The needle problem approach to non-periodic homogenization

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Abstract: We suggest a new approach to homogenization of non-periodic problems and illustrate the approach with the elliptic equation $-\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = f$. Our assumption on the coefficients a^ε is the following: for solutions u^ε of homogeneous ε-problems on simplices with average slope $\xi \in \mathbb{R}^n$, the flux-averages $\fint a^\varepsilon \nabla u^\varepsilon \in \mathbb{R}^n$ converge, for $\varepsilon \to 0$, to some limit $a^*(\xi)$, which is independent of the simplex. If this assumption is satisfied, we conclude the homogenization result for general domains and arbitrary f. The proof uses a new auxiliary problem, the *needle problem.* Solutions of the needle problem depend on a triangulation of the domain, they solve an ε-problem in each simplex and are affine on faces.

1 Introduction

Due to its relevance in many applications, homogenization theory is nowadays an important field of mathematical analysis. To give a very general description, homogenization is concerned with solutions u^ε of partial differential equations $\mathcal{A}^\varepsilon(u^\varepsilon) = f$, where f are given data and $\mathcal{A}^\varepsilon$ is a differential operator with oscillatory coefficients that vary on a scale of order $\varepsilon > 0$. The task is to determine a homogenized operator $\mathcal{A}^*$ such that solutions u^* of $\mathcal{A}^* u^* = f$ are approximations of the oscillatory solutions u^ε in the sense that $u^\varepsilon \to u^*$ for $\varepsilon \to 0$ in some norm.

The characterization of the effective operator $\mathcal{A}^*$ is usually given with a cell problem. Let us be more specific and describe the idea in the standard case of the operator $(\mathcal{A}^\varepsilon u)(x) = -\nabla \cdot (a^\varepsilon(x) \nabla u(x))$ for $u \in H^1_0(Q)$, understood in the weak sense on $Q \subset \mathbb{R}^n$. For periodic coefficients a^ε, the homogenized operator turns out to be $\mathcal{A}^* u = -\nabla \cdot (a^* \nabla u(x))$ with a matrix $a^* \in \mathbb{R}^{n\times n}$. The averaged coefficient a^* can be determined with the help of the following property: if a solution sequence u^ε of $\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = 0$ has the average slope $\xi \in \mathbb{R}^n$, then the corresponding fluxes $a^\varepsilon \nabla u^\varepsilon$ have the average value $a^*\xi$,

$$\nabla u^\varepsilon \rightharpoonup \xi \qquad \Rightarrow \qquad a^\varepsilon \nabla u^\varepsilon \rightharpoonup a^* \xi. \tag{1.1}$$

This property is usually encoded with a cell problem.

The aim of the contribution at hand is, loosely speaking, to conclude from property (1.1) the homogenization result $u^\varepsilon \to u^*$ for arbitrary Q and f. Theorem 1.2 investigates solutions u^ε and u^* on domains $Q \subset \mathbb{R}^2$ or $Q \subset \mathbb{R}^3$ and provides the convergence

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$u^\varepsilon \to u^*$, weakly in $H^1(Q)$. The important new feature of our result is that we do neither use periodicity of the coefficients nor a specific stochastic construction.

The above problem was treated and solved for periodic coefficients [4, 5, 12, 19, 23], with the method of two-scale convergence [2], with the periodic unfolding method [8], and in the stochastic case [7, 11, 15, 16]. Regarding homogenization of other equations we mention [1, 20, 22, 24, 25], regarding a further analysis of the homogenization limit or the homogenized problem [17, 26]. Recent results typically regard large coefficients or singular geometries [3, 6]. Numerical studies are concerned with the construction of fast methods that resolve the fine scale only on small sub-domains. One common method is the heterogeneous multiscale method [13, 14].

Homogenization as a two-step procedure. The new approach presented here is inspired by numerical methods and, more generally, by the principle of representative volume elements (RVEs). A loose description of such approaches is the following: the macroscopic domain is discretized with a triangulation as if a homogenized problem was available. In order to find the effective coefficients in each volume element of size h, a representative volume element is chosen with diameter large compared to ε, but small compared to h. The solution of an ε-problem on the RVE provides the effective coefficients in the volume element.

The heterogeneous multiscale method follows this idea, convergence results for the elliptic problem are obtained e.g. in [14]. The authors use an error e(HMM) which measures how well the homogenized matrix can be recovered by solving problems on RVEs. Theorem 1.1 of [14] shows that, without any assumptions on the coefficients, e(HMM) and the grid size control the error of the scheme. Further theorems provide the smallness of e(HMM) with appropriate bounds in several cases: in the periodic case, and in a stochastic case with mixing properties in dimensions 1 and 3.

We show a rigorous result in this spirit: we assume that homogeneous solutions on simple domains with affine boundary conditions corresponding to slope ξ have an averaged flux $a^*\xi$, independent of the domain. Our result is that then a^* is the matrix of homogenized coefficients in general boundary value problems. The needle problem introduces intermediate solutions that can be regarded as the analog to discrete solutions in the heterogeneous multiscale method.

We regard the homogenization of an equation as a two-step procedure: in a first step one has to understand the behavior of solutions u^ε that approximate an affine function. These are the functions that are usually considered in cell problems. For such functions, the constitutive relation (e.g. between flux $a^\varepsilon\nabla u^\varepsilon$ and gradient ∇u^ε) must be investigated and an averaged constitutive relation for weak limits must be derived. In our case, this averaged relation is given by the matrix a^* in (1.1). In a second step, the data of the concrete problem are incorporated. One considers no longer affine boundary data on simple domains and homogeneous solutions, but solutions u^ε to given data Q and f. The aim in this second step is to show that the averaged constitutive relation defines indeed the averaged operator $\mathcal{A}^*$. Our contribution regards entirely the second step, our aim is to assume as little as possible about the first step.

With this aim, we will not even use the weak convergences of (1.1), but impose only a property of averages. Our stabilization result provides (1.1) as a consequence of the weaker assumption of Definition 1.1. The main difficulty in the verification of that assumption is to show that the limit of the averages exists and that it is independent of the simplex. In the context of stochastic coefficients, these properties can be regarded as an ergodicity and stationarity assumption on the coefficients. We emphasize that, in the standard stochastic setting, all our assumptions are satisfied, see Appendix A.

In the forthcoming contribution [21] we apply the method to a parabolic problem. Since our new approach is very general, we believe that it allows furthermore to perform the second step of the homogenization procedure for more complex operators such as e.g. hysteresis operators of plasticity equations.

The technique of the needle problem approach. The usual way to perform step 2 in the above program is to start from solutions of the cell problem and to construct test-functions. Our aim is not to use cell problem solutions, since they might not be available. As a replacement, we use solutions to the needle problem. The needle problem is the original problem with coefficients a^ε, introducing a side condition: we search for functions u_h^ε that are affine on the faces of a grid $\mathscr{T}_h$ and solutions in each simplex. The side condition implies that our general assumption on solutions to affine boundary data of Definition 1.1 is applicable. On the other hand, for small h, the side condition is not a severe restriction, and we find that $u^\varepsilon - u_h^\varepsilon$ is small. The combination of these two facts allows to conclude the homogenization result.

The main technical problem in our new method is that we need a div-curl-Lemma in each simplex of the triangulation. Since in the simplices of the triangulation we do not have prescribed boundary conditions for u^ε, the standard div-curl-lemma does not apply. We can provide a div-curl-lemma under the assumption that the grid is adapted to the sequence u^ε. To give a first idea of that property, we observe the following: Since the sequence ∇u^ε is bounded in $L^2(Q)$, on almost every hyperplane E through Q, the sequence $\nabla u^\varepsilon|_E$ is also bounded. Then the trace $u^\varepsilon|_E$ is not only controlled in $H^{1/2}(E)$, but also in $H^1(E)$. The corresponding compactness allows to conclude the div-curl-lemma for adapted grids. We emphasize that the lengthy construction of adapted grids in Section 4 can be used for any $H^1(Q)$-bounded sequence u^ε and is therefore completely independent of the equation; the results of Section 4 can be used in any other homogenization problem. The construction of the adapted grids has similarities with the constructions of [9, 10].

To conclude this introduction, we emphasize that the new aspects of our main theorem are: 1) the assumption on the sequence a^ε is very general, it includes periodic coefficients and ergodic stochastic coefficients. 2) the proof introduces the *needle problem* and relies on a div-curl-Lemma with boundary, which holds on adapted grids.

1.1 Main result

Let $Q \subset \mathbb{R}^n$ be bounded, open, with Lipschitz boundary, and let the family of coefficients $(a^\varepsilon)_\varepsilon$, with $a^\varepsilon \in L^\infty(Q;\mathbb{R}^{n\times n})$ for $\varepsilon > 0$, satisfy the uniform ellipticity and boundedness condition

$$\alpha_1|\eta|^2 \leq a^\varepsilon(x)\eta\cdot\eta \leq \alpha_2|\eta|^2, \qquad \forall\,\eta\in\mathbb{R}^n,\ \text{for a.e.}\,x\in\mathbb{R}^n, \tag{1.2}$$

for constants $0 < \alpha_1 < \alpha_2$. In the next condition we use a simplex $T \subset Q$ and, for $\xi \in \mathbb{R}^n$ and $b \in \mathbb{R}$, the affine function $U_\xi(x) := \xi\cdot x + b$ on T to prescribe boundary conditions. To these data, we study the unique weak solution $u^\varepsilon : T \to \mathbb{R}$ of the problem

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= 0 \qquad \text{in } T, \\ u^\varepsilon &= U_\xi \qquad \text{on } \partial T. \end{aligned} \tag{1.3}$$

In the subsequent definition we use the notation $\fint_A f := |A|^{-1} \int_A f$ for averages of an integrable function f on a domain A.

Definition 1.1. *We say that the coefficients a^ε* allow averaging of the constitutive relation with the matrix $a^* \in \mathbb{R}^{n\times n}$ *if the following is satisfied: for every simplex $T \subset Q$ and every $\xi \in \mathbb{R}^n$, $b \in \mathbb{R}$, the solutions u^ε of* (1.3) *satisfy*

$$\lim_{\varepsilon\to 0} \fint_T a^\varepsilon \nabla u^\varepsilon = a^* \xi \,. \tag{1.4}$$

As mentioned before, the property (1.4) is satisfied for periodic coefficients a^ε and for ergodic stochastic coefficients. Regarding the latter, we mention in Appendix A a theorem which is derived in [15] and which implies that ergodic stochastic coefficients allow averaging of the constitutive relation.

It would be slightly more general to write on the right hand side of (1.4) a general function $a^*(\xi)$ with $a^* : \mathbb{R}^n \to \mathbb{R}^n$. Since the problems are linear in ξ, we actually know that the limit (if it exists) must also be linear in ξ. The important assumption is therefore that the limit exists and that it is independent of T.

Our main result is the following homogenization theorem.

Theorem 1.2. *Let $Q \subset \mathbb{R}^n$ be an n-dimensional bounded domain with Lipschitz boundary and $n = 2$ or $n = 3$. Let $f \in L^2(Q)$ be arbitrary and let $\psi \in H^1(Q)$ be affine. We assume that the coefficients $(a^\varepsilon)_\varepsilon$ satisfy the ellipticity relation* (1.2) *and that they allow averaging of the constitutive relation with the matrix a^* in the sense of Definition 1.1. Then the sequence $(u^\varepsilon)_\varepsilon$ of weak solutions of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= f \qquad \text{in } Q, \\ u^\varepsilon &= \psi \qquad \text{on } \partial Q, \end{aligned} \tag{1.5}$$

satisfies

$$u^\varepsilon \rightharpoonup u^* \qquad \text{weakly in } H^1(Q), \tag{1.6}$$

$$a^\varepsilon \nabla u^\varepsilon \rightharpoonup a^* \nabla u^* \qquad \text{weakly in } L^2(Q), \tag{1.7}$$

where u^* is the weak solution of

$$\begin{aligned} -\nabla \cdot (a^* \nabla u^*) &= f \qquad \text{in } Q, \\ u^* &= \psi \qquad \text{on } \partial Q. \end{aligned} \tag{1.8}$$

The theorem is given here only for space dimension $n = 2$ and $n = 3$. The proof of the theorem, given in Sections 2 and 3, is independent of the dimension, but it uses the adapted grids of Theorem 4.8. The construction of adapted grids is performed here only in these lower dimensional cases. We have no doubt that Theorem 4.8 remains valid in higher dimension, but the notation is much more involved in the general case.

By an approximation, the condition $f \in L^2(Q)$ can easily be relaxed to $f \in H^{-1}(Q)$. The above theorem is stated for an affine boundary condition ψ. A general Dirichlet condition with $\psi \in H^1(Q)$ can be treated with slightly more notational effort. We note that the boundary condition $u^* = \psi$ on ∂Q is automatically satisfied for $H^1(Q)$-weak limits u^*. Therefore, we only have to verify the elliptic relation of (1.8) in the interior of Q.

1.2 Description of the needle problem method

Our method is based on a discretization of Q. The discretization introduces a mesh $\mathscr{T}_h$, the parameter h stands for the mesh-size. Given the triangulation, we consider two auxiliary problems. The first problem is the standard finite element discretization of the homogenized problem (1.8) with a solution U_h, introduced in Subsection 2.1. The solution U_h is used additionally in (2.8) to substitute the given right hand side f with an equivalent jump condition across the interfaces of the mesh.

The second auxiliary problem is the *needle problem* and we refer to Subsection 2.2 for its definition. Solutions are denoted as u_h^ε, these functions are affine on the interfaces introduced by $\mathscr{T}_h$, and they solve $-\nabla \cdot (a^\varepsilon \nabla u_h^\varepsilon) = 0$ in the simplices. These conditions help to conclude $u_h^\varepsilon \rightharpoonup U_h$ weakly in $H^1(Q)$, for $\varepsilon \to 0$. The homogenization program follows the scheme

$$\begin{array}{ccc} u_h^\varepsilon & \xrightarrow[\varepsilon]{L.\ 3.4} & U_h \\ {\scriptstyle \varepsilon, h}\big\updownarrow {\scriptstyle P.\ 2.6} & & {\scriptstyle h}\big\downarrow {\scriptstyle L.\ 2.1} \\ u^\varepsilon & & u^* \end{array}$$

The diagram illustrates the following results: $\lim_{h\to 0} \lim_{\varepsilon\to 0} \|u^\varepsilon - u_h^\varepsilon\|_{H^1(Q)} = 0$ of Proposition 2.6, the weak-$H^1(Q)$ convergence $u_h^\varepsilon \rightharpoonup U_h$ for $\varepsilon \to 0$ of Lemma 3.4, and $U_h \rightharpoonup u^*$ in $H^1(Q)$ for $h \to 0$ of Lemma 2.1. The combination of these results provides, since h is arbitrary, the weak-$H^1(Q)$ convergence $u^\varepsilon \rightharpoonup u^*$. In the diagram, the arrow on the right is a standard result for finite element discretizations. The arrow on the left is done by energy methods and reflects the testing procedure in common homogenization approaches; our new div-curl lemma is used here. The arrow on top is based on the averaging assumption of Definition 1.1. It involves a stabilization result, namely that indeed ∇u^ε and $a^\varepsilon \nabla u^\varepsilon$ converge weakly in $L^2(Q)$ to constant functions as in (1.1).

2 Two auxiliary problems

2.1 Discretization and the solution U_h

For arbitrary $h > 0$ we want to discretize Q with simplices. Since Q is, in general, not a polygonal domain, we discretize only a smaller, polygonal domain $Q_h \subset Q$. We demand that

$$\begin{aligned} &\mathscr{T}_h := \{T_k\}_{k\in\Lambda_h} \quad \text{is a triangulation of } Q_h, \quad \operatorname{diam}(T_k) < h \quad \forall\, T_k \in \mathscr{T}_h, \\ &Q_h \text{ has the property that } x \in Q, \operatorname{dist}(x, \partial Q) \geq h \text{ implies } x \in Q_h\,, \end{aligned} \tag{2.1}$$

where T_k are disjoint open simplices and Λ_h is a finite set of indices. We consider the finite element space of continuous and piecewise linear functions with vanishing boundary values,

$$Y_h := \left\{ \phi \in H_0^1(Q) \ : \ \phi|_{T_k} \text{ is affine } \forall\, T_k \in \mathscr{T}_h,\ \phi \equiv 0 \text{ on } Q \setminus Q_h \right\}.$$

With the matrix $a^* \in \mathbb{R}^{n\times n}$ of Definition 1.1, with $f \in L^2(Q)$ and the affine boundary condition ψ, we consider the following approximate problem.

$$\text{Find} \quad U_h \in \psi + Y_h \text{ with } \int_Q (a^* \nabla U_h) \cdot \nabla\phi = \int_Q f\phi, \qquad \forall\, \phi \in Y_h. \tag{2.2}$$

The following comparison is a standard observation for finite element approximations.

Lemma 2.1 (Comparison of U_h and u^*)**.** *There exists a unique solution U_h of* (2.2)*. For an affine boundary condition ψ there holds*

$$U_h \rightharpoonup u^* \text{ in } H^1(Q) \tag{2.3}$$

for $h \to 0$, where u^* is the solution of (1.8)*.*

Proof. Existence and uniqueness of solutions U_h together with uniform estimates in $H^1(Q)$ follow from the Lax-Milgram theorem, applied in the space Y_h. Weak convergence of a subsequence follows by compactness. The unique characterization of the limit is a consequence of the fact that the L^2-orthogonal projections $P_h : H_0^1(Q) \to Y_h \subset H_0^1(Q)$ satisfy $P_h(\phi) \to \phi$ for $h \to 0$, strongly in $H^1(Q)$, for all $\phi \in H_0^1(Q)$. □

Our next aim is to transform the right hand side f into jump conditions across edges of the grid $\mathscr{T}_h$. We will extract the relevant information on jumps from the finite element solution U_h of system (2.2). We denote the set of interior interfaces by Γ_h and the interface of two simplices T_k and T_j by Γ_{kj},

$$\Gamma_h := \left(\bigcup_k \partial T_k \right) \setminus \partial Q_h = \bigcup_{k<j} \Gamma_{kj}, \qquad \Gamma_{kj} := \overline{T}_k \cap \overline{T}_j\,.$$

We furthermore use the notation $\nu_{(k)}$ for the outer normal to T_k on ∂T_k. For a function $f \in L^2(Q;\mathbb{R}^n)$, such that $f|_{T_k}$ has a trace on ∂T_k for all k, the jump across Γ_{kj} is defined as

$$[\![f]\!]_{kj} := f|_{T_k} \cdot \nu_{(k)} + f|_{T_j} \cdot \nu_{(j)} = \left(f|_{T_k} - f|_{T_j} \right) \cdot \nu_{(k)}.$$

By definition, there holds $[\![f]\!]_{kj} = [\![f]\!]_{jk}$. We consider the jump as a scalar function on Γ_h. With the solution U_h of (2.2), we define $g_h : \Gamma_h \to \mathbb{R}$ as the function

$$g_h|_{\Gamma_{kj}} := [\![a^* \nabla U_h]\!]_{kj}. \tag{2.4}$$

The gradients ∇U_h are constant in each simplex T_k, hence $g_h : \Gamma_h \to \mathbb{R}$ is constant on each interface Γ_{kj}.

Remark 2.2. *The finite element solution U_h was defined in* (2.2) *with f. We can equivalently characterize U_h with g_h as the unique solution of*

$$U_h \in \psi + Y_h, \qquad \textit{with} \qquad [\![a^* \nabla U_h]\!]_{kj} = g_h|_{\Gamma_{kj}} \ \forall k < j. \tag{2.5}$$

Problem (2.5) *is equivalent to problem* (2.2)*. This is a consequence of the fact that the jump conditions determine piecewise affine functions uniquely: for all $U, V \in Y_h$*

$$[\![\nabla U]\!]_{kj} = [\![\nabla V]\!]_{kj},\ \forall\, k \neq j \qquad \textit{implies} \qquad U \equiv V.$$

The remark indicates that the right hand side f has been transformed into the jump condition g_h. This is even more clear with the observation that, for all $\phi \in Y_h$,

$$\int_Q f\phi = \int_Q a^*\nabla U_h \cdot \nabla\phi = \sum_k \int_{\partial T_k} (a^*\nabla U_h \cdot \nu_{(k)})\phi = \sum_{k<j} \int_{\Gamma_{kj}} [\![a^*\nabla U_h]\!]_{kj}\phi = \int_{\Gamma_h} g_h\phi, \tag{2.6}$$

since $a^*\nabla U_h$ is constant in each T_k. Considering only functions $\phi \in Y_h$, we have therefore equivalently replaced $f \in L^2(Q)$ by $g_h\,\mathcal{H}^{n-1}|_{\Gamma_h} \in H^{-1}(Q)$.

2.2 The needle problem

Until now, we considered the original problem with solution u^ε and a discrete problem with solution U_h. The needle problem lies in between: we search for a function u_h^ε which solves the original problem in each simplex, but we demand that it is affine on all interfaces. The above transformation of f into jump conditions g_h is made in order to reduce the problem to harmonic solutions in each simplex. In the subsequent definition we assume that a discretization of $Q_h \subset Q$ is given as in (2.1).

Definition 2.3 (The needle problem)**.** *We are given a Lipschitz domain $Q \subset \mathbb{R}^n$, a triangulation $\mathscr{T}_h$ of $Q_h \subset Q$ with interior interfaces Γ_h, and a piecewise affine function ψ prescibing a boundary condition. We introduce the function space*

$$\mathcal{N}_h := \left\{\phi \in H_0^1(Q) \,:\, \phi|_{\partial T_k} \text{ is affine for all } T_k \in \mathscr{T}_h,\ \phi \equiv 0 \text{ on } Q \setminus Q_h\right\}.$$

For a given function $g_h : \Gamma_h \to \mathbb{R}$, the needle problem *is to find $u_h^\varepsilon \in \psi + \mathcal{N}_h$ such that*

$$\int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla\phi = \int_{\Gamma_h} g_h\phi \qquad \forall\phi \in \mathcal{N}_h\,. \tag{2.7}$$

We observe that, for $g_h \in L^2(\Gamma_h, \mathbb{R})$, the trace theorem implies $g_h\mathcal{H}^{n-1}|_{\Gamma_h} \in H^{-1}(Q)$. In particular, in that case, the Lax-Milgram theorem is applicable and yields the unique existence of a solution $u_h^\varepsilon \in \psi + \mathcal{N}_h$ of the needle problem.

A formulation of (2.7) on single simplices is as follows: we search for $u_h^\varepsilon \in \psi + \mathcal{N}_h$ with

$$\begin{aligned} -\nabla\cdot(a^\varepsilon\nabla u_h^\varepsilon) &= 0 \qquad \text{in } T_k, \quad \forall\, T_k \in \mathscr{T}_h\,, \\ \int_{\Gamma_h} \left([\![a^\varepsilon \nabla u_h^\varepsilon]\!] - g_h\right)\phi &= 0 \qquad \forall\,\phi \in \mathcal{N}_h. \end{aligned} \tag{2.8}$$

Indeed, from equation (2.8) we calculate for $\phi \in \mathcal{N}_h$

$$\int_Q a^\varepsilon\nabla u_h^\varepsilon \cdot \nabla\phi = \sum_k \int_{T_k} a^\varepsilon\nabla u_h^\varepsilon \cdot \nabla\phi = \int_{\Gamma_h} [\![a^\varepsilon\nabla u_h^\varepsilon]\!]\phi = \int_{\Gamma_h} g_h\phi.$$

A similar calculation shows that every solution of (2.7) solves (2.8).

The name *needle problem* is chosen for the following reason. We think of a two-dimensional domain Q and of functions $u : Q \to \mathbb{R}$, which we consider as height functions that describe a two-dimensional surface above Q. In the needle problem we search for a surface that minimizes the Dirichlet energy corresponding to a^ε, but we want the surface to contain a straight segment above each Γ_{kj}. We imagine the surface like a soap-film containing thin needles which force the free boundary to follow straight segments at certain places.

Definition 2.4. *We introduce projections $\mathscr{F}_h : \mathcal{N}_h \to Y_h \subset \mathcal{N}_h$ as follows: a function $u \in \mathcal{N}_h$ (which is piecewise affine on edges) is mapped to the piecewise affine extension of the values of u on edges. More precisely, $\mathscr{F}_h(u) : Q \to \mathbb{R}$ is the function*

$$\mathscr{F}_h(u) \in Y_h, \quad \mathscr{F}_h(u)|_{\Gamma_h} = u|_{\Gamma_h} . \tag{2.9}$$

We use the construction also in affine spaces and define $\mathscr{F}_h^\psi : \psi + \mathcal{N}_h \to \psi + Y_h$ as $\mathscr{F}_h^\psi(u) := \psi + \mathscr{F}_h(u - \psi)$.

Some useful properties of the projections $\mathscr{F}_h$ are collected in Lemma 2.5 below. At this point, we want to observe the following consequence of the above constructions: for solutions u_h^ε of the needle problem and arbitrary $\phi \in \mathcal{N}_h$ holds

$$\int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla \phi \overset{(2.7)}{=} \int_{\Gamma_h} g_h \phi \overset{(2.9)}{=} \int_{\Gamma_h} g_h \mathscr{F}_h(\phi) \overset{(2.6)}{=} \int_Q f \mathscr{F}_h(\phi). \tag{2.10}$$

This shows once more that the needle problem (2.7) can be regarded as a variant of the original problem with right hand side f in the space $\mathcal{N}_h$.

Lemma 2.5. *We study the projections $\mathscr{F}_h : \mathcal{N}_h \to Y_h \subset \mathcal{N}_h$ of Definition 2.4. These projections and their affine counterparts $\mathscr{F}_h^\psi$ have the following properties.*

1. $\nabla \mathscr{F}_h(u)(x) = \fint_{T_k} \nabla u$ *for* $x \in T_k$.
2. *Let $u^\varepsilon \in \mathcal{N}_h$, $u^\varepsilon \rightharpoonup u$ weakly in $H^1(Q)$ for fixed $h > 0$. Then*
$$\mathscr{F}_h(u^\varepsilon) \underset{\varepsilon}{\rightharpoonup} \mathscr{F}_h(u), \quad \text{weakly in } H^1(Q).$$
3. *Let $u_h \in \mathcal{N}_h$, $u_h \rightharpoonup u$ weakly in $H^1(Q)$ for $h \to 0$. Then*
$$\mathscr{F}_h(u_h) \underset{h}{\rightharpoonup} u, \quad \text{weakly in } H^1(Q).$$

Proof. Concerning property 1, we first note that $\nabla \mathscr{F}_h(u)$ is indeed a constant vector in each simplex. The claim follows from the following calculation for a direction e_j, $j = 1, ..., n$, and a simplex T_k with exterior normal ν,

$$\fint_{T_k} \partial_j \mathscr{F}_h(u) = \frac{1}{|T_k|} \int_{\partial T_k} \mathscr{F}_h(u)\, e_j \cdot \nu = \frac{1}{|T_k|} \int_{\partial T_k} u\, e_j \cdot \nu = \fint_{T_k} \partial_j u\,.$$

For property 2 we note that the projection is bounded in $H^1(Q)$. Indeed, for $u \in \mathcal{N}_h$, by Poincaré's and Jensen's inequalities

$$\|\mathscr{F}_h(u)\|^2_{H^1(Q)} \le C \|\nabla \mathscr{F}_h(u)\|^2_{L^2(Q)} = C \sum_k \int_{T_k} \left| \fint_{T_k} \nabla u \right|^2 \le C \int_Q |\nabla u|^2.$$

In particular, for sequences $u^\varepsilon \in \mathcal{N}_h$, $u^\varepsilon \rightharpoonup u$ weakly in $H^1(Q)$ for $\varepsilon \to 0$, we find a subsequence of $\mathscr{F}_h(u^\varepsilon)$ which converges weakly in $H^1(Q)$ to a limit $F \in Y_h$. We used here that Y_h is weakly closed in $H^1(Q)$. We can identify the limit to be $F = \mathscr{F}_h(u)$ by noting that, for all $T_k \in \mathscr{T}_h$ and all $x \in T_k$

$$\nabla \mathscr{F}_h(u^\varepsilon)(x) = \fint_{T_k} \nabla u^\varepsilon \underset{\varepsilon}{\to} \fint_{T_k} \nabla u = \nabla \mathscr{F}_h(u)(x).$$

In order to show property 3, let $\mathcal{N}_h \ni u_h \rightharpoonup u$ weakly in $H^1(Q)$. As noted above, the sequence $\mathscr{F}_h(u_h)$ is also bounded in $H^1(Q)$. We can thus find a subsequence such that $\mathscr{F}_{h_l}(u_{h_l}) \rightharpoonup F$ in $H^1(Q)$.

In order to identify the limit as $F = u$, we choose an arbitrary test-function $\phi \in C_c^\infty(Q; \mathbb{R}^n)$. By density of the piecewise constant functions in L^2, we find a sequence (ϕ_h) of piecewise constant functions with $\phi_h \to \phi$ strongly in $L^2(Q; \mathbb{R}^n)$. We compute

$$\begin{aligned}
&\left| \int_Q \nabla \mathscr{F}_h(u_h) \cdot \phi - \int_Q \nabla u \cdot \phi \right| \\
&\quad = \left| \int_Q \nabla \mathscr{F}_h(u_h) \cdot \phi_h + \int_Q \nabla \mathscr{F}_h(u_h) \cdot (\phi - \phi_h) - \int_Q \nabla u \cdot \phi \right| \\
&\quad \leq \left| \int_Q \nabla u_h \cdot \phi_h - \int_Q \nabla u \cdot \phi \right| + \|\nabla \mathscr{F}_h(u_h)\|_{L^2} \|\phi - \phi_h\|_{L^2}.
\end{aligned}$$

The first term on the right-hand side converges to zero since $\nabla u_h \rightharpoonup \nabla u$ weakly and $\phi_h \to \phi$ strongly in $L^2(Q; \mathbb{R}^n)$, the second term vanishes by boundedness of the first factor and strong convergence of ϕ_h. We can therefore conclude $F = u$.

The definition of $\mathscr{F}_h^\psi$ implies that properties remain valid on affine subspaces. □

Our next aim is to compare the original solution u^ε with the needle problem solution u_h^ε. This comparison is provided with the following Proposition.

Proposition 2.6 (Comparison of u_h^ε and u^ε)**.** *Let coefficients $a^\varepsilon \in L^\infty(Q; \mathbb{R}^{n\times n})$ satisfy the ellipticity* (1.2) *and let ψ be an affine function. Let $u^\varepsilon \in H^1(Q)$ be the weak solution of the original problem* (1.5)*, and let $u_h^\varepsilon \in \psi + \mathcal{N}_h$ be solutions to the needle problem* (2.7) *with g_h of* (2.4)*. Furthermore, we assume that the grids $\mathscr{T}_h$ are adapted grids for $(u^\varepsilon)_\varepsilon$ according to Definition 4.7. Then there holds*

$$\lim_{h\to 0} \lim_{\varepsilon \to 0} \|u_h^\varepsilon - u^\varepsilon\|_{H^1(Q)} = 0. \tag{2.11}$$

The idea of the proof is to use $(u^\varepsilon - u_h^\varepsilon)$ as a test-function for the original problem (1.5) and in the needle problem (2.7), and to take the difference. We note that this test function satisfies a homogeneous Dirichlet condition. By ellipticity of a^ε, the result provides an upper bound for $\|u^\varepsilon - u_h^\varepsilon\|^2_{H^1(T)}$. It remains to show that the upper bound vanishes in the limit as $\varepsilon \to 0$ and then $h \to 0$.

Proof. All solution sequences of the proposition are bounded in $H^1(Q)$. This allows to choose a subsequence and limit functions such that, as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup u, \quad u_h^\varepsilon \rightharpoonup u_h \quad \text{weakly in } H^1(Q), \tag{2.12}$$

$$\nabla u_h^\varepsilon \rightharpoonup \nabla u_h, \quad q_h^\varepsilon := a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup q_h \quad \text{weakly in } L^2(Q). \tag{2.13}$$

We note that the distributional divergence of q_h^ε vanishes in each simplex T_k by (2.8).

Since the needle problem does not allow to use u^ε as a test function, we must apply a projection. We use the $L^2(Q)$-orthogonal projection $P_h : L^2(Q) \to Y_h \subset L^2(Q)$ and the affine counterpart $P_h^\psi : L^2(Q) \to \psi + Y_h$ defined by $P_h^\psi(u) := \psi + P_h(u - \psi)$. As a consequence of (2.12), we have the strong convergence $u^\varepsilon \to u$ in $L^2(Q)$, and hence also $P_h^\psi(u^\varepsilon) \to P_h^\psi(u)$ in $L^2(Q)$. Since P_h^ψ maps into a space of finite dimension, the convergence is in all norms, in particular, as $\varepsilon \to 0$, also

$$P_h^\psi(u^\varepsilon) \to P_h^\psi(u) \qquad \text{in } H^1(Q).$$

We can now start the computations. For some $\alpha_0 > 0$ that combines the ellipticity constant $\alpha_1 > 0$ and the constant from Poincaré's inequality, we find

$$\begin{aligned}
\alpha_0 \|u^\varepsilon - u_h^\varepsilon\|_{H^1(Q)}^2 &\le \int_Q a^\varepsilon \nabla(u^\varepsilon - u_h^\varepsilon) \cdot \nabla(u^\varepsilon - u_h^\varepsilon) \\
&= \int_Q a^\varepsilon \nabla u^\varepsilon \cdot \nabla(u^\varepsilon - u_h^\varepsilon) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(u^\varepsilon - u_h^\varepsilon) \\
&\overset{(1.5)}{=} \int_Q f\,(u^\varepsilon - u_h^\varepsilon) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(P_h^\psi(u^\varepsilon) - u_h^\varepsilon) \\
&\overset{(2.10)}{=} \int_Q f\,(u^\varepsilon - u_h^\varepsilon) - \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) - \int_Q f\, \mathscr{F}_h(P_h^\psi(u^\varepsilon) - u_h^\varepsilon) \\
&= \int_Q f\,(u^\varepsilon - P_h^\psi(u^\varepsilon)) + \int_Q f\,(\mathscr{F}_h^\psi(u_h^\varepsilon) - u_h^\varepsilon) - \int_Q q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)).
\end{aligned}$$

In the last line we only re-ordered terms and used $\mathscr{F}_h^\psi \circ P_h^\psi(u^\varepsilon) = P_h^\psi(u^\varepsilon)$. Our aim is to show that the right hand side vanishes as $\varepsilon \to 0$, and then $h \to 0$. Concerning the first integral we have

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \int_Q f\,(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \lim_{h\to 0} \int_Q f\,(u - P_h^\psi(u)) = 0.$$

In order to treat the second integral we select a subsequence $h \to 0$ such that $u_h \rightharpoonup \tilde{u}$ for $h \to 0$, weakly in $H^1(Q)$ for some limit $\tilde{u}$. This allows to use Lemma 2.5, first property 2 together with (2.12), and then property 3. We find

$$\lim_{\varepsilon\to 0} \int_Q f\,(\mathscr{F}_h(u_h^\varepsilon) - u_h^\varepsilon) = \int_Q f\,(\mathscr{F}_h(u_h) - u_h) \to 0 \qquad \text{for } h \to 0.$$

Concerning the third integral, we must use a div-curl lemma. The integrand is the product of the functions $q_h^\varepsilon = a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup q_h$ in $L^2(Q)$, and of $\nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) \rightharpoonup \nabla(u - P_h^\psi(u))$ weakly in $L^2(Q)$, both convergences for $\varepsilon \to 0$. On the other hand, we treat the product of a weakly convergent sequence q_h^ε satisfying $\nabla \cdot q_h^\varepsilon = 0$ with a weakly convergent sequence of gradients. Since the grid is adapted to the sequence u^ε, the hypothesis of the div-curl Theorem 4.8 are satisfied. Relation (4.24) allows to calculate the limit

$$\lim_{\varepsilon\to 0} \int_Q q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \lim_{\varepsilon\to 0} \int_{Q_h} q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \int_{Q_h} q_h \cdot \nabla(u - P_h^\psi(u)).$$

We now use that q_h is bounded in $L^2(Q)$ and $P_h^\psi(u) \to u$ converges strongly in $H^1(Q)$ to conclude

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \int_Q q_h^\varepsilon \cdot \nabla(u^\varepsilon - P_h^\psi(u^\varepsilon)) = \lim_{h\to 0} \int_Q q_h \cdot \nabla(u - P_h^\psi(u)) = 0.$$

This implies smallness of the third integral and verifies the claim of the proposition. □

We note that, at this point, we have already verified the smallness conditions regarding vertical arrows in the diagram of Subsection 1.2, namely $\lim_{h\to 0}\lim_{\varepsilon\to 0}\|u^\varepsilon - u_h^\varepsilon\|_{H^1(Q)} = 0$ of the above Proposition, and $U_h \underset{h}{\rightharpoonup} u^*$ in $H^1(Q)$ in Lemma 2.1. We emphasize that we used one non-trivial ingredient: the fact that the triangulation can be chosen adapted to the sequence u^ε and the corresponding div-curl Theorem 4.8. That theorem, stated and proved in Section 4, yields that adapted triangulations always exist in two and three space dimensions and that the div-curl compensated compactness holds.

3 Stabilization result and proof of Theorem 1.2

To conclude our approach, it remains to verify the weak H^1-convergence $u_h^\varepsilon \underset{\varepsilon}{\rightharpoonup} U_h$. This convergence result is quite straightforward once we know, using the notation of Definition 1.1, the L^2-convergence $\nabla u^\varepsilon \rightharpoonup \xi$ and $a^\varepsilon \nabla u^\varepsilon \rightharpoonup a^*\xi$. The important point here is that the weak limits are constant functions; we refer to this fact as stabilization. The verification of the stabilization is the main purpose of this section. After that, the conclusion of Theorem 1.2 is performed easily with Lemma 3.4.

As a preparation, we observe that the averaging property (1.4) extends to sequences of affine boundary conditions.

Remark 3.1. *Let the coefficients a^ε allow averaging of the constitutive relation with the matrix a^*. Then, for every simplex $T \subset Q$ and every sequence $U_{\xi^\varepsilon}(x) = \xi^\varepsilon \cdot x + b^\varepsilon \to U_\xi(x) = \xi \cdot x + b$, the solutions u^ε of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= 0 \qquad \text{in } T, \\ u^\varepsilon &= U_{\xi^\varepsilon} \qquad \text{on } \partial T, \end{aligned} \tag{3.1}$$

satisfy

$$\lim_{\varepsilon\to 0} \fint_T a^\varepsilon \nabla u^\varepsilon = a^* \xi. \tag{3.2}$$

Proof. It suffices to consider the solution u^ε to boundary data U_{ξ^ε} and the solution $\tilde{u}^\varepsilon$ to boundary data U_ξ. For $\tilde{u}^\varepsilon$, the convergence (3.2) is precisely the averaging property (1.4). It therefore suffices to show that the difference $u^\varepsilon - \tilde{u}^\varepsilon$ is small in $H^1(T)$. This smallness follows by linearity and ellipticity of the equation. □

Proposition 3.2 (Stabilization). *Let the coefficients $a^\varepsilon \in L^\infty(Q;\mathbb{R}^{n\times n})$ satisfy* (1.2) *and allow averaging with matrix a^* in the sense of Definition 1.1. Let $T \subset \mathbb{R}^n$ be a simplex, $U_\xi(x) = \xi \cdot x + b$ an affine function, and u^ε a sequence of weak solutions of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= 0 \qquad \text{in } T, \\ u^\varepsilon &= U_\xi \qquad \text{on } \partial T. \end{aligned} \tag{3.3}$$

We denote the limits of functions and fluxes by u and q, i.e. we assume

$$\begin{aligned} u^\varepsilon &\rightharpoonup u \qquad \text{weakly in } H^1(T,\mathbb{R}), \\ q^\varepsilon := a^\varepsilon \nabla u^\varepsilon &\rightharpoonup q \qquad \text{weakly in } L^2(T;\mathbb{R}^n). \end{aligned}$$

Then u is affine and q is constant. More precisely, there holds

$$\nabla u \equiv \xi \quad in\ T, \tag{3.4}$$

$$q \equiv a^* \xi \quad in\ T. \tag{3.5}$$

Proof. In this proof, we consider sequences u^ε on a fixed simplex T. The simplex T now plays the role of the arbitrary domain Q of Section 2, and our aim is to use the results obtained so far. We fix a sequence $h \searrow 0$. We choose polygonal domains $T_h \subset T$ and triangulations of T_h,

$$\mathcal{S}_h := \{S_k\}_{k\in\Lambda_h} \quad \text{be a triangulation of } T_h,$$

where S_k are simplices such that $\max\{\text{diam}(S_k)|\ k \in \Lambda_h\} < h$ and $T_h \subset T$ as in (2.1). By Theorem 4.8 we may assume that, for all h, the subdivision $\mathcal{S}_h$ is an adapted grid for u^ε according to Definition 4.7.

Let $(u_h^\varepsilon)_\varepsilon$ be a subsequence of solutions of the needle problem (2.7) on T with vanishing jump conditions $g \equiv 0$ and with boundary condition $\psi = U_\xi$. We select a subsequence $\varepsilon \to 0$ and limit functions u_h such that, for all h in the sequence, $u_h^\varepsilon \rightharpoonup u_h$ for $\varepsilon \to 0$, weakly in $H^1(T)$. We note that all functions u_h^ε, and thus also u_h, are affine on all ∂S_k. The needle problem comparison result of Proposition 2.6 yields $\|u - u_h\|_{H^1}^2 \le \limsup_{\varepsilon\to 0} \|u^\varepsilon - u_h^\varepsilon\|_{H^1}^2 \le \eta(h) \to 0$ for $h \to 0$.

Proof of relation (3.4). Corresponding to the needle problem solution u_h^ε, we consider the piecewise affine functions $\bar u_h^\varepsilon := \mathscr{F}_h^\psi(u_h^\varepsilon)$, and (after selection of a weakly convergent subsequence) their weak limits $\bar u_h \in H^1(T)$. We use the abbreviations $\xi_k^\varepsilon := \nabla \bar u_h^\varepsilon|_{S_k} \to \nabla \bar u_h|_{S_k} =: \xi_k$. For fixed h, we consider a test-function ϕ in the corresponding needle space: ϕ is continuous on $\bar T$, vanishes on $T \setminus T_h$, and is piecewise affine on every simplex S_k. We calculate, exploiting that $\nabla\phi$ is constant on each simplex S_k, for $\varepsilon \to 0$,

$$0 \overset{(2.7)}{=} \int_T a^\varepsilon \nabla u_h^\varepsilon \nabla\phi = \sum_k \int_{S_k} a^\varepsilon \nabla u_h^\varepsilon \nabla\phi \overset{(3.2)}{\to} \sum_k \int_{S_k} a^* \xi_k \nabla\phi = \int_T a^* \nabla \bar u_h \nabla\phi\,.$$

We conclude that $\bar u_h$ is a finite element solution of $-\nabla \cdot (a^* \nabla \bar u_h) = 0$ with affine boundary condition U_ξ, which implies $\bar u_h = U_\xi$. Property 2 of Lemma 2.5 implies $\bar u_h^\varepsilon = \mathscr{F}_h^\psi(u_h^\varepsilon) \rightharpoonup \mathscr{F}_h^\psi(u_h)$ in H^1, hence $U_\xi = \bar u_h = \mathscr{F}_h^\psi(u_h)$. The convergence $u_h \to u$ in $H^1(T)$ from the needle problem estimate allows to conclude, using property 3 of Lemma 2.5, $\mathscr{F}_h^\psi(u_h) \rightharpoonup u$ in H^1 for $h \to 0$, and hence $u = U_\xi$. This shows (3.4).

Proof of relation (3.5). We consider, after selection of a subsequence, the limiting fluxes $q^\varepsilon = a^\varepsilon \nabla u^\varepsilon \rightharpoonup q$ and $q_h^\varepsilon := a^\varepsilon \nabla u_h^\varepsilon \rightharpoonup q_h$, with weak convergence in $L^2(T)$ for $\varepsilon \to 0$. Lower semi-continuity of the norm and the estimate for the needle problem of Proposition 2.6 yields $\lim_{h\to 0} \|q - q_h\|_{L^2} \le \lim_{h\to 0} \liminf_{\varepsilon\to 0} \|a^\varepsilon \nabla u^\varepsilon - a^\varepsilon \nabla u_h^\varepsilon\|_{L^2} = 0$. Our aim is to show $q \equiv a^*\xi$.

We use an arbitrary function $\psi \in C_c^1(T)$, which we approximate by functions $\psi_h : T \to \mathbb{R}$ that vanish on $T \setminus T_h$ and are piece-wise constant in each simplex $S_k \subset T$ (for the triangulation corresponding to h), with $\psi_h \to \psi$ strongly in $L^2(T)$ for $h \to 0$. We use once more Remark 3.1 in each S_k, where u_h^ε satisfies affine boundary conditions with slope $\xi_k^\varepsilon \to \xi$. We calculate, for $\varepsilon \to 0$,

$$\int_T q_h \psi_h \leftarrow \int_T a^\varepsilon \nabla u_h^\varepsilon \psi_h = \sum_k \int_{S_k} (a^\varepsilon \nabla u_h^\varepsilon)\psi_h \to \sum_k \int_{S_k} a^* \xi \psi_h = \int_T a^* \xi \psi_h\,.$$

The strong L^2-convergences $q_h \to q$ and $\psi_h \to \psi$ yield $q \equiv a^*\xi$, since ψ was arbitrary. This concludes the proof of Proposition 3.2. $\square$

The result of the above proposition remains valid for a convergent sequence of affine boundary conditions. We note this direct consequence for later use in the proof of our main theorem.

Corollary 3.3. *Let the coefficients a^ε satisfy* (1.2) *and allow averaging with matrix a^* in the sense of Definition 1.1. We study a simplex T and a convergent sequence of affine functions $U_{\xi^\varepsilon}(x) = \xi^\varepsilon \cdot x + b^\varepsilon \to U_\xi(x) = \xi \cdot x + b$. Then, the solutions (w^ε) of*

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla w^\varepsilon) &= 0 && \text{in } T \\ w^\varepsilon &= U_{\xi^\varepsilon} && \text{on } \partial T \end{aligned}$$

satisfy

$$\begin{aligned} \nabla w^\varepsilon &\rightharpoonup \xi && \text{weakly in } L^2(T), \\ a^\varepsilon \nabla w^\varepsilon &\rightharpoonup a^*\xi && \text{weakly in } L^2(T, \mathbb{R}^n). \end{aligned}$$

Proof. We use the solutions u^ε of

$$\begin{aligned} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) &= 0 && \text{in } T \\ u^\varepsilon &= U_\xi && \text{on } \partial T \end{aligned}$$

as studied in Proposition 3.2. In view of that proposition, it suffices to derive smallness in $H^1(T)$ of $u^\varepsilon - w^\varepsilon$. We multiply the equation for $u^\varepsilon - w^\varepsilon$ with $(u^\varepsilon - U_\xi) - (w^\varepsilon - U_{\xi^\varepsilon})$, which vanishes on the boundary ∂T. By Hölder's inequality and uniform ellipticity of a^ε, there exists $C > 0$ such that

$$\|u^\varepsilon - w^\varepsilon\|^2_{H^1(T)} \le C\|U_\xi - U_{\xi^\varepsilon}\|^2_{H^1(T)} \to 0.$$

This yields the claim. $\square$

The subsequent lemma shows the missing convergence in the diagram of Subsection 1.2. It hence concludes the proof of Theorem 1.2.

Lemma 3.4 (Comparison of needle problem and discretized problem)**.** *Let the domain Q, coefficients a^ε, f and ψ be as in Theorem 1.2. Let $h > 0$ be fixed, U_h the solution of the auxiliary problem* (2.2) *and g_h as in* (2.4)*. Let u_h^ε be the solution of the needle problem* (2.7)*. Then, as $\varepsilon \to 0$,*

$$\begin{aligned} u_h^\varepsilon &\rightharpoonup U_h && \text{weakly in } H^1(Q, \mathbb{R}), \\ a^\varepsilon \nabla u_h^\varepsilon &\rightharpoonup a^*\nabla U_h && \text{weakly in } L^2(Q, \mathbb{R}^n). \end{aligned}$$

Proof. Let u_h^ε be the solution of (2.7) and let u_h be any $H^1(Q)$-weak limit point of $(u_h^\varepsilon)_\varepsilon$, as $\varepsilon \to 0$. As solutions of the needle problem, the functions u_h^ε are affine on the boundaries of each simplex. For fixed h and fixed simplex T_k, we denote the corresponding affine function by $U^{(k)}_{\xi_k^\varepsilon}$, and find further subsequences $\varepsilon \to 0$ such that these functions converge for each simplex to affine functions $U^{(k)}_{\xi_k}$. Corollary 3.3 implies, for all $T_k \in \mathscr{T}_h$, as $\varepsilon \to 0$,

$$\begin{aligned} \nabla u_h^\varepsilon &\rightharpoonup \xi_k && \text{weakly in } L^2(T_k), \\ a^\varepsilon \nabla u_h^\varepsilon &\rightharpoonup a^*\xi_k && \text{weakly in } L^2(T_k). \end{aligned}$$

In particular, $u_h \in Y_h$. We now use an arbitrary test-function $\phi \in Y_h$ and use the needle problem characterization (2.10) to find, for $\varepsilon \to 0$,

$$\int_Q f\phi = \int_Q a^\varepsilon \nabla u_h^\varepsilon \cdot \nabla\phi \to \int_Q a^* \nabla u_h \cdot \nabla \phi.$$

By uniqueness of solutions of the discrete problem (2.2), we find $u_h = U_h$ and have thus verified the claim. □

4 The adapted grid

In this section, we consider an n-dimensional domain Ω, a fixed sequence $\varepsilon = (\varepsilon_k)_k \to 0$ for $\mathbb{N} \ni k \to \infty$, and a fixed family of functions $u^\varepsilon : \Omega \to \mathbb{R}$, bounded in $H^1(\Omega)$. Since we will treat integrals over objects of different dimensions, we write $\mathcal{L}^m$ and $\mathcal{H}^m$ for the m-dimensional Lebesgue- and Hausdorff-measure. Our assumption on the sequence u^ε is then written as

$$\int_\Omega |u^\varepsilon(z)|^2 \, d\mathcal{L}^n(z) + \int_\Omega |\nabla u^\varepsilon(z)|^2 \, d\mathcal{L}^n(z) \le C_0 \qquad \forall \varepsilon, \tag{4.1}$$

for some $C_0 > 0$. Our interest in this section is to find (many) simplices contained in Ω, such that, loosely speaking, ∇u^ε is L^2-bounded on the faces. Such a boundedness implies compactness of the boundary values in $H^{1/2}$ and allows to construct extensions of the boundary values that are strongly convergent in H^1. The fact that on almost all $(n-1)$-dimensional hyperplanes the functions ∇u^ε are L^2-bounded is a consequence of Fubini's theorem.

In the construction of strongly convergent extensions we must be careful in the treatment of the $(n-2)$-dimensional edges of the simplices, the boundaries of the $(n-1)$-dimensional faces. In order to treat these boundaries, we demand additionally that the averages of $|\nabla u^\varepsilon|^2$ over small neighborhoods of edges are bounded. To make such a property precise, we use a sequence of positive numbers $\delta_k \to 0$, these numbers will be radii of small balls or cylinders. For the rest of this work we may choose $\delta_k = \frac{1}{k}$.

4.1 Adapted grids in two dimensions

This subsection is devoted to the construction of adapted grids for case $n = 2$. Some concepts are independent of the dimension and are treated here for general dimension as a preparation for $n = 3$. We always assume that we are given two sequences of positive numbers, $\varepsilon_k \to 0$ and $\delta_k \to 0$, and a sequence of functions $u^\varepsilon : \Omega \to \mathbb{R}$ satisfying (4.1).

Definition 4.1 (Points of typical average). *We say that $x \in \Omega$ is a* point with typical averages *for $(\varepsilon_k)_k$, $(\delta_k)_k$, and $(u^\varepsilon)_\varepsilon$, if the following holds. There exists a subsequence $k_j \to \infty$ and real numbers c_x and M_x such that*

$$⨍_{B_{\delta_k}(x)} |\nabla u^{\varepsilon_k}(z)|^2 \, d\mathcal{L}^n(z) \le M_x \quad \forall\, k = k_j \tag{4.2}$$

$$c_x^k := ⨍_{B_{\delta_k}(x)} u^{\varepsilon_k}(z) \, d\mathcal{L}^n(z) \;\to c_x \quad \text{for } k = k_j \to \infty. \tag{4.3}$$

We say that $(k_j)_j$ is a good subsequence *for the point x when (4.2) and (4.3) are satisfied along this subsequence.*

In the above definition and in all proofs we use the convention that integrals $\int_B$ denote integrals $\int_{B\cap\Omega}$. For inner points $x \in \Omega$, because of $\delta_k \to 0$, the balls $B_{\delta_k}(x)$ are contained in Ω for large k.

We note that a point of typical average is similar to a Lebesgue point — but it is chosen for a whole sequence of functions.

Lemma 4.2 (Many points of typical average). *Let $\Omega \subset \mathbb{R}^n$, $(\varepsilon_k)_k$, $(\delta_k)_k$ be as above, and let $(u^\varepsilon)_\varepsilon$ be a bounded family in $H^1(\Omega, \mathbb{R})$. Then almost every point $x \in \Omega$ is a point of typical average.*

Proof. It is sufficient to show the following: For arbitrary $\vartheta > 0$ there exists an exceptional set $E \subset \Omega$ with Lebesgue measure $|E| \le \vartheta$, such that all points $x \in \Omega \setminus E$ are points of typical average. We fix $\vartheta > 0$ and assume, for a contradiction argument, that there exists an exceptional set $E \subset \Omega$ with $|E| > \vartheta$, consisting of points that are not of typical average. We fix now $M > 3^{n+1}C_0/\vartheta$, where C_0 is the $H^1(\Omega)$-bound of the sequence u^ε.

Let $x \in E$ be one of the exceptional points. Then, for all subsequences k_j, the integrals of (4.2) are unbounded. In particular, for every $x \in E$, there exists $K(x) \in \mathbb{N}$ such that

$$\fint_{B_{\delta_k}(x)} |\nabla u^{\varepsilon_k}|^2 \ge M \qquad \text{for all } k \ge K(x). \tag{4.4}$$

We choose, for every $x \in E$, the minimal $K(x)$ with this property. Then $K : \Omega \to \mathbb{N}$ is lower semi-continuous, since the integral on the left is continuous in x for every k. In particular, K is (Borel-)measurable. We now consider the measurable sets

$$E_N := \{x \in E : K(x) \le N\},$$

such that

$$E = \bigcup_{N\in\mathbb{N}} E_N, \quad E_{N+1} \supset E_N, \quad \text{and hence } |E| = \lim_{N\to\infty} |E_N|. \tag{4.5}$$

By hypothesis we have $|E| > \vartheta$, thus we find $N \in \mathbb{N}$ with $|E_N| > \vartheta/2$. By measurability of E_N, there exists a compact set $\tilde{E}_N$ satisfying

$$\tilde{E}_N \subset E_N, \qquad |\tilde{E}_N| > \frac{\vartheta}{3}. \tag{4.6}$$

Corresponding to the covering

$$\tilde{E}_N \subset \bigcup_{x\in\tilde{E}_N} B_{\delta_N}(x)$$

we find a finite sub-covering by compactness of $\tilde{E}_N$. We can apply an elementary covering lemma (see, e.g., [18], Lemma 7.3) to select a finite set of points $(x_m)_m$ such that

$$\tilde{E}_N \subset \bigcup_m B_{3\delta_N}(x_m), \qquad B_{\delta_N}(x_{m_1}) \cap B_{\delta_N}(x_{m_2}) = \emptyset, \quad \text{for all } m_1 \ne m_2\,. \tag{4.7}$$

Recalling the H^1-boundedness (4.1) of the sequence, we can now calculate with $k = N$

$$\begin{aligned} C_0 &\geq \int_\Omega |\nabla u^{\varepsilon_k}|^2 \geq \int_{\bigcup_m B_{\delta_N}(x_m)} |\nabla u^{\varepsilon_k}|^2 \stackrel{(4.7)}{=} \sum_m \int_{B_{\delta_N}(x_m)} |\nabla u^{\varepsilon_k}|^2 \\ &\stackrel{(4.4)}{\geq} \sum_m |B_{\delta_N}(x_m)| M \geq \left| \bigcup_m B_{3\delta_N}(x_m) \right| \frac{M}{3^n} \stackrel{(4.7)}{\geq} |\tilde{E}_N| \frac{M}{3^n} \stackrel{(4.6)}{\geq} M \frac{\vartheta}{3^{n+1}} > C_0, \end{aligned}$$

where we used $M > 3^{n+1} C_0/\vartheta$ in the last step. This provides the desired contradiction. We used in the above calculation that $x_m \in \tilde{E}_N \subset E_N$, such that for $k = N$ inequality (4.4) holds.

The fact that averages of the functions as in (4.3) do not diverge for almost every x can be shown along the above lines. Upon a selection of a further subsequence and appropriate c_x we find (4.3). □

We next study conditions for segments. For points $x, y \in \mathbb{R}^n$ we use the notation $[x, y] := \{\theta x + (1-\theta) y : \theta \in [0,1]\}$ and refer to $[x, y]$ as the segment to the pair (x, y). Loosely speaking, we want to show that, for most segments $\Gamma \subset \Omega$, the sequence of gradients $\nabla u^\varepsilon|_\Gamma$ is bounded in $L^2(\Gamma)$.

Let us start with a general comment on the construction. With u^ε as above, the $L^2(\Omega)$-function ∇u^ε is specified almost everywhere, hence the values of the function on segments Γ are specified almost everywhere on the segment, at least for almost every segment. In this sense, we can consider integrals of the gradient over segments.

Later on, we want to relate the gradient to traces. For $n = 2$, given a segment Γ, we consider the $H^{1/2}(\Gamma)$-functions $u^\varepsilon|_\Gamma$ and their distributional (tangential) gradients $\nabla_\tau u^\varepsilon|_\Gamma$. For smooth functions, these coincide with the projection of ∇u^ε to the tangential space of the segment Γ. With smooth test-functions and an integration over families of parallel segments one can verify that the two constructions yield the same function $\nabla_\tau u^\varepsilon|_\Gamma$ for almost all segments Γ.

Definition 4.3 (Typical segments)**.** *For a set $\Omega \subset \mathbb{R}^n$, given sequences $\delta_k \to 0$, $\varepsilon_k \to 0$, and a bounded sequence $(u^\varepsilon)_\varepsilon \in H^1(\Omega)$, we say that a segment $\Gamma = [x, y]$ is a* typical segment *if the following holds: There exists a subsequence $k_j \to \infty$ and a constant $M_\Gamma > 0$ such that, for $k = k_j$,*

$$\|u^{\varepsilon_k}|_\Gamma\|^2_{L^2(\Gamma)} + \|\nabla_\tau u^{\varepsilon_k}|_\Gamma\|^2_{L^2(\Gamma)} \leq M_\Gamma. \tag{4.8}$$

We furthermore demand that the end-points x and y are points of typical average and that the subsequence $(k_j)_j$ is a good subsequence for x and for y.

A subsequence $(k_j)_j$ with the above properties is called a good subsequence *for the segment Γ.*

Lemma 4.4 (Many typical segments)**.** *Let $\Omega \subset \mathbb{R}^n$ be a convex domain, $\delta_k \to 0$ and $\varepsilon_k \to 0$, and let $(u^\varepsilon) \subset H^1(\Omega)$ be a bounded family. Then, for almost every $x \in \Omega$, there is a good set $\mathscr{G}_x \subset \Omega$ of full measure $|\mathscr{G}_x| = |\Omega|$, such that for all $y \in \mathscr{G}_x$ the segment $[x, y]$ is a typical segment according to Definition 4.3.*

Proof. Let us first observe that almost every $x \in \Omega$ is a point of typical average by Lemma 4.2. Now we apply the Lemma again to ε_{k_j} and δ_{k_j}. We find that almost every $y \in \Omega$ is a typical point for that sequence. This means that we find a further subsequence which is a good sequence for both x and y.

We additionally have to verify that almost every segment (chosen in the described way) satisfies (4.8). We abbreviate the integrands as $f^k(x) := |u^{\varepsilon_k}|^2(x) + |\nabla u^{\varepsilon_k}|^2(x)$, a sequence of non-negative functions that are defined almost everywhere. The family f^k satisfies $\int_\Omega f^k \le C_0$. With the diameter $\mathrm{diam}(\Omega)$ of Ω we calculate for segments

$$\begin{aligned}
&\int_\Omega \int_\Omega \int_{[x,y]} f^k(z)\, d\mathcal{H}^1(z)\, dy\, dx \le \mathrm{diam}(\Omega) \int_\Omega \int_\Omega \int_0^1 f^k(\theta x + (1-\theta)y)\, d\theta\, dy\, dx \\
&= \mathrm{diam}(\Omega) \int_0^{1/2} \int_\Omega \left\{ \int_\Omega f^k(\theta x + (1-\theta)y)\, dy \right\} dx\, d\theta \\
&\quad + \mathrm{diam}(\Omega) \int_{1/2}^1 \int_\Omega \left\{ \int_\Omega f^k(\theta x + (1-\theta)y)\, dx \right\} dy\, d\theta \\
&\le \mathrm{diam}(\Omega) \int_0^{1/2} \int_\Omega 2^n C_0\, dx\, d\theta + \mathrm{diam}(\Omega) \int_{1/2}^1 \int_\Omega 2^n C_0\, dx\, d\theta = \mathrm{diam}(\Omega)|\Omega| 2^n C_0.
\end{aligned}$$

This calculation provides that the family of maps

$$F^k : \Omega \times \Omega \to \mathbb{R}, (x,y) \mapsto \int_{[x,y]} f^k(z)\, d\mathcal{H}^1(z)$$

is bounded by some constant $C_1 > 0$ in $L^1(\Omega \times \Omega)$. Let $E \subset \Omega \times \Omega$ be the (exceptional) set of pairs (x,y) such that there is no subsequence $(k_j)_j$ and no constant M_Γ with $F^k((x,y)) \le M_\Gamma$. Let $M > 0$ be arbitrary. We consider the sets $E_N := \{(x,y) \in \Omega \times \Omega : F^k((x,y)) \ge M\ \forall k \ge N\}$. These sets satisfy $E \subset \bigcup_N E_N$, $E_{N+1} \supset E_N$, and $|E_N| \le C_1/M$, hence also $|E| \le C_1/M$. Since M was arbitrary, this shows that E has measure 0. □

For triangles $T \subset \mathbb{R}^2$ with three typical segments as sides, we can now show the main tool for the compensated compactness result.

Proposition 4.5 (Strongly convergent extensions in $\mathbb{R}^2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a convex domain, $\delta_k \to 0$ and $\varepsilon_k \to 0$ fixed, and let $(u^\varepsilon) \subset H^1(\Omega)$ be a bounded family. Let T be a triangle, given by a triple (x_1, x_2, x_3), such that all segments $[x_l, x_m]$, $l \ne m$, are typical segments for u^ε, and let $(k_j)_j$ be a good subsequence for the three segments. Then, for $\varepsilon = \varepsilon_{k_j}$, there exists a family of functions $v^\varepsilon \in H^1(T)$ and a limit function $v \in H^1(T)$ such that*

$$v^\varepsilon = u^\varepsilon \text{ on } \partial T, \tag{4.9}$$

$$v^\varepsilon \to v \text{ strongly in } H^1(T). \tag{4.10}$$

Proof. Let T be a triangle as described and $\varepsilon = \varepsilon_{k_j} \to 0$. Our aim is to construct the extensions v^ε on the basis of the fact that (4.2), (4.3), and (4.8) are satisfied for the corner points and the sides

Without loss of generality, we can assume in the sequel that $c^k_{x_l} = c_{x_l} = 0$ for all k and $l = 1, 2, 3$, where $c^k_{x_l}$ and c_{x_l} are the averages around corner x_l as in (4.3). Indeed, in the general case, we replace u^{ε_k} by $\tilde{u}^{\varepsilon_k} = u^{\varepsilon_k} - \alpha^k$, where α^k is the affine function satisfying

$$c^k_{x_l} = ⨍_{B_{\delta_k}(x_l)} \alpha^k(z)\, d\mathcal{L}^2(z). \tag{4.11}$$

Since the sequences $c^k_{x_l}$ converge in $\mathbb{R}$, the functions α^k converge strongly in $H^1(\Omega)$. If $\tilde v^{\varepsilon_k}$ is the strongly converging sequence for $\tilde u^\varepsilon$ as in the thesis of Proposition 4.5, we can set $v^{\varepsilon_k} := \tilde v^{\varepsilon_k} + \alpha^k$.

Let $\phi_k \in C^\infty(\mathbb{R}^2, [0,1])$ be a sequence of cut-off functions with

$$\operatorname{supp}\phi_k \subset \bigcup_{l=1}^3 B_{\delta_k}(x_l), \quad \phi_k(\xi) \equiv 1 \text{ on } \bigcup_{l=1}^3 B_{\delta_k/2}(x_l), \quad \|\nabla\phi_k\| \le \frac{3}{\delta_k}. \tag{4.12}$$

We set $\psi_k := 1 - \phi_k$ and write $u^{\varepsilon_k} = u^{\varepsilon_k}\phi_k + u^{\varepsilon_k}\psi_k$. The idea of the proof is to show that $u^{\varepsilon_k}\psi_k$ admits a strongly convergent extension with the help of a compact extension operator $E : H^1_0([x_i, x_l]) \to H^1(T)$. Concerning an extension of $(u^{\varepsilon_k}\phi_k)|_{\partial T}$, we will show that the family $u^{\varepsilon_k}\phi_k$ itself vanishes strongly in $H^1(T)$.

Claim 1. We treat one of the sides, $\Gamma = [x_i, x_l]$. Our aim is to show that there exists $C > 0$ such that

$$\|(u^{\varepsilon_k}\psi_k)|_\Gamma\|_{H^1(\Gamma)} \le C. \tag{4.13}$$

For $\delta > 0$, a set $B \subset \mathbb{R}^n$, let $B_\delta := \delta B = \{x \in \mathbb{R}^n : x/\delta \in B\}$. By a simple rescaling argument applied to the classical trace and Poincaré inequalities, for all bounded open sets $B \subset \mathbb{R}^n$ with Lipschitz boundary, there exists a constant $K = K(B)$ such that

$$\delta \int_{\partial B_\delta} |u|^2 + \int_{B_\delta} |u|^2 \le \delta^2 K \int_{B_\delta} |\nabla u|^2, \tag{4.14}$$

for all $\delta > 0$ and for all functions $u \in H^1(B_\delta)$ such that $\int_{B_\delta} u = 0$. The same estimate holds when the boundary integral over ∂B_δ is replaced by an integral over another $(n-1)$-dimensional submanifold δS, $S \subset B$.

We now consider the left hand side in (4.13). Regarding the L^2-norm we note that $\|(u^{\varepsilon_k}\psi_k)|_\Gamma\|_{L^2(\Gamma)} \le \|u^{\varepsilon_k}|_\Gamma\|_{L^2(\Gamma)} \le C$, holds by (4.8). Regarding the gradient, we compute

$$\nabla_\tau(u^{\varepsilon_k}\psi_k) = \psi_k \nabla_\tau u^{\varepsilon_k} + u^{\varepsilon_k}\nabla_\tau\psi_k, \tag{4.15}$$

and note that

$$\|\psi_k \nabla_\tau u^{\varepsilon_k}\|_{L^2(\Gamma)} \le \|\nabla_\tau u^{\varepsilon_k}\|_{L^2(\Gamma)} \le C, \tag{4.16}$$

again by (4.8). For the other term we find, using (4.12),

$$\|u^{\varepsilon_k}\nabla_\tau\psi_k\|^2_{L^2(\Gamma)} \le \sum_{l=1}^3 \|u^{\varepsilon_k}\nabla_\tau\psi_k\|^2_{L^2(B_{\delta_k}(x_l)\cap\Gamma)} \le \frac{9}{\delta_k^2}\sum_{l=1}^3 \int_{B_{\delta_k}(x_l)\cap\Gamma} |u^{\varepsilon_k}|^2 d\mathcal{H}^1 .$$

With (4.14), exploiting $c^k_{x_l} = 0$, we can calculate

$$\frac{1}{\delta_k^2}\int_{B_{\delta_k}(x_l)\cap\Gamma} |u^{\varepsilon_k}|^2 \le \frac{K}{\delta_k}\int_{B_{\delta_k}(x_l)} |\nabla u^{\varepsilon_k}|^2 = \delta_k K\,|B_1(0)| \fint_{B_{\delta_k}(x_l)} |\nabla u^{\varepsilon_k}|^2 \le CK\delta_k,$$

where we used (4.2) in the last inequality, exploiting that x_l is a point of typical average. This concludes the proof of (4.13).

Claim 2. We now construct a strongly convergent extension of $u^{\varepsilon_k}\psi_k$. Using affine coordinate transformations, it is sufficient to show the following: Let Γ be the horizontal segment $\Gamma = [(0,0),(\pi,0)] \equiv [0,\pi] \subset \mathbb{R}^2$, let $\ell > 0$ be given and let R be the rectangle $(0,\pi)\times(0,\ell)$. Let $w_k \in H^1(\Gamma)$ be a bounded sequence with $w_k \equiv 0$ in $\delta_k/2$-neighborhoods of the end-points of Γ. Then there exist extensions $w_k : R \to \mathbb{R}$ with $w_k \equiv 0$ on $\partial R \setminus \Gamma$ and a limit function w such that

$$w_k \to w \qquad \text{strongly in } H^1(R). \tag{4.17}$$

We sketch a proof for this extension result with a Fourier expansion argument. In order to take Fourier series, we extend the domain with $\tilde\Gamma = (0,2\pi)$ to $\tilde R = \tilde\Gamma \times (0,\ell)$ and take the odd extension of $w_k|_\Gamma$ to $\tilde\Gamma$, which is bounded in $H^1(\tilde\Gamma)$. Once we have constructed a 2π-periodic, odd extension $\tilde w_k : \tilde R \to \mathbb{R}$, the restriction to $w_k = \tilde w_k|_R$ is the desired function which vanishes on lateral boundaries.

Performing all calculations on the original domains we write

$$w_k|_\Gamma(s) = \sum_{m\in\mathbb{Z}} a_m^k e^{ims},$$

which satisfies, using an appropriate equivalent norm,

$$\|(u^{\varepsilon_k}\psi_k)|_\Gamma\|^2_{H^1(\Gamma)} = \sum_{m\in\mathbb{Z}} |a_m^k|^2\,|m|^2 \le C. \tag{4.18}$$

The harmonic extension $(w_k)|_\Gamma$ to $R = \Gamma\times(0,\ell)$ is then

$$w_k(s,t) := \sum_{m\in\mathbb{Z}} a_m^k e^{ims} e^{-mt}.$$

This sequence is bounded in $H^1(\Gamma\times(0,\ell))$, as can be shown by a direct calculation. We choose a subsequence $k\to\infty$ such that all coefficients a_m^k converge. The corresponding formal limit function is w,

$$w(s,t) := \sum_{m\in\mathbb{Z}} a_m e^{ims} e^{-mt}, \qquad \text{where } a_m = \lim_{k\to\infty} a_m^k. \tag{4.19}$$

We claim that the strong convergence $w_k \to w$ in $H^1(\Gamma\times(0,\ell))$ holds. We compute for an arbitrary $N\in\mathbb{N}$

$$\begin{aligned}
\int_0^\pi\!\int_0^\ell |\nabla w_k(s,t) - \nabla w(s,t)|^2\,ds\,dt &\le C\int_0^\pi\!\int_0^\ell \sum_{m\in\mathbb{Z}} |a_m^k - a_m|^2 |m|^2 e^{-2mt} ds\,dt\\
&\le C\sum_{m\in\mathbb{Z}} |a_m^k - a_m|^2|m|^2\frac{1}{|m|} \le C\sum_{|m|\le N} |a_m^k - a_m|^2|m| + \frac{C}{N}\left(\|w_k\|^2_{H^1} + \|w\|^2_{H^1}\right)\\
&\le C\sum_{|m|\le N} |a_m^k - a_m|^2|m| + \frac{C}{N}.
\end{aligned}$$

Passing to the limit as $k\to\infty$, owing to (4.19), we find

$$\lim_{k\to\infty} \|\nabla w_k - \nabla w\|^2_{L^2(\Gamma\times(0,\ell))} \le \frac{C}{N}.$$

Since $N\in\mathbb{N}$ was arbitrary, this concludes the proof of (4.17). Multiplication of all w_k and of w with a cut-off function provides additionally vanishing boundary values at the upper boundary $(0,\pi)\times\{\ell\}$.

Claim 3. We finally claim that the extensions $u^{\varepsilon_k}\phi_k$ of $(u^{\varepsilon_k}\phi_k)|_{\partial T}$ converges strongly to 0 in $H^1(T)$. Indeed, we can compute

$$\nabla(u^{\varepsilon_k}\phi_k) = \phi_k\nabla u^{\varepsilon_k} + u^{\varepsilon_k}\nabla\phi_k, \tag{4.20}$$

and use (4.2) to find

$$\fint_{B_{\delta_k}(x_l)} |\nabla u^{\varepsilon_k}|^2|\phi_k|^2\, d\mathcal{L}^2 \le \fint_{B_{\delta_k}(x_l)} |\nabla u^{\varepsilon_k}|^2\, d\mathcal{L}^2 \le CM_l\delta_k^2.$$

For the term $u^{\varepsilon_k}\nabla\phi_k$ we use (4.12), the Poincaré inequality (4.14), and (4.2),

$$\fint_{B_{\delta_k}(x_l)} |u^{\varepsilon_k}|^2|\nabla\phi_k|^2 \le \frac{9}{\delta_k^2}\fint_{B_{\delta_k}(x_l)} |u^{\varepsilon_k}|^2 \le 9K\fint_{B_{\delta_k}(x_l)} |\nabla u^{\varepsilon_k}|^2 \le CM_l\delta_k^2.$$

This yields the thesis of Claim 3 and concludes the proof of the proposition. □

We wish to emphasize that the extension of $w_k|_\Gamma$ with a Fourier series exploits that w_k vanishes in the corners. It was in order to cut out the corners in the above proof that we introduced the notion of a point of typical average.

As a preparation for the three-dimensional case we make a remark on another possible extension.

Remark 4.6. *The extensions v^ε can be chosen such that all segments $\Gamma = [x_i, x_l]$, $i \ne l$, are also typical segments for v^ε, and such that v^ε satisfies, for some $M_\Gamma > 0$,*

$$\fint_{B_{\delta_k}(\Gamma)} |\nabla v^{\varepsilon_k}(z)|^2\, d\mathcal{L}^2(z) \le M_\Gamma. \tag{4.21}$$

Proof. One part of the extended function v^ε is $u^{\varepsilon_k}\phi_k$. For these contributions, the boundedness (4.21) was actually shown in Claim 3.

The extension of $w_k|_{(0,\pi)}$ to functions w_k on $R = (0,\pi)\times(0,\ell)$ was performed with Fourier series. The construction can be altered by using the original function $w_k|_{(0,\pi)}$ in a δ_k-strip and then the extension of the above proof, i.e.

$$\tilde{w}_k(s,t) = \begin{cases} w_k(s,0) & \text{if } t < \delta_k \\ w_k(s, t-\delta_k) & \text{else.} \end{cases}$$

With this choice, in $B_{\delta_k}(\Gamma)$, the values $|\nabla\tilde{w}_k(x)|$ are bounded by multiples of corresponding point-values of $|\nabla_\tau w_k|_\Gamma|$ and $|w_k|_\Gamma|$. These are bounded by (4.18).

One easily verifies that the segment Γ is a typical segment also for v^ε. □

Definition 4.7 (Adapted grid for $n = 2$)**.** *Let $Q \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $(u^\varepsilon)_\varepsilon$ a bounded sequence in $H^1(Q)$ for $\varepsilon = \varepsilon_k \searrow 0$, $h > 0$ fixed and $\delta_k \searrow 0$. We say that a family $\mathscr{T}_h = \{T_k\}_{k\in\Lambda_h}$ of triangles is an* adapted grid *for $(u^\varepsilon)_\varepsilon$ if the boundaries of all triangles are typical segments according to Definition 4.3. We furthermore assume that one subsequence $(k_j)_j$ is a good subsequence for all segments.*

The above observations on typical points, typical segments, and strongly convergent extensions provide the main result of this section, the compensated compactness result that was already used in the proof of the main theorem.

Theorem 4.8. *Let $Q \subset \mathbb{R}^n$, $n = 2$ or $n = 3$ be a bounded Lipschitz domain, $(u^\varepsilon)_\varepsilon$ be a bounded sequence in $H^1(Q)$, and $\delta_k \searrow 0$.*

1. *To arbitrary $h > 0$ there exists $Q_h \subset Q$ and a triangulation $\mathscr{T}_h$ of Q_h as in (2.1), such that $\mathscr{T}_h$ is an* adapted grid for $(u^\varepsilon)_\varepsilon$.

2. *Let $(u^\varepsilon)_\varepsilon$ be a sequence with $u^\varepsilon \rightharpoonup u$ weakly in $H^1(Q)$ and let $\mathscr{T}_h$ be an adapted grid for $(u^\varepsilon)_\varepsilon$. Furthermore, let $(q^\varepsilon)_\varepsilon$ be a sequence in $L^2(Q, \mathbb{R}^n)$ satisfying*

$$q^\varepsilon \rightharpoonup q \qquad \text{weakly in } L^2(Q), \tag{4.22}$$

$$f^\varepsilon := \nabla \cdot q^\varepsilon \to f \qquad \text{strongly in } H^{-1}(T), \quad \text{for all } T \in \mathscr{T}_h. \tag{4.23}$$

Then there holds

$$\lim_{\varepsilon \to 0} \int_{Q_h} q^\varepsilon \cdot \nabla u^\varepsilon \, dx = \int_{Q_h} q \cdot \nabla u \, dx. \tag{4.24}$$

Since adapted grids in three space dimensions are constructed only in the next Subsection, we postpone the proof for $n = 3$ to Subsection 4.2. We note already here that the proof of item 2 is independent of the dimension.

Proof of Theorem 4.8 for $n = 2$. Item 1. Existence of adapted grids. The grid can be chosen by subsequently adding grid-points. Every corner x is chosen as a point of typical average and such that almost every segment with x as an end-point is a typical segment. Since almost every x has both properties by Lemmas 4.2 and 4.4, we can construct a grid to prescribed $h > 0$ in this way.

Item 2. Compensated compactness. It is sufficient to consider a single triangle T. For the fixed triangle (a simplex in general space dimension) we use the strongly $H^1(T)$-convergent extension v^ε of the boundary values of u^ε, constructed in Proposition 4.5, $v^\varepsilon \to v$ in $H^1(T)$. The boundary values are always expressed through the trace theorem, hence, by definition of identical traces, we have

$$\int_T q^\varepsilon \cdot \nabla u^\varepsilon + \int_T \nabla \cdot q^\varepsilon u^\varepsilon = \int_T q^\varepsilon \cdot \nabla v^\varepsilon + \int_T \nabla \cdot q^\varepsilon v^\varepsilon \,.$$

We can therefore calculate

$$\int_T q^\varepsilon \cdot \nabla u^\varepsilon = \int_T q^\varepsilon \cdot \nabla v^\varepsilon - \langle f^\varepsilon, u^\varepsilon - v^\varepsilon \rangle_{H^{-1}, H_0^1} \to \int_T q \cdot \nabla v - \langle f, u - v \rangle_{H^{-1}, H_0^1} \,.$$

We use here the weak L^2-convergence of q^ε and the strong L^2-convergence of ∇v^ε. In the term containing f, we use the weak H_0^1-convergence $u^\varepsilon - v^\varepsilon \to u - v$ and the strong H^{-1}-convergence $f^\varepsilon \to f$.

Performing the above interpretation of identical boundary values again for u and v instead of u^ε and v^ε provides

$$\int_T q^\varepsilon \cdot \nabla u^\varepsilon \to \int_T q \cdot \nabla v - \langle f, u - v \rangle_{H^{-1}, H_0^1} = \int_T q \cdot \nabla u,$$

and thus, after a summation over all triangles, the claim (4.24). $\square$

4.2 Adapted grids in three dimensions

We are again given sequences $(\varepsilon_k)_k$ and $u^\varepsilon \in H^1(\Omega)$, now with $\Omega \subset \mathbb{R}^3$. Our aim is to show that almost all simplices S contained in the domain Ω are "typical" in the sense that $u^\varepsilon|_{\partial S}$ has a strongly convergent extension for a subsequence $(k_j)_j$. Since objects of different dimensions appear in the sequel, we find it convenient to indicate the dimension with a superscript. We will typically use Γ^1 for segments, E^2 for planes, and S^3 for three-dimensional simplices.

In two space dimensions, we considered typical segments and points of typical average. Regarding segments we demanded boundedness of u^ε on the segment, regarding points, we demanded more, namely a boundedness property in a neighborhood. Transfering these concepts to three space dimensions, we will demand that u^ε is bounded on triangles T^2, and that averages of u^ε are bounded in neighborhood of segments Γ^1. We therefore introduce below *segments of typical average*, which has stronger requirements than a typical segment.

Definition 4.9 (Segments of typical average and typical triangles)**.** *Let $n = 3$ and $\Gamma^1 = [x, y] \subset \Omega$ be a segment, contained in a two-dimensional plane $E^2 \subset \mathbb{R}^3$. We say that Γ^1 is a* segment of typical average for $(u^\varepsilon)_\varepsilon$ and E^2, *if $u^\varepsilon|_{E^2}$ is an H^1-bounded sequence and if*

1. *The segment Γ^1 is a typical segment in E^2 according to Definition 4.3.*
2. *Along the same subsequence $(k_j)_j$, for a constant $M_0 > 0$, holds*

$$\fint_{B_{\delta_k}(\Gamma^1)\cap\Omega} |u^{\varepsilon_k}(z)|^2 + |\nabla u^{\varepsilon_k}(z)|^2 \, d\mathcal{L}^3(z) \le M_0, \tag{4.25}$$

$$\fint_{B_{\delta_k}(\Gamma^1)\cap E^2} |u^{\varepsilon_k}(z)|^2 + |\nabla u^{\varepsilon_k}(z)|^2 \, d\mathcal{L}^2(z) \le M_0. \tag{4.26}$$

We say that a triangle $T^2 \subset \mathbb{R}^3$ is a typical triangle, *if the three sides are segments of typical average for the plane E^2 containing T^2, for the same subsequence $(k_j)_j$.*

We note that, by definition of a typical triangle, for some $M_0 > 0$,

$$\|u^{\varepsilon_k}|_{T^2}\|^2_{L^2(T^2)} + \|\nabla_\tau u^{\varepsilon_k}|_{T^2}\|^2_{L^2(T^2)} \le M_0. \tag{4.27}$$

Lemma 4.10 (Many typical triangles)**.** *Let $\Omega \subset \mathbb{R}^3$ be a convex domain, $\delta_k \to 0$ and $\varepsilon_k \to 0$ fixed, and $(u^\varepsilon)_\varepsilon$ be an $H^1(\Omega)$-bounded sequence. Then, successively chosen, for almost all $x_1 \in \Omega$, for almost all $x_2 \in \Omega$, for almost all $x_3 \in \Omega$, the triangle T^2 given by (x_1, x_2, x_3) is a typical triangle.*

Sketch of proof. For almost every plane E^2 defined by (x_1, x_2, x_3), the family $u^\varepsilon|_{E^2}$ is bounded in $H^1(E^2)$. This follows from Fubini's theorem, arguing as in Lemma 4.4.

Let E^2 be such a plane. Then, by Lemma 4.4, applied with $n = 2$, almost all segments in E^2 are typical segments in E^2. This provides the property of item 1.

It remains to check properties (4.25) and (4.26) of item 2 for almost every choice of (x_1, x_2, x_3). Let $0 \neq \gamma \in \mathbb{R}^3$ be an arbitrary vector such that $\Gamma_x := [x, x + \gamma]$ defines a segment in $\mathbb{R}^3$ for every $x \in \mathbb{R}^3$. With fixed γ, we now consider

$$f^\varepsilon : \mathbb{R}^3 \to \mathbb{R}, \quad f^\varepsilon(x) = \int_{(x+\mathbb{R}\gamma)\cap\Omega} |u^\varepsilon|^2 + |\nabla u^\varepsilon|^2.$$

Let $F^2 \subset \mathbb{R}^3$ be an arbitrary plane orthogonal to γ. We consider the restriction $f^\varepsilon : F^2 \to \mathbb{R}$, which is a bounded family in $L^1(F^2)$. Arguing as in the proof of Lemma 4.2, we conclude that for almost all $x \in F^2$, the δ_k-averages of f^{ε_k} are bounded. This implies (4.25).

The estimate (4.26) follows in the same way when we choose a line $F^1 \subset E^2$, which is orthogonal to γ. □

Lemma 4.11 (Strongly convergent extensions in $\mathbb{R}^3$). *Let $\Omega \subset \mathbb{R}^3$, $\delta_k \searrow 0$ and $\delta_k \searrow 0$, and let $(u^\varepsilon)_\varepsilon$ be a bounded sequence in $H^1(\Omega)$. Let $S^3 \subset \Omega$ be a simplex such that the four sides T^2_m, $m = 1, 2, 3, 4$, are typical triangles. Then there exists a subsequence $(k_j)_j$ and extensions $v^\varepsilon \in H^1(S^3)$ of the boundary values $u^\varepsilon|_{\partial S^3}$ such that, for a limit function $v \in H^1(S^3)$,*

$$v^\varepsilon = u^\varepsilon \text{ on } \partial S^3, \tag{4.28}$$

$$v^\varepsilon \to v \text{ strongly in } H^1(S^3) \text{ along the subsequence.} \tag{4.29}$$

Proof. Step 1. Modification of u^ε to $\tilde{u}^\varepsilon$ with vanishing values along the edges. Our first aim is to modify u^ε such that we only have to treat functions that vanish on the edges Γ^1_i, $i = 1, ..., 6$. To this end we note that, since every side T^2_m, $m = 1, ..., 4$, is a typical triangle, we may use the two-dimensional result of Proposition 4.5 on each face. This provides extensions $w^\varepsilon : T^2_m \to \mathbb{R}$ with $w^\varepsilon|_{\Gamma^1_i} = u^\varepsilon|_{\Gamma^1_i}$ that are strongly convergent in $H^1(T^2_m)$. With a rotation of the functions w^ε around Γ^1_i, using additionally linear transformations and cut-off functions, we can construct extensions

$$\tilde{w}^\varepsilon : S^3 \to \mathbb{R}, \quad \tilde{w}^\varepsilon|_{T^2_m} = w^\varepsilon, \quad \tilde{w}^\varepsilon \text{ strongly convergent in } H^1(S^3).$$

The last property follows from the strong convergence of w^ε in $H^1(T^2_m)$. By Remark 4.6, we can achieve that each edge Γ^1_i is a segment with typical averages not only for the sequence u^ε, but also for the sequence $\tilde{w}^\varepsilon$ (compare Definition 4.9 and estimate (4.21), which remains valid after the extension by rotation).

We now consider the modified sequence of functions $\tilde{u}^\varepsilon := u^\varepsilon - \tilde{w}^\varepsilon$. This sequence has vanishing values on all edges Γ^1_i. Since the sequences $\tilde{w}^\varepsilon$ converges strongly in $H^1(S^3)$, it is sufficient to show for $\tilde{u}^\varepsilon$ the existence of a strongly $H^1(S^3)$-convergent subsequence. It is important to note that our construction guarantees that the edges Γ^1_i are segments of typical averages also for the sequence $\tilde{u}^\varepsilon$.

Step 2. Extension of $\tilde{u}^\varepsilon$. We treat one of the faces T^2, let $\Gamma^1 \subset \partial T^2$ be one edge. We use a family of smooth cut-off functions $\phi_k : \mathbb{R}^3 \to [0, 1]$ with $\operatorname{supp}(\phi_k) \subset B_{\delta_k}(\Gamma^1)$ and $\|\nabla \phi_k\|_\infty \le C/\delta_k$, such that $\phi_k \equiv 1$ on $B_{\delta_k/2}(\Gamma^1) \subset \mathbb{R}^3$. Analogous to Proposition 4.5, we want to extend the trace $[(1 - \phi_k)\tilde{u}^{\varepsilon_k}]|_{T^2}$ as a harmonic function to S^3. We calculate

$$\int_{T^2} |\nabla_\tau [(1 - \phi_k)\tilde{u}^{\varepsilon_k}]|^2 \, d\mathcal{L}^2 \le C \frac{1}{\delta_k^2} \int_{B_{\delta_k}(\Gamma^1) \cap T^2} |\tilde{u}^{\varepsilon_k}|^2 \, d\mathcal{L}^2 + C \int_{T^2} |\nabla_\tau \tilde{u}^{\varepsilon_k}|^2 \, d\mathcal{L}^2.$$

The last integral is bounded by (4.27). For the other integral on the right hand side we use the boundedness of the gradient in $B_{\delta_k}(\Gamma^1) \cap T^2$ and Poincaré's inequality, exploiting $\tilde{u}^{\varepsilon_k} \equiv 0$ on Γ^1. We find that $[(1-\phi_k)\tilde{u}^{\varepsilon_k}]|_{T^2}$ is a bounded sequence in $H^1(T^2)$, which vanishes in a neighborhood of the boundary. This allows to extend the function harmonically to S^3 with vanishing values on $\partial S^3 \setminus T^2$. As calculated for Proposition 4.5, the harmonic extension posesses a strongly $H^1(S^3)$-convergent subsequence.

It remains to verify the smallness in $H^1(S^3)$ of the functions $\phi_k \tilde{u}^{\varepsilon_k}$. We calculate

$$\begin{aligned}\int_{S^3} |\nabla(\phi_k \tilde{u}^{\varepsilon_k})|^2 \, d\mathcal{L}^3 &\le C \frac{1}{\delta_k^2} \int_{B_{\delta_k}(\Gamma^1)\cap S^3} |\tilde{u}^{\varepsilon_k}|^2 \, d\mathcal{L}^3 + C \int_{S^3} |\phi_k|^2 |\nabla \tilde{u}^{\varepsilon_k}|^2 \, d\mathcal{L}^3 \\ &\le C ⨍_{B_{\delta_k}(\Gamma^1)\cap S^3} |\tilde{u}^{\varepsilon_k}|^2 \, d\mathcal{L}^3 + C\delta_k^2 ⨍_{B_{\delta_k}(\Gamma^1)\cap S^3} |\nabla \tilde{u}^{\varepsilon_k}|^2 \, d\mathcal{L}^3 \to 0.\end{aligned}$$

The convergence to 0 of the second term is an immediate consequence of the boundedness of the integral, which follows from property (4.25) of segments with typical averages. For the first term we use once more Poincaré's inequality: the gradients are bounded on planes and in space by (4.26) and (4.25), the vanishing values $\tilde{u}^{\varepsilon_k} \equiv 0$ on Γ^1 imply smallness of averages in the neighborhood. □

In order to make the statements in the three-dimensional case precise, we include the following definition.

Definition 4.12 (Adapted grid in three dimensions)**.** *Let $Q \subset \mathbb{R}^3$ be a bounded domain and let $(u^\varepsilon)_\varepsilon$ be a bounded sequence in $H^1(Q)$. We say that a subdivision $\mathscr{T}_h = \{S_k\}_{k\in\Lambda_h}$ of $Q_h \subset Q$ in simplices S_k is an* adapted grid *for $(u^\varepsilon)_\varepsilon$ if all sides T_m^2 of the simplices are typical triangles with the same subsequence $(k_j)_j$ according to Definition 4.9.*

With this definition, Theorem 4.8 is valid also in the case $n = 3$. The proof of Theorem 4.8 in the case $n = 3$ is identical to the two-dimensional case, using the three-dimensional Lemmata above instead of the corresponding two-dimensional results.

A Ergodic homogenization cell problem

In [15], a probability space setting is introduced to treat homogenization of stochastic coefficients. The authors use dynamical systems (corresponding to translations) $T_{x/\varepsilon} : \omega \to T_{x/\varepsilon}(\omega)$ on the probability space $(\Omega_{\mathcal{P}}, \mathcal{P})$ to construct coefficients $a^\varepsilon(x) = \tilde{a}(x/\varepsilon; \omega)$. Under ergodicity assumptions, they obtain the following result.

Theorem A.1. *Under ergodicity assumptions on the coefficients $\tilde{a}(x;\omega)$, the following holds. There exists a matrix $a^* \in \mathbb{R}^{n\times n}$ such that for $\mathcal{P}$-almost every ω exists $\psi_k(.;\omega) : \mathbb{R}^n \to \mathbb{R}^n$ with*

$$\nabla_y \cdot (\tilde{a}(y)\psi_k(y)) = 0 \quad \text{on } \mathbb{R}^n, \tag{1.1}$$

$$\text{curl } \psi_k = 0 \quad \text{on } \mathbb{R}^n, \tag{1.2}$$

such that the average of ψ_k is e_k and the average of $\tilde{a} \cdot \psi_k$ is $a^* \cdot e_k$, in the following sense: For every subset $K \subset \mathbb{R}^n$ holds

$$\psi_k(./\varepsilon;\omega) \rightharpoonup e_k \qquad \text{in } L^2(K), \tag{1.3}$$

$$\tilde{a}(./\varepsilon;\omega)\,\psi_k(./\varepsilon;\omega) \rightharpoonup a^* \cdot e_k \ \text{ in } L^2(K). \tag{1.4}$$

From this theorem, one easily deduces the property of Definition 1.1. We conclude that stochastic coefficients as constructed in [15] allow averaging of the constitutive relation.

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