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GENERALIZED DUALITY FOR k-FORMS

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Abstract. We give the definition of a duality that is applicable to arbitrary k-forms. The operator that defines the duality depends on a fixed form Ω. Our definition extends in a very natural way the Hodge duality of n-forms in $2n$ dimensional spaces and the generalized duality of two-forms. We discuss the properties of the duality in the case where Ω is invariant with respect to a subalgebra of $\mathfrak{so}(V)$. Furthermore, we give examples for the invariant case as well as for the case of discrete symmetry.

1. Introduction: Self duality and Ω-duality

Given a Riemannian or semi-Riemannian space (V, g) of dimension D, the metric g induces isomorphisms $* : \Lambda^k V \to \Lambda^{D-k} V$. This so called Hodge operator has the property $*^2 = \varepsilon \mathbb{1}$ where the sign ε depends on the dimension and the signature of the metric by $*^2\big|_{\Lambda^k V} = (-)^{t+k(D-k)}\mathbb{1}$. If the dimension of V is even, $D = 2n$, we have a particular automorphism $* : \Lambda^n V \to \Lambda^n V$. For $\varepsilon = 1$, i.e. $D \equiv 0 \mod 4$, we call the n-form F self dual and anti-self dual if it is an eigenform of $*$ to the eigenvalue 1 and -1, respectively, i.e.

$$* F = \pm F\,. \tag{1}$$

Duality relations are in particular interesting for two-forms. Consider a vector bundle E over the Riemannian base (M, g). The curvature tensor of a connection on E is a two-form on M with values in the endomorphism bundle of E. So for $\dim M = 4$ we may consider connections with (anti-)self dual curvature tensor. In the case of $E = TM$ (anti-)self duality is connected to complex structures on M, see [3].

In dimension four we may use the volume form $\mathrm{vol} = *1$ to rewrite (1) as

$$*\,(*\mathrm{vol} \wedge F) = \pm F\,. \tag{2}$$

This motivates the introduction of Ω-duality of two-forms in arbitrary dimension, see for example [1, 2, 4, 7] and [9]. It is defined as follows. Let Ω be a four-form on V and consider the symmetric operator $*_\Omega : \Lambda^2 V \to \Lambda^2 V$ with

$$*_\Omega F := *(*\Omega \wedge F)\,. \tag{3}$$

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Let us suppose that $*_\Omega$ admits real eigenvalues, then a two-form F is called (Ω,β)-dual if it obeys

$$*\,(*\Omega\wedge F)=\beta F\,, \tag{4}$$

see [1]. In local coordinates with $\Omega=\Omega_{ijkl}$ and $F=F_{ij}$ the left hand side of (4) is given by $*(*\Omega\wedge F)_{ij}=\frac{1}{2}\Omega_{ijkl}F^{kl}$.

Example 1.1. Consider the three-form θ in seven dimensions that is given by $\theta_{ijk}=1$ for $(ijk)=(123)$, (435), (471), (516), (572), (624), (673). We associate to this the four-form $\bar\theta:=*\theta$ in seven and the four-form $\Theta:=\bar\theta+\theta\wedge e_8$ in eight dimensions. The latter is self-dual, i.e. $\Theta=*_8\Theta$.[1] The forms above are strongly related to the discussion of $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$. In particular, the duality relations yield the decompositions of the adjoint representations of $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$ into irreducible representations of $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$, respectively:

- $*_{\bar\theta}:\Lambda^2\mathbb{R}^7\to\Lambda^2\mathbb{R}^7$ has eigenvalues 1 and -2 and the eigenspace decomposition corresponds to the decomposition of $\Lambda^2\mathbb{R}^7$ with respect to $\mathfrak{g}_2$. In particular $E(1,*_{\bar\theta})=\mathbf{14}$ is the adjoint representation and $E(-2,*_{\bar\theta})=\mathbf{7}$ is the vector representation of $\mathfrak{g}_2$.
- $*_\Theta:\Lambda^2\mathbb{R}^8\to\Lambda^2\mathbb{R}^8$ has eigenvalues 1 and -3. The eigenspace decomposition corresponds to the decomposition of $\Lambda^2\mathbb{R}^8$ with respect to $\mathfrak{spin}(7)$. In particular $E(1,*\Theta)=\mathbf{21}$ is the adjoint representation and $E(-3,*_\Theta)=\mathbf{7}$ is the vector representation of $\mathfrak{spin}(7)$.

Example 1.2. Consider the globally defined parallel four-form $\Omega:=\omega_1\wedge\omega_1+\omega_2\wedge\omega_2+\omega_3\wedge\omega_3$ on the quaternionic-Kähler manifold $(M,\omega_1,\omega_2,\omega_3)$.

Then the operator $*_\Omega$ on $\Lambda^2TM\otimes\mathbb{C}$ has eigenvalues 1, $-\frac{1}{3}$ and $-\frac{2m+1}{3}$ corresponding to the eigenspaces $E(1,*_\Omega)=S^2Y\otimes\sigma_Z$, $E(-\frac{1}{3},*_\Omega)=\Lambda^2_0Y\otimes S^2Z$ and $E(-\frac{2m+1}{3},*_\Omega)=\sigma_Y\otimes S^2Z$. Here $TM\otimes\mathbb{C}=Y\otimes Z$ is the (local) decomposition of the complexified tangent bundle into a rank-$2m$ and a rank-2 bundle with respect to the $Sp(m)Sp(1)$-structure, and $g=\sigma_Y\otimes\sigma_Z$ the corresponding decomposition of the complexified Riemannian metric on M. The 1-eigenspace is connected to half-flatness introduced by the authors in [1, 2].

Remark 1.3. Equation (4) can be generalized further in an straight forward way. Let Ω be a $2k$-Form and F be a k-Form. Then $*_\Omega(F):=*(*\Omega\wedge F)$ is also a k form and the question whether or not $*_\Omega$ has real eigenvalues is reasonable. Such operators is discussed in [5] and examples are given in [10] for $k=4,6$ in dimension ten.

2. Duality of k-forms

All examples in the previous section have in common that the ℓ-form Ω yields a duality relation on the space $\Lambda^{\frac{\ell}{2}}V$ only. It would be preferable to give for one fixed Ω a duality relation on each Λ^kV. As we will see in Lemma 2.5 this is possible at least up to some mild restrictions.

[1]The invariant four-orms are explicitly given by $\bar\theta_{ijkl}=1$ for $(ijkl)=(1245)$, (1276), (1346), (1357), (2356), (2437), (4567) and $\Theta_{ijkl}=1$ for $(ijkl)=(1245)$, (1276), (1346), (1357), (2356), (2437), (4567), (1238), (4358), (4718), (5168), (5728), (6248), (6738).

Definition 2.1. Let $\Omega \in \Lambda^\ell V$ be an ℓ-form on $V = \mathbb{R}^D$. The *duality operator* b_Ω is defined by

$$b_\Omega : \Lambda^k V \to \Lambda^k V\,, \quad F \mapsto \pi_k(\Omega \otimes F)\,. \tag{5}$$

Here $\pi_k : \Lambda^\ell V \otimes \Lambda^k V \to \Lambda^k V$ denotes the projection in the decomposition of $\Lambda^\ell V \otimes \Lambda^k V$ with respect to $\mathfrak{so}(V)$, see (49).[2] We call the duality operator b_Ω OF ORDER N if it admits N distinct eigenvalues.

Example 2.2. In [10] the authors discuss two operators on three- and two-forms in dimension $D = 10$. These two are covered by the special case $\ell = 2k$ in Definition 2.1.

Example 2.3. A very basic example is the following. Let Ω be a complex structure on $V = \mathbb{R}^{2n}$ interpreted as two-form, i.e. $\Omega_{ij}\Omega^j{}_k = -\delta_{ik}$. Then b_Ω on $\Lambda^1 V$ has eigenvalues $\pm i$ with eigenspaces $\Lambda^1_{(i)}V = \Lambda^{1,0}V$ and $\Lambda^1_{(-i)}V = \Lambda^{0,1}V$. The eigenvalues of b_Ω on $\Lambda^k V$ for $k \leq n$ are then given by $\frac{k-2q}{k} i$ for $q = 0, \dots, k$ with eigenspaces $\Lambda^k_{\left(\frac{k-2q}{k} i\right)}V = \Lambda^{k-q}(\Lambda^{1,0}V) \otimes \Lambda^q(\Lambda^{0,1}V) = \Lambda^{k-q,q}V$. For $k > n$ we refer the reader to Remark 2.12. For instance, the Hodge dual to $\Lambda^{n-k}_{\left(\frac{n-k-2q}{n-k} i\right)}V = \Lambda^{n-k-q,q}V$ is $\Lambda^{n+k}_{\left(\frac{n-k-2q}{n+k} i\right)}V = \Lambda^{n-q,k+q}V$.

Remark 2.4. The preceding example can be generalized. For $\Omega \in \Lambda^2 V$ the action of b_Ω on $\Lambda^k V$ is just a the action of $\frac{1}{k}\Omega \in \mathfrak{so}(V)$ on $\Lambda^k V$.

Lemma 2.5. *Let $\Omega \in \Lambda^\ell V$. Then $b_\Omega \neq 0$ only if ℓ is even and $\ell \leq 2k$.*

Proof. Consider the $\mathfrak{so}(V)$-decomposition as given in (49). Then we have

$$\Lambda^k V \subset \Lambda^\ell V \otimes \Lambda^k V = \bigoplus_{i=0}^{\min\{k,\ell\}} \bigoplus_{j=0}^{i-1} [\![k+\ell-i-j, i-j]\!]_0 \oplus \bigoplus_{i=0}^{\min\{k,\ell\}} \Lambda^{k+\ell-2i} V \tag{6}$$

if and only if $2i = \ell$ for some $i \in \{0, \dots, \min\{k,\ell\}\}$. This is ℓ even and $\frac{\ell}{2} \leq \min\{k,\ell\}$ or $\ell \leq 2k$. □

Remark 2.6. Because of the restriction given in Lemma 2.5 it would be preferable, that ℓ is not too big. Therefore, the case $\ell = 4$ is of particular interest. The main examples which have been cited so far are connected to this value.

Example 2.7. The examples from section 1, the Hodge duality and the Ω-dualities, are operators of the form b_Ω. They are of order two (Hodge-duality and Example 1.1) or order three (Example 1.2) with $\ell = 2k$.

Lemma 2.8. *In local coordinates be may write $\Omega = (\Omega_{i_1 \dots i_{2m}})$. Then b_Ω and b^2_Ω are given by*

$$(b_\Omega)^{d_1 \dots d_k}{}_{i_1 \dots i_k} = \Omega^{[d_1 \dots d_m}{}_{[i_1 \dots i_m} \delta^{d_{m+1} \dots d_k]}_{i_{m+1} \dots i_k]} \tag{7}$$

and

$$(b^2_\Omega)^{d_1 \dots d_k}{}_{i_1 \dots i_k} = \delta^{b_1 \dots b_m [d_{m+1} \dots d_k}_{j_1 \dots j_m [i_{m+1} \dots i_k} \Omega^{|j_1 \dots j_m|}{}_{i_1 \dots i_m]} \Omega^{d_1 \dots d_m]}{}_{b_1 \dots b_m} \tag{8}$$

respectively. In particular b_Ω is trace free.

[2] To simplify the formulas we feel free to consider the projection up to a constant factor. This leads to the fact, that for example $b_\Omega = 2*_\Omega$ for a four-form Ω acting on $\Lambda^2 V$, compare Example 1.1 and Lemma 2.8.

Proof. Consider $\Omega = (\Omega_{i_1\dots i_{2m}})$ and $F = (F_{i_1\dots i_k})$. Then the definition of $b_\Omega(F)$ as projection on $\Lambda^k V$ in (6) yields

$$\begin{aligned}(b_\Omega(F))_{i_1\dots i_k} &= \Omega_{j_1\dots j_m[i_1\dots i_m} F^{j_1\dots j_m}{}_{i_{m+1}\dots i_k]}\\ &= \delta^{a_1\dots a_k}_{i_1\dots i_k}\delta^{d_1\dots d_m d_{m+1}\dots d_k}_{j_1\dots j_m a_{m+1}\dots a_k}\Omega^{j_1\dots j_m}{}_{a_1\dots a_m}F_{d_1\dots d_k}\\ &= \delta^{[d_{m+1}\dots d_k}_{[i_{m+1}\dots i_k}\Omega^{d_1\dots d_m]}{}_{i_1\dots i_m]}F_{d_1\dots d_k}\end{aligned}$$

The square of b_Ω obeys

$$\begin{aligned}(b^2_\Omega(F))_{i_1\dots i_k} &= \delta^{a_1\dots a_k}_{i_1\dots i_k}\Omega^{j_1\dots j_m}{}_{a_1\dots a_m}(b_\Omega(F))_{j_1\dots j_m a_{m+1}\dots a_k}\\ &= \delta^{a_1\dots a_k}_{i_1\dots i_k}\delta^{b_1\dots b_m b_{m+1}\dots b_k}_{j_1\dots j_m a_{m+1}\dots a_k}\Omega^{j_1\dots j_m}{}_{a_1\dots a_m}\Omega_{c_1\dots c_m b_1\dots b_m}\cdot\\ &\quad\cdot F^{c_1\dots c_m}{}_{b_{m+1}\dots b_k}\\ &= \delta^{a_1\dots a_m a_{m+1}\dots a_k}_{i_1\dots i_m i_{m+1}\dots i_k}\delta^{d_1\dots d_m d_{m+1}\dots d_k}_{c_1\dots c_m b_{m+1}\dots b_k}\delta^{b_1\dots b_m b_{m+1}\dots b_k}_{j_1\dots j_m a_{m+1}\dots a_k}\cdot\\ &\quad\cdot\Omega^{j_1\dots j_m}{}_{a_1\dots a_m}\Omega^{c_1\dots c_m}{}_{b_1\dots b_m}F_{d_1\dots d_k}\\ &= \delta^{b_1\dots b_m d_{m+1}\dots d_k}_{j_1\dots j_m[i_{m+1}\dots i_k}\Omega^{j_1\dots j_m}{}_{i_1\dots i_m]}\Omega^{d_1\dots d_m}{}_{b_1\dots b_m}F_{d_1\dots d_k}\end{aligned}$$

If we use (7) we see, that the trace of b_Ω is given by

$$\operatorname{tr}(b_\Omega) = \Omega^{i_1\dots i_m}{}_{[i_1\dots i_m}\delta^{i_{m+1}\dots i_k}_{i_{m+1}\dots i_k]} \propto \Omega^{i_1\dots i_m}{}_{i_1\dots i_m} = 0\,.$$

□

Remark 2.9. Let Ω be an ℓ-Form with $\ell = 2m$. From (7) we see that the linear operator b_Ω is skew symmetric if m is odd and that it is symmetric if m is even. In particular, b_Ω is diagonalizable with purely imaginary eigenvalues if m is odd and real eigenvalues if m is even.

If b_Ω is of order N with different eigenvalues $\beta_1, \dots, \beta_N$, then b_Ω solves its minimal polynomial $\lambda^N - (\beta_1 + \dots + \beta_N)\lambda^{N-1} + \dots + (-)^N\beta_1\cdots\beta_N = 0$.

Because b^2_Ω is symmetric, it is contained in $S^2(\Lambda^k V)$. So the right hand side of (8) is an element in $\Lambda^\ell V \otimes \Lambda^\ell V$ that is embedded in $S^2(\Lambda^k V)$ via some δ-tensor. If b_Ω is of order two with eigenvalues $\beta_1 \neq -\beta_2$ then b_Ω has to be symmetric, too. This is enough to state the following result on duality operators of order two.

Proposition 2.10. *Let Ω be an ℓ-form on V. The operator b_Ω is of order two with two eigenvalues $\beta_1 \neq -\beta_2$ only if $\Lambda^\ell V \subset S^2(\Lambda^k V)$. In particular $\ell \equiv 0 \mod 4$.*

Moreover, the projections on the two respective eigenspaces are given by

$$\begin{aligned}\pi_{\beta_1} &= \frac{\beta_2}{\beta_2 - \beta_1}\big(\mathbb{1} - \frac{1}{\beta_2}b_\Omega\big)\\ \pi_{\beta_2} &= \frac{\beta_1}{\beta_1 - \beta_2}\big(\mathbb{1} - \frac{1}{\beta_1}b_\Omega\big)\end{aligned} \tag{9}$$

Remark 2.11. The restriction to ℓ in Proposition 2.10 is a consequence of the symmetry of b_Ω or, equivalently, of (50). This is not a contradiction to example 2.7 where the Hodge duality operator is of degree 2 but $\ell = \dim V$ may be equal to 2 mod 4, because in this case we have $\beta_1 = -\beta_2 = 1$.

We emphasize on the following compatibility of the duality operator with the Hodge operator.

Remark 2.12.

- Consider V to be of dimension D and let $\Omega \in \Lambda^{2m}V$ such that b_Ω is defined on $\Lambda^k V$ as well as on $\Lambda^{D-k}V$. Then we have

$$\binom{D-k}{m} * \, b_\Omega * = (-1)^{k(D-k)} \binom{k}{m} b_\Omega \tag{10}$$

where the action is on $\Lambda^k V$.

In particular, if $F \in \Lambda^k V$ is an eigenform of b_Ω to the eigenvalue β, then $*F \in \Lambda^{D-k}V$ is an eigenform of b_Ω to the eigenvalue $\beta' = \frac{\binom{k}{m}}{\binom{D-k}{m}}\beta$, i.e. $\Lambda^k_{(\beta)} \approx \Lambda^{n-k}_{(\beta')}$ via $*$.

- If we consider V of dimension $4m$ and $\Omega \in \Lambda^{2m}V$ then for all $F \in \Lambda^{2m}V$ we have

$$* \, b_\Omega(F) = b_{*\Omega}(F) = b_\Omega(*F)\,. \tag{11}$$

In the case that Ω is either self-dual or anti self-dual, i.e. $*\Omega = \pm\Omega$ we have $*b_\Omega(F) = \pm b_\Omega(F)$. Therefore, for $F \in \Lambda^{2m}_{(\beta)}$ we have $*F = \pm F$ or $\beta = 0$, i.e. $(\Lambda^{2m}V)^{\mp} \subset \Lambda^{2m}_{(0)}$. We will recall this fact in Proposition 3.5.

3. Invariant duality operators

3.1. **General properties of invariant duality operators.** Let $b_\Omega : \Lambda^k V \to \Lambda^k V$ be a duality operator of order N associated to $\Omega \in \Lambda^\ell V$. Consider Ω to be invariant with respect to a subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)$. Then b_Ω is invariant under $\mathfrak{h}$ as well. If $\Lambda^k V = W_1 \oplus \ldots \oplus W_r$ is the decomposition into irreducible representation spaces with respect to $\mathfrak{h}$, then

$$b_\Omega\big|_{W_\alpha} = \beta_\alpha \mathbb{1}$$

for some number β_α due to Schur's Lemma, i.e. $W_\alpha \subset \Lambda^k_{(\beta_\alpha)}$. In particular, b_Ω is diagonalizable with (not necessarily distinct) eigenvalues $\beta_1, \ldots, \beta_r$. Because b_Ω is trace free, we have in this special situation

$$\sum_{\alpha=1}^{r} \beta_\alpha \dim(W_\alpha) = 0\,. \tag{12}$$

Definition 3.1. Let $\Omega \in \Lambda^\ell V$ be invariant under a subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)$. Then $b_\Omega : \Lambda^k V \to \Lambda^k V$ is called PERFECT if it is of order r where r is the number of irreducible submodules of $\Lambda^k V$.

If $\Omega \in \Lambda^\ell V$ is invariant with respect to a subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)$, then this is the same as to say that it spans a singlet within the decomposition of the $\mathfrak{so}(V)$-representation $\Lambda^\ell V$ into irreducible $\mathfrak{h}$-representations.

As noticed before the case $\ell = 4$ is of particular interest. On the one hand due to the Ω-duality of two-forms as in (4), on the other hand due to the restriction cf. Lemma 2.5. An $\mathfrak{h}$-invariant four-form may be constructed via an $\mathfrak{h}$-invariant metric as the $\Lambda^4 V$-part of $S^2\mathfrak{h} \subset S^2(\Lambda^2 V)$. This is in particular possible in the cases where $\mathfrak{h}$ is a holonomy-algebra, see [2]. The four-forms from the examples in section 1, that deal with $\mathfrak{spin}(7)$, $\mathfrak{g}_2$, and $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$, are of this type. How they occur as a singlet and that they are unique up to a multiple can be seen as follows.

For instance, the four-form $\bar\theta \in \Lambda^4\mathbb{R}^7$, or it Hodge-dual $\theta \in \Lambda^3\mathbb{R}^7$, considered in Example 1.1 is the singlet in the $\mathfrak{g}_2$-decomposition $\Lambda^4\mathbb{R}^8 \simeq \Lambda^3\mathbb{R}^7 = \mathbf{27}\oplus\mathbf{7}\oplus\mathbf{1}$. The

same is true for the four-form $\Theta \in \Lambda^4\mathbb{R}^8$ also from Example 1.1. It represents the singlet in the $\mathfrak{spin}(7)$-decomposition $\Lambda^4\mathbb{R}^8 = \mathbf{35} \oplus \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{1}$. Moreover the four-form from example 1.2 represents the singlet in the $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$-decomposition of $\Lambda^4 V$ for $V = \mathbb{C}^{4n}$. Let us recall the splitting $V = E \otimes H$ with $H = \mathbb{C}^2$ and $H = \mathbb{C}^{2n}$, then it can be located in the following way. The splitting yields $\Lambda^2 V = \Lambda^2(H \otimes E) = (\mathbf{1} \otimes S^2 E) \oplus (S^2 H \otimes \Lambda^2_* H) \oplus (S^2 E \otimes \mathbf{1})$. Then the singlet in $\Lambda^4 V$ coincides with singlet in $\mathbf{1} \otimes \mathbf{1} \subset (\mathbf{1} \otimes S^2 E) \otimes (\mathbf{1} \otimes S^2 E) \subset \Lambda^2 V \otimes \Lambda^2 V$ coming from the trace in $S^2 E \otimes S^2 E$. In particular, these three examples yield perfect duality operators on the space of two forms.

A list and the explicit construction of invariant four-forms in dimension $D \leq 8$ for subgroups of $\mathfrak{so}(D)$ is given in [8]. In particular, the authors give a four-form depending on three real parameters, that yield the decomposition of $\Lambda^2\mathbb{R}^8$ for $\mathfrak{h} = \mathfrak{u}(4) = \mathfrak{su}(4) \oplus \mathfrak{u}(1)$, and $\mathfrak{spin}(7)$, as well as $\mathfrak{su}(4)$. The decomposition for $\mathfrak{u}(4)$ is not perfect, whereas the remaining two are.

The authors in [10] discuss the Ω-duality in dimension $D = 10$ in the generalized sense cf. Remark 1.3. They construct a six-form and its associated Hodge-dual four-form invariant under $\mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{so}(8) \oplus \mathfrak{u}(1) \subset \mathfrak{so}(10)$. The corresponding eigenspace decompositions of $\Lambda^3\mathbb{R}^{10}$ and $\Lambda^2\mathbb{R}^{10}$ are not perfect in the sense of Definition 3.1.

3.2. **The $\mathfrak{spin}(7)$- and $\mathfrak{g}_2$-duality.** The first two examples of this section, i.e. Propositions 3.2 and 3.5 make use of the $\mathfrak{spin}(7)$-invariant four-form to give the eigenspace decomposition of $\Lambda^3\mathbb{R}^8$ and $\Lambda^4\mathbb{R}^8$. In particular, the duality-operator is perfect in both cases and therefore, the eigenspace decomposition coincides with the decomposition into irreducible representations. This extends the result from Example 1.1 to all forms on $\mathbb{R}^8$. Of course, these $\mathfrak{spin}(7)$-decompositions in terms of the invariant tensor Θ are not new, but very common in the literature, see e.g. [13, 17, 11] or [6].[3] Nevertheless, they yield nice examples how the known results fit in our duality framework.

Proposition 3.2. *Consider the $\mathfrak{spin}(7)$-invariant four-form Θ on $V = \mathbb{R}^8$. Then $b_\Theta : \Lambda^3 V \to \Lambda^3 V$ with $(b_\Theta)_{lmn}{}^{ijk} = \Theta_{[lm}{}^{[ij}\delta^{k]}_{n]}$ is a perfect duality operator of order two that obeys*

$$(b_\Omega)^2 = \frac{8}{3}\mathrm{id} - \frac{10}{3} b_\Omega.$$

The eigenvalues are -4 and $\frac{2}{3}$ corresponding to the eight-dimensional and 48-dimensional $\mathfrak{spin}(7)$-invariant subspaces of $\Lambda^3 V$.[4]

[3] For his description the author in [11] uses the concept of vector cross products, of which a nice discussion and classification is given in [12].

[4] **8** is the spin representation and the **48** is the spin-$\frac{3}{2}$ representation of $\mathfrak{so}(7)$. The latter is given by vector-spinors which obey $\gamma^\mu \psi_\mu = 0$.

Proof. The traces of the eight-tensor $\widetilde{\Theta} = \Theta \otimes \Theta$ have components in the skew-symmetric parts of $S^2(\Lambda^4\mathbb{R}^8))$ only. They are explicitely given by

$$\begin{aligned}
\Theta_{ijko}\Theta^{lmno} &= 6\delta^{lmn}_{ijk} - 9\Theta_{[ij}{}^{[lm}\,\delta^{n]}_{k]}\,,\\
\Theta_{ijmn}\Theta^{klmn} &= 12\delta^{kl}_{ij} - 4\Theta_{ij}{}^{kl}\,,\\
\Theta_{iklm}\Theta^{jklm} &= 42\delta^{j}_{i}\,,\\
\Theta_{ijkl}\Theta^{ijkl} &= 336\,.
\end{aligned} \tag{13}$$

This gives

$$\begin{aligned}
b^2_\Theta(F)_{i_1i_2i_3} &= \delta^{b_1b_2d_3}_{j_1j_2[i_3}\,\Theta^{j_1j_2}{}_{i_1i_2]}\,\Theta^{d_1d_2}{}_{b_1b_2}F_{d_1d_2d_3}\\
&= \tfrac{1}{3}\big(\delta^{b_1b_2}_{j_1j_2}\delta^{d_3}_{[i_3}\,\Theta^{j_1j_2}{}_{i_1i_2]}\,\Theta^{d_1d_2}{}_{b_1b_2}F_{d_1d_2d_3}\\
&\quad + 2\delta^{d_3b_1}_{j_1j_2}\delta^{b_2}_{[i_3}\,\Theta^{j_1j_2}{}_{i_1i_2]}\,\Theta^{d_1d_2}{}_{b_1b_2}F_{d_1d_2d_3}\big)\\
&= \tfrac{1}{3}\big(\Theta_{d_1d_2j_1j_2}\,\Theta^{j_1j_2}{}_{[i_1i_2}F_{i_3]d_1d_2} - 2\Theta_{j_2}{}^{d_1d_2m}\,\Theta_{j_2j_1[i_1i_2}g_{i_3]m}F_{d_1d_2}{}^{j_1}\big)\\
&= \tfrac{1}{3}\big(12\delta^{d_1d_2}_{[i_1i_2} - 4\Theta^{d_1d_2}{}_{[i_1i_2}\big)F_{i_3]}{}^{d_1d_2}\\
&\quad - \tfrac{2}{3}\big(6\delta^{d_1d_2m}_{j_1[i_1i_2} - 9\Theta^{[d_1d_2}{}_{b_1b_2}\,\delta^{m]}_{b_3}\delta^{b_1b_2b_3}_{j_1[i_1i_2}\big)g_{i_3]m}F_{d_1d_2}{}^{j_1}\\
&= 4F_{i_1i_2i_3} - \tfrac{4}{3}(b_\Theta(F))_{i_1i_2i_3} - 4\big(\tfrac{1}{3}\delta^m_{j_1}\delta^{d_1d_2}_{[i_1i_2}g_{i_3]m}F_{d_1d_2}{}^{j_1}\\
&\quad + \tfrac{2}{3}\delta^{d_1}_{j_1}\delta^{d_2m}_{[i_1i_2}g_{i_3]m}F_{d_1d_2}{}^{j_1}\big)\\
&\quad + 6\big(\tfrac{1}{3}\Theta^{d_1d_2}{}_{b_1b_2}\,\delta^m_{b_3} + \tfrac{2}{3}\Theta^{md_1}{}_{b_1b_2}\,\delta^{d_2}_{b_3}\big)\delta^{b_1b_2b_3}_{j_1[i_1i_2}g_{i_3]m}F_{d_1d_2}{}^{j_1}\\
&= 4F_{i_1i_2i_3} - \tfrac{4}{3}(b_\Theta(F))_{i_1i_2i_3} - \tfrac{4}{3}F_{i_1i_2i_3}\\
&\quad + 2\big(\tfrac{2}{3}\Theta^{d_1d_2}{}_{j_1[i_1}\,\delta^m_{i_2} + \tfrac{1}{3}\delta^m_{j_1}\,\Theta^{d_1d_2}{}_{i_1i_2}\big)g_{i_3]m}F_{d_1d_2}{}^{j_1}\\
&\quad + 4\big(\tfrac{2}{3}\Theta^{md_1}{}_{j_1[i_1}\,\delta^{d_2}_{i_2} + \tfrac{1}{3}\delta^{d_2}_{j_1}\,\Theta^{md_1}{}_{i_1i_2}\big)g_{i_3]m}F_{d_1d_2}{}^{j_1}\\
&= \tfrac{8}{3}F_{i_1i_2i_3} - \tfrac{4}{3}(b_\Theta(F))_{i_1i_2i_3} + \tfrac{2}{3}\Theta_{d_1d_2[i_1i_2}F^{d_1d_2}{}_{i_3]} - \tfrac{8}{3}\Theta^{d_1d_2}{}_{[i_1i_2}F_{i_3]d_1d_2}\\
&= \tfrac{8}{3}F_{i_1i_2i_3} - \tfrac{10}{3}(b_\Theta(F))_{i_1i_2i_3}\,.
\end{aligned}$$

The eigenvalues of b_Θ are the zeros of $\beta^2 + \frac{10}{3}\beta - \frac{8}{3}$ that are $\frac{2}{3}$ and -4. The eigenspaces are given by $\Lambda^3_{(\frac{2}{3})}V = \mathbf{48}$ and $\Lambda^3_{(-4)}V = \mathbf{8}$ due to $(-4).8 + \frac{2}{3}.48 = 0$. □

Lemma 3.3. *Let Θ and V as before and consider the duality map $b_\Theta : \Lambda^4V \to \Lambda^4V$ given by $b_\Theta(F)_{ijkl} = \Theta^{mn}{}_{[ij}F_{kl]mn}$. This operator obeys*[5]

$$b^2_\Theta(F)_{ijkl} = \tfrac{1}{6}\Theta^{mn}{}_{[ij}\Theta^{op}{}_{kl]}F_{mnop} + \tfrac{2}{3}F_{ijkl} - \tfrac{8}{3}b_\Theta(F)_{ijkl}\,, \tag{14}$$

$$b^3_\Theta(F)_{ijkl} = \tfrac{4}{3}F_{ijkl} + \tfrac{2}{3}b_\Theta(F)_{ijkl} - \tfrac{10}{3}b^2_\Theta(F)_{ijkl} + \tfrac{2}{9}\Theta_{[ijk}{}^p\Theta^{rsn}{}_{l]}F_{pnrs}\,, \tag{15}$$

$$b^4_\Theta(F)_{ijkl} = 4b_\Theta(F)_{ijkl} - \tfrac{8}{3}b^2_\Theta(F)_{ijkl} - \tfrac{13}{3}b^3_\Theta(F)_{ijkl} + \tfrac{1}{9}\Theta_{ijkl}\Theta^{prsn}F_{prsn}\,, \tag{16}$$

$$b^5_\Theta(F)_{ijkl} = -\tfrac{25}{3}b^4_\Theta(F)_{ijkl} - 20b^3_\Theta(F)_{ijkl} - \tfrac{20}{3}b^2_\Theta(F)_{ijkl} + 16b_\Theta(F)_{ijkl}\,. \tag{17}$$

Remark 3.4. Equation (17) yields, that b_Θ is a null of the polynomial

$$\beta^5 + \tfrac{25}{3}\beta^4 + 20\beta^3 + \tfrac{20}{3}\beta^2 - 16\beta = \beta(\beta+4)(\beta+3)(\beta+2)(\beta-\tfrac{2}{3})\,. \tag{18}$$

so that the possible eigenvalues are $\beta = 0, -2, -3, -4$, and $\frac{2}{3}$.

[5]We postpone the calculations to Appendix B.

Proposition 3.5. *Let Θ and V as before. The duality operator $b_\Theta : \Lambda^4 V \to \Lambda^4 V$ with $(b_\Theta)_{i_1i_2i_3i_4}{}^{j_1j_2j_3j_4} = \Theta_{[i_1i_2}{}^{[j_1j_2}\delta_{i_3i_4]}^{j_3j_4]}$ is a perfect duality operator of order four. The eigenspaces of b_Θ and the irreducible representations of $\Lambda^4 V$ with respect to $\mathfrak{spin}(7)$ correspond in the following way:*

$$\Lambda^4_{(0)}V = \mathbf{35}\,, \quad \Lambda^4_{(-4)}V = \mathbf{1}\,, \quad \Lambda^4_{(-2)}V = \mathbf{7}\,, \quad \Lambda^4_{(\frac{2}{3})}V = \mathbf{27}\,. \tag{19}$$

The minimal polynomial is consequently given by

$$\beta(\beta+2)(\beta+4)(\beta-\tfrac{2}{3}) = \beta^4 + \tfrac{16}{3}\beta^3 + 4\beta^2 - \tfrac{16}{3}\beta\,. \tag{20}$$

We know that $\Lambda^4 V$ decomposes into four irreducible representations of dimension 1, 7, 27 and 35 with respect to $\mathfrak{spin}(7)$. Therefore, one of the values from Remark 3.4 is not an eigenvalue. In principle, we do not need this information to sort one of the values out. Nevertheless, the following proof of Proposition 3.5 will implicitly make use of it.

Proof. First we show that -4, 0 and -2 occur as eigenvalues and that the spaces of the right hand sides of (19) are subsets of the respective eigenspaces.

In particular, at least on part of the zero-eigenspace is given by $\mathbf{35} = (\Lambda^4 V)^- \subset \Lambda^4_{(0)}V$ due to the self-duality of Θ and Remark 2.12.

From (13) we immediately get $b_\Theta(\Theta) = -4\Theta$, such that $\mathbf{1} = \mathbb{R}\Theta \subseteq \Lambda^4_{(-4)}V$.

The next element we insert into b_Θ is $F_{ijkl} = \alpha_{m[i}\Theta^m{}_{jkl]}$ for $\alpha \in \Lambda^2 V$:

$$\begin{aligned}
b_\Theta(F)_{ijkl} &= \Theta^{mn}{}_{[ij}\delta^{abcd}_{kl]mn}\alpha_{oa}\Theta^o{}_{bcd} \\
&= \tfrac{1}{2}\Theta^{mn}{}_{[ij}\big(\delta^a_k\delta^{bcd}_{l]mn} + \delta^{bcd}_{kl]n}\delta^a_m\big)\alpha_{oa}\Theta^o{}_{bcd} \\
&= \tfrac{1}{2}\Theta^{mn}{}_{[ij}\alpha_k{}^o\Theta_{l]omn} + \tfrac{1}{2}\Theta^{mn}{}_{[ij}\Theta^o{}_{kl]n}\alpha_{om} \\
&= \tfrac{1}{2}\delta^{abcd}_{ijkl}g_{dd'}\Theta^{mn}{}_{ab}\alpha_{co}\Theta_{mn}{}^{d'o} + \tfrac{1}{2}\delta^{abcd}_{ijkl}g_{aa'}g_{bb'}\Theta^{ma'b'n}\Theta_{ocdn}\alpha^o{}_m \\
&= \tfrac{1}{2}\delta^{abcd}_{ijkl}g_{dd'}\big(12\delta^{d'o}_{ab} - 4\Theta_{ab}{}^{d'o}\big)\alpha_{co} + \tfrac{1}{2}\delta^{abcd}_{ijkl}g_{aa'}g_{bb'}\big(6\delta^{ma'b'}_{ocd} \\
&\quad - 9\Theta_{[oc}{}^{[ma'}\delta^{b']}_{d]}\big)\alpha^o{}_m \\
&= -2\alpha_{o[i}\Theta^o{}_{jkl]} - \tfrac{1}{2}\delta^{abcd}_{ijkl}g_{aa'}g_{bb'}\alpha^o{}_m\big(4\Theta_{oc}{}^{ma'}\delta^{b'}_d + 2\Theta_{oc}{}^{a'b'}\delta^m_d \\
&\quad + 2\Theta_{cd}{}^{ma'}\delta^{b'}_o + \Theta_{cd}{}^{a'b'}\delta^m_o\big) \\
&= -2\alpha_{o[i}\Theta^o{}_{jkl]} - \delta^{abcd}_{ijkl}\big(-\alpha_{o[i}\Theta^o{}_{jkl]} + \alpha_{m[i}\Theta^m{}_{jkl]}\big) \\
&= -2\alpha_{o[i}\Theta^o{}_{jkl]}
\end{aligned}$$

therefore[6] $\mathbf{7} = \big\{\alpha_{m[i}\Theta^m{}_{jkl]}\,;\,\alpha \in \Lambda^2_{(-6)}V\big\} \subset \Lambda^4_{(-2)}V$.

There is a space of dimension 27 left, which can not be decomposed further without getting more singlets in $\Lambda^4 V$. Therefore it is irreducible, and has to be a subspace of one of the eigenspaces. The trace formula $0\cdot 35 + (-4)\cdot 1 + (-2)\cdot 7 + \beta\cdot 27 = 0$ is only solved by $\beta = \frac{2}{3}$. Such that equality in (19) follows.

[6] We recall the decomposition of $\Lambda^2 V$ as given in Example 1.1 and that we have to double the eigenvalues given there, when we consider b_Θ. In particular, $\alpha_{m[i}\Theta^m{}_{jkl]} = 0$ for $\alpha \in \Lambda^2_{(2)}V$.

The above calculations and (14)-(16) yield the following decomposition of $\widetilde{\Theta} = \Theta \otimes \Theta$

$$\Theta_{ijkl}\Theta^{mnop} = -42\Theta^{[mn}{}_{[ij}\delta^{op]}_{kl]} + 2\Theta_{[ijk}{}^{[m}\Theta_{l]}{}^{nop]} + 3\Theta_{[ij}{}^{[mn}\Theta_{kl]}{}^{op]}\,. \tag{21}$$

In contrast to its traces, the full eight-tensor $\widetilde{\Theta}$ has contributions not only from the skew-symmetric parts $\Lambda^8 V$, $\Lambda^4 V$, and $\Lambda^0 V$ but also from $[\![6,2]\!]_0$ and $[\![4,4]\!]_0$. □

Remark 3.6. We complete the discussion of the invariant $\mathfrak{spin}(7)$-four-form by adding the missing result for the closely related invariant $\mathfrak{g}_2$-four-form, $\bar{\theta}$, see Example 1.1.

The minimal polynomial of $b_{\hat{\theta}} : \Lambda^3\mathbb{R}^7 \to \Lambda^3\mathbb{R}^7$ is $\beta^3 + \frac{16}{3}\beta^2 + 4\beta - \frac{16}{3}$ and the eigenspaces are $\Lambda^3_{(-4)}\mathbb{R}^7 = \mathbf{1}$, $\Lambda^3_{(-2)}\mathbb{R}^7 = \mathbf{7}$, and $\Lambda^3_{(\frac{2}{3})}\mathbb{R}^7 = \mathbf{27}$.

3.3. **Lifting to higher dimensions.** There are two straightforward ways to lift an ℓ-form Ω on $\mathbb{R}^n$ to $\mathbb{R}^D$ for $D > n$. First we consider the trivial lift given by an ℓ-form that lives only on the n-space perpendicular to a specified $(D-n)$-plane. We denote this first lift by the same Symbol Ω. Secondly, we consider the $*_D$-dual to this first lift, i.e. the $(D-\ell)$-form $\hat{\Omega} = *_D\Omega$. If Ω is $\mathfrak{g}$-invariant, theses lifts are invariant with respect to $\mathfrak{g} \oplus \mathfrak{so}(D-n)$. We will discuss these two constructions for the $\mathfrak{spin}(7)$-invariant four-form in dimension eight from the preceding section and its lifts to dimension ten. The maximal invariant subalgebra is $\mathfrak{spin}(7) \oplus \mathfrak{so}(2) = \mathfrak{spin}(7) \oplus \mathfrak{u}(1)$.

We specify the $e_9 \wedge e_{10}$-plane and we consider Θ to live on $\operatorname{span}\{e_i\}_{i\leq 8} = \mathbb{R}^8$. With respect to the decomposition $\mathbb{R}^{10} = \mathbb{R}^8 \oplus \mathbb{R}^2$ the k-forms split as

$$\Lambda^k\mathbb{R}^{10} = \Lambda^k\mathbb{R}^8 \oplus \Lambda^{k-1}\mathbb{R}^8 \otimes \mathbb{R}^2 \oplus \Lambda^{k-2}\mathbb{R}^8 \otimes \Lambda^2\mathbb{R}^2\,. \tag{22}$$

The trivial lift of Θ now yields for $k \geq 3$ a duality operator which is given by $b_\Theta = b_\Theta \otimes \mathbb{1}$ on each summand. The eigenspace decomposition for $k = 3, 4$ can immediately be read from the preceding sections. Moreover, in the case $k = 5$ we can furthermore use the symmetry $*_{10}(\Lambda^5\mathbb{R}^8) = \Lambda^3\mathbb{R}^8 \otimes \Lambda^2\mathbb{R}^2$ such that the missing decomposition follows from b_Θ on $\Lambda^3\mathbb{R}^8$ alone, and the eigenvalues and eigenspaces correspond as in (10) from Remark 2.12. In particular, the duality operator is not perfect in all cases, due to the doubling from the second summand in the right hand side of (22).

Secondly we consider the six-form $*_{10}\Theta$. Because Θ lives on $\mathbb{R}^8 \subset \mathbb{R}^{10}$ we have $*_{10}\Theta = *_8\Theta \wedge \epsilon = \Theta \wedge \epsilon$ which we will denote by $\hat{\Theta}$. Here ϵ denotes the volume-form on $\mathbb{R}^2 \subset \mathbb{R}^{10}$. Although this six-form is directly connected to the one before, we get a different behaviour of the eigenspaces. In fact, it turns out, that the restriction of $b_{\hat{\Theta}}$ to $\Lambda^k\mathbb{R}^{10}/\ker(b_{\hat{\Theta}})$ is perfect for $k = 3, 4$. For $k = 5$ the operator is not perfect, but the two basic $\mathfrak{spin}(7)$-representations of dimension seven and eight correspond to the same non-vanishing eigenvalue.

We will state the results for $k = 5, 4, 3$ and again postpone the calculations for the case $k = 5$ to the appendix. That hopefully will convince the reader that the calculations for the remaning cases can be peformed similarly.

For the case $k = 5$ we need the following lemma.

Lemma 3.7. *We consider the maps*

$$\begin{aligned} d_\Theta &: \Lambda^5\mathbb{R}^8 \to \Lambda^3\mathbb{R}^8, \quad d_\Theta(F)_{lmn} = \Theta_{ijk[l}F^{ijk}{}_{mn]}\,, \\ \tilde{d}_\Theta &: \Lambda^3\mathbb{R}^8 \to \Lambda^5\mathbb{R}^8, \quad \tilde{d}_\Theta(F)_{jklmn} = \Theta_{i[jkl}F^{i}{}_{mn]}\,. \end{aligned} \tag{23}$$

These maps are isomorphisms and connected to b_Θ and to the Hodge operator via

$$d_\Theta \circ \tilde{d}_\Theta = -\frac{6}{5}id + \frac{3}{2}b_\Theta(F)\,, \quad \text{and} \quad *\, d_\Theta * = -20\tilde{d}_\Theta\,. \tag{24}$$

A consequence of this is $\tilde{d}_\Theta \circ d_\Theta = - * d_\Theta \circ \tilde{d}_\Theta, *$.

Proof. The identities in (24) are verified in the appendix. Due to Schur's Lemma d_Θ and $\tilde{d}_\Theta$ are proportional to the identity when restricted to the eigenspaces of b_Θ and moreover they are non-vanishing due to (24). □

If we use lemma 3.7 and the calculations from the appendix we get the next result.

Proposition 3.8. *Let $\hat{\Theta}$ be the lift of Θ to ten dimension given by $\hat{\Theta} = \Theta \wedge \epsilon$. If we consider the decomposition of $\Lambda^5\mathbb{R}^{10}$ given by (22), then $b_{\hat{\Theta}} : \Lambda^5\mathbb{R}^{10} \to \Lambda^5\mathbb{R}^{10}$ is given by*

$$b_{\hat{\Theta}} = \begin{pmatrix} & & 6\tilde{d}_\Theta \otimes * \\ & \frac{9}{5}b_\Theta \otimes * & \\ \frac{3}{10}d_\Theta \otimes * & & \end{pmatrix}. \tag{25}$$

If we denote the $\pm i$-eigenspaces of $*_2$ on $\mathbb{R}^2$ by $\mathbb{R}_\pm$, the eigenvalues and eigenspaces of $b_{\hat{\Theta}}$ and their dimensions are given by

0	$\Lambda^4_{(0)}\mathbb{R}^8 \otimes \mathbb{R}_+ \oplus \Lambda^4_{(0)}\mathbb{R}^8 \otimes \mathbb{R}_-$	$35 + 35 = 70$
$\pm\frac{32}{5}i$	$\Lambda^4_{(-4)}\mathbb{R}^8 \otimes \mathbb{R}_\mp$	2×1
$\pm\frac{18}{5}i$	$\left\{\left(\mp\frac{5}{3}i\,\tilde{d}_\Theta(F), F \wedge \epsilon\right) \middle\vert F \in \Lambda^3_{(-4)}\mathbb{R}^8\right\} \oplus \Lambda^4_{(-2)}\mathbb{R}^8 \otimes \mathbb{R}_\mp$	$2 \times (8 + 7)$
$\pm\frac{6}{5}i$	$\Lambda^4_{(\frac{2}{3})}\mathbb{R}^8 \otimes \mathbb{R}_\pm$	2×27
$\pm\frac{3}{5}i$	$\left\{\left(\mp 10i\,\tilde{d}_\Theta(F), F \wedge \epsilon\right) \middle\vert F \in \Lambda^3_{(\frac{2}{3})}\mathbb{R}^8\right\}$	2×48

The first summand in the third row and the space in the last row are subspaces of $\Lambda^5_{(-\frac{6}{5})}\mathbb{R}^8 \oplus \Lambda^3_{(-4)}\mathbb{R}^8 \otimes \epsilon$ and $\Lambda^5_{(\frac{1}{5})}\mathbb{R}^8 \oplus \Lambda^3_{(\frac{2}{3})}\mathbb{R}^8 \otimes \epsilon$, respectively.

Similar to Lemma 3.7 we get the following.

Lemma 3.9. *Conisider the maps*

$$\begin{aligned} c_\Theta &: \Lambda^4\mathbb{R}^8 \to \Lambda^2\mathbb{R}^8, \quad c_\Theta(F)_{ij} = \Theta_{klm[i}F^{klm}{}_{j]}\,, \\ \tilde{c}_\Theta &: \Lambda^2\mathbb{R}^8 \to \Lambda^4\mathbb{R}^8, \quad \tilde{c}_\Theta(F)_{ijkl} = \Theta_{m[ijk}F^{m}{}_{l]}\,. \end{aligned} \tag{26}$$

Their kernels are $\ker(c_\Theta) = \Lambda^4_{(0)}\mathbb{R}^8 \oplus \Lambda^4_{(-4)}\mathbb{R}^8 \oplus \Lambda^4_{(\frac{2}{3})}\mathbb{R}^8$ and $\ker(\tilde{c}_\Theta) = \Lambda^2_{(2)}\mathbb{R}^8$ and the restrictions to $\Lambda^4_{(-2)}\mathbb{R}^8$ and $\Lambda^2_{(-6)}\mathbb{R}^8$ obey

$$c_\Theta \circ \tilde{c}_\Theta\big|_{\Lambda^2_{(-6)}\mathbb{R}^8} = -24id \quad \text{and} \quad \tilde{c}_\Theta \circ c_\Theta\big|_{\Lambda^4_{(-2)}\mathbb{R}^8} = -24id\,. \tag{27}$$

Proof. The statements follow from calculations similar to those for the case $k = 5$ and from Schur's Lemma together with the decompositions in Example 1.1 and Proposition 3.5. □

From Lemma 3.9 we get a result similar to the previous Proposition.

Proposition 3.10. *Let $\hat{\Theta}$ be the lift of Θ as before and consider the decomposition of $\Lambda^4\mathbb{R}^{10}$ given by (22). Then $b_{\hat{\Theta}} : \Lambda^4\mathbb{R}^{10} \to \Lambda^4\mathbb{R}^{10}$ is given by*

$$b_{\hat{\Theta}} = \begin{pmatrix} & & 6\tilde{c}_\Theta \otimes * \\ & \frac{9}{4} b_\Theta \otimes * & \\ \frac{1}{2} c_\Theta \otimes * & & \end{pmatrix} . \tag{28}$$

the eigenvalues and eigenspaces of $b_{\hat{\Theta}}$ as well as their dimensions are given by

0	$\Lambda^4_{(0)}\mathbb{R}^8 \oplus \Lambda^4_{(-4)}\mathbb{R}^8 \oplus \Lambda^4_{(\frac{2}{3})}\mathbb{R}^8 \oplus \Lambda^2_{(2)}\mathbb{R}^8 \otimes \epsilon$	$35 + 1 + 27 + 21 = 84$
$\pm 9i$	$\Lambda^3_{(-4)}\mathbb{R}^8 \otimes \mathbb{R}_\mp$	2×8
$\pm \frac{3}{2} i$	$\Lambda^3_{(\frac{2}{3})}\mathbb{R}^8 \otimes \mathbb{R}_\pm$	2×48
$\pm 6\sqrt{2} i$	$\left\{ \left(\mp \frac{i}{\sqrt{2}} \tilde{c}_\Theta(F), F \wedge \epsilon \right) \middle\vert F \in \Lambda^2_{(-6)}\mathbb{R}^8 \right\}$	2×7

The space in the last row is a subspace of $\Lambda^4_{(-2)}\mathbb{R}^8 \oplus \Lambda^2_{(-6)}\mathbb{R}^8 \otimes \epsilon$ to and can also be written as $\left\{ \left(F, \mp \frac{i}{12\sqrt{2}} c_\Theta(F) \wedge \epsilon \right) \middle\vert F \in \Lambda^4_{(-2)}\mathbb{R}^8 \right\}$ due to (27).

To complete the discussion we add the result for $k = 3$.

Proposition 3.11. *With $\hat{\Theta}$ as before and with (22) the operator $b_{\hat{\Theta}} : \Lambda^3\mathbb{R}^{10} \to \Lambda^3\mathbb{R}^{10}$ is given by*

$$b_{\hat{\Theta}} = \begin{pmatrix} & & 6\tilde{e}_\Theta \otimes * \\ & -3 b_\Theta \otimes * & \\ e_\Theta \otimes * & & \end{pmatrix} . \tag{29}$$

Its eigenvalues, eigenspaces and their dimensions are

0	$\Lambda^3_{(\frac{2}{3})}\mathbb{R}^8$	48
$\pm 18i$	$\Lambda^2_{(-6)}\mathbb{R}^8 \otimes \mathbb{R}_\pm$	2×7
$\pm 6i$	$\Lambda^2_{(2)}\mathbb{R}^8 \otimes \mathbb{R}_\mp$	2×21
$\pm 6\sqrt{7} i$	$\left\{ \left(\mp \frac{i}{\sqrt{7}} \tilde{e}_\Theta(F), F \wedge \epsilon \right) \middle\vert F \in \mathbb{R}^8 \right\}$	2×8

Here we used the following Lemma similar to Lemmas 3.7 and 3.9.

Lemma 3.12. *The maps*

$$\begin{aligned} e_\Theta &: \Lambda^3\mathbb{R}^8 \to \mathbb{R}^8 , \quad e_\Theta(F)_l = \Theta_{ijkl} F^{ijk} \\ \tilde{e}_\Theta &: \mathbb{R}^8 \to \Lambda^3\mathbb{R}^8 , \quad \tilde{e}_\Theta(F)_{jkl} = \Theta_{ijkl} F^i \end{aligned} \tag{30}$$

obey

$$e_\Theta\big|_{\Lambda^3_{(\frac{2}{3})}\mathbb{R}^8} = 0\,, \quad \tilde{e}_\Theta \circ e_\Theta\big|_{\Lambda^3_{(-4)}\mathbb{R}^8} = -24id\,, \quad e_\Theta \circ \tilde{e}_\Theta = -24id\,. \tag{31}$$

4. An example with discrete symmetry

On $V = \mathbb{R}^8$ we consider the four form[7]

$$\Omega = e_{1234} + e_{2345} + e_{3456} + e_{4567} + e_{5678} + e_{6781} + e_{7812} + e_{8123}\,. \tag{32}$$

This four-form is invariant under the action of $\mathbb{Z}_8$ on $\Lambda^k V$ which is given by $\sigma_a(e_{i_1\ldots i_k}) := e_{i_1+a\ldots i_k+a}$. We will denote the generator by $\sigma := \sigma_1$. The

b_Ω is defined on $\Lambda^2 V$, $\Lambda^3 V$ and $\Lambda^4 V$. A careful calculation yields the following results.

[$\mathbf{k = 2}$]. The minimal polynomial of $b_\Omega : \Lambda^2 V \to \Lambda^2 V$ is given by

$$p(t) = t(t^2-1)(t^2-4)(t^2-2)(t^2-(1+\sqrt{2})^2)(t^2-(1-\sqrt{2})^2)$$

and the eigenvalues of σ on $\Lambda^2 V$ have multiplicities 3 for ± 1 and $\pm i$, and 4 for $\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$.

The eigenspaces V_β for $\beta = 0$, ± 1, ± 2, $\pm\sqrt{2}$, and $\pm 1 \pm \sqrt{2}$ as well as their behavior under $\sigma \in \mathbb{Z}_8$ are explicitly given as follows.

$$\begin{aligned} V_{\pm 1} = \operatorname{span}\big\{ & v^1_\pm = (e_{56} - e_{12}) \pm (e_{38} + e_{47})\,, v^2_\pm = (e_{67} - e_{23}) \pm (e_{58} - e_{14})\,, \\ & v^3_\pm = (e_{78} - e_{34}) \mp (e_{16} + e_{25})\,, v^4_\pm = -(e_{18} + e_{45}) \mp (e_{27} + e_{36})\big\} \end{aligned} \tag{33}$$

with $v^1_\pm \xrightarrow{\sigma} v^2_\pm \xrightarrow{\sigma} v^3_\pm \xrightarrow{\sigma} v^4_\pm \xrightarrow{\sigma} -v^1_\pm$ such that $\sigma^4 + \mathbb{1}$ is the minimal equation on $V_{\pm 1}$.

$$V_{\pm 2} = \operatorname{span}\{v_\pm = e_{13} - e_{17} + e_{35} + e_{57} \mp (e_{24} - e_{28} + e_{46} + e_{68})\} \tag{34}$$

with $\sigma(v_\pm) = \mp v_\pm$ such that $\sigma \pm \mathbb{1} = 0$ on $V_{\pm 2}$.

$$\begin{aligned} V_{\pm\sqrt{2}} = \operatorname{span}\big\{ & v^\pm_1 = -e_{13} + e_{17} + e_{35} + e_{57} \mp \sqrt{2}(e_{28} + e_{46}), \\ & v^\pm_2 = -e_{24} - e_{28} - e_{46} + e_{68} \pm \sqrt{2}(e_{17} + e_{35})\big\} \end{aligned} \tag{35}$$

with $\sigma(v^\pm_1) = \mp\sqrt{2} v^\pm_1 + v^\pm_2$ and $\sigma(v^\pm_2) = -v^\pm_1$, i.e. $\sigma^2 \pm \sqrt{2}\,\sigma + \mathbb{1} = 0$ is the minimal equation on $V_{\pm\sqrt{2}}$.

Moreover, for $\epsilon, \eta \in \{\pm 1\}$ we have

$$\begin{aligned} V_{\epsilon+\eta\sqrt{2}} = \operatorname{span}\big\{ & v^\eta_\epsilon = e_{14} - \epsilon e_{27} + \epsilon e_{36} + e_{58} + (\epsilon + \eta\sqrt{2})(e_{23} - \epsilon e_{18} + \epsilon e_{45} + e_{67}), \\ & w^\eta_\epsilon = \epsilon e_{25} - \epsilon e_{16} - e_{38} + e_{47} + (\epsilon + \eta\sqrt{2})(e_{12} + \epsilon e_{34} + e_{56} + \epsilon e_{78})\big\} \end{aligned} \tag{36}$$

with $v^\eta_\epsilon \xrightarrow{\sigma} \epsilon w^\eta_\epsilon \xrightarrow{\sigma} \epsilon v^\eta_\epsilon$ such that $\sigma^2 - \epsilon\mathbb{1} = 0$ is the minimal equation on $V_{\epsilon+\eta\sqrt{2}}$.

[7] We use the short notation $e_{ijkl} = e_i \wedge e_j \wedge e_k \wedge e_l$.

Last but not least,

$$\begin{aligned} V_0 = \operatorname{span}\big\{ & w_1 = e_{24} + e_{28} - e_{46} + e_{68}, w_2 = e_{13} + e_{17} - e_{35} + e_{57}, \\ & e_{15}, e_{26}, e_{37}, e_{48} \big\} \end{aligned} \tag{37}$$

with $e_{15} \xrightarrow{\sigma} e_{26} \xrightarrow{\sigma} e_{37} \xrightarrow{\sigma} e_{48} \xrightarrow{\sigma} -e_{15}$ and $w_1 \xrightarrow{\sigma} -w_2 \xrightarrow{\sigma} -w_1$. I.e. $\sigma^4 + \mathbb{1} = 0$ and $\sigma^2 + \mathbb{1} = 0$ are the minimal equations on $E = \operatorname{span}\{e_{15}, e_{26}, e_{37}, e_{48}\}$ and $W = \operatorname{span}\{w_1, w_2\}$, respectively.

[$\mathbf{k = 3}$]. On $\Lambda^3 V$ the duality operator $3b_\Omega$ has minimal polynomial

$$p(t) = t(t^2 - 4)(t^2 - 2)(t^4 - 14t^2 + 16) \tag{38}$$

such that the eigenvalues are given by 0, ± 2, $\pm\sqrt{2}$, and $\pm\frac{\sqrt{22}}{2} \pm \frac{\sqrt{6}}{2}$. Moreover, the eigenvalues of σ have multiplicities 7 each.

The respective eigenspaces and the action of σ are given as follows.

$$\begin{aligned} V_{\pm 2} = \operatorname{span}\big\{ & w_1^\pm = e_{237} - e_{125} - e_{156} + e_{367} \pm (e_{138} - e_{134} + e_{457} - e_{578}), \\ & w_2^\pm = e_{348} - e_{236} - e_{267} + e_{478} \pm (e_{124} - e_{168} - e_{245} + e_{568}), \\ & w_3^\pm = e_{145} + e_{158} - e_{347} - e_{378} \pm (e_{167} - e_{127} + e_{235} - e_{356}), \\ & w_4^\pm = e_{126} - e_{148} + e_{256} - e_{458} \pm (e_{278} - e_{238} + e_{346} - e_{467}), \\ & u_1^\pm = e_{257} - e_{123} - e_{136} + e_{567} \pm (e_{158} - e_{145} + e_{347} - e_{378}), \\ & u_2^\pm = e_{368} - e_{234} - e_{247} + e_{678} \pm (e_{126} - e_{148} - e_{256} + e_{458}), \\ & u_3^\pm = e_{147} + e_{178} - e_{345} - e_{358} \pm (e_{156} - e_{125} + e_{237} - e_{367}), \\ & u_4^\pm = e_{128} - e_{146} + e_{258} - e_{456} \pm (e_{267} - e_{236} + e_{348} - e_{478}) \big\}. \end{aligned} \tag{39}$$

This basis is well adapted in the way that $w_1^\pm \xrightarrow{\sigma} w_2^\pm \xrightarrow{\sigma} w_3^\pm \xrightarrow{\sigma} w_4^\pm \xrightarrow{\sigma} w_1^\pm$ and $u_1^\pm \xrightarrow{\sigma} u_2^\pm \xrightarrow{\sigma} u_3^\pm \xrightarrow{\sigma} u_4^\pm \xrightarrow{\sigma} -u_1^\pm$, i.e. $\sigma^4 - \mathbb{1} = 0$ and $\sigma^4 + \mathbb{1} = 0$ are the respective minimal equations on $W^\pm = \operatorname{span}\{w_i^\pm\}$ and $U^\pm = \operatorname{span}\{u_i^\pm\}$.

For the zero eigenvalue we have

$$\begin{aligned} V_0 = \ \operatorname{span}\big\{ & x_1 = e_{236} - e_{267} + e_{348} - e_{478}\,, x_2 = e_{145} - e_{158} + e_{347} - e_{378}\,, \\ & x_3 = e_{256} - e_{126} - e_{148} + e_{458}\,, x_4 = e_{156} - e_{125} - e_{237} + e_{367}\,, \\ & y_1 = e_{123} - e_{136} + e_{257} - e_{567}\,, y_2 = e_{234} - e_{247} + e_{368} - e_{678}\,, \\ & y_3 = e_{147} - e_{178} + e_{345} - e_{358}\,, y_4 = e_{258} - e_{128} - e_{146} + e_{456}\,, \\ & u_1 = e_{278} - e_{238} - e_{346} + e_{467}\,, u_2 = e_{138} - e_{134} - e_{457} + e_{578}\,, \\ & u_3 = e_{124} + e_{168} - e_{245} - e_{568}\,, u_4 = e_{127} - e_{167} + e_{235} - e_{356}\,, \\ & v_1 = e_{127} - e_{123} - e_{134} - e_{136} - e_{138} + e_{147} + e_{167} + e_{178} \\ & \qquad + e_{235} - e_{257} + e_{345} + e_{356} + e_{358} - e_{457} - e_{567} - e_{578}\,, \\ & v_2 = e_{128} - e_{124} + e_{146} - e_{168} - e_{234} + e_{238} - e_{245} - e_{247} \\ & \qquad + e_{258} + e_{278} + e_{346} - e_{368} + e_{456} + e_{467} - e_{568} - e_{678}\,, \\ & w_1 = e_{127} - e_{123} + e_{134} - e_{136} + e_{138} - e_{147} + e_{167} - e_{178} \\ & \qquad + e_{235} - e_{257} - e_{345} + e_{356} - e_{358} + e_{457} - e_{567} + e_{578} \\ & w_2 = e_{124} - e_{128} - e_{146} + e_{168} - e_{234} + e_{238} + e_{245} - e_{247} \\ & \qquad - e_{258} + e_{278} + e_{346} - e_{368} - e_{456} + e_{467} + e_{568} - e_{678} \big\}. \end{aligned} \tag{40}$$

The above basis obeys $x_1 \xrightarrow{\sigma} x_2 \xrightarrow{\sigma} x_3 \xrightarrow{\sigma} x_4 \xrightarrow{\sigma} -x_1$, $y_1 \xrightarrow{\sigma} y_2 \xrightarrow{\sigma} y_3 \xrightarrow{\sigma} y_4 \xrightarrow{\sigma} -y_1$ and $u_1 \xrightarrow{\sigma} u_2 \xrightarrow{\sigma} u_3 \xrightarrow{\sigma} u_4 \xrightarrow{\sigma} -u_1$ as well as $v_1 \xrightarrow{\sigma} v_2 \xrightarrow{\sigma} -v_1$ and $w_1 \xrightarrow{\sigma} w_2 \xrightarrow{\sigma} w_1$. Therefore, the minimal equations are $\sigma^4 + \mathbb{1} = 0$ on $X = \operatorname{span}\{x_i\}$, $Y = \operatorname{span}\{y_i\}$ and $U = \operatorname{span}\{u_i\}$ as well as $\sigma^4 - \mathbb{1} = 0$ on $V \oplus W$ for $V = \operatorname{span}\{v_1, v_2\}$ and $W = \operatorname{span}\{w_1, w_2\}$ – more precisely $\sigma^2 + \mathbb{1} = 0$ on V and $\sigma \mp \mathbb{1} = 0$ on $W^{\pm} = \operatorname{span}\{w_1 \pm w_2\}$.

Furthermore,

$$
\begin{aligned}
V_{\pm\sqrt{2}} = \operatorname{span}\big\{ & v_1^{\pm} = e_{168} - e_{124} - e_{245} + e_{568} \pm \sqrt{2}(e_{135} - e_{157}), \\
& v_2^{\pm} = e_{127} + e_{167} - e_{235} - e_{356} \pm \sqrt{2}(e_{246} - e_{268}), \\
& v_3^{\pm} = e_{238} + e_{278} - e_{346} - e_{467} \pm \sqrt{2}(e_{357} - e_{137}), \\
& v_4^{\pm} = e_{134} + e_{138} - e_{457} - e_{578} \pm \sqrt{2}(e_{468} - e_{248}) \big\}.
\end{aligned} \tag{41}
$$

This basis is chosen in such a way that $v_1^{\pm} \xrightarrow{\sigma} v_2^{\pm} \xrightarrow{\sigma} v_3^{\pm} \xrightarrow{\sigma} v_4^{\pm} \xrightarrow{\sigma} -v_1^{\pm}$. Therefore, the minimal equation is $\sigma^4 + \mathbb{1} = 0$ on both spaces.

Last but not least for $\beta \in \{\pm\frac{\sqrt{22}}{2} \pm \frac{\sqrt{6}}{2}\}$ we have

(42)

$$
\begin{aligned}
V_\beta = \operatorname{span}\Big\{ v_1^\beta = {} & (e_{126} + e_{148} + e_{256} + e_{458}) + \frac{\beta}{4}(e_{238} + e_{278} + e_{346} + e_{467}) \\
& + \frac{8 - \beta^2}{4}(e_{137} + e_{357}) + \frac{2}{\beta}(e_{234} + e_{678}) + \frac{\beta^2 - 4}{2\beta}(e_{247} + e_{368}), \\
v_2^\beta = {} & (e_{125} + e_{156} + e_{237} + e_{367}) + \frac{\beta}{4}(e_{134} + e_{138} + e_{457} + e_{578}) \\
& + \frac{8 - \beta^2}{4}(e_{248} + e_{468}) + \frac{2}{\beta}(e_{178} + e_{345}) + \frac{\beta^2 - 4}{2\beta}(e_{147} + e_{358}), \\
v_3^\beta = {} & (e_{236} + e_{267} + e_{348} + e_{478}) + \frac{\beta}{4}(e_{124} + e_{168} + e_{245} + e_{568}) \\
& + \frac{8 - \beta^2}{4}(e_{135} + e_{157}) + \frac{2}{\beta}(e_{128} + e_{456}) + \frac{\beta^2 - 4}{2\beta}(e_{146} + e_{258}), \\
v_4^\beta = {} & (e_{145} + e_{158} + e_{347} + e_{378}) + \frac{\beta}{4}(e_{127} + e_{167} + e_{235} + e_{356}) \\
& + \frac{8 - \beta^2}{4}(e_{246} + e_{268}) + \frac{2}{\beta}(e_{123} + e_{567}) + \frac{\beta^2 - 4}{2\beta}(e_{136} + e_{257}) \Big\}.
\end{aligned}
$$

This choice of basis obeys $v_1^\beta \xrightarrow{\sigma} v_2^\beta \xrightarrow{\sigma} v_3^\beta \xrightarrow{\sigma} v_4^\beta \xrightarrow{\sigma} -v_1^\beta$, such that $\sigma^4 - \mathbb{1} = 0$ is the minimal equation for σ on V_β.

[$\mathbf{k} = \mathbf{4}$]. On $\Lambda^4 V$ the minimal polynomial of $6b_\Omega$ is given by

$$
p(t) = t(t^2 - 4)(t^2 - 16)(t^2 - 8) \tag{43}
$$

and the eigenvalues 0, ± 2, ± 4, and $\pm 2\sqrt{2}$ have multiplicities 26, 16, 4 and 2, respectively. Moreover, the multiplicities of the eigenvalues of σ are 10 for $\pm i$, 9 for ± 1, and 8 for $\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. We will list here the low dimensional eigenspaces and we will show, how Ω is related to the eigenvalues $\pm 2\sqrt{2}$.

The eigenspaces to the eigenvalues ± 4 are given by

(44)
$$\begin{aligned} V_{\pm 4} = \operatorname{span}\Big\{ & v_1^\pm = e_{1257} + e_{1356} + e_{2478} + e_{3468} \pm (e_{1347} - e_{1246} + e_{2568} - e_{3578}), \\ & v_2^\pm = e_{2368} + e_{2467} - e_{1358} - e_{1457} \mp (e_{1367} - e_{1468} + e_{2357} - e_{2458}), \\ & w_1^\pm = e_{1357} - e_{1458} - e_{2367} + e_{2468} \mp (e_{1368} + e_{1467} + e_{2358} + e_{2457}), \\ & w_2^\pm = e_{1256} - e_{1357} + e_{2468} - e_{3478} \pm (e_{1247} + e_{1346} - e_{2578} - e_{3568}) \Big\} \end{aligned}$$

with $v_1^\pm \xrightarrow{\sigma} v_2^\pm \xrightarrow{\sigma} \mp v_1^\pm$ and $w_1^\pm \xrightarrow{\sigma} -w_2^\pm \xrightarrow{\sigma} -w_1^\pm$ such that the minimal equation of σ is $\sigma^2 + \mathbb{1} = 0$ on V_4, and $\sigma^4 - \mathbb{1} = 0$ on V_{-4}.

The eigenspaces to the eigenvalues $\pm 2\sqrt{2}$ are given by

(45)
$$\begin{aligned} V_{\pm 2\sqrt{2}} = \operatorname{span}\Big\{ & u_1^\pm = e_{2345} - e_{1238} - e_{1678} + e_{4567} \pm \sqrt{2}(e_{2367} - e_{1458}), \\ & u_2^\pm = e_{1234} + e_{1278} + e_{3456} + e_{5678} \pm \sqrt{2}(e_{1256} - e_{3478}) \Big\} \end{aligned}$$

with $u_1^\pm \xrightarrow{\sigma} u_2^\pm \xrightarrow{\sigma} u_1^\pm$ such that σ has eigenvalues ± 1 on $V_{\pm 2\sqrt{2}}$.

Remark 4.1. The two-dimensional $+1$-eigenspace of σ within $V_{2\sqrt{2}} \oplus V_{-2\sqrt{2}}$ is given by $\operatorname{span}\{\Omega, \omega\}$ where

(46)
$$\begin{aligned} \Omega &= \frac{1}{2}(u_1^+ + u_1^- + u_2^+ + u_2^-)\,, \\ \omega &:= \frac{1}{2}(u_1^+ - u_1^- + u_2^+ - u_2^-)\,. \end{aligned}$$

These forms fulfill $b_\Omega(\Omega) = \frac{\sqrt{2}}{3}\omega$ and $b_\Omega(\omega) = \frac{\sqrt{2}}{3}\Omega$. In particular, Ω itself is not an eigenform with respect to b_Ω, in contrast to the discussion following Definition 3.1.

We conclude this example by adding some comments on the eigenspaces of b_Ω to the remaining eigenvalues 0 and ± 2 which we as usual denote by V_0 and $V_{\pm 2}$. This explains the so far unusual asymmetry in the behavior of σ on $V_{\pm 4}$.

The map σ acting V_0 has eigenvalues $\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$ with multiplicity 4, $\pm i$ with multiplicity 3, as well as ± 1 with multiplicity 2. Restricted to $V_{\pm 2}$ the eight eigenvalues of σ come with multiplicity 2, each.

5. Outlook

The duality operator we defined here in flat space can be defined in the same way on a Riemannian or semi-Riemannian manifold. In particular, all that has been discussed for $\mathfrak{g}$-invariant duality operators can be transfered to manifolds with a $\mathfrak{g}$-structure. In this case the $\mathfrak{g}$-invariant differential form $\Omega \in \Omega^\ell(M)$ is parallel with respect to a connection associated to the given $\mathfrak{g}$-structure.

One application of our duality relations may be the following. Let the manifold under consideration be spin, and take a connection on the spinor bundle S on M. This connection and its curvature are locally described by elements in the exterior algebra of M, the so called k-form potentials and fluxes, see for example [14, 15]. The duality relation presented here then is a candidate to generalize the duality for metric connections on the base manifold M.

Appendix A. Useful Decompositions

We are interested in the decomposition of certain tensor products of irreducible representations of $\mathfrak{so}(n)$. We recall the decomposition of the tensor product of anti-symmetric powers of $V = \mathbb{R}^n$ into irreducible $\mathfrak{gl}(n)$-modules. Let $k, \ell \leq \frac{n}{2}$ then

$$\Lambda^\ell V \otimes \Lambda^k V = \bigoplus_{i=0}^{\min\{k,\ell\}} [\![k+\ell-i, i]\!]\,. \tag{47}$$

Here $[\![k+\ell-i,i]\!]$ denotes the irreducible representation space of weight $e_i + e_{k+\ell-i}$. With respect to $\mathfrak{so}(n)$ these spaces are reducible for $i \neq 0$. The irreducible components are obtained by contraction with the metric. If we denote the trace free parts by $[\![\cdot,\cdot]\!]_0$ we get

$$[\![k+\ell-i,i]\!] = \bigoplus_{j=0}^{i} [\![k+\ell-i-j, i-j]\!]_0 \tag{48}$$

which yields the final $\mathfrak{so}(n)$-decomposition

$$\Lambda^\ell V \otimes \Lambda^k V = \bigoplus_{i=0}^{\min\{k,\ell\}} \bigoplus_{j=0}^{i} [\![k+\ell-i-j, i-j]\!]_0\,. \tag{49}$$

Due to Hodge duality the preceding formula can be used for, say, $\ell > \frac{n}{2}$, too, we only have to insert $\Lambda^{n-\ell}V \approx \Lambda^\ell$ instead. For a more systematic treatment of such decompositions we refer the reader to the nice article [16].

By π_m we denote the projection $\Lambda^k V \otimes \Lambda^\ell V \to \Lambda^m V = [\![m,0]\!]$.

We are in particular interested in the second symmetric power of $\Lambda^k V$. With the above notation for $k = \ell$ we have the following $\mathfrak{so}(n)$-decomposition

$$\begin{aligned} S^2(\Lambda^k V) &= \bigoplus_{j=0}^{[\frac{k}{2}]} \Big(\bigoplus_{i=0}^{k-2j} [\![k+2j-i, k-2j-i]\!]_0 \Big) \\ &= \bigoplus_{j=0}^{[\frac{k}{2}]} \Big(\bigoplus_{i=0}^{k-2j-1} [\![k+2j-i, k-2j-i]\!]_0 \Big) \oplus \bigoplus_{j=0}^{[\frac{k}{2}]} \Lambda^{4j} V\,. \end{aligned} \tag{50}$$

In particular $\Lambda^\ell V \subset S^2(\Lambda^k V)$ only if $\ell \equiv 0 \mod 4$.

Appendix B. Some Calculations

In this appendix we add the calculations for equations (14) to (17) that we left out in Lemma 3.3 as well as the calculations for Lemma 3.7 and Proposition 3.8.

B.1. **Calculations for Lemma 3.3.** We recall the content of Lemma 3.3: The duality map $b_\Theta : \Lambda^4\mathbb{R}^8 \to \Lambda^4\mathbb{R}^8$ given by $b_\Theta(F)_{ijkl} = \Theta^{mn}{}_{[ij}F_{kl]mn}$. obeys

(14) $b^2_\Theta(F)_{ijkl} = \frac{1}{6}\Theta^{mn}{}_{[ij}\Theta^{op}{}_{kl]}F_{mnop} + \frac{2}{3}F_{ijkl} - \frac{8}{3}b_\Theta(F)_{ijkl}$

(15) $b^3_\Theta(F)_{ijkl} = \frac{4}{3}F_{ijkl} + \frac{2}{3}b_\Theta(F)_{ijkl} - \frac{10}{3}b^2_\Theta(F)_{ijkl} + \frac{2}{9}\Theta_{[ijk}{}^p\Theta^{rsn}{}_{l]}F_{pnrs}$

(16) $b^4_\Theta(F)_{ijkl} = 4b_\Theta(F)_{ijkl} - \frac{8}{3}b^2_\Theta(F)_{ijkl} - \frac{13}{3}b^3_\Theta(F)_{ijkl} + \frac{1}{9}\Theta_{ijkl}\Theta^{prsn}F_{prsn}$

$$b^5_\Theta(F)_{ijkl} = -\tfrac{25}{3} b^4_\Theta(F)_{ijkl} - 20 b^3_\Theta(F)_{ijkl} - \tfrac{20}{3} b^2_\Theta(F)_{ijkl} + 16 b_\Theta(F)_{ijkl} \tag{17}$$

To get (14) we calculate

$$\begin{aligned}
b^2_\Theta(F)_{ijkl} &= \Theta^{mn}{}_{[ij} b_\Theta(F)_{kl]mn} \\
&= \Theta^{mn}{}_{[ij} \delta^{abcd}_{kl]mn} \Theta^{op}{}_{ab} F_{cdop} \\
&= \tfrac{1}{6} \Theta^{mn}{}_{[ij} \delta^{ab}_{kl]} \delta^{cd}_{mn} \Theta^{op}{}_{ab} F_{cdop} + \tfrac{1}{6} \Theta^{mn}{}_{[ij} \delta^{cd}_{kl]} \delta^{ab}_{mn} \Theta^{op}{}_{ab} F_{cdop} \\
&\quad - \tfrac{2}{3} \Theta^{mn}{}_{[ij} \delta^{ac}_{kl]} \delta^{bd}_{mn} \Theta^{op}{}_{ab} F_{cdop} \\
&= \tfrac{1}{6} \Theta^{mn}{}_{[ij} \Theta^{op}{}_{kl]} F_{mnop} + \tfrac{1}{6} \Big(12 \delta^{op}_{[ij} - 4 \Theta^{op}{}_{[ij} \Big) F_{kl]op} \\
&\quad + \tfrac{2}{3} \delta^{i'j'k'l'}_{ijkl} \Big(6 \delta^{opq}_{ni'j'} - 9 \Theta_{[ni'}{}^{[op} \delta^{q]}_{j']} \Big) g_{k'q} F_{l'}{}^{n}{}_{op} \\
&= \tfrac{1}{6} \Theta^{mn}{}_{[ij} \Theta^{op}{}_{kl]} F_{mnop} + 2 F_{ijkl} - \tfrac{2}{3} b_\Theta(F)_{ijkl} \\
&\quad + 4 \delta^{i'j'k'l'}_{ijkl} \Big(\tfrac{1}{3} \delta^q_n \delta^{op}_{i'j'} + \tfrac{2}{3} \delta^o_n \delta^{pq}_{i'j'} \Big) g_{k'q} F_{l'}{}^{n}{}_{op} \\
&\quad - 6 \delta^{i'j'k'l'}_{ijkl} \Big(\tfrac{1}{3} \cdot \tfrac{1}{3} \Theta_{i'j'}{}^{op} \delta^q_n + \tfrac{1}{3} \cdot \tfrac{2}{3} \Theta_{ni'}{}^{op} \delta^q_{j'} + \tfrac{2}{3} \cdot \tfrac{1}{3} \Theta_{i'j'}{}^{qo} \delta^p_n \\
&\quad + \tfrac{2}{3} \cdot \tfrac{2}{3} \Theta_{ni'}{}^{qo} \delta^p_{j'} \Big) g_{k'q} F_{l'}{}^{n}{}_{op} \\
&= \tfrac{1}{6} \Theta^{mn}{}_{[ij} \Theta^{op}{}_{kl]} F_{mnop} + \tfrac{2}{3} F_{ijkl} - \tfrac{2}{3} b_\Theta(F)_{ijkl} \\
&\quad + \tfrac{2}{3} \Theta_{[ij}{}^{op} F_{kl]op} - \tfrac{8}{3} \Theta_{mn[ij} F_{kl]}{}^{mn} \\
&= \tfrac{1}{6} \Theta^{mn}{}_{[ij} \Theta^{op}{}_{kl]} F_{mnop} + \tfrac{2}{3} F_{ijkl} - \tfrac{8}{3} b_\Theta(F)_{ijkl} \,.
\end{aligned}$$

To get (15) we need the image of the first summand in (14) under b_Θ.

$$\begin{aligned}
&\Theta^{mn}{}_{[ij} \delta^{abcd}_{kl]mn} \Theta^{pq}{}_{ab} \Theta^{rs}{}_{cd} F_{pqrs} \\
&\quad = \tfrac{1}{6} \Theta^{mn}{}_{[ij} \delta^{ab}_{kl]} \delta^{cd}_{mn} \Theta^{pq}{}_{ab} \Theta^{rs}{}_{cd} F_{pqrs} + \tfrac{1}{6} \Theta^{mn}{}_{[ij} \delta^{cd}_{kl]} \delta^{ab}_{mn} \Theta^{pq}{}_{ab} \Theta^{rs}{}_{cd} F_{pqrs} \\
&\qquad - \tfrac{2}{3} \Theta^{mn}{}_{[ij} \delta^{ac}_{kl]} \delta^{bd}_{mn} \Theta^{pq}{}_{ab} \Theta^{rs}{}_{cd} F_{pqrs} \\
&\quad = \tfrac{1}{3} \Big(12 \delta^{rs}_{[ij} - 4 \Theta^{rs}{}_{[ij} \Big) \Theta^{pq}{}_{kl]} F_{pqrs} - \tfrac{2}{3} \delta^{i'j'k'l'}_{ijkl} \Theta_{mni'j'} \Theta^{mpqt} \Theta^{rsn}{}_{l'} g_{k't} F_{pqrs} \\
&\quad = 4 \Theta_{pq[ij} F_{kl]}{}^{pq} - \tfrac{4}{3} \Theta^{rs}{}_{[ij} \Theta^{pq}{}_{kl]} F_{pqrs} \\
&\qquad - \tfrac{2}{3} \delta^{i'j'k'l'}_{ijkl} \Big(6 \delta^{pqt}_{ni'j'} - 9 \Theta_{[ni'}{}^{[pq} \delta^{t]}_{j']} \Big) \Theta^{rsn}{}_{l'} g_{k't} F_{pqrs} \\
&\quad = 4 b_\Theta(F)_{ijkl} - \tfrac{4}{3} \Theta^{rs}{}_{[ij} \Theta^{pq}{}_{kl]} F_{pqrs} - \tfrac{4}{3} \Theta^{rs}{}_{[ij} F_{kl]rs} \\
&\qquad + \tfrac{2}{3} \delta^{i'j'k'l'}_{ijkl} \Theta_{i'j'}{}^{pq} \delta^t_n \Theta^{rsn}{}_{l'} g_{k't} F_{pqrs} + \tfrac{4}{3} \delta^{i'j'k'l'}_{ijkl} \Theta_{ni'}{}^{pq} \delta^t_{j'} \Theta^{rsn}{}_{l'} g_{k't} F_{pqrs} \\
&\qquad + \tfrac{4}{3} \delta^{i'j'k'l'}_{ijkl} \Theta_{i'j'}{}^{tp} \delta^q_n \Theta^{rsn}{}_{l'} g_{k't} F_{pqrs} + \tfrac{8}{3} \delta^{i'j'k'l'}_{ijkl} \Theta_{ni'}{}^{tp} \delta^q_{j'} \Theta^{rsn}{}_{l'} g_{k't} F_{pqrs} \\
&\quad = \tfrac{8}{3} b_\Theta(F)_{ijkl} - \tfrac{4}{3} \Theta^{rs}{}_{[ij} \Theta^{pq}{}_{kl]} F_{pqrs} + \tfrac{2}{3} \Theta^{pq}{}_{[ij} \Theta^{rs}{}_{kl]} F_{pqrs} \\
&\qquad + \tfrac{4}{3} \Theta_{[ijk}{}^{p} \Theta^{rsn}{}_{l]} F_{pnrs} + \tfrac{8}{3} \delta^{i'j'k'l'}_{ijkl} \Theta_{npi'k'} \Theta^{nrst} g_{l't} F^{p}{}_{j'rs} \\
&\quad = \tfrac{8}{3} b_\Theta(F)_{ijkl} - \tfrac{2}{3} \Theta^{rs}{}_{[ij} \Theta^{pq}{}_{kl]} F_{pqrs} + \tfrac{4}{3} \Theta_{[ijk}{}^{p} \Theta^{rsn}{}_{l]} F_{pnrs} \\
&\qquad + \tfrac{8}{3} \delta^{i'j'k'l'}_{ijkl} \Big(6 \delta^{rst}_{pi'k'} - 9 \Theta_{[pi'}{}^{[rs} \delta^{t]}_{k'] } \Big) g_{l't} F^{p}{}_{j'rs} \\
&\quad = \tfrac{8}{3} b_\Theta(F)_{ijkl} - \tfrac{2}{3} \Theta^{rs}{}_{[ij} \Theta^{pq}{}_{kl]} F_{pqrs} + \tfrac{4}{3} \Theta_{[ijk}{}^{p} \Theta^{rsn}{}_{l]} F_{pnrs}
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{16}{3}\delta^{i'j'k'l'}_{ijkl}\delta^{rs}_{i'k'}\delta^{t}_{p}g_{l't}F^{p}{}_{j'rs} + \tfrac{32}{3}\delta^{i'j'k'l'}_{ijkl}\delta^{tr}_{i'k'}\delta^{s}_{p}g_{l't}F^{p}{}_{j'rs}\\
&- \tfrac{24}{3}\cdot\tfrac{1}{3}\delta^{i'j'k'l'}_{ijkl}\Theta_{i'k'}{}^{rs}\delta^{t}_{p}g_{l't}F^{p}{}_{j'rs} - \tfrac{24}{3}\cdot\tfrac{2}{3}\delta^{i'j'k'l'}_{ijkl}\Theta_{pi'}{}^{rs}\delta^{t}_{k'}g_{l't}F^{p}{}_{j'rs}\\
&- \tfrac{48}{3}\cdot\tfrac{1}{3}\delta^{i'j'k'l'}_{ijkl}\Theta_{i'k'}{}^{tr}\delta^{s}_{p}g_{l't}F^{p}{}_{j'rs} - \tfrac{48}{3}\cdot\tfrac{2}{3}\delta^{i'j'k'l'}_{ijkl}\Theta_{pi'}{}^{tr}\delta^{s}_{k'}g_{l't}F^{p}{}_{j'rs}\\
&= \tfrac{8}{3}b_\Theta(F)_{ijkl} - \tfrac{2}{3}\Theta^{rs}{}_{[ij}\Theta^{pq}{}_{kl]}F_{pqrs} + \tfrac{4}{3}\Theta_{[ijk}{}^{p}\Theta^{rsn}{}_{l]}F_{pnrs}\\
&\quad + \tfrac{16}{3}F_{ijkl} - \tfrac{8}{3}b_\Theta(F)_{ijkl} + \tfrac{32}{3}b_\Theta(F)_{ijkl}\\
&= \tfrac{16}{3}F_{ijkl} + \tfrac{32}{3}b_\Theta(F)_{ijkl} - \tfrac{2}{3}\Theta^{rs}{}_{[ij}\Theta^{pq}{}_{kl]}F_{pqrs} + \tfrac{4}{3}\Theta_{[ijk}{}^{p}\Theta^{rsn}{}_{l]}F_{pnrs}\,.
\end{aligned}
$$

So we get for the third power of b_Θ

$$
\begin{aligned}
b^3_\Theta(F)_{ijkl} &= b_\Theta(b^2_\Theta(F))_{ijkl}\\
&= \tfrac{2}{3}b_\Theta(F)_{ijkl} - \tfrac{8}{3}b^2_\Theta(F)_{ijkl}\\
&\quad + \tfrac{1}{6}\Big(\tfrac{16}{3}F_{ijkl} + \tfrac{32}{3}b_\Theta(F)_{ijkl} - \tfrac{2}{3}\Theta^{rs}{}_{[ij}\Theta^{pq}{}_{kl]}F_{pqrs}\\
&\quad + \tfrac{4}{3}\Theta_{[ijk}{}^{p}\Theta^{rsn}{}_{l]}F_{pnrs}\Big)\\
&= \tfrac{2}{3}b_\Theta(F)_{ijkl} - \tfrac{8}{3}b^2_\Theta(F)_{ijkl}\\
&\quad + \tfrac{8}{9}F_{ijkl} + \tfrac{16}{9}b_\Theta(F)_{ijkl} - \tfrac{2}{3}\Big(b^2_\Theta(F)_{ijkl} - \tfrac{2}{3}F_{ijkl} + \tfrac{8}{3}b_\Theta(F)_{ijkl}\Big)\\
&\quad + \tfrac{2}{9}\Theta_{[ijk}{}^{p}\Theta^{rsn}{}_{l]}F_{pnrs}\\
&= \tfrac{4}{3}F_{ijkl} + \tfrac{2}{3}b_\Theta(F)_{ijkl} - \tfrac{10}{3}b^2_\Theta(F)_{ijkl} + \tfrac{2}{9}\Theta_{[ijk}{}^{p}\Theta^{rsn}{}_{l]}F_{pnrs}\,.
\end{aligned}
$$

To evaluate b^4_Θ, i.e. (16), we need the image of $\Theta_{[ijk}{}^{p}\Theta^{rsn}{}_{l]}F_{pnrs}$ under b_Θ,

$$
\begin{aligned}
&\Theta^{mn}{}_{[ij}\delta^{abcd}_{kl]mn}\Theta_{abc}{}^{o}\Theta^{prs}{}_{d}F_{oprs}\\
&\quad= \tfrac{1}{4}\Theta^{mn}{}_{[ij}\delta^{abc}_{kl]m}\delta^{d}_{n}\Theta_{abc}{}^{o}\Theta^{prs}{}_{d}F_{oprs} - \tfrac{3}{4}\Theta^{mn}{}_{[ij}\delta^{abd}_{kl]m}\delta^{c}_{n}\Theta_{abc}{}^{o}\Theta^{prs}{}_{d}F_{oprs}\\
&\quad= \tfrac{1}{4}\Theta^{mn}{}_{[ij}\Theta_{kl]m}{}^{o}\Theta^{prs}{}_{n}F_{oprs}\\
&\qquad - \tfrac{3}{4}\Theta^{mn}{}_{[ij}\Big(\tfrac{1}{3}\delta^{ab}_{kl]}\delta^{d}_{m} + \tfrac{2}{3}\delta^{da}_{kl]}\delta^{b}_{m}\Big)\delta^{c}_{n}\Theta_{abc}{}^{o}\Theta^{prs}{}_{d}F_{oprs}\\
&\quad= \tfrac{1}{2}\Theta^{mn}{}_{[ij}\Theta_{kl]m}{}^{o}\Theta^{prs}{}_{n}F_{oprs} + \tfrac{1}{2}\Theta_{mn[ij}\Theta_{k}{}^{mno}\Theta^{prs}{}_{l]}F_{oprs}\\
&\quad= \tfrac{1}{2}\delta^{i'j'k'l'}_{ijkl}\Big(6\delta^{tuo}_{ni'j'} - 9\Theta_{[ni'}{}^{[tu}\delta^{o]}_{j']}\Big)\Theta^{prsn}g_{tk'}g_{ul'}F_{oprs}\\
&\qquad + \tfrac{1}{2}\delta^{i'j'k'l'}_{ijkl}\Big(12\delta^{to}_{i'j'} - 4\Theta_{i'j'}{}^{to}\Big)\Theta^{prs}{}_{l'}g_{k't}F_{oprs}\\
&\quad= -2\Theta_{[ijk}{}^{o}\Theta^{prs}{}_{l]}F_{oprs} - \delta^{i'j'k'l'}_{ijkl}\Big(\tfrac{1}{2}\Theta_{i'j'k'l'}\Theta^{prsn}F_{nprs}\\
&\qquad + \Theta_{ni'k'l'}\Theta^{prsn}F_{j'prs} - \Theta_{i'j'k'}{}^{o}\Theta^{prs}{}_{l'}F_{oprs}\Big)\\
&\quad= -\Theta_{[ijk}{}^{o}\Theta^{prs}{}_{l]}F_{oprs} - \tfrac{1}{2}\Theta_{ijkl}\Theta^{prsn}F_{nprs}\\
&\qquad + \delta^{i'j'k'l'}_{ijkl}\Big(6\delta^{prs}_{i'j'k'} - 9\Theta_{[i'j'}{}^{[pr}\delta^{s]}_{k']}\Big)F_{l'prs}\\
&\quad= -\Theta_{[ijk}{}^{o}\Theta^{prs}{}_{l]}F_{oprs} + \tfrac{1}{2}\Theta_{ijkl}\Theta^{prsn}F_{prsn} - 6F_{ijkl} + 9b_\Theta(F)_{ijkl}\,.
\end{aligned}
$$

This is

$$\begin{aligned}
b^4_\Theta(F)_{ijkl} &= b_\Theta(b^3_\Theta(F))_{ijkl}\\
&= \tfrac{4}{3}b_\Theta(F)_{ijkl} + \tfrac{2}{3}b^2_\Theta(F)_{ijkl} - \tfrac{10}{3}b^3_\Theta(F)_{ijkl}\\
&\quad + \tfrac{2}{9}\Big(-\Theta_{[ijk}{}^o\Theta^{prs}{}_{l]}F_{oprs} + \tfrac{1}{2}\Theta_{ijkl}\Theta^{prsn}F_{prsn} - 6F_{ijkl} + 9b_\Theta(F)_{ijkl}\Big)\\
&= -\tfrac{4}{3}F_{ijkl} + \tfrac{10}{3}b_\Theta(F)_{ijkl} + \tfrac{2}{3}b^2_\Theta(F)_{ijkl} - \tfrac{10}{3}b^3_\Theta(F)_{ijkl}\\
&\quad - \Big(b^3_\Theta(F)_{ijkl} - \tfrac{4}{3}F_{ijkl} - \tfrac{2}{3}b_\Theta(F)_{ijkl} + \tfrac{10}{3}b^2_\Theta(F)_{ijkl}\Big)\\
&\quad + \tfrac{1}{9}\Theta_{ijkl}\Theta^{prsn}F_{prsn}\\
&= 4b_\Theta(F)_{ijkl} - \tfrac{8}{3}b^2_\Theta(F)_{ijkl} - \tfrac{13}{3}b^3_\Theta(F)_{ijkl} + \tfrac{1}{9}\Theta_{ijkl}\Theta^{prsn}F_{prsn}\,.
\end{aligned}$$

The last step is easy. For b^5_Θ we need the image of $\Theta_{ijkl}\Theta^{prsn}F_{prsn}$. This is is a multiple of Θ_{ijkl} for which we have $b_\Theta(\Theta)_{ijkl} = \Theta_{mn[ij}\Theta^{mn}{}_{kl]} = -4\Theta_{ijkl}$. This yields

$$\begin{aligned}
b^5_\Theta(F)_{ijkl} &= b_\Theta(b^4_\Theta(F))_{ijkl}\\
&= 4b^2_\Theta(F)_{ijkl} - \tfrac{8}{3}b^3_\Theta(F)_{ijkl} - \tfrac{13}{3}b^4_\Theta(F)_{ijkl} - \tfrac{4}{9}\Theta_{ijkl}\Theta^{opqr}F_{opqr}\\
&= -\tfrac{25}{3}b^4_\Theta(F)_{ijkl} - 20b^3_\Theta(F)_{ijkl} - \tfrac{20}{3}b^2_\Theta(F)_{ijkl} + 16b_\Theta(F)_{ijkl}\,.
\end{aligned}$$

B.2. **Calculations for Lemma 3.7 and Proposition 3.8.** The proof of Lemma 3.7 is a straightforward calculation. The maps

$$\begin{aligned}
&d_\Theta : \Lambda^5\mathbb{R}^8 \to \Lambda^3\mathbb{R}^8, \quad d_\Theta(F)_{lmn} = \Theta_{ijk[l}F^{ijk}{}_{mn}\,,\\
&\tilde{d}_\Theta : \Lambda^3\mathbb{R}^8 \to \Lambda^5\mathbb{R}^8, \quad \tilde{d}_\Theta(F)_{jklmn} = \Theta_{i[jkl}F^{i}{}_{mn]}\,.
\end{aligned} \tag{23}$$

are isomorphisms and connected to b_Θ and to the Hodge operator via

$$d_\Theta \circ \tilde{d}_\Theta = -\frac{6}{5}id + \frac{3}{2}b_\Theta(F)\,, \quad \text{and} \quad *\, d_\Theta * = -20\tilde{d}_\Theta\,. \tag{24}$$

A consequence of this is $\tilde{d}_\Theta \circ d_\Theta = -*\, d_\Theta \circ \tilde{d}_\Theta, *$.

We make use of (13) and get

$$\begin{aligned}
d_\Theta\tilde{d}_\Theta(F)_{lmn} &= \Theta_{ijk[l}\tilde{d}_\Theta(F)^{ijk}{}_{mn]}\\
&= \delta^{abc}_{lmn}\Theta^{ijk}{}_a\Theta_{s[ijk}F^s{}_{bc]}\\
&= \delta^{abc}_{lmn}\big(\tfrac{1}{10}\Theta^{ijk}{}_a\Theta_{sijk}F^s{}_{bc} + \tfrac{3}{10}\Theta^{ijk}{}_a\Theta_{sbci}F^s{}_{jk}\\
&\quad - \tfrac{6}{10}\Theta^{ijk}{}_a\Theta_{sbij}F^a{}_{kc}\big)\\
&= -\tfrac{21}{5}F_{lmn} - \tfrac{3}{5}\delta^{abc}_{lmn}\big(-6\delta_{m'k}\delta_{sl'} - 4\Theta_{sklm}\big)F^{sk}{}_{n'}\\
&\quad - \tfrac{3}{10}\delta^{abc}_{lmn}\big(2\delta^{jk}_{bc}\delta_{sa} - 4\Theta_a{}^j{}_{sb}\delta^k_c - \Theta^{jk}{}_{bc}\delta_{sa}\big)F^s{}_{jk}\\
&= -\tfrac{21}{5}F_{lmn} - \tfrac{3}{5}F_{lmn} - \tfrac{6}{5}\Theta_{jk[lm}F^{jk}{}_{n]} + \tfrac{3}{10}\Theta_{jk[lm}F^{jk}{}_{n]}\\
&\quad + \tfrac{18}{5}F_{lmn} + \tfrac{12}{5}\Theta_{jk[lm}F^{jk}{}_{n]}\\
&= -\tfrac{6}{5}F_{lmn} + \tfrac{3}{2}b_\Theta(F)_{lmn}\,.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
*d_\Theta(*F)_{ijklm} =& -\tfrac{1}{6}\epsilon_{ijklmnop}d_\Theta(*F)^{nop} = -\tfrac{1}{6}\epsilon_{ijklmnop}\Theta_{abc}{}^{n}(*F)^{abcop}\\
=& \frac{1}{36}\epsilon_{ijklmnop}\epsilon^{abcoprst}\Theta_{abc}{}^{n}F_{rst} = 40\delta^{abcrst}_{ijklmn}\Theta_{abc}{}^{n}F_{rst}\\
=& 20\delta^{abcrs}_{ijklm}\delta^{t}_{n}\Theta_{abc}{}^{n}F_{rst} = -20\Theta_{n[ijk}F^{n}{}_{lm]}\\
=& -20\tilde{d}_\Theta(F)_{ijklm}
\end{aligned}$$

The proof of Proposition 3.8 is divided into three cases $b_{\hat\Theta}(\hat F)$ where we consider $\hat F = F^5$, $\hat F = F^4\wedge V$, and $\hat F = F^3\wedge\epsilon$ with $F^k\in\Lambda^k\mathbb{R}^8$ and $V\in\mathbb{R}^2$, separately.

First, we consider $\hat F = F$ and get

$$\begin{aligned}
b_{\hat\Theta}(F)_{lmnop} &= \hat\Theta_{ijk[lmn}F^{ijk}{}_{op]}\\
&= 15\Theta_{[ijka}\epsilon_{bc]}F^{ijk}{}_{de}\delta^{abcde}_{lmnop} = 3\Theta_{ijk[l}\epsilon_{mn}F^{ijk}{}_{op]}\\
&= 3b_\Theta(F)_{[lmn}\epsilon_{op]} = \tfrac{3}{10}(d_\Theta(F)\wedge\epsilon)_{lmnop}\\
&= \tfrac{3}{10}(d_\Theta\otimes *)(F)_{lmnop}
\end{aligned}$$

Second, we insert $\hat F = F\wedge\epsilon$ which yields

$$\begin{aligned}
b_{\hat\Theta}(F\wedge\epsilon)_{lmnop} &= \hat\Theta_{ijk[lmn}(F\wedge\epsilon)^{ijk}{}_{op]}\\
&= 15\Theta_{[ijka}\epsilon_{bc]}(F\wedge\epsilon)^{ijk}{}_{de}\delta^{abcde}_{lmnop}\\
&= 30\big(\Theta^{ijk}{}_{a}\epsilon_{bc} + 3\Theta_{ab}{}^{ij}\epsilon^{k}{}_{c} - \Theta_{abc}{}^{i}\epsilon^{jk}\big)F_{[ijk}\epsilon_{de]}\delta^{abcde}_{lmnop}\\
&= 3\Theta_{ijk[l}\epsilon_{mn}F^{ijk}\epsilon_{op]} + 54\Theta_{ij[lm}F^{[ij}{}_{n}\epsilon^{k]}{}_{o}\epsilon_{p]k} - 9\Theta_{[lmn}{}^{i}\epsilon^{jk}F_{op][i}\epsilon_{jk]}\\
&= -3\Theta_{[lmn}{}^{i}\epsilon^{jk}F_{op]i}\epsilon_{jk} = 6\tilde{d}_\Theta(F)_{lmnop}\\
&= 6(\tilde{d}_\Theta\otimes *)(F\wedge\epsilon)_{lmnop}
\end{aligned}$$

Last but not least, for $\hat F = F\wedge V$ we get

$$\begin{aligned}
b_{\hat\Theta}(F\wedge V)_{lmnop} &= \hat\Theta_{ijk[lmn}(F\wedge V)^{ijk}{}_{op]}\\
&= 15\Theta_{[ijka}\epsilon_{bc]}(F\wedge V)^{ijk}{}_{de}\delta^{abcde}_{lmnop}\\
&= 15\big(\Theta^{ijk}{}_{a}\epsilon_{bc} + 3\Theta_{ab}{}^{ij}\epsilon^{k}{}_{c} - \Theta_{abc}{}^{i}\epsilon^{jk}\big)F_{[ijkd}V_{e]}\delta^{abcde}_{lmnop}\\
&= \tfrac{3}{2}\big(4\Theta^{ijk}{}_{a}\epsilon_{bc}F_{[ijk]d}V_e + 18\Theta_{ab}{}^{ij}\epsilon^{k}{}_{c}F_{de[ij}V_{k]}\big)\delta^{abcde}_{lmnop}\\
&= 9\Theta_{ij[lm}F^{ij}{}_{no}\epsilon^{k}{}_{p]}V_k = 9b_\Theta(F)_{[lmno}(*V)_{p]}\\
&= \tfrac{9}{5}(b_\Theta\otimes *)(F\wedge V)_{lmnop}
\end{aligned}$$

The result on the eigenspaces and eigenvalues may be checked by applying $b_{\hat\Theta}$ and using Lemma 24.

References

[1] D.V. Alekseevsky, V. Cortes, and C. Devchand. Partially-flat gauge fields on manifolds of dimension greater than four. `hep-th/0205030`, 2002.

[2] D.V. Alekseevsky, V. Cortes, and C. Devchand. Yang-Mills connections over manifolds with Grassmann structure. *J. Math. Phys.* 44 (2003), no. 12, 6047-6076.

[3] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A* 362 (1978), no. 1711, 425-461.

[4] Laurent Baulieu, Hiroaki Kanno, and I. M. Singer. Special quantum field theories in eight and other dimensions. *Commun. Math. Phys.* 194 (1998), no. 1, 149-175.
[5] Laurent Baulieu and Céline Laroche On generalized self-duality equations towards supersymmetric quantum field theories of forms. *Modern Phys. Lett. A* 13 (1998), no. 14, 11151132.
[6] Robert L. Bryant. Metrics with exeptional holonomy. Ann. Math. 126 (1987), no. 3, 525-576.
[7] E. Corrigan, P. Goddard, and A. Kent. Some comemnts on the ADHM construction in $4k$ dimensions. *Commun. Math. Phys.* 100 (1985), no. 1, 1-13.
[8] E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuyts. First-order equations for gauge fields in spaces of dimension greater than four. *Nucl. Phys. B* 214 (1983), 452-464.
[9] Y. Brihaye, C. Devchand, and J. Nuyts. Self-duality for eight-dimensional gauge theories. *Phys. Rev. D (3)* 32 (1985), no. 4, 990-994.
[10] C. Devchand, J. Nuyts, and G. Weingart. Matryoshka of Special Democratic Forms. *Comm. Math. Phys.* 293 (2010), no. 2, 545-562.
[11] M. Fernandez. A classification of Riemannian manifolds with structure group $Spin(7)$. *Ann. Mat. Pura Appl. (4)* 143 (1986), no. 1, 101-122.
[12] R. B. Brown and A. Gray. Vector cross products. *Comment. Math. Helv.* 42 (1967), 222-236.
[13] Jerome P. Gauntlett, Jan B. Gutowski, and Stathis Pakis. The geometry of $D = 11$ null Killing spinors. *J. High Energy Phys.* 2003, no. 12, 049, 29 pp.
[14] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest. Aspects of spinorial geometry. *Modern Phys. Lett. A* 22 (2007), no. 1, 116.
[15] F. Klinker. The torsion of spinor connection and related structures. *SIGMA Symmetry Integrability Geom. Methods Appl.* 2 (2006), Paper 077, 28 pp.
[16] Kazuhiko Koike and Itaru Terada. Young-Diagrammatic Methods for the Representation Theory of the Classical Groups of type B_n, C_n, D_n. *J. Algebra* 107 (1987), no. 2, 466-511.
[17] Dimitros Tsimpis. M-theory on eight-manifolds revisited: $\mathcal{N} = 1$ supersymmetry and generalized $Spin(7)$ structures. *J. High Energy Phys.* 2006, no. 4, 027, 26 pp.
[18] R. S. Ward. Completely solvable gauge-field equations in dimensions greater than four. *Nucl. Phys.* B236 (1984), 381-396.

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