technische universität
dortmund

The Richards equation with hysteresis and degenerate capillary pressure

Ben Schweizer

Preprint 2011-10 August 2011

Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

The Richards equation with hysteresis and degenerate capillary pressure

Ben Schweizer[1]

August 5, 2011

Abstract: We study the Richards equation with a dynamic capillary pressure, including hysteresis. We provide existence and approximation results for degenerate capillary pressure curves p_c, treating two cases. In the first case, the permeability function k can be degenerate, but the initial saturation does not take the critical values. In the second case, the permeability function k is strictly positive, but the capillary pressure function can be multi-valued. In both cases, the degenerate behavior of p_c leads to the physically desired uniform bounds for the saturation variable. Our approach exploits maximum principles and relies on the corresponding uniform bounds for pressure and saturation. A new compactness result for the saturation variable allows to take limits in nonlinear terms. The solution concept uses tools of convex analysis.

key-words: Non-equilibrium Richards equation, nonlinear pseudo-parabolic system, capillary hysteresis, maximum principle

mathematical subject classification: 76S05, 35K65

1 Introduction

We investigate the flow of two incompressible and immiscible phases in a porous medium. The principal modelling assumption, eventually leading us to the Richards equation, is that one of the two phases need not be modelled, its pressure is assumed to be constant. We denote by $\Omega \subset \mathbb{R}^n$ the bounded domain which is occupied by the porous material, $[0, T] \subset \mathbb{R}$ is the time interval of interest, we set $\Omega_T := \Omega \times (0, T)$. Denoting the pressure of the relevant fluid by $p : \Omega_T \to \mathbb{R}$ and its saturation by $s : \Omega_T \to [0, 1]$ (the volume fraction of pore space filled with this fluid), the combination of mass conservation and Darcy's law for the velocity yields

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + g]) \,. \tag{1.1}$$

In this equation, a normalization of porosity and density is performed, gravity acts in direction $-e_n$ such that the constant vector g points in direction $+e_n$. The permeability $k = k(s)$ is given as a function $k : [0, 1] \to [0, \infty)$, it may additionally depend explicitly on the spatial position $x \in \Omega$.

[1]Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, D-44227 Dortmund, Germany.

The interesting modelling problem regards the relation between saturation s and pressure p. When we assume a constant pressure for one of the two phases (typically the air), then p is given by the capillary pressure. A widely used model is the functional dependence and to demand $p = \varphi(s)$. We will use in the following a function $\varphi : [0, 1] \to \mathbb{R}$ to describe capillary pressure and avoid the letter p_c. This is done for two reasons: one is brevity of formulas, the other is that we prefer to work with monotonically increasing coefficient functions. In fact, the standard convention regarding the capillary pressure is slightly asymmetric: s is the water saturation, p_c is air pressure minus water pressure. This asymmetric definition leads to a monotonically decreasing function p_c, we prefer to work with the water pressure and use $\varphi(\theta) = -p_c(\theta)$.

When hysteresis and dynamical effects are relevant, one replaces the algebraic relation between p and s with a dynamic relation. We will study the following relation which includes dynamic capillary pressure and hysteresis,

$$\partial_t s \in \psi(p - \varphi(s)). \tag{1.2}$$

A relevant example for relation (1.2) is the following play-type hysteresis model with dynamical effects,

$$p \in \varphi(s) + \gamma \operatorname{sign}(\partial_t s) + \tau \partial_t s. \tag{1.3}$$

In this relation, sign is the multi-valued function with $\operatorname{sign}(\xi) := \pm 1$ for $\pm\xi > 0$ and $\operatorname{sign}(0) := [-1, 1]$. The numbers $\tau, \gamma \geq 0$ are parameters of the hysteresis relation, γ indicates the width of a hysteresis loop, τ indicates the relevant time scale in a dynamic adaption of the saturation. The model was suggested in [6] and receives considerable attention, compare e.g. [11, 14, 16]. An important feature of the hysteresis model is that it can explain fingering effects, see [22] and references therein. For $\tau > 0$, the multi-valued function $\xi \mapsto \tau\xi + \gamma\operatorname{sign}(\xi)$ can be inverted. If we denote the Lipschitz continuous inverse by $\psi : \mathbb{R} \to \mathbb{R}$, equation (1.3) transforms into (1.2).

In this contribution, we derive existence results for system (1.1)–(1.2). We only treat the case with a Lipschitz continuous function ψ, which means that τ must be positive in model (1.3). Our analysis is based on a maximum principle for system (1.1)–(1.2) and on compactness results for approximate solutions. We treat two different cases. (P_1) is a formally degenerate system in which the permeability vanishes in one point, but the saturation cannot reach the corresponding critical value. (P_2) is a degenerate system with positive permeability for all saturation values, but with a multi-valued capillary pressure function φ.

Comparison with the literature

The case of an algebraic relation between s and p. Even with an algebraic relation $p \in \varphi(s)$ instead of (1.2), i.e. in the case without dynamic effects and without hysteresis, the Richards equation is an interesting mathematical object due to the degenerate behavior of the permeability k and the capillary pressure φ. Typically, one assumes a vanishing permeability $k(s)$ for some value of s. Regarding the capillary pressure one often assumes $\varphi(s) \to \pm\infty$ for s tending to critical saturation values. Another choice, which is closer to the physical background, is to prescribe φ multi-valued in the critical points. In the first model, critical saturation values cannot be reached if the pressure is bounded. The model with a multi-valued function φ allows critical saturation values at finite pressure.

With an algebraic relation $p = \varphi(s)$ between p and s, the main tool for the analysis of the Richards equation is the Kirchhoff transformation. Constructing a primitive $\Phi_K : [0,1] \to \mathbb{R}$ satisfying $\Phi_K'(s) = k(s)\varphi'(s)$, the calculation $\nabla[\Phi_K(s)] = k(s)\varphi'(s)\nabla s = k(s)\nabla[\varphi(s)] = k(s)\nabla p$ allows to transform equation (1.1) into the system

$$\partial_t s = \Delta u + \nabla \cdot [k(s)g], \qquad u = \Phi_K(s).$$

If possible, one often inverts the monotonically increasing function Φ_K with $b = \Phi_K^{-1}$ and writes the system as $\partial_t[b(u)] = \Delta u + \nabla \cdot [k(b(u))g]$. Existence results are obtained for such equations in the classical articles [2] and [3], uniqueness is treated e.g. in [10, 18], physical outflow boundary conditions are treated e.g. in [1] and [20].

Concerning the analytical treatment of the system, the algebraic pressure-saturation relation has two advantages. One advantage is that information on time derivatives of s and information on spatial derivatives of p can be combined in order to conclude compactness results. The other advantage is that the limit problem can be formulated as above with the expression Δu, such that the equation is meaningful in the distributional sense for every locally integrable function u. For the dynamic capillary pressure, it can actually be difficult to give a meaningful definition of the velocity term $-k(s)\nabla p$.

At first sight, the inclusion of a time derivative in relation (1.2) seems to regularize the problem and one expects simpler existence results. In fact, a positive parameter τ allows to conclude stronger a priori estimates, we refer to (2.14) which provides an $L^2(\Omega_T)$-bound for $\partial_t s$. On the other hand, the Kirchhoff transformation is not available. For this reason, the dynamic capillarity makes compactness and existence results more difficult.

Hysteresis models and dynamic capillary pressure. The play-type hysteresis model is described in (1.3) with a singular ordinary differential equation. Even without the coupling to a partial differential equation, the functional analytic description of this hysteresis relation is interesting, we refer to [24] for the corresponding discussion. In both cases, the rate-independent case $\tau = 0$ and the rate-dependent case $\tau > 0$, the hysteresis relation may be considered as a functional relation $s(t) = \mathcal{B}(t, p|_{[0,t]})$, where $\mathcal{B}$ maps the history of the pressure values to a saturation value, where we assume that initial values s_0 are given. We must regard p as an input and s as an output, determined by (1.3) or by (1.2).

Concerning purely static hysteresis, which means $\tau = 0$ in the above model, an existence result for the Richards equation was provided in [19] under the assumption that the partial differential equation is linear, i.e. in the case that $k(.)$ is not depending on s and that $\varphi(.)$ is an affine function. For other hysteresis models, existence results for nonlinear Richards equations have been obtained in [4, 5].

Slightly more is known if the dynamic effect is included by assuming $\tau > 0$. In [16], an existence result was derived for this model, which includes static play-type hysteresis. The restriction of that result is that the permeability must be bounded from below by a positive number. We improve [16] in the direction that the function φ can be degenerate and multi-valued; one relevant consequence of this extension is that the saturation remains for all times in the physically relevant range, $0 \leq s \leq 1$.

Furthermore, at least in one of the two models, k may be degenerate in one point. We note that, on the other hand, the existence result of [16] is valid for initial and boundary conditions in the natural energy spaces, while we assume here more regularity in order to verify the maximum principle and in order to have easier compactness proofs. In the case $\tau > 0$ and with strictly positive permeability, even the two-phase flow equations can be treated by similar methods, see [15].

Degenerate permeability functions are treated in [8] and [17]. The situation in these articles can be compared to our problem (P_1), with the restriction that the special function $\psi = \mathrm{id}$ is analyzed, hence static hysteresis is not covered. In both contributions, critical saturation values are essentially excluded by an integral condition.

We are not aware of literature that could be compared to our analysis of problem (P_2). In that problem, the maximum principle does not exclude critical saturation values. The analysis of the system is more involved and a solution concept based on variational inequalities must be used. We investigate the case that the capillary pressure is multi-valued, but the permeability is not degenerate.

2 Preliminaries and main results

2.1 Assumptions on the coefficients

Initial and boundary conditions. The unknowns in the porous media model (1.1)–(1.2) are $s, p : \Omega \times [0, T) \to \mathbb{R}$. We prescribe initial values for the saturation s with a function $s_0 \in L^2(\Omega)$. Regarding boundary conditions, we assume that $\partial\Omega$ is decomposed as $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$ with Γ and Σ disjoint, relatively open subsets of $\partial\Omega$. We impose a homogeneous Neumann condition for p on Γ and a Dirichlet condition for p on Σ, for which we assume positivity of the Hausdorff measure, $\mathcal{H}^{n-1}(\Sigma) > 0$. The Dirichlet conditions on $\Sigma \times (0, T)$ are prescribed through a function $p_0 \in L^2(0, T; H^1(\Omega))$.

Coefficient functions. Given are coefficient functions of the form

$$\varphi \subset \mathbb{R} \times \mathbb{R} \text{ a maximal monotone graph,} \tag{2.1}$$

$$\psi : \mathbb{R} \to \mathbb{R} \text{ Lipschitz continuous, monotonically increasing, } \psi(0) = 0, \tag{2.2}$$

$$k : \mathbb{R} \to [0, \infty) \text{ Lipschitz continuous.} \tag{2.3}$$

We recall that $\varphi = \varphi(s)$ is a capillary pressure function, $k = k(s)$ is a permeability coefficient, physically relevant arguments are saturation values $s \in [0, 1]$. The function $\psi = \psi(\zeta)$ encodes the hysteretic behavior of the system, the argument ζ has the units of a pressure.

When we think of the play-type model (1.3), assumption (2.2) is satisfied if and only if $\tau > 0$. In this article, we study (1.1)–(1.2) in two settings, made precise below as problem (P_1) and problem (P_2). In problem (P_1), the permeability can vanish on an interval $s \in [0, a]$. On the other hand, the degeneracy of φ in $s = a$ has the effect that the saturation s does not reach the value a. In problem (P_2), we consider a capillary pressure function φ that is multi-valued in a. In this setting, the saturation s can take the value a, hence the degeneracy of φ is indeed visible in the evolution. In the setting of (P_2) we assume that the permeability is strictly positive.

$(\mathbf{P_1})$ The formally degenerate problem. We denote the interval of relevant saturation values by $[a,b] \subset [0,1]$ and assume that the function φ is singular in the end-points,

$$\begin{aligned} &\varphi \in C^1((a,b),\mathbb{R}) \text{ strictly increasing with} \\ &\varphi(\xi) \to -\infty \text{ for } \xi \searrow a \text{ and } \varphi(\xi) \to +\infty \text{ for } \xi \nearrow b, \end{aligned} \tag{2.4}$$

$$k(s) > 0 \; \forall\, s \in (a,b). \tag{2.5}$$

We assume that the initial saturation s_0 satisfies $a + \varepsilon \le s_0 \le b - \varepsilon$ on Ω for some $\varepsilon > 0$. With the help of a maximum principle we will show for problem (P_1) that the saturation $s = s(x,t)$ remains at a distance from the end-points of (a,b) for all times. In this sense, the problem is only formally degenerate. Once that the maximum principle is available, we can conclude the existence theorem essentially from results of [16].

To simplify notations, we will later identify a single-valued function $\varphi : (a,b) \to \mathbb{R}$ with the multi-valued function $(a,b) \ni s \mapsto \{\varphi(s)\} \subset \mathbb{R}$.

$(\mathbf{P_2})$ The problem with a multi-valued capillary pressure. We consider a multivalued function φ and assume for $[a,b] \subset [0,1]$

$$\begin{aligned} &\text{there exists } \tilde\varphi \in C^{0,1}([a,b],\mathbb{R}), \text{ strictly increasing, such that} \\ &\varphi(\xi) = \{\tilde\varphi(\xi)\} \; \forall \xi \in (a,b), \quad \varphi(a) = (-\infty, \tilde\varphi(a)], \quad \varphi(b) = [\tilde\varphi(b), \infty), \end{aligned} \tag{2.6}$$

$$k(s) > 0 \quad \forall\; s \in [a,b]. \tag{2.7}$$

We do not assume anything on initial and boundary values that prevents the saturation from taking one of the critical values a and b. Unfortunately, we must compensate this generality by the non-degeneracy assumption on the permeability, assumption (2.7) implies $k \ge \kappa_0$ on $[a,b]$ for some $\kappa_0 > 0$.

On the generality of the assumptions, further cases. Other relevant cases concern a mixed scenario with one behavior of φ at a and another behavior of φ at b, for example the van Genuchten model, where φ behaves at a as in (2.4), but it behaves at b as in (2.6). The methods of this article can be used also in such mixed cases.

For φ as in (2.4), we assume that the Dirichlet values of the pressure p_0 are uniformly bounded and that the initial saturation s_0 has a bounded corresponding pressure $\varphi(s_0)$. The case of a multi-valued capillary pressure as in (2.6) and $k(a) = 0$ is covered by our methods if the Dirichlet values of the pressure p_0 are contained in a compact subinterval of $(\tilde\varphi(a), \tilde\varphi(b))$ and if, additionally, the initial saturation s_0 has its values in a compact subinterval of (a,b). In this case, the critical values are not attained by the saturation.

The interval $[0,1]$ can be replaced by any other compact interval. Furthermore, in the existence results, the choice of the whole line $s \in \mathbb{R}$ as a domain for the coefficient functions is a possible choice; the existence results remain valid for monotone capillary pressure function $\varphi : \mathbb{R} \to \mathbb{R}$ as in (2.4) with $a = -\infty$ and $b = +\infty$.

Remarks on the physical interpretation of the multi-valued model $(\mathbf{P_2})$. The multi-valued capillary pressure function φ provides an adequate and elegant description of the physical situation in the case of extreme saturation values. This can be exploited also in the analysis of interface conditions, see [7, 9, 21]. The physical argument to use a multi-valued function φ is best described with the situation of full water saturation, $s \equiv 1$. In this situation, the pressure can be increased arbitrarily without changes in the water saturation. Multi-valued capillary pressure functions p_c can sometimes be avoided by using the inverse $S = (p_c)^{-1}$, which may have flat parts.

2.2 Existence result for problem $(\mathbf{P_1})$.

We search for $p \in L^2(0,T;H^1(\Omega))$ and use a standard weak solution concept.

Definition 2.1. *We say that a pair* (s,p) *with* $s : \Omega_T \to [a,b]$ *and* $p : \Omega_T \to \mathbb{R}$ *is a* weak solution *of* (1.1)–(1.2)*, if the following is satisfied.*

1. The functions have the regularity

$$p \in L^2(0,T;H^1(\Omega)), \quad s, \partial_t s \in L^2(0,T;L^2(\Omega)). \tag{2.8}$$

2. Relation (1.1) *and the no-flux condition are satisfied in the sense that*

$$\int_{\Omega_T} \partial_t s\, \phi + \int_{\Omega_T} k(s)[\nabla p + g]\nabla\phi = 0 \tag{2.9}$$

holds for every $\phi \in L^2(0,T;H^1(\Omega))$ *with* $\phi = 0$ *on* Σ.

3. The hysteresis relation (1.2)*,* $\partial_t s = \psi(p - \varphi(s))$*, holds pointwise a.e. in* Ω_T*.*
We furthermore demand that the initial and boundary conditions $s = s_0$ *on* $\Omega \times \{0\}$ *and* $p = p_0$ *on* $\Sigma \times (0,T)$ *are satisfied in the sense of traces.*

We now formulate with Theorem 2.2 our existence result for problem (P_1). In contrast to [17] and [8] we include static hysteresis. Furthermore, we exploit a maximum principle and can therefore obtain a slightly stronger solution concept.

Theorem 2.2 (Existence for the formally degenerate problem (P_1))**.** *Let* Ω *be a parallelepiped in* $\mathbb{R}^n$ *with* $n \leq 3$*, let* Γ *be a union of sides of* Ω*. Let* $T > 0$ *and let the coefficients* φ*,* k*, and* ψ *satisfy the general assumptions* (2.1)–(2.3) *and the* (P_1)*-assumptions* (2.4)–(2.5)*. Let initial and boundary data be given by* $s_0 \in H^1(\Omega,(a,b))$ *and* $p_0, \partial_t p_0 \in L^2(0,T;H^1(\Omega))$ *with*

$$p_0 \in L^\infty(\Omega_T), \quad \varphi(s_0) \in L^\infty(\Omega). \tag{2.10}$$

Then there exists a weak solution (s,p) *to* (1.1)–(1.2) *as described in Definition 2.1.*

Theorem 2.2 is shown in several steps. We consider a regularized system with a small parameter $\delta > 0$ in Subsection 3.1. For the regularized system we derive a maximum principle in Subsection 3.2. We apply results of [16] to conclude the existence of solutions to the regularized system. A compactness lemma, shown in Subsection 3.3, allows to perform the limit $\delta \to 0$. Theorem 2.2 is concluded in Subsection 4.1.

Remarks on the assumptions of Theorem 2.2. The theorem is formulated only for parallelepipeds Ω. This geometric restriction is made only for one reason, namely in order to use elliptic regularity results. With those results we verify the regularity of solutions to the regularized system. The elliptic regularity results remain valid also for bounded domains Ω with boundary of class $C^{2,\alpha}$. Furthermore, since also regularizations of the domain can be considered, it would be sufficient to assume that Ω can be approximated by such $C^{2,\alpha}$-domains Ω_δ. Special care must be taken regarding the subsets Γ and Σ of $\partial\Omega$. We need that Dirichlet- Neumann-problems can be solved by smooth functions on Ω or, at least, on regularized domains Ω_δ with boundary parts Γ_δ and Σ_δ.

2.3 Existence result for problem $(\mathbf{P_2})$.

For a multi-valued capillary pressure function φ, the formulation of the hysteresis relation $\partial_t s \in \psi(p - \varphi(s))$ of (1.2) is not trivial. We will demand that there exists a scalar field $\rho = \rho(x,t)$ with $\rho \in \varphi(s)$ such that $\partial_t s = \psi(p-\rho)$ holds in a weak sense.

The latter condition is formulated with tools of convex analysis. We denote the primitive of ψ by $F : \mathbb{R} \to \mathbb{R}$; more precisely, we demand $F' \equiv \partial F = \psi$ and $F(0) = 0$. Here ∂ denotes the subdifferential. We furthermore use the convex conjugate of the convex function F, the function $F^* : \mathbb{R} \to \mathbb{R}$, $F^*(\sigma) := \sup_{p\in\mathbb{R}} \{\sigma \cdot p - F(p)\}$. The evolution equation reads $\partial_t s \in \partial F(p-\rho)$ and is, by the Fenchel relations, equivalent to $F(p-\rho) + F^*(\partial_t s) \le (p-\rho)\,\partial_t s$.

Additionally, we use in Definition 2.3 a primitive $\tilde\Phi$ of $\tilde\varphi$. More precisely, let $\tilde\Phi : [a,b] \to \mathbb{R}$ be the convex and differentiable function with $\tilde\Phi'(s) = \tilde\varphi(s)$ for all $s \in (a,b)$, normalized with $\tilde\Phi((a+b)/2) = 0$.

Definition 2.3. *We say that a triple (s,p,ρ) with $s : \Omega_T \to [a,b]$, $p,\rho : \Omega_T \to \mathbb{R}$ is a* variational weak solution *to* (1.1)–(1.2)*, if the following conditions are satisfied.*

1. *The functions have the regularity*

$$p \in L^2(0,T;H^1(\Omega)), \quad s, \partial_t s \in L^2(0,T;L^2(\Omega)), \quad \rho \in L^\infty(0,T;L^2(\Omega)). \tag{2.11}$$

2. $\partial_t s = \nabla\cdot(k(s)[\nabla p + g])$ *and the no-flux condition hold in the weak sense of* (2.9)*.*

3. $\rho(x,t) \in \varphi(s(x,t))$ *holds a.e. on* Ω_T

4. *The variational inequality*

$$\begin{aligned} 0 \ge & \int_\Omega \tilde\Phi(s(x,t))\,dx\Big|_{t=0}^{t=T} - \int_0^T\int_\Omega p_0(x,t)\,\partial_t s(x,t)\,dx\,dt \\ &+ \int_0^T\int_\Omega \{F(p(x,t)-\rho(x,t)) + F^*(\partial_t s(x,t))\}\,dx\,dt \\ &+ \int_0^T\int_\Omega k(x,s(x,t))\,(\nabla p(x,t)+g)\,\nabla(p(x,t)-p_0(x,t))\,dx\,dt \end{aligned} \tag{2.12}$$

holds.

Concerning initial and boundary values we assume $s = s_0$ on $\Omega\times\{0\}$ and $p = p_0$ on $\partial\Omega\times(0,T)$ in the sense of traces.

In order to justify our solution concept, we show that, under an additional regularity assumption, every variational weak solution satisfies relation (1.2) almost everywhere.

Lemma 2.4. *Let $\varphi = \tilde{\varphi}$ be a single-valued, Lipschitz continuous function and let (s,p) be a weak solution of problem* (1.1)–(1.2) *in the sense of Definition 2.1. Then the triple $(s,p,\varphi(s))$ is a variational weak solution.*

Vice versa, let φ satisfy condition (2.6) *and let (s,p,ρ) be a variational weak solution as described in Definition 2.3. We assume that the additional regularity $\rho \in L^2(0,T;H^1(\Omega))$ is satisfied. Then the hysteresis evolution relation $\partial_t s = \psi(p-\rho)$ is satisfied almost everywhere.*

Proof. The function $\rho = \varphi(s)$ has the integrability properties of s, hence item 1 of Definition 2.1 is satisfied. The weak formulation of the evolution equation (1.1) is identical in both solution concepts, hence item 2 is satisfied. By the choice of ρ, item 3 is trivially satisfied. Concerning item 4, it suffices to use $\phi := p - p_0$ as a test-function in (2.9), the regularity (2.8) allows to use this test-function. We recognize immediately the second and the fourth integral of (2.12). In order to recognize the term $\int p\,\partial_t s$, we have to exploit the Fenchel inequality. The hysteresis relation (1.2), i.e. $\partial_t s \in \partial F(p-\varphi(s))$, is satisfied almost everywhere by the weak solution, hence there also holds $F(p-\varphi(s)) + F^*(\partial_t s) \le (p-\varphi(s))\partial_t s$. This allows to calculate

$$\int_0^T\int_\Omega p\,\partial_t s \ge \int_0^T\int_\Omega \varphi(s)\,\partial_t s + \int_0^T\int_\Omega \left\{F(p-\varphi(s)) + F^*(\partial_t s)\right\}.$$

The chain rule can be applied to the primitive $\tilde{\Phi}$ of $\varphi = \tilde{\varphi}$ with argument s. This provides the variational inequality (2.12) and shows that every weak solution is a variational weak solution.

In order to show the opposite implication, let now (s,p,ρ) be a variational weak solution with the additional regularity $\rho \in L^2(0,T;H^1(\Omega))$. We use the sets $M_a := \{(x,t) \in \Omega_T | s(x,t) = a\}$ and $M_b := \{(x,t) \in \Omega_T | s(x,t) = b\}$. The Lemma of Stampacchia and Fubini's theorem imply $\partial_t s = 0$ on $M_a \cup M_b$, we therefore have

$$\int_{\Omega_T} (\rho - \tilde{\varphi}(s))\partial_t s = \int_{\Omega_T\setminus(M_a\cup M_b)} (\rho - \tilde{\varphi}(s))\partial_t s = 0,$$

since by property 3 of variational weak solutions $\rho - \tilde{\varphi}(s) = 0$ holds on $\Omega_T \setminus (M_a \cup M_b)$. We use this observation in order to write the first integral of the variational inequality (2.12) as

$$\int_\Omega \tilde{\Phi}(s)\Big|_{t=0}^{t=T} = \int_{\Omega_T} \partial_t[\tilde{\Phi}(s)] = \int_{\Omega_T} \tilde{\varphi}(s)\partial_t s = \int_{\Omega_T} \rho\partial_t s.$$

The last integral of (2.12) can be integrated by parts, which transforms the integrand into $-\partial_t s\,(p-p_0)$. The two integrals over $\pm\partial_t s\,p_0$ cancel and (2.12) reads

$$0 \ge \int_0^T\int_\Omega (\rho - p)\,\partial_t s + \int_0^T\int_\Omega \left\{F(p-\rho) + F^*(\partial_t s)\right\}.$$

The definition of the Fenchel conjugate implies $F(p-\rho) + F^*(\partial_t s) \ge (p-\rho)\,\partial_t s$. Therefore, the integral inequality is indeed an equality and, as a consequence, the equality $F(p-\rho) + F^*(\partial_t s) = (p-\rho)\,\partial_t s$ is satisfied pointwise almost everywhere. As observed before, this equality implies the inclusion $\partial_t s \in \partial F(p-\rho)$ almost everywhere, which is identical to $\partial_t s = \psi(p-\rho)$. □

Remark. Lemma 2.4 essentially implies that items 2-4 of Definition 2.3 encode the hysteresis relation (1.2). An interesing aspect of this observation is that the multi-valued behavior of φ in the end-points a and b does not appear in the variational inequality; only the regular part $\tilde{\varphi}$ of φ appears through its primitive $\tilde{\Phi}$ in (2.12).

Theorem 2.5 (Existence for the degenerate problem (P_2))**.** *Let Ω be a parallelepiped in $\mathbb{R}^n$ with $n \le 3$, let Γ be a union of sides of Ω. Let $T > 0$ and let the coefficients φ, k, and ψ satisfy the general assumptions* (2.1)–(2.3) *and the* (P_2)*-assumptions* (2.6)–(2.7)*. Let initial and boundary data be given by $s_0 \in H^1(\Omega)$, $s_0 : \Omega \to [a, b]$, and $p_0 \in L^2(0, T; H^1(\Omega)) \cap L^\infty(\Omega_T)$. Then there exists a variational weak solution to* (1.1)–(1.2) *as described in Definition 2.3.*

Theorem 2.5 is shown in Section 4, Subsection 4.2. The proof uses Theorem 2.2, which provides the existence of solutions to a regularized system with capillary pressure functions φ_δ. Theorem 2.5 follows by performing the limit procedure $\delta \to 0$, exploiting a maximum principle and a compactness result.

The regularity assumption $s_0 \in H^1(\Omega)$ of Theorem 2.5 is not essential. The assumption is made here in order have with Lemma 3.3 a simple compactness proof for the saturation variable.

A heuristic uniqueness argument. We made the step from a single-valued to a multi-valued function φ. For a single-valued function φ, the relation $\rho(x, t) \in \varphi(s(x, t))$ of Definition 2.3 determines ρ once that s is known. By contrast, for a multi-valued function φ, the relation $\rho(x, t) \in \varphi(s(x, t))$ leaves more freedom for the function ρ. We should convince ourselves, that we can still expect uniqueness of solutions in the multi-valued case.

Let us give a heuristic argument that uniqueness should hold also in the multi-valued case, at least for strictly increasing ψ. Given $s(., t)$, a monotone elliptic relation such as $\Delta p - \psi(p - \rho) = 0$ defines a map $\rho \mapsto p$. In particular, given $\rho(., t)$, the evolution equation $\partial_t s = \Delta p$ determines uniquely a saturation increment. The increment of ρ is determined by two relations: if the saturation values are not extreme, $s \neq a$ and $s \neq b$, then we must satisfy $\rho = \tilde{\varphi}(s)$, which determines the ρ-increment. If, on the other hand, s takes an extreme value, we expect $0 = \partial_t s = \psi(p - \rho)$ and are forced to set $\rho = p$.

2.4 Natural function spaces

Before we start the rigorous analysis of system (1.1)–(1.2), we present the corresponding a priori energy estimate. Following the usual pathway to existence results, we will define in Section 3 a set of regularized differential equations, solve this approximate system, and obtain a solution of the original problem as a weak limit of the approximate solutions. The appropriate function spaces in this process are dictated by the formal energy estimate. We note that the energy estimate could also be obtained from inequality (2.12), but we present here the calculation from scratch rather than analyzing the integrand $F^*(\partial_t s)$.

A multiplication of (1.1) with $p - p_0$ and an integration over Ω provides

$$\int_\Omega (p - p_0)\, \partial_t s + \int_\Omega k(s)[\nabla p + g]\nabla(p - p_0) = 0. \tag{2.13}$$

The second integral can provide an estimate for $|\nabla p|^2$. In the first integral we write

$$\int_\Omega p\,\partial_t s = \int_\Omega (p-\varphi(s))\,\partial_t s + \int_\Omega \varphi(s)\,\partial_t s$$
$$= \int_\Omega (p-\varphi(s))\,\psi(p-\varphi(s)) + \int_\Omega \partial_t[\Phi(s)],$$

where $\Phi = \tilde{\Phi}$ is the primitive with $\Phi' = \varphi$. We omit the tilde symbol since we think of problem (P_1) here.

We can now exploit the Lipschitz continuity of ψ. Denoting the Lipschitz constant by $1/\tau_0$, using monotonicity of ψ and $\psi(0)=0$, we find $|\psi(\zeta)| \le \tau_0^{-1}|\zeta|$ and $\zeta\,\psi(\zeta) \ge \tau_0|\psi(\zeta)|^2$ for every $\zeta\in\mathbb{R}$. This allows to calculate

$$\int_\Omega p\,\partial_t s \ge \tau_0 \int_\Omega |\psi(p-\varphi(s))|^2 + \partial_t \int_\Omega \Phi(s) = \tau_0 \int_\Omega |\partial_t s|^2 + \partial_t \int_\Omega \Phi(s).$$

With this observation, we have recognized several positive terms in equation (2.13). We obtain

$$\tau_0 \int_\Omega |\partial_t s|^2 + \partial_t \int_\Omega \Phi(s) + \int_\Omega k(s)|\nabla p|^2 \le \int_\Omega \left\{ p_0 \partial_t s + k(s)[\nabla p + g]\nabla p_0 - k(s) g \nabla p \right\}.$$

We finally integrate over $t\in[0,T]$. On the data we assume $p_0 \in L^2(0,T;H^1(\Omega))$ and that s_0 satisfies $\Phi(s_0)\in L^1(\Omega)$. On the coefficient functions we assume boundedness of k and that Φ is bounded from below. With the usual application of the Cauchy-Schwarz and the Young inequality we obtain the energy estimate

$$\int_{\Omega_T} k(s)|\nabla p|^2 + \tau_0|\partial_t s|^2 \le C_0. \tag{2.14}$$

In this estimate, the constant C_0 depends on g, p_0, and s_0. It is independent of the shape of φ and ψ, it only depends on the general properties that are listed in (2.1)–(2.3).

The energy estimate (2.14) is very valuable. The estimate holds also for a regularized system, independent of the regularization parameter δ. We have therefore spatial estimates for p and temporal estimates for s at our disposal. On the other hand, the estimate for the pressure contains the factor $k(s)$, which can be small in degenerate systems. Furthermore, we lack spatial regularity for the saturation variable s. This latter problem can be compensated by (1.2), which allows to derive spatial regularity of s from spatial regularity of p.

We believe that, without a positive lower bound for the permeability k, it is not possible to obtain compactness for families $s\in L^2(\Omega_T)$ based only on (2.14) and (1.2).

3 Regularization, maximum principle, compactness

3.1 The regularized system

Our aim is to derive, in addition to the energy estimate (2.14), uniform estimates for solutions with a maximum principle. Since we use a geometric approach to the maximum principle, we need regular solutions to the system, say of class C^2. As we will see below, smooth solutions can be obtained, at least if the coefficient functions φ, ψ, and k, and the data p_0, s_0, and g are sufficiently smooth.

We choose a sequence $\delta = \delta_j \searrow 0$ for $j \in \mathbb{N}$ and use δ as an index for regularized functions. On the regularized coefficient functions we assume

$$k_\delta \in C^\infty(\mathbb{R}, (0,\infty)), \quad k_\delta \geq \delta. \tag{3.1}$$

$$\varphi_\delta : \mathbb{R} \to \mathbb{R} \text{ is } C^\infty \text{ with } \partial_s \varphi_\delta \geq \delta, \tag{3.2}$$

$$\psi_\delta : \mathbb{R} \to \mathbb{R} \text{ is } C^\infty \text{ with } \partial_\varsigma \psi_\delta \geq \delta\,. \tag{3.3}$$

We assume that these new coefficient functions are approximations in the sense that

$$k_\delta \to k \text{ uniformly on } \mathbb{R}, \tag{3.4}$$

$$\varphi_\delta \to \varphi \text{ uniformly on compact subsets of } (a,b), \tag{3.5}$$

$$\varphi_\delta(\xi) \to \infty \text{ for } \xi > b \text{ and } \varphi_\delta(\xi) \to -\infty \text{ for } \xi < a, \tag{3.6}$$

$$\psi_\delta \to \psi \text{ uniformly on compact subsets of } \mathbb{R}\,. \tag{3.7}$$

For simplicity of notation we assume here that the data p_0, s_0 and g are given smooth functions. The general case can be treated with an additional regularization.

Lemma 3.1 (Existence and regularity for the regularized system)**.** *Let $\Omega \subset \mathbb{R}^n$ and $\Gamma \subset \partial\Omega$ be as in Theorem 2.2 and let the regularized coefficients satisfy* (3.1)–(3.7)*. Then the system*

$$\partial_t s^\delta = \nabla \cdot \big(k_\delta(s^\delta)[\nabla p^\delta + g]\big) \tag{3.8}$$

$$\partial_t s^\delta = \psi_\delta(p^\delta - \varphi_\delta(s^\delta)) \tag{3.9}$$

together with smooth initial and boundary data, posesses a classical solution (s^δ, p^δ) of class $C^{2,\alpha}(\Omega_T)$. The solutions satisfy the energy estimate

$$\int_{\Omega_T} k_\delta(s^\delta)|\nabla p^\delta|^2 + \tau_0|\partial_t s^\delta|^2 \leq C_0, \tag{3.10}$$

where C_0 does not depend on δ.

Proof. Step 1. Weak solutions and initial regularity. The existence of a weak solution (s^δ, p^δ) was shown in [16] with the help of a Galerkin discretization. In that work, a spatial discretization with parameter $h > 0$ is introduced and discrete solutions $(s^{\delta,h}, p^{\delta,h})$ are defined. These solutions satisfy the uniform energy estimate (3.10), which is identical to the formally derived estimate (2.14). By strict positivity of k_δ, the energy bound provides an h-independent estimate for $p^{\delta,h} \in L^2(0,T;H^1(\Omega))$. The essential step is then a compactness result. The hysteresis relation (3.9) transmits spatial regularity of $p^{\delta,h}$ to $s^{\delta,h}$ and permits to conclude the pre-compactness of the sequence $s^{\delta,h} \in L^2(\Omega_T)$. From the strong convergence $s^{\delta,h} \to s^\delta$ for $h \to 0$ one concludes that every limit function (s^δ, p^δ) is indeed a weak solution of the hysteresis system.

To be precise, we mention that [16] is concerned with the special function ψ corresponding to relation (1.3). But, as noted in [16] before the main theorem, a general Lipschitz continuous function ψ_δ can be treated as well, at least concerning results on a priori estimates and compactness. An additional regularity is observed in Section 4 of [16]: for regular initial and boundary data an energy type estimate can be obtained for time derivatives of the solution. Equation (4.4) of [16] provides

$$\|\partial_t p^\delta\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t^2 s^\delta\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{3.11}$$

where $C \in \mathbb{R}$ depends on Ω, T, and δ. This estimate is derived in the above mentioned equation (4.4) for the discrete solutions $(s^{\delta,h}, p^{\delta,h})$ with C independent of h. We note that the estimate actually implies directly the compactness statement, additionally the pre-compactness of $p^{\delta,h}$, and hence the solution property for (s^δ, p^δ). In particular, the existence of a solution (s^δ, p^δ) satisfying (3.11) follows from [16] also for general ψ_δ as in (3.3).

Step 2. Hölder regularity. In order to conclude additional regularity of solutions, we next derive Hölder estimates. We will make use of deep regularity results for elliptic equations. In order to obtain an initial regularity, we consider (3.8) for every $t \in [0,T]$ as an elliptic equation on Ω,

$$\nabla \cdot \big(k(s^\delta)[\nabla p^\delta + g]\big) = f := \partial_t s^\delta(.,t). \tag{3.12}$$

We want to show that the right hand side f is bounded in some space $L^q(\Omega)$ with $q > n$. Such an estimate is not contained in (3.11).

In up to three space dimensions, $n \leq 3$, the embedding $H^1(\Omega) \subset L^q(\Omega)$ is continuous for $q = 4$. From (3.11) and this embedding, we infer a t-independent bound for the pressure $p^\delta(.,t) \in L^q(\Omega)$. The evolution equation (3.9) transfers this regularity estimate to the saturation s^δ and its time derivative, compare Lemma 3.3 of [16]. We find that both $s^\delta(.,t)$ and $\partial_t s^\delta(.,t)$ are uniformly bounded in $L^q(\Omega)$. This provides the boundedness of $f \in L^q(\Omega)$, independent of $t \in [0,T]$.

With this observation, we can now exploit the fundamental regularity result of De Giorgi for elliptic equations without a continuity assumption on the coefficients. In (3.12), the coefficient $k(s^\delta(.,t))$ is measurable, bounded from above and strictly positive. A De Giorgi result as in [12] provides an estimate for the solution p^δ in the space $C^\alpha(\Omega)$ for some $\alpha > 0$. Since we treat the case with $f \neq 0$, we must use the inhomogeneous result of Stampacchia, see [23], Theorem 4.2. In this step, the inequality $q > n$ and the scalar character of the equation is exploited.

Step 3. Classical solutions. At this point, we have a t-independent estimate for $p^\delta(.,t) \in C^\alpha(\Omega)$. Once more, for smooth initial data s_0, the evolution equation (3.9) transfers this estimate to the saturation s^δ, a direct argument exploiting the theory of ordinary differential equations shows this result also in Hölder spaces. Accordingly, we now consider with (3.12) an equation in which the right hand side and the coefficients $k(s^\delta(.,t))$ are Hölder continuous with uniform bounds. This allows to use regularity estimates for systems in divergence form, we refer to Giaquinta, [13] Chapter III, Theorem 3.2 and the comments on global estimates after the theorem. We infer a t-independent estimate for $p^\delta(.,t) \in C^{1,\alpha}(\Omega)$.

We can now iterate the arguments and improve the regularity to arbitrary order. The $L^\infty([0,T], C^{1,\alpha}(\Omega))$-regularity of p^δ implies through (3.9) the same regularity for s^δ and $\partial_t s^\delta$. With the help of [13], Chapter III, Theorem 3.3 we achieve an arbitrary order of regularity. This implies the claim, the existence of a solution (s^δ, p^δ) of class $C^{2,\alpha}(\Omega_T)$ to the regularized system (3.8)–(3.9).

Step 4. Remarks on boundary conditions. We used three classical theorems on elliptic regularity. None of these theorems treats mixed boundary conditions. We therefore restricted this discussion to parallelepipeds Ω with homogeneous Neumann conditions on entire sides. For such domains, symmetric extension of the solution across Neumann sides allows to treat Neumann boundary points as inner points. $\square$

3.2 The maximum principle

Lemma 3.2 below provides a maximum principle for smooth solutions of (1.1)–(1.2).

Let us note already here that the maximum principle can be transferred to weak solutions of (1.1)–(1.2). We regularize the system as above, using the regularization parameter $\delta > 0$. By Lemma 3.1 the regularized system posesses a smooth solution (s^δ, p^δ). Lemma 3.2 provides uniform bounds for p^δ and $\varphi_\delta(s^\delta)$. Performing the limit $\delta \to 0$, we obtain weak solutions (s, p) as limits of (s^δ, p^δ). The uniform bounds for the regular solutions remain valid also for weak limits, hence we obtain a maximum principle for every weak solution of the original problem that was obtained with a regularization procedure.

In order to avoid the sub- and superscipts δ, we assume in this subsection that $k \in C^\infty(\mathbb{R}, (0,\infty))$ is strictly positive and that $\varphi, \psi \in C^\infty(\mathbb{R}, \mathbb{R})$ have positive lower bounds for the derivatives.

Lemma 3.2 (Maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded with piecewise C^1-boundary and let $[0,T]$ be a time interval. We assume that the coefficient functions are as specified in* (2.1)*–*(2.3)*, with strict inequalities $k, \partial_s\varphi, \partial_\zeta\psi > 0$. Let the coefficient functions, the initial and the boundary data be such that there exists a solution (s, p) of system* (1.1)*–*(1.2) *of class $C^2(\bar\Omega_T)$. Let $M > 0$ be a constant such that*

$$|p_0(x,t) + g\cdot x| < M \ \forall (x,t) \in \bar\Omega_T,$$
$$|\varphi(s_0(x)) + g\cdot x| < M \ \forall x \in \bar\Omega.$$

Then there holds, for all $(x,t) \in \Omega_T$,

$$|p(x,t) + g\cdot x| < M,$$
$$|\varphi(s(x,t)) + g\cdot x| < M.$$

Proof. The estimate for p. We use the geometric approach to prove the maximum principle. We introduce the functions $\hat p(x,t) := p(x,t) + g\cdot x$ and $\hat p_0(x,t) := p_0(x,t) + g\cdot x$. We perform the calculations for the upper bounds, the lower bounds are derived in an analogous fashion.

For a contradiction argument we consider the first time instance $t_0 \in [0,T]$ such that $\max_{x\in\Omega} \hat p(x,t_0) = M$. We choose one maximum $x_0 \in \bar\Omega$ such that $\hat p(x_0,t_0) = M$.

Claim A: $\partial_t s(x_0,t_0)$ is positive. We derive the positivity of the time derivative from (1.2). In the case $t_0 = 0$, it suffices to calculate $\partial_t s(x_0,t_0) = \psi(p - \varphi(s))(x_0,t_0) = \psi(M - g\cdot x_0 - \varphi(s_0(x))) > \psi(0) = 0$.

In the case $t_0 > 0$ we consider the segment $\{x_0\} \times [0,t_0] \subset \bar\Omega_T$. We claim that $\varphi(s(x_0,t)) + g\cdot x_0 < M$ holds for all $t \in [0,t_0]$. Indeed, for $t = 0$ holds $\varphi(s(x_0,t)) + g\cdot x_0 = \varphi(s_0(x_0)) + g\cdot x_0 < M$ by assumption. Let now $t_1 \in (0,t_0]$ be the minimal time instance with $\varphi(s(x_0,t_1)) + g\cdot x_0 = M$. In this point holds $\partial_t s(x_0,t_1) = \psi(p - \varphi(s))(x_0,t_1) =$

$\psi(\hat{p}(x_0,t_1) - g \cdot x_0 - \varphi(s(x_0,t_1))) = \psi(\hat{p}(x_0,t_1) - M) < \psi(0) = 0$, in contradiction to minimality of t_1. This shows $\varphi(s(x_0,t)) + g \cdot x_0 < M$ for all $t \in [0,t_0]$.

We can now use this property to calculate $\partial_t s(x_0,t_0) = \psi(p-\varphi(s))(x_0,t_0) > \psi(M - g \cdot x_0 - M + g \cdot x_0) = \psi(0) = 0$. This shows Claim A.

Conclusion of the contradiction argument. Using Claim A, we now continue the analysis of the point (x_0,t_0). Equation (1.1) in the point (x_0,t_0) reads

$$\partial_t s(x_0,t_0) = \nabla \cdot (k(s)\nabla\hat{p})(x_0,t_0),$$

and we recall that x_0 is, by construction, a maximum of $\hat{p}(.,t_0)$.

Case 1. x_0 is an interior point, i.e. $x_0 \in \Omega$. In this case, the geometrical condition of a maximum yields $\nabla \cdot (k(s)\nabla\hat{p})(x_0,t_0) \le 0$ for the right hand side. This is in contradiction with Claim A that provides the positivity $\partial_t s(x_0,t_0) > 0$ of the left hand side.

Case 2. $x_0 \in \partial\Omega$ is a boundary point. We note that x_0 cannot lie on the Dirichlet boundary or its closure. This is a consequence of our assumption on the boundary values, $\hat{p}_0 < M$ on $\bar{\Omega}_T$.

It remains to study the case that $x_0 \in \Gamma$ is a point on the Neumann boundary. Regarding this case we first observe that, by continuity of $\partial_t s$, the positivity $\partial_t s > 0$ holds also in a neighborhood of (x_0,t_0). This implies the positivity $\nabla \cdot (k(s)\nabla\hat{p}) > 0$ in this neighborhood, hence $\hat{p}(.,t_0)$ is a subsolution for the elliptic operator $L = -\nabla \cdot (k(s(.,t_0))\nabla$ in this neighborhood, $L\hat{p}(.,t_0) \le 0$. We can therefore apply the Hopf lemma for subsolutions, treating strict maxima on the boundary. We conclude for the exterior normal vector ν the inequality $\nu \cdot \nabla\hat{p}(x_0,t_0) > 0$. This is in contradiction with the homogeneous Neumann condition $\nu \cdot k(s)\nabla\hat{p} = 0$ on $\Gamma \times [0,T]$.

We found a contradiction to the assumption that $\hat{p}(x_0,t_0) = M$. This contradiction implies $\hat{p} < M$.

The estimate for $\varphi(s)$. The estimate for $\varphi(s)$ follows with a similar contradiction argument. We denote by $t_1 \in (0,T]$ the first time instance such that $\varphi(s(x,t)) + g \cdot x = M$ for some point $x \in \Omega$. In such a point $x = x_0$ we can calculate, using the uniform pressure estimate of the first step, $\partial_t s = \psi(p - \varphi(s)) < \psi(M - g \cdot x_0 - M + g \cdot x_0) = 0$. This provides the contradiction to minimality of t_1. □

3.3 Compactness

Compactness for the saturation in problems (P_1) and (P_2)

Lemma 3.3 (Saturation compactness)**.** *Let $\Omega \subset \mathbb{R}^n$ be as in Lemma 3.2, $T > 0$ and $C_0, \kappa_0 > 0$ real numbers. Let the family of coefficient functions φ_δ, k_δ, and ψ_δ satisfy* (3.1)–(3.3) *for $\delta = \delta_j \to 0$. Let p_0^δ and s_0^δ be regularized boundary data with $p_0^\delta \to p_0$ in $L^2(0,T;H^1(\Omega))$ and $s_0^\delta \to s_0$ in $H^1(\Omega)$. We assume that p_0^δ and $\varphi_\delta(s_0^\delta)$ are uniformly bounded by C_0. Let (s^δ,p^δ) be classical solutions of class $C^2(\bar{\Omega}_T)$ to the regularized system* (3.8)–(3.9)*. We assume that for all $\delta = \delta_j$ holds*

$$0 < \partial_\zeta\psi_\delta(\zeta) \le C_0 \;\forall\zeta \in \mathbb{R}, \quad 0 < \partial_s\varphi_\delta(s) \;\forall s \in \mathbb{R}, \quad 0 < \kappa_0 \le k_\delta(s^\delta) \text{ on } \Omega_T. \tag{3.13}$$

Then the solution sequence has the property that

$$(s^\delta)_\delta \text{ is pre-compact in } L^2(0,T;L^2(\Omega)). \tag{3.14}$$

We recall that the regular solution exists by Lemma 3.1 for parallelepipeds Ω. Every solution (s^δ, p^δ) satisfies uniform bounds by Lemma 3.2.

Proof. We first consider the time derivative $\partial_t s^\delta$. We note that uniform bounds for $\partial_t s^\delta \in L^2(\Omega_T)$ could be concluded from the energy estimate (3.10) (which holds also for general Lipschitz domains), but this estimate can be improved to uniform bounds in the situation of this lemma. We use the hysteresis relation (3.9), which reads

$$\partial_t s^\delta = \psi_\delta(p^\delta - \varphi_\delta(s^\delta)), \tag{3.15}$$

and combine it with the maximum principle. Lemma 3.2 implies that the right hand side of (3.15) is uniformly bounded, hence we find boundedness of $\|\partial_t s^\delta\|_{L^\infty(\Omega_T)}$.

In order to show the compactness statement (3.14), we want to derive additionally spatial regularity estimates for s^δ. We take the gradient of both sides of (3.15), which is possible by our regularity assumptions. We obtain

$$\partial_t \nabla s^\delta = (\partial_\zeta \psi_\delta)|_{(p^\delta - \varphi_\delta(s^\delta))} \cdot (\nabla p^\delta - (\partial_s \varphi_\delta)|_{s^\delta} \nabla s^\delta). \tag{3.16}$$

We multiply this relation with ∇s^δ. Using $0 \le \partial_\zeta \psi_\delta \le C_0$ and $0 \le \partial_s \varphi_\delta$, we obtain

$$\begin{aligned} \partial_t \frac{1}{2} \int_\Omega |\nabla s^\delta|^2 &= \int_\Omega \partial_t \nabla s^\delta \cdot \nabla s^\delta = \int_\Omega (\partial_\zeta \psi_\delta)|_{p^\delta - \varphi_\delta(s^\delta)} \cdot (\nabla p^\delta - (\partial_s \varphi_\delta)|_{s^\delta} \nabla s^\delta) \cdot \nabla s^\delta \\ &\le C \int_\Omega \nabla p^\delta \cdot \nabla s^\delta \le C \|\nabla p^\delta\|_{L^2(\Omega)} \|\nabla s^\delta\|_{L^2(\Omega)} . \end{aligned}$$

We obtain that $y_\delta(t) := \left(\int_\Omega |\nabla s^\delta(.,t)|^2\right)^{1/2}$ satisfies

$$y_\delta(0) = \|\nabla s_0^\delta\|_{L^2(\Omega)} \le C, \qquad \partial_t y_\delta(t) \le C \|\nabla p^\delta\|_{L^2(\Omega)} .$$

Because of the lower bound $k_\delta(s^\delta) \ge \kappa_0 > 0$ along solutions, the energy estimate (3.10) implies the boundedness of $\nabla p^\delta \in L^2(\Omega_T)$. Therefore, the map $t \mapsto \|\nabla p^\delta\|_{L^2(\Omega)}$ is bounded in $L^1(0,T;\mathbb{R})$, and we conclude

$$\sup_{t\in[0,T]} \int_\Omega |\nabla s^\delta|^2 \le C. \tag{3.17}$$

Since both the temporal derivatives $\partial_t s^\delta$ and the spatial derivatives ∇s^δ are bounded in $L^2(\Omega_T)$, the Rellich embedding theorem provides the pre-compactness of the family s^δ in $L^2(\Omega_T)$. □

Compactness for the pressure in the formally degenerate problem (P_1)

Our second compactness result concerns the pressure variable in problem (P_1). Even though we consider, in general, permeabilities k that take the value 0 in one point, we are, effectively, in a non-degenerate situation. This is a consequence of the maximum principle and the behavior of φ. The maximum principle implies that the pressure function p and the expression $\varphi(s)$ are uniformly bounded for solutions (s,p). As a consequence, the degeneracy of φ implies that the saturation s stays away from the critical values. This leads to the fact that, along solutions, the derivative $\partial_s \varphi(s)$ is bounded and the function $k(s)$ is strictly positive.

Once this effective non-degeneracy is shown, we can derive higher order estimates for the pressure. These estimates imply immediately the compactness. We remark that analogous higher order estimates are also observed in Proposition 4.2 of [16] and that they have already been used for fixed $\delta > 0$ in the regularity proof of Lemma 3.1.

Lemma 3.4 (Pressure compactness for (P_1))**.** *Let the situation be as in Theorem 2.2, in particular with the formally degenerate coefficient functions of* (P_1)*. Let the regularizations of the coefficient functions be as in* (3.1)–(3.7) *for* $\delta = \delta_j \to 0$*. Let the regularized boundary data satisfy* $p_0^\delta \to p_0$ *and* $\partial_t p_0^\delta \to \partial_t p_0$ *in* $L^2(0,T;H^1(\Omega))$*, and* $s_0^\delta \to s_0$ *in* $H^1(\Omega)$*, furthermore we assume that* p_0^δ *and* $\varphi_\delta(s_0^\delta)$ *are uniformly bounded.*

On the regularized coefficients we additionally assume that for every $\varepsilon > 0$ *there exist constants* $C, C_\varepsilon > 0$ *such that*

$$0 < \partial_s \varphi_\delta(\xi) \le C_\varepsilon \quad \forall \delta = \delta_j, \xi \in [a+\varepsilon, b-\varepsilon], \tag{3.18}$$

$$0 < \partial_\zeta \psi_\delta(\zeta) \le C \quad \forall \delta = \delta_j, \zeta \in \mathbb{R}, \tag{3.19}$$

$$0 < k_\delta(s), \partial_s k_\delta(s) \le C \quad \forall \delta = \delta_j, s \in [a,b]. \tag{3.20}$$

Let (s^δ, p^δ) *be classical solutions of class* $C^2(\bar\Omega_T)$ *to the regularized system* (3.8)–(3.9)*, which exist by Lemma 3.1 and which satisfy uniform bounds by Lemma 3.2. Then*

$$(p^\delta)_\delta \text{ is pre-compact in } L^2(0,T;L^2(\Omega)). \tag{3.21}$$

We note already here that arbitrary coefficient functions φ, ψ, k of problem (P_1) can be approximated such that the regularized function $\varphi_\delta, \psi_\delta, k_\delta$ satisfy the conditions (3.1)–(3.7) and (3.18)–(3.20).

Proof. We derive an higher order a priori estimate for p^δ essentially by testing the time derivative of the evolution equation with time derivatives of p^δ. Since the gravity force g does not depend on t, the time derivative of (3.8) is

$$\partial_t^2 s^\delta = \nabla \cdot \left(k_\delta(s^\delta) \nabla \partial_t p^\delta + \partial_s k_\delta(s^\delta) \partial_t s^\delta [\nabla p^\delta + g] \right). \tag{3.22}$$

We recall that the solutions are classical solutions such that all expressions are meaningful. The second derivative $\partial_t^2 s^\delta$ on the left hand side can also be evaluated using the time derivative of (3.9),

$$\partial_t^2 s^\delta = \partial_\zeta \psi_\delta|_{(p^\delta - \varphi_\delta(s^\delta))} \cdot (\partial_t p^\delta - \partial_s \varphi_\delta(s^\delta) \cdot \partial_t s^\delta). \tag{3.23}$$

Multiplication of (3.22) with $\partial_t p^\delta - \partial_t p_0^\delta$ provides, exploiting (3.23) and omitting the arguments of the coefficient functions,

$$\begin{aligned} &\int_{\Omega_T} k_\delta |\nabla \partial_t p^\delta|^2 + \int_{\Omega_T} (\nabla \partial_t p^\delta - \nabla \partial_t p_0^\delta) \cdot \partial_s k_\delta \, \partial_t s^\delta \, [\nabla p^\delta + g] \\ &\quad - \int_{\Omega_T} k_\delta \nabla \partial_t p^\delta \cdot \nabla \partial_t p_0^\delta + \int_{\Omega_T} \partial_\zeta \psi_\delta \cdot (\partial_t p^\delta - \partial_s \varphi_\delta \cdot \partial_t s^\delta) \, (\partial_t p^\delta - \partial_t p_0^\delta) = 0. \end{aligned} \tag{3.24}$$

This equation can provide an estimate for $\nabla \partial_t p^\delta \in L^2(\Omega_T)$ in the case that $\partial_s \varphi_\delta$ is bounded. In the situation of this lemma, the functions p^δ and $\varphi_\delta(s^\delta)$ are uniformly bounded by the maximum principle. The degeneracy (2.4) of φ at a and b together with the approximation property (3.5) implies that, for some $\varepsilon > 0$, the values of s^δ are restricted to the interval $[a+\varepsilon, b-\varepsilon]$. Therefore, (3.18) implies the uniform bound $\partial_s \varphi_\delta(s^\delta) \le C_\varepsilon$. Using additionally the non-negativity of the integrand $\partial_\zeta \psi_\delta |\partial_t p^\delta|^2$ and the strict positivity $k_\delta(s^\delta) \ge \kappa_\varepsilon > 0$, which follows from (2.5) and (3.4), we obtain

$$\kappa_\varepsilon \int_{\Omega_T} |\nabla \partial_t p^\delta|^2 \le \int_{\Omega_T} |\nabla \partial_t p^\delta - \nabla \partial_t p_0^\delta| \cdot |\partial_s k_\delta| \, |\partial_t s^\delta| \, |\nabla p^\delta + g| \tag{3.25}$$
$$+ \int_{\Omega_T} k_\delta |\nabla \partial_t p^\delta| \, |\nabla \partial_t p_0^\delta| + C\, C_\varepsilon \int_{\Omega_T} |\partial_t s^\delta| \, |\partial_t p^\delta| + \int_{\Omega_T} |\partial_t p^\delta - \partial_s \varphi_\delta \cdot \partial_t s^\delta| \, |\partial_t p_0^\delta|.$$

Regarding the different expressions on the right hand side we observe that k_δ and $\partial_s k_\delta$ are bounded, by assumption the boundary values are bounded functions $\nabla \partial_t p_0^\delta \in L^2(\Omega_T)$. Furthermore, the solution (s^δ, p^δ) satisfies the a priori estimates in energy spaces as in (3.10), i.e. uniform bounds for $\nabla p^\delta \in L^2(\Omega_T)$ and $\partial_t s^\delta \in L^2(\Omega_T)$.

In order to treat the right hand side of the above inequality, we must make an additional observation. The maximum principle of Lemma 3.2 provides uniform bounds for p^δ and $\varphi_\delta(s^\delta)$. This can be exploited in the evolution equation $\partial_t s^\delta = \psi_\delta(p^\delta - \varphi_\delta(s^\delta))$. The uniform global Lipschitz constant for ψ_δ implies that also $\partial_t s^\delta$ is uniformly bounded, $\|\partial_t s^\delta\|_{L^\infty(\Omega_T)} \le C$. With this additional information, exploiting the Cauchy-Schwarz and the Poincaré inequality, we conclude from (3.25)

$$\kappa_\varepsilon \int_{\Omega_T} |\nabla \partial_t p^\delta|^2 \le C + C \|\nabla \partial_t p^\delta\|_{L^2(\Omega_T)}.$$

We use Young's inequality and absorb terms into the left hand side to obtain

$$\int_{\Omega_T} |\nabla \partial_t p^\delta|^2 \le C, \tag{3.26}$$

with a constant that is independent of δ. This implies, in particular, the compactness of the sequence p^δ in $L^2(\Omega_T)$. □

4 Derivation of the existence results

4.1 Existence in the formally degenerate case ($\mathbf{P_1}$)

Proof of Theorem 2.2. Let the situation be as described in Theorem 2.2. We approximate the coefficient functions as in Subsection 3.1 with coefficient functions φ_δ, k_δ, and ψ_δ as in (3.4)–(3.7). We can choose regularizations that satisfy the assumptions of Lemma 3.4. Furthermore, we approximate p_0 and s_0 by smooth functions p_0^δ and $s_0^\delta \to s_0$ as required in Lemma 3.4. The resulting regularized system has a smooth solution (s^δ, p^δ) by Lemma 3.1. The energy estimate (3.10) allows to choose a subsequence $\delta \to 0$ and limit functions such that

$$(s^\delta, \partial_t s^\delta, p^\delta, \nabla p^\delta) \rightharpoonup (s, \partial_t s, p, \nabla p) \quad \text{in } L^2(\Omega_T). \tag{4.1}$$

The weak convergence of ∇p^δ and $\partial_t s^\delta$ implies that the initial and boundary conditions are satisfied in the sense of traces by the limit functions.

We have additional information on the approximating sequence: the maximum principle of Lemma 3.2 provides a constant $M_0 > 0$ such that $|p^\delta| \le M_0$ and $|\varphi_\delta(s^\delta)| \le M_0$ on $\bar\Omega_T$, independent of δ. As in the last proof, the singularity property (2.4) for φ then provides uniform bounds for s^δ: For some $\varepsilon > 0$ there holds $s^\delta \in [a+\varepsilon, b-\varepsilon]$ on $\bar\Omega_T$ for all δ. As a consequence, the permeability $k_\delta(s^\delta)$ is bounded from below by some constant $\kappa_\varepsilon > 0$ and $\partial_s\varphi_\delta(s^\delta)$ is uniformly bounded.

This observation allows to apply the compactness results of Lemma 3.3 and Lemma 3.4. We conclude that $(s^\delta, p^\delta) \to (s,p)$ strongly in $L^2(\Omega_T)$ and, upon choice of a subsequence, pointwise almost everywhere. This allows to take the weak limit $k_\delta(s^\delta)\nabla p^\delta \rightharpoonup k(s)\nabla p$ in (1.1). Furthermore, using once more the maximum principle, it implies the strong convergence $\varphi_\delta(s^\delta) \to \varphi(s)$. This strong convergence and the strong convergence of p^δ, together with the uniform convergence $\psi_\delta \to \psi$ on compact sets allows to perform the limit in relation (3.9); we obtain that (1.2) holds pointwise almost everywhere. This verifies the weak solution properties of the limit pair (s,p) and concludes the proof of Theorem 2.2. □

4.2 Existence in the multi-valued case ($\mathbf{P_2}$)

In the multi-valued case, the saturation values a and b may be attained. This has the consequence that we have no uniform bound for $\partial_s\varphi$ along solutions. A compactness result for the pressure functions p^δ is not available, Lemma 3.4 can not be applied.

Our way to circumvent this problem is to use a weaker solution concept in the multi-valued case. We introduced variational weak solutions in Definition 2.3; our aim is now to derive that every weak limit of regularized solutions is a variational weak solution.

Proof of Theorem 2.5. Let the situation be as described in Theorem 2.5, φ_δ, k_δ, and ψ_δ approximations of the coefficient functions as in (3.4)–(3.7), satisfying additionally the assumptions of Lemma 3.3. The boundary data p_0 and s_0 are approximated by $p_0^\delta \to p_0$ in $L^2(0,T;H^1(\Omega))$ and $s_0^\delta \to s_0$ in $H^1(\Omega)$ as before. The regularized system has a smooth solution (s^δ, p^δ). As in the last proof, we choose a subsequence and a limit pair (s,p) such that (4.1) holds. We recall that the maximum principle of Lemma 3.2 provides $M_0 > 0$ such that $|p^\delta| \le M_0$ and $|\varphi_\delta(s^\delta)| \le M_0$ holds. In particular, we can assume on the regularization ψ_δ that $\psi = \psi_\delta$ outside of a compact subset of $\mathbb{R}$. Lemma 3.3 can be applied and provides the strong convergence $s^\delta \to s$ in $L^2(\Omega_T)$ and we can assume the convergence pointwise almost everywhere along the subsequence.

For a further subsequence we can define ρ as a limit of $\varphi_\delta(s^\delta)$, more precisely $\varphi_\delta(s^\delta) \to \rho \in L^\infty(\Omega_T)$ weak-$*$. We have to verify items 1-4 of Definition 2.3 of variational weak solutions.

Item 1. By construction as weak limits, the function spaces for the limit functions p, s, and ρ are as required.

Item 2. As in the last proof, we have to verify (2.9), the weak form of the evolution (1.1). The verification of this relation poses no problems. We fix a test-function $\phi \in L^2(0,T;H^1(\Omega))$ with $\phi = 0$ on Σ. The regularized solutions satisfy

$$\int_{\Omega_T} \partial_t s^\delta\, \phi + \int_{\Omega_T} k_\delta(s^\delta)[\nabla p^\delta + g]\nabla\phi = 0. \tag{4.2}$$

The weak convergences $\partial_t s^\delta \rightharpoonup \partial_t s$ and $\nabla p^\delta \rightharpoonup \nabla p$ together with the strong convergence of s^δ implies that the limit functions satisfy (2.9).

Item 3. We must verify that the inclusion $\rho(x,t) \in \varphi(s(x,t))$ holds almost everywhere on Ω_T. We derive this inclusion easily from $\rho \leftharpoonup \rho^\delta := \varphi_\delta(s^\delta)$ weakly in $L^2(\Omega_T)$, the strong convergence $s^\delta \to s$ and the uniform convergence $\varphi_\delta \to \varphi$ on compact subsets of (a,b). We present here an elementary proof without reference to theory on maximal monotone functions. For a subsequence, we have the pointwise convergence $s^\delta(x,t) \to s(x,t)$ for almost every (x,t). By the Theorem of Egorov, the convergence is uniform on arbitrarily large subsets of Ω_T.

In a first step, we consider the set $A_\varepsilon = \{(x,t) \in \Omega_T | s(x,t) \in [a+\varepsilon, b-\varepsilon]\}$. On an arbitrarily large subset of $\tilde{A}_\varepsilon \subset A_\varepsilon$, there holds $s^\delta(x,t) \in [a+\varepsilon/2, b-\varepsilon/2]$ for all sufficiently small δ. We therefore find the pointwise convergence $\varphi_\delta(s^\delta) \to \tilde{\varphi}(s)$ on this set. For an arbitrary test-function $\phi \in C_c^\infty(\Omega_T)$ we can calculate with the Lebesgue convergence theorem

$$\int_{\tilde{A}_\varepsilon} \rho\phi \leftarrow \int_{\tilde{A}_\varepsilon} \rho^\delta \phi = \int_{\tilde{A}_\varepsilon} \varphi_\delta(s^\delta)\phi \to \int_{\tilde{A}_\varepsilon} \tilde{\varphi}(s)\phi.$$

Since the measure of $A_\varepsilon \setminus \tilde{A}_\varepsilon$ is arbitrarily small there holds $\rho = \tilde{\varphi}(s)$ almost everywhere on A_ε. Since $\varepsilon > 0$ was arbitrary, we conclude $\rho = \tilde{\varphi}(s)$ almost everywhere on $\{(x,t) | s(x,t) \in (a,b)\}$.

We next consider points $B := \{(x,t) \in \Omega_T | s(x,t) = b\}$. We consider a subset $\tilde{B} \subset B$ such that uniform convergence $s^\delta(x,t) \to s(x,t)$ holds on $\tilde{B}$. The set $B \setminus \tilde{B}$ can be chosen arbitrarily small. For arbitrary $\varepsilon > 0$, we find $\varphi_\delta(s^\delta) \geq \tilde{\varphi}(b) - \varepsilon$ on the set $\tilde{B}$ for every sufficiently small δ. This provides, in the limit $\delta \to 0$, the relation $\rho \geq \tilde{\varphi}(b) - \varepsilon$ on $\tilde{B}$. Since $\varepsilon > 0$ was arbitrary and the subset $\tilde{B}$ can be chosen large, we conclude $\rho \geq \tilde{\varphi}(b)$ on B, hence $\rho \in \varphi(s)$ on B. Since the point $s = a$ can be treated in the same way, we conclude $\rho \in \varphi(s)$ on Ω_T. This concludes the property of item 3.

Item 4. It remains to derive the variational inequality (2.12). Since (s^δ, p^δ) is a weak solution of the regularized problem, by Lemma 2.4, the pair is also a variational weak solution. In particular, denoting the primitive of φ_δ by Φ_δ and the primitive of ψ_δ by F_δ, there holds, with $\rho^\delta = \varphi_\delta(s^\delta)$, the variational inequality

$$\begin{aligned} 0 \geq \int_\Omega \Phi_\delta(s^\delta(x,t))\,dx\Big|_{t=0}^{t=T} &- \int_0^T \int_\Omega p_0^\delta(x,t)\,\partial_t s^\delta(x,t)\,dx\,dt \\ &+ \int_0^T \int_\Omega \left\{F_\delta(p^\delta - \rho^\delta) + F_\delta^*(\partial_t s^\delta(x,t))\right\}\,dx\,dt \\ &+ \int_0^T \int_\Omega k_\delta(x, s^\delta(x,t))\left(\nabla p^\delta(x,t) + g\right)\nabla(p^\delta(x,t) - p_0^\delta(x,t))\,dx\,dt. \end{aligned} \tag{4.3}$$

In some of the integrals of (4.3), the limit can be performed directly by the weak convergences of $\partial_t s^\delta$ and ∇p^δ and the strong convergence of s^δ and, hence, of $k_\delta(s^\delta)$. This concerns the second integral and all terms of the last integral, except for the quadratic term in the pressure.

In other integrals, we have to exploit weak lower semi-continuity. We claim that

$$\liminf_{\delta} \int_{\Omega_T} F_\delta(p^\delta - \rho^\delta) + F_\delta^*(\partial_t s^\delta) + k_\delta(s^\delta)|\nabla p^\delta|^2 \\ \geq \int_{\Omega_T} F(p-\rho) + F^*(\partial_t s) + k(s)|\nabla p|^2 \,. \tag{4.4}$$

Inequality (4.4) is a consequence of the weak convergences of $\partial_t s^\delta$, p^δ, ∇p^δ, and ρ^δ in $L^2(\Omega_T)$, and of the convexity of the functions $F, F^* : \mathbb{R} \to \mathbb{R}$ and $|.|^2 : \mathbb{R}^n \to \mathbb{R}$. Concerning the δ-dependence of the coefficients we can exploit the uniform convergences $F_\delta \to F$ and $F_\delta^* \to F^*$. Concerning the permeability, we use the pointwise convergence of s^δ, which implies the pointwise convergence $0 \leq k_\delta(s^\delta) \to k(s)$. Using once more Egorov's theorem, we conclude the lower semi-continuity of the pressure-term and obtain (4.4).

It remains to verify, for fixed $t \in [0,T]$ and as $\delta \to 0$,

$$\liminf_{\delta} \int_\Omega \Phi_\delta(s^\delta(x,t))\, dx \geq \int_\Omega \tilde{\Phi}(s(x,t))\, dx, \tag{4.5}$$

with equality for $t = 0$. Due to the $L^2(\Omega_T)$ estimates for $\partial_t s^\delta$ and the weak convergence in this space, the functions $s^\delta(.,t)$ and $s(.,t)$ are well-defined in the space $L^2(\Omega)$ in the sense of traces, and we have $s^\delta(.,t) \to s(.,t)$ weakly in $L^2(\Omega)$. In the special point $t = 0$, the convergence is strong, $s^\delta(.,0) = s_0^\delta \to s_0 = s(.,0)$ strongly in $L^2(\Omega)$.

Relation (4.5) will be a consequence of the following claim for arbitrary numbers $M_0 > 0$ and $\varepsilon > 0$:

$$\exists \delta_0 > 0\ \forall \delta \in (0,\delta_0), \xi \in \mathbb{R} : \qquad |\varphi_\delta(\xi)| \leq M_0 \Rightarrow |\Phi_\delta(\xi) - \tilde{\Phi}(\xi)| \leq \varepsilon. \tag{4.6}$$

Let us first show that property (4.6) implies (4.5). We choose $M_0 > 0$ from the maximum principle such that $|\varphi_\delta(s^\delta)| \leq M_0$ holds for all solutions of the regularized equation. The error $\varepsilon > 0$ is fixed arbitrarily small. We observe that the condition of (4.6) is always satisfied for $\xi = s^\delta(x,t)$, hence we can replace the function Φ_δ by the function $\tilde{\Phi}$, introducing only a small error of order ε on the left hand side of (4.5). The function $\tilde{\Phi}$ is convex, hence the lower semi-continuity is immediate. In the case $t = 0$, we exploit again that Φ_δ can be replaced by $\tilde{\Phi}$ with a small error and the strong convergence of $s^\delta(.,0)$ in $L^2(\Omega)$. This provides equality in (4.5) for $t = 0$.

It remains to prove property (4.6). With $\varepsilon > 0$ and $M_0 > 0$ given, we set $\eta := \min\{\varepsilon/(4M_0), \varepsilon/(4\|\tilde{\varphi}\|_\infty)\}$. We choose $\delta_0 > 0$ such that $|\Phi_\delta(\xi) - \tilde{\Phi}(\xi)| \leq \varepsilon/2$ for all $\xi \in [a+\eta, b-\eta]$, which is possible by uniform convergence $\varphi_\delta \to \varphi$ on compact subintervals of (a,b), compare (3.5). We can now check the assertion of (4.6) by distinguishing two cases for the argument $\xi \in \mathbb{R}$. For $\xi \in [a+\eta, b-\eta]$, the smallness holds by our choice of δ_0. For $\xi > b - \eta$ we calculate, using $\varphi_\delta(s) \leq \varphi_\delta(\xi) \leq M_0$ for every $s \in [b-\eta, \xi]$,

$$\begin{aligned} |\Phi_\delta(\xi) - \tilde{\Phi}(\xi)| &\leq |\Phi_\delta(\xi) - \Phi_\delta(b-\eta)| + |\Phi_\delta(b-\eta) - \tilde{\Phi}(b-\eta)| + |\tilde{\Phi}(b-\eta) - \tilde{\Phi}(\xi)| \\ &\leq M_0\eta + \frac{\varepsilon}{2} + \|\tilde{\varphi}\|_\infty \eta \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Arguments $\xi < a + \eta$ are treated in an analogous way. This shows (4.6) and hence the variational inequality for the limit functions.

The maximum principle implies the boundedness of $\varphi_\delta(s^\delta)$. On the other hand, the regularization was chosen in such a way that $|\varphi_\delta(\xi)| \to \infty$ for $\xi \notin [a,b]$, compare (3.6). This shows that the limit function s takes only values in $[a,b]$. The proof of Theorem 2.5 is complete. □

Concluding remarks

We have shown an existence result for the hysteresis system in two settings, one allows degeneracy of k in one point, the other allows to use the physically relevant multi-valued capillary pressure functions. For the second setting, a weak solution concept was introduced with the help of a variational inequality. The solution property for limit functions is verified with a compactness property of approximate saturation values s^δ.

We do not have a solution concept that can be used without a strong convergence property of the saturations in $L^2(\Omega_T)$. This is one of the reasons why we cannot treat the case $k(a) = 0$ in the multi-valued setting.

Acknowledgement

Helpful discussions with Christof Melcher on elliptic regularity issues are gratefully acknowledged. This work has been supported by DFG-grant SCHW 639/3-1.

References

[1] H. W. Alt and E. DiBenedetto. Nonsteady flow of water and oil through inhomogeneous porous media. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):335–392, 1985.

[2] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[3] H. W. Alt, S. Luckhaus, and A. Visintin. On nonstationary flow through porous media. *Ann. Mat. Pura Appl. (4)*, 136:303–316, 1984.

[4] F. Bagagiolo and A. Visintin. Hysteresis in filtration through porous media. *Z. Anal. Anwendungen*, 19(4):977–997, 2000.

[5] F. Bagagiolo and A. Visintin. Porous media filtration with hysteresis. *Adv. Math. Sci. Appl.*, 14(2):379–403, 2004.

[6] A. Y. Beliaev and S. M. Hassanizadeh. A theoretical model of hysteresis and dynamic effects in the capillary relation for two-phase flow in porous media. *Transp. Porous Media*, 43(3):487–510, 2001.

[7] F. Buzzi, M. Lenzinger, and B. Schweizer. Interface conditions for degenerate two-phase flow equations in one space dimension. *Analysis (Munich)*, 29(3):299–316, 2009.

[8] C. Cancès, C. Choquet, Y. Fan, and I. Pop. Existence of weak solutions to a degenerate pseudo-parabolic equation modeling two-phase flow in porous media. CASA-Report 10-75, 2010.

[9] C. Cancès and M. Pierre. An existence result for multidimensional immiscible two-phase flows with discontinuous capillary pressure field. HAL :hal-00518219, 2011.

[10] J. Carrillo and P. Wittbold. Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems. *J. Differential Equations*, 156(1):93–121, 1999.

[11] C. Cuesta and J. Hulshof. A model problem for groundwater flow with dynamic capillary pressure: stability of travelling waves. *Nonlinear Anal.*, 52(4):1199–1218, 2003.

[12] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[13] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, volume 105 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1983.

[14] R. Helmig, A. Weiss, and B. Wohlmuth. Dynamic capillary effects in heterogeneous porous media. *Comput. Geosci.*, 11(3):261–274, 2007.

[15] J. Koch, A. Rätz, and B. Schweizer. Two-phase flow equations with a dynamic capillary pressure. Preprint TU Dortmund, in preparation, 2011.

[16] A. Lamacz, A. Rätz, and B. Schweizer. A well-posed hysteresis model for flows in porous media and applications to fingering effects. *Adv. Math. Sci. Appl.*, 2011.

[17] A. Mikelić. A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure. *J. Differential Equations*, 248(6):1561–1577, 2010.

[18] F. Otto. L^1-contraction and uniqueness for unstationary saturated-unsaturated porous media flow. *Adv. Math. Sci. Appl.*, 7(2):537–553, 1997.

[19] B. Schweizer. Averaging of flows with capillary hysteresis in stochastic porous media. *European J. Appl. Math.*, 18(3):389–415, 2007.

[20] B. Schweizer. Regularization of outflow problems in unsaturated porous media with dry regions. *J. Differential Equations*, 237(2):278–306, 2007.

[21] B. Schweizer. Homogenization of degenerate two-phase flow equations with oil trapping. *SIAM J. Math. Anal.*, 39(6):1740–1763, 2008.

[22] B. Schweizer. Instability of gravity wetting fronts for richards equations with hysteresis. Preprint TU Dortmund, 2010.

[23] G. Stampacchia. Problemi al contorno ellitici, con dati discontinui, dotati di soluzionie hölderiane. *Ann. Mat. Pura Appl. (4)*, 51:1–37, 1960.

[24] A. Visintin. *Differential models of hysteresis*, volume 111 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1994.

Preprints ab 2009/10

2011-10 **Ben Schweizer**
The Richards equation with hysteresis and degenerate capillary pressure

2011-09 **Andreas Rätz and Matthias Röger**
Turing instabilities in a mathematical model for signaling networks

2011-08 **Matthias Röger and Reiner Schätzle**
Control of the isoperimetric deficit by the Willmore deficit

2011-07 **Frank Klinker**
Generalized duality for k-forms

2011-06 **Sebastian Aland, Andreas Rätz, Matthias Röger, and Axel Voigt**
Buckling instability of viral capsides - a continuum approach

2011-05 **Wilfried Hazod**
The concentration function problem for locally compact groups revisited: Non-dissipating space-time random walks, τ-decomposable laws and their continuous time analogues

2011-04 **Wilfried Hazod, Katrin Kosfeld**
Multiple decomposability of probabilities on contractible locally compact groups

2011-03 **Alexandra Monzner* and Frol Zapolsky†**
A comparison of symplectic homogenization and Calabi quasi-states

2011-02 **Stefan Jäschke, Karl Friedrich Siburg and Pavel A. Stoimenov**
Modelling dependence of extreme events in energy markets using tail copulas

2011-01 **Ben Schweizer and Marco Veneroni**
The needle problem approach to non-periodic homogenization

2010-16 **Sebastian Engelke and Jeannette H.C. Woerner**
A unifying approach to fractional Lévy processes

2010-15 **Alexander Schnurr and Jeannette H.C. Woerner**
Well-balanced Lévy Driven Ornstein-Uhlenbeck Processes

2010-14 **Lorenz J. Schwachhöfer**
On the Solvability of the Transvection group of Extrinsic
Symplectic Symmetric Spaces

2010-13 **Marco Veneroni**
Stochastic homogenization of subdifferential inclusions via scale integration

2010-12 **Agnes Lamacz, Andreas Rätz, and Ben Schweizer**
A well-posed hysteresis model for flows in porous media and
applications to fingering effects

2010-11 **Luca Lussardi and Annibale Magni**
Γ-limits of convolution functionals

2010-10 **Patrick W. Dondl, Luca Mugnai, and Matthias Röger**
Confined elastic curves

2010-09 **Matthias Röger and Hendrik Weber**
Tightness for a stochastic Allen–Cahn equation

2010-08 **Michael Voit**
Multidimensional Heisenberg convolutions and product formulas
for multivariate Laguerre polynomials

2010-07 **Ben Schweizer**
Instability of gravity wetting fronts for Richards equations with hysteresis

2010-06 **Lorenz J. Schwachhöfer**
Holonomy Groups and Algebras

2010-05 **Agnes Lamacz**
Dispersive effective models for waves in heterogeneous media

2010-04 **Ben Schweizer and Marco Veneroni**
Periodic homogenization of Prandtl-Reuss plasticity equations in arbitrary dimension

2010-03 **Holger Dette and Karl Friedrich Siburg and Pavel A. Stoimenov**
A copula-based nonparametric measure of regression dependence

2010-02 **René L. Schilling and Alexander Schnurr**
The Symbol Associated with the Solution of a Stochastic Differential Equation

2010-01 **Henryk Zähle**
Rates of almost sure convergence of plug-in estimates for distortion risk measures

2009-16 **Lorenz J. Schwachhöfer**
Nonnegative curvature on disk bundles

2009-15 **Iuliu Pop and Ben Schweizer**
Regularization schemes for degenerate Richards equations and outflow conditions

2009-14 **Guy Bouchitté and Ben Schweizer**
Cloaking of small objects by anomalous localized resonance

2009-13 **Tom Krantz, Lorenz J. Schwachhöfer**
Extrinsically Immersed Symplectic Symmetric Spaces

2009-12 **Alexander Kaplun**
Continuous time Ehrenfest process in term structure modelling

2009-11 **Henryk Zähle**
Ein aktuarielles Modell für die Portabilität der Alterungsrückstellungen
in der PKV

2009-10 **Andreas Neuenkirch and Henryk Zähle**
Asymptotic error distribution of the Euler method for SDEs with
non-Lipschitz coefficients