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Understanding default risk premia on public debt

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Abstract
We model the pricing of public debt in a quantitative macroeconomic model with government default risk. Default occurs if a shift in the state of the economy leads to a build-up of debt that exceeds the government's ability to repay. Investors are unwilling to engage in a Ponzi game and withdraw lending in this case and thus force default at an endogenously determined fractional repayment rate. Interest rates on government bonds reflect expectations of this event. There may exist multiple bond prices compatible with a rational expectations equilibrium. At high debt-to-output ratios, small changes in fundamentals lead to steeply rising risk premia. Key determinants of the level of indebtedness at which this occurs are the perceived amount of aggregate risk, the feasibility of revenue maximizing tax rates, and the maturity of bonds.

Keywords: Sovereign default; government debt; asset pricing; fiscal policy
JEL: E62, G12, H6, E32

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1 Introduction

The recent financial crisis has turned into a fiscal crisis in several European countries. An unusually large adverse shock has reduced tax revenues and lead to higher government spending in an attempt to mitigate the consequences of the shock for aggregate output and employment. The resulting boost in public deficits has produced unprecedented levels of government debt, which are already above 100% of yearly GDP in some countries and are predicted to rise to even higher levels in the near future. Sizeable yield spreads between government bonds of member countries of the European Monetary Union have emerged, and for some countries (in particular for Greece and Portugal), yield spreads increased dramatically in the first quarter of 2010. While these spreads arguably reflect the risk that governments default on their debt obligations, the magnitude and the dynamics of bond spreads are hardly understood.

According to conventional wisdom, sovereign default risk premia should increase with the level of public debt, which is also found by several empirical articles that study the relation between interest rate spreads of public bonds over some risk-free benchmark to the level of a sovereign issuer's indebtedness (e.g. Manganelli and Wolswijk, 2009, Codogno et al., 2003, Bernoth et al., 2006, Akitobi and Stratmann, 2008, Schuknecht et al., 2008). However, there is a stunning variation in the relation between levels of government debt and observed interest rate spreads on public bonds. For instance, in the last quarter of 2009, Greece had a debt-to-gdp ratio of 115% and its bonds in the first quarter of 2010 paid an interest rate spread over German bonds of 3.06 percentage points. At the same time, public debt was 96.8% of gdp in Belgium with Belgian bonds yielding only 0.23 percentage points above German ones.[3] Thus the question arises why similar levels of public indebtedness can lead to wildly divergent levels of risk premia that manifest themselves in large differences in interest rate spreads. Moreover, recent events, in particular in Greece, have shown that spreads can rise strongly and very rapidly, without much short-run change in fundamentals, if investors fear impending default. Another question thus is what drives the dynamics in risk spreads and why they may virtually explode at some point.

In this paper, we argue that empirical government bond price movements can be interpreted as being driven by shifts in expectations, where the ultimate source of these shifts can be either non-fundamental or fundamental events. In particular, our point is that investors' willingness to lend, and thus to roll over debt, is essential for the expectations of sovereign default. Hence, if – for any reason – default is expected to be very likely, lenders demand a high interest rate premium to be compensated for default risk, which raises the

[3] The spreads data pertain to ten year government bond yields, OECD main economic indicators, while the debt-to-gdp figures are from Eurostat.

debt burden even more such that the probability of default actually increases. Phrased differently, default expectations can be self-fulfilling. On the other hand, the probability of default – and thereby risk premia – also depend on fundamentals and expectations thereof. A government's debt repayment capacity increases with the present value of government surpluses, such that its credibility to raise future revenues is essential for the perceived probability of default. Lenders' expectations about a government's capacity to repay debt out of current and future surpluses are decisive for the premia they demand as a compensation for the risk of sovereign default. Likewise, if future macroeconomic developments are more uncertain, the perceived default probability and risk premia increase.

The paper presents a simple model which accounts for these mechanisms. We apply a dynamic general equilibrium framework and consider an indebted government that fails to guarantee repayment of debt in all periods, while it nevertheless aims at avoiding default as far as possible. Specifically, we consider a government that levies a proportional tax on labor income (there are no lump-sum taxes available). It issues non-state contingent one-period debt contracts to finance a given stream of real government expenditures, while uncertainty is due to aggregate productivity shocks.[4] The government repays its debt as far as possible. In case of default, lenders can just seize current net revenues from the government (a situation that differs from private credit relations where the lender may become a claimant on future profit streams).[5]

What determines sovereign default and risk premia in this model? Consider an adverse productivity shock. If the shock makes the present value of future surpluses fall short of covering the level of outstanding debt, even if the revenue maximizing tax rate – which is well defined here, because with only labor income taxation there is a tax Laffer curve with an interior maximum –, is levied for the entire future, the government's debt repayment capacity is exceeded. A potential household-lender who realizes that he would support a Ponzi game if he invested in government bonds will stop lending to the government. In this case, default becomes inevitable and current surpluses are distributed to bond holders, who therefore experience only a partial redemption of their investments. Each individual lender assesses the probability that this event will occur in the next period and consequently demands a default risk premium as a compensation for expected losses. Concisely, default occurs if current debt exceeds the debt repayment capacity, while the repayment rate is residually determined by available revenues of the government that cannot roll over debt. This may give rise to self-fulfilling price expectations: if investors

[4] Section 4.4 also studies the changes that arise if the government issues debt of longer maturity.

[5] The only risk associated with investments in public debt is default risk, since we assume that bonds are real so that debt revaluations via price level shifts (which are the focus in the fiscal theory of the price level, see Woodford, 1994, Sims, 1994, or Niepelt, 2004) are impossible.

assign a higher probability to sovereign default, they demand a higher risk premium, which raises debt servicing costs and indeed tends to lower the repayment rate.

Our approach to model sovereign default is related to Uribe's (2006) 'Fiscal Theory of Sovereign Default'. He considers nominal debt and exogenous surpluses in an endowment economy to demonstrate that default is inevitable under certain monetary-fiscal policy regimes. Apart from these details, our strategy to determine default substantially differs from his approach. As shown in Schabert (2010), the intertemporal budget constraint is not sufficient as a criterion to determine the default rate, i.e. the fractional rate of repayment of outstanding debt in the case of default. For the case which is most closely related to our set-up, Uribe (2006) introduces an additional 'fiscal policy constraint restricting the behavior of the default rate' (p. 1869), which allows to uniquely determine the equilibrium interest rate. For example, he considers a policy rule whereby the government decides to default if the tax-to-debt ratio falls below a certain threshold. In the present paper, in contrast, we instead introduce the assumption that investors stop lending in the case where a government Ponzi-game becomes inevitable, which allows us to determine an entire sequence of default rates without any additional restriction on the government's behavior.[6]

Our approach to model default risk further differs from a large body of theoretical literature on sovereign default that focuses on external debt in open economies. In this literature, default is modelled as an optimal decision of the government that trades off costs and benefits of not serving external debt (see Eaton and Gersovitz, 1981, or Arellano, 2008, among others). While this assumption has proven to be useful for the case of external debt of emerging market economies, we view it as less suited to explain risk premia in economies where governments have not been observed defaulting on their debt in the recent past.[7]

The main results are as follows. Generally, there exist multiple equilibrium prices for government debt. In particular, two interest rates on government bonds can exist in equilibrium: both a combination of high interest rates, high default risk, and high public debt, as well as one of low interest rates, low default risk, and low public debt can be compatible with the expected rate of return of investors and with the government's demand for external funds. Default immediately occurs if the lenders coordinate their expectations on a high risk equilibrium, thereby imposing an unsustainable financing burden on the government through high risk premia in the period of maturity. The mechanism that links

[6] Without such an assumption or Uribe's (2006) fiscal closing rules, default rates (and rational expectations thereof) can only be determined in the initial period, as shown in Schabert (2010). Bi (2010), who also applies Uribe's framework, assumes that the default rate is a policy choice variable and sets it exogenously.

[7] Our approach can further be motivated by the empiricial evidence in Reinhart and Rogoff (2008), who find that in surprisingly many cases default does not involve external debt.

high interest rate premia due to default expectations to the actual default probability is similar to the one in Calvo (1988), with the difference that in our model the government does not default voluntarily.

Furthermore, even abstracting from self-fulfilling default expectations, we show that under certain conditions the government's maximum debt repayment capacity may be so stringent that default becomes inevitable for fundamental reasons. When we focus on the low equilibrium interest rate, which exhibits plausible comparative static properties, default premia are monotonically increasing in the initial debt level and depend negatively on productivity, as expected. We analyze the model's predictions with respect to default premia in terms of the implied equilibrium bond pricing curve that gives the relation between the beginning-of-period debt level and the interest rate that market participants demand over the risk free interest rate in equilibrium. We show that this equilibrium pricing curve can be extremely steep above certain critical levels of debt to gdp.

For the baseline parameterization of the model, in which parameters are chosen to capture some relevant quantitative features of the average of European Monetary Union member countries in a stylized way, we show that non-negligible interest rate spreads would emerge only for very high levels of debt around 200% of gdp. For the average Eurozone country, thus, the model predicts that fiscal spare capacity is ample, and thus that risk premia should be negligible at observed debt-to-gdp levels, which is consistent with empirical evidence. However, we also demonstrate the influences on the equilibrium pricing curve, and hence the factors that lower the critical debt levels above which risk spreads begin to rise steeply. In particular, we point out three such influences.

The first is the perceived amount of aggregate risk. If investors believe that aggregate risk increases in the future, the perceived probability that a future adverse shock forces government default is higher, all else equal, such that investors will claim higher risk premia. Second, the level of debt that market participants consider to entail a non-negligible default risk depends on investors' perception of the political ability of the government to raise distortionary taxes. If the politically feasible tax rate is perceived to be substantially lower than the revenue maximizing tax rate, there can be high default risk premia even at relatively low levels of debt to gdp. The third influence is the maturity of debt. In the baseline model, all government debt is in the form of one period debt. In practice, however, governments typically issue bonds with longer maturity. These are subject to default risk in several periods ahead and are thus more vulnerable in the presence of serially correlated productivity shocks. We show that in an example with two period government bonds, spreads on annualized yields are higher than in the case of one period bonds only.

We use these theoretical results to ask what our model can contribute to the understanding of movements in risk premia on Eurozone government bonds as recently observed. Specifically, we parameterize the model to capture some essential empirical properties of fiscal policy and of aggregate risk in Greece. We find that risk premia at the level of several dozens of basis points that have been observed in Greece prior to and well into 2008 can be explained by the model, if we take into account the severity of the recent recession and the fact that bonds with a maturity of several years are common. However, the extreme rise in risk premia that have very recently been observed must be blamed on, according to the logic of our model, either a loss of confidence in the political ability of the Greek government to raise taxes sufficiently, or even non-fundamentally induced shifts in expectations.

The remainder is organized as follows. Section 2 introduces the model. Section 3 describes the determination of equilibrium bond prices. Section 4 presents quantitative results and section 5 concludes.

2 The model

In this section we present a simple real dynamic general equilibrium model where the government levies income taxes and issues non-state contingent one period debt. Labor supply is endogenous, which gives rise to a Laffer curve that bounds equilibrium tax revenues. We consider the case where fiscal policy does not guarantee that the government never runs a Ponzi-game.[8] Households are assumed to stop lending to the government when they realize that a Ponzi scheme is inevitable. Without further access to credit, the government defaults while lenders can seize current net revenues. Households know that this event is possible when adverse productivity shocks lead to a build-up of public debt. They form expectations of the future fractional rate of repayment of government debt. Accordingly, in an arbitrage-free equilibrium risk premia exist that compensate household-lenders for the risk of government default.

2.1 The private sector

There exists a continuum of infinitely lived and identical households of mass one. Their utility increases in consumption c_t and decreases in working time l_t, the latter variable being bounded by a unit time endowment such that $l_t \in (0, 1)$. The objective of a representative household is given by

$$\max E_s \sum_{t=0}^{\infty} \beta^t \left[\ln c_{t+s} + \frac{1 - l_{t+s}}{\gamma} \right], \qquad \text{with } \beta \in (0, 1), \gamma > 0, \tag{1}$$

[8] This assumption is analoguous to the fiscal policy specification in Uribe (2006) and in the fiscal theory of the price level (see Sims, 1994, and Woodford, 1994). In contrast to these studies, in our purely real model the price level is irrelevant.

where β denotes the discount factor. Households borrow and lend among each other via one-period private debt contracts. Private debt is introduced here to define a risk free interest rate R_t^{rf}. Let d_{t-1} denote the beginning of period net private asset position and $1/R_t^{rf}$ the period-t-price for a payoff of one unit of output in period $t+1$. We restrict our attention to the case where private debt contracts are enforceable and households satisfy the borrowing constraint

$$\lim_{t\to\infty}\left(d_{t+s}/R_{t+s}^{rf}\right)\prod_{i=1}^{t}1/R_{s+i-1}^{rf}\geq 0. \tag{2}$$

Utility maximization subject to the borrowing constraint (2) requires the following first order condition for borrowing and lending in terms of private debt (i.e. the consumption Euler equation) to be satisfied

$$c_t^{-1}=R_t^{rf}\beta E_t\left(c_{t+1}^{-1}\right), \tag{3}$$

as well as the transversality condition

$$\lim_{t\to\infty}E_s\left(d_{t+s}/R_{t+s}^{rf}\right)\prod_{i=1}^{t}1/R_{s+i-1}^{rf}=0. \tag{4}$$

Households can further invest in one-period government bonds b_t, subject to $b_{-1}>0$ and $b_t\geq 0$. The government offers one-period debt contracts at the price $1/R_t$ in period t that promise to deliver one unit of output in period $t+1$. In contrast to private borrowers, the government does not guarantee full debt repayment. In case of default the lenders will proportionally be served with current net revenues. It should be noted that this differs from the case of lending to a firm, where default typically leads to lenders' taking over the firm as a claimant on future profit streams through a debt-to-equity swap.

If current and discounted future surpluses are expected to be large enough to repay outstanding debt, the household optimality condition for investment in government bonds would be the analogue to the Euler equation (3), namely, $c_t^{-1}=R_t\beta E_t\left(c_{t+1}^{-1}\right)$. The requirement $b_t\geq 0$ further requires that in the household optimum the transversality condition

$$\lim_{t\to\infty}E_s\left(b_{t+s}/R_{t+s}\right)\prod_{i=1}^{t}1/R_{s+i-1}^{rf}=0, \tag{5}$$

holds, where $R_{t+s}=R_{t+s}^{rf}$ when the government fully services its debt obligations. If beginning-of-period public debt exceeds a level that is too high to be repayable even for the maximum present value of budget surpluses (see section 2.2 for a definition), the government runs into a Ponzi game, which would be inconsistent with the households' transversality condition (5). In this case, households are assumed to stop lending to the government, which necessarily implies that the government defaults in period t, i.e. can honor only a fraction of its debt obligations out of current surpluses.

Since households are assumed to have rational expectations, they realize the possibility of partial default on government bonds and account for the probability of default (of course, since households are atomistic, an individual investor does not take into consideration the influence of his behavior on the probability of default). Let $1 - \delta_t$ denote the fraction of government bonds that is redeemed and $\delta_t \in [0, 1]$ the default rate. The household flow budget constraint then reads

$$c_t + (b_t/R_t) + \left(d_t/R_t^{rf}\right) \leq (1 - \tau_t)w_t l_t + (1 - \delta_t)\, b_{t-1} + d_{t-1} + \pi_t,$$

where π_t are firms' profits, and labor income $w_t l_t$ (with the real wage rate w_t) is subject to a proportional tax rate $\tau_t \in (0, 1)$. The household optimum is characterized by the first order conditions (3),

$$c_t = \gamma\,(1 - \tau_t)\, w_t, \tag{6}$$

$$c_t^{-1} = R_t \beta E_t \left(c_{t+1}^{-1}\,(1 - \delta_{t+1})\right), \tag{7}$$

and the transversality conditions (4) and (5). Note that the Euler equation for risky government debt, (7), differs from the one for risk-free private debt (3), in that the pricing of government bonds is affected by the fact that repayment is expected to be only partial because of possible future default.

If debt b_{t+s-1} at the beginning of some period $t+s$ is too large such that a Ponzi game becomes inevitable, households do not lend to the government, i.e. the end of period debt equals zero, $b_{t+s} = 0$, and the government defaults. Lending may resume, however, in the subsequent periods, when partial default has ameliorated the fiscal position.

Perfectly competitive firms produce the output good y_t with a simple linear technology

$$y_t = a_t l_t, \tag{8}$$

where labor productivity a_t is generated by

$$a_t = \rho a_{t-1} + (1 - \rho)\overline{a} + \varepsilon_t; \tag{9}$$

here $\overline{a} > 0$ is a constant long-run average productivity level, the coefficient of autocorrelation is ρ, and ε_t is an i.i.d. zero mean random variable. Labor demand satisfies

$$w_t = a_t. \tag{10}$$

2.2 The public sector

The government does not have access to lump-sum taxation. It raises revenues by issuing debt and taxing labor income, and purchases an exogenously given amount g_t of the final good in each period. Throughout, we assume government spending to be constant, $g_t = g > 0$. The underlying assumption is that political constraints make a certain amount of government spending inevitable. The flow budget constraint is given by

$$b_t R_t^{-1} + s_t = (1 - \delta_t)\, b_{t-1}, \tag{11}$$

where the surpluses s_t equal tax revenues net of expenditures,

$$s_t = \tau_t w_t l_t - g. \tag{12}$$

The government does not guarantee to fully service debt. We assume that the government does not preclude that public debt might evolve on a path that implies a Ponzi scheme. Since households are not willing to engage in such schemes, they may stop lending and (temporarily) disrupt the government from access to credit.

To see this, consider, for a moment, the default free case, i.e. presume the non-repayment rate δ_{t+k} were equal to zero for all $k \geq 0$. In this case, one would obtain by iterating the government flow budget constraint (11) forward and taking expectations, $\delta_{t+k} = 0 \quad \forall k \geq 0 \Rightarrow$

$$b_{t-1} = E_t \sum_{k=0}^{\infty} s_{t+k} \prod_{i=1}^{k} (1/R_{t+i-1}) + \lim_{k\to\infty} E_t b_{t+k} R_{t+k}^{-1} \prod_{i=1}^{k} \frac{1}{R_{t+i-1}}. \tag{13}$$

Now suppose that outstanding debt b_{t-1} exceeds the present value of future surpluses, i.e. the first term on the right hand side of (13). Then, the limit term would exceed zero, $\lim_{k\to\infty} E_t b_{t+k} R_{t+k}^{-1} \prod_{i=1}^{k} 1/R_{t+i-1} > 0$. By definition, the government would then run into a Ponzi game. But this, together with $R_{t+k} = R_{t+k}^{rf}\ \forall k \geq 0$ for $\delta_{t+k} = 0$ (see 3 and 7) would be inconsistent with the households' transversality condition (5). As mentioned above, we assume that households will then stop lending to the government, such that $b_t = 0$ in that period. The only way for the government budget constraint (11) to be satisfied in this case is through default in the sense $\delta_t > 0$.

As a specific way to implement a fiscal policy that entails default risk in this sense, we assume that the government keeps the tax rate constant, $\tau_t = \tau$. This is a prominent example of a large class of fiscal rules that do not incorporate enough self-corrective behavior on the part of the government as to avoid Ponzi schemes in each period of time.[9]

[9] This assumption rules out the debt stabilizing behavior that has been found by Bohn (1998) to characterize US fiscal policy empirically.

However, it can also be viewed as a natural benchmark in this framework: if government bonds were state contingent, it is well-established that in this type of model an optimal income tax rate under commitment (and without default) would have to be constant and sufficiently large to finance initial outstanding debt and future expenditures (see e.g. Ljungqvist and Sargent, 2004). In this paper, government bonds are however non-state contingent, which implies that this type of tax policy is in general not consistent with a set of 'measurability constraints' for each period that relate the present value of future surpluses to the beginning of period stock of public debt to rule out Ponzi games (see Ayiagari et al., 2002). The choice of a constant tax rate can thus, besides being a simple example, be seen as the strategy of a government that ignores this subtle difference and sets the tax rate as if debt was state contingent.

Note that there exists a maximum value for the present value of future surpluses, which we call the maximum debt repayment capacity. The latter is the maximum amount of debt that the government would be able to repay if it imposed the revenue maximizing tax rate for the entire future. A well defined revenue maximizing tax rate, τ^*, exists because with proportional labor income taxation there is a tax Laffer curve with an interior maximum (see section 3 for an explicit derivation). We denote the period t value of the maximum debt repayment capacity by Ψ_t, defined as

$$\Psi_t = E_t \sum_{k=0}^{\infty} s^*_{t+k} \prod_{i=1}^{k} 1/R^{rf}_{t+i-1}. \tag{14}$$

Here, $s^*_{t+k} = \tau^* w^*_{t+k} l^*_{t+k} - g$ is the maximum period surplus that is obtained if the revenue maximizing tax rate τ^* is applied. This leads to corresponding levels of labor income denoted $w^*_{t+k} l^*_{t+k}$ and the risk free rate R^{rf}_{t+k} is applied for discounting.[10] Note that households will account for the maximum debt repayment capacity for their lending decision in equilibrium. We thereby allow for the case where the current tax rate differs from the revenue maximizing tax rate, which could in principle be implemented by future governments.

The maximum initial debt level that can be expected to be repaid without default is thus characterized by $b_{t-1} = \Psi_t$. The government will fully serve debt obligations if $b_{t-1} \leq \Psi_t$. As long as this is the case, no government default occurs. Default, however,

[10] Note that the maximum debt repayment capacity bears a resemblance to Aiyagari's (1994) natural debt limit for consumers. Private households cannot accumulate more debt than would be expected to be repaid by pledging the entire stream of future incomes. While households are assumed to respect the natural private debt limit (as a borrowing constraint), the government is not constrained in an analogous way, which is why default may occasionally occur in our model.

becomes inevitable if the current stock of debt exceeds the maximum repayment capacity:

$$b_{t-1} > \Psi_t. \tag{15}$$

If this is the case, tax rates are not able to generate enough current and future revenues to enable full repayment of outstanding debt.

In the case where (15) is satisfied, (13) with $R_{t+k} = R^{rf}_{t+k}$ $\forall k \geq 0$ is inconsistent with the transversality condition (5) and no individual household is willing to lend to the government. The consequence is that aggregate lending to the government comes to a halt, such that end-of-period debt equals zero, $b_t = 0$, in the current period. The government is then unable to fully honor its obligations and redeems as much as possible of its outstanding debt out of current surpluses. As a consequence, repayment will only be partial. The non-repayment or default rate δ_t in the case (15) satisfies (see 11):

$$\delta_t = 1 - s_t/b_{t-1} \tag{16}$$

To sum up, if beginning-of-period debt b_{t-1} is smaller than Ψ_t, households are willing to lend to the government according to (7), while the government does not default in period t, $\delta_t = 0$, and borrows to balance its budget such that end-of-period debt equals $b_t = (b_{t-1} - s_t)\, R_t$. The price of debt, $1/R_t$, then reflects the probability of default in $t+1$. If, however, beginning-of-period debt is too high such that (15) is satisfied, households stop lending. The government then has to default and repays debt as far as possible, with a default rate given by (16). In the period subsequent to a default event, the stock of government debt is zero and default is not possible in the next period, such that households are again willing to lend to the government.

2.3 Equilibrium

In equilibrium, prices adjust to clear markets for goods, labor, and assets and the net stock of risk-free private debt d_t is zero in the aggregate. Households' initial asset endowments are assumed to be positive, i.e. the government is initially indebted. A rational expectations equilibrium is a set of sequences $\{c_t,\, l_t \in [0,1],\, y_t,\, w_t,\, b_t \geq 0,\, \delta_t \in [0,1], R^{rf}_t,$ $R_t, s_t\}_{t=0}^{\infty}$ satisfying (3), (6), (7), (8), (10), (12) and

$$y_t = c_t + g_t, \tag{17}$$

$$b_t = \begin{cases} (b_{t-1} - s_t)\, R_t & \text{if } \ \Psi_t \geq b_{t-1} \\ 0 & \text{if } \ \Psi_t < b_{t-1} \end{cases}, \tag{18}$$

$$\delta_t = \begin{cases} 0 & \text{if } \ \Psi_t \geq b_{t-1} \\ 1 - s_t/b_{t-1} & \text{if } \ \Psi_t < b_{t-1} \end{cases}, \tag{19}$$

(4), (5), and (14), a fiscal policy setting $\tau \in [0,1]$, given $\{a_t\}_{t=0}^{\infty}$, $g > 0$, and initial debt $b_{-1} > 0$.

The equilibrium allocation is not directly affected by public debt and the (expected) default rate. The first property is due to the fact that the labor income tax is assumed not to be contingent on the fiscal stance. The second property follows from the fact that default does not lead to resource losses or distortions. Of course, the price of government bonds will depend on the expected default rate, which can be seen from the asset pricing equation (7). This reflection of the probability of future default in the interest rate on government bonds is our main object of study.

The equilibrium sequences of consumption, working time, output, the wage rate, the risk free rate and government surpluses $\{c_t,\, l_t,\, y_t,\, w_t,\, R_t^{rf},\, s_t\}_{t=0}^{\infty}$ are determined for given g and $\{a_t\}_{t=0}^{\infty}$ by (6), (8), (10), (12) and (17), which can be summarized by

$$c_t = c\left(a_t, \tau\right) := \gamma\left(1-\tau\right) a_t \tag{20}$$

$$l_t = l\left(a_t, \tau\right) := \left(c\left(a_t, \tau\right) + g\right)/a_t \tag{21}$$

$$s_t = s\left(a_t, \tau\right) := \tau c\left(a_t, \tau\right) - (1-\tau) g \tag{22}$$

$$R_t^{rf} = c\left(a_t, \tau\right)^{-1} \beta^{-1} / E_t\left(c\left(a_{t+1}, \tau\right)^{-1}\right) \tag{23}$$

as well as $w_t = a_t$ and $y_t = a_t l\left(a_t, \tau\right)$.[11]

While the equilibrium sequences $\{c_t,\, l_t,\, y_t,\, w_t,\, s_t\}_{t=0}^{\infty}$ are not affected by sovereign default, these variables are of course correlated with the default rate δ_t due to changes in the state a_t. In any case, they will be stationary, given that the state a_t is stationary.

With the above solutions, we can easily identify a time-invariant tax rate to compute the maximum debt repayment capacity (14). We look for a feasible tax rate $\tau^* \in (0,1)$ that maximizes tax revenues for the case where the state equals its mean ($a_t = \overline{a}$), $\tau w l = \tau[\gamma\left(1-\tau\right)\overline{a} + g]$. This tax rate satisfies $F(\tau^*) := g + \overline{a}\gamma\left(1-2\tau^*\right) = 0$, such that the unique tax rate τ^* that maximizes tax revenues is given by

$$\tau^* = \frac{1}{2} + \frac{g}{2\overline{a}\gamma}.$$

In order to determine the bond prices we need to compute expectations about future defaults. As can be seen from (36), the maximum debt repayment capacity is solely a function of (policy and preference) parameters and of the current and future exogenous states of the economy. Given that it contains expectations of a non-linear function of future states, we apply a second order approximation of Ψ_t. Though public debt might

[11] If default occurs ($\delta_t = 1 - s_t/b_{t-1}$) the budget constraints imply $c_t = (1-\tau_t) w_t l_t + (1-\delta_t)\, b_{t-1} = (1-\tau_t) w_t l_t + s_t$ and thus $y_t = a_t l_t = c_t + g$.

not be stationary, we can exploit the fact that the exogenous state variable a_t is stationary and apply a local approximation of Ψ_t at the unconditional mean $\overline{a}$. In appendix 7.1, we show that Ψ_t can be approximated by:

$$\Psi\left(a_t, \sigma_\varepsilon, \rho, \tau^*\right) \simeq \gamma\left(1-\tau^*\right) a_t \cdot \left\{ \frac{f\left(\overline{a}\right)}{1-\beta} + \frac{f'\left(\overline{a}\right)}{1-\beta\rho}\left(a_t - \overline{a}\right) \right. \tag{24}$$
$$\left. + \frac{\frac{1}{2} f''\left(\overline{a}\right)}{1-\rho^2}\left(\left(\frac{\sigma_\varepsilon^2}{1-\beta} - \frac{\sigma_\varepsilon^2}{1-\beta\rho^2}\right) + \frac{\left(1-\rho^2\right)\left(a_t-\overline{a}\right)^2}{1-\beta\rho^2}\right)\right\}.$$

According to (24), Ψ_t is a function only of today's state and time invariant parameter values. Due to this property we can easily compute equilibrium values for the expected default rate, public debt, and the bond price.

The expected default rate, public debt, and the bond price have to be determined simultaneously using the equilibrium conditions (7), (18), and (19). In order to identify these solutions, we have to consider the probabilities of the two distinct cases $\Psi_t \geq b_{t-1}$ and $\Psi_t < b_{t-1}$.

Let a_t^* be the productivity level that leads to a maximum debt repayment capacity Ψ_t that exactly equals beginning-of-period debt b_{t-1},

$$a_t^* : \Psi\left(a_t^*, \sigma_\varepsilon, \rho, \tau^*\right) = b_{t-1}. \tag{25}$$

Thus, a_t^* is the minimum productivity level that allows full debt repayment and thus precludes default; we will refer to this as the productivity threshold. Further, let $\pi_t\left(a_{t+1}\right) = \pi\left(a_{t+1}|a_t\right)$ be the probability of a particular value a_{t+1} conditional on a_t. Then, the probabilities of default and of non-default in $t+1$ conditional on the information in t are

$$prob\left(\Psi_{t+1} < b_t|a_t, b_t\right) = \int_{-\infty}^{a_{t+1}^*} \pi_t\left(a_{t+1}\right) da_{t+1},$$
$$prob\left(\Psi_{t+1} \geq b_t|a_t, b_t\right) = \int_{a_{t+1}^*}^{\infty} \pi_t\left(a_{t+1}\right) da_{t+1}.$$

We use the asset pricing equation (7), which includes the expectation term $E_t\left[c_{t+1}^{-1}\left(1-\delta_{t+1}\right)\right]$. We thereby account for the possibility that consumption and the default rate are not independent. According to the assumptions in section 2.2, the default rate δ_{t+1} equals zero if $\Psi_{t+1} \geq b_t$, and $\delta_{t+1} = 1 - s_{t+1}/b_t$ if $\Psi_{t+1} < b_t$. Hence, $E_t\left[c_{t+1}^{-1}\left(1-\delta_{t+1}\right)\right]$ is given by

$$E_t\left[c_{t+1}^{-1}\left(1-\delta_{t+1}\right)\right]$$
$$= \int_{-\infty}^{a_{t+1}^*} \pi_t\left(a_{t+1}\right)\left[c_{t+1}^{-1} \cdot \left(s_{t+1}/b_t\right)\right] da_{t+1} + \int_{a_{t+1}^*}^{\infty} \pi_t\left(a_{t+1}\right)\left[c_{t+1}^{-1} \cdot (1-0)\right] da_{t+1}$$

Using the solutions (20) and (22), the asset pricing equation (7) can thus be written as

$$1/R_t = \frac{\beta}{c_t^{-1}} \left[\begin{array}{c} b_t^{-1} \int_{-\infty}^{a_{t+1}^*} \pi_t\left(a_{t+1}\right) \left[c\left(a_{t+1}, \tau\right)^{-1} s\left(a_{t+1}, \tau\right) \right] da_{t+1} \\ + \int_{a_{t+1}^*}^{\infty} \pi_t\left(a_{t+1}\right) \left[c\left(a_{t+1}, \tau\right)^{-1} \right] da_{t+1} \end{array} \right]. \tag{26}$$

Risk premia can be computed as follows (further details can be found in appendix 7.2): At the beginning of period t, b_{t-1} is known and the shock to a_t realizes. We get solutions $\{c_t, s_t\}$ from (20) and (22). Then, we can compute the maximum debt repayment capacity using (24). If $\Psi_t < b_{t-1}$, the government defaults, while bonds are not traded. For $\Psi_t \geq b_{t-1}$, the government does not default in period t. The bond price $1/R_t$, end-of-period debt b_t, and the productivity threshold a_{t+1}^* then simultaneously solve (26), the updated version of (25) which reads $b_t = \Psi\left(a_{t+1}^*, \sigma_\varepsilon, \rho, \tau^*\right)$, and the government's flow budget identity

$$b_t/R_t = b_{t-1} - s_t. \tag{27}$$

After the equilibrium bond price $1/R_t$ is derived, we compute the sovereign risk premium using $R_t - R_t^{rf}$ (using 23), which is non-zero only if $\Psi_t \geq b_{t-1}$.

3 Multiple Equilibrium Bond Prices

In this section we examine the determination of bond prices and show that multiple equilibrium bond prices can exist. For this, we apply a simplified version of the model. To lighten the notation in this section, we drop the time index and define $a = a_t$, $a' = a_{t+1}$, $a^* = a_{t+1}^*$ for all $a \in (a_l, a_h)$, where a_l and a_h are positive constants. We assume that the innovations ε are uniformly distributed between $a_l - \overline{a}$ and $a_h - \overline{a}$ and that the productivity level is not serially correlated ($\rho = 0$). To further simplify the derivation of analytical results, we assume that only the first-order terms of the maximum debt capacity (24) are non-negligible.

With these assumptions, consumption, surpluses, and maximum repayable debt are linear functions of the current exogenous state a:

$$\Psi\left(a\right) = \left(1 - \tau^*\right)\left(\gamma\tau^* - \overline{a}^{-1} g\right)\left(1 - \beta\right)^{-1} a = \theta_1 a, \tag{28}$$

$$c\left(a\right) = \gamma\left(1 - \tau\right) a = \theta_2 a, \tag{29}$$

$$s\left(a\right) = \tau\gamma\left(1 - \tau\right) a - \left(1 - \tau\right) g = \theta_3 a - \theta_4, \tag{30}$$

where in each line the second equality sign defines the composite parameters $\theta_{1,2,3,4} > 0$. Further, end of period debt satisfies $b = \Psi\left(a^*\right) = \theta_1 a^*$ (see 25), and the government budget (27) demands $1/R = \left(b_{-1} - s\right)/b = \left(b_{-1} - \theta_3 a + \theta_4\right)/\theta_1 a^*$. The asset pricing equation (26) can then be written as

$$1/R = \beta a \left\{ (\theta_1 a^*)^{-1} \left[\theta_3 \int_{a_l}^{a^*} \pi\left(a'\right) da' - \theta_4 \int_{a_l}^{a^*} \pi\left(a'\right)\left(1/a'\right) da' \right] + \int_{a^*}^{a_h} \pi\left(a'\right)\left(1/a'\right) da' \right\}.$$

With uniformly distributed productivity levels, we get the asset pricing equation

$$1/R = \beta \frac{a}{a_h - a_l} \left\{ \frac{\theta_3 \left(b\theta_1^{-1} - a_l\right) - \theta_4 \left(\log b - \log \theta_1 - \log a_l\right)}{b} + \left(\log a_h - \log b - \log \theta_1\right) \right\}, \tag{31}$$

where we used that the solvency threshold is defined as $a^* = b/\theta_1$ (see 28).

Thus, condition (31), which can be interpreted as a credit supply condition, describes the bond price $1/R$ as a function of end-of-period debt b for a given exogenous state a. Further, the government's demand for credit is described by the period budget constraint (27), which reads $b/R = (b_{-1} - s)$ or using (30), it can be written as

$$1/R = \left(b_{-1} - \theta_3 a + \theta_4\right)/b. \tag{32}$$

Credit supply (31) and demand (32) provide two conditions that determine the price $1/R$ and the quantity of debt b issued in period t. It can be shown that there are either no or two equilibrium bond prices, which is summarized in the following proposition.

Proposition 1 *Suppose that* $\beta \to 1$. *Then, two equilibrium bond prices* $1/R$ *can exist.*

Proof. Define the RHS of (31) as $G(b)$, such that $1/R = G(b)$. The derivatives of $G(b)$ are given by $G'(b) = -\kappa\{1 + (\theta_1 a^*)^{-1}(\theta_4 \ln a_l - \theta_4 \ln a^* - \theta_3 a_l + \theta_4)\}b^{-1}$ and $G''(b) = \kappa\{1 + (\theta_1 a^*)^{-1}[\theta_4 - 2(\theta_3 a_l - \theta_4) + 2\theta_4(\ln a_l - \ln a^*)]\}b^{-2}$, where $\kappa = a\beta/(a_h - a_l)$. Given that $\beta \to 1 \Rightarrow \theta_1 \to 0$ (see 28), $G'(b) < 0$ and $G''(b) > 0$ if $\beta \to 1$. Define the RHS of (32) as $H(b)$, such that $1/R = H(b)$, where $H'(b) < 0$ and $H''(b) > 0$. Since both functions are $G(b)$ and $H(b)$ are decreasing and convex, they generally exhibit two or no intersections. ■

Credit demand (32) implies end-of-period debt to be proportional to the interest rate for a given stock of debt at the beginning-of-period b_{-1} and the exogenous state a, which reflects the fact that the government has to issue more debt if the interest rate is higher. At the same time, credit supply (31) is also upward sloping, since future surpluses that suffice to repay debt become less likely for higher thresholds a^* ($= b/\theta_1$), which tends to reduce the expected return from bonds (since it increases the probability of default) and investors to demand a higher interest rate for compensation. Yet, with higher end-of-period debt levels the interest rate increases more than proportionally. Hence, equilibrium credit demand (32) and credit supply (31) imply that two equilibrium interest rates can exist.

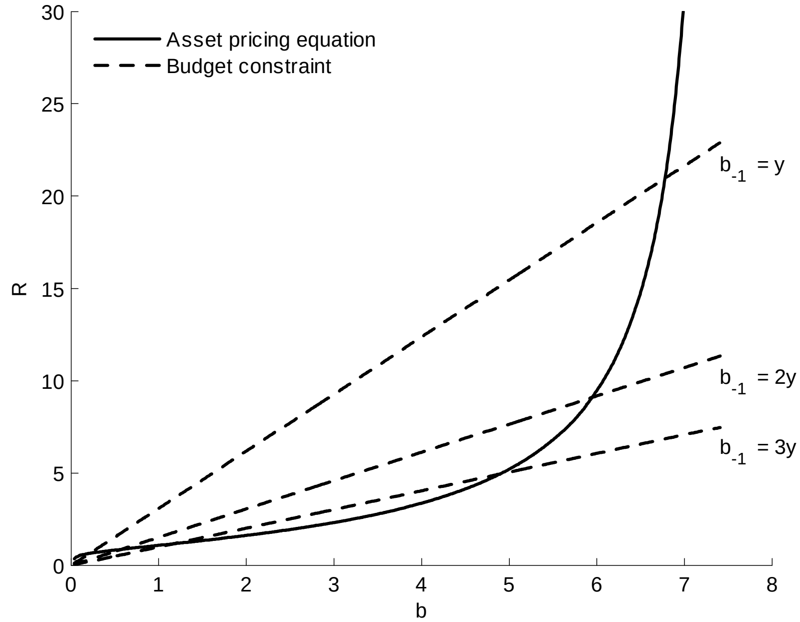

Figure 1: Multiple equilibrium bond prices

Depending on how investors coordinate their expectations, a high or a low equilibrium bond price can emerge and self-fulfilling default expectations are possible. The mechanism is that if lenders fear default, they will demand high risk premia as a compensation, which adds to the government's fiscal burden and may indeed force default. The argument is analogous to the one made with respect to an optimizing government in Calvo (1988). Self-fulfilling default expectations are thus one way to explain recent empirical developments where risk premia in some European countries surged very quickly and coincident with the lowering of ratings for government bonds.

To illustrate the emergence of multiple equilibrium bond prices, we apply the parameter values $\tau = 0.38$, $g/y = 0.35$, $\beta = 0.99$, and $\gamma = 0.35$, and we assume that the uniform distribution for the productivity level is characterized by $a_h = 1.99$ and $a_l = 0.01$. These parameters are merely illustrative examples; see the next section for a parameterization intended to match certain characteristics of European data in the context of a more realistic specification of the process governing aggregate risk. We choose initial debt levels to match a debt-to-gdp ratio (at the mean of the productivity level) equal to 1, 2 and 3, respectively. For these parameter values we plot the interest rate R as a function of the end-of-period debt level b (which equals $\theta_1 a^*$) in figure 1, using the pricing equation (31) (the solid line) and the budget constraint $R = (b_{-1} - s)^{-1} b$ (the dashed lines correspond to the three initial debt levels considered).

As figure 1 shows, the lower equilibrium interest rate increases with a higher stock of initially outstanding debt b_{-1}. In contrast, the high equilibrium interest rate decreases with higher initial debt. Given this implausible comparative static property of the high equilibrium rate, we will focus on the lower equilibrium interest rate throughout the subsequent analysis.[12] Thus, assuming that capital market participants coordinate their expectations on low equilibrium interest rates, we will examine how the sovereign risk premium behaves in response to a change in the state of the economy. This will give rise to an implied equilibrium pricing rule, which gives the interest rate spread on risky government bonds as a function of the beginning-of-period ratio of debt to output. As figure 1 suggests, the risk premium increases monotonically in the debt to output ratio. Yet, risk premia implied by the uniform productivity distribution are extremely large (by empirical standards). We will therefore drop the simplifying assumption of a uniform productivity distribution in the next section.

4 Quantitative Results

In this section, we calibrate the model to derive quantitative results. In particular, we consider normally distributed productivity shocks. We will further examine the role of expectations about aggregate risk and the maximum feasible tax rate for default risk premia. Later in this section, we consider the case of two-period bonds to assess the influence of the maturity of debt on default risk premia.

4.1 The calibrated model version

We relax the simplifications made in the previous section and solve the model numerically for a more realistic parametrization. We will concentrate on the lower equilibrium interest rate and show under which conditions risk spreads can emerge and reach high levels without a self-fulfilling expectation of a switch to the higher interest rate equilibrium.

We assume that the productivity process in (9) can be serially correlated, $\rho > 0$, and that innovations ε_t are normally distributed. The parameters are chosen as follows. We interpret one period as a year. The discount rate is therefore set at $\beta = 0.97$ to match a standard average value for a risk free annual real interest rate. The tax rate τ and the share of government spending in output g/y are parameterized based on empirical averages of data from the 16 countries currently forming the Eurozone (EURO-16). The data are obtained from the European Commission's Annual Macroeconomic Database (AMECO) and cover the time span from 1995 (the earliest period for which all data are available)

[12] In simulations, we found that the realization of the high equilibrium interest rate would immediately force default (see Juessen et al., 2009).

to 2009. We measure the tax rate τ as the ratio of the total tax burden (including actual and excluding imputed social security contributions) over gdp at current market prices. Likewise, we calculate g/y as the ratio of total current expenditure (excluding interest payments) of the general government over gdp at current market prices (thus, our measure of government expenditure includes transfers). Calculating averages of these variables over the sample period, we arrive at $\tau = 0.404$ and $g/y = 0.405$ for the EURO-16. For later use, we also calculate the corresponding values for Greece as a particularly interesting example; the results are $\tau^{GR} = 0.326$ and $(g/y)^{GR} = 0.345$.

Further, we set the mean working time share equal to $l = 1/3$ (and adjust γ accordingly). To calibrate the standard deviation of productivity shocks, we regress the log of annual real gdp for the EURO-16 countries on a constant and a linear time trend.[13] The estimated standard deviations of real output range from $\sigma_y = 3.9\%$ for the Netherlands to $\sigma_y = 15.7\%$ for Greece, with an average of 7.3%. For the benchmark case, we choose the innovation variance in our model such that for $\rho = 0.9$, which accords to the average of the autocorrelation in the sample of countries, the standard deviation of $\widehat{y}_t = \log(y_t/\overline{y})$ from stochastically simulated model runs conforms with this average value.

Figure 2 shows the model's pricing rule for government bonds for the benchmark (EURO-16) parameterization, as a relation between the interest rate spread of risky government bonds over the riskless interest rate and the beginning-of-period ratio of debt to output.[14] The solid line displays risk premia for the steady state productivity level ($a = 1$). The figure shows that with normally distributed and autocorrelated productivity levels, sizeable risk spreads would only occur for extremely high debt ratios exceeding about 200% of gdp. However, productivity realizations below the mean lead to higher premia that occur at somewhat lower debt ratios, as can be seen from the dotted line in the figure which represents a situation where productivity is ten percent below its steady state value ($a = 0.9$). In this case, which corresponds to a particularly severe adverse state, risk spreads rise strongly if debt exceeds about 170% of gdp. Furthermore, the equilibrium pricing rule is extremely steep, suggesting that above a certain critical value – that itself depends on the current aggregate state of the economy a – even small further increases in debt can lead to rapidly increasing risk spreads.

Thus, the benchmark model predicts that market participants become extremely sensitive to changes in debt at high debt-to-gdp ratios. Large risk premia will occur as a consequence of a cyclical downturn that reduces the debt capacity and thus shifts the

[13] Data are from the European Commission's AMECO database; the time span covered is 1960-2008 for 11 out of the 16 countries, but shorter for Malta, Cyprus, Slovakia, Slovenia, and unified Germany, due to data availability.

[14] Appendix 7.2 presents details on the computation of equilibrium bond prices.

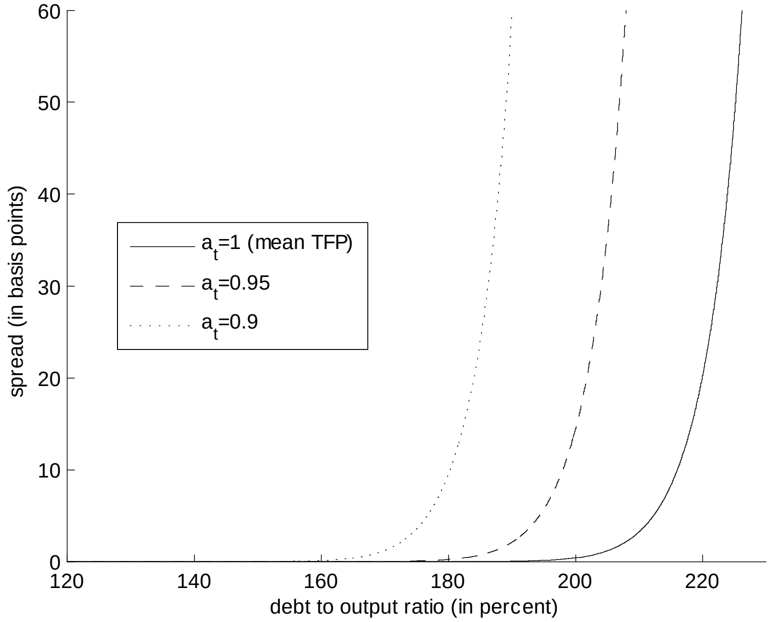

Figure 2: Default risk spreads, benchmark parameterization.

equilibrium pricing curve to the left. This is rationally anticipated by households who are *thus* willing to lend to the government at or very close to the risk free interest rate, unless the debt-to-gdp ratio becomes very high. Thus, the model's predictions are so far consistent with evidence for the average of EURO-16 countries, where current debt-to-gdp is far below the levels of debt at which non-negligible yield spreads arise according to figure 2.[15] In the following sections, we explore the factors that would lead to sizeable spreads emerging at lower debt-to-output ratios.

4.2 Perceived risk

We start by first considering investors to believe in future aggregate risk that is higher than measured in historical data, $\sigma_y^p > \sigma_y$. Figure 3 shows risk spreads in relation to the debt-to-output ratio for a variance of the productivity process that is larger than the one underlying the preceding figure 2. Note that the variance of productivity shocks assumed in figure 2 above was chosen to match the average historical European Monetary Union experience. Twice this value, which is depicted in figure 3, is roughly the historical volatility value for the most strongly fluctuating economy in the sample, namely Greece.

The dashed and dotted lines in figure 3 show that a higher variance of shocks can

[15] According to Eurostat data, the average debt-to-gdp ratio of EURO-16 countries was roughly 66% in 2009.

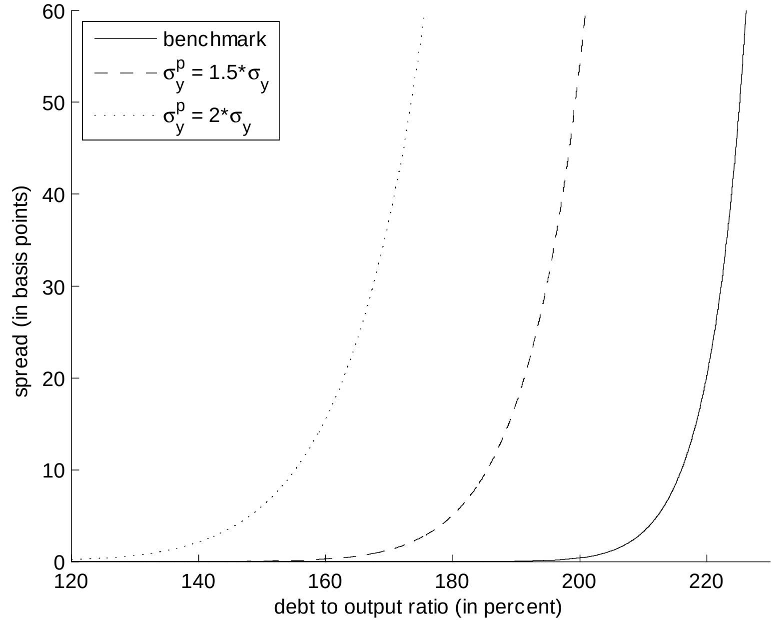

Figure 3: Default risk spreads, high aggregate risk parameterization.

substantially lower the critical debt value above which premia become sizeable (both lines in the figure are drawn for $a = 1$; of course, the equilibrium pricing curves would shift still further to the left in the case of a cyclical downturn $a < 1$). While the debt levels at which sizeable yield spreads occur are still higher than what is currently observed for the average Eurozone country, the analysis shows that, for sizeable risk premia to emerge, it is sufficient that perceived aggregate risk increases. The reason is that higher cyclical volatility leads to a larger probability of a series of serially correlated adverse shocks that make the fiscal position unsustainable in the future.

4.3 Maximum tax rates

So far, we have assumed that when assessing the maximum debt repayment capacity investors use the revenue maximizing tax rate τ^*, which is the tax rate that delivers tax revenues at the peak of the Laffer curve. In our benchmark parameterization, this revenue maximizing tax rate equals $\tau^* = 0.703$ and is thus substantially higher than tax rates that have hitherto been observed even in the most indebted countries. It might therefore be the case that investors do not believe that the government has enough political strength to claim a tax rate this high. As a consequence, investors might use a different, lower maximum feasible tax rate, τ^p say, that is below the revenue maximizing rate, but which is for example the maximum politically feasible rate or the maximum rate that would be

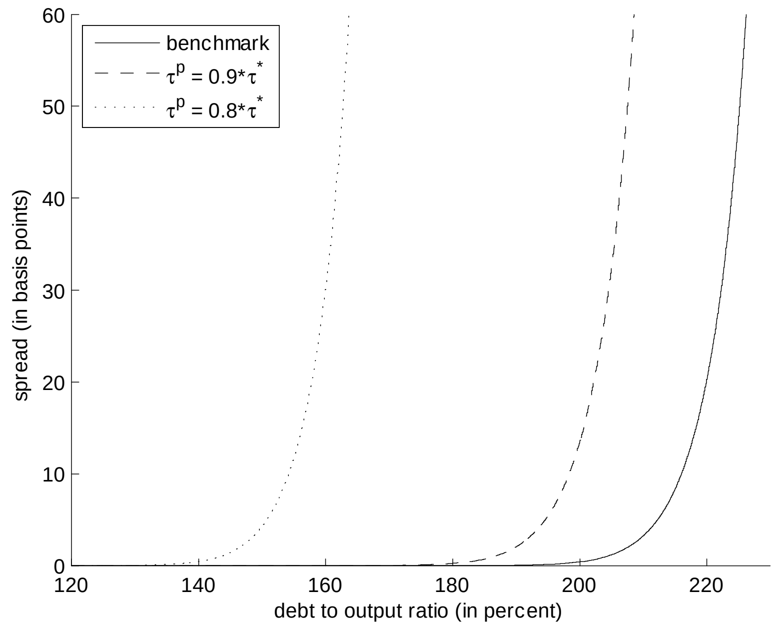

Figure 4: Default risk spreads, lower maximum tax rate.

compatible with the survival of the government in future elections.

The difference between τ^p and τ^* is thus an indicator of the perceived ability of the government to conduct a strict austerity program. How does this difference influence the pricing of government debt? Figure 4 shows bond prices for the cases where the debt capacity is based on tax rates equal to 90% and 80% of the Laffer curve maximizer τ^* (the dashed and dotted lines, respectively). For the volatility of productivity innovations, we have used the baseline value underlying figure 2. Comparing this with the results from the benchmark model that uses the revenue maximizing tax rate, see the solid line in figure 4, one can see that risk premia start to emerge for substantially lower debt levels, as expected. The reason is that the maximum debt capacity directly depends on the taxing ability of the government; if the taxing ability is low because extremely high tax rates are not credible, a given debt level entails a higher risk of non-repayment, such that interest rate premia must be larger in equilibrium.

4.4 Two period debt

In this section, we examine how default premia are affected when the maturity of government debt exceeds one period. For simplicity, we consider the case where the government issues two-period bonds only. Its period-by-period budget constraint then reads $b_{t,t+2}R_{t,t+2}^{-1} + s_t = (1-\delta_t)\, b_{t-2,t}$, where $b_{t,t+2}$ denotes bonds issued in t maturing in $t+2$

and $R_{t,t+2}^{-1}$ its period t price. Households invest in government bonds and trade non-maturing bonds issued in $t-1$ among each other at the period t price $q_{t-1,t}$. The market value of total current debt, i.e. maturing and outstanding, can then be written as

$$\widetilde{b}_t = b_{t-2,t} + q_{t-1,t} b_{t-1,t+1}. \tag{33}$$

Like in the one-period bonds case, we apply the intertemporal budget constraint as a criterion for default: If total debt, as defined in (33), exceeds the maximum repayment capacity $\Psi_t = E_t \sum_{k=0}^{\infty} s_{t+k}^* \prod_{i=1}^{k} 1/R_{t+i-1}^{rf}$, debt cannot be repaid and household stop lending. Thus, default occurs if $\widetilde{b}_t > \Psi_t$. Consistent with this criterion, we assume that the government defaults equally on maturing and outstanding bonds, where the latter are priced according to their current market value. Hence, the default rate is then endogenously determined by

$$\widetilde{b}_t > \Psi_t : (1-\delta_t)\,\widetilde{b}_t = s_t. \tag{34}$$

while $\delta_t = 0$ if $\widetilde{b}_t \leq \Psi_t$. Like in the one-period debt case, households take into account that the government might default, which is now relevant not only for the period of maturity but also one period before. Thus, the period t price of a two-period bond $R_{t,t+2}^{-1}$ (which matures in $t+2$) depends on the probabilities of default in $t+2$ and in $t+1$. As before, we introduce a lower bound for the productivity level a_t^* at which the government does not default $\widetilde{b}_t = \Psi(a_t^*)$. Further, let $\pi(a_{t+1}|a_t)$ denote the period t probability of a_{t+1} and $\pi(a_{t+2}|a_{t+1})$ the probability of a_{t+2} given a_{t+1}. Thus, an investor will demand the price of two-period bonds $R_{t,t+2}^{-1}$ to satisfy

$$\begin{aligned} \frac{c_t^{-1}}{R_{t,t+2}} &= \beta^2 \left[\int_{-\infty}^{a_{t+1}^*} \pi(a_{t+1}|a_t)(1-\delta_{t+1})\, da_{t+1} + \int_{a_{t+1}^*}^{\infty} \pi(a_{t+1}|a_t)\, da_{t+1} \right] \\ &\cdot \left[\int_{-\infty}^{a_{t+2}^*} \pi\left(a_{t+2}|a_{t+1} \geq a_{t+1}^*\right)(1-\delta_{t+2})\, c_{t+2}^{-1} da_{t+2} + \int_{a_{t+2}^*}^{\infty} \pi\left(a_{t+2}|a_{t+1} \geq a_{t+1}^*\right) c_{t+2}^{-1} da_{t+2} \right]. \end{aligned} \tag{35}$$

Hence, the risk premium on bonds maturing in $t+2$ tends to rise also with the probability of default in $t+1$. It should be noted that the default rates δ_{t+1} and δ_{t+2} depend on the market value of total debt (see 34), and thereby on future prices of non-maturing debt. In appendix 7.3, we fully characterize the determination of the price for two-period bonds.

Figure 5 shows the impact of the maturity structure of debt. While the solid line is the same as in the benchmark model, compare figure 2[16], the dashed line gives the resulting default risk premia – computed as the annualized interest rate spread over the riskless

[16] The pricing curves in figure 5 have been calculated for a smaller number of grid points than in figure 2, because with two period debt, two rather than one critical productivity thresholds have to be determined.

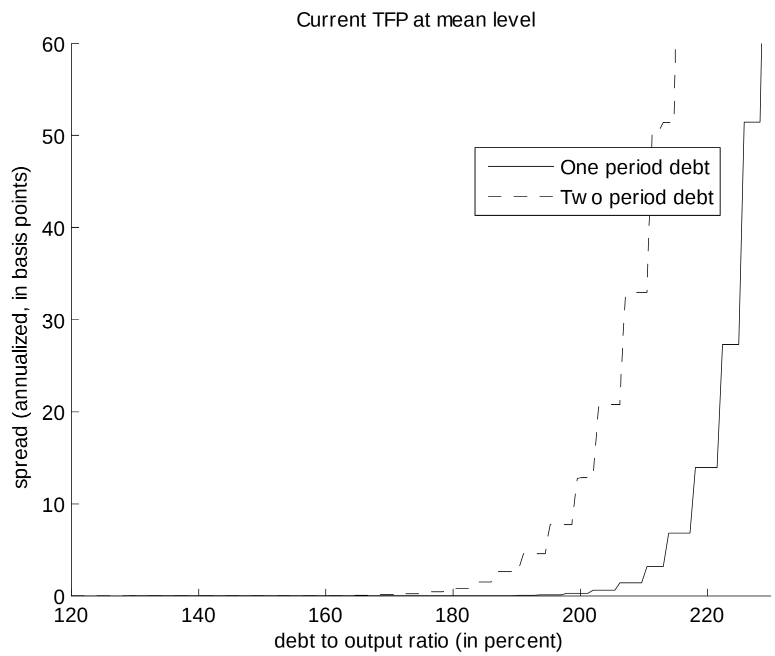

Figure 5: Default risk spreads, two period vs one period bonds.

rate – if all debt is issued in the form of two period bonds (the figure assumes that half the outstanding debt matures in the current period, while the other half matures in the following period).

As figure 5 shows, if the government issues debt of longer maturity, the default risk premium is higher at any given level of total outstanding debt. In the context of our model, this implies that investors' risk associated with government debt increases. Since productivity levels are serially correlated, an adverse shock in period $t+1$ can lead to low productivity levels also in $t+2$ such that the probability of default in either of the two periods increases. Hence, lenders will demand higher risk premia if governments use longer term maturities. As a consequence, since the higher risk premia add to the fiscal burden, default becomes more likely in this case. The model thus states that government finances may in practice, i.e. taking into account the higher risk through longer term maturities, be substantially more vulnerable than the above arguments based on one period bond financing suggested.

4.5 Fiscal stance

Finally, we examine how a fiscal stance (τ, g/y) that differs from the benchmark case affects pricing of government bonds. The exercise can be viewed as an attempt to explain the recent dynamics in Greek bond prices. According to official statistics, Greece had a

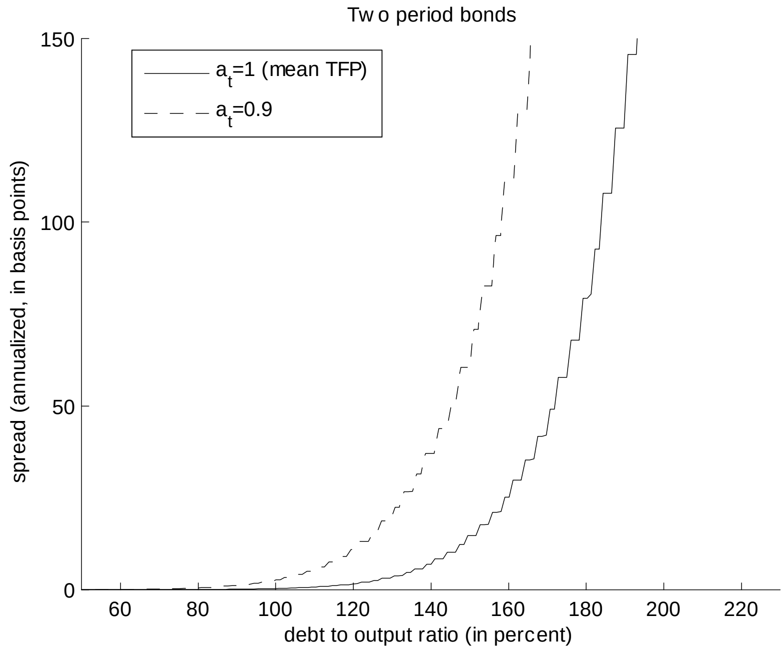

Figure 6: Default risk spreads, two period bonds, Greek parameterization.

substantial primary deficit on average over the past decade. When we calculate average tax rates and government spending shares for Greece using the same methods and data sources as before for the EURO-16 average, we get parameters values $\tau^{GR} = 0.326$ and $(g/y)^{GR} = 0.345$, respectively. As mentioned above, macroeconomic fluctuations in Greece have been more severe historically, implying about twice the level of aggregate risk than for the Eurozone average.

Figure 6 shows that our model would predict non-negligible default risk premia to emerge for two period bonds at debt-to-gdp ratios higher than about 100% or 120%, respectively. Both lines in figure 6 are based on the revenue maximizing tax rate τ^*; the solid line is for the mean productivity level, while the dashed line is computed for the case of a severe recession $a = 0.9$. To put these results in perspective, it is instructive to look at empirical risk premia. Figure 7 shows annualized yield spreads between Greek and German government bonds with maturities of three and ten years.[17] While the spreads are slightly larger for ten year bonds, the overall patterns are similar. From 2005 to 2008, spreads have been less than 20 basis points, and in the first half of 2008 at around 40 basis points. Thereafter, yield spreads increased to more than 200 basis points, returned to lower

[17] Data are from the websites of the Bank of Greece and the Bundesbank. Maturities of two years that would exactly fit our theoretical model were not available; we used three year maturities instead.

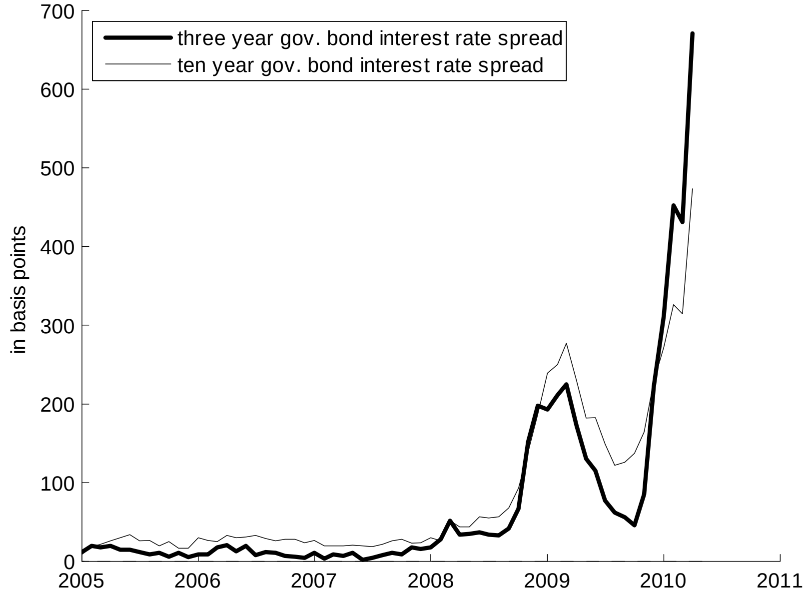

Figure 7: Empirical Spreads on Greek vs. German Government Bond Yields

levels in the second half of 2009, and went up heavily to several hundred basis points in the first quarter of 2010. At that time, Greek government debt stood at about 115% of gdp. Hence, our model is roughly consistent with the spreads observed before mid-2008. However, the extremely high risk premia in 2009 and 2010 cannot be rationalized with fundamentals alone. Thus, the analysis suggests that shifts in expectations as suggested in sections 4.2 and 4.3, or even a coordination of expectations on the high risk equilibrium were responsible for these developments.

5 Conclusion

This paper has asked how the dynamic behavior of risk premia on public bonds can be understood under the assumption that the government is committed to repay its debt as far as possible. We have presented a model where sovereign default is the result of lenders' withdrawal of funding to the government when the government's maximum debt capacity is exceeded. The risk premium on public bonds depends on the expected probability of this event and on the expected rate of partial repayment in the case of the government's inability to fully repay debt.

We have shown that the generic existence of two equilibrium bond prices can give rise to self-fulfilling default expectations, since the probability of default depends on the fiscal burden of interest payments, which themselves depend on the expected probability of default. Furthermore, even in the case where lenders expect the lower interest rate equilibrium, sizeable risk premia can arise. Crucially, the maximum debt repayment capacity depends on fundamental factors. In particular, we have pointed out that the perceived level of aggregate risk, the existence of longer-term maturities of public bonds, and the possible infeasibility of revenue maximizing tax rates increase the default risk premium. The model provides a rationale why different economies with similar levels of government debt in relation to gdp have experienced very different risk premia on their public bonds. Importantly, one implication is that risk premia may be very sensitive to small changes in debt at high debt-to-gdp ratios.

When we ask in how far the model helps in explaining the recent surge in risk spreads in some Eurozone member countries, the answer is mixed. On the one hand, yield spreads below 50 basis points, as experienced for example by Greece prior to autumn 2008, are very much in line with what the model predicts for a country that is characterized by Greece's fiscal stance. On the other hand, the recent explosion of risk premia in this country can only be explained by either a loss of confidence in the political ability of the Greek government to raise taxes sufficiently, or by non-fundamentally induced shifts in expectations. In the latter case, there is obviously a useful role of interventions in the market for public debt that steers expectations away from a high default risk equilibrium.

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7 Appendix

7.1 Local approximation of the maximum debt capacity

In this appendix, we apply a second order approximation of the maximum debt capacity (14). For this, we transform Ψ_t in the following way. The surpluses in (14) refer to the no-default case $\delta_{t+k} = 0 \; \forall k \geq t$, where the Euler equation reads $1/R^{rf}_{t+k} = \beta E_t \left(c^{-1}_{t+k+1} / c^{-1}_{t+k} \right)$.

Further, using the law of iterated expectations $\Pi_{i=1}^{k}(1/R_{t+i-1}^{rf}) = (1/R_t^{rf})(1/R_{t+1}^{rf})... = \beta E_t \left(c_{t+1}^{-1}/c_t^{-1}\right) \beta E_{t+1} \left(c_{t+2}^{-1}/c_{t+1}^{-1}\right) ... = \beta^k E_t \left(c_{t+k}^{-1}/c_t^{-1}\right)$, we can write

$$\Psi_t = E_t \sum_{k=0}^{\infty} \beta^k \frac{c_{t+k}^{*-1}}{c_t^{*-1}} s_{t+k}^*,$$

where $c_t^* = c(a_t, \tau^*)$ denotes consumption as a function of the state and the revenue maximizing tax rate τ^*. Using the solutions for consumption and government surpluses (20) and (22), we have

$$\Psi_t = c(a_t, \tau^*) E_t \sum_{k=0}^{\infty} \beta^k c(a_{t+k}, \tau^*)^{-1} s(a_{t+k}, \tau^*),$$

and summarizing terms we get

$$\Psi_t = \Psi(a_t, \sigma_\varepsilon, \rho, \tau^*) = \gamma(1-\tau^*) a_t E_t \sum_{k=0}^{\infty} \beta^k f(a_{t+k}), \tag{36}$$
$$\text{where} \quad f(a_{t+k}) = \tau^* - a_{t+k}^{-1}\gamma^{-1}g.$$

Using that the exogenous state variable a_t is generated by a stationary process, we apply a second order Taylor expansion of $E_t f(a_{t+k})$ at $\overline{a}$, which yields

$$E_t f(a_{t+k}) \simeq f(\overline{a}) + f'(\overline{a}) E_t(a_{t+k} - \overline{a}) + \frac{1}{2} f''(\overline{a}) E_t(a_{t+k} - \overline{a})^2, \tag{37}$$

where $E_t f(a_{t+k}) = E_t\left(\tau^* - a_{t+k}^{-1}\gamma^{-1}g\right)$ and

$$f(\overline{a}) = \tau^* - \overline{a}^{-1}\gamma^{-1}g, \quad f'(\overline{a}) = \overline{a}^{-2}\gamma^{-1}g, \quad f''(\overline{a}) = -2\overline{a}^{-3}\gamma^{-1}g.$$

Next, we use that a_{t+k} can be written as

$$\begin{aligned} a_{t+k} &= \rho a_{t+k-1} + (1-\rho)\overline{a} + \varepsilon_{t+k} = \rho^k a_t + \sum_{i=0}^{k-1} \rho^i (1-\rho)\overline{a} + \sum_{i=0}^{k-1} \rho^i \varepsilon_{t+k-i} \\ &= \rho^k a_t + \overline{a}\left(1-\rho^k\right) + \sum_{i=0}^{k-1} \rho^i \varepsilon_{t+k-i}. \end{aligned} \tag{38}$$

Hence, the mean and the variance of a_{t+k} conditional on information in period t, $E_t a_{t+k}$ and $var_t a_{t+k} = E_t[(a_{t+k})^2] - [E_t a_{t+k}]^2$ are given by

$$E_t a_{t+k} = \rho^k a_t + \overline{a}\left(1-\rho^k\right) \tag{39}$$

$$var_t a_{t+k} = E_t a_{t+k}^2 - \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2. \tag{40}$$

The term in (40) can, by substituting out a_{t+k} with (38), be simplified to

$$\begin{aligned}
var_t a_{t+k} &= E_t\left[\left(\rho^k a_t + \overline{a}\left(1-\rho^k\right) + \sum_{i=0}^{k-1}\rho^i \varepsilon_{t+k-i}\right)^2\right] - \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2 \\
&= \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2 + E_t\left(\sum_{i=0}^{k-1}\rho^i \varepsilon_{t+k-i}\right)^2 - \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2 \\
&= E_t\left(\sum_{i=0}^{k-1}\rho^i \varepsilon_{t+k-i}\right)^2 = \frac{1-\rho^{2k}}{1-\rho^2}\sigma_\varepsilon^2.
\end{aligned}$$

Using (39), we rewrite (37) as

$$E_t f(a_{t+k}) \simeq f(\overline{a}) + f'(\overline{a})\, E_t\left(\rho^k a_t + \overline{a}\left(1-\rho^k\right) - \overline{a}\right) + \frac{1}{2} f''(\overline{a})\left(E_t a_{t+k}^2 - 2\overline{a} E_t a_{t+k} + \overline{a}^2\right).$$

Further, using $E_t a_{t+k}^2 = var_t a_{t+k} + \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2 = \frac{1-\rho^{2k}}{1-\rho^2}\sigma_\varepsilon^2 + \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2$, we can simplify $E_t f(a_{t+k})$ to

$$\begin{aligned}
E_t f(a_{t+k}) \simeq{} & f(\overline{a}) + f'(\overline{a})\, E_t\left(\rho^k a_t + \overline{a}\left(1-\rho^k\right) - \overline{a}\right) \\
& + \frac{1}{2} f''(\overline{a})\left(\frac{1-\rho^{2k}}{1-\rho^2}\sigma_\varepsilon^2 + \left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right]^2 - 2\overline{a}\left[\rho^k a_t + \overline{a}\left(1-\rho^k\right)\right] + \overline{a}^2\right) \\
={} & f(\overline{a}) + f'(\overline{a})\,\rho^k (a_t - \overline{a}) + \frac{1}{2} f''(\overline{a})\left[\frac{1-\rho^{2k}}{1-\rho^2}\sigma_\varepsilon^2 + \rho^{2k}\left(a_t - \overline{a}\right)^2\right]. \qquad (41)
\end{aligned}$$

Summing up the discounted values of $E_t f(a_{t+k})$ for $k = 0$ to ∞, and using (41), we get

$$\begin{aligned}
\sum_{k=0}^{\infty}\beta^k E_t f(a_{t+k}) \simeq{} & \sum_{k=0}^{\infty}\beta^k f(\overline{a}) + \sum_{k=0}^{\infty}\beta^k f'(\overline{a})\,\rho^k (a_t - \overline{a}) \\
& + \sum_{k=0}^{\infty}\beta^k \frac{1}{2} f''(\overline{a})\,\frac{1-\rho^{2k}}{1-\rho^2}\sigma_\varepsilon^2 + \sum_{k=0}^{\infty}\beta^k \frac{1}{2} f''(\overline{a})\,\rho^{2k}\left(a_t - \overline{a}\right)^2 \\
={} & \frac{1}{1-\beta} f(\overline{a}) + f'(\overline{a})\left(a_t - \overline{a}\right)\sum_{k=0}^{\infty}\beta^k\rho^k \\
& + \frac{\frac{1}{2} f''(\overline{a})\,\sigma_\varepsilon^2}{1-\rho^2}\left(\sum_{k=0}^{\infty}\beta^k - \sum_{k=0}^{\infty}\beta^k\rho^{2k}\right) + \frac{1}{2} f''(\overline{a})\left(a_t - \overline{a}\right)^2 \sum_{k=0}^{\infty}\beta^k\rho^{2k}
\end{aligned}$$

Since β and ρ lie inside the unit circle, the infinite sums converge to finite values:

$$\begin{aligned}
\sum_{k=0}^{\infty}\beta^k E_t f(a_{t+k}) \simeq{} & \frac{1}{1-\beta} f(\overline{a}) + \frac{f'(\overline{a})}{1-\beta\rho}(a_t - \overline{a}) \\
& + \frac{\frac{1}{2} f''(\overline{a})}{1-\rho^2}\left(\sigma_\varepsilon^2\left(\frac{1}{1-\beta} - \frac{1}{1-\beta\rho^2}\right) + \frac{1-\rho^2}{1-\beta\rho^2}\left(a_t - \overline{a}\right)^2\right).
\end{aligned}$$

Hence, the maximum debt capacity Ψ can be approximated as

$$\begin{aligned}
&\Psi\left(a_t,\sigma_\varepsilon,\rho,g,\tau^*\right)\\
&\simeq \gamma\left(1-\tau^*\right)a_t\cdot\left\{\frac{1}{1-\beta}f\left(\overline{a}\right)+\frac{f'\left(\overline{a}\right)}{1-\beta\rho}\left(a_t-\overline{a}\right)\right.\\
&\left.+\frac{\frac{1}{2}f''\left(\overline{a}\right)}{1-\rho^2}\left(\sigma_\varepsilon^2\left(\frac{1}{1-\beta}-\frac{1}{1-\beta\rho^2}\right)+\frac{1-\rho^2}{1-\beta\rho^2}\left(a_t-\overline{a}\right)^2\right)\right\}.
\end{aligned}$$

7.2 Computation of equilibrium bond prices

We replace the original problem presented in sections 2 and 3 by a discrete valued problem, i.e. we assume that the model's state space consists of a finite number of discrete points. Choose the following parameters of the model:

Parameter	Description	Benchmark Calibration
Preferences		
$\bar{l}$	Labor in steady state	1/3
β	Discount factor	0.97
γ	Preference parameter	$\left(\left(1-\frac{g}{y}\right)\bar{l}\right)/\left(1-\tau\right)=0.3324$
Productivity process and state space		
ρ	Autocorrelation	0.9
σ_ε	Std. of productivity shocks	0.0550
$\bar{a}$	Unconditional mean of TFP	1
n	Number of TFP states	4001
m	Number of initial debt states	4001
Government		
g/y	Government share	0.4053
τ	Tax rate	0.4037
g	Level of government exp.	$y^*\cdot(g/y)=0.1351$
τ^*	Laffer curve maximizer	$0.5+g/(2\bar{a}\gamma)=0.7032$

We use Tauchen's (1982) algorithm to approximate the continuous valued AR(1)-process for productivity (see (9)) by a discrete valued Markov chain. We provide the size of the interval $I_a=[a_1,a_n]$ and the number of grid points, n. Tauchen's algorithm then delivers the exogenous state space of the model[18]

$$S = \{a_1, a_2, ..., a_n\}, a_i < a_{i+1}, i = 1, 2, ..., n-1,$$

and the associated transition probability matrix $P = (p_{ij})$, whose row i and column j element is the probability of moving from state a_i state to state a_j. Given ρ, the interval I_a is chosen to include ± 4 standard deviations of the productivity process.

For a given combination of initial debt b_{t-1} and current productivity level a_t, the equilibrium interest rate spread on government bonds is determined as follows:

1. At the beginning of a period t, the initial debt level, b_{t-1}, and the current productivity level, a_t, are given. Current consumption $c_t = c(a_t, \gamma, \tau)$, surpluses $s_t = s(a_t, \gamma, \tau)$, and the maximum debt repayment capacity of the current period, $\Psi_t = \Psi\left(a_t, \sigma_\varepsilon^2, \rho, \gamma, \tau^*\right)$ are known (see $(20), (22)$, and (24)).

2. Calculate the risk free rate, which is given by

$$R_t^{rf} = \frac{c_t^{-1}}{\beta E_t c_{t+1}^{-1}}.$$

   In this expression, the conditional expectation $E_t c_{t+1}^{-1}$ is calculated as

$$E_t c_{t+1}^{-1} = \sum_{j=1}^{n} p_{ij} \cdot c(a_j, \gamma, \tau)^{-1},$$

   where i denotes the index number for today's stochastic state, a_t.

3. Check whether the government defaults in period t or not.

   (a) If $\Psi_t < b_{t-1}$, the government defaults, end-of-period debt equals zero, $b_t = 0$, and the algorithm ends.

   (b) If $\Psi_t > b_{t-1}$, the government does not default in period t and the algorithm continues with step 4.

4. If the government does not default in period t (case 3b applied), the bond price $1/R_t$ and end-of-period debt b_t have to be solved simultaneously. Replacing the integrals in (26) by sums over the finite number of states, the asset pricing equation reads

$$\frac{b_{t-1} - s_t}{b_t} = \frac{\beta}{c_t^{-1}} \begin{bmatrix} b_t^{-1} \sum_{a_{t+1}=a_1}^{a_{t+1}^*} \pi_t(a_{t+1}) \left[c(a_{t+1})^{-1} s(a_{t+1})\right] \\ + \sum_{a_{t+1}=a_{t+1}^*}^{a_n} \pi_t(a_{t+1}) \left[c(a_{t+1})^{-1}\right] \end{bmatrix} \tag{42}$$

[18] We use equally spaced points $\Delta = a_{i+1} - a_i$ for all $i = 1, 2, ..., n-1$.

Use the updated version of (25) $b_t = \Psi\left(a_{t+1}^*, \sigma_\varepsilon, \rho, \gamma, \tau^*\right)$ to replace b_t in (42) :

$$b_{t-1} - s_t = \frac{\Psi\left(a_{t+1}^*, \sigma_\varepsilon, \rho, \gamma, \tau^*\right)\beta}{c_t^{-1}} \cdot \left[\begin{array}{c} \Psi\left(a_{t+1}^*, \sigma_\varepsilon, \rho, \gamma, \tau^*\right)^{-1} \sum_{a_{t+1}=a_1}^{a_{t+1}^*} \pi_t\left(a_{t+1}\right)\left[c\left(a_{t+1}\right)^{-1} s\left(a_{t+1}\right)\right] \\ + \sum_{a_{t+1}=a_{t+1}^*}^{a_n} \pi_t\left(a_{t+1}\right)\left[c\left(a_{t+1}\right)^{-1}\right] \end{array}\right]. \tag{43}$$

5. Equation (43) is solved for the unknown productivity threshold in the next period, a_{t+1}^*, which is its only unknown. If there are multiple solutions for a_{t+1}^* we choose the lower value a_{t+1}^* as our solution. This equilibrium corresponds to the low interest rate equilibrium (see the discussion on multiple equilibria in section 3).

6. Given the solution for a_{t+1}^*, next-period's debt level b_t and the asset price $1/R_t$ are determined by

$$b_t = \Psi\left(a_{t+1}^*, \sigma_\varepsilon, \rho, \gamma, \tau^*\right)$$

and

$$\frac{1}{R_t} = \frac{\beta}{c_t^{-1}}\left[\begin{array}{c} \Psi\left(a_{t+1}^*, \sigma_\varepsilon, \rho, \gamma, \tau^*\right)^{-1} \sum_{a_{t+1}=a_1}^{a_{t+1}^*} \pi_t\left(a_{t+1}\right)\left[c\left(a_{t+1}\right)^{-1} s\left(a_{t+1}\right)\right] \\ + \sum_{a_{t+1}=a_{t+1}^*}^{a_n} \pi_t\left(a_{t+1}\right)\left[c\left(a_{t+1}\right)^{-1}\right] \end{array}\right]$$

7. The risk premium on government bonds for given states b_{t-1} and a_t is calculated as $R_t - R_t^{rf}$.

7.3 The case of two-period debt

The period t price of two-period debt issued in period t, $1/R_{t,t+2}$, depends on the expected pay-off in period $t+2$, which is a function of the default rate δ_{t+2} and on the termination value of the debt contract in $t+1$ for the case where the government defaults in period $t+1$ (see 35). Hence, determination of the period t price $1/R_{t,t+2}$ requires the joint determination of (expected) default rates in period $t+1$ and $t+2$, δ_{t+1} and δ_{t+2}. The default rate further depend on the market value of total debt in both periods, $\widetilde{b}_{t+1}$ and $\widetilde{b}_{t+2}$, and thus on the prices for outstanding debt, $q_{t,t+1}$ and $q_{t+1,t+2}$, and the price of debt newly issued in $t+1$, $1/R_{t+1,t+3}$. Specifically, the default rates satisfy

$$\delta_{t+1} = 1 - s_{t+1}/\widetilde{b}_{t+1} \tag{44}$$

$$\delta_{t+2} = 1 - s_{t+2}/\widetilde{b}_{t+2} \tag{45}$$

while the market values of total debt in $t+1$ and $t+2$, defined as $\widetilde{b}_{t+1} = b_{t-1,t+1} + q_{t,t+1} b_{t,t+2}$ and $\widetilde{b}_{t+2} = b_{t,t+2} + q_{t+1,t+2} b_{t+1,t+3}$ are given by

$$\widetilde{b}_{t+1} = b_{t-1,t+1} + q_{t,t+1} R_{t,t+2} \left(b_{t-2,t} - s_t\right) \tag{46}$$

$$\widetilde{b}_{t+2} = R_{t,t+2} \left(b_{t-2,t} - s_t\right) + \begin{cases} q_{t+1,t+2} R_{t+1,t+3} \left(b_{t-1,t+1} - s_{t+1}\right) & \text{for} \quad \widetilde{b}_{t+1} \leq \Psi_{t+1} \\ 0 & \text{for} \quad \widetilde{b}_{t+1} > \Psi_{t+1} \end{cases} \tag{47}$$

where we used the government budget constraints $b_{t,t+2}/R_{t,t+2} = b_{t-2,t} - s_t$ and $b_{t+1,t+3}/R_{t+1,t+3} = b_{t-1,t+1} - s_{t+1}$. Given that outstanding debt is traded within a period before the default is realized, the outstanding debt in $t+1$ and $t+2$ is priced according to

$$q_{t,t+1} = \beta \left(\begin{array}{c} \int_{-\infty}^{a^*_{t+2}} \pi\left(a_{t+2} | a_{t+1} \geq a^*_{t+1}\right) \left(1 - \delta_{t+2}\right) \frac{c^{-1}_{t+2}}{c^{-1}_{t+1}} da_{t+2} \\ + \int_{a^*_{t+2}}^{\infty} \pi\left(a_{t+2} | a_{t+1} \geq a^*_{t+1}\right) \frac{c^{-1}_{t+2}}{c^{-1}_{t+1}} da_{t+2} \end{array} \right) \tag{48}$$

$$q_{t+1,t+2} = \beta \int_{-\infty}^{\infty} \pi\left(a_{t+3} | a_{t+2} \geq a^*_{t+2}, a_{t+1} \geq a^*_{t+1}\right) \frac{c^{-1}_{t+3}}{c^{-1}_{t+2}} da_{t+3} \tag{49}$$

where we assumed, for simplicity, that investors neglects the possibility of default in period $t+3$. Given this assumption, debt issued in $t+1$, is priced according to

$$\begin{aligned} &\frac{1}{R_{t+1,t+3}} \\ &= \beta^2 \left(\begin{array}{c} \int_{-\infty}^{a^*_{t+2}} \pi\left(a_{t+2} | a_{t+1} \geq a^*_{t+1}\right) c_{t+1} \left(1 - \delta_{t+2}\right) \int_{-\infty}^{\infty} \pi\left(a_{t+3} | a_{t+2}\right) c^{-1}_{t+3} da_{t+3} da_{t+2} \\ + \int_{a^*_{t+2}}^{\infty} \pi\left(a_{t+2} | a_{t+1} \geq a^*_{t+1}\right) c_{t+1} \int_{-\infty}^{\infty} \pi\left(a_{t+3} | a_{t+2}\right) c^{-1}_{t+3} da_{t+3} da_{t+2} \end{array} \right) \end{aligned} \tag{50}$$

Hence, in each period, after the exogenous state is realized, the unknowns δ_{t+1}, δ_{t+2}, $\widetilde{b}_{t+1}$, $\widetilde{b}_{t+2}$, $R_{t,t+2}$, $q_{t,t+1}$, $q_{t+1,t+2}$ and $R_{t+1,t+3}$ can be determined by (44)-(50) where $c_t = c\left(a_t, \tau\right)$ and $s_t = s\left(a_t, \tau\right)$ and a^*_{t+1} and a^*_{t+2} are defined as

$$\widetilde{b}_{t+1} = \Psi\left(a^*_{t+1}\right) \quad \text{and} \quad \widetilde{b}_{t+2} = \Psi\left(a^*_{t+2}\right) \tag{51}$$

given a sequence $\{a_t\}_{t=0}^{\infty}$, and the predetermined endogenous state variables $b_{t-2,t} > 0$ and $b_{t-1,t+1} > 0$.

Substituting out δ_{t+1}, δ_{t+2}, $q_{t,t+1}$, $q_{t+1,t+2}$, $1/R_{t+1,t+3}$, $\widetilde{b}_{t+1}$, and $\widetilde{b}_{t+2}$ with (44), (45), (48), (49), (50), and (51) in (35), (46), and (47), and conditioning on period t information, we end up with the following system in a^*_{t+1}, a^*_{t+2} and $R_{t,t+2}$

$$\begin{aligned} \frac{c^{-1}_t}{R_{t,t+2}} &= \beta^2 \left[\int_{-\infty}^{a^*_{t+1}} \pi\left(a_{t+1} | a_t\right) \left(s_{t+1}/\Psi\left(a^*_{t+1}\right)\right) da_{t+1} + \int_{a^*_{t+1}}^{\infty} \pi\left(a_{t+1} | a_t\right) da_{t+1} \right] \\ &\cdot \left[\begin{array}{c} \int_{a^*_{t+1}}^{\infty} \pi\left(a_{t+1} | a_t\right) \left(\int_{-\infty}^{a^*_{t+2}} \pi\left(a_{t+2} | a_{t+1}\right) \left(s_{t+2}/\Psi\left(a^*_{t+2}\right)\right) c^{-1}_{t+2} da_{t+2} \right) da_{t+1} \\ + \int_{a^*_{t+1}}^{\infty} \pi\left(a_{t+1} | a_t\right) \left(\int_{a^*_{t+2}}^{\infty} \pi\left(a_{t+2} | a_{t+1}\right) c^{-1}_{t+2} da_{t+2} \right) da_{t+1} \end{array} \right] \end{aligned} \tag{52}$$

$$\Psi\left(a_{t+1}^{*}\right) = b_{t-1,t+1} + \tag{53}$$
$$\beta \int_{-\infty}^{\infty} \pi\left(a_{t+1}|a_t\right) \left(\begin{array}{c} \int_{-\infty}^{a_{t+2}^{*}} \pi\left(a_{t+2}|a_{t+1} \geq a_{t+1}^{*}\right) \left(s_{t+2}/\Psi\left(a_{t+2}^{*}\right)\right) \frac{c_{t+2}^{-1}}{c_{t+1}^{-1}} da_{t+2} \\ + \int_{a_{t+2}^{*}}^{\infty} \pi\left(a_{t+2}|a_{t+1} \geq a_{t+1}^{*}\right) \frac{c_{t+2}^{-1}}{c_{t+1}^{-1}} da_{t+2} \end{array} \right) R_{t,t+2}\left(b_{t-2,t} - s_t\right) da_{t+1}$$

$$\Psi\left(a_{t+2}^{*}\right) = R_{t,t+2}\left(b_{t-2,t} - s_t\right) + \tag{54}$$
$$\int_{a_{t+1}^{*}}^{\infty} \pi\left(a_{t+1}|a_t\right) \frac{\beta\left(b_{t-1,t+1} - s_{t+1}\right) \int_{a_{t+2}^{*}}^{\infty} \pi\left(a_{t+2}|a_{t+1}\right) \int_{-\infty}^{\infty} \pi\left(a_{t+3}|a_{t+2}\right) \frac{c_{t+3}^{-1}}{c_{t+2}^{-1}} da_{t+3} da_{t+2}}{\beta^2 \left(\begin{array}{c} \int_{-\infty}^{a_{t+2}^{*}} \pi\left(a_{t+2}|a_{t+1}\right) c_{t+1} \left(s_{t+2}/\Psi\left(a_{t+2}^{*}\right)\right) \int_{-\infty}^{\infty} \pi\left(a_{t+3}|a_{t+2}\right) c_{t+3}^{-1} da_{t+3} da_{t+2} \\ + \int_{a_{t+2}^{*}}^{\infty} \pi\left(a_{t+2}|a_{t+1}\right) c_{t+1} \int_{-\infty}^{\infty} \pi\left(a_{t+3}|a_{t+2}\right) c_{t+3}^{-1} da_{t+3} da_{t+2} \end{array} \right)} da_{t+1}$$

where $c_t = c(a_t, \tau)$ and $s_t = s(a_t, \tau)$, given a sequence $\{a_t\}_{t=0}^{\infty}$, and $b_{t-2,t} > 0$ and $b_{t-1,t+1} > 0$.

Computation of bond prices based on (52)-(54) To find the equilibrium price of two-period bonds $R_{t,t+2}^{-1}$, we have to solve the system (52)-(54) in a_{t+1}^{*}, a_{t+2}^{*} and $R_{t,t+2}$. We do so numerically. At the beginning of a period t, the exogenous state a_t and the endogenous states $b_{t-2,t} > 0$ and $b_{t-1,t+1} > 0$ are given. For a given a_t, the maximum debt repayment capacity $\Psi\left(a_t^{*}, \sigma_\varepsilon, \rho, \gamma, \tau^{*}\right)$, consumption c_t, and surpluses s_t are determined.

To find the equilibrium price $R_{t,t+2}^{-1}$, we evaluate the system of equations (52)-(54) for all possible combinations of productivity thresholds in periods $t+1$ and $t+2$, respectively, from the productivity grid S. This means that we evaluate the system of equations using all combinations of candidate (still unknown) values for a_{t+1}^{*} and a_{t+2}^{*} that result from the given productivity grid. Thereby, we use equation (52) to substitute for $R_{t,t+2}$ in equations (53) and (54) so that the system reduces to two equations in the two unknowns a_{t+1}^{*} and a_{t+2}^{*}.

When evaluating the two equations at the candidate values for a_{t+1}^{*} and a_{t+2}^{*}, the integrals are replaced by sums over the finite number of states (as in the case of one-period debt, see above). Accordingly, the nested integrals in (52)-(54) are calculated as nested sums. As before, the conditional probabilities π are replaced by the respective entries in the transition probability matrix $P = (p_{ij})$.

For a given combination of candidate solutions for the two productivity thresholds, we substract the left-hand side from the right-hand side in equations (53) and (54), respectively, square the respective values, and add them. We repeat this calculation for all possible candidate solutions for a_{t+1}^{*} and a_{t+2}^{*} on the grid and store the respective residuals. The equilibrium values a_{t+1}^{*} and a_{t+2}^{*} are the ones for which the residual is closest to zero.

As in the case of one-period debt, there may be multiple equilibria. We choose the equilibrium that is associated with the smallest values for a^*_{t+1} and a^*_{t+2}. This equilibrium corresponds to the low interest rate equilibrium. The solution for $R^{-1}_{t,t+2}$ is recovered from equation (52), given the equilibrium values for a^*_{t+1} and a^*_{t+2}. The annualized two-period interest rate is given by the square root of $R_{t,t+2}$.

The risk-free rate for the two-period case is calculated as

$$R^{rf}_{t,t+2} = \frac{c_t^{-1}}{\beta^2 E_t c^{-1}_{t+2}}.$$

In this expression, the conditional expectation $E_t c^{-1}_{t+2}$ is calculated as

$$E_t c^{-1}_{t+2} = \sum_{j=1}^{n} \tilde{p}_{ij} \cdot c\left(a_j\right)^{-1},$$

where i denotes the index number for today's stochastic state, a_t, and $\tilde{p}_i$ is the ith row of the two-period transition probability matrix $\tilde{P} = P \cdot P$. The annualized interest rate spread is then given by $\sqrt{R_{t,t+2}} - \sqrt{R^{rf}_{t,t+2}}$.