

Tightness for a stochastic Allen–Cahn equation

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ABSTRACT. We study an Allen–Cahn equation perturbed by a multiplicative stochastic noise that is white in time and correlated in space. Formally this equation approximates a stochastically forced mean curvature flow. We derive a uniform bound for the diffuse surface area, prove the tightness of solutions in the sharp interface limit, and show the convergence to phase-indicator functions.

1. INTRODUCTION

The Allen–Cahn equation

$$\varepsilon \partial_t u_\varepsilon = \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} F'(u_\varepsilon) \quad (1.1)$$

is an important prototype for phase separation processes in melts or alloys that is of fundamental interest both for theory and applications. It describes an evolution of non-conserved phases driven by the surface area reduction of their common interface. The Allen–Cahn equation is a *diffuse interface* model, *i.e.* phases are indicated by smooth fields, assuming a partial mixing of the phases. It is well-known [13, 19, 25] that in the *sharp interface limit* $\varepsilon \rightarrow 0$ solutions of the Allen–Cahn equation converge to an evolution of hypersurfaces $(\Gamma_t)_{t \in (0, T)}$ by *mean curvature flow* (MCF)

$$v(t, \cdot) = H(t, \cdot), \quad (1.2)$$

where v describes the velocity vector of the evolution and $H(t, \cdot)$ denotes the mean curvature vector of Γ_t .

Our goal is to introduce a stochastic perturbation of the Allen–Cahn equation that formally approximates a stochastic mean curvature type flow

$$v(t, \cdot) = H(t, \cdot) + X(t, \cdot), \quad (1.3)$$

where X now is a random vector-field in the ambient space. More specifically we are considering is the following Stratonovich stochastic partial differential equation (SPDE):

$$du_\varepsilon = \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} F'(u_\varepsilon) \right) dt + \nabla u_\varepsilon \cdot X(x, \circ dt), \quad (1.4)$$

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where X is a vectorfield valued Brownian motion. A particular case of such a Brownian motion is

$$X(t, x) = X^{(0)}(x)t + \sum_{k=1}^N X^{(k)}(x)W_k(t), \quad (1.5)$$

where the $X^{(k)}$ are fixed vectorfields and W_k are independent standard Brownian motions. In this case (1.4) reduces to the Stratonovich SPDE

$$du_\varepsilon = \left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2} F'(u_\varepsilon) + \nabla u_\varepsilon \cdot X^{(0)} \right) dt + \sum_{k=1}^N \nabla u_\varepsilon \cdot X^{(k)} \circ dW_k(t). \quad (1.6)$$

Our setting is more general as it allows for infinite sums of Brownian motions. See below for a more detailed discussion. We complement (1.4) by deterministic initial and zero Neumann-boundary data,

$$u_\varepsilon(0, \cdot) = u_\varepsilon^0 \quad \text{in } U, \quad (1.7)$$

$$\nabla u_\varepsilon \cdot \nu_\Omega = 0 \quad \text{on } (0, T) \times \partial U. \quad (1.8)$$

Our main result is the tightness of the solutions $(u_\varepsilon)_{\varepsilon>0}$ of (1.4) and the convergence to an evolution of (random) phase indicator functions $u(t, \cdot) \in BV(U)$. In particular we prove a uniform control (in $\varepsilon > 0$) of the diffuse surface area of u_ε .

In the next sections we briefly review the analysis of the deterministic Allen–Cahn equation and report on stochastic extensions. In Section 3 we state our main assumptions and recall some notations for stochastic flows. Our main results are stated in Section 4. In Section 5 we prove an existence result for (1.4). Finally in Section 6 we derive the estimates for the diffuse surface area and prove the tightness of solutions.

2. BACKGROUND

2.1. Deterministic sharp interface limit. As many other diffuse interface models the Allen–Cahn equation (1.1) is based on the *Van der Waals–Cahn–Hilliard* energy

$$E_\varepsilon(u_\varepsilon) := \int_U \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) dx \quad \text{for } u_\varepsilon : U \rightarrow \mathbb{R}. \quad (2.1)$$

The energy E_ε favors a decomposition of the spatial domain U into two regions (phases) where $u_\varepsilon \approx -1$ and $u_\varepsilon \approx 1$, separated by a transition layer (diffuse interface) with a thickness of order ε . Modica and Mortola [35, 33] proved that E_ε Gamma-converges (with respect to L^1 -convergence) to a constant multiple of the perimeter functional \mathcal{P} , restricted to phase indicator functions,

$$E_\varepsilon \rightarrow c_0 \mathcal{P}, \quad \mathcal{P}(u) := \begin{cases} \frac{1}{2} \int_U d|\nabla u| & \text{if } u \in BV(U, \{-1, 1\}), \\ \infty & \text{otherwise.} \end{cases}$$

\mathcal{P} measures the surface-area of the phase boundary $\partial^* \{u = 1\} \cap U$. In this sense E_ε describes a diffuse approximation of the surface-area functional.

The Allen–Cahn equation (1.1) in fact is the (accelerated) L^2 -gradient flow of E_ε . It is proved in different formulations [13, 19, 25] that (1.1) converges to motion by

mean curvature. Since mean curvature flow in general allows for the formation of singularities in finite time it is necessary to consider suitable generalized formulations of (1.2), as for example in the sense of viscosity solutions [2, 9, 10, 19, 20], De Giorgi’s barriers [3, 4, 8, 14], or geometric measure theory. The first approaches rely on the maximum principle, the latter was pioneered by Brakke [6] and is based on the *localized* energy equality

$$\begin{aligned} & \frac{d}{dt} \int_{\Gamma_t} \eta(x) d\mathcal{H}^{n-1}(x) \\ &= \int_{\Gamma_t} \nabla \eta(t, x) \cdot V(t, x) d\mathcal{H}^{n-1}(x) - \int_{\Gamma_t} H(t, x) \cdot V(t, x) \eta(t, x) d\mathcal{H}^{n-1}(x) \end{aligned} \quad (2.2)$$

that holds for arbitrary $\eta \in C_c^1(U)$ and for any classical solutions $(\Gamma_t)_{t \in (0, T)}$ of mean curvature flow. Ilmanen [25] proved the convergence of the Allen–Cahn equation to mean curvature flow in the formulation of Brakke, using a diffuse analog of the (localized) energy equality (2.2). By similar methods Mugnai and the first author [36] proved the convergence of (deterministically) perturbed Allen–Cahn equations.

One of the key results of the present paper is an energy inequality for the stochastic Allen–Cahn equation (1.4). By Itô’s formula the stochastic drift produces some extra terms in the time-derivative of the diffuse surface energy $E_\varepsilon(u_\varepsilon)$. These ‘bad’ terms are exactly compensated by the additional terms in (1.4) which are hidden in the Stratonovich formalism.

2.2. Stochastic perturbations of the Allen–Cahn equation and MCF. Additive perturbations of the Allen–Cahn equation were studied in the one-dimensional case in [22, 7] and in the higher-dimensional case in [23, 38, 31]. Note that perturbation results such as [36] do not apply to the stochastic case as one typically perturbs with a white noise i.e. the time-derivative of a C^α function for $\alpha < \frac{1}{2}$, which is not covered by most techniques.

In the one-dimensional case the equation was studied with an additive space-time white noise and at least for the case where the interface consists of a single kink the sharp interface limit was described rigorously [22, 7]. In higher dimensions the picture is much less complete. For instance, the Allen–Cahn equation with space-time white noise is in general not well-posed: the noise term is so rough that for $n \geq 2$ solutions to the stochastic heat equation attain values only in Sobolev spaces of negative order, and on such spaces the nonlinear potential can a priori not be defined. This existence problem can be avoided if one introduces spatial correlations as we do in (1.4). In all of the above papers conditions on the stochastic perturbations are much more restrictive than in our approach. In fact it is always assumed that the noise is constant in space and smoothed in time with a correlation length that is coupled to the interface width ε and goes to zero for $\varepsilon \downarrow 0$. All of these papers rely on a construction of the limit dynamics by different means and then an explicit construction of sub- and supersolutions making use of the maximum principle. Our approach is based only on energy estimates. On the other hand, we only prove tightness of the approximations and do not obtain an evolution law for limit points.

The restriction on spatially constant noise in previous papers and our problem to derive the stochastic motion law in the limit are closely related to the lack of existence results and generalized formulations for stochastically forced mean curvature flow. Up to now there are only results for spatially constant forcing [15, 23, 31] or in the case of evolution of graphs in $1 + 1$ dimensions [16].

Our approach is closely related to Yip's construction [40] of a time-discrete stochastically forced mean curvature flow. Yip follows the deterministic scheme of [1, 32], where for a given time step $\delta > 0$ a sequence of sets of bounded perimeter is constructed iteratively. The heart of the construction is the minimization of a functional that is given by the perimeter plus a suitable distance from the previous set. Yip [40] introduces randomness to this scheme by performing a stochastic flow in between two minimization steps. For the resulting time-discrete evolution of sets Yip proves uniform bounds (in δ) for the perimeter and shows tightness of the time-discrete solutions with $\delta \rightarrow 0$. As in our case, a characterization of the limiting evolution is not given. If one applies Yip's scheme to the Allen–Cahn equation (substituting the perimeter functional by the diffuse surface area energy and using a rescaled L^2 -distance between phase fields) one in fact would obtain our stochastic Allen–Cahn equation (1.4) in the limit $\delta \rightarrow 0$.

Noisy perturbations of the Allen–Cahn equation were studied from a different point of view in [27]. There the authors study the action functional which appears if one first applies Freidlin–Wentzel theory to the Allen–Cahn equation with an additive noise that is white in time and spatially correlated, and then formally takes the spatial correlation to zero. Then the sharp interface limit $\varepsilon \downarrow 0$ is studied on the level of action functionals and a *reduced action functional* as a possible Γ -limit is derived. See [39, 37] for a subsequent analysis.

3. ASSUMPTIONS AND STOCHASTIC FLOWS

3.1. Notation and assumptions. Let $U \subset \mathbb{R}^n$ be an open bounded subset of \mathbb{R}^n with smooth boundary, let $T > 0$, and set $U_T := (0, T) \times U$. We denote by $x \in U$ and $t \in (0, T)$ the space- and time-variables respectively; ∇ and Δ denote the spatial gradient and Laplacian.

We assume the potential F to be smooth and verify the following assumptions:

$$\left. \begin{aligned} F(r) &\geq 0 \quad \text{and} \quad F(r) = 0 \quad \text{iff} \quad r = \pm 1, \\ F' &\text{ admits exactly three zeros } \{\pm 1, 0\} \text{ and } F''(0) < 0, F''(\pm 1) > 0, \\ F &\text{ is symmetric, } \forall r \geq 0 \quad F(r) = F(-r), \\ F(r) &\geq C|r|^{2+\delta} \quad \text{for some } \delta > 0 \text{ and } |r| \text{ sufficiently large.} \end{aligned} \right\} \quad (3.1)$$

The standard choice for F is

$$F(r) = \frac{1}{4}(1 - r^2)^2,$$

such that the nonlinearity in (1.1) becomes $F'(r) = -r(1 - r^2)$.

Next we give some geometric meaning to u_ε . We define the *normal direction* with respect to u_ε by

$$\nu_\varepsilon(t, x) := \begin{cases} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}(t, x) & \text{if } |\nabla u_\varepsilon(t, x)| \neq 0, \\ \vec{e} & \text{else,} \end{cases} \quad (3.2)$$

where \vec{e} is an arbitrary fixed unit vector. We define the *diffuse surface area measures*

$$\mu_\varepsilon^t(\eta) := \int_U \eta \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(t, \cdot)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(t, \cdot)) \right) dx \quad (3.3)$$

for $\eta \in C_c^0(U)$. We denote the *diffuse mean curvature* by

$$w_\varepsilon := -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} F'(u_\varepsilon). \quad (3.4)$$

For the initial data we assume that u_ε^0 is smooth and that

$$E_\varepsilon(u_\varepsilon^0) \leq \Lambda \quad (3.5)$$

holds for all $\varepsilon > 0$ and a fixed $\Lambda > 0$. Note that by [25, page 423] the boundary of every open set that verifies a density bound and that can be approximated in BV by smooth hypersurfaces can be approximated by phase fields with uniformly bounded diffuse surface area. On the other hand (3.5) implies by [34, 35] that the sequence u_ε^0 is compact in $L^1(U_T)$ and that every limit belongs to the space of phase indicator functions $BV(U, \{\pm 1\})$.

3.2. Stochastic Flows. Let us briefly introduce some notations for stochastic flows. We refer the reader to Kunita's book [29] Chapter 3 and Section 2,5 and 6 in Chapter 4 for further background.

Let (Ω, \mathcal{F}, P) be a probability space with a filtration $\{\mathcal{F}_{s,t}\}_{0 \leq s \leq t \leq T}$ satisfying the usual conditions. Let $(X(t, x), t \in [0, T], x \in U)$ be a continuous vectorfield valued semimartingale with local characteristic $(\tilde{a}_{ij}(t, x, y), b_i(t, x))$ on (Ω, \mathcal{F}, P) . This means that for every $x \in U$ the process $X(t, x)$ is a continuous \mathbb{R}^n valued semimartingale with finite variation process $\int_0^t b(s, x) ds$ and quadratic variation

$$\langle X_i(t, x), X_j(t, y) \rangle = \int_0^t \tilde{a}_{ij}(s, x, y) ds. \quad (3.6)$$

We assume that for every $(x, y) \in U \times U$ the function \tilde{a} is continuous in time and of class $C^{4,\alpha}$ in both space variables, and that $x \mapsto b^i(t, x)$ is of class $C^{3,\alpha}$ for some $\alpha > 0$. Finally we assume that \tilde{a} and b have compact support in $U \times U$ resp. in U .

Denote by $(\varphi_{s,t}, s < t)$ the Stratonovich-Flow associated to $-X$. This means that almost surely $(\varphi_{s,t}, s < t)$ is a two parameter family of diffeomorphisms of U fixing the boundary and verifying the flow property

$$\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t} \quad \text{for } r \leq s \leq t. \quad (3.7)$$

Furthermore, for every x and every $s \in [0, T)$ the process $(\varphi_{s,t}(x), t \geq s)$ is a solution of the stochastic differential equation

$$\begin{aligned} d\varphi_{s,t}(x) &= -X(\circ dt, \varphi_{s,t}(x)) \\ \varphi_{s,s}(x) &= x. \end{aligned} \quad (3.8)$$

Under the above regularity assumption for all $s \leq t$ the mapping $\varphi_{s,t}$ is a $C^{3,\beta}$ diffeomorphism of U , for all $\beta < \alpha$.

A particular example is that of a stochastic flow given by a usual Stratonovich-differential equation. If $X^{(k)}(t, \cdot), k = 0, \dots, N$ are smooth time dependent vectorfields on U and B_1, \dots, B_N are independent standard Brownian motions, then

$$X(t, x) = \sum_{k=1}^N \int_0^t X^{(k)}(s, x) dB_k(s) + \int_0^t X^{(0)}(s, x) ds, \quad (3.9)$$

is a vectorfield valued Brownian motion as considered above. Its local characteristic is given by

$$\begin{aligned} \tilde{a}_{ij}(s, x, y) &= \sum_{k=1}^N X_i^{(k)}(s, x) X_j^{(k)}(s, y) \\ b_i(s, x) &= X_i^{(0)}(s, x). \end{aligned} \quad (3.10)$$

In this case the stochastic differential equation (3.8) reduces to the more familiar

$$\begin{aligned} d\varphi_{s,t}(x) &= X^{(0)}(t, \varphi_{s,t}(x))dt + \sum_{k=1}^N X^{(k)}(t, \varphi_{s,t}(x)) \circ dB_k(t) \\ \varphi_{s,s}(x) &= x. \end{aligned} \quad (3.11)$$

The advantage of Kunita's framework is that it allows for infinite sums in the noise part i.e. for noise fields of the form

$$X(t, x) = \sum_{i=1}^{\infty} \int_0^t X^{(i)}(s, x) \circ dB_i(s) + \int_0^t X^{(0)}(s, x) ds, \quad (3.12)$$

for vectorfields with the right summability properties. We prefer this approach as a restriction to finite dimensional noises is unnecessary and a severe restriction.

4. RESULTS

In this section we state our main results. For the proofs see the subsequent sections. We first address the question of existence and uniqueness of solutions for (1.4). There are some classical existence and uniqueness results for equations similar to (1.4), see for example [11, p. 212 ff.], [28], [21]. In those references either mild or weak variational solutions are constructed. Using the technique from [29] we obtain here Hölder-continuous strong solutions.

Theorem 4.1. *Let u_ε^0 , F , and X satisfy the assumptions (3.1), (3.5) and the smoothness conditions stated in Section 3.2. Then for every $\varepsilon > 0$ there exists a unique solution u_ε of*

$$u_\varepsilon(t, x) = u_\varepsilon^0(x) + \int_0^t \left(\Delta u_\varepsilon(s, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(s, x)) \right) ds + \int_0^t \nabla u_\varepsilon(s, x) \cdot X(x, \circ ds) \quad (4.1)$$

$$\nabla u_\varepsilon \cdot \nu_\Omega = 0 \quad \text{on } (0, T) \times \partial U. \quad (4.2)$$

The function $u_\varepsilon(t, \cdot)$ is a continuous $C^{3,\beta}(\bar{U})$ -valued semimartingale. Furthermore, we have the following bound for the spatial derivatives:

$$\mathbb{E}[\sup_{x \in U} |\partial^\gamma u_\varepsilon(t, x)|^p] < \infty. \quad (4.3)$$

for any multi-index γ with $|\gamma| \leq 3$ and all $t \in (0, T)$.

Our main result concerns the tightness of the solutions u_ε in the limit $\varepsilon \rightarrow 0$. In addition we show that limit points are concentrated on the space of phase indicator function of bounded variation. The key step in the proof of these results is a uniform bound on the diffuse surface area.

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied and let u_ε be the solution of (4.1)-(4.2) for $\varepsilon > 0$. Then the following statements hold:*

(1) *Uniform bounds on the energy: For every $T > 0$ and every $p \geq 1$ we have*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t, \cdot))^p \right] < \infty. \quad (4.4)$$

(2) *Uniform bounds on the diffuse mean curvature: For every $T > 0$ and every $p \geq 1$ we have*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 dx dt \right)^p \right] < \infty. \quad (4.5)$$

(3) *Tightness of the sequence: Let \mathbb{Q}^ε be the distribution of the solution (1.4). Then the family \mathbb{Q}^ε is tight on $C([0, T], L^1(U))$. In particular, there exists a sequence $\varepsilon_i \downarrow 0$ such that the processes u_{ε_i} can jointly be realized on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and converge $\tilde{\mathbb{P}}$ -almost-surely in $C([0, T], L^1(U))$ to a limiting process u . For almost all $t \in (0, T)$ we have $u(t, \cdot) \in BV(U, \{\pm 1\})$ almost surely and*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{BV(U)}^p \right] < \infty$$

holds for every $T > 0$ and every $p \geq 1$.

In most of the sequel we will use the Itô-form of (4.1), which is by [29, Section 6.2] given as

$$\begin{aligned} u_\varepsilon(t, x) = u_\varepsilon(0, x) &+ \int_0^t \left(\Delta u_\varepsilon(s, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(s, x)) \right) ds + \int_0^t \nabla u_\varepsilon(s, x) \cdot X(ds, x) \\ &+ \frac{1}{2} \int_0^t \left(A(s, x) : D^2 u_\varepsilon(s, x) + c(s, x) \cdot \nabla u_\varepsilon(s, x) \right) ds. \end{aligned} \quad (4.6)$$

The Itô-Stratonovich correction terms in (4.6) are given by the matrix $A = (a_{ij})_{i,j=1,\dots,n}$ and the vector field $c = (c_i)_{i=1,\dots,n}$,

$$a_{ij}(t, x) = \tilde{a}_{ij}(t, x, x), \quad (4.7)$$

$$c_j(t, x) = \partial_{y_i} \tilde{a}_{ij}(t, x, y)|_{y=x}, \quad (4.8)$$

where we sum here and in the following over repeated indices. Note that the extra term $A : D^2 u$ is of highest order, such that it changes the diffusion coefficient in (4.1). In particular, the stochastic-parabolicity condition (see e.g. [21, condition (1)]), which is often needed in the case of gradient dependent noise, is always satisfied.

5. EXISTENCE AND UNIQUENESS

In this section we prove Theorem 4.1 by reducing the existence of solutions of (1.4) to an existence statement for a deterministic reaction-diffusion equation with random coefficients. This technique is borrowed from [29].

Proof of Theorem 4.1. As above denote by $\varphi_{s,t}$ the stochastic flow generated by $-X$. For a function $u : U \rightarrow \mathbb{R}$ define the transformation $w_t(u)(x) = u(\varphi_{0,t}^{-1}(x))$. By the regularity of the stochastic flow it is clear that w_t maps $C^{3,\beta}(U)$ into itself. Denote by \mathcal{L} the nonlinear operator

$$\mathcal{L}(u) = \Delta u - \frac{1}{\varepsilon^2} F'(u)$$

and by \mathcal{L}_t^w the operator $w_t^{-1} \mathcal{L} w_t$. Then a direct computation shows that \mathcal{L}_t^w is given by

$$\mathcal{L}_t^w u(t, x) = \sum_{i,j=1}^n R_w^{i,j}(t, x) \frac{\partial^2}{\partial x^i \partial x^j} u(t, x) + \sum_{i=1}^n S_w^i(t, x) \frac{\partial}{\partial x^i} u(t, x) - \frac{1}{\varepsilon^2} F'(u(t, x)), \quad (5.1)$$

with coefficients

$$R_w^{i,j}(t, x) = \sum_{k,l} \partial_k \left(\varphi_{0,t}^{-1} \right)^i (\varphi_{0,t}(x)) \partial_l \left(\varphi_{0,t}^{-1} \right)^j (\varphi_{0,t}(x)) \quad (5.2)$$

$$S_w^i(t, x) = \sum_{k,l} \partial_k \partial_l \left(\varphi_{0,t}^{-1} \right)^i (\varphi_{0,t}(x)) \quad (5.3)$$

In particular, the coefficients are random, the $R_w^{i,j}$ are of class $C^{3,\beta}$ in space and continuous C^γ in time for every $\gamma < \frac{1}{2}$, and the S^i are of class $C^{2,\beta}$ in space and C^γ

in time. Furthermore, note that $R^{i,j} = \delta^{i,j}$ and $S = 0$ close to the boundary. Similar to Lemma 6.2.3 in [29] it can be seen by another straightforward computation that a smooth semimartingal u is a solution to 4.1 if and only if $u' = w_t^{-1}u$ is a solution to

$$\frac{\partial}{\partial t} v'(t, x) = \mathcal{L}_t^w v(t, x). \quad (5.4)$$

Existence and uniqueness of smooth solutions to reaction diffusion equation like (5.4) can be derived in a standard way: For example, apply Schaefer's Fixed Point Theorem [18, Theorem 9.2.4] in combination with Schauder-estimates [30, Theorem IV.5.3] for the linear part of (5.4) and a-priori estimates by the maximum principle. This yields existence of solutions that are $C^{2,\alpha}$ in space and $C^{1,\alpha/2}$ in time. Differentiation with respect to space and another application of [30, Theorem IV.5.3] proves $C^{4,\alpha}$ -regularity in space. To derive (4.3) note that by (5.4) the derivatives of u up to order 3 can be bounded in terms of the derivatives of v and φ^{-1} . The bounds on v follow from the Schauder-estimates [30, Theorem IV.5.3] applied to the random coefficients $R^{i,j}$ and S^j . The bounds on these coefficients as well as on the derivatives of φ follow from [29, Theorem 6.1.10]. \square

6. TIGHTNESS

In this section we derive estimates for the diffuse surface area. For future use we include a possible localization of the energy.

Proposition 6.1. *Let u_ε satisfy (1.4). Then for all $0 \leq t_0 < t_1$ and all $\eta \in C^2(\bar{U})$ we have*

$$\begin{aligned} & \mu_\varepsilon^{t_1}(\eta) - \mu_\varepsilon^{t_0}(\eta) \\ &= - \int_{t_0}^{t_1} \int_U \eta(x) \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt + \int_{t_0}^{t_1} \int_U w_\varepsilon(t, x) \nabla \eta(t, x) \cdot \nabla u_\varepsilon(t, x) dx dt \\ & \quad + \int_U \int_{t_0}^{t_1} \eta(x) w_\varepsilon(x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\ & \quad - \int_U \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\ & \quad + \int_{t_0}^{t_1} \int_U R(t, x, \eta(x), \nabla \eta(x), D^2 \eta(x)) : \varepsilon \nabla u_\varepsilon(t, x) \otimes \nabla u_\varepsilon(t, x) dx dt \\ & \quad + \int_{t_0}^{t_1} \int_U S(t, x, \eta(x), \nabla \eta(x), D^2 \eta(x)) \frac{1}{\varepsilon} F(u_\varepsilon) dx dt. \end{aligned} \quad (6.1)$$

The functions R, S are affine linear in the $\eta, \nabla \eta, D^2 \eta$ components with coefficients that are bounded in $C^0(\bar{U}_T)$ by a constant that only depends on $\|A, \tilde{A}\|_{C^0([0,T], C^2(\bar{U}))}$.

Remark 6.2. According to our assumption in Section 3.2 $X(t, \cdot)$ has compact support in U . Using this and the Neumann boundary condition for u_ε one sees that we do not need to impose η to have compact support in order to avoid boundary terms appearing in partial integrations.

Proof. Let $\eta \in C^2(\overline{U})$ be a smooth function. We compute the differential of the localized diffuse surface area

$$\mu_t^\varepsilon(\eta) = \int_U \eta(x) \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(t, x)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) \right) dx. \quad (6.2)$$

The density of the cross variation $q(x, y, s)$ of the stochastic drift term $\nabla u_\varepsilon \cdot X$ in (4.6) can by [29, Theorem 2.3.2] be computed from the local characteristics as

$$q(s, x, y) = \nabla u_\varepsilon(s, x) \cdot \tilde{A}(s, x, y) \nabla u_\varepsilon(s, y). \quad (6.3)$$

To compute the differential of the first term in (6.2) we also need the density $Q(x, y, s)$ of the cross variation of $\nabla(\nabla u_\varepsilon \cdot X)$. Using [29, Theorem 3.1.3] this is given by

$$\begin{aligned} & Q_{kl}(s, x, x) \\ &= \partial_k \partial_l' q(s, x, y)|_{y=x} \\ &= \partial_k \nabla u(s, x) \cdot A(x, s) \partial_l \nabla u_\varepsilon(s, x) + \partial_k \nabla u_\varepsilon(s, x) \cdot \partial_l' \tilde{A}(s, x, x) \nabla u_\varepsilon(s, x) \\ &\quad + \nabla u_\varepsilon(s, x) \cdot \partial_k \tilde{A}(s, x, x) \partial_l' \nabla u_\varepsilon(s, x) + \nabla u_\varepsilon(s, x) \cdot \partial_k \partial_l' \tilde{A}(s, x, x) \nabla u_\varepsilon(s, x), \end{aligned} \quad (6.4)$$

where ∂_k and ∂_l' denote the derivatives with respect to the x_k and y_l component, respectively.

We now obtain from (6.2) and Itô's formula that

$$\begin{aligned} \mu_\varepsilon^{t_1}(\eta) - \mu_\varepsilon^{t_0}(\eta) &= \int_U \left(\int_{t_0}^{t_1} \eta w_\varepsilon(t, \cdot) du_\varepsilon(t, \cdot) - \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon \cdot \nabla \eta du_\varepsilon(t, \cdot) \right) dx \\ &\quad + \frac{1}{2} \int_U \int_{t_0}^{t_1} \varepsilon \eta(x) \operatorname{tr} Q(t, x, x) + \eta(x) \frac{1}{\varepsilon} F''(u_\varepsilon(t, x)) q(t, x, x) dt dx. \end{aligned} \quad (6.5)$$

When evaluating the right hand-side of this equation we obtain one 'good' term, which corresponds to minus the integral over the squared diffuse mean curvature in the purely deterministic case. Additional terms are due to the stochastic drift and Itô terms in (4.6), and the Itô terms in the last line. The objective is to show that these term can finally be controlled by the good mean curvature term or by a Gronwall argument. With this aim we derive by a series of partial integration the representation (6.1).

We start with the first term in (6.5) and deduce from (4.6) that

$$\begin{aligned}
& \int_{t_0}^{t_1} \eta w_\varepsilon(t, x) du_\varepsilon(t, x) \\
&= \int_{t_0}^{t_1} \eta(x) \left(-\frac{1}{\varepsilon} w_\varepsilon^2(t, x) + w_\varepsilon(t, x) \frac{1}{2} A(t, x) : D^2 u_\varepsilon(t, x) \right) dt \\
&\quad + \int_{t_0}^{t_1} \eta(x) w_\varepsilon(t, x) \frac{1}{2} c(t, x) \cdot \nabla u_\varepsilon(t, x) dt \\
&\quad + \int_{t_0}^{t_1} \eta(x) w_\varepsilon(x) \nabla u_\varepsilon(t, x) \cdot X(x, dt) \\
&= - \int_{t_0}^{t_1} \eta(x) \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dt + \int_{t_0}^{t_1} \eta(x) w_\varepsilon(x) \nabla u_\varepsilon(t, x) \cdot X(x, dt) \\
&\quad + \int_{t_0}^{t_1} \left(R_1(t, x) + T_1(t, x) + T_2(t, x) \right) dt, \tag{6.6}
\end{aligned}$$

where

$$R_1(t, x) := \eta(x) w_\varepsilon(t, x) \frac{1}{2} c(t, x) \cdot \nabla u_\varepsilon(t, x), \tag{6.7}$$

and

$$T_1(t, x) := -\frac{\varepsilon}{2} \eta(x) \Delta u_\varepsilon(t, x) A(t, x) : D^2 u_\varepsilon(t, x), \tag{6.8}$$

$$T_2(x) := \frac{1}{2\varepsilon} \eta(x) F'(u_\varepsilon(t, x)) A(t, x) : D^2 u_\varepsilon(t, x). \tag{6.9}$$

The second term in (6.5) is given by

$$\begin{aligned}
& - \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) du_\varepsilon(t, x) \\
&= \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) \left(\frac{1}{\varepsilon} w_\varepsilon(t, x) - \frac{1}{2} A(t, x) : D^2 u_\varepsilon(t, x) \right. \\
&\quad \left. - \frac{1}{2} c(t, x) \cdot \nabla u_\varepsilon(t, x) \right) dt \\
&\quad - \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) \\
&= \int_{t_0}^{t_1} w_\varepsilon(t, x) \nabla \eta(x) \cdot \nabla u_\varepsilon(t, x) dt + \int_{t_0}^{t_1} \left(R_2(t, x) + T_3(t, x) \right) dt \\
&\quad - \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) \nabla u_\varepsilon(t, x) \cdot X(dt, x), \tag{6.10}
\end{aligned}$$

where

$$R_2(t, x) := -\frac{\varepsilon}{2} \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) c(t, x) \cdot \nabla u_\varepsilon(t, x) \tag{6.11}$$

and

$$T_3(t, x) := -\frac{\varepsilon}{2} \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) A(t, x) : D^2 u_\varepsilon(t, x). \quad (6.12)$$

Using (6.4) we compute for the integrand in the third term of (6.5)

$$\frac{\varepsilon}{2} \eta(x) \operatorname{tr} Q(x, x, t) = T_4(t, x) + T_5(t, x) + T_6(t, x) + R_3(t, x) \quad (6.13)$$

where

$$R_3(t, x) := \frac{\varepsilon}{2} \eta(x) \nabla u_\varepsilon(t, x) \cdot \partial_k \partial'_k \tilde{A}(t, x, x) \nabla u_\varepsilon(t, x) \quad (6.14)$$

and

$$T_4(t, x) := \frac{\varepsilon}{2} \eta(x) \partial_k \nabla u_\varepsilon(t, x) \cdot A(t, x) \partial_k \nabla u_\varepsilon(t, x) \quad (6.15)$$

$$T_5(t, x) := \frac{\varepsilon}{2} \eta(x) \partial_k \nabla u_\varepsilon(t, x) \cdot \partial'_k \tilde{A}(t, x, x) \nabla u_\varepsilon(t, x) \quad (6.16)$$

$$T_6(t, x) := \frac{\varepsilon}{2} \eta(x) \nabla u_\varepsilon(t, x) \cdot \partial_k \tilde{A}(t, x, x) \partial_k \nabla u_\varepsilon(t, x). \quad (6.17)$$

Finally, for the last integrand in (6.5) we have

$$\begin{aligned} T_7(t, x) &:= \eta(x) \frac{1}{2\varepsilon} F''(u_\varepsilon(t, x)) q(t, x, x) \\ &= \eta(x) \frac{1}{2\varepsilon} F''(u_\varepsilon(t, x)) \nabla u_\varepsilon(t, x) \cdot A(t, x) \nabla u_\varepsilon(t, x). \end{aligned} \quad (6.18)$$

In the next step we manipulate the terms T_1, \dots, T_7 and show that they combine to expressions that again can be controlled. We start with the terms involving derivatives of F . For T_7 we derive, noting that $F''(u_\varepsilon) \nabla u_\varepsilon = \nabla F'(u_\varepsilon)$ and $F'(u_\varepsilon) \nabla u_\varepsilon = \nabla F(u_\varepsilon)$

$$\begin{aligned} &\int_U T_7(t, x) dx \\ &= \frac{1}{2\varepsilon} \int_U \eta(x) F''(u_\varepsilon(t, x)) \nabla u_\varepsilon(t, x) \cdot A(t, x) \nabla u_\varepsilon(t, x) dx \\ &= -\frac{1}{2\varepsilon} \int_U \left(F'(u_\varepsilon) \nabla \eta \cdot A \nabla u_\varepsilon + \eta F'(u_\varepsilon) (\nabla \cdot A) \nabla u_\varepsilon + \eta F'(u_\varepsilon) A : D^2 u_\varepsilon \right) dx \\ &= \int_U \frac{1}{2\varepsilon} F(u_\varepsilon) \left(A : D^2 \eta + 2 \nabla \eta \nabla \cdot A + \eta \partial_i \partial_j a_{ij} \right) dx - \int_U T_2(t, x) dx. \end{aligned} \quad (6.19)$$

In particular

$$\int_U T_2(t, x) + T_7(t, x) dx = \int_U R_4(t, x) dx, \quad (6.20)$$

where

$$R_4(t, x) = \frac{1}{2\varepsilon} F(u_\varepsilon) \left(A : D^2 \eta + 2 \nabla \eta \cdot \nabla \cdot A + \eta \partial_i \partial_j a_{ij} \right). \quad (6.21)$$

Now we consider the terms not involving the potential F . We rewrite T_4 and perform a partial integration with respect to x_j ,

$$\begin{aligned}
& \int_U T_4(t, x) dx \\
&= \int_U \frac{\varepsilon}{2} \eta \partial_k \partial_i u_\varepsilon a_{ij} \partial_k \partial_j u_\varepsilon dx \\
&= - \int_U \frac{\varepsilon}{2} \partial_k u_\varepsilon \partial_k \partial_j u_\varepsilon (\eta \partial_i a_{ij} + \partial_i \eta a_{ij}) dx - \int_U \frac{\varepsilon}{2} \eta \partial_k u_\varepsilon a_{ij} \partial_i \partial_j \partial_k u_\varepsilon dx \\
&= \int_U \frac{\varepsilon}{4} |\nabla u_\varepsilon|^2 \left(A : D^2 \eta + 2 \nabla \eta \cdot (\nabla \cdot A) + \eta \partial_i \partial_j a_{ij} \right) dx \\
&\quad + \int_U \frac{\varepsilon}{2} \eta A : D^2 u_\varepsilon \Delta u_\varepsilon dx + \int_U \frac{\varepsilon}{2} \nabla \eta \cdot \nabla u_\varepsilon A : D^2 u_\varepsilon dx \\
&\quad + \int_U \frac{\varepsilon}{2} \eta \partial_i \partial_j u_\varepsilon \partial_k a_{ij} \partial_k u_\varepsilon dx \\
&= \int_U \frac{\varepsilon}{4} |\nabla u_\varepsilon|^2 \left(A : D^2 \eta + 2 \nabla \eta \cdot (\nabla \cdot A) + \eta \partial_i \partial_j a_{ij} \right) dx \\
&\quad - \int_U T_1(t, x) dx - \int_U T_3(t, x) dx + \int_U \frac{\varepsilon}{2} \eta \partial_i \partial_j u_\varepsilon \partial_k a_{ij} \partial_k u_\varepsilon dx, \tag{6.22}
\end{aligned}$$

where in the third equality we have used that $\partial_k u_\varepsilon \partial_k \partial_j u_\varepsilon = \partial_j \frac{1}{2} |\nabla u_\varepsilon|^2$ and a partial integration of the $\partial_i \partial_j \partial_k u_\varepsilon$ term with respect to x_k . The last term in (6.22) can be further manipulated by an x_i partial integration

$$\begin{aligned}
\int_U \frac{\varepsilon}{2} \eta \partial_i \partial_j u_\varepsilon \partial_k a_{ij} \partial_k u_\varepsilon &= - \int_U \frac{\varepsilon}{2} \eta \partial_j u_\varepsilon \partial_k a_{ij} \partial_i \partial_k u_\varepsilon dx \\
&\quad - \int_U \frac{\varepsilon}{2} \eta \partial_j u_\varepsilon \partial_i \partial_k a_{ij} \partial_k u_\varepsilon - \frac{\varepsilon}{2} \partial_i \eta \partial_j u_\varepsilon \partial_k a_{ij} \partial_k u_\varepsilon dx \\
&= - \int_U T_5 dx - \int_U T_6 dx \\
&\quad - \int_U \frac{\varepsilon}{2} \eta \partial_j u_\varepsilon \partial_i \partial_k a_{ij} \partial_k u_\varepsilon dx + \frac{\varepsilon}{2} \partial_i \eta \partial_j u_\varepsilon \partial_k a_{ij} \partial_k u_\varepsilon dx, \tag{6.23}
\end{aligned}$$

where in the last equality we have used that

$$\begin{aligned}
-\frac{\varepsilon}{2} \eta \partial_j u_\varepsilon \partial_k a_{ij} \partial_i \partial_k u_\varepsilon + T_5 + T_6 &= \frac{\varepsilon}{2} \eta \partial_j u_\varepsilon \partial_i \partial_k u_\varepsilon \left[- (\partial_k \tilde{a}_{ij} + \partial'_k \tilde{a}_{ij}) + \partial'_k \tilde{a}_{ij} + \partial_k \tilde{a}_{ji} \right] \\
&= \varepsilon \eta \partial_j u_\varepsilon \partial_i \partial_k u_\varepsilon \left[- \partial_k \tilde{a}_{ij} + \partial_k \tilde{a}_{ji} \right] \\
&= 0,
\end{aligned}$$

since the factor in the brackets is antisymmetric and the other factor symmetric in i, j . Putting together (6.22) and (6.23) we obtain the identity

$$\int_U T_1(t, x) + T_3(t, x) + T_4(t, x) + T_5(t, x) + T_6(t, x) dx = \int_U R_5(t, x) dx, \tag{6.24}$$

where

$$\begin{aligned} R_5(t, x) &:= \frac{\varepsilon}{4} |\nabla u_\varepsilon|^2 \left(A : D^2 \eta + 2 \nabla \eta \cdot (\nabla \cdot A) + \eta \partial_i \partial_j a_{ij} \right) \\ &\quad - \frac{\varepsilon}{2} \eta \partial_j u_\varepsilon \partial_i \partial_k a_{ij} \partial_k u_\varepsilon - \frac{\varepsilon}{2} \partial_i \eta \partial_j u_\varepsilon \partial_k a_{ij} \partial_k u_\varepsilon \end{aligned} \quad (6.25)$$

Finally, we manipulate the term R_1 in (6.7). For this we first observe that

$$\begin{aligned} \nabla \cdot (\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon) &= \varepsilon \Delta u_\varepsilon \nabla u_\varepsilon + \varepsilon D^2 u_\varepsilon \nabla u_\varepsilon \\ &= \varepsilon \Delta u_\varepsilon \nabla u_\varepsilon + \nabla \cdot \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 \right) \end{aligned} \quad (6.26)$$

and therefore

$$\begin{aligned} w_\varepsilon \nabla u_\varepsilon &= -\varepsilon \Delta u_\varepsilon \nabla u_\varepsilon + \frac{1}{\varepsilon} F'(u_\varepsilon) \nabla u_\varepsilon \\ &= -\nabla \cdot (\varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon) + \nabla \cdot \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right). \end{aligned} \quad (6.27)$$

Using this equality we derive

$$\begin{aligned} &\int_U R_1(t, x) dx \\ &= \int_U \eta(x) w_\varepsilon(t, x) \frac{1}{2} c(t, x) \cdot \nabla u_\varepsilon(t, x) dx \\ &= \frac{1}{2} \int_U \left(\nabla \eta(x) \otimes c(t, x) + \eta(t, x) Dc(t, x) \right) : \varepsilon \nabla u_\varepsilon(t, x) \otimes \nabla u_\varepsilon(t, x) dx \\ &\quad - \frac{1}{2} \int_U \left(\nabla \eta(t, x) \cdot c(t, x) + \eta(t, x) \nabla \cdot c(t, x) \right) \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(t, x)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) \right) dx. \end{aligned} \quad (6.28)$$

Evaluating now (6.5), inserting (6.6), (6.10), (6.13), (6.18), and using (6.20), (6.24) we deduce that

$$\begin{aligned} \mu_\varepsilon^{t_1}(\eta) - \mu_\varepsilon^{t_0}(\eta) &= - \int_{t_0}^{t_1} \int_U \eta(x) \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt + \int_{t_0}^{t_1} \int_U w_\varepsilon \nabla \eta \cdot \nabla u_\varepsilon dx dt \\ &\quad + \int_U \int_{t_0}^{t_1} \eta(x) w_\varepsilon(x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\ &\quad - \int_U \int_{t_0}^{t_1} \varepsilon \nabla u_\varepsilon(t, x) \cdot \nabla \eta(x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\ &\quad + \int_{t_0}^{t_1} \int_U \left(R_1(t, x) + R_2(t, x) + R_3(t, x) + R_4(t, x) + R_5(t, x) \right) dx dt. \end{aligned} \quad (6.29)$$

By the definition of R_1, \dots, R_5 and (6.28) we obtain

$$\begin{aligned}
& \int_U \left(R_1 + R_2 + R_3 + R_4 + R_5 \right) dx \\
&= \int_U \left[\eta \frac{1}{2} \left(Dc - (\nabla \cdot c) \text{Id} + \partial_k \partial'_k \tilde{A} + \nabla \cdot \nabla \cdot A \text{Id} - D(\nabla \cdot A) \right) : \varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon \right. \\
&\quad + \eta \frac{1}{2} \left(-\nabla \cdot c + \nabla \cdot (\nabla \cdot A) \right) \frac{1}{\varepsilon} F(u_\varepsilon) \\
&\quad + \frac{1}{2} \left(-\nabla \eta \cdot c \text{Id} - \partial_i \eta \nabla a_i + \left(\frac{1}{2} A : D^2 \eta + \nabla \eta \cdot (\nabla \cdot A) \right) \text{Id} \right) : \varepsilon \nabla u_\varepsilon \otimes \nabla u_\varepsilon \\
&\quad \left. + \left(-\frac{1}{2} \nabla \eta \cdot c + \frac{1}{2} D^2 \eta : A + \nabla \eta \cdot (\nabla \cdot A) \right) \frac{1}{\varepsilon} F(u_\varepsilon) \right] dx. \tag{6.30}
\end{aligned}$$

and by (6.29) this shows (6.1). \square

In the following we only need an estimate for the global energy.

Corollary 6.3. *For all $\varepsilon > 0$ and all $0 < t_0 < t_1$*

$$\begin{aligned}
& E_\varepsilon(u_\varepsilon(t_1, \cdot)) - E_\varepsilon(u_\varepsilon(t_0, \cdot)) \\
&= - \int_{t_0}^{t_1} \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt + \int_U \int_{t_0}^{t_1} w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\
&\quad + \int_{t_0}^{t_1} \int_U \varepsilon \nabla u_\varepsilon(t, x) \cdot \Psi(t, x) \nabla u_\varepsilon(t, x) + \psi(t, x) \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) dx dt \tag{6.31}
\end{aligned}$$

holds with

$$\begin{aligned}
\Psi(t, x) &= \frac{1}{2} \left(Dc - (\nabla \cdot c) \text{Id} + \partial_k \partial'_k \tilde{A} + (\nabla \cdot \nabla \cdot A) \text{Id} - D(\nabla \cdot A) \right), \\
\psi &= \frac{1}{2} \left(-\nabla \cdot c + \nabla \cdot \nabla \cdot A \right). \tag{6.32}
\end{aligned}$$

Proof. Taking $\eta = 1$ in (6.1) and evaluating (6.30) we immediately obtain the desired estimate. \square

Proposition 6.4. *Let $T > 0$ be given. Then for every $p \in \mathbb{N}$ there exists $C_p = C_p(\|b\|_{C^0([0, T] \times U)}, \|\tilde{A}\|_{C^0([0, T]; C^2(U)})$ such that the following estimates hold: For every $\varepsilon > 0$, and for all $0 \leq s \leq t \leq T$*

$$\mathbb{E} \left[\left(E_\varepsilon(u_\varepsilon(t, \cdot)) \right)^p \middle| \mathcal{F}_s \right] \leq E_\varepsilon(u_\varepsilon(s, \cdot))^p \exp(C_p(t - s)) \tag{6.33}$$

and

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[\int_0^t \left(E_\varepsilon(u_\varepsilon(s, \cdot)) \right)^{p-1} \int_U \frac{1}{\varepsilon} w_\varepsilon^2(s, x) dx ds \middle| \mathcal{F}_s \right] \leq C_p. \tag{6.34}$$

Proof. We treat the case $p = 1$ first. Taking the expectation in (6.31) and using that the martingale part of the stochastic term has vanishing conditional expectation

yields that

$$\begin{aligned}
& \mathbb{E} \left[E_\varepsilon(u_\varepsilon(t, \cdot)) + \int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon^2(r, x) dx dr \middle| \mathcal{F}_s \right] \\
&= E_\varepsilon(u_\varepsilon(s, \cdot)) + \mathbb{E} \left[\int_s^t \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot b(r, x) dx dr \middle| \mathcal{F}_s \right] \\
&+ \mathbb{E} \left[\int_s^t \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(r, x) \cdot \Psi(r, x) \nabla u_\varepsilon(r, x) + \psi(r, x) \frac{1}{2\varepsilon} F(u_\varepsilon(r, x)) dx dr \middle| \mathcal{F}_s \right],
\end{aligned} \tag{6.35}$$

with ψ, Ψ as defined in (6.32). The first integral on the right hand side can be bounded using Young's inequality:

$$\begin{aligned}
& \left| \int_s^t \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot b(r, x) dx dr \right| \\
&\leq \frac{1}{2} \int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr + \frac{1}{2} \|b\|_{C^0([0, T] \times U)}^2 \int_s^t \int_U \varepsilon |\nabla u_\varepsilon(r, x)|^2 dx dr.
\end{aligned} \tag{6.36}$$

For the second integral we obtain that

$$\begin{aligned}
& \left| \int_s^t \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(r, x) \cdot \Psi(r, x) \nabla u_\varepsilon(r, x) + \psi(r, x) \frac{1}{2\varepsilon} F(u_\varepsilon(r, x)) dx dr \right| \\
&\leq C(\|\tilde{A}\|_{C^0([0, T]; C^2(U))}) \int_s^t E_\varepsilon(u_\varepsilon(r, \cdot)) dr.
\end{aligned} \tag{6.37}$$

Therefore, we obtain from (6.35) - (6.37)

$$\begin{aligned}
& \mathbb{E} \left[\left(E_\varepsilon(u_\varepsilon(t, \cdot)) + \int_s^t \int_U \frac{1}{2\varepsilon} w_\varepsilon(r, t)^2 dx dr \right) \middle| \mathcal{F}_s \right] \\
&\leq E_\varepsilon(u_\varepsilon(s, \cdot)) + C(\|b\|_{C^0([0, T] \times U)}^2, \|\tilde{A}\|_{C^0([0, T]; C^2(U))}) \int_s^t \mathbb{E} \left[E_\varepsilon(u_\varepsilon(r, \cdot)) \middle| \mathcal{F}_s \right] dr.
\end{aligned} \tag{6.38}$$

Thus Gronwall's Lemma first yields (6.33) for $p = 1$. Then (6.38) and (6.33) imply (6.34) for $p = 1$.

Let us now treat the case $p \geq 2$. Note that Itô's formula implies

$$dE_\varepsilon^p = pE_\varepsilon^{p-1} dE_\varepsilon + \frac{p(p-1)}{2} E_\varepsilon^{p-2} d\langle E_\varepsilon \rangle. \tag{6.39}$$

From (6.31) we see that

$$d\langle E_\varepsilon \rangle_t = \int_U \int_U w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \tilde{A}(t, x, y) w_\varepsilon(t, y) \nabla u_\varepsilon(t, y) dx dy dt. \tag{6.40}$$

Therefore, we have

$$\begin{aligned}
dE_\varepsilon^p &= pE_\varepsilon^{p-1} \left(\int_U -\frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt + \int_U w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \right. \\
&\quad \left. + \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(t, x) \cdot \Psi(t, x) \nabla u_\varepsilon(t, x) dt + \psi(t, x) \frac{1}{2\varepsilon} F(u_\varepsilon(t, x)) dx dt \right) \\
&\quad + \frac{p(p-1)}{2} E_\varepsilon^{p-2} \left(\int_U \int_U w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \tilde{A}(t, x, y) w_\varepsilon(t, y) \nabla u_\varepsilon(t, y) dx dy \right) dt.
\end{aligned} \tag{6.41}$$

Noting as above that the conditional expectation of the martingale part of the stochastic integral vanishes, we get

$$\begin{aligned}
&\mathbb{E} \left[E_\varepsilon(u_\varepsilon(t, \cdot))^p + p \int_s^t \left(E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \int_U \frac{1}{\varepsilon} w_\varepsilon^2(r, x) dx \right) dr \middle| \mathcal{F}_s \right] \\
&= E_\varepsilon(u_\varepsilon(s, \cdot))^p + \mathbb{E} \left[\int_s^t p E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \left(\int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot b(r, x) dx \right) dr \right. \\
&\quad \left. + \int_s^t p E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \right. \\
&\quad \quad \left(\int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(r, x) \cdot \Psi(r, x) \nabla u_\varepsilon(r, x) + \psi(r, x) \frac{1}{2\varepsilon} F(u_\varepsilon(r, x)) dx \right) dr \\
&\quad \left. + \int_s^t \frac{p(p-1)}{2} E_\varepsilon^{p-2} \right. \\
&\quad \quad \left. \left(\int_U \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot \tilde{A}(r, x, y) w_\varepsilon(r, y) \nabla u_\varepsilon(r, y) dx dy \right) dr \middle| \mathcal{F}_s \right].
\end{aligned} \tag{6.42}$$

We now give bounds on the individual terms on the right hand side of (6.42): Using Young's inequality one obtains for the first integral for any $\delta > 0$:

$$\begin{aligned}
&\left| \int_s^t p E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \left(\int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot b(r, x) dx \right) dr \right| \\
&\leq \|b(r, x)\|_{C^0([0, T] \times U)} \int_s^t p E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \\
&\quad \left(\frac{\delta}{2} \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx + \frac{1}{2\delta} \int_U \varepsilon |\nabla u_\varepsilon(r, x)|^2 dx \right) dr \\
&\leq \|b(r, x)\|_{C^0([0, T] \times U)} \int_s^t p E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \left(\frac{\delta}{2} \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx + \frac{1}{\delta} E_\varepsilon(u_\varepsilon(r, \cdot)) \right) dr.
\end{aligned} \tag{6.43}$$

The terms in the second line can directly be bounded:

$$\begin{aligned}
&\int_s^t p E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \left(\int_U \frac{\varepsilon}{2} \nabla u_\varepsilon(r, x) \cdot \Psi(r, x) \nabla u_\varepsilon(r, x) + \psi(r, x) \frac{1}{2\varepsilon} F(u_\varepsilon(r, x)) dx \right) dr \\
&\leq p \left(\|\Psi\|_{C^0([0, T] \times U)} + \|\psi\|_{C^0([0, T] \times U)} \right) \int_s^t \mathbb{E} \left[E_\varepsilon(u_\varepsilon(r, \cdot))^p \right] dr.
\end{aligned} \tag{6.44}$$

To bound the last term in (6.42) we use (6.27); a partial integration yields

$$\int_U \tilde{A}(r, x, y) w_\varepsilon(r, y) \nabla u_\varepsilon(r, y) dy \quad (6.45)$$

$$\begin{aligned} &= \int_U \nabla_y \tilde{A}(r, x, y) (\varepsilon \nabla u_\varepsilon(r, y) \otimes \nabla u_\varepsilon(r, y)) \\ &\quad - \nabla_y \tilde{A}(r, x, y) \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon(r, y)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(r, y)) \right) dy. \end{aligned} \quad (6.46)$$

Thus repeating the same partial integration in the x -variable we can conclude that

$$\begin{aligned} &\left| \int_U \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot \tilde{A}(r, x, y) w_\varepsilon(r, y) \nabla u_\varepsilon(r, y) dx dy \right| dr \\ &\leq 4 \|\tilde{A}\|_{C^0([0, T]; C^2(\bar{U} \times \bar{U}))} E_\varepsilon(u_\varepsilon(t, \cdot))^2 \end{aligned} \quad (6.47)$$

So the last term in (6.42) can be bounded by

$$\begin{aligned} &\mathbb{E} \left[\int_s^t \frac{p(p-1)}{2} E_\varepsilon^{p-2} \left(\int_U \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \tilde{A}(r, x, y) w_\varepsilon(r, y) \nabla u_\varepsilon(r, y) dx dy \right) dr \middle| \mathcal{F}_s \right] \\ &\leq 2p(p-1) \|\tilde{A}\|_{C^0([0, T], C^2(\bar{U} \times \bar{U}))} \int_s^t \mathbb{E} [E_\varepsilon^p(u_\varepsilon(t, \cdot)) | \mathcal{F}_s] ds \end{aligned} \quad (6.48)$$

Therefore, we get from (6.42)-(6.48) taking $\delta = \left(\|b\|_{C^0([0, T] \times U)} \right)^{-1}$ that:

$$\begin{aligned} &\mathbb{E} \left[E_\varepsilon(u_\varepsilon(t, \cdot))^p + \frac{p}{2} \int_s^t \left(E_\varepsilon(u_\varepsilon(r, \cdot))^{p-1} \int_U \frac{1}{\varepsilon} w_\varepsilon^2(r, x) dx \right) dr \middle| \mathcal{F}_s \right] \\ &\leq E_\varepsilon(u_\varepsilon(t, \cdot))^p + C(p, \|b\|_{C^0([0, T] \times U)}, \|\tilde{A}\|_{C^0([0, T]; C^2(\bar{U} \times \bar{U}))}) \int_s^t \mathbb{E} \left[E_\varepsilon(u_\varepsilon(r, \cdot))^p \middle| \mathcal{F}_s \right] dr. \end{aligned} \quad (6.49)$$

Thus (6.33) follows from another application of Gronwall's Lemma. In order to deduce (6.34) we use (6.49) and note that the terms on the right hand side can be bounded using (6.33). \square

Lemma 6.5. *For every $p \in \mathbb{N}$ we have*

$$\lim_{\lambda \rightarrow \infty} \sup_{\varepsilon > 0} \mathbb{P} \left[\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t, \cdot))^p > \lambda \right] = 0, \quad (6.50)$$

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t, \cdot))^p \right] < \infty \quad (6.51)$$

Proof. Fix $p \in \mathbb{N}$. For $t \geq 0$ set $Y_p^\varepsilon(t) = \exp(-C_p t) E_\varepsilon(u_\varepsilon(t, \cdot))$. From Proposition 6.4 we conclude that $Y_p^\varepsilon(t)$ is a supermartingale. Thus Doob's maximal inequality [26, Theorem 3.8 (ii) on page 14] implies

$$\begin{aligned} \mathbb{P} \left[\sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon)^p > \lambda \right] &\leq \mathbb{P} \left[\sup_{0 \leq t \leq T} Y_p^\varepsilon(t) > \lambda \exp(-C_p T) \right] \\ &\leq \lambda^{-1} \exp(C_p T) E_\varepsilon(u_\varepsilon^0)^p. \end{aligned} \quad (6.52)$$

Due to (3.5) the last term converges to 0 for $\lambda \rightarrow \infty$ uniformly in ε . Assertion (6.51) follows in the same way using the integrated version of the maximal inequality [26, Theorem 3.8 (iv) on page 14]. \square

We will need the following bound for the diffuse mean curvature.

Lemma 6.6. *For all $p \in \mathbb{N}$ and all $T > 0$ we have:*

$$\sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt \right)^p \right] < \infty \quad (6.53)$$

Proof. Equation (6.31) implies:

$$\begin{aligned} & \int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) dx dt \\ &= E_\varepsilon(u_\varepsilon^0) - E_\varepsilon(u_\varepsilon(T, \cdot)) + \int_U \int_0^T w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \\ & \quad + \int_0^T \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon \cdot \Psi \nabla u_\varepsilon + \psi \frac{1}{\varepsilon} F(u_\varepsilon) dx dt. \end{aligned} \quad (6.54)$$

We then compute that

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 dx dt \right)^p \right] \\ & \leq 2^{p-1} \mathbb{E} \left[\left| E_\varepsilon(u_\varepsilon^0) - E_\varepsilon(u_\varepsilon(T, \cdot)) \right|^p \right] \\ & \quad + \left| \int_0^T \int_U \frac{\varepsilon}{2} \nabla u_\varepsilon \cdot \Psi \nabla u_\varepsilon + \psi \frac{1}{\varepsilon} F(u_\varepsilon) dx dt \right|^p \\ & \quad + 2^{p-1} \mathbb{E} \left[\left| \int_U \int_0^T w_\varepsilon \nabla u_\varepsilon \cdot X(dt, x) dx \right|^p \right]. \end{aligned} \quad (6.55)$$

By (6.51) the terms in the second and third line of (6.55) are bounded uniformly in ε . For the last term one writes using Burkholder-Davis-Gundy inequality ([26, Theorem 3.28 on page 166]):

$$\begin{aligned} & \mathbb{E} \left[\left| \int_U \int_0^T w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot X(dt, x) dx \right|^p \right] \\ & \leq 2^{p-1} \mathbb{E} \left[\left| \int_0^T \int_U w_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \cdot b(t, x) dx dt \right|^p \right] \\ & \quad + 2^{p-1} \mathbb{E} \left[\left| \int_0^T \int_U \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot \tilde{A}(r, x, y) \nabla u_\varepsilon(r, y) w_\varepsilon(r, y) dx dy dt \right|^{p/2} \right]. \end{aligned} \quad (6.56)$$

Using Young's inequality the first term on the right hand side of (6.56) can be estimated by

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^T \int_U w_\varepsilon \nabla u_\varepsilon \cdot b \, dx \, dt \right|^p \right] \\
& \leq \|b\|_{C^0([0,T] \times U)}^p \mathbb{E} \left[\left| \int_0^T \int_U \frac{\delta}{2\varepsilon} w_\varepsilon^2 + \frac{\varepsilon}{2\delta} |\nabla u_\varepsilon|^2 \, dx \, dt \right|^p \right] \\
& \leq 2^{p-1} \|b\|_{C^0([0,T] \times U)}^p \mathbb{E} \left[\delta \left| \int_0^T \int_U \frac{1}{2\varepsilon} w_\varepsilon^2 \, dx \, dt \right|^p + \left| \int_0^T \int_U \frac{\varepsilon}{2\delta} |\nabla u_\varepsilon|^2 \, dx \, dt \right|^p \right] \quad (6.57)
\end{aligned}$$

For the second term in (6.56) we write

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^T \int_U \int_U w_\varepsilon(r, x) \nabla u_\varepsilon(r, x) \cdot \tilde{A}(r, x, y) w_\varepsilon(r, y) \nabla u_\varepsilon(r, y) \, dx \right|^{p/2} \right] \\
& \leq \|\tilde{A}\|_{C^0([0,T] \times U \times U)}^{p/2} \mathbb{E} \left[\left| \int_0^T \left(\int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) \, dx \right) \left(\int_U \varepsilon |\nabla u_\varepsilon(t, x)|^2 \, dx \right) dt \right|^{p/2} \right] \\
& \leq \|\tilde{A}\|_{C^0([0,T] \times U \times U)}^{p/2} \mathbb{E} \left[\left| \sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t, \cdot)) \right|^{p/2} \left| \int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) \, dx \, ds \right|^{p/2} \right] \\
& \leq \delta \mathbb{E} \left[\left| \int_0^T \int_U \frac{1}{\varepsilon} w_\varepsilon^2(t, x) \, dx \, ds \right|^p \right] + \|\tilde{A}\|_{C^0([0,T] \times U \times U)}^{p/2} \frac{1}{4\delta} \mathbb{E} \left[\left| \sup_{0 \leq t \leq T} E_\varepsilon(u_\varepsilon(t, \cdot)) \right|^p \right]. \quad (6.58)
\end{aligned}$$

Choosing $\delta = \delta(\|\tilde{A}\|_{C^0([0,T] \times U \times U)}, \|b\|_{C^0([0,T] \times U)})$ small enough the first term on the right hand side of (6.57) and (6.58) respectively can be absorbed into the left hand side of (6.55). The other terms are bounded by (6.51). This finishes the proof. \square

Denote by G the function $G(r) = \int_0^r \sqrt{2F(r)} \, dr$. Note that G is smooth, increasing and that $G(0) = 0$. We will need the following bound on the increments of $G(u_\varepsilon(t))$:

Lemma 6.7. *For every smooth testfunction $\varphi \in C^\infty(U)$ and every $p \in \mathbb{N}$ we have for all $0 \leq s < t \leq T$*

$$\mathbb{E} \left[\left| \int_U G(u_\varepsilon(t, x)) \varphi(x) \, dx - \int_U G(u_\varepsilon(s, x)) \varphi(x) \, dx \right|^{2p} \right] \leq C |t - s|^p, \quad (6.59)$$

where $C = C(p, \|\tilde{A}\|_{C^0([0,T] \times U \times U)}, \|A\|_{C^0([0,T]; C^1(U))}, \|b, c\|_{C^0([0,T] \times U)}, \|\varphi\|_{C^1(U)})$.

Proof. Noting that $G'(u_\varepsilon) = \sqrt{2F(u_\varepsilon)}$, Itô's formula and (4.6), (6.3) imply that

$$\begin{aligned}
& \int_U G(u_\varepsilon(t, x)) \varphi(x) dx - \int_U G(u_\varepsilon(s, x)) \varphi(x) dx \\
&= \int_U \left[\int_s^t \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left[\left(\Delta u_\varepsilon(r, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(r, x)) \right) dr \right. \right. \\
&\quad \left. \left. + \int_s^t \nabla u_\varepsilon(r, x) \cdot X(dr, x) \right. \right. \\
&\quad \left. \left. + \int_s^t \frac{1}{2} A(r, x) : D^2 u_\varepsilon(r, x) dr + \frac{1}{2} c(r, x) \cdot \nabla u_\varepsilon(r, x) dr \right] dx \right. \\
&\quad \left. + \frac{1}{2} \int_s^t \int_U G''(u_\varepsilon(r, x)) \varphi(x) \nabla u_\varepsilon(r, x) \cdot A(r, x) \nabla u_\varepsilon(r, x) dx dr. \right. \tag{6.60}
\end{aligned}$$

Thus for $p \in \mathbb{N}$ one can write

$$\mathbb{E} \left[\left| \int_U G(u_\varepsilon(t)) \varphi dx - \int_U G(u_\varepsilon(s)) \varphi dx \right|^{2p} \right] \leq 4^{p-1} (I_1 + I_2 + I_3 + I_4). \tag{6.61}$$

Let us bound the individual terms:

$$\begin{aligned}
I_1 &= \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left[\Delta u_\varepsilon(r, x) - \frac{1}{\varepsilon^2} F'(u_\varepsilon(r, x)) \right] dx dr \right|^{2p} \right] \\
&\leq \|\varphi\|_{C^0(U)}^{2p} \mathbb{E} \left[\left(\int_s^t \int_U \frac{1}{\varepsilon} F(u_\varepsilon(r, x)) dx dr \right)^p \left(\int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr \right)^p \right] \\
&\leq \|\varphi\|_{C^0(U)}^{2p} \mathbb{E} \left[\left(\int_s^t E_\varepsilon(u_\varepsilon(r, \cdot)) dr \right)^{2p} \right]^{1/2} \mathbb{E} \left[\left(\int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr \right)^{2p} \right]^{1/2} \\
&\leq \|\varphi\|_{C^0(U)}^{2p} (t-s)^p \mathbb{E} \left[\left(\sup_{s \leq r \leq t} E_\varepsilon(u_\varepsilon(r, \cdot)) \right)^{2p} \right]^{1/2} \mathbb{E} \left[\left(\int_s^t \int_U \frac{1}{\varepsilon} w_\varepsilon(r, x)^2 dx dr \right)^{2p} \right]^{1/2} \\
&\leq C(t-s)^p. \tag{6.62}
\end{aligned}$$

Here in the second and third line we have used Cauchy-Schwarz inequality. In the last line we have used (6.51) and (6.53). The second term can be bounded using Young's inequality:

$$\begin{aligned}
I_2 &= \mathbb{E} \left[\left| \int_s^t \int_U \varphi \sqrt{2F(u_\varepsilon(r, x))} \left[\nabla u_\varepsilon(r, x) \cdot \left(\frac{1}{2} c(r, x) + b(r, x) \right) \right] dx dr \right|^{2p} \right] \\
&\leq \|\varphi\|_{C^0(U)}^{2p} \|c + b\|_{C^0([0, T] \times U)}^{2p} \mathbb{E} \left[\left| \int_s^t E_\varepsilon(u_\varepsilon(r, \cdot)) dr \right|^{2p} \right] \\
&\leq C(t-s)^{2p}. \tag{6.63}
\end{aligned}$$

Here we have used the inequalities $|\nabla u_\varepsilon \sqrt{2F(u_\varepsilon)}| \leq \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon)$ and (6.51) in the last line. For the martingale term we get using Burkholder-Davis-Gundy

inequality in the second line and then Youngs inequality in the third line:

$$\begin{aligned}
I_3 &= \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left[\nabla u_\varepsilon(r, x) \cdot (X(dr, x) - b(r, x)dr) \right] dx \right|^{2p} \right] \\
&\leq \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \nabla u_\varepsilon(r, x) \cdot \right. \right. \\
&\quad \left. \left. \cdot \tilde{A}(r, x, y) \nabla u_\varepsilon(r, y) \sqrt{2F(u_\varepsilon(r, y))} \varphi(y) dx dy dr \right|^p \right] \\
&\leq \|\varphi\|_{C^0(U)}^{2p} \|\tilde{A}\|_{C^0([0, T] \times U)}^p \mathbb{E} \left[\left(\int_s^t \left(\int_U \sqrt{2F(u_\varepsilon(r, x))} |\nabla u_\varepsilon(r, x)| dx \right)^2 dr \right)^p \right] \\
&\leq \|\varphi\|_{C^0(U)}^{2p} \|\tilde{A}\|_{C^0([0, T] \times U)}^p \mathbb{E} \left[\left| \int_s^t E_\varepsilon(u_\varepsilon(r, \cdot))^2 dr \right|^p \right] \\
&\leq C(t-s)^p. \tag{6.64}
\end{aligned}$$

For the fourth term we write:

$$\begin{aligned}
I_4 &= \frac{1}{2} \mathbb{E} \left[\left| \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} \left(A(r, x) : D^2 u_\varepsilon(r, x) \right) \right. \right. \\
&\quad \left. \left. + \varphi(x) G''(u_\varepsilon(r, x)) \nabla u_\varepsilon(r, x) \cdot A(r, x) \nabla u_\varepsilon(r, x) dx dr \right|^{2p} \right]. \tag{6.65}
\end{aligned}$$

After a partial integration the second summand can be written as:

$$\begin{aligned}
&\int_U \varphi(x) G''(u_\varepsilon(r, x)) \nabla u_\varepsilon(x) \cdot A(r, x) \nabla u_\varepsilon(x) dx \\
&= \int_U \varphi(x) \nabla \left(\sqrt{2F(u_\varepsilon(r, x))} \right) \cdot A(r, x) \nabla u_\varepsilon(x) dx \\
&= - \int_U \nabla \varphi(x) \cdot A(r, x) \nabla u_\varepsilon(x) \sqrt{2F(u_\varepsilon(r, x))} dx \\
&\quad - \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} (\nabla \cdot A(r, x)) \nabla u_\varepsilon(r, x) dx \\
&\quad - \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} A(r, x) : D^2 u_\varepsilon(r, x) dx. \tag{6.66}
\end{aligned}$$

Noting that the terms involving $D^2 u$ in (6.65) and (6.66) cancel it remains to bound:

$$\begin{aligned}
&\mathbb{E} \left[\left| \int_s^t \int_U \nabla \varphi(x) \cdot A(r, x) \nabla u_\varepsilon(x) \sqrt{2F(u_\varepsilon(r, x))} dx \right. \right. \\
&\quad \left. \left. + \int_s^t \int_U \varphi(x) \sqrt{2F(u_\varepsilon(r, x))} (\nabla \cdot A(r, x)) \nabla u_\varepsilon(r, x) dx ds \right|^{2p} \right] \\
&\leq \left(2 \|A\|_{C^0([0, T], C^1(U))} \|\varphi\|_{C^1(U)} \right)^{2p} \mathbb{E} \left[\left| \int_s^t E_\varepsilon(u_\varepsilon(r)) dr \right|^{2p} \right] \\
&\leq C(t-s)^{2p}. \tag{6.67}
\end{aligned}$$

This finishes the proof. \square

Now we are ready to prove our main theorem.

Proof of Theorem 4.2. As a first step we will show that the distributions of $G(u_\varepsilon)$ are tight on $C([0, T], L^1(U))$. To this end it suffices to show the following two conditions (see e.g. [12, Theorem 3.6.4, page 54] together with [5, Theorem 8.3, page 56]. Note that conditions (8.3) and (8.4) in [5] are implied by our stronger assumption (ii)):

- (i) (*Compact Containment*) For every $\delta > 0$ there exists a compact set $K^\delta \subseteq L^1(U)$ such that for all $\varepsilon > 0$

$$\mathbb{P}\left[G(u_\varepsilon(t, \cdot)) \in K^\delta \text{ for all } 0 \leq t \leq T\right] \geq 1 - \delta \quad (6.68)$$

- (ii) (*Weak tightness*) For every smooth testfunction $\varphi \in C^\infty(U)$ there exist positive α, β, C such that for all $0 \leq s < t \leq T$

$$\mathbb{E}\left[\left|\int_U G(u_\varepsilon(t, x))\varphi(x) dx - \int_U G(u_\varepsilon(s, x))\varphi(x) dx\right|^\alpha\right] \leq C|t - s|^{1+\beta}. \quad (6.69)$$

To prove the first statement note that

$$\begin{aligned} \int_U |\nabla G(u_\varepsilon(t, x))| dx &= \int_U |\sqrt{2F(u_\varepsilon(t, x))}| |\nabla u_\varepsilon(t, x)| dx \\ &\leq \int_U \frac{\varepsilon}{2} |\nabla u_\varepsilon(t, x)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(t, x)) dx. \end{aligned} \quad (6.70)$$

Furthermore, (3.1) implies $G(r) \leq C(1 + F(r))$ such that

$$\int_U |G(u_\varepsilon(t, x))| dx \leq C \int_U F(u_\varepsilon(t, x)) dx + C|U|. \quad (6.71)$$

Thus Lemma 6.5 yields that

$$\mathbb{P}\left[\sup_{0 \leq t \leq T} \|G(u_\varepsilon(t, \cdot))\|_{W^{1,1}(U)} \geq \lambda\right] \rightarrow 0, \quad (6.72)$$

for $\lambda \rightarrow \infty$ which together with Rellich's Theorem implies condition (i).

The second condition (ii) follows from (6.59). The tightness of $G(u_\varepsilon)$ is thus proved.

As a second step we prove that the distributions of u_ε are tight in $C([0, T], L^1(U))$. Denote by G^{-1} the inverse function of G . Then similar to [34, page 139] we observe that the operator $v \mapsto G^{-1} \circ v$ is continuous from $L^1(U)$ to itself. In fact assume $v_i \rightarrow v$ as $i \rightarrow \infty$ in $L^1(U)$. The growth condition in (3.1) implies that G^{-1} is uniformly continuous on \mathbb{R} . This implies convergence in measure and convergence pointwise in U for the sequence $G^{-1} \circ v_i$. Furthermore, using the growth condition once more one can see that $|G^{-1} \circ v_i| \leq C(|v_i| + 1)$ which then implies by Vitali's Convergence Theorem that $G^{-1}(v_i) \rightarrow G^{-1}(v)$ in $L^1(U)$. Thus using the following Lemma 6.8 we can conclude that the distributions of u_ε are tight on $C([0, T], L^1(U))$ as well.

In particular, there exists a decreasing sequence $\varepsilon_k \downarrow 0$ such that the distributions of u_ε converge weakly to a limiting measure on $C([0, T], L^1(U))$. We may now use Skorohod's observation that we can find a subsequence $\varepsilon_k \downarrow 0$ such that the random

functions u_{ε_k} can be realized on a single probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. On this space the u_{ε_k} converge almost surely in $C^0([0, T]; L^1(U))$ towards u (see [24, page 9]). By (6.51) and Fatou's Lemma we can for almost all $\omega \in \Omega, t \in (0, T)$ select a subsequence $\varepsilon'_k \rightarrow 0$ such that $\sup_{\varepsilon'_k} E_{\varepsilon'_k}(u_{\varepsilon'_k}(t, \cdot)) < \infty$. Thus using the Gamma convergence of the functionals E_ε we can conclude that $u_{\varepsilon'_k}(t, \cdot)$ converges for a subsequence strongly in $L^1(U)$ to a limit $v \in BV(U; \{-1, 1\})$ and that

$$\|v\| \leq \liminf_{\varepsilon'_k \rightarrow 0} E_{\varepsilon'_k}(u_{\varepsilon'_k}(t, \cdot)) \quad (6.73)$$

But since $u_{\varepsilon_k} \rightarrow u$ almost surely in $C([0, T], L^1(U))$ we have $v = u(t, \cdot)$. This proves that $u(t, \cdot) \in BV(U; \{-1, 1\})$; (6.51), (6.73) yield

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{BV(U)}^p \right] < \infty.$$

which concludes the proof. \square

Lemma 6.8. *Let $((X_t^\varepsilon, t \in [0, T]), \varepsilon > 0)$ be a family of stochastic processes taking values in a separable metric space E . Let \tilde{E} be another separable metric space and $F: E \rightarrow \tilde{E}$ a continuous function. Suppose the family of distributions of X_t^ε is tight on $C([0, T], E)$. Then the family of distributions of $F(X_t^\varepsilon)$ is tight on $C([0, T], \tilde{E})$.*

Proof. According to [17, Thm. 7.2 and Rem. 7.3 on page 128] the distributions of X_t^ε are tight if and only if

- (1) For all $\eta > 0$ there exists a compact set $\Gamma_\eta \subseteq E$ such that for all ε

$$\mathbb{P} \left[X_t^\varepsilon \in \Gamma_\eta, \quad \forall 0 \leq t \leq T \right] \geq 1 - \eta \quad (6.74)$$

- (2) For all $\eta > 0$ there exists $\delta > 0$ such that

$$\sup_t \mathbb{P} \left[\omega(X_t^\varepsilon, \delta) \geq \eta \right] \leq \eta, \quad (6.75)$$

where the modulus of continuity is defined as $\omega(X_t, \delta) = \sup_{t-\delta \leq s \leq t+\delta} d(X_t, X_s)$.

It is clear from the continuity of F that the $F(X_t^\varepsilon)$ satisfy the property corresponding to (6.74) if X_t does.

To see that $F(X_t)$ also satisfies (6.75) fix $\eta > 0$ and the set $\Gamma_{\eta/2}$ such that

$$\mathbb{P} \left[X_t^\varepsilon \in \Gamma_{\eta/2}, \quad \forall 0 \leq t \leq T \right] \geq 1 - \eta/2.$$

As $\Gamma_{\eta/2}$ is compact F is uniformly continuous when restricted to $\Gamma_{\eta/2}$. Thus one can choose η' such that $d(x, y) \leq \eta'$ implies $d(F(x), F(y)) \leq \eta$ for all $x, y \in \Gamma_{\eta/2}$. Thus choosing δ small enough such that $\sup_t \mathbb{P} \left[\omega(X_t, \delta) \geq \eta' \right] \leq \eta/2$ one obtains

$$\begin{aligned} \sup_t \mathbb{P} \left[\omega(F(X_t), \delta) \geq \eta \right] &\leq \sup_t \mathbb{P} \left[X_t \notin \Gamma_{\eta/2} \right] + \sup_t \mathbb{P} \left[\omega(X_t, \delta) \geq \eta' \right] \\ &\leq \frac{\eta}{2} + \frac{\eta}{2}. \end{aligned} \quad (6.76)$$

This finishes the proof. \square

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