SFB 823

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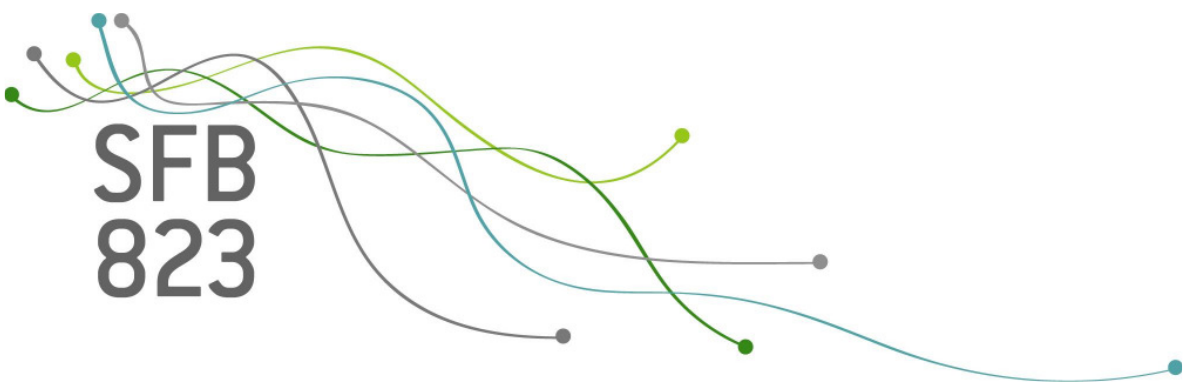
Nr. 38/2010

Discussion Paper



Stock market confidence and copula-based Markov models*

Mario Jovanović†

September 28, 2010

Abstract

This paper presents a descriptive model of stock market confidence conditional on stock market uncertainty in a first-order copula-based Markov approach. By using monthly closing prices of the VIX as a stock market uncertainty proxy for the United States and the copula of Fang et al. (2000) a stable nonlinear relation between confidence and uncertainty is derived. Based on the existence of a specific dependence structure uncertainty-reducing policies by US institutions which are intended to recover stock market confidence are evaluated with respect to its efficiency. The model implication for high uncertainty regimes is an aggressive uncertainty-reducing policy in order to avoid sticking in an uncertainty trap. Furthermore, uncertainty driving profit expectations force an uncertainty level which does not correspond to high confidence and calls for regulatory actions. Additionally, the methodological approach is appropriate for conditional quantile forecasts and a potential tool in risk management.

JEL Classification: C12, C22, E44
Keywords: Uncertainty, confidence, temporal dependence

*I would like to thank Sue Man Fan, Manfred Lösch and Michael Roos for extensive discussions. Furthermore, I would like to thank Holger Dette and Walter Krämer for the publication of this work in the Discussion Paper Series of the Collaborative Research Center 823 "Statistical modelling of nonlinear dynamic processes" funded by the German Research Foundation.

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1 Introduction

Missing confidence is the core problem of the latest financial market crisis started in the US and spilled over the whole world and led, for instance, to a drop of interbank transactions. In general, economic transactions which are based on confidence towards business partners are strongly effected by a state of misconfidence. In response much effort has been placed on the recovery of confidence in financial markets by means of an uncertainty-reducing policy. For example, between the emergence of the recent financial crisis in the summer of 2007 and January of 2009, the federal funds rate was brought down by 325 basis points. "In historical comparison, this policy response stands out as exceptionally rapid and proactive" (Bernanke, 2009). If an uncertainty-reducing policy increases confidence in financial markets, a stable relationship between uncertainty and confidence must exist. The aim of this paper is the derivation of a descriptive model which allows for the evaluation of such an uncertainty policy.

In a recent paper, Bloom (2009) identifies the so-called VIX index, which deals with implied volatility, as a canonical proxy for financial market uncertainty. The index is designed to measure the market's expectation of 30-day variability implied by at-the-money S&P 500 option prices and is published by the Chicago Board Options Exchange since 1990. Especially the temporal dependence structure of the VIX is important with respect to confidence and will be analyzed in this paper. Regarding variance risk premium measurement the temporal dependence of high-frequency intra-day VIX data is an issue for real-time trading and is investigated by Bollerslev et al. (2009). However, investors with a medium-run strategy, say one month, interested in futures and options contracts based on implied volatility have a stake in the monthly development of the VIX. From a central bank's perspective intra-day uncertainty measures are also less important with respect to monetary policy, due to the focus on uncertainty trends. Following Clarida et al. (1998, 2000) a forward-looking Taylor rule is the best available framework for evaluation and simulation of monetary policy. The incorporated economic indicators expected inflation, interest rate and output gap are available in a monthly frequency. Hence, an uncertainty augmented forward-looking Taylor rule should contain monthly uncertainty proxies. Jovanović and Zimmermann (2010) show that pacifying financial markets by interest rate cuts is part of the US Federal Reserve Bank (Fed) monetary reaction function for more than 25 years. Therefore, in case of a description of temporal dependence of the VIX it is possible to generate uncertainty forecasts as a decision support for monetary policy. Consequently, the derivation of a descriptive model of temporal dependence of the monthly VIX is also interesting for

medium-run investors and institutions.

In order to derive a descriptive model of stock market confidence copula-based Markov models are applied as the methodological framework. By the theorem of Sklar (1959) any multivariate distribution can be expressed in terms of its marginal distributions and its copula function. A copula function is a multivariate distribution function with standard uniform marginals, which captures the scale-free dependence structure of the multivariate distribution function. The copula-based approach has the advantage of separating the information about the marginal distributions from the scale-free dependence structure. Darsow et al. (1992) extent this approach to Markov processes. By coupling different marginal distributions with different copula functions, copula-based time series models are able to model a wide variety of marginal behaviors (such as skewness and fat tails) and dependence properties (such as clusters, positive or negative tail dependence). Chen and Fan (2006) develop a two-step estimation procedure for parametric copula functions and make this methodological approach usable for economic applications. By the formulation of a so called generalized semiparametric regression transformation model they derive the basis for a wide range of applications in behavioral economics.

The rest of the paper is organized as follows. Section 2 reviews the methodological concept of copula-based Markov processes and the two-step estimation procedure of Chen and Fan (2006). Furthermore, a definition of conditional variability of temporal dependence and a definition of conditional temporal dependence is derived. Section 3 presents an economic model of stock market confidence and section 4 deals with the VIX data. Section 5 outlines the statistical results, whereas section 6 presents economic implications. Section 7 concludes. Tables and technical details are relegated to the appendix.

2 Methodology

Copula-based first-order Markov processes build the main methodological approach for the economic model and its implications. Based on Sklar's general copula theory (see Sklar (1959)) Darsow et al. (1992) convey this methodology to first-order Markov processes. Chen and Fan (2006) derived a consistent estimation procedure of the first-order case and Ibragimov (2009) provides a charecterization for higher-order Markov processes.

Let $\{Y_t\}$ be a stationary first-order Markov process with continuous state space. Then the joint distribution function $H(y_{t-1}, y_t) = P(Y_{t-1} \leq y_{t-1}, Y_t \leq y_t)$, $(y_{t-1}, y_t) \in \mathbb{R}^2$, of Y_{t-1} and Y_t completely determines the stochastic

properties of $\{Y_t\}$. Due to Sklar's theorem, it is possible to express $H(y_{t-1}, y_t)$ in terms of the marginal distribution $G(y_t) = P(Y_t \leq y_t)$, $y_t \in \mathbb{R}$, of Y_t and the dependence function of Y_{t-1} and Y_t. This dependence function

$$C(G(y_{t-1}), G(y_t)) = H(y_{t-1}, y_t) \tag{1}$$

is known under the name of copula. Hence, $C(u_{t-1}, u_t) = P(U_{t-1} \leq u_{t-1}, U_t \leq u_t)$, $(u_{t-1}, u_t) \in [0,1]^2$, is the joint distribution function of the two random variables $U_{t-1} = G(Y_{t-1})$ and $U_t = G(Y_t)$. $h(\cdot,\cdot)$, $c(\cdot,\cdot)$ and $g(\cdot)$ are the associated (joint) density functions. We will consider in this paper three frequently used copulas (Gauss, Clayton, Frank) and one rarely used copula (Fang). For details see the appendix. One obvious feature of the copula-based time series approach is the possibility to seperate the time dependence structure from the marginal distribution. Especially in Economics this issue becomes important, due to plenty of economic information reflected by the marginal distribution (see section 4). We consider the following set of assumptions:

(A1) $\{Y_t\}_{t=1}^n$ is a sample of a stationary first-order Markov process generated from the true marginal distribution $G(\cdot)$ - which is invariant and absolutely continuous with respect to the Lebesgue measure on the real line - and the true parametric copula $C(\cdot,\cdot;\alpha)$ - which is absolutely continuous with respect to the Lebesgue measure on $[0,1]^2$.
(A2) $G(\cdot)$ and the d-dimensional copula parameter $\alpha \in \mathbb{R}^d$ are unknown.
(A3) $C(\cdot,\cdot;\alpha)$ is neither the Fréchet-Hoeffding upper bound ($C(u_{t-1}, u_t) = \min(u_{t-1}, u_t)$) nor the lower bound ($C(u_{t-1}, u_t) = \max(u_{t-1} + u_t - 1, 0)$).

If (A3) would not be true, it is well-known that Y_t would be almost surely a monotonic function of Y_{t-1}. Therefore, the resulting time series would be deterministic and in case of stationarity, $Y_t = Y_{t-1}$ for the upper bound and $Y_t = G^{-1}(1 - G(Y_{t-1}))$ for the lower bound would follow. We abandon from these cases to focus on stochastic samples of stationary first-order Markov processes. Due to Sklar's Theorem of equation (1) the copula density function $c(u_{t-1}, u_t; \alpha) = \frac{\partial^2 C(u_{t-1}, u_t; \alpha)}{\partial u_{t-1} \partial u_t}$ equals $\frac{h(y_{t-1}, y_t)}{g(y_{t-1}) \cdot g(y_t)}$. Hence, the conditional density of y_t given $y_{t-1}, \ldots, y_1$ is

$$h(y_t | y_{t-1}) = g(y_t) c(G(y_{t-1}), G(y_t); \alpha) \ . \tag{2}$$

As far as the conditional density is a function of the copula and the marginal, the v_t-th, $v_t \in [0,1]$, conditional quantile Q_{v_t} of y_t given y_{t-1} is a function of the copula and the marginal,

$$Q_{v_t}(y_t | y_{t-1}) = G^{-1}\left(C_{t|t-1}^{-1}[v_t | G(y_{t-1}); \alpha]\right) \ . \tag{3}$$

$C_{t|t-1}(u_t|u_{t-1};\alpha) = P(U_t \leq u_t|U_{t-1} = u_{t-1}) = \frac{\partial C(u_{t-1},u_t;\alpha)}{\partial u_{t-1}}$ denotes the conditional distribution of U_t given $U_{t-1} = u_{t-1}$, which we assume to exist. Therefore, $C^{-1}_{t|t-1}[v_t|G(y_{t-1});\alpha]$ is the v_t-th conditional quantile of u_t given u_{t-1}. Considering assumption (A2) the unknown marginal distribution $G(\cdot)$ and the unknown copula parameter vector α have to be estimated. Chen and Fan (2006)[1] derive the following semiparametric two-step procedure:

Step 1: Estimate $G(y)$ by the rescaled empirical distribution

$$\widehat{G}(y) = \frac{1}{n+1}\sum_{t=1}^{n} 1\{Y_t \leq y\} \ . \tag{4}$$

Step 2: Estimate the copula parameter vector by

$$\widehat{\alpha} = \arg\max_{\alpha} \frac{1}{n}\sum_{t=2}^{n} \log c(\widehat{G}(Y_{t-1}), \widehat{G}(Y_t);\alpha) \ . \tag{5}$$

$\widehat{\alpha}$ is root-n consistent and has approximately a normal distribution.

According to Chen and Fan (2006) the following generalized semiparametric regression transformation model exists:

$$\Lambda_1(G(Y_t)) = \Lambda_2(G(Y_{t-1})) + \nu_t \ , \ \ E(\nu_t|Y_{t-1}) = 0 \tag{6}$$

$\Lambda_1(\cdot)$ is a parametric increasing function of U_t, $\Lambda_2(u_{t-1}) := E(\Lambda_1(U_t)|U_{t-1} = u_{t-1})$, and the conditional density of ν_t given $U_{t-1} = u_{t-1}$ is

$$f_{\nu_t|U_{t-1}=u_{t-1}}(\nu_t) = c(u_{t-1}, \Lambda_1^{-1}(\nu_t + \Lambda_2(u_{t-1}));\alpha) \Big/ \frac{d\Lambda_1(\nu_t + \Lambda_2(u_{t-1}))}{d\nu_t} \ . \tag{7}$$

It follows in general

$$\Lambda_2(u_{t-1}) = E(\Lambda_1(U_t)|U_{t-1} = u_{t-1}) = \int_0^1 \Lambda_1(u_t)c(u_{t-1}, u_t;\alpha)du_t \tag{8}$$

and for the special case of identity mapping $\Lambda_1(u_t) = u_t$

$$\Lambda_2(u_{t-1}) = E(U_t|U_{t-1} = u_{t-1}) = 1 - \int_0^1 C_{t|t-1}(u_t|u_{t-1};\alpha)du_t \ . \tag{9}$$

In particular, the variability of temporal dependence between the random variables U_{t-1} and U_t is important in copula-based time series Econometrics. Generally, by selecting a bivariate copula it is possible to describe the variability of the dependence structure between two random variables X and Y conditional on the realization level of these random variables. Analogously, in case of Markov processes copulas describe the conditional temporal dependence structure. Conditional on the realization level

[1]Instead of using the rescaled empirical distribution function, one could use an adequate kernel estimator of the distribution function. Furthermore, they offer an appropriate bootstrap method to construct statistical inference procedures for the estimated quantiles.

of U_{t-1} and U_t the variability of temporal dependence varies, i.e. temporal dependence is heteroskedastic. Bivariate tail dependence is one way to focus on variability of temporal dependence. This concept relates to the amount of dependence in the lower-quadrant tail or the upper-quadrant tail of a bivariate distribution (see e.g. Joe (1997)) and is relevant for dependence in extreme values. A copula has lower tail dependence if $\lambda_L \in (0, 1]$, where $\lambda_L = \lim_{u \to 0} P(U_{t-1} \leq u | U_t \leq u)$, and no lower tail dependence if $\lambda_L = 0$. Similarly, a copula has upper tail dependence if $\lambda_U \in (0, 1]$, where $\lambda_U = \lim_{u \to 1} P(U_{t-1} > u | U_t > u)$, and no upper tail dependence if $\lambda_U = 0$.

In order to receive an intuitive impression of the variability of the dependence between two random variables in the entire domain of attraction (and not only in extreme regions like the tail dependence concept examines), contour plots of the copula density are frequently used (see e.g. Härdle and Okhrin (2010)). According to the intuitive approach we use a proxy for variability of temporal dependence in a probabilistic context.

Consider the unit square with realizations u_{t-1} of the random variable U_{t-1} on the abscissa and realizations u_t of the random variable U_t on the ordinate. The diagonal through the origin separates the unit square into two regions. "A" marks the region above the diagonal and "B" stands for the region below the diagonal. The contour line of the bivariate copula density function on a specific density level, say p, in Region "A" can be mirrored by the diagonal to region "B". A straightforward proxy for conditional variability of temporal dependence between U_{t-1} and U_t given a value u_{t-1} is the distance from point $b = (u_{t-1}; u_t)$ on the contour line in Region "B" and its mirrored point $a = (u_t; u_{t-1})$ on the contour line in Region "A". Assume that u_{t-1}, the copula C and p are given, the calculation of $u_t = C^{-1}_{t|t-1}(p|u_{t-1}; \alpha)$ is possible. Because of the mirrored symmetry of the contour lines around the diagonal, the distance between "a" and "b" is the length of the diagonal of a square with side length $u_{t-1} - u_t$, $u_t = C^{-1}_{t|t-1}(p|u_{t-1}; \alpha)$, and equals a conditional first difference. Furthermore, the following assumption will be considered:

(A4) The regarded contour lines do not tangent the diagonal of the unit square.

If (A4) is not true the variability according to the following proxy always converges to zero for values of u_{t-1} and u_t which cause contour lines near by the diagonal. If (A4) is true the contour line of Region "B" tangents the abscissa in point $(u^*_{t-1}; 0)$. Analogously, the contour line of Region "A" tangents the ordinate in point $(0; u^*_t)$. Consequently, the following definition of conditional variability of temporal dependence will be considered:

Definition 1 :
The proxy for conditional variability of temporal dependence between the random variables U_{t-1} and U_t given u_{t-1}, a copula C and a p-density level is defined by:

$$var(U_{t-1}, U_t|u_{t-1}, C, p) := \begin{cases} |u_{t-1} - C^{-1}_{t|t-1}(p|u_{t-1}; \alpha)|\sqrt{2} & , \text{ if } u_{t-1} \geq u^*_{t-1} \\ |u_{t-1} - C^{-1}_{t|t-1}(p|u^*_{t-1}; \alpha)|\sqrt{2} & , \text{ if } u_{t-1} < u^*_{t-1} \end{cases}$$

u^*_{t-1} marks the value on the u_{t-1} axis of the unit square of u_{t-1} and u_t, where the contour line tangents the u_{t-1} axis.

Hence, by selecting a specific copula and density level p it is possible to describe the variability of temporal dependence between U_{t-1} and U_t conditional on the level of u_{t-1}. The higher the variability of conditional temporal dependence the lower the temporal dependence itself. Therefore, the following definition of conditional temporal dependence will be considered:

Definition 2 :
The proxy for conditional temporal dependence between the random variables U_{t-1} and U_t given u_{t-1}, a copula C and a p-density level is defined by:

$$dep(U_{t-1}, U_t|u_{t-1}, C, p) := var(U_{t-1}, U_t|u_{t-1}, C, p)^{-1}$$

Although the copula parameters - which can be transformed to the correlation coefficient according to Kendall or Spearman - are treated as time invariant (α and not α_t) the copula itself allows for a variation of temporal dependence conditional on the level of u_{t-1}.

3 Economic model

Confidence in terms of reasonable expectation is a long-standing issue in Economics. The higher the confidence, the easier to find a transaction partner and superiority of market conditions. Especially the latest financial market crisis - which could be described as a confidence crisis - bare the fundamental function of confidence in Economics. Much effort has been placed on the recovery of confidence in the stock market by means of stock market uncertainty reduction. If uncertainty reduction is intended to increase confidence,

a stable relationship between these variables should exist and motivate uncertainty reduction. To the best of my knowledge the existence of a stable relationship justified by real world data is not described in the literature. Therefore, the aim of this paper is the derivation of a descriptive model of stock market confidence which is empirically stable.

Consider the random variable $Y_{i,t}^{\star}$ which stands for the sensation of stock market uncertainty of investor $i = 1, \ldots, m$ at time $t = 1, \ldots, n$. Due to the heterogeneity of investors different valuation rules of uncertainty are possible. Hence, a microeconomic founded structural description of $Y_{i,t}^{\star}$ for all i seems not to exist. Instead of a structural form we examine a time series approach. The current individual feeling of uncertainty is dependent on previous uncertainty $Y_{i,t-1}^{\star}$ and an unpredictable uncertainty shock which is induced by changing stock market conditions. Applying the marginal distribution G_i of $Y_{i,t}^{\star}$ leads to the quantile $U_{i,t}^{\star} = G_i(Y_{i,t}^{\star})$. Without loss of generalization the generalized semiparametric regression transformation model of equation (6) can be applied for the case of identity mapping in the following way,

$$U_{i,t}^{\star} = \Lambda_3(U_{i,t-1}^{\star}) + e_{i,t}^{\star} \ , \tag{10}$$

where $\Lambda_3(u_{i,t-1}^{\star}) := E(U_{i,t}^{\star}|U_{i,t-1}^{\star} = u_{i,t-1}^{\star}) = 1 - \int_0^1 C_{t|t-1}(u_{i,t}^{\star}|u_{i,t-1}^{\star}; \alpha_i)du_{i,t}^{\star}$ and $E(e_{i,t}^{\star}|Y_{i,t-1}^{\star}) = 0$ holds. It is assumed that the copula is invariant across individuals i and time t, whereas the copula parameter individually varies. Consequently, the individual sensation of stock market uncertainty is explained by a first-order copula-based Markov process.

With respect to the level of individual sensation of stock market uncertainty the market participants choose an investment strategy. By selecting a specific strategy the investor believes that this strategy is relative beneficial to treat available information and to achieve relative high profits. Hence, uncertainty itself is not evaluated but acts like a determinant of the strategy choice. Regarding the number of alternative investment strategies in face of stock market uncertainty we reduce the dimension to two. Aoki and Yoshikawa (2007) show that about 95 percent of the total market participants belong to the two largest subgroups of agents by types. With two largest clusters, there are two regimes; one with a cluster of investors with strategy 1 as the largest share, and the other with a cluster of investors using strategy 2 as the largest share. Namely, fundamentalists dominate the market in regime 1 and chartists dominate the market in regime 2. Corresponding to Fama (1970) these strategies reflect theoretically inefficient and weak efficient markets. In case of inefficient markets asset prices do not reflect historical price information and lead to more stock market uncertainty. Thus, it is possible to earn excess returns by being a chartist in case of higher uncertainty. On

the other hand, if the market is rather weak efficient, asset prices reflect historical price information and leads to less stock market uncertainty. Then it is possible to achieve excess returns by being a fundamentalist in case of lower uncertainty. Hence, the decision in period t of a market participant i being a fundamentalist ($y_{i,t} = 1$) or a chartist ($y_{i,t} = 0$) is determined by the individual sensation of stock market uncertainty $y^{\star}_{i,t}$:

$$y_{i,t} = \begin{cases} 0 & , \text{ if } y^{\star}_{i,t} \geq \varphi_i \\ 1 & , \text{ if } y^{\star}_{i,t} < \varphi_i \end{cases}$$

φ_i marks the individual threshold for being a chartist or a fundamentalist. If investors have confidence in their strategy $y_{i,t}$ they will not change their strategy. From this it follows that the variability of $y_{i,t}$ over time is low. A low variability of the evolution of $y_{i,t}$ implies low variability of the progress of $y^{\star}_{i,t}$. Hence, low variability of the process $y^{\star}_{i,t}$ is achieved, if the conditional variance of $e^{\star}_{i,t}$, $V(e^{\star}_{i,t}|Y^{\star}_{i,t-1})$, is low. Therefore, $V(e^{\star}_{i,t}|Y^{\star}_{i,t-1})$ is dependent on the level of $y^{\star}_{i,t}$ and the process $y^{\star}_{i,t}$ itself is autoregressive and heteroskedastic. If the conditional variance of $e^{\star}_{i,t}$, $V(e^{\star}_{i,t}|Y^{\star}_{i,t-1})$, is low changes of market conditions are moderate. This leads to the conclusion that high individual stock market confidence (the believe of moderate changes of market conditions) is connected to low variability of stock market uncertainty which causes high temporal dependence of individual stock market uncertainty, and vice versa. This argumentation is in line with the general definition of confidence according to Luhmann (2000) and is understood as the assumption of an expected development (reasonable expectation) with the availability of different alternatives of taking actions. Analog to Definition 1 and 2 the definition of individual stock market confidence is:

Definition 3 :
The proxy for individual stock market confidence $\kappa_{i,t|t-1}$ equals the conditional temporal dependence between individual stock market uncertainty $Y^{\star}_{i,t-1}$ and $Y^{\star}_{i,t}$ given $y^{\star}_{i,t-1}$, a copula C, a p-density level and is defined by:

$$\kappa_{i,t|t-1} := dep(G_i(Y^{\star}_{i,t-1}), G_i(Y^{\star}_{i,t})|G_i(y^{\star}_{i,t-1}), C, p)$$

G_i marks the marginal distribution of $Y^{\star}_{i,t}$.

An analytical derivation of a specific structure of heteroskedasticity and with it a specific structure of conditional temporal dependence necessarily needs behavioral assumptions about investors. Various assumptions could be plausible, so we concentrate on the methodological and economic framework which will be driven by data. In that sense this investigation is descriptive.

Leaving the individual level by averaging over all market participants leads to the market structure S_t, $0 \leq S_t \leq 1$, which is determined by $S_t = m^{-1} \sum_{i=1}^{m} y_{i,t}$. $S_t = 0$ stands for a zero share of fundamentalists and $S_t = 1$ stands for a zero share of chartists. If $S_t < 0.5$ chartists dominate the market and if $S_t > 0.5$ fundamentalists dominate the market. According to Aoki and Yoshikawa (2007) the fluctuation of stock prices becomes greater when chartists dominate than when fundamentalists dominate the market. Hence, the fluctuation of stock prices STD_t is a decreasing function f of the market structure S_t

$$STD_t = f(S_t) \ . \tag{11}$$

Section 4 provides a definition of stock market uncertainty over the whole market according to Bloom (2009) as a random variable Y_t in terms of expected (E_t) future stock market variability (STD_{t+1}), hence,

$$Y_t = E_t(STD_{t+1}) \ . \tag{12}$$

The quantile of Y_t is $U_t = G(Y_t)$, whereas G marks the marginal distribution of Y_t. If the market participants make rational expectations, the process of market wide uncertainty Y_t mimic the process of individual uncertainty $Y_{i,t}^{\star}$ of equation 10. Consequently, stock market uncertainty over the whole market equals a first-order Markov process

$$U_t = \Lambda_4(U_{t-1}) + e_t \ , \tag{13}$$

where $\Lambda_4(u_{t-1}) := E(U_t|U_{t-1} = u_{t-1}) = 1 - \int_0^1 C_{t|t-1}(u_t|u_{t-1};\alpha)du_t$ and $E(e_t|Y_{t-1}) = 0$ holds. The individual description of stock market uncertainty of equation (10) leads canonical to a market wide description of stock market uncertainty of equation (13). Analog to Definition 3 the following definition of market wide stock market confidence will be considered:

Definition 4 :
The proxy for market wide stock market confidence $\kappa_{t|t-1}$ equals the conditional temporal dependence between market wide stock market uncertainty Y_{t-1} and Y_t given y_{t-1}, a copula C, a p-density level and is defined by:

$$\kappa_{t|t-1} := dep(G(Y_{t-1}), G(Y_t)|G(y_{t-1}), C, p)$$

G marks the marginal distribution of Y_t.

Knowing the correct copula leads to the description of stock market confidence. Hence, copula-based Markov models seem to be useful tools in confidence theory.

4 US stock market uncertainty

The empirical application of this paper deals with the issue of US stock market confidence. By Definition 2 stock market confidence equals the conditional temporal dependence of stock market uncertainty. According to Bloom (2009) a proxy for stock market uncertainty is the volatility index VIX of the S&P 500 created by the Chicago Board Options Exchange (CBOE). The VIX reflects the market expectation in period t of future stock market variability, $t+1$, whereas stock market variability can be quantified by the standard deviation of the S&P 500.

We use data from Thompson Datastream for the period January 1990 to August 2010. Define the logarithmic growth $r_\tau := \log sp_\tau - \log sp_{\tau-1}$ as the daily closing return of the S&P 500, sp_τ, with daily time index $\tau = 1, 2, \ldots, m$. STD_t marks the standard deviation of the daily returns during a month $t = 1, 2, \ldots, n$. VIX_t denotes the monthly closing price of the VIX. Hence, the number of observed month is $n = 248$.

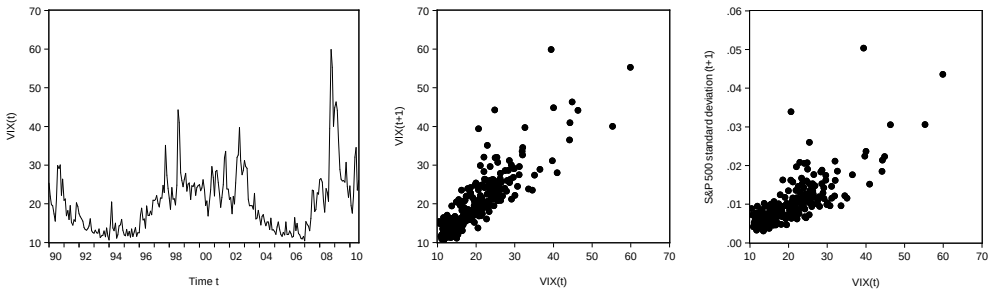

Figure 1: Line graph of the VIX (left) and scatter plots of the level data (middle: $\widehat{\rho}_s = 0.885$; right: $\widehat{\rho}_s = 0.803$)

In Figure 1 $\widehat{\rho}_s$ stands for the estimated Spearman correlation coefficient. By inspecting the scatter plots of level data, one merges information about marginal distributions and temporal dependence structures, respectively, and considers scale-conditioned dependence structures. Transforming the scale-dependent level data into scale-free data by using the empirical distribution leads to empirical quantiles. The scatter plots of the empirical quantiles (Figure 2) concentrate on the pure dependence structures (not influenced by marginal distributions). The left scatter plot gives an impression of the temporal dependence structure between VIX_t and VIX_{t+1} and the right scatter plot shows the cross dependence structure between VIX_t and STD_{t+1}.

Comparing Figure 1 and 2 it is obvious that the marginal distributions

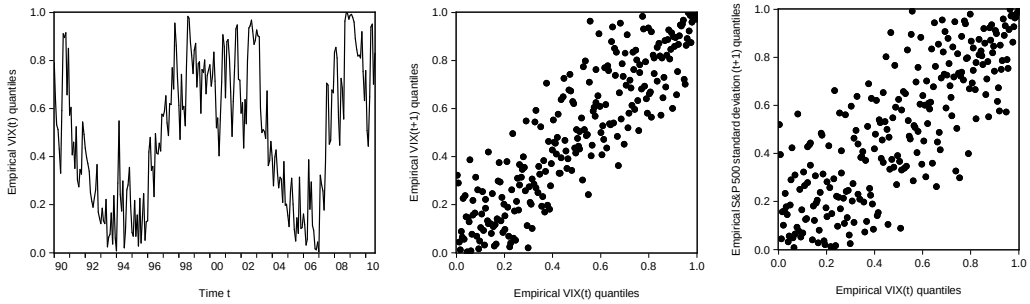

Figure 2: Line graph of the empirical VIX quantiles (left) and scatter plots of the empirical quantiles

contain information which are not important with regard to the temporal dependence structures. Basically the uncertainty proxy VIX depends on options and belongs to the category of assets, whereas stocks are the driving forces for options. Consequently, the marginal distributions of the VIX as well as the STD incorporate information about the stock prices.

According to neoclassical theory stock prices are essentially dependent on the real economy. The correct stock price equals the future profits of the stock (present value), which are emitted by firms. Therefore, neoclassical theory postulates a general equilibrium of stock prices determined by supply and demand of the real economy (Diamond (1967)). From that point of view stock prices, as well as consumption and production, are dependent on preferences and technologies and are connected to the real economy. The causality runs from the real economy to stock prices under neoclassical assumptions and implies no stock market bubbles. This consequence obviously thwarts reality.

In return the literature concerning "financial accelerator" (see Minsky (1957), Bernanke and Gertler (1989), Bernanke et al. (1996)) emphasizes the importance of financial markets for the real economy. Macroeconomic risk always emerges when valuation risk exceeds a certain level and spills over to the real economy. The resulting economic downturn leads to even higher uncertainty so that risk spreads even rise and a further economic downturn emerges. The adverse feedback loop between financial markets and the real economy is the so-called financial accelerator. How strong this feedback effect is, depends, among other factors, on the quality of assets that serve as collateral for liabilities. In the recent financial crisis the quality of these assets was often poor and the economic downturn diminished the value of these assets further.

Not only in light of the financial accelerator literature the neoclassical the-

ory appears to be theoretically doubtful. Even from an empirical perspective the neoclassic seems to be unrealistic. Based on the "variance-bound tests" Shiller (1981) concludes that due to too high stock price volatility the property of exponential growth of real economy variables is not observed in financial markets. Accordingly, the discussion of "excess volatility" is boosted by empirical facts. Since Mandelbrot (1963) the distribution of stock prices is an important issue in the literature. At present it is widely accepted that stock price changes follow power-law distributions (see e.g. Gabaix et al. (2003, 2007), Huang and Solomon (2001)) and not exponential distributions like real economy variables. Because of the great number of market participants with individual strategies and the great number of market transactions the quantity of micro growth events on stock markets is high. This issue leads to power-law distributions of returns. On the other hand the quantity of growth events in the real economy is rather small and causes exponential distributions of real indicators. Hence, Figure 1 is strongly affected by the power-law property of the return distribution. In order to extract the pure temporal dependence structure, one has to abstract from economic founded information of the power-law property of the marginal distribution.

5 Statistical results

Table 1 of the appendix contains estimation results and further technical details concerning the regarded copulas. The hypothesis that the Fang copula captures the time dependence structure of the VIX can not be rejected on any plausible level of significance. Based on empirical tests the correctness of the remaining copulas can be rejected. Hence, the Fang copula seems to be the only correct copula in the set of copulas.

In order to test the correctness of a copula in a first-order Markov framework, consider the following hypothesis test of interest:

H_0: $\{Y_t\}$ is a first-order Markov process with copula C

H_0 is equivalent to

$H_0' : V_t = C_{t|t-1}(U_t|U_{t-1};\alpha)$ is uniformly on $[0,1]$ distributed and not autocorrelated

We reject H_0 if H_0' is rejected. According to the distributional transform the random variable V_t must be uniformly distributed on $[0,1]$ (see e.g. Ferguson (1967)) and the quantile transform - which is exactly the inverse of the

distributional transform - satisfies $U_t = C^{-1}_{t|t-1}(V_t|U_{t-1};\alpha)$, a.s.. In case of a misspecified copula $C^\star$ the resulting conditional copula $C^\star_{t|t-1}$ is misspecified. Consequently, $V_t^\star = C^\star_{t|t-1}(U_t|U_{t-1};\alpha)$ is not anymore the true distributional transform of U_t given U_{t-1} and $V_t^\star$ contains residual information concerning the time dependence between U_t and U_{t-1}. Hence, the residual time dependence yields to time dependence between $V_t^\star$ and $V_{t-l}^\star$, $l \in \{1, 2, \ldots, n\}$. In a practical situation realizations for U_t are unobservable and will be substituted due to the empirical distribution, as it is known to converge to the true distribution. Denote the nonparametrical estimation of the realizations based on the empirical distribution with $\widehat{u}_t = \frac{n+1}{n}\widehat{G}(y)$, where $\widehat{G}(y)$ stands for the rescaled empirical distribution of equation (4). The unknown copula parameter vector α can be substituted by the consistent ML-estimator $\widehat{\alpha}$ described in section 2. Straightforward, $\widehat{V}_t = C_{t|t-1}(\widehat{U}_t|\widehat{U}_{t-1};\widehat{\alpha})$ is uniformly distributed on $[0,1]$ and is an adequate substitute for V_t. Even in a quasi maximum likelihood situation, where $\widehat{\alpha}$ is consistent, the fact of a misspecified copula leads to the same conclusions for $\widehat{V}_t^\star$ like $V_t^\star$ because of the incorrect modelling of temporal dependence.

Based on the goodness-of-fit test (see Table 1) a discrimination of the non-Fang copulas is impossible. The remaining time dependence between the realizations $\widehat{v}_t = C_{t|t-1}(\widehat{u}_t|\widehat{u}_{t-1};\widehat{\alpha})$ and $\widehat{v}_{t-1} = C_{t|t-1}(\widehat{u}_{t-1}|\widehat{u}_{t-2};\widehat{\alpha})$ leads to the unique selection of the Fang copula.

To control for the robustness of the superiority of the Fang copula ascertained by the joint hypothesis test, we use the results and technical notes of Table 2 (see appendix). Consider the nonparametric estimated conditional quantiles $\widehat{u}_t$, which contain no information about a parametric copula. On the other hand if a parametric copula is selected, it is possible to calculate copula implied conditional quantiles which are used to construct a copula-based confidence interval of the conditional quantiles. Regarding the level of significance ϵ it follows for the upper interval bound

$$\widehat{u}_{t,\overline{\epsilon}} = C^{-1}_{t|t-1}(1-\epsilon/2|\widehat{u}_{t-1};\widehat{\alpha}) \tag{14}$$

and for the lower interval bound

$$\widehat{u}_{t,\underline{\epsilon}} = C^{-1}_{t|t-1}(\epsilon/2|\widehat{u}_{t-1};\widehat{\alpha})\,. \tag{15}$$

The ,,overall region" of Table 2 reports the estimated error rates for all conditional quantiles $\widehat{u}_t$, $t = 2, \ldots, n$. Therefore, given $\widehat{u}_t$, $\widehat{u}_{t,\overline{\epsilon}}$ and $\widehat{u}_{t,\underline{\epsilon}}$ copula-based error rates are:

$$\widehat{\epsilon}_{overall} = 1 - \left(\frac{1}{n-1}\sum_{t=2}^{n} 1\{\widehat{u}_{t,\underline{\epsilon}} \leq \widehat{u}_t \leq \widehat{u}_{t,\overline{\epsilon}}\}\right) \tag{16}$$

Focusing the tails of the bivariate copula leads to further information about the copula adequacy. The calculation of the estimated error rates of the ,,lower region" of Table 2 is analog to (16), but only valid for lower $\widehat{u}_t$. We define the region for lower quantiles by $\widehat{u}_t < \pi$ with $\pi = 1/3$.[2] According to

$$\widehat{\epsilon}_{lower} = 1 - \left(\frac{1}{\underline{n}} \sum_{t=2}^{\underline{n}} 1\{\widehat{u}_{t,\underline{\epsilon}} \leq \widehat{u}_t \leq \widehat{u}_{t,\overline{\epsilon}} \text{ and } \widehat{u}_t < \pi\} \right) \tag{17}$$

the estimated error rate for the lower region are computed. Consequently, for the ,,upper region"

$$\widehat{\epsilon}_{upper} = 1 - \left(\frac{1}{\overline{n}} \sum_{t=2}^{\overline{n}} 1\{\widehat{u}_{t,\underline{\epsilon}} \leq \widehat{u}_t \leq \widehat{u}_{t,\overline{\epsilon}} \text{ and } \widehat{u}_t > 1 - \pi\} \right) \tag{18}$$

holds. $\underline{n}$ stands for the cases with $\widehat{u}_t < \pi$ and $\overline{n}$ for the cases with $\widehat{u}_t > 1 - \pi$. Table 2 shows additionally the root mean squared error of the true and estimated error rates separated according to different regions. The Fang copula is also superior with respect to this criterion. Obviously, the data

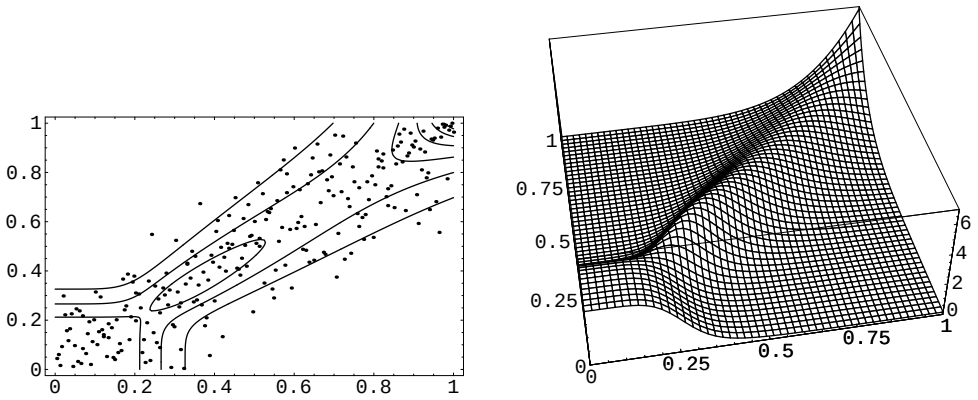

Figure 3: Contour with scatter plot of the empirical quantiles (left) and perspective plot (right) of the Fang copula, with $\widehat{\alpha} = 0.173$ and $\widehat{\beta} = 0.9994$

obey tail dispersion, which can not be modelled by the Gauss and Clayton copula. Even the Frank copula as a representative of a copula with symmetric tail dispersion is inferior in comparison to the Fang copula. Only the Fang copula is able to deal with asymmetric tail dispersion.

Summing up the hypothesis tests and the robustness checks the correctness of the Fang copula is indicated. This is likewise supported by the copula implied Spearman correlation coefficient of the Fang copula according to

[2]Also for varying π similar error rates are observed.

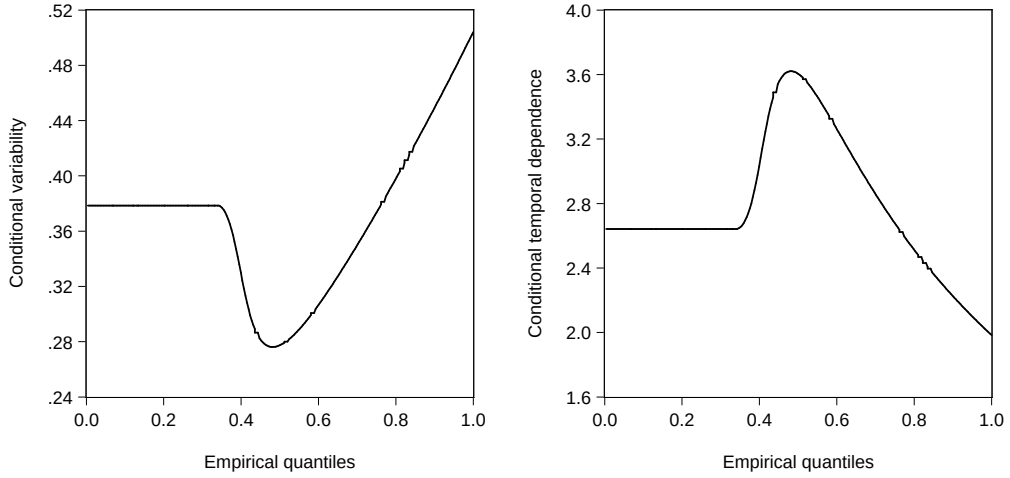

Figure 4: Conditional variability $\widehat{var}(U_{t-1}, U_t|\widehat{u}_{t-1}, C, p)$, conditional temporal dependence $\widehat{dep}(U_{t-1}, U_t|\widehat{u}_{t-1}, C, p)$, $p = 0.05$, and empirical quantiles $\widehat{u}_{t-1}$

equation (22), $\widehat{\rho}_s(\widehat{\alpha}, \widehat{\beta}) = 0.81$, which is in the neighbourhood of the non-parametric estimated coefficient, $\widehat{\rho}_s(\widehat{u}_{t-1}; \widehat{u}_t) = 0.88$.

Identifying the Fang copula as the appropriate copula, Figure 3 allows for a graphical inspection of its density based on the parameter estimates (see Table 1). The copula shows more density mass in in the lower and upper tails and seems to exhibit asymmetric tail dispersion. Formalizing this impression in terms of the conditional variability proxy of Definition 1, Figure 4 confirms this intuition on the $p = 0.05$ density level.[3] The estimated conditional variability proxy $\widehat{var}(U_{t-1}, U_t|\widehat{u}_{t-1}, C, p)$ is computed according to

$$\begin{cases} |\widehat{u}_{t-1} - C^{-1}_{t|t-1}(p|\widehat{u}_{t-1}; \widehat{\alpha}, \widehat{\beta})|\sqrt{2} & , \text{ if } \widehat{u}_{t-1} \geq \widehat{u}^*_{t-1} \\ |\widehat{u}_{t-1} - C^{-1}_{t|t-1}(p|\widehat{u}^*_{t-1}; \widehat{\alpha}, \widehat{\beta})|\sqrt{2} & , \text{ if } \widehat{u}_{t-1} < \widehat{u}^*_{t-1} \end{cases} \tag{19}$$

whereas $\widehat{u}_{t-1}$ marks the ascending sorted empirical quantiles and $\widehat{u}^*_{t-1} = 0.3785$ holds. According to Definition 2 the probabilistic conditional temporal dependence proxy is calculated as

$$\widehat{dep}(U_{t-1}, U_t|\widehat{u}_{t-1}, C, p) = \widehat{var}(U_{t-1}, U_t|\widehat{u}_{t-1}, C, p)^{-1}. \tag{20}$$

[3]For alternative density levels similar results can be derived.

Due to the fact that the inverse of the conditional distribution $C_{t|t-1}^{-1}$ does not exist in closed form, $u_t = C_{t|t-1}^{-1}(p|u_{t-1}; \alpha, \beta)$ can be obtained from the equation $p = C_{t|t-1}(u_t|u_{t-1}; \alpha, \beta)$ using a numerical root-finding routine (here: Newton's procedure). Hence, numerical imprecisenesses of the root-finding routine lead to small jags in the curve progression of Figure 4 and can be neglected. Figure 4 is a very important statistical result and have deep impact on economic implications of section 6.

6 Economic implications

The main economic implication of the economic model and the statistical investigation is that US stock market confidence is dependent on the level of US stock market uncertainty in a stable and nonlinear manner.[4] Transforming the quantile treatment based on the empirical distribution of Figure 4 to level data of US stock market uncertainty in terms of the VIX, the following Figure 5 summarizes the dependence structure between confidence and uncertainty. Conditional on the uncertainty level confidence varies. The latest

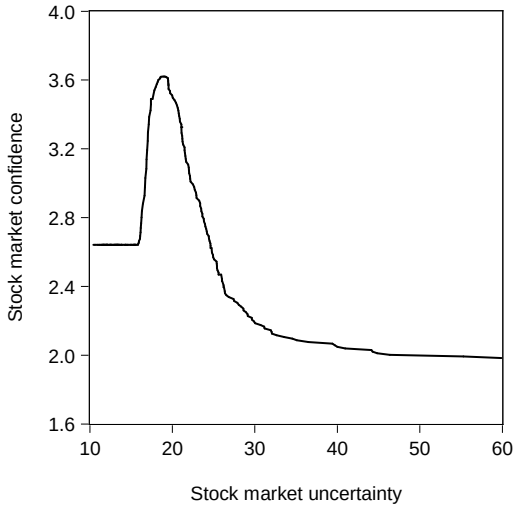

Figure 5: Dependence between stock market uncertainty (VIX) and stock market confidence, $p = 0.05$

financial crisis which can be characterized as a confidence crisis shows actu-

[4]Concerning the definition of stock market confidence see section 3 and for the empirical implementation see section 5.

ally the importance of confidence in the financial market. Very low interest rates by the US Federal Reserve Bank, a planned increase of the minimum reserve of banks and temporary suspension of short-sales are only prominent examples of institutional reactions which are intended to reduce uncertainty and finally increase confidence. This policy is empirically indicated by the descriptive model.

For low uncertainty levels an uncertainty-reducing policy with the aim of recovery of confidence is inefficient. Lower uncertainty leads not to higher confidence due to the rigidity of confidence toward uncertainty. Based on the Fang copula estimation and a density level $p = 0.05$ visualized in Figure 5 the region of inefficient policy is up to a VIX value of approximately 16. The region of uncertainty levels between 16 and 19 is also characterized by an inefficient and counterproductive uncertainty-reducing policy. Lower uncertainty leads to lower confidence. Values larger than 19 agree with an efficient uncertainty-confidence policy. A decrease of uncertainty increases confidence. Furthermore, the confidence behavior of the market is asymmetric in terms of extreme uncertainty values. Uncertainty values larger than 25 correspond to confidence below regimes characterized by very low uncertainty and saturated confidence. In that sense this region of uncertainty is abnormal and could be interpreted as a call for institutional actions. This situation is currently observable and according to the model and empiricism justified. For very large uncertainty values, say greater than 30, there is a kind of uncertainty trap evident. In that region a diffident uncertainty-reducing policy - which causes only a small uncertainty decline - is relative inefficient, because of low sensitivity of confidence. Hence, in times of "excess misconfidence" an aggressive uncertainty-reducing policy is indicated. Even this model implication supports the current way of policy making.

A second implication of the model is shown in Figure 3. Much density mass is observable for small and large uncertainty values VIX_{t-1} and VIX_t. This is a result of the well known issue of volatility clusters of financial market data. More interesting is the fact that the high confidence region corresponds to low density mass. Although, confidence is high VIX values feature no cluster in the high confidence region. Uncertainty as an outcome of investment behaviour is apparently not driven by stock market confidence. According to basic financial theory profit expectations are the driving forces of investment behavior and finally uncertainty. Hence, the uncertainty level linked to high confidence is not compatible with the uncertainty level of the expected profits desired by the market. The political implication of this circumstance is that financial market rules should be established, which provoke a coincident of confidence and expected profits on the same uncertainty level. That would mean that profit expectations push the market to an uncertainty

level which is in line with low changes of market conditions (stable market) approximated by the stock market confidence proxy. This is undoubtedly an important issue on the current and future regulation agenda of financial markets.

A by-product of the first-order copula-based Markov characterization of the VIX is the possibility to calculate conditional quantiles for risk management. As shown by Jovanović and Zimmermann (2010) pacifying financial markets by interest rate cuts is part of the US Federal Reserve Bank monetary reaction function for more than 25 years. Hence, in order to implement forward-looking risk management support the statistical framework could be used.

7 Conclusions

The main result of the paper is the derivation of a descriptive model of stock market confidence in a first-order copula-based Markov approach. Based on monthly closing prices of the US volatility index VIX of the Chicago Board Options Exchange - which acts like a proxy for US stock market uncertainty - an empirically stable nonlinear relationship between stock market confidence and stock market uncertainty exists. Hereby the economic interpretation of conditional temporal dependence of uncertainty - defined by the inverse of the variance of temporal dependence conditional on the uncertainty level - is confidence. The parametric copula developed by Fang et al. (2000) captures the structure of heteroskedasticity of the autoregressive VIX process and allows for a description of a stock market confidence proxy.

An uncertainty-reducing policy which intends to increase US stock market confidence is efficient or inefficient dependent of the level of uncertainty itself. For VIX values smaller than 16 an uncertainty-reducing policy is inefficient due to the rigidity of confidence. VIX values between 16 and 19 are also connected to policy inefficiency and moreover counterproductive due to decreasing confidence in face of decreasing uncertainty. Policy efficiency is evident for VIX values larger than 19. In case of uncertainty levels larger than 25 confidence follows which is below the saturation level of low uncertainty regimes and can be characterized as abnormal. This behavior of confidence implies a kind of asymmetry regarding very small and large uncertainty values. Excess misconfidence is evident for uncertainty values larger than 30 and leads to an uncertainty trap. Small uncertainty changes effect confidence only marginal. Hence, in times of excess uncertainty an aggressive uncertainty-reducing policy is indicated. This economic implication of the descriptive model justifies the current way of uncertainty-reducing US

policy which is intended to recover confidence in financial markets.

Another conclusion of the model is that the uncertainty level corresponding to high confidence does not coincide with the uncertainty level which corresponds to profit expectations. Therefore, profit expectations cause market uncertainty which corresponds not to high confidence, i.e. stable market conditions. Regulatory changes should try to establish profit expectations which lead to high confidence.

As a by-product of the paper the copula-based Markov approach is adequate for conditional quantile forecasts and could be used in risk management.

Finally, due to the possibility of a structural description of heteroskedasticity of autoregressive processes commonly found in Economics the methodological framework - particularly the generalized semiparametric regression transformation model of Chen and Fan (2006) - of copula-based Markov processes seems to be a useful tool in behavioral economics in general and could build the basis for behavioral econometrics. If the economic model leads to an autoregressive representation, the description of economic behavior is data driven. Especially the Fang copula allows for the description of various forms of heteroskedasticity constituted by data. Hence, data driven behavioral assumptions could be used in behavioral economics instead of theoretical behavioral assumptions and would bring more reality in behavioral economics.

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Appendix

Copula review:

I. The Gauss copula (e.g. Joe (1997))

$$C(u_{t-1}, u_t; \alpha) = \Phi_\alpha[\Phi^{-1}(u_{t-1}), \Phi^{-1}(u_t)]$$

with the standard normal distribution function $\Phi(\cdot)$, the bivariate normal distribution function $\Phi_\alpha(\cdot, \cdot)$ with means zero and variances 1 and the correlation coefficient $|\alpha| < 1$ is an elliptical copula. Its lower tail dependence parameter is $\lambda_L = 0$ und its upper tail dependence parameter is $\lambda_U = 0$. Therefore, it exhibits neither dependence in the negative tail nor in the positive tail. The copula density function $c(u_{t-1}, u_t; \cdot)$ is:

$$(1-\alpha^2)^{-1/2} \exp\left\{-\frac{1}{2}(1-\alpha^2)^{-1}[u_{t-1}^2 + u_t^2 - 2\alpha u_{t-1}u_t]\right\} \exp\left\{\frac{1}{2}[u_{t-1}^2 + u_t^2]\right\}$$

Due to the linearity of the Gauss copula according to Chen and Fan (2006) $\Phi^{-1}(u_t) = \alpha\Phi^{-1}(u_{t-1}) + \varepsilon_t$ with $\varepsilon_t \sim N(0; \sqrt{1-\alpha^2})$ follows. Consequently, $u_t = \Phi(\alpha\Phi^{-1}(u_{t-1}) + \varepsilon_t)$ and $v_t = \Phi(\varepsilon_t/\sqrt{1-\alpha^2})$ follows.

II. The Clayton copula (Clayton (1978))

$$C(u_{t-1}, u_t; \alpha) = \left(u_{t-1}^{-\alpha} + u_t^{-\alpha} - 1\right)^{-\frac{1}{\alpha}} \quad ,$$

$\alpha > 0$, is an asymmetric Archimedean copula. Its lower tail dependence parameter is $\lambda_L = 2^{-\frac{1}{\alpha}}$ und its upper tail dependence parameter is $\lambda_U = 0$. Therefore, it exhibits greater dependence in the negative tail than in the positive tail. The copula density function is:

$$c(u_{t-1}, u_t; \alpha) = (1+\alpha)\,(u_{t-1}u_t)^{-\alpha-1}\left({u_{t-1}}^{-\alpha} + {u_t}^{-\alpha} - 1\right)^{-2-1/\alpha}$$

The inverse of the conditional distribution is:

$$C_{t|t-1}^{-1}(v_t|u_{t-1}; \alpha) = u_t = [(v_t^{-\alpha/(1+\alpha)} - 1)u_{t-1}^{-\alpha} + 1]^{-1/\alpha}$$

III. The Frank copula (Frank (1979))

$$C(u_{t-1}, u_t; \alpha) = -\frac{1}{\alpha} \log \left(1 + \frac{(e^{-\alpha u_{t-1}} - 1)(e^{-\alpha u_t} - 1)}{(e^{-\alpha} - 1)} \right) \ ,$$

$\alpha = (-\infty, +\infty) \backslash \{0\}$, is a symmetric Archimedean copula. Its lower tail dependence parameter is $\lambda_L = 0$ and its upper tail dependence parameter is $\lambda_U = 0$. Therefore, it exhibits neither dependence in the negative tail nor in the positive tail and shows more tail dispersion than the Gauss copula. The copula density function is:

$$c(u_{t-1}, u_t; \alpha) = \alpha \eta e^{-\alpha(u_{t-1}+u_t)} / [\eta - (1 - e^{-\alpha u_{t-1}})(1 - e^{-\alpha u_t})]^2, \ \eta = 1 - e^{-\alpha}$$

The inverse of the conditional distribution is:

$$C^{-1}_{t|t-1}(v_t | u_{t-1}; \alpha) = u_t = -\alpha^{-1} \log\{1 - (1 - e^{-\alpha}) / [(v_t^{-1} - 1) e^{-\alpha u_{t-1}} + 1]\}$$

In order to allow for a more flexible copula specification the following two parameter copula will be applied.

IV. The Fang copula (Fang et al. (2000))

$$C(u_{t-1}, u_t; \alpha, \beta) = \frac{u_{t-1} u_t}{\left[1 - \beta \left(1 - {u_{t-1}}^{\frac{1}{\alpha}} \right) \left(1 - {u_t}^{\frac{1}{\alpha}} \right) \right]^{\alpha}} \tag{21}$$

considers the parameters $\alpha > 0$ and $0 \leq \beta \leq 1$. When $\beta = 0$, U_{t-1} and U_t are independent. When $\beta = 1$, $C(u_{t-1}, u_t; \alpha, 1)$ in (21) becomes the bivariate Clayton copula. As $\alpha = 1$, $C(u_{t-1}, u_t; 1, \beta)$ is the Ali-Mikhail-Haq copula (Ali et al. (1978)) and the generalized Eyraud-Farlie-Gumbel-Morgenstern copula (Cambanis (1977)). By means of some stochastic transforms, some bivariate distributions can be induced by the Fang copula, such as the generalization of Gumbel's bivariate logistic distribution given by Satterthwaite and Hutchinson (1978). Moreover, it can be shown that if $\beta < 1$, $\lim_{\alpha \to 0} C(u_{t-1}, u_t; \alpha, \beta) = \lim_{\alpha \to \infty} C(u_{t-1}, u_t; \alpha, \beta) = u_{t-1} u_t$. Therefore, U_{t-1} and U_t are independent as $\alpha \to 0$ and $\alpha \to \infty$. To asses the correlation between two random variables, copulas can be used to define Spearman's ρ_s (see Joe (1997)) in general. Analog to the general case the Spearman's correlation coefficient of the Fang copula between U_{t-1} and U_t is representable by a hypergeometric function. A hypergeometric function of x is defined as

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} \ ,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ and $a_1, \ldots a_p, b_1, \ldots, b_q$ are parameters. $\Gamma(z)$ stands for the gamma function $\int_0^\infty e^{-t} t^{z-1} dt$. Then, the Spearman's correlation coefficient $\rho_s(\alpha, \beta)$ of the Fang copula in (21) between U_{t-1} and U_t is given by

$$\rho_s(\alpha, \beta) = 3({}_3F_2(1, 1, \alpha; 1 + 2\alpha, 1 + 2\alpha; \beta) - 1) \ . \tag{22}$$

The copula density function is:

$$c(u_{t-1}, u_t; \alpha, \beta) = \frac{(\beta^2 + \beta/\alpha)(u_{t-1}u_t)^{1/\alpha} + (\beta - \beta^2)(u_{t-1}^{1/\alpha} + u_t^{1/\alpha}) + (1-\beta)^2}{[1 - \beta(1 - {u_{t-1}}^{1/\alpha})(1 - {u_t}^{1/\alpha})]^{\alpha+2}}$$

$C_{t|t-1}^{-1}$ does not exist in closed form. $u_t = C_{t|t-1}^{-1}(v_t|u_{t-1}; \alpha, \beta)$ can be obtained from the equation $v_t = C_{t|t-1}(u_t|u_{t-1}; \alpha, \beta)$ using a numerical root-finding routine (here: Newton's procedure).

Table 1: VIX estimation results of the copula models

Copula	ML-estimation		Estimated autocorrelation				G-o-f
	α	β	1	2	3	4	
Gauss	0.851		-0.173*	0.034	-0.084	0.032	0.834
	(0.030)		(0.007)	(0.601)	(0.197)	(0.625)	(0.490)
Clayton	1.994		0.053	0.242*	0.068	0.121	1.271
	(0.118)		(0.410)	(0.000)	(0.290)	(0.061)	(0.079)
Frank	10.898		-0.132*	0.029	-0.069	0.055	0.574
	(0.793)		(0.038)	(0.654)	(0.290)	(0.392)	(0.897)
Fang	0.173	0.9994	-0.084	0.047	-0.032	0.058	0.629
	(0.015)	(0.0004)	(0.191)	(0.461)	(0.614)	(0.367)	(0.823)

Notes: Sample: 1990:1-2010:8 • Initial value of the one parameter copulas is 1 and of the Fang copula are $\widehat{\alpha}_1 = 0.4$ and $\widehat{\beta}_1 = 1$ • ML-estimates are different from zero at any level of significance (standard error in brackets) • Spearman's correlation coefficients and p-values of the hypothesis $\rho_s(V_t, V_{t-l}) = 0$, $l = 1, 2, 3, 4$, in brackets • * indicates a significant autocorrelation on the 5% overall error rate using Bonferroni's adjustment (see e.g. Sokal and Rohlf (1995)) • 2 is the number of tests performed (correlation test up to a specific lag and goodness-of-fit (G-o-f) test) • Finite sample adjustment of the Kolmogorov test and corresponding p-values of the hypothesis $V_t \sim U[0, 1]$ in brackets (see e.g. D'Agostino and Stephens (1986))

Table 2: Estimated conditional quantile error rates of the VIX

Copula	Lower region		Upper region		Overall region	
	ϵ		ϵ		ϵ	
	0.10	0.05	0.10	0.05	0.10	0.05
Gauss	0.17	0.11	0.09	0.07	0.09	0.07
	(0.06)		(0.02)		(0.01)	
Clayton	0.22	0.17	0.07	0.04	0.10	0.07
	(0.12)		(0.02)		(0.01)	
Frank	0.10	0.01	0.13	0.07	0.10	0.03
	(0.03)		(0.03)		(0.02)	
Fang	0.07	0.02	0.11	0.02	0.09	0.04
	(0.03)		(0.02)		(0.01)	

Notes: Sample: 1990:1-2010:8 • The estimated conditional quantiles $\widehat{u}_t$ are computed by the empirical distribution. By assuming a certain parametric copula a level of significance ϵ determines a $(1-\epsilon)$ confidence interval of the nonparametric estimated conditional quantiles $\widehat{u}_t$. With respect to the inverse conditional distributions for the upper interval bound $v_t = 1 - \epsilon/2$ and for the lower bound $v_t = \epsilon/2$ holds. The unknown copula parameters are substituted by appropriate ML-estimates according to Table 1. • The copula specific numbers are the relative frequencies for the nonparametric estimated conditional quantiles outside the parametric confidence interval. The lower quantile region is defined by quantiles in a range of $(0; 1/3)$. For the upper quantile region $(2/3; 1)$ holds. • Root mean squared errors of the regions in brackets