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The concentration function problem for locally compact groups revisited: Non-dissipating space-time random walks, τ-decomposable laws and their continuous time analogues

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THE CONCENTRATION FUNCTION PROBLEM FOR LOCALLY COMPACT GROUPS REVISITED: NON-DISSIPATING SPACE-TIME RANDOM WALKS, τ-DECOMPOSABLE LAWS AND THEIR CONTINUOUS TIME ANALOGUES

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Abstract. The concentration function problem for locally compact groups, i.e., the structure of groups admitting adapted non-dissipating random walks, is closely related to relatively compact M- or skew semigroups and corresponding space-time random walks, resp. to τ-decomposable laws, where τ denotes an automorphism. Analogous results are obtained in the case of continuous time: Non-dissipating Lévy processes are related to relatively compact distributions of generalized Ornstein Uhlenbeck processes and corresponding space-time processes, resp. T-decomposable laws, $T = (\tau_t)$ denoting a continuous group of automorphisms acting on groups of the form $N = C_K(T)$.

Introduction

Let $\mathbb{G}$ be a locally compact group, $\lambda \in \mathcal{M}^1(\mathbb{G})$ a (w.l.o.g.) adapted probability measure. λ, more precisely, the random walk $\left\{\lambda^k\right\}_{k\geq 0}$, is called non-dissipating (or non scattering) if for some compact subset $C \subseteq \mathbb{G}$ the (right) concentration functions $f_{\lambda^k}(C) := \sup_{x\in\mathbb{G}} \lambda^k(Cx)$ fail to converge to 0, with time $k \to \infty$. Analogously one could define left concentration functions as $g_{\lambda^k}(C) := \sup_{x\in\mathbb{G}} \lambda^k(xC)$. Note that the behaviour of left and right concentration functions may differ in characteristic manner. (Cf. e.g., Example 1.10.) If the random walk is non-dissipating, $\left\{\lambda^k\right\}_{k\geq 0}$ is relatively shift compact, equivalently, $\left\{\lambda^k * \widetilde{\lambda}^k\right\}_{k\geq 0}$ is relatively compact. Furthermore, if $N = N_\lambda$ denotes the smallest closed normal subgroup containing the support $\mathrm{supp}\lambda * \widetilde{\lambda}$, then $\mathbb{G}/N \cong \mathbb{Z}$. Hence there exist $x \in \mathbb{G}$ such that $\lambda = \nu * \varepsilon_x$ with $\mathrm{supp}\nu \subseteq N$.

Denoting the restriction of the inner automorphism i_x to N by $\tau := i_x\,|_N$, we obtain: $G \cong N \rtimes_\tau \mathbb{Z}$, and λ is representable as $\lambda = \nu(1) \otimes \varepsilon_1$, with $\nu = \nu(1) \in \mathcal{M}^1(N)$, hence the random walk is representable as $\lambda^k = \nu(k) \otimes \varepsilon_k$, all $k \in \mathbb{Z}_+$, ($\nu(k) \in \mathcal{M}^1(N)$, $\nu(0) := \varepsilon_0$).

For the history of the concentration function problem on locally compact groups the reader is referred to the survey of W. Jaworski [27] showing previous developments and a recent state of investigations: Beginning with the pioneer works [4], [24] to the investigations [27, 28, 25]. This is closely related to parallel investigations of M-semigroups and τ-decomposability:

To avoid trivialities, throughout $\mathbb{G}$ is supposed to be non-compact.

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As immediately seen, $\nu(k)$ is representable as $\nu(k) = \mathop{\ast}\limits_{j=0}^{k-1} \tau^j(\nu(1)),\ k \in \mathbb{N}$, and hence satisfies the the relation

$$\nu(k+\ell) = \nu(k) * \tau^k(\nu(\ell),\ k, \ell \in \mathbb{Z}_+ \tag{0.1}$$

A sequence $\{\nu(k)\}_{k\in\mathbb{Z}_+}$ satisfying (0.1) is called discrete time M-(or Mehler-)semigroup (in fact, the (discrete) semigroup of transition kernels is frequently called discrete Mehler semigroup), also called skew semigroup or τ-semigroup. Moreover, this M-semigroup is relatively compact. Conversely, discrete M-semigroups $\{\nu(k)\}$ define *space-time* random walks $\{\lambda^k = \nu(k) \otimes \varepsilon_k\}$ on the *space-time building* $\mathbb{G} = N \rtimes_\tau \mathbb{Z}$. So non-dissipating random walks (in $\mathbb{G}$) correspond in a 1-1-manner to relatively compact M-semigroups (on N). (Cf. Theorem 1.6 below.)

It is shown in [27] that N has an interesting structure: $N = C_K(\tau)$, where $C(\tau) := \{x \in N : \tau^n(x) \to e\}$ and $C_K(\tau) := \{x \in N : \tau^n(x) \to e \mod K\}$ denote the contractible subgroup resp. the K-contractible subgroup, K is a compact subgroup of N. And moreover, $C_K(\tau) = C(\tau) \cdot K$, at least in the case of Lie groups or totally disconnected groups. In Theorem 1.6 we show that $\rho := \lim\limits_{k\to\infty} \nu(k) \star \omega_K$ exists and is τ-decomposable, i.e., $\rho = \nu(k) * \tau^k(\rho)$ for *cofactors* $\nu(k)$, $k \in \mathbb{Z}_+$; furthermore, ρ is right K-invariant. Conversely, for any such measure we have $\lim \nu(k) \star \omega_K = \rho$, and hence ρ generates a relatively compact M-semigroup of cofactors $\{\nu(k)\}$. Thus ρ resp. the cofactors generate a non-dissipating random walk.

The second part of the paper is concerned with the continuous time analogues. The random walk is replaced by a continuous convolution semigroup $\{\lambda_t\}_{t\in\mathbb{R}_+}$, the distributions of a Lévy process. First we show that $\{\lambda_t\}_{t\in\mathbb{R}_+}$ is non-dissipating if some (hence all) *skeleton random walk* $\{\lambda_{t_0}^k = \lambda_{t_0 k}\}_{k\in\mathbb{Z}_+}$ is non-dissipating and that the subgroup $N = N_{\lambda_{t_0}}$ is independent of $t_0 > 0$ and is a normal subgroup in $\mathbb{G}$ (not only within the group $\mathbb{G}_{t_0}$ generated by $\mathrm{supp}\lambda_{t_0}$). Thus the results for discrete times apply easily to the continuous time setup. Furthermore, $\mathbb{G}/N \cong \mathbb{R}$, and there exists a continuous one-parameter group $(x(t))_{t\in\mathbb{R}}$ such that $T = \left(\tau_t := i_{x(t)}\,|_N\right) \subseteq \mathrm{Aut}(N)$ and $\mathbb{G} \cong N \rtimes_T \mathbb{R}$.

In Theorem 2.7 we show that non-dissipating continuous convolution semigroups and relatively compact (continuous time) M-semigroups $\{\nu(t)\}_{t\in\mathbb{R}_+}$ correspond in a 1-1 way. Furthermore, $\rho := \lim\limits_{t\to\infty} \nu(t) \star \omega_K$ exists, is T-self-decomposable, i.e., for some cofactors $\nu(t) \in \mathcal{M}^1(N)$, $\rho = \nu(t) \star \tau_t(\rho)$ for all $t \in \mathbb{R}_+$ and ρ is right K-invariant. Conversely, given such a measure, then there exists a *continuous* M-semigroup of cofactors $\{\nu(t)\}_{t\in\mathbb{R}_+} \subseteq \mathcal{M}^1(N)$ defining a space-time continuous convolution semigroup $\{\lambda_t = \nu(t) \otimes \varepsilon_t\}_{t\in\mathbb{R}_+}$. And since $\{\nu(t)\}$ is relatively compact, $\{\lambda_t\}$ is non-dissipating.

The latter property, relative compactness of the M-semigroups, can be characterized by the existence of logarithmic moments. For continuous time this (and some equivalent assertions) is shown in Theorem 2.8.

For continuous time, M-(Mehler-) or skew semigroups and corresponding space-time continuous convolution semigroups had been investigated in the past in different papers. Beginning with the pioneer work [23] (with slightly different representations) to [14, 13, 21]. See also [3] for Mehler hemigroups and embedding of discrete time models into continuous time ones. In [14] the reader will find some more hints to the literature, in particular to vector spaces beyond the locally compact group case. Both branches of investigations, the concentration function problem and (semi)-stability and (self)-decomposability resp. M-semigroups lead to investigations of the structure of (K)-contractible subgroups $C(\tau)$ resp. $C_K(\tau)$ of locally compact groups. See e.g., [39, 40], [20, 19], [21], Ch. III, [34, 35, 7, 6], [27, 28, 25], [10], [1] and the literature mentioned there.

1. Discrete time: Non-dissipating random walks

Recall the following notations: $\mathbb{G}$ denotes a locally compact group, for $\lambda \in \mathcal{M}^1(\mathbb{G})$, the set of probabilities, let $\widetilde{\lambda}$ be the image of λ under the inverse mapping. $*$ denotes convolution on $\mathbb{G}$, λ^k is the k-th convolution power, $\lambda^0 := \varepsilon_e$, where ε_x denotes the point measure in $x \in \mathbb{G}$. W.l.o.g. λ is supposed to be adapted, i.e., $\mathbb{G}$ is the closed group generated by the support suppλ. $N = N_\lambda$ denotes the smallest closed normal subgroup containing supp$\lambda * \widetilde{\lambda}$. The concentration function of the random walk $\{\lambda^k\}_{k\in\mathbb{Z}_+}$ is defined as $\mathbb{Z}_+ \ni k \mapsto f_{\lambda^k}(C) := \sup_{x\in\mathbb{G}} \lambda^k(Cx)$ for compact $C \subseteq \mathbb{G}$. λ or $\{\lambda^k\}$ is called non-dissipating if $f_{\lambda^k}(C)$ fails to converge to 0 with $k \to \infty$, for some compact C.

To distinguish, in the sequel '$*$' will denote convolution on $\mathbb{G}$ while convolution on N is denoted by '$\star$'.

We collect some properties:

Facts 1.1. *a)* $\{\lambda^k\}$ *is non-dissipating iff* $\{\lambda^k\}$ *is relatively shift compact, i.e., for some* $x_k \in \mathbb{G}\backslash\{e\}$, $\left\{\lambda^k * \varepsilon_{x_k^{-1}}\right\}$ *is relatively compact. Equivalently, iff* $\left\{\lambda^k * \widetilde{\lambda}^k\right\}$ *is relatively compact.*

In that case, if $\mathbb{G}$ *is non-compact, the following assertions b)–f) hold:*
b) $\mathbb{G}/N \cong \mathbb{Z}$, *hence the shifts can be chosen as* $x_k = x^k$ *for some* $x \in \mathbb{G}\backslash N$.
c) The restriction to N *of the inner automorphism* $\tau := i_x|_N$ *is considered as automorphism of* N, *and hence* $\mathbb{G}$ *has a canonical representation* $\mathbb{G} = N \rtimes_\tau \mathbb{Z}$ *(with product* $(g,k)(h,\ell) = (g\tau(h), k+\ell)$ *for* $g,h \in N,\ k,\ell \in \mathbb{Z}$, *where* $x = (e,1)$*).*
d) There exists a compact subgroup $K \subseteq N$ *such that* $N = C_K(\tau) := \left\{x \in N : \tau^k(x) \to e \mod K\right\}$. *Moreover, with* $C(\tau) = C_{\{e\}}(\tau)$ *we have* $C_K(\tau) = C(\tau)\cdot K$, *at least in the case of Lie groups or totally disconnected groups. If* N *is compact then* $N = K$ *as* τ *is compactly contracting.*
e) λ *and* $\nu := \lambda * \varepsilon_{x^{-1}}$ *are representable as* $\lambda = \nu \otimes \varepsilon_1 \in \mathcal{M}^1(N \rtimes_\tau \mathbb{Z})$, $\nu =: \nu(1)$ *identified with a probability of* $\mathcal{M}^1(N)$. *Hence by induction,* $\lambda^k = \nu(k) \otimes \varepsilon_k$, $k \in \mathbb{Z}_+$, *with* $\nu(0) = \varepsilon_e$, $\nu(1) = \nu$, $\nu(k) = \bigstar_{j=0}^{k-1} \tau^j(\nu) \in \mathcal{M}^1(N)$ *and* x *identified with* $(e,1)$. *Furthermore,* $\{\nu(k)\}_{k\in\mathbb{Z}_+}$ *is relatively compact.*
f) Put $L := \{\text{supp}\nu(k) : k \in \mathbb{Z}_+\}$. *Then* N *is the smallest* τ*-invariant normal subgroup of* N *containing* L. *In general,* L *will not generate* N.

⟦ a) see [27], Corollary 3.2, and the literature mentioned there. In particular, [9], [26], theorem 1.

b)c) See [27], theorem 3.4, [9].

d) See [27], theorem 3.5, 3.9. For the representation $C_K(\tau) = C(\tau)K$ see [28], see also [1], [10], [25] for previous results. For Lie groups see [19], [21], theorem 3.2.13.

e) is immediately verified.

f) λ is adapted, hence $\mathbb{G}$ is generated by $\mathrm{supp}\nu(1) \otimes 1$. $N \triangleleft \mathbb{G}$, hence N is τ-invariant and $N \supseteq L$. By definition, N is the smallest subgroup with this property. It is easily shown, cf. e.g., Example 1.10, that N may be larger than the group generated by L. ⟧

As already mentioned, $\mathbb{G}$ is always supposed to be non-compact, else any random walk would be non-dissipating. If N is compact then $N = K$ and thus any space-time random walk $\{\lambda^k = \nu(k) \otimes \varepsilon_k\}$ is non-dissipating.

Definition 1.2. *A probability μ on a locally compact group $\mathbb{H}$ possesses finite first order moments resp. finite logarithmic moments if $\int_{\mathbb{H}} f \mathrm{d}\mu < \infty$ resp. $\int_{\mathbb{H}} \log(1 + f)\mathrm{d}\mu < \infty$ for all sub-additive Borel functions $f : \mathbb{H} \to \mathbb{R}_+$.*

Similarily we define for nonnegative measures η on $\mathbb{H}$ which are bounded outside any neighbourhood $W \in \mathfrak{U}(e)$: η possesses finite first order moments resp. finite logarithmic moments if $\int_{\mathbb{H}\setminus W} f \mathrm{d}\mu < \infty$ resp. $\int_{\mathbb{H}\setminus W} \log(1+f)\mathrm{d}\mu < \infty$ for all sub-additive Borel functions $f : \mathbb{H} \to \mathbb{R}_+$.

Facts 1.3. *If N is a second countable locally compact group, let (X_k) denote iid random variables in N with distribution $X_k(P) = \nu(1)$. Put $Z_n := X_1\tau(X_2) \cdots \tau^{n-1}(X_n)$ with distribution $Z_n(P) = \nu(n)$. Then the following assertions are equivalent:*

(i) $\nu(k)$ *is weakly convergent* mod K, *i.e.,* $\lim \nu(k) \star \omega_K$ *exists*

(ii) $\{Z_k\}$ *is stochastically convergent* mod K

(iii) $\{Z_k\}$ *is almost surely convergent* mod K

(iv) $\nu(1)$ *possesses finite logarithmic moments.*

(v) λ *possesses finite first order moments.*

⟦ Obviously we have $Z_k(P) = \nu(k)$. The equivalence of the assertions $(i) - (iii)$ follows by Lévy's equivalence theorem for groups, see [22], theorem 2.2.14, or, in context of invariant metrics on N/K, [27].

The equivalence of the conditions $(i)-(iv)$ is folklore for vector spaces (cf. e.g. [29]), for homogeneous groups [18], [21], 2.14.24, for general contractible groups [17]. For K-contractible groups, $N = C_K(\tau)$, cf. [27], proposition 4.3, (in the context of τ-functions and invariant metrics.)

$(iv) \Leftrightarrow (v)$ cf. [27], corollary 2.15. ⟧

Remark 1.4. *The equivalence $(i) \iff (iv)$ in Facts 1.3 holds true without separability assumptions.*

Proof. (Cf. also [27], proof of theorem 3.9).
$\mathbb{G}$, $\mathrm{supp}\lambda$, and hence N are σ-compact. Hence (cf. [5], page 101, ex.

11, or [41], theorem 5.2) representable as projective limits of metrizable quotients, $N = \varprojlim N/\Gamma_\alpha$ resp. $\mathbb{G} = \varprojlim N/\Gamma_\alpha \rtimes_\tau \mathbb{Z}$ with compact, τ-invariant subgroups $\Gamma_\alpha \lhd N$. Obviously, $\nu(k) \star \omega_K$ is convergent resp. $\nu(1)$ possesses finite logarithmic moments iff for all α the the projections to the quotients share this property. □

Definition 1.5. *a) A sequence $\{\nu(k)\}_{k\in\mathbb{Z}_+}$ in $\mathcal{M}^1(N)$ satisfying*

$$\nu(0) = \varepsilon_e,\ \nu(1) =: \nu,\ \nu(k+\ell) = \nu(k) \star \tau^k(\nu(\ell));\quad k,\ell \in \mathbb{Z}_+ \tag{1.1}$$

is called discrete M-semigroup, (also called Mehler semigroup, τ-semigroup, skew semigroup etc).

b) $\rho \in \mathcal{M}^1(N)$ is called τ-decomposable if for some cofactor $\nu = \nu(1) \in \mathcal{M}^1(N)$ *we have $\rho = \nu \star \tau(\rho)$. Then by induction, $\rho = \nu(k) \star \tau^k(\rho)$.* $\mathrm{Cof}_\rho(\tau^k)$ *denotes the set of cofactors. According to the shift-compactness theorem (*[33]*, III, theorems 2.1, 2.2 (for metrizable groups),* [22]*, theorem 1.2.21), the sets of cofactors are compact for all k.*

c) ρ is right K-invariant if $\rho \star \omega_K = \rho$, where ω_K denotes the normalized Haar measure on a compact subgroup $K \subseteq N$.

d) For short: ρ is $K-\tau$-decomposable if ρ is τ-decomposable and right K-invariant.

e) A 2-parameter family $\{\nu(k,\ell)\}_{k,\ell\in\mathbb{Z}_+}$ is called discrete hemigroup (or distribution of an additive process) if for all $k,\ell,r \in \mathbb{Z}_+$ we have $\nu(k, k+\ell+r) = \nu(k,k+\ell) \star \nu(k+\ell, k+\ell+r)$. It is a τ-hemigroup, if in addition $\tau(\nu(k,\ell)) = \nu(k+1,\ell+1)$. Then obviously, $\nu(k,k+\ell+r) = \nu(k,k+\ell) \star \tau^\ell(\nu(k,k+r))$.

Theorem 1.6. *The following assertions a)–c) are equivalent:*

a) $\{\lambda^n\}_{n\in\mathbb{Z}_+}$ is a non-dissipating random walk on $\mathbb{G}$, hence representable as $\lambda^k = \nu(k) \otimes \varepsilon_k$, with $\nu(k) = \bigstar_{j=0}^{k-1} \tau^j(\nu) \in \mathcal{M}^1(N)$, $\nu(0) = \varepsilon_e$, $\nu = \nu(1)$.

b) $\{\nu(k)\}_{k\in\mathbb{Z}_+}$ is a relatively compact discrete M-semigroup in $\mathcal{M}^1(N)$ (cf. Definition 1.5).

c) If $N = C_K(\tau)$, then $\rho := \lim_{k\to\infty} \nu(k) \star \omega_K$ exists and is $K-\tau$-decomposable (cf. Definition 1.5).

d) Conversely, if ρ is $K-\tau$-decomposable, the cofactors may be chosen as $\nu(k) = \bigstar_{j=0}^{k-1} \tau^j(\nu)$, $\nu = \nu(1) \in$ $\mathrm{Cof}_\rho(\tau)$ *and satisfying (1.1), hence form a M-semigroup (of cofactors) with $\rho = \lim \nu(k) \star \omega_K$.*

Therefore, $\{\nu(k)\}$ is relatively compact and hence the corresponding space-time *random walk $\{\lambda^k\}$ is non-dissipating.*

e) A M-semigroup $\{\nu(k)\}$ is relatively compact iff $\nu(1)$ possesses finite logarithmic moments resp. λ possesses finite first order moments.

f) $\left\{\nu(k,k+\ell) := \bigstar_{j=k}^{k+\ell-1} \tau^j(\nu(1)) = \tau^k(\nu(\ell))\right\}_{k,\ell\in\mathbb{Z}_+}$ is a discrete, relatively compact τ-hemigroup, and conversely, any discrete, relatively compact τ-hemigroup defines a relatively compact M-semigroup $\nu(\ell) := \nu(0,\ell)$, $\ell \in \mathbb{Z}_+$.

Proof. For $'a) \Leftrightarrow b)'$ see Facts 1.1 e).

'$b \Leftrightarrow c)$' Let $\{\nu(k)\}$ be a relatively compact M-semigroup on $N = C_K(\tau)$. Since τ^k is uniformly K-contracting on compact subsets, the accumulation points of $\left\{\tau^k(\nu(n))\right\}_{n\in\mathbb{Z}_+}$ are supported by K. Assume, for some subnets, $\nu(k_n) \to \alpha$ and $\nu(\ell_n) \to \beta$ and, w.l.o.g., $k_n \leq \ell_n$ for all n. Then $\nu(\ell_n) \star \omega_K = \nu(k_n) \star \tau^{k_n}(\nu(\ell_n - k_n)) \star \omega_K \to \beta \star \omega_K$, on the one hand, and $\lim \tau^{k_n}(\nu(\ell_n - k_n)) \star \omega_K = \omega_K$ on the other. Whence $\alpha = \beta$ follows.

I.e., $\rho = \lim \nu(k) \star \omega_K$ exists and is obviously right K-invariant. Conversely, if ρ exists, the M-semigroup is relatively compact.

Furthermore, $\rho = \lim \nu(k+1) \star \omega_K = \lim \nu(1) \star \tau(\nu(k)) \star \omega_K = \nu(1) \star \tau(\rho)$ yields $K-\tau$-decomposability of ρ.

Conversely, assume ρ to be $K-\tau$-decomposable. $\left\{\tau^k(\rho)\right\}$ is relatively compact as τ is K-contracting, and all accumulation points are supported by K. Right K-invariance implies $\tau^k(\rho) \to \omega_K$. Hence $\rho = \nu(k) \star \tau^k(\rho)$ yields $\nu(k) \star \omega_K \to \rho$ according to the shift compactness theorem ([33], III, theorems 2.1, 2.2 (for metrizable groups), [22], theorem 1.2.21).

d) By induction, if ρ is τ-decomposable, we can choose $\nu(k) = \mathop{\bigstar}\limits_{j=0}^{k-1} \tau^j(\nu), \nu = \nu(1) \in \mathrm{Cof}_\rho(\tau)$, hence as relatively compact M-semigroup. Therefore, according to Facts 1.1, a space-time random walk on $N \rtimes_\tau \mathbb{Z}$ is defined, which is non-dissipating, since $\{\nu(k)\}$ is relatively compact.

e) See Facts 1.3, Remark 1.4, or see [27], theorem 3.9.

f) Obviously, with $\nu = \nu(1)$, $\nu(k, k+\ell) := \mathop{\bigstar}\limits_{j=k}^{k+\ell-1} \tau^j(\nu) = \tau^k(\nu(\ell))$ a τ-hemigroup is defined. The converse follows along the same lines: $\nu(0, k+\ell) = \nu(0,k) \star \nu(k, k+\ell) = \nu(0,k) \star \tau^k(\nu(0,\ell))$. $\square$

Remarks 1.7. *a) The connection between τ-decomposability and existence of logarithmic moments in Theorem 1.6 e) is folklore for vector spaces, see e.g., for continuous time, the monograph* [29]. *For contractible Lie groups (homogeneous groups) cf.* [18], [21], *for general contractible groups* [17]. *For the general case, $N = C_K(\tau)$, see* [27]. *(For logarithmic moments see also the discussion before Theorem 2.8.)*

b) As in the continuous time case, Section 2, the interplay between τ-hemigroups and M-semigroups is well known. We listed up property f) in Theorem 1.6 for sake of completeness. For stable hemigroups (in the continuous time case) the reader is referred, e.g., to [2].

Note that in Theorem 1.6, if $N = C_K(\tau)$ and $K \neq \{e\}$, $\nu(k)$ will in general not be convergent. See e.g., the example 3.16 in [27], with compact $N = \mathbb{T}^2$ and an infinite number of accumulation points. A further type of examples is obtained in the following way:

Example 1.8. *Let M be a contractible locally compact group, with contracting $\sigma \in \mathrm{Aut}(M)$, hence $M = C(\sigma)$. Let $\{\mu(k)\}$ be a relatively compact M-semigroup, $\mu(k+\ell) = \mu(k) \star \sigma^k(\mu(\ell))$. As $\sigma^k \{\mu(\ell) : \ell \geq 0\} \stackrel{k\to\infty}{\longrightarrow} \{\varepsilon_e\}$, $\lim \mu(k) =: \rho_1$ exists. Let K be a finite cyclic group, let $x_0 \in K$ generating K with $\mathrm{ord}(x_0) > 2$, and assume for some $\xi \in \mathrm{Aut}(K)$ that $\xi(x_0) = x_0^{-1}$. (E.g. $\xi : x \mapsto x^{-1}$). Put $N := M \otimes K$, define $\tau \in \mathrm{Aut}(N)$ as $\tau = \sigma \otimes \xi$, and put finally $\nu = \nu(1) := \mu(1) \otimes \varepsilon_{x_0}$. Then $\left\{\nu(k) = \mu(k) \otimes \varepsilon_{y(k)}\right\}$ is a relatively compact M-semigroup (w.r.t. τ)*

in $\mathcal{M}^1(N)$, *where* $y(k) = \prod_{j=0}^{k-1} \xi^j(x_0)$. *But infinitely often* $y(k) = e$ *and* $y(k) = x_0$, *hence* $\nu(k)$ *is not convergent.*

Investigations of the structure of contractible and $K-$contractible groups had been pushed forward in connection with investigations of (semi-)stable laws. See e.g., [20], [19], [21], [6], and the literature mentioned there. However, there the concentration functions were not used as an essential tool (only in connection with random time substitutions and *geometric* (semi-)stability, cf. [16], [21]). Nevertheless it is worth to point out that semistable laws provide interesting examples of relatively compact M-semigroups, hence of non-dissipating random walks:

Let $\{\rho_t\}$ be a continuous convolution semigroup in $\mathcal{M}^1(N)$, N a locally compact group. Let $\tau \in \mathrm{Aut}(N)$ and $0 < c < 1$. $\{\rho_t\}$ is (τ, c)-semistable if for all $t \geq 0$, $\tau(\rho_t) = \rho_{t \cdot c}$. The idempotent $\rho_0 = \omega_H$ is a normalized Haar measure on a compact τ-invariant subgroup H. If N is second countable, the contraction subgroups $C(\tau)$ and $C_H(\tau)$ are Borel sets, and we have: $\rho_t(C_H(\tau)) = 1$ for all t. Hence we assume $N = (C_H(\tau))^-$. If $C(\tau)$ is closed, then $C_H(\tau) = C(\tau) \rtimes H$ is closed, hence $H = K$ and $N = C_K(\tau)$. If N is a Lie group, or if $C(\tau)$ is closed, ρ_t may be identified with a H-invariant semistable continuous convolution semigroup on $C(\tau)$ with idempotent $\rho_0 = \varepsilon_e$. Cf. e.g., [21], proposition 3.4.4, theorem 3.4.5. ff.

$C(\tau)$ is known to be closed if there exist contracting continuous one-parameter groups of automorphisms ([19]), moreover, for p-adic Lie groups ([42]), and more generally, for totally disconnected groups if τ is a *tidy* automorphism ([10]). If $C(\tau)$ is not closed and w.l.o.g. $N = (C_H(\tau))^-$, then τ is *weakly contracting* mod$H$ on N, and hence $N = C_K(\tau)$ for some compact, τ-invariant subgroup $K \supseteq H$ (cf.[26], theorem 5).

Example 1.9. *Let now* $\rho := \rho_1$. *Then* $\rho = \rho_{1-c^n} \star \rho_{c^n} = \rho_{1-c^n} \star \tau^n(\rho)$. *Hence* ρ *is* $H-\tau$*-decomposable with cofactors* $\nu(n) := \rho_{1-c^n} \in \mathrm{Cof}_\rho(\tau^n)$. *In particular,* ρ_{1-c^n} *(and hence all)* ρ_t *possess finite logarithmic moments. (This could also be proved by direct calculation). In that example,* $\lim \nu(n) = \rho$ *exists. (And thus trivially also* $\lim \nu(n) \star \omega_K = \rho \star \omega_K$*).*

As mentioned, it is well known that left and right concentration functions, hence (right) concentration functions of $\{\lambda^k\}$ and $\{\widetilde{\lambda}^k\}$ may differ in characteristic manner. Already mentioned in [4] e.g. for Lie groups. Here we discuss an example of totally disconnected groups (cf. [39, 40], [21], 3.1.9, 3.1.10):

Example 1.10. *Let* N *be totally disconnected and* $\tau \in \mathrm{Aut}(N)$, *let* $(U_n)_{n \in \mathbb{Z}}$ *be a filtration with compact open subgroups, i.e.,* $U_n \supseteq U_{n+1}$, $U_{n+1} = \tau(U_n)$, $\bigcup U_n = N$, $\bigcap U_n = \{e\}$. *Obviously we have* $N = C(\tau)$, *and on the other hand,* $C(\tau^{-1}) = \{e\}$ *(since e.g. for all* $x \neq e$ *we have* $\tau^{-n}(x) \notin U_0$ *for sufficiently large* n*). Furthermore,* $\{e\}$ *is the only* τ*-invariant compact subgroup.*

Let $\mathbb{G} = N \rtimes_\tau \mathbb{Z}$, *put* $\nu := \omega_{U_0}$, $\lambda = \nu \otimes \varepsilon_1$. *Obviously* λ *is adapted. We observe* $\nu(k) = \bigstar_{j=0}^{k-1} \omega_{\tau^j(U_0)} = \omega_{U_0}$, *hence* $\{\lambda^k\}$ *is non-dissipating. On the other hand,* $\widetilde{\lambda} = \omega_{\tau^{-1}(U_0)} \otimes \varepsilon_{-1}$, *hence* $\widetilde{\lambda}^k =: \mu(k) \otimes \varepsilon_{-k}$ *with*

$\mu(k) = \omega_{U_{-k}}$. One can easily show that $\{\mu(k)\}$ is not relatively compact, hence $\left\{\widetilde{\lambda}^k\right\}$ is dissipating.

This can also be proved in the following way: Assume that $\left\{\widetilde{\lambda}^k\right\}$ is non-dissipating. Then $\mu(k)(C_K(\tau^{-1})) = 1$ for all $k \geq 0$, for some compact τ-invariant subgroup K. But $K = \{e\}$ and $C(\tau^{-1}) = \{e\}$, as mentioned above, hence $C_K(\tau^{-1}) = C(\tau^{-1}) \cdot K = \{e\}$. Thus $\widetilde{\lambda} = \varepsilon_e \otimes \varepsilon_{-1}$, and therefore $\lambda = \varepsilon_e \otimes \varepsilon_1$, a contradiction.

The following result will explain more detailed the interplay between limit behaviour of relatively compact M-semigroups and τ-decomposability.

Proposition 1.11. *Let $\{\nu(k)\}$ be a relatively compact M-semigroup. To avoid measurability problems, N is supposed to be second countable. Put $\mathcal{A} := \mathrm{LIM}\{\nu(k) : k \to \infty\}$ and $\mathcal{S} := \mathrm{LIM}\{\tau^n(\nu(k_n)) : n \to \infty, (k_n) \subseteq \mathbb{Z}_+\}$. (*LIM *denoting the set of accumulation points.)*

Then we have:

a) $\mathcal{A} \subseteq \rho \star \mathcal{S}$

b) $\tau(\mathcal{S}) = \mathcal{S}$

c) If $\{\nu(k)\} \subseteq \mathcal{M}$, a commutative $\star$-sub-semigroup of $\mathcal{M}^1(N)$, then any $\rho \in \mathcal{A}$ is τ-decomposable with $\nu(k)\star\alpha_k \in \mathrm{Cof}_\rho(\tau^k)$ for some $\alpha_k \in \mathcal{S}$, $k \in \mathbb{Z}_+$.

d) If for some compact τ-invariant subgroup $H \subseteq K$, $\nu(k)\,(C_H(\tau)) = 1$ and $\nu(k) \star \omega_H = \nu(k)$ for all k, then $\mathcal{S} = \{\omega_H\}$, hence $\mathcal{A} = \rho$ and $\lim \nu(k) = \rho$. Moreover, ρ is τ-decomposabel with $\nu(k) \in \mathrm{Cof}_\rho(\tau^k)$. Note that these conditions are satisfied in the semi-stable case, cf. Example 1.9. (It is not supposed that $C_h(\tau)$ is closed.)

Proof. a) Let $\rho, \sigma \in \mathcal{A}$, $\nu(k_n) \to \rho$ and $\nu(\ell_n) \to \sigma$. Assume w.l.o.g. $\ell_n \geq k_n$ for all n (else pass to a subsequence). Then $\nu(\ell_n) = \nu(k_n) \star \tau^{k_n}(\nu(\ell_n - k_n)) \to \rho\star\alpha$, $\alpha \in \mathcal{S}$. Hence $\sigma = \rho\star\alpha \in \rho\star\mathcal{S}$, and analogously, $\rho \in \sigma \star \mathcal{S}$ follows.

b) Assume $\tau^{\ell_n}(\nu(m_n)) \to \gamma \in \mathcal{S}$. Them $\tau(\gamma) = \lim \tau^{\ell_n+1}(\nu(m_n))$, hence $\tau(\gamma) \in \mathcal{S}$, and analogously $\tau^{-1}(\gamma) \in \mathcal{S}$ follows.

c) If $\{\nu(k)\} \subseteq \mathcal{M}$ then $\mathcal{A} \subseteq \mathcal{M}^-$, a closed commutative sub-semigroup. Let $\rho \in \mathcal{A}$, $\nu(k_n) \to \rho$. Then $\nu(k_n - 1) \to \rho \star \alpha$ (along a sub-sequence) for some $\alpha \in \mathcal{S}$. Hence $\nu(k_n) = \nu(1) \star \tau(\rho) \star \tau(\alpha) = (\nu(1) \star \tau(\alpha)) \star \tau(\rho)$. I.e., $\nu(1) \star \tau(\alpha) \in \mathrm{Cof}_\rho(\tau)$.

d) τ is contracting mod H on $C_H(\tau)$ and $\nu(k)\,(C_H(\tau)) = 1$ for all k, hence $\alpha(C_H(\tau)) = 1$ for $\alpha \in \mathcal{S}$. Furthermore, $\alpha \star \omega_H = \alpha$ yields $\mathcal{S} = \{\omega_H\}$. Obviously, ρ is right H-invariant, therefore $\rho \star \mathcal{S} = \{\rho\}$, i.e., $\lim \nu(k) = \rho$. □

2. Continuous time: Non-dissipating continuous convolution semigroups

Next we replace the random walk by a continuous convolution semigroup $\{\lambda_t\}_{t\in\mathbb{R}_+}$ (the distribution of a Lévy process on $\mathbb{G}$, if $\mathbb{G}$ is metrizable). W.l.o.g. we assume that $\mathbb{G}$ is generated by $\{\mathrm{supp}(\lambda_t) : t \geq 0\}$. For short, $\{\lambda_t\}$ is called adapted then. Note that this does not imply that a single λ_t is adapted. $\{\lambda_t\}_{t\in\mathbb{R}_+}$ is non-dissipating if the concentration functions $f_{\lambda_t}(C) := \sup_{x\in\mathbb{G}} \lambda_t(Cx)$ do not converge to 0 for some compact C (for $t \to \infty$). For any $t_0 > 0$ the random walk $\left\{\lambda_{t_0}^k = \lambda_{t_0 k}\right\}$ (called *skeleton random walk*) is a non-dissipating random walk as Proposition 2.1 below shows.

First we compare the behaviour of concentration functions of continuous convolution semigroups and their skeleton random walks:

Proposition 2.1. *Let $\{\lambda_t\} \subseteq \mathcal{M}^1(\mathbb{G})$ be a continuous convolution semigroup. Then the following assertions are equivalent:*
(i) $\{\lambda_t\}$ is non-dissipating
(ii) For all [(ii') for some] $t_0 > 0$ the skeleton $\left\{\lambda_{t_0}^k\right\}$ is non-dissipating
(iii) For all [(iii') for some] $t_0 > 0$ the skeleton $\left\{\lambda_{t_0}^k\right\}$ is relatively shift compact
(iv) For all [(iv') for some] $t_0 > 0$ $\left\{\lambda_{t_0}^k * \widetilde{\lambda}_{t_0}^k\right\}$ is relatively compact
(v) $\{\lambda_t\}$ is relatively shift compact
(vi) $\left\{\lambda_t * \widetilde{\lambda}_t\right\}$ is relatively compact
(vii) For all [(vii') for some] $t_0 > 0$ λ_{t_0} has finite first moments
(viii) For all [(viii') for some] $t_0 > 0$ $\nu(t_0)$ has finite logarithmic moments

Proof. Obviously, we have $(i) \Rightarrow (ii), (ii) \Rightarrow (ii'), (iii) \Rightarrow (iii'), (iv) \Rightarrow (iv'), (iii) \Leftrightarrow (iv), (iii') \Leftrightarrow (iv'), (v) \Leftrightarrow (vi)$, furthermore, $(v) \Rightarrow (iii)$. $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and $(ii') \Leftrightarrow (iii') \Leftrightarrow (iv')$ follow by the results mentioned in Facts 1.1 a).

$'(iii') \Rightarrow (v)'$ Let $x_k \in \mathbb{G}$ such that $\left\{\lambda_{t_0 k} * \varepsilon_{x_k^{-1}}\right\}$ is relatively compact. Then, $\{\lambda_t\}$ being a continuous convolution semigroup, the set $\{\lambda_r : 0 \le r \le t_0\} * \left\{\lambda_{t_0 k} * \varepsilon_{x_k^{-1}} : k \in \mathbb{Z}_+\right\} = \{\lambda_t : t \ge 0\}$ is relatively compact. Whence (v) follows.

$'(v) \Rightarrow (i)'$ $\left\{\lambda_t * \varepsilon_{x(t)^{-1}}\right\}$ is relatively compact. Hence for any $\epsilon > 0$ there exists a compact C such that for all $t \in \mathbb{R}_+$, $\lambda_t(Cx(t)^{-1}) \ge \epsilon$. Therefore, $f_{\lambda_t}(C)$ fails to converge to 0.

$'(ii') \Leftrightarrow (vii') \Leftrightarrow (viii')'$ and $'(ii) \Leftrightarrow (vii) \Leftrightarrow (viii)'$ follow by Theorem 1.6 □

As mentioned before, a skeleton random walk need not be adapted on $\mathbb{G}$, hence we introduce for $t_0 > 0$ the subgroups $\mathbb{G}_{t_0}$ as smallest closed subgroup containing suppλ_{t_0}. Hence λ_{t_0} is adapted on $\mathbb{G}_{t_0}$. Furthermore, a priori $N_{t_0} := N_{\lambda_{t_0}}$, the smallest closed normal subgroup of $\mathbb{G}_{t_0}$ containing supp$\lambda_{t_0} * \widetilde{\lambda}_{t_0}$, might not be normal in $\mathbb{G}$ and might depend on t_0. In order to apply the results of Section 1 we have to overcome these difficulties.

Proposition 2.2. *We have $N_{t_0} = N_1 =: N$ for all $t_0 > 0$ and $N \lhd \mathbb{G}$.*

Proof. Obviously, for all $t > 0$, $\mathbb{G}_t = \bigcup_{k\in\mathbb{Z}} N_t x_t^k$ for some (all) $x_t \in$ suppλ_t. First we consider dyadic numbers t, and then proceed to real t applying continuity of $\{\lambda_t\}$.

Let $t = 1$. Then obviously $\mathbb{G}_1 \subseteq \mathbb{G}_{1/2}$ since $\lambda_1 = \lambda_{1/2}^2$. For all $x, x_{1/2} \in \text{supp}\lambda_{1/2}$ it follows $xx_{1/2}, x_{1/2}^2 \in \text{supp}\lambda_1 \subseteq \mathbb{G}_1$, hence

$$xx_{1/2}N_1 = N_1xx_{1/2} \text{ and } xx_{1/2}N_1 = x_{1/2}^2N_1 = N_1x_{1/2}^2 = N_1xx_{1/2} \tag{2.1}$$

$\llbracket$ In fact, $\text{supp}\lambda_1 \subseteq x_{1/2}^2N_1 \Rightarrow xx_{1/2} = x_{1/2}^2v = wx_{1/2}^2$ for some $v, w \in N_1$, whence (2.1) follows. $\rrbracket$

Therefore,

$$x^{-1}x_{1/2}^2N_1 = x^{-1}xx_{1/2}N_1 = x_{1/2}N_1 \quad \text{and} \tag{2.2}$$

$$N_1x_{1/2}^2x^{-1} = N_1x_{1/2}xx^{-1} = N_1x_{1/2} \tag{2.3}$$

Claim: $\lambda_{1/2}(x_{1/2}N_1) = \lambda_{1/2}(N_1x_{1/2}) = 1$ and hence – with $N_1^* := N_1 \cap x_{1/2}N_1x_{1/2}^{-1}$ – it follows

$$\lambda_{1/2}(N_1^*x_{1/2}) = \lambda_{1/2}(x_{1/2}N_1^*) = 1 \text{ and } N_1^*x_{1/2} = x_{1/2}N_1^* \tag{2.4}$$

$\llbracket$ Proof of the claim: By (2.1) and (2.2) we have

$$\begin{aligned} 1 &= \lambda_1(N_1x_{1/2}^2) = \int \lambda_{1/2}(x^{-1}N_1x_{1/2}^2)\mathrm{d}\lambda_{1/2}(x) \\ &\overset{(2.1)}{=} \int \lambda_{1/2}(x^{-1}x_{1/2}^2N_1)\mathrm{d}\lambda_{1/2}(x) \overset{(2.2)}{=} \int \lambda_{1/2}(x_{1/2}N_1)\mathrm{d}\lambda_{1/2}(x) \\ &= \lambda_{1/2}(N_1x_{1/2}) = \lambda_{1/2}(x_{1/2}N_1) \end{aligned}$$

Analogously, $1 = \int \lambda_{1/2}(N_1x_{1/2}^2x^{-1})\mathrm{d}\lambda_{1/2}(x) = \lambda_{1/2}(N_1x_{1/2})$. Whence the first assertion follows.

Hence we also have $\lambda_{1/2}(N_1^*x_{1/2}) = \lambda_{1/2}(x_{1/2}N_1^*) = 1$. Since $x_{1/2}^2 \in \mathbb{G}_1$, $N_1 \triangleleft \mathbb{G}_1$, it follows $x_{1/2}N_1^*x_{1/2}^{-1} = x_{1/2}N_1x_{1/2}^{-1} \cap x_{1/2}(x_{1/2}N_1x_{1/2}^{-1})x_{1/2}^{-1} = x_{1/2}N_1x_{1/2}^{-1} \cap x_{1/2}^2N_1x_{1/2}^{-2} \overset{(2.1)}{=} N_1^*$. Hence also the second assertion of (2.4) follows. $\rrbracket$

Consequently, we have $\mathbb{G}_{1/2} \subseteq \bigcup_{k\in\mathbb{Z}} x_{1/2}^kN_1^* = \bigcup_{k\in\mathbb{Z}} N_1^*x_{1/2}^k$ (as $\text{supp}\lambda_{1/2} \subseteq x_{1/2}N_1^* = N_1^*x_{1/2}$), according to (2.4).

Claim: $N_1^* = zN_1^*z^{-1}$ for all $z \in \mathbb{G}_{1/2}$. $\llbracket$ Let $z \in \mathbb{G}_{1/2}$. Then there exist $v, w \in N_1^*$, $k \in \mathbb{Z}$, such that $z = x_{1/2}^kv = wx_{1/2}^k$. Hence $z^{-1}N_1^*z = x_{1/2}^{-k}w^{-1}N_1^*wx_{1/2}^k = x_{1/2}^{-k}N_1^*x_{1/2}^k \overset{(2.4)}{=} N_1^*$. $\rrbracket$

Claim: $N_{1/2} \subseteq N_1^*$. $\llbracket$ $N_1^* \cap N_{1/2} =: N^*$ is a closed normal subgroup of $\mathbb{G}_{1/2}$ such that $1 = \lambda_{1/2}(x_{1/2}N^*) = \lambda_{1/2}(N^*x_{1/2})$. But $N_{1/2}$ is minimal with this property. Whence $N_{1/2} \subseteq N^* \subseteq N_1^*$. $\rrbracket$

Claim: $N_1^* = N^* = N_{1/2} = N_1$. $\llbracket$ $N_{1/2} \triangleleft \mathbb{G}_{1/2}$ yields $N_{1/2} \triangleleft \mathbb{G}_1$ (since $N_1^* \subseteq \mathbb{G}_1 \subseteq \mathbb{G}_{1/2}$). And $\lambda_1(x_{1/2}^2N_{1/2}) = \int \lambda_{1/2}(x^{-1}x_{1/2}^2N_{1/2})\mathrm{d}\lambda_{1/2}(x) = \lambda_{1/2}(x_{1/2}N_{1/2}) = \lambda_{1/2}(N_{1/2}x_{1/2}) = 1$. In fact, for $x \in \text{supp}\lambda_{1/2}$, $x = x_{1/2}w = vx_{1/2}$, with $w, v \in N_{1/2}$ we have $x^{-1}x_{1/2}^2N_{1/2} = w^{-1}x_{1/2}N_{1/2} = w^{-1}N_{1/2}x_{1/2} = N_{1/2}x_{1/2} = x_{1/2}N_{1/2}$. Hence, according to the definition, $N_1 \subseteq N_{1/2}$ follows. Together we obtain $N_1 \subseteq N_{1/2} \subseteq N_1^* \subseteq N_1$, whence the assertion follows. $\rrbracket$

By induction, we obtain for all $n \in \mathbb{Z}_+$: $N := N_1 = N_{1/2^n}$ and hence $N = N_t$ for all $t \in D_+ = \{k/2^n : k, n \in \mathbb{Z}_+\}$. Furthermore, $n \mapsto \mathbb{G}_{1/2^n}$ is increasing. Finally, continuity of $t \mapsto \lambda_t$ shows that $N = N_t$ for all $t \in \mathbb{R}_+ \backslash \{0\}$. And thus $\mathbb{G} = \left(\bigcup_{n,k \in \mathbb{Z}_+} x^k_{1/2^n} N \right)^- = \bigcup_{t \in \mathbb{R}_+} x_t N$ for suitable $x_t \in \mathbb{G}$. $\square$

Proposition 2.3. *Let $\mathbb{G}$ be non-compact and let $\{\lambda_t\}$ be non-dissipating. Then, with the afore introduced notations, there exists a continuous one-parameter group $\{x(t)\}_{t \in \mathbb{R}} \subseteq \mathbb{G}$ such that $\mathbb{G}/N \cong \mathbb{R}$ and* $\mathrm{supp}\lambda_t \subseteq x(t)N,\ t \geq 0$.

Let, for $t \in \mathbb{R}$, $\tau_t := i_{x(t)}|_N$ denote the restriction of the inner automorphism to N. Then $T = (\tau_t)$ is a continuous one-parameter group in $\mathrm{Aut}(N)$. *And we have: $\mathbb{G} \cong N \rtimes_T \mathbb{R}$ (with group operation $(g,t)(h,s) = (g\tau_t(h), t+s),\ g,h \in N,\ t,s \in \mathbb{R}$). Furthermore, λ_t may be represented as $\lambda_t = \nu(t) \otimes \varepsilon_t$, where $t \mapsto \nu(t) \in \mathcal{M}^1(N)$ is a continuous M-semigroup (w.r.t. T) (cf. Definition 2.6 below).*

Proof. As shown before, there exist $x_t \in \mathbb{G}$ such that $\mathrm{supp}\lambda_t \subseteq x_t N$ (for all $t \in \mathbb{R}_+$). Hence, $\pi : \mathbb{G} \to \mathbb{G}/N$ denoting the canonical homomorphism, we obtain $\pi(\lambda_t) = \varepsilon_{z(t)}$ with $z(t) = x_t N$. Hence $\{\varepsilon_{z(t)}\}_{t \in \mathbb{R}_+}$ and therefore $\{z(t)\}_{t \in \mathbb{R}_+}$ are continuous one-parameter semigroups, extendible to groups, and thus $\mathbb{G}/N \cong \mathbb{R}$. Finally there exists a continuous one-parameter group $\{x(t)\}_{t \in \mathbb{R}}$ in $\mathbb{G}$ with $\pi(x(t)) = z(t)$, whence the assertion follows. (Cf. e.g. [30]).

To show that $\mathbb{G}$ splits as a semi-direct product, assume $N \cap S =: L$ to be non-trivial, where $S := \{x(t) : t \in \mathbb{R}\}$. The subgroup $N \subseteq \mathbb{G}$ is invariant under the inner automorphisms $i_{x(t)}$, $L \subseteq N$ is a subgroup and $i_{x(t)}$ acts trivially on S. Hence $L = S$, thus $S \subseteq N$. Furthermore, any $\tau_{t_1} = i_{x(t_1)}$ is K-contracting, whence $S \subseteq K$; in particular, $S = \{x(t)\}$ is relatively compact.

Therefore, $\{\lambda_t\}$ is relatively compact since $\{\lambda_t * \varepsilon_{x(t)^{-1}}\}$ is. But then $\mathbb{G}$ must be compact, a contradiction to the assumption. (In fact, by [31], theorem 2, applied to a skeleton, $\mathbb{G}$ must be compact. The metrizability condition there is easily seen to be superfluous, since $\mathbb{G}$ can be approximated by metrizable gropups.) $\square$

Again, to avoid trivialities, throughout in the sequel $\mathbb{G}$ is assumed to be non-compact. If N is compact, hence $N = K$, any continuous convolution semigroup $\{\lambda_t = \nu(t) \otimes \varepsilon_t\}$ is non-dissipating.

In the discrete time case it was essentially used that $N \cong C_K(\tau)$. This results follows immediately in the continuous time case, if we consider the skeletons $\{\lambda^k_{t_0}\}$. However, in the continuous time-case the structure of N is even nicer: Put again, $C(T) = \left\{ x \in N :\ \tau_t(x) \overset{t\to\infty}{\longrightarrow} e \right\}$ and $C_K(T) = \left\{ x \in N :\ \tau_t \overset{t\to\infty}{\longrightarrow} e \mod K \right\}$ for some compact, T-invariant subgroup K.

Proposition 2.4. *a) With the notations above we have $N \cong C_K(T)$*

In fact, for any locally compact group N admitting a continuous one parameter group $T = (\tau_t) \subseteq$ $\mathrm{Aut}(N)$ *and a τ-invariant compact subgroup K we have:*

b) For all $t > 0$, $C_K(T) = C_K(\tau_t)$, in particular, $C(T) = C(\tau_t)$.

c) $C(T)$ is closed, T-invariant, contractible, connected and isomorphic to a contractible Lie group (hence a homogeneous group), $C_K(T)$ is a closed T-invariant subgroup of N. If N is compact then $C_K(T) = K$.

d) $C(T) \lhd C_K(T)$ and there exists continuous homomorphism $\beta : \mathrm{Aut}(K) \to \mathrm{Aut}(C(T))$, $\beta(\kappa)(g) := \kappa^{-1} g \kappa$, $\kappa \in K, g \in C(T)$, such that $C_K(T) = C(T) \rtimes_\beta K$

e) The restrictions $S := T\,\big|_{C(T)} = (\sigma_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(C(T))$ and $\Gamma := T\,|_K = (\gamma_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(K)$ are continuous one-parameter groups, Γ consisting of inner automorphisms of K.

f) $T = (\tau_t = (\sigma_t, \gamma_t))$ satisfies the following consistency conditions

$(\sigma_t \circ \beta(\kappa)) = (\beta(\gamma_t(\kappa)) \circ \sigma_t)$, $\kappa \in K$, $t \in \mathbb{R}$.

Conversely, any group T arises in that way.

Proof. In fact, a) follows applying the results of Section 1, Facts 1.1 d) for any skeleton, and by b)–f).

b) See [39], [21], Lemma 3.2.6.

c), d) Cf. [21], Theorem 3.2.32, [20], e) [19], Lemma 3.3, [21] and f) [21], 3.3.4 [20]. □

Remark 2.5. *As mentioned before (Facts 1.1 d)), also in the discrete time case, we have $C_K(\tau) = C(\tau)K$, at lest in the case of Lie- or totally disconnected groups. However, $C(\tau)$ will in general not be closed. Not even on a 2-dimensional torus. (Cf. e.g., [19], [21], Example 3.12.5.) As mentioned, $C(\tau)$ is closed and hence $C_K(\tau)$ splits semi-directly if $\mathbb{G}$ is a p-adic Lie group ([42], [34], [8]), more generally, a totally disconnected group, and if τ is a* tidy *automorphism ([1], [10], [26]).*

Now we define in analogy to the discrete time case:

Definition 2.6. *a) $\{\nu(t)\}_{t\in\mathbb{R}_+} \subseteq \mathcal{M}^1(N)$ is a (continuous time) M-semigroup (w.r.t. $T = (\tau_t)$) – also called skew semigroup, T-semigroup, or distribution of a generalized Ornstein Uhlenbeck process – if $t \mapsto \nu(t)$ is continuous and if the following cocycle equation is satisfied:*

$$\nu(t+s) = \nu(t) \star \tau_t(\nu(s)) \quad \textit{for all } t, s \in \mathbb{R}_+ \tag{2.5}$$

In the following we are interested in relatively compact M-semigroups.

b) $\rho \in \mathcal{M}^1(N)$ is T-decomposable or T-self-decomposable, if for all $t \in \mathbb{R}_+$, $\rho = \nu(t) \star \tau_t(\rho)$ with cofactors $\nu(t)$. $\mathrm{Cof}_\rho(\tau_t)$ denotes the set of cofactors. $\rho \in \mathcal{M}^1(N)$ is $K - T$-decomposable if in addition ρ is right K-invariant.

c) A 2-parameter family $\{\nu(t, t+s)\}_{t,s\geq 0} \subseteq \mathcal{M}^1(N)$ is called T-stable hemigroup if $(t, s) \mapsto \nu(t, t+s)$ is continuous, for all $r, s, t \in \mathbb{R}_+$ $\nu(t, t+s+r) = \nu(t, t+s) \star \nu(t+s, t+s+r)$ and $\tau_t(\nu(s, s+r)) = \nu(s+t, s+r+t)$.

Theorem 2.7. *a) $\{\nu(s)\}_{s\in\mathbb{R}_+}$ is a continuous M-semigroup in $\mathcal{M}^1(N)$ iff $\{\nu(t, t+s) := \tau_t(\nu(s))\}_{t,s\in\mathbb{R}_+}$ is a T-stable hemigroup.*

b) $\{\nu(s)\}_{s\in\mathbb{R}_+}$ is a continuous M-semigroup in $\mathcal{M}^1(N)$ iff $\{\lambda_t := \nu(t) \otimes \varepsilon_t\}_{t\in\mathbb{R}_+}$ is a continuous convolution semigroup in $\mathcal{M}^1(\mathbb{G})$, $\mathbb{G} \cong N \rtimes_T \mathbb{R}$.

$\{\nu(s)\}_{s\in\mathbb{R}_+}$ in a), b) is relatively compact iff $\{\lambda_s\}_{s\in\mathbb{R}_+}$ is non-dissipating.

c) Let $\{\nu(s)\}_{s\in\mathbb{R}_+}$ be relatively compact and $N = C_K(T)$. Then $\rho := \lim_{t\to\infty} \nu(t)\star\omega_K$ exists and is $K-T$-decomposable, with cofactors $\nu(t) \in \mathrm{Cof}_\rho(\tau_t)$, $t\in\mathbb{R}_+$.

d) Conversely, let $\rho\in\mathcal{M}^1(N)$ be $K-T$-decomposable. Then there exists a non-dissipating continuous convolution semigroup $\{\lambda_t\}\subseteq\mathcal{M}^1(\mathbb{G})$, with $\lambda_t = \nu(t)\otimes\varepsilon_t$, and hence a relatively compact continuous M-semigroup $\{\nu(t)\}$ in $\mathcal{M}^1(N)$ with $\rho := \lim_{t\to\infty}\nu(t)\star\omega_K$

d) $\{\nu(s)\}$ is relatively compact iff for all (some) $t_0 > 0$ λ_{t_0} has finite first order moments, equivalently, iff $\nu(t_0)$ has finite logarithmic moments.

Proof. a), b) are proved as in the discrete time case, Theorem 1.6 f). (For aperiodic groups see [14], proposition A, [13], theorem 3.16.)

c) Assume $\nu(t_n)\to\alpha, \nu(s_n)\to\beta$ and w.l.o.g. $t_n\geq s_n$. Then $\nu(t_n)\star\omega_K = \nu(s_n)\star\tau_{s_n}(\nu(t_n-s_n))\star\omega_K \to \beta\star\omega_K$, since τ_{s_n} is compactly contracting $\mod K$. Hence $\alpha\star\omega_K = \beta\star\omega_K$ for all accumulation points, whence the assertion follows.

d) To prove the converse, i.e., the existence of a *continuous* solution $\{\nu(t)\}$, we need some tools from the *embedding problem* for infinitely divisible laws on groups. (For aperiodic groups, in particular simply connected nilpotent Lie group, see [13], proposition 3.6, see also [14], theorem B.)

Let again $D_+ := \{k/2^n : k,n\in\mathbb{Z}_+\}$ denote the dyadic numbers. For all $n\in\mathbb{Z}_+$ choose $\nu_{1,n}\in\mathrm{Cof}_\rho(\tau_{1/2^n})$. As immediately seen, for $\mu_i\in\mathrm{Cof}_\rho(a_i)$, $i=1,2$, we have $\mu_1\star a_1(\mu_2)\in\mathrm{Cof}_\rho(a_1a_2)$. Hence by induction, $\nu_{k,n} := \bigstar_{j=0}^{k-1}\tau_{j/2^n}(\nu_{1,n})\in\mathrm{Cof}_\rho(\tau_{k/2^n})$. Put $\lambda^{(n)}_{1/2^n} := \nu_{1,n}\otimes\varepsilon_{1/2^n}$, and $\lambda^{(n)}_1 = \left(\lambda^{(n)}_{1/2^n}\right)^{2^n} = \left(\bigstar_{j=0}^{2^n-1}\tau_{j/2^n}(\nu_{1,n})\right)\otimes\varepsilon_1 = \nu_{2^n,n}\otimes\varepsilon_1$. In fact, for all $k\in\mathbb{Z}_+$ we have $\left(\lambda^{(n)}_{1/2^n}\right)^k = \nu_{k,n}\otimes\varepsilon_{k/2^n}$ and $\nu_{k,n}\in\mathrm{Cof}_\rho(\tau_{k/2^n})$. Indeed, as immediately verified, $\left\{\sigma : \sigma^k = \lambda = \mu\otimes\varepsilon_k\right\} = \left\{\sigma = \nu\otimes\varepsilon_1 :\ \nu\in\mathrm{Cof}_\lambda(\tau_{1/k})\right\}$.

For all fixed n, $\{\nu_{k,n}\}_{k\geq 0}$ satisfies the discrete cocycle (or M-semigroup) equation: Put $\nu^{(n)}(k/2^n) := \nu_{k,n}$ then for $t = k/2^n, s = \ell/2^n$ we have $\nu^{(n)}(t+s) = \nu^{(n)}(t)\star\tau_t(\nu^{(n)}(s))$. Hence $\left\{\lambda^{(n)}_1\right\}_{n\in\mathbb{Z}_+}$ is relatively compact with $\left\{\nu^{(n)}_1\right\}_{n\in\mathbb{N}}\subseteq\mathrm{Cof}_\rho(\tau_1)$ and has relatively compact 2^m-th root sets $\left\{\lambda^{(n)}_{1/2^m}\right\}_{n\in\mathbb{N}}$ with $\left\{\nu^{(n)}(1/2^m)\right\}\subseteq\mathrm{Cof}_\rho(\tau_{1/2^m})$. According to Tychonov's theorem, we can choose a convergent subnet (n') such that for all $m\in\mathbb{N}$, $\lim_{(n')}\left(\lambda^{(n)}_{1/2^{m\cdot n}}\right)^n =: \lambda_{1/2^m} = \nu(1/2^m)\otimes\varepsilon_{1/2^m}$ exists, and by construction, $\left(\lambda_{1/2^{mk}}\right)^k = \lambda_{1/2^m}$ for all m,k. Thus, $D_+\ni t = k/2^n\mapsto\lambda_t := \lambda_{k/2^n}$ is a convolution semigroup with parameter set D_+.

We have to show that a continuous version may be selected. Put $\mathcal{R}_0 := \{\lambda_t : t\in[0,1]\cap\mathbb{D}_+\}^-$.

Claim: $\mathcal{R}_0$ is compact in $\mathcal{M}^1(\mathbb{G})$ with all accumulation points of the form $\nu(t)\otimes\varepsilon_t$. $\Big[\!\Big[$ Let $t_n\in[0,1]\cap\mathbb{D}_+, t_n\to t\in[0,1]$ along a subnet, hence $\tau_{t_n}(\rho) \to \tau_t(\rho)$. Furthermore, $\rho = \nu(t_n)\star\tau_{t_n}(\rho)$ yields that $\{\nu(t_n)\}$ is relatively compact with all accumulation points $\nu(t) \in \mathrm{Cof}_\rho(\tau_t)$. $\rrbracket$

Let Φ_0 denote the set of accumulation points of $\{\lambda_t \in \mathcal{R}_0 : t \to 0\}$. Therefore, $\sigma \in \Phi_0$ is representable as $\sigma = u \otimes \varepsilon_0$ for $u \in \mathrm{Cof}_\rho(\tau_0)$, i.e., $u \star \rho = \rho$.

A straight forward generalization of E. Siebert's embedding theorem ([36], 5. Satz 1, 6. Satz 1, [22], Theorem 3.5.4) shows the existence of a continuous convolution semigroup $\{\lambda'_t = \nu'(t) \otimes \varepsilon_t\}_{t\geq 0}$ with $\lambda'_t = \sigma_t\star\lambda_t$, $\sigma_t = u_t \otimes \varepsilon_t \in \Phi_0$: In fact, one of the essential steps in the proof is an extension of a semigroup homomorphism from $\mathbb{Q}_+$ to $\mathbb{R}_+$ ([36], 4. Satz 1, [22], Theorem 3.5.1). And $\mathbb{Q}_+$ may be replaced by any sub-monogeneous semigroup, hence e.g., by D_+. This is shown in the diploma thesis [32]. (For a sketch of a proof cf. [12], Lemma 2.4).

Therefore $\left\{\lambda'_t\right\}$ is a continuous convolution semigroup, and thus $\{\nu'(t)\}$ is a continuous M-semigroup of cofactors of ρ, as asserted. $\square$

To obtain in the continuous time case necessary and sufficient conditions for the existence of logarithmic moments in analogy to [27], proposition 2.14, corollary 2.15, we have to recall some notations and facts: Let $\mathbb{H}$ be a locally compact group. A Borel function $f : \mathbb{H} \to \mathbb{R}_+$ is sub-additive (resp. a 'jauge') if $f(xy) \leq f(x)+f(y)$ (resp. $\leq C+f(x)+f(y)$ for some $C \geq 0$). f is sub-multiplicative if $f(xy) \leq f(x)f(y)$. As immediately seen, if f is sub-additive (and ≥ 0) then $g := 1 + f$ is sub-multiplicative and ≥ 1 and if h is sub-multiplicative and ≥ 1 then $\log(h)$ is sub-additive and ≥ 0. Thus $\log(1 + f)$ is sub-additive and $1 + \log(1 + f)$ is sub-multiplicative.

We shall always suppose that for some $\epsilon > 0$ the set $\{f \leq \epsilon\}$ is a neighbourhood of e.

Let $\mathbb{H}$ be compactly generated, $\mathbb{H} = \bigcup_{n\geq 0} V^n$ for some symmetric compact $V \subseteq \mathfrak{U}(e)$. Then we define a sub-additive function $\delta_V : x \mapsto \inf\{n \in \mathbb{Z}_+ : x \in V^n\}$. According to [11], proposition 1, in that case any jauge, in particular any sub-additive function f is dominated by δ_V, i.e.,

$$f \leq A\delta_V + B \quad \text{for some constants } A > 0, B \geq 0 \tag{2.6}$$

If $\{\lambda_t\}$ is a continuous convolution semigroup, the Lévy measure η is defined by $\int g\mathrm{d}\eta = \frac{\mathrm{d}_+}{\mathrm{d}t}|_{t=0} \int g\mathrm{d}\lambda_t$, for $g \in C^b(H)$ vanishing in a neighbourhood of the unit. If as before, $\mathbb{H} = \mathbb{G} = \mathbb{N} \rtimes \mathbb{R}$ and $\lambda_t = \nu(t) \otimes \varepsilon_t$, then, as easily seen, η is supported by N and $\int_N g\mathrm{d}\eta = \frac{\mathrm{d}_+}{\mathrm{d}t}|_{t=0} \int_N g\mathrm{d}\nu(t)$. Recall that a Lévy measure is bounded outside any neighbourhood of e. According to a well-known Result of E. Siebert, [37], theorem 1, [38], theorem 5, for a sub-multiplicative function g such that for some $\epsilon > 0$, $\{g \leq \epsilon\} \in \mathfrak{U}(e)$, we have:

$$\int g\mathrm{d}\lambda_t < \infty \;\forall t \geq 0 \text{ iff } \int_{\mathbb{G}\setminus U} g\mathrm{d}\eta < \infty,\; U \in \mathfrak{U}(e) \tag{2.7}$$

Hence, for a sub-additive function f we obtain $\int 1 + f\mathrm{d}\lambda_t < \infty$, equivalently $\int f\mathrm{d}\lambda_t < \infty$ iff $\int_{\mathbb{G}\setminus U} 1 + f\mathrm{d}\eta < \infty$, equivalently, $\int_{\mathbb{G}\setminus U} f\mathrm{d}\eta < \infty$.

In [27], definition 2.6, particular sub-additive functions $\varphi : N \to \mathbb{R}_+$, called *regular τ-functions* are defined (depending on τ and K) and it is shown ([27], proposition 2.14) that for symmetric $U \in \mathfrak{U}(e)$ (in N)

and $V_0 := U \otimes \{\pm 1, 0\} \subseteq \mathbb{G}_1 = N \rtimes \mathbb{Z}$ the restriction of δ_{V_0} to N and $\log(1+\varphi)$ are equivalent, i.e.,

$$\delta_{V_0}(\cdot, 0) \leq A \log(1+\varphi(\cdot) + B \text{ and } \log(1+\varphi(\cdot) \geq C\delta_{V_0}(\cdot, 0) \quad (2.8)$$

Put $V := U \otimes [-1, 1] \subseteq \mathbb{G}$, then we observe: $\bigcup V_0^n = \mathbb{G}_1$ and $\bigcup V^n = \mathbb{G}$. Furthermore, $V_0 \subseteq V$, hence $\delta_V(x, t) \leq \delta_{V_0}(x, t)$ for $(x, t) \in \mathbb{G}_1$, i.e., for $t \in \mathbb{Z}$. In particular, for $t = 0$, $\delta_V|_N \leq \delta_{V_0}|_N$.

Note that $\delta_V|_N(x) = \delta_V(x, 0) = \inf\{n : \exists y_i \in N, t_i \in [-1, 1], t_0 = 0,$ such that $\prod_0^{n-1} \tau_{t_i}(y_i) = x, \sum_1^n t_i = 0\}$

Theorem 2.8. *The following assertions are equivalent:*

(a) $\{\lambda_t\}$ *is non-dissipating*

(b) For some (any) $t_0 > 0$, λ_{t_0} *has finite first moments*

(c) η *has finite first moments*

(d) $\int_{\mathbb{G}} \delta_V \mathrm{d}\lambda_{t_0} < \infty$

(e) $\int_{\mathbb{G}\setminus W} \delta_V \mathrm{d}\eta < \infty$ *for some (any)* $W \in \mathfrak{U}(e)$

(f) For some (any) $t_0 > 0$ $\nu(t_0)$ *has finite logarithmic moments*

(g) η *(considered as measure on* N*) has finite logarithmic moments*

(h) For some (any) $t_0 > 0$ $\int_N \delta_V(x, 0)\mathrm{d}\nu(t_0)(x) < \infty$

(i) $\int_{N\setminus W'} \delta_V(x, 0)\mathrm{d}\eta(x) < \infty$ *for some (any) neighbourhood* W' *of the unit in* N.

Proof. $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (f)$ follow by Theorem 2.7, $(b) \Leftrightarrow (c)$ by the above mentioned Result of E. Siebert, cf.(2.7). $(c) \Rightarrow (d)$ is obvious and $(d) \Rightarrow (c)$ follows since any sub-additive function (on $\mathbb{G}$) is dominated by δ_V, as mentioned above, cf.(2.6). $(d) \Leftrightarrow (e)$ again by Siebert's result, as afore.

$(d) \Leftrightarrow (f) \Leftrightarrow (h)$ and $(e) \Leftrightarrow (g) \Leftrightarrow (i)$: Let φ be a regular τ_1-function. In particular, sub-additive. Then, as easily verified, $\psi : \mathbb{G} \to \mathbb{R}_+$, $(x, t) \mapsto 1 + \varphi(x) + ||\tau_t||$ is sub-multiplicative, where $||\tau_t|| := \sup\{\varphi(\tau_t(x))/\varphi(x) : x \notin K\}$. (For $N = C(T)$ see e.g., [21], proposition 2.14.28.) Hence, as mentioned before, $\log(1+\psi)$ is dominated by δ_V; moreover, $\delta_{V_0}|_N$ is dominated by $\log(1+\varphi) = \log(1+\psi)|_N$. Furthermore $\delta_V|_N \leq \delta_{V_0}|_N$. Finally, arguing as before, we obtain for any sub-additive function $f : N \to \mathbb{R}_+$ that $\log(1+f)$ is dominated by $\delta_V|_N$ and hence by $\log(1+\varphi)$. Whence the assertions immediately follow. □

We close with two examples in analogy with the case of discrete times. The first shows that $\lim \nu(t)$ need not to exist if $N = C_K(T)$, $K \neq \{e\}$. (For $K = \{e\}$, i.e., $N = C(T)$, $\lim \nu(t)$ exists, cf. [14], proposition A.)

Example 2.9. *Let* M *be a contractible, hence simply connected nilpotent Lie group with contracting one-parameter group* $S = (\sigma_t)_{t>0}$ *in* $\mathrm{Aut}(M)$, *let* K *be a solenoidal compact group with dense one-parameter subgroup* $(x(t))_{t\in\mathbb{R}}$. *Put* $N := M \otimes K$ *and define* $T := (\tau_t) \subseteq \mathrm{Aut}(N)$ *by* $\tau_t = \sigma_t \otimes id$, $t \in \mathbb{R}$. *Hence* $N = C_K(T)$. *Let* $\{\mu(t)\}$ *be a continuous M-semigroup (w.r.t. S) in* $\mathcal{M}^1(M)$ *such that* $\rho = \lim \mu(t)$ *exists. As* $\xi_t := \tau_t|_K = id$, *M-semigroups in* $\mathcal{M}^1(K)$ *(w.r.t.* (ξ_t)*) are just continuous convolution semigroups, in particular,* $\{\varepsilon_{x(t)}\}$ *is a M-semigroup. It is immediately verified that* $\{\nu(t) := \mu(t) \otimes \varepsilon_{x(t)}\}$ *is a relatively compact M-semigroup in* $\mathcal{M}^1(N)$ *and* $\{\lambda_t := \nu(t) \otimes \varepsilon_t\}$ *is* *a non-dissipating continuous convolution semigroup in* $\mathcal{M}^1(N \rtimes_T \mathbb{R})$. *As* $(x(t))$ *is dense in* K, *the set of accumulation points of* $\nu(t), t \to \infty$, *consists of* $\{\rho \otimes \varepsilon_\kappa : \kappa \in K\}$. *In particular,* $\lim \nu(t)$ *does not exist.*

In the next example we point out the connections between stable laws and non-dissipating continuous convolution semigroups.

Example 2.10. *Stable laws are particular self-decomposable laws. To show this we have to switch between additive and multiplicative parametrizations of continuous one-parameter groups:*

Let N *be a simply connected nilpotent Lie group and* $S = (\sigma_t) \subseteq \mathrm{Aut}(N)$ *be a contracting continuous group with multiplicative parametrization,* $\sigma_t\sigma_s = \sigma_{t\cdot s}$ *for* $t, s > 0$, *and* $\lim_{t\to 0} \sigma_t(x) = e$ *for all* $x \in N$. *A continuous convolution semigroup* $\{\rho_t\}$ *in* $\mathcal{M}^1(N)$ *is called* S*-stable if* $\sigma_t(\rho_1) = \rho_t$, *for all* $t > 0$, *equivalently,* $\sigma_t(\rho_s) = \rho_{t\cdot s}$ *for* $t > 0, s \geq 0$. *Put* $\rho := \rho_1$. *We have* $\rho = \rho_{1-s} \star \sigma_s(\rho)$, *hence* ρ *is* σ_s*-decomposable with cofactors* $\rho_{1-s} \in \mathrm{Cof}_\rho(\sigma_s)$, *for all* $0 < s \leq 1$.

To obtain a continuous M-semigroup of cofactors we have to switch to additive parametrization: $T := (\tau_t := \sigma_{\mathrm{e}^{-t}})_{t\in\mathbb{R}}$ *is a continuous one-parameter group satisfying* $\tau_{t+s} = \tau_t\tau_s$ *and* $\lim_{t\to\infty} \tau_t(x) = e$ *for all* $x \in N$. *And with this notations we obtain* $\rho = \rho_{1-\mathrm{e}^{-t}} \star \tau_t(\rho)$ *for* $t \geq 0$.

As immediately verified, $\{\nu(t) := \rho_{1-\mathrm{e}^{-t}}\}_{t\in\mathbb{R}_+}$ *is a relatively compact continuous M-semigroup with* $\nu(t) \in \mathrm{Cof}_\rho(\tau_t)$ *and* $\rho = \lim \nu(t)$, *hence defines a non-dissipating continuous convolution semigroup in* $\mathcal{M}^1(\mathbb{G})$, $\mathbb{G} = N \rtimes_T \mathbb{R}$. *Note that in this (trivial) case,* $\nu(t) = \rho_{1-e^{-t}} = \sigma_{1-e^{-t}}(\rho) = \tau_{\log(e^{-t}-1)}(\rho)$.

In analogous way, stable laws on $N = C_K(T)$ *with idempotent* ω_K *could be treated.*

Of course, also Proposition 1.11 has a counterpart for continuous time M-semigroups. However, less instructive, since $H = K$ and $C_H(T)$ is closed and $= C_K(T) = N$.

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