solutions of the natural frequency optimization problems for rotating bodies. Similar results for problems for the non-rotating case can be found in [6].

For the formulation of the discretized natural frequency optimization problem the set of admissible functions U_h is restricted about the constraint on the eigenvalues and is

$$U_{h,c} \quad = \quad \{r \in U_h |\, \lambda_h(r) \geq \lambda^*\}.$$

In practice the eigenvalue constraint is only set for few specific modes. Similar to Chapter 3 we consider only one constraint function for the analysis. Other constraints can be included in the same way. The natural frequency optimization problem with a continuous objective function J writes as

$$\begin{array}{l} \min_{r_h} J(r_h) \\ \text{subject to} \\ r_h \in U_{h,c}. \end{array} \tag{24}$$

We now show that a sequence of optimal solutions of (24) converges towards an optimal solution of (19).

Theorem 4.3 *Let r_h^* be a sequence of optimal solutions of (24), $h \to 0$. Then one can find a subsequence such that there exists a function $r^* \in U$ and*

$$r_h^* \to r^* \quad in \quad L^\infty(I)$$

and r^* is an optimal solution of (19). In addition, any accumulation point of r_h^* possesses this property.

Proof. Let $r_h \in U_h, h \to 0$ be an arbitrary sequence. With any r_h a continuous piecewise linear function $\hat{r}_h$ is defined on the partition $\{b_i\}_{i=0,\dots,n+1}$, where

$$0 = b_0 = a_0 < b_1 < a_1 < \ldots < a_{n-1} < b_n < a_n = b_{n+1}$$

and b_i is the midpoint of the interval $[a_{i-1}, a_i]$,

$$b_i = \frac{a_i + a_{i-1}}{2}, \quad i = 1, \dots, n.$$

The function $\hat{r}_h$ is given by

$$\hat{r}_h(b_i) = r_h(b_i), \quad i = 0, \dots, n \quad \text{and} \quad \hat{r}_h(b_{n+1}) = r_h(a_n)$$

and

$$\hat{r}_h|_{[b_{i-1},b_i]} \in P_1([b_{i-1}, b_i]), \quad i = 1, \dots, n+1.$$

This definition implies that

$$\underline{r} \leq \hat{r}_h \leq \bar{r} \quad \text{in}\, I$$

and

$$|\hat{r}_h'| \leq L_0 \quad \text{in}\, I.$$

We have that $\hat{r}_h \in U$. Since U is compact there exists a subsequence $\hat{r}_h$ and a function $\hat{r} \in U$ such that

$$\|\hat{r}_h - \hat{r}\|_{L^\infty(I)} \to 0 \quad \text{as} \quad h \to 0.$$

The function r_h can be viewed as piecewise constant interpolant of $\hat{r}_h$ implying that

$$\|r_h - \hat{r}_h\|_{L^\infty(I)} \leq L_0 h.$$

Using the triangle inequality we can now show that r_h converges towards the function $\hat{r}$

$$\|r_h - \hat{r}\|_{L^\infty(I)} \leq \|r_h - \hat{r}_h\|_{L^\infty(I)} + \|\hat{r}_h - \hat{r}\|_{L^\infty(I)} \to 0.$$

So far, we only have $\hat{r} \in U$. But of course, the constraint on the eigenvalue should also be fulfilled for the limit function. This follows straightforwardly from the continuous dependence of λ on r, i.e $\lambda_h(r_h) \overset{h\to 0}{\to} \lambda(r)$ which was shown in Theorem 1. Hence we have $\hat{r} \in U_c$.

The density of $\cup_{h\to 0} U_h$ in U in the L^∞-norm can be shown as follows.
Let $r \in U$ and define r_h as

$$r_h = \sum_{i=1}^{n} \left(\frac{1}{|a_i - a_{i-1}|} \int_{a_{i-1}}^{a_i} r(s)\, ds \right) \chi_i,$$

where χ_i is the characteristic function of $[a_{i-1}, a_i]$, $i = 1, \ldots, n$. We have $r_h \in U_h$ and $r_h \to r$ in $L^\infty(I)$ if $h \to 0$. Restricting both spaces about the eigenvalue constraint the density result also holds for the spaces $\cup_{h\to 0} U_{h,c}$ in U_c.

We now consider a sequence of optimal solutions r_h^* to problems (24) and denote its limit function by r^*. It remains to show that r^* is an optimal solution of (19). Therefore, we consider an arbitrary $\tilde{r} \in U_c$. A sequence $\tilde{r}_h \in U_{h,c}$ can be found such that

$$\|\tilde{r}_h - \tilde{r}\|_{L^\infty(I)} \to 0.$$

Since r_h^* is an optimal solution of problem (24) we have

$$J(r_h^*) \leq J(\tilde{r}_h).$$

Since $\|r_h^* - r^*\|_{L^\infty(I)} \to 0$ and $\|\tilde{r}_h - \tilde{r}\|_{L^\infty(I)} \to 0$ and J is continuous in r for suitable functions we obtain in the limit

$$J(r^*) \leq J(\tilde{r})$$

for any $\tilde{r} \in U$. This shows that r^* is an optimal solution of (19). □

5 A priori estimates

In this section we show a priori estimates for the convergence of the eigenvalues. Our development is based on results found in Kolata [8] and Babuška & Osborn [2].

In the previous chapter we have introduced the weak formulation of the operators by (20) and (21). We have $a : V \times \tilde{V} \to \mathbb{C}^2$ and $b : V \times \tilde{V} \to \mathbb{C}^2$ and for the eigenvalue problem

$$\lambda a(\phi, \eta) = b(\phi, \eta) \quad \text{for } \eta = (\eta_1, \eta_2) \in \tilde{V}. \tag{25}$$

To obtain a form mapping into the scalar field $\mathbb{C}$ we now set

$$\tilde{a}(\phi, \eta) = m(\phi_1, \eta_1) + m(\phi_2, \eta_2)$$

and

$$\tilde{b}(\phi, \eta) = l(\phi_1, \phi_2, \eta_1) + m(\phi_1, \eta_2),$$

where $\phi \in V, \eta \in \tilde{V}$ and

$$m(\phi_1, \eta_1) = \int \mu \phi_1 \bar{\eta}_1 \, ds + \frac{1}{2} \int I_p \phi_1' \bar{\eta}_1' \, ds,$$

$$l(\phi_1, \phi_2, \eta_1) = i \int I_p \omega \phi_1' \bar{\eta}_1' \, ds - \int E I_a \phi_2'' \bar{\eta}_1'' \, ds.$$

Using suitable functions $\eta \in \tilde{V}$ we obtain the two equations of our system (25). Indeed for $\eta = (\phi_2, 0)$ we have

$$\lambda m(\phi_1, \phi_2) = l(\phi_1, \phi_2, \phi_2)$$

and for $\eta = (0, \phi_1)$

$$\lambda m(\phi_2, \phi_1) = m(\phi_1, \phi_1).$$

We now want to verify the continuity and positivity assumptions stated in Kolata [8] which enables subsequently the desired estimates. For our case the assumptions are

1. $|\tilde{a}(u, v)| \leq \|u\|_V \|v\|_{\tilde{V}}, \quad |\tilde{b}(u, v)| \leq \|u\|_V \|v\|_{\tilde{V}},$
2. $\inf_{u \in V} \sup_{v \in \tilde{V}} |\tilde{b}(u, v)| = \beta_1 > 0,$
3. $\sup_{u \in V} |\tilde{b}(u, v)| > 0, \quad 0 \neq v \in \tilde{V}.$

From Lemma 1 and 2 assumption (1) follows straightforwardly. To verify that the assumptions (2) and (3) hold we use suitable test functions. For an arbitrary $\phi = (\phi_1, \phi_2) \in V$ we consider $\eta = (0, \phi_1) \in \tilde{V}$. Then

$$\tilde{b}(\phi, \eta) = m(\phi_1, \phi_1).$$

The latter form is coercive as was shown in the proof of Lemma 1 (equation 13) and this holds for any $\phi \in V$. Hence assumption (2) is fulfilled.
Conversely, for any $0 \neq \eta = (\eta_1, \eta_2) \in \tilde{V}$ we choose $\phi = (0, \eta_1)$. Then we have

$$\tilde{b}(\phi, \eta) = k(\eta_1, \eta_1).$$

as defined in Lemma 2. In this lemma the coercivity of the form k is shown (equation 16) which guarantees that assumption (3) is satisfied.

We define a projection operator $\tilde{P}_h$, $V \to V^h$, analogously to Definition 1 for the form $\tilde{b}$ as

$$\tilde{b}_h(\tilde{P}_h \phi, \eta_h) = \tilde{b}(\phi, \eta_h), \quad \forall \eta \in \tilde{V}^h.$$

Indeed, we have $\tilde{P}_h = P_h$. since by definition

$$l(\phi_1^h, \phi_2^h, \eta_1^h) + m(\phi_1^h, \phi_2^h, \eta_2^h) = l(\phi_1, \phi_2, \eta_1^h) + m(\phi_1, \eta_2^h)$$

which holds for all test functions $\eta \in \tilde{V}_h$. Choosing the test functions as above, i.e. $\eta^h = (\phi_2^h, 0)$ and $\eta^h = (0, \phi_1^h)$ we obtain the two equations of the Definition of P_h for the form b.

Now we can apply the result of Kolata and we can deduce

$$\|u - P_h u\| \leq c \inf_{\chi \in V_h} \|u - \chi\|.$$

We now set $u = T\phi$ with T being the compact operator as introduced previously and $\phi \in V$. We obtain

$$\|(T - T_h)\phi\| = \|(I - P_h)T\phi\| \leq c \inf_{\chi \in V_h} \|T\phi - \chi\|. \tag{26}$$

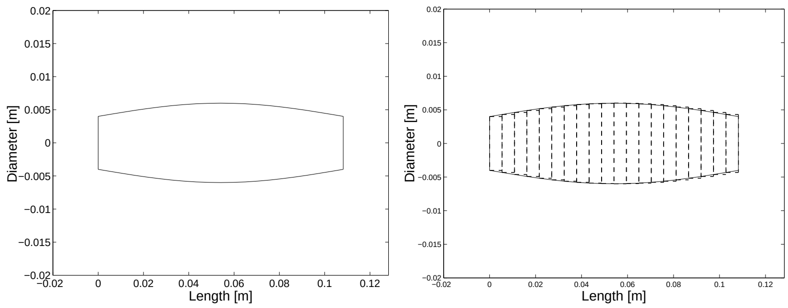

Figure 2: Shape of test rotor and beam element approximation.

Under adequate regularity properties for the operator T, we may apply the standard interpolation results due to Bramble-Hilbert (see [2] for more details) which lead to

$$\inf_{\chi \in V_h} \|T\phi - \chi\| \leq ch. \tag{27}$$

Based on the inequalities (26) and (27) the a priori theory developed by Babuška and Osborn [2] can be directly used in our framework and lead to the following approximation result.

Theorem 5.1 *Let λ be an eigenvalue of (25) with algebraic multiplicity σ. Then, for sufficiently small h, there are exactly σ eigenvalues $\{\lambda_{h,i}\}_{i=1,\cdots,\sigma}$ of the discrete problem (23) counted according to their algebraic multiplicity, such that*

$$\left| \lambda - \frac{1}{\sigma} \sum_{i=1}^{\sigma} \lambda_{h,i} \right| \leq c_\lambda h^2. \tag{28}$$

This convergence result is illustrated by a numerical example. We consider a rotor which is described by the shape function $r(s) = a \sin(\pi s/l) + c$ with total length l, shape parameters $a, c > 0$ and $s \in [0, l]$. The continuous rotor is now divided into several beam elements. For the discretized rotor the eigenvalue problem is formulated and solved numerically. We focus on the smallest two eigenvalues and look at the relative error for each discretization. As error function for the relative error $e_{\text{rel},i}$ for eigenvalue λ_i we consider

$$e_{\text{rel},i} = \frac{|\lambda^* - \lambda_i^h|}{|\lambda^*|}.$$

The 'exact' value λ^* is obtained by calculating once the eigenvalues considering a discretization which is fine enough. In Figure 3 we see the behaviour of the logarithmic error function. To show the quadratic convergence in terms of the discretization parameter proven in (28), we add the graph of $\log(ch^2)$ to the figure, (where $h = 1/N$ and N the number of elements). Indeed for a suitable c, in our case e.g. $c = 2$, we observe in Figure 3 the quadratic convergence.

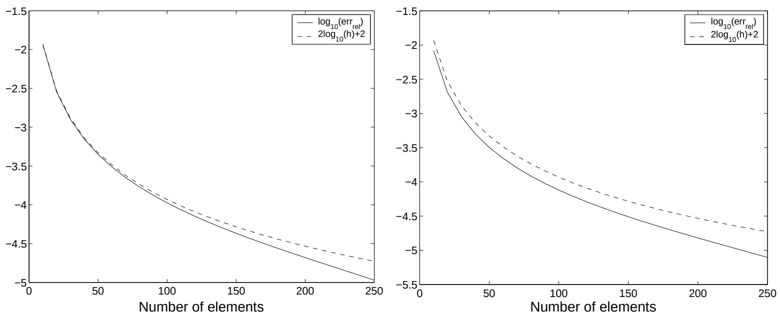

Figure 3: Error function of first (left) and second (right) smallest eigenvalue.

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