Waves in heterogeneous media: Long time behavior and dispersive models

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Introduction

1 Introduction to homogenization theory

The development of the mathematical theory of *homogenization* is strongly related to the requirement to describe the behavior of composite materials.

Composite materials consist of two or more individual constituents. They are finely mixed and look almost homogeneous from a macroscopic point of view. On a much smaller microscopic scale, the ingredients are separated. In other words, the heterogeneities of a composite determine a specific length scale, the microscopic scale, which is very small compared to the global dimension of the material, which in turn characterizes the macroscopic scale.

According to the fact that composite materials in general exhibit better properties than their ingredients, they are widely used in industry, see for instance pavement in roadways or superconducting multi filamentary composites in optical fibers. However, especially from the numerical point of view the small heterogeneities are very hard to treat. They produce a wide range of fluctuations and oscillations, which considerably affect the global behavior of the material.

The aim of the *homogenization* theory is to describe the global properties of a given composite medium. More precisely, the aim is to replace the highly oscillating characteristics of the composite in question by constant, usually referred to as *effective* quantities, which in turn correspond to a homogeneous material, called the effective material.

In what follows we will always suppose that the heterogeneities of the composite in question are evenly distributed, which is a perfectly appropriate assumption for a wide range of applications. One natural way to express this assumption in a mathematical model is to consider periodically inhomogeneous media, where the periodicity length (and thus the characteristic length of the micro-scale) is represented by a small parameter $\varepsilon > 0$, see Fig 1.

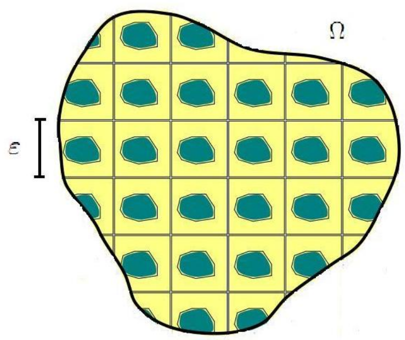

Figure 1: An ε-periodic composite material occupying a domain Ω. The material consists of two ingredients.

The notion of *mathematical periodic homogenization* indicates the process of taking $\varepsilon \to 0$ and the study of solutions u_ε of corresponding ε-problems in this limit.

In the last 40 years several books have been devoted to the periodic and non-periodic homogenization theory, see for instance [5, 10, 18] for a general overview. In this introductory section we will present the most fundamental classical results and methods in this field.

1.1 Elliptic homogenization problem

In this subsection we introduce the most elementary periodic homogenization problem of investigating solutions u_ε to the elliptic problem

$$-\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) = f(x). \tag{1.1}$$

It is made more precise in Definition 1.2 below.

In fact, Eq.(1.1) is a widely studied model case. On the one hand it models thermal, electrical and elastic properties of composites, which are encoded in the ε-periodic matrix $A\left(\frac{\cdot}{\varepsilon}\right)$. It is thus relevant for many applications. On the other hand, already in this relatively simple setting the main mathematical difficulties in the homogenization process, $\varepsilon \to 0$, become obvious.

Let us first of all introduce a class of admissible matrices to guarantee the well-posedness of problem (1.1).

Definition 1.1 (Class of admissible matrices)**.** *Let $N \in \mathbb{N}$ and let $0 < \alpha < \beta$. We denote by $M(\alpha, \beta)$ the set of all matrices $A \in \mathbb{R}^{N\times N}$ such that for every $\lambda \in \mathbb{R}^N$ there holds*

$$\begin{aligned} \langle A\lambda, \lambda\rangle &\geq \alpha|\lambda|^2, \\ |A\lambda| &\leq \beta|\lambda|, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$ and $|\lambda|$ is the length of λ.

We are now in the position to introduce the classical elliptic homogenization problem with Dirichlet boundary conditions.

Definition 1.2 (Elliptic homogenization problem)**.** *Let $0 < \alpha < \beta$ and let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $A(\cdot) = (a_{ij}(\cdot))_{1\leq i,j\leq N} \in C^\infty(\Omega, \mathbb{R}^{N\times N})$ be such that $A(x) \in M(\alpha,\beta)$ for every $x \in \Omega$. Moreover, let $A(\cdot)$ be $(0,1)^N$-periodic, $A(x+e_i) = A(x)$ for every $i = 1, ..., N$, where e_i denotes the i-th unit vector in $\mathbb{R}^N$. We call $u_\varepsilon \in H^1(\Omega)$ a solution to the elliptic homogenization problem if*

$$\begin{aligned} -\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) &= f && \text{in } \Omega, \\ u_\varepsilon &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

in the weak sense for f given in $H^{-1}(\Omega)$, the dual space of $H^1_0(\Omega)$.

We remark that due to the $(0,1)^N$-periodicity of the matrix $A(\cdot)$, the coefficient $A\left(\frac{\cdot}{\varepsilon}\right)$ is $(0,\varepsilon)^N$-periodic and thus highly oscillating.

By the Lax-Milgram theorem there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ to the elliptic problem of Definition 1.2. Moreover, the following uniform (in ε) estimate holds

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{1.3}$$

Consequently, there exists some $u \in H_0^1(\Omega)$ such that, up to a subsequence, u_ε converges weakly to u as ε goes to zero, $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$.

At this point, two natural questions arise.

1. Is u uniquely determined in the sense that every subsequence of u_ε converges to the same limit function u?
2. Which *effective* problem is solved by u?

Already in the 1970s both questions have been answered, see for instance Sanchez-Palencia [26, 27] or Bensoussan, Lions and Papanicolaou [5]. The result, see Theorem 1.3 in the next subsection, is now standard.

1.2 Elliptic homogenization result

In this subsection we state the classical homogenization result for elliptic problems and briefly present the three classical homogenization methods in the periodic framework:

1. Formal asymptotic expansions
2. Oscillating test functions
3. Two-scale convergence

While the first method is just a formal approach, the second and the third one are rigorous and provide proofs of the classical homogenization result stated below.

Theorem 1.3 (Classical homogenization result for elliptic problems)**.** *Let u_ε be the solution to the elliptic homogenization problem of Definition 1.2. Then*

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{in } H_0^1(\Omega), \\ A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon &\rightharpoonup A^*\nabla u_0 && \text{in } (L^2(\Omega))^N, \end{aligned}$$

where the limit function $u_0 \in H_0^1(\Omega)$ is the unique solution to the effective constant coefficient problem

$$\begin{aligned} -\nabla\cdot(A^*\nabla u_0) &= f && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

The matrix $A^* = (a^*_{ij})_{1\leq i,j\leq N}$ is given through

$$a^*_{ij} = \int_{(0,1)^N} a_{ij}(x)\,dx - \int_{(0,1)^N}\sum_{k=1}^N a_{ik}(x)\frac{\partial\hat{\chi}_j}{\partial y_k}(x)\,dx, \tag{1.5}$$

where the functions $\hat{\chi}_j(\cdot)$, often referred to as correctors, are $(0,1)^N$-periodic and solve specific auxiliary cell problems. They are defined in (1.8).

Theorem 1.3 suggests that the whole sequence (question 1 in the previous subsection) converges to the function u_0. The limit function u_0 solves a constant coefficient problem of exactly the same type as the original problem (question 2).

Let us draw the readers attention to a peculiarity of the homogenization process. Contrary to what one would expect at first sight, the effective matrix A^* is not just the mean value of the oscillating coefficient $A(\cdot)$. Formula (1.5) suggests that in fact the mean value of $A(\cdot)$ has to be corrected by additional terms which include the gradients of specific auxiliary functions.

In what follows we will not perform the proof of Theorem 1.3 in detail, which can for instance be found in [10]. Instead we will briefly present the three classical methods mentioned above. By means of asymptotic expansions we will show how formula (1.5) can be formally justified. By means of oscillating test functions and two-scale convergence we will sketch two quite different ways to prove Theorem 1.3.

Formal asymptotic expansions. The method is based on the existence of two distinct scales. The macroscopic variable x describes the global position of a point in the domain Ω. The microscopic variable $y := \frac{x}{\varepsilon}$ describes the position of a point in the rescaled periodicity cell $(0,1)^N$. The idea is to look for an asymptotic expansion of the form

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + ..., \tag{1.6}$$

where $u_j(x,y)$ is $(0,1)^N$-periodic in the second variable.

Plugging the ansatz in Eq. (1.1) and comparing power-like terms of ε, one derives an infinite system of equations. Without going into details we remark that the specific structure of the system permits to determine the unknowns u_j successively.

The first equation of the system, the equation at order $(1/\varepsilon^2)$, yields that u_0 is independent of y. Hence, $u_0(x)$ is expected to be the solution to the effective problem (1.4). The second equation in the system, the equation at order $(1/\varepsilon)$, provides

$$u_1(x,y) = -\sum_{j=1}^{N} \hat{\chi}_j(y) \frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x), \tag{1.7}$$

where each $\hat{\chi}_j$ is $(0,1)^N$-periodic and solves the following auxiliary problem

$$\begin{aligned} -\nabla \cdot (A(y)\nabla\hat{\chi}_j(y)) &= -\nabla \cdot (A(y)e_j)\,, \\ \int_{(0,1)^N} \hat{\chi}_j(y)\, dy &= 0. \end{aligned} \tag{1.8}$$

The problem is well posed since the mean value (in y) of the right hand side of (1.8) is equal to zero.

We investigate one more equation in the system, the equation at order (1). It determines u_2 through

$$-\nabla_y \cdot (A(y)\nabla_y u_2(x,y)) = F_1(x,y), \tag{1.9}$$

where F_1 is written in terms of u_0, u_1 and f. Problem (1.9) is well posed if and only if the mean value (in y) of the right hand side vanishes. It is exactly this condition which, using (1.7), gives the effective equation (1.4) for u_0 and formally justifies formula (1.5).

Before discussing the method of oscillating test functions and the concept of two-scale convergence let us firstly demonstrate the main difficulty in the proof of Theorem 1.3.

On the one hand, the uniform bound in (1.3) yields that there exists some $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_\varepsilon \to u$ in $L^2(\Omega)$. On the other hand, setting $\xi_\varepsilon(x) := A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)$ one discovers that ξ_ε is bounded in $(L^2(\Omega))^N$ with $-\nabla\cdot\xi_\varepsilon = f$. Hence, there exists some $\xi \in (L^2(\Omega))^N$ such that, again up to a subsequence, $\xi_\varepsilon \rightharpoonup \xi$ in $(L^2(\Omega))^N$. The weak limit ξ satisfies $-\nabla\cdot\xi = f$.

We remark that the proof of Theorem 1.3 is done, if one can show that

$$\xi(x) = A^*\nabla u(x) \tag{1.10}$$

with A^* as in formula (1.5). Unfortunately, the flux $\xi_\varepsilon = A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon$ is a product of only weakly converging sequences and thus the individual limits do not provide any information about the weak limit of the product. In what follows we will show how this difficulty is treated by the method of oscillating test functions and the method of two-scale convergence.

Oscillating test functions. The method has been proposed by Tartar [35] in the late seventies and is based on the construction of special test functions by means of the adjoint operator $-\nabla\cdot\big(A^T(y)\nabla\big)$. The particular structure of the test functions effects that all terms containing a product of only weakly converging sequences, i.e. that terms where a direct passage to the limit is not possible, cancel out.

Let $j = 1,...N$. Let each $w_\varepsilon^j \in H^1(\Omega)$ be a particular solution, see (1.14) below, to the problem

$$\int_\Omega \phi(x)\left(A^T\left(\frac{x}{\varepsilon}\right)\nabla w_\varepsilon^j(x)\right)\cdot\nabla u_\varepsilon(x)\; dx + \int_\Omega u_\varepsilon(x)\left(A^T\left(\frac{x}{\varepsilon}\right)\nabla w_\varepsilon^j(x)\right)\cdot\nabla\phi(x)\, dx = 0 \tag{1.11}$$

for every $\phi \in C_c^\infty(\Omega)$. Using $w_\varepsilon^j\phi \in H_0^1(\Omega)$ as a test function in $-\nabla\cdot\xi_\varepsilon = f$ one obtains

$$\begin{aligned}&\int_\Omega \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right)\cdot\nabla w_\varepsilon^j(x)\,\phi(x)\; dx + \int_\Omega \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right)\cdot\nabla\phi(x)\, w_\varepsilon^j(x)\; dx\\ &= \langle f, w_\varepsilon^j(x)\,\phi(x)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned} \tag{1.12}$$

Due to the duality of A and A^T the first terms on the left hand side of (1.11) and (1.12) are equal. They cancel by subtraction,

$$\begin{aligned}&\int_\Omega \left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right)\cdot\nabla\phi(x)\, w_\varepsilon^j(x)\; dx - \int_\Omega u_\varepsilon(x)\left(A^T\left(\frac{x}{\varepsilon}\right)\nabla w_\varepsilon^j(x)\right)\cdot\nabla\phi(x)\, dx\\ &= \langle f, w_\varepsilon^j(x)\,\phi(x)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned} \tag{1.13}$$

The goal is to pass to the limit in (1.13).

At this point, let us make the choice of $w_\varepsilon^j(x)$ more precise. We set

$$w_\varepsilon^j(x) := e_j\cdot x - \varepsilon\chi_j\left(\frac{x}{\varepsilon}\right), \tag{1.14}$$

where χ_j solves the auxiliary problem (1.8) with the adjoint matrix $A^T(\cdot)$ instead of $A(\cdot)$. With this choice of w_ε^j one can easily show that w_ε^j is a particular solution to problem (1.11) and that

$$\begin{aligned} w_\varepsilon^j &\to e_j \cdot x && \text{in } L^2(\Omega), \\ A^T\left(\frac{x}{\varepsilon}\right) \nabla w_\varepsilon^j(x) &\rightharpoonup \int_{(0,1)^N} A^T(y)\,(e_j - \nabla \chi_j(y))\; dy = (A^*)^T e_j && \text{in } L^2(\Omega), \end{aligned}$$

see [10] for details. We remark that the last convergence holds due to the fact that oscillating periodic functions converge weakly to their mean value.

We are now in the position to pass to the limit in (1.13), since all terms on the left hand side of (1.13) are products of a weakly converging and a strongly converging sequence. The passage to the limit in the right hand side is straightforward and we arrive at

$$\begin{aligned} &\int_\Omega \xi(x) \cdot \nabla\phi(x)\,(e_j \cdot x)\; dx - \int_\Omega u(x)((A^*)^T e_j) \cdot \nabla\phi(x)\, dx \\ &= \langle f, (e_j \cdot x)\,\phi(x)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \tag{1.15}$$

Finally, using $-\nabla \cdot \xi = f$, we rewrite the first term on the left hand side of (1.15) as

$$\int_\Omega \xi(x) \cdot \nabla\phi(x)\,(e_j \cdot x)\; dx = \langle f, (e_j \cdot x)\,\phi(x)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \xi(x) \cdot e_j\,\phi(x)\; dx$$

and apply integration by parts in the second term. Consequently,

$$\xi(x) \cdot e_j = (A^* \nabla u(x)) \cdot e_j$$

for $j = 1, ...N$ and thus relation (1.10) follows.

Two-scale convergence. The concept of two-scale convergence, introduced by Nguetseng [21] in 1989 and further developed by Allaire [1], establishes an adapted notion of convergence, which in particular rigorously justifies the formal asymptotic expansion presented above. Several applications and features of this powerful method can be found in [1].

Definition 1.4 (Two-scale convergence)**.** *Let $Y := (0,1)^N$. A sequence of functions $u_\varepsilon \in L^2(\Omega)$ is said to two-scale converge to a limit function $u \in L^2(\Omega \times Y)$ if*

$$\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)\; dx \to \int_\Omega \int_Y u(x,y)\phi(x,y)\; dy\, dx \tag{1.16}$$

for every $\phi \in L^2(\Omega; C_{per}(Y))$. The subscript per indicates subsets of periodic functions.

We remark that the notion of two-scale convergence is equipped with the following compactness properties, see [1] for a proof.

1. For each bounded sequence v_ε in $L^2(\Omega)$ there exists a function $v_0 \in L^2(\Omega \times Y)$ such that, up to a subsequence, v_ε two-scale converges to v_0.

2. For each bounded sequence v_ε in $H^1(\Omega)$ with $v_\varepsilon \rightharpoonup v_0$ in $H^1(\Omega)$ there exists a function $v_1 \in L^2(\Omega; H^1_{per}(\Omega))$ such that, up to a subsequence, ∇v_ε two-scale converges to $\nabla v_0 + \nabla_y v_1$.

Let u_ε be the solution to the elliptic homogenization problem of Definition 1.2. Our aim is to pass to the limit in the "bad" term $\xi_\varepsilon = A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon$, which is a product of only weakly converging sequences. In particular, there exists some $u \in H^1(\Omega)$ such that, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$. By the compactness result stated above, there exists some $u_1 \in L^2(\Omega; H^1_{per}(\Omega))$ such that, again up to a subsequence, ∇u_ε two-scale converges to $\nabla u + \nabla_y u_1$.

The key point in the proof of Theorem 1.3 is the specific structure of admissible test functions in the definition of two-scale convergence. It permits to regard $A\left(\frac{x}{\varepsilon}\right)$ in the term $A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon$ as part of an admissible test function and to pass to the two-scale limit. Let us make this idea more precise.

Consider $v_0 \in C_c^\infty(\Omega)$ and $v_1 \in C_c^\infty(\Omega; C^\infty_{per}(Y))$. Then $v_0(\cdot) + \varepsilon v_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \in H^1_0(\Omega)$. Consequently,

$$\begin{aligned}\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\cdot\left[\nabla v_0(x) + \nabla_y v_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon\nabla_x v_1\left(x,\frac{x}{\varepsilon}\right)\right]\,dx \\ = \left\langle f, v_0(x) + \varepsilon v_1\left(x,\frac{x}{\varepsilon}\right)\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)},\end{aligned} \tag{1.17}$$

since u_ε solves the elliptic homogenization problem of Definition 1.2.

Our aim is to pass to the limit in (1.17). Indeed, the limit procedure in the right hand side of (1.17) is straightforward. We rewrite the left hand side of (1.17) as

$$\int_\Omega \nabla u_\varepsilon(x)\cdot A^T\left(\frac{x}{\varepsilon}\right)\left[\nabla v_0(x) + \nabla_y v_1\left(x,\frac{x}{\varepsilon}\right)\right]\,dx + \varepsilon\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\cdot\nabla_x v_1\left(x,\frac{x}{\varepsilon}\right)\,dx.$$

Since $A^T\left(\frac{x}{\varepsilon}\right)\left[\nabla v_0(x) + \nabla_y v_1\left(x,\frac{x}{\varepsilon}\right)\right]$ is an admissible test function in the framework of two-scale convergence, we can directly pass to the two-scale limit in the first term. The second term is of order ε and vanishes in the limit. We arrive at the following effective problem with unknowns u and u_1

$$\begin{aligned}\int_\Omega\int_Y \left(\nabla_x u(x) + \nabla_y u_1(x,y)\right) A^T(y)\left(\nabla v_0(x) + \nabla_y v_1(x,y)\right)\,dx\,dy \\ = \langle f, v_0(x)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\end{aligned} \tag{1.18}$$

for $v_0 \in C_c^\infty(\Omega)$ and $v_1 \in C_c^\infty(\Omega; C^\infty_{per}(Y))$.

In [10] it is shown that (1.18) is equivalent to the effective problem of Theorem 1.3.

The classical homogenization methods presented above are very flexible and can also be applied in the time-dependent framework. They provide analogous homogenization results for parabolic (heat equation) as well as for hyperbolic (wave equation) PDEs.

1.3 Homogenization of the wave equation

This subsection is devoted to the homogenization of the wave equation for an arbitrary bounded domain $\Omega \subset \mathbb{R}^N$ and an arbitrary *fixed* time T. The homogenization result, see Proposition 1.6 below, can be labeled as standard. It is obtained by a reduction of the time-dependent problem to the elliptic setting. Its proof can be found in [10].

However, the wave equation exhibits an interesting peculiarity which is not present in the elliptic and in the parabolic framework. Brahim-Otsmane, Francfort and Murat, see Ref.[8], pointed out that the energy E_ε corresponding to $\bar{u}_\varepsilon$, solution to the homogenization problem of Definition 1.5 below, does not in general converge to the energy corresponding to the limit function $\bar{u}$.

Let us first of all introduce the hyperbolic homogenization problem in divergence form.

Definition 1.5 (Hyperbolic homogenization problem)**.** *Let $A(\cdot)$ be as in Definition 1.2 with $A(\cdot) = A^T(\cdot)$ and let $c_0 \in H_0^1(\Omega)$ and $d_0 \in L^2(\Omega)$. We call $\bar{u}_\varepsilon$ a solution to the hyperbolic homogenization problem if $\bar{u}_\varepsilon \in L^2(0,T;H_0^1(\Omega)), \partial_\tau \bar{u}_\varepsilon \in L^2(0,T;L^2(\Omega))$ and*

$$\begin{aligned} \partial_\tau^2 \bar{u}_\varepsilon(x,\tau) &= \nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right) \nabla \bar{u}_\varepsilon(x,\tau) \right), \\ \bar{u}_\varepsilon(x,0) &= c_0(x), \\ \partial_\tau \bar{u}_\varepsilon(x,0) &= d_0(x). \end{aligned} \tag{1.19}$$

By reduction to the elliptic setting the following homogenization result is obtained.

Proposition 1.6 (Hyperbolic homogenization result)**.** *Let $\bar{u}_\varepsilon$ be the solution to the hyperbolic homogenization problem of Definition 1.5. Then there holds*

$$\begin{aligned} \bar{u}_\varepsilon &\overset{*}{\rightharpoonup} \bar{u} \ \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ \partial_\tau \bar{u}_\varepsilon &\overset{*}{\rightharpoonup} \partial_\tau \bar{u} \ \text{ in } L^\infty(0,T;L^2(\Omega)), \\ A\left(\frac{x}{\varepsilon}\right) \nabla \bar{u}_\varepsilon &\rightharpoonup A^* \nabla \bar{u} \ \text{ in } \left(L^2(0,T;L^2(\Omega))\right)^N, \end{aligned}$$

where $\bar{u}$ is the unique solution to the effective wave equation

$$\begin{aligned} \partial_\tau^2 \bar{u}(x,\tau) &= \nabla \cdot \left(A^* \nabla \bar{u}(x,\tau) \right), \\ \bar{u}(x,0) &= c_0(x), \\ \partial_\tau \bar{u}(x,0) &= d_0(x) \end{aligned} \tag{1.20}$$

and A^* is the effective matrix of Theorem 1.3.

The proposition suggests that in the *fixed time* homogenization process of wave equations the time variable τ plays just the role of a parameter. We remark that an analogous result is available also for the parabolic framework. Nevertheless, the wave equation stands out due to the lack of convergence of the energy, which is discussed in the following.

Suppose that $\bar{u}_\varepsilon$ is the solution to the hyperbolic homogenization problem of Definition 1.5. By a testing procedure it is easily shown that $\bar{u}_\varepsilon$ satisfies the principle of energy conservation,

$$\begin{aligned} &\int_\Omega \left[(\partial_\tau \bar{u}_\varepsilon)^2 + \left(A\left(\frac{x}{\varepsilon}\right) \nabla \bar{u}_\varepsilon \right) \cdot \nabla \bar{u}_\varepsilon \right] (x,t)\, dx \\ &= \int_\Omega (d_0(x))^2 + \left(A\left(\frac{x}{\varepsilon}\right) \nabla c_0(x) \right) \cdot \nabla c_0(x)\, dx =: E_\varepsilon(\bar{u}_\varepsilon). \end{aligned} \tag{1.21}$$

In the same manner, the limit function $\bar{u}$, solution to the effective wave equation (1.20), satisfies

$$\int_\Omega \left[(\partial_\tau \bar{u})^2 + (A^*\nabla\bar{u})\cdot\nabla\bar{u}\right](x,t)\ dx = \int_\Omega (d_0(x))^2 + (A^*\nabla c_0(x))\cdot\nabla c_0(x)\ dx =: E(\bar{u}).$$

Since $A\left(\frac{\cdot}{\varepsilon}\right)$ converges weakly to the mean value $\int_Y A(y)\ dy \neq A^*$, see Formula (1.5), one directly concludes that the energy $E_\varepsilon(\bar{u}_\varepsilon)$ does not converge to $E(\bar{u})$.

This phenomenon has been extensively studied by Brahim-Otsmane, Francfort and Murat. In [8] the authors show that in fact the original solution $\bar{u}_\varepsilon$ can be decomposed into two parts, $\bar{u}_\varepsilon := \tilde{u}_\varepsilon + v_\varepsilon$. The principal part $\tilde{u}_\varepsilon$ solves the hyperbolic homogenization problem with suitable initial data, which are constructed such that $E_\varepsilon(\tilde{u}_\varepsilon) \to E(\bar{u})$. The remainder term v_ε is proved to converge weakly to zero. Nevertheless, the energy associated to v_ε does not vanish in the limit.

Confirmed by various numerical results, see [12, 13, 14, 15], this striking fact is one of the reasons to expect that the *long time* behavior of $\bar{u}_\varepsilon$ is not well described by the effective wave equation of Proposition 1.6.

Indeed, various articles deal with the effective equation for long time intervals, where already the titles of the papers mention dispersive effects. Nevertheless, a clear mathematical statement concerning an effective model is not yet available.

With this thesis we want to fill this gap by providing a complete description of the long time behavior in the one-dimensional setting. The multi-dimensional case is investigated as well and estimates for the effective propagation speed are derived.

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2 Long time homogenization of waves and main results

In recent years, many approaches have been developed for the homogenization of waves in periodic media. Powerful methods such as two-scale convergence, the method of oscillating test-functions or compensated compactness, see Ref.[18], have been used to prove rigorous convergence results.

By now, it is well known that the homogenization limit ($\varepsilon \to 0$) of

$$\partial_\tau^2 \bar{u}_\varepsilon(x,\tau) = \nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right) \nabla \bar{u}_\varepsilon(x,\tau) \right)$$

is a linear wave equation with a constant coefficient A^*, called the effective coefficient. In other words, the problem fits the following picture: On bounded domains and fixed time intervals $\tau \in (0,T)$, the original solution $\bar{u}_\varepsilon$ coincides up to corrections of order $O(\varepsilon)$ with a limit function $\bar{u}$, which solves the effective wave equation $\partial_\tau^2 \bar{u} - \nabla \cdot (A^* \nabla \bar{u}) = 0$, see Proposition 1.6. Nevertheless, the limit $\bar{u}_\varepsilon \to \bar{u}$ is a weak convergence and, in particular, the energy of the ε-solutions need not converge to the energy of $\bar{u}$.

In the present work we are interested in very long time scales, i.e. observation times of order $O(1/\varepsilon^2)$. Numerical results and formal asymptotic expansions, see Ref.[12], suggest that the shape of the propagating wave is considerably modified at long sight. Our aim is to catch this effect with a uniformly valid dispersive model. The difficulty is, taking into account the long time scale $t := \varepsilon^2 \tau$, that we are now dealing with a true three-scale problem whose homogenization limit is not given by the solution to the effective wave equation.

The interest in this question is not new, see Ref.[3] for dispersive limits in the case of large potentials and Ref.[4] in the case of high frequency initial data. Likewise, works such as [13, 15, 16, 28] deal with the long time behavior of waves and give formal calculations. Nevertheless, it seems that a rigorous mathematical statement on the homogenization limit (or even an effective equation) is still missing.

2.1 Main results in the one-dimensional case

In this thesis, we propose two different dispersive models and prove that they both approximate the original one-dimensional problem for long observation times. For the sake of completeness, we also show the well-posedness of the models. While the first model (weakly dispersive equation) still depends on powers of ε, the second model (linearized Korteweg-de-Vries equation) is ε-independent. The general concept of our homogenization proofs can be described with the following three steps. 1) We state and solve the homogenized system. 2) We modify the solution of the homogenized system to construct an approximate solution to the original system. 3) We show by a testing procedure that the result of this construction is close to the original solution. This principle is flexible and can be applied in complex applications, e.g. to another three-scale problem,[32] or to problems with hysteresis,[31, 33, 34].

Let us start with a detailed description of the original one-dimensional problem. We assume that the coefficient $A(\cdot) \in C^\infty(\mathbb{R})$ is periodic and admissible in the sense of Definition 1.1. To be more precise, we assume that there exist $\alpha, \beta > 0$ such that $a(\cdot) := A(\cdot) \in C^\infty(\mathbb{R})$ with $0 < \alpha \le a(y) \le \beta$ and $a(y+1) = a(y)$ for all $y \in \mathbb{R}$. Moreover, we are dealing with smooth initial data with compact support being perturbated at order $O(\varepsilon)$ by high frequency terms. To sum up, we consider the following problem on the long-time interval $(0, T/\varepsilon^2)$.

Definition 2.1 (Homogenization problem). *Let $T, R > 0$. Denote by $\bar{u}_\varepsilon(x,\tau)$ the unique solution to the wave equation*

$$\begin{aligned} \partial_\tau^2 \bar{u}_\varepsilon(x,\tau) &= \partial_x \left(a\left(\frac{x}{\varepsilon}\right) \partial_x \bar{u}_\varepsilon(x,\tau) \right), \\ \bar{u}_\varepsilon(x,0) &= c_0(x) + \varepsilon L_1\left(\frac{x}{\varepsilon}\right) \partial_x c_0(x) + \varepsilon^2 L_2\left(\frac{x}{\varepsilon}\right) \partial_x^2 c_0(x), \\ \partial_\tau \bar{u}_\varepsilon(x,0) &= d_0(x) + \varepsilon L_1\left(\frac{x}{\varepsilon}\right) \partial_x d_0(x) \end{aligned}$$

for $(x,\tau) \in \mathbb{R} \times (0, T/\varepsilon^2)$. We assume that the initial data are smooth and compactly supported, $c_0, d_0 \in C_c^\infty\big((-R,R)\big)$, and that the initial time derivative d_0 has zero mean value, $\int_{-R}^R d_0(x)\,dx = 0$.

The special functions $L_i(y) \in C^2(\mathbb{R})$ are Y-periodic and solve auxiliary cell problems. They are defined in Definition 5.1 for $1 \le i \le 5$. We remark that L_1 and L_2 are determined by classical cell problems, which are known from elliptic homogenization theory. By contrast, the functions L_3, L_4, L_5, which are used in the construction of the adaption operator in section 5, do not correspond to the classical auxiliary functions.

The existence of solutions $\bar{u}_\varepsilon$ is a standard result, solutions can be constructed as weak, strong or classical solutions. We use the weak setting in the following and work with $\bar{u}_\varepsilon \in L^\infty(0, T/\varepsilon^2; H^2(\mathbb{R}))$ and $\partial_\tau \bar{u}_\varepsilon \in L^\infty(0, T/\varepsilon^2; H^1(\mathbb{R}))$.

We are now in the position to introduce the three different long time problems and to state our main results in the one-dimensional scalar setting. We remark that the major part of the one-dimensional results has already been accepted for publication, see Ref. [19].

We start with a time-scaled version of the homogenization problem. Considering the long time variable $t := \varepsilon^2 \tau$ and setting $u_\varepsilon(x,t) := \bar{u}_\varepsilon(x, t/\varepsilon^2)$, one arrives at the following definition.

Definition 2.2 (Time-scaled homogenization problem). *Let $(x,t) \in \mathbb{R} \times (0,T)$. Denote by $u_\varepsilon(x,t)$ the unique solution to*

$$\begin{aligned} \varepsilon^4 \partial_t^2 u_\varepsilon(x,t) &= \partial_x \left(a\left(\frac{x}{\varepsilon}\right) \partial_x u_\varepsilon(x,t) \right), \\ u_\varepsilon(x,0) &= c_0(x) + \varepsilon L_1\left(\frac{x}{\varepsilon}\right) \partial_x c_0(x) + \varepsilon^2 L_2\left(\frac{x}{\varepsilon}\right) \partial_x^2 c_0(x), \\ \partial_t u_\varepsilon(x,0) &= \frac{1}{\varepsilon^2} \Big(d_0(x) + \varepsilon L_1\left(\frac{x}{\varepsilon}\right) \partial_x d_0(x) \Big). \end{aligned}$$

Let us discuss here what we expect in view of the classical homogenization results. The effective coefficient $a^* := A^* \in \mathbb{R}$ is, in the one-dimensional case, given by the harmonic mean

$$a^* = \left(\int_Y \frac{1}{a(y)}\, dy \right)^{-1} > 0. \tag{2.1}$$

The associated homogenized wave speed is $c^* := \sqrt{a^*}$. Classical homogenization hence suggests that waves $\bar{u}_\varepsilon$ move with asymptotic speed c^*. Accordingly, in the time-scaled version of Definition 2.2, we expect that waves u_ε propagate with an asymptotic speed c^*/ε^2.

In the next step, we introduce a fourth-order weakly dispersive solution. A proof of existence and uniqueness as well as energy estimates can be found in Section 4.

Definition 2.3 (Weakly dispersive problem)**.** *Denote by $v_\varepsilon(x,t)$ the unique solution to the following problem*

$$\begin{aligned} \varepsilon^4 \partial_t^2 v_\varepsilon(x,t) - a^* \partial_x^2 v_\varepsilon(x,t) - \varepsilon^6 \frac{a_2^*}{a^*} \partial_t^2 \partial_x^2 v_\varepsilon(x,t) = 0,& \\ v_\varepsilon(x,0) = c_0(x),& \\ \partial_t v_\varepsilon(x,0) = \frac{1}{\varepsilon^2} d_0(x)& \end{aligned}$$

for $(x,t) \in \mathbb{R} \times (0,T)$ and $a_2^* > 0$ introduced in Definition 5.1.

Finally, after a decomposition of the initial data into a right-going and a left-going part $c_0 = c_0^+ + c_0^-$ (strongly depending on the initial time derivative d_0, see Section 3 for details), let us now define the ε-independent linearized Korteweg-de-Vries (lKdV) equations as follows.

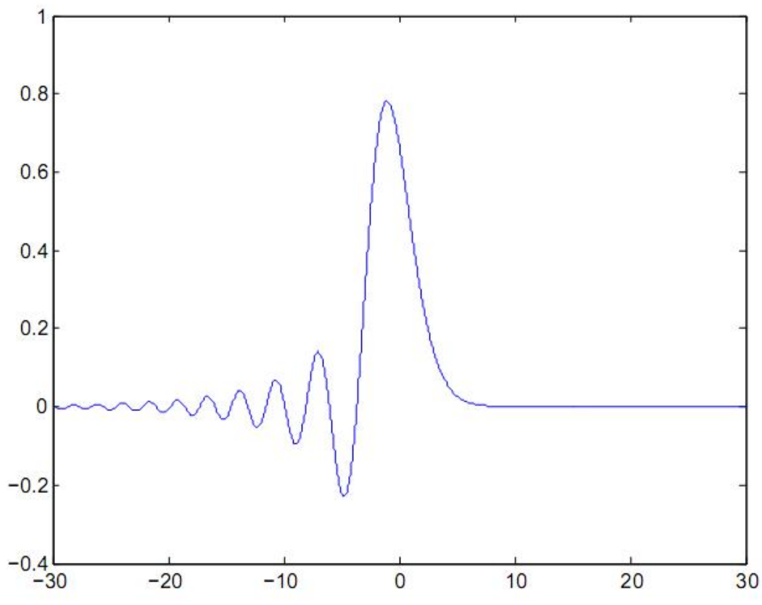

Figure 2: Numerical solution W^+ to the right going lKdV-problem, evaluated in $t = 1$. It is obtained with a finite difference scheme for initial data $W^+(x,0) = sech(x) = \frac{2}{e^x + e^{-x}}$. The solution W^- is obtained by symmetry.

Definition 2.4 (The lKdV equations)**.** *We distinguish between a right moving and a left moving wave. Denote by $W^{\pm}(x,t)$ the unique solution to*

$$\begin{aligned}\partial_t W^{\pm}(x,t) \pm \frac{a_2^*}{2c^*}\partial_x^3 W^{\pm}(x,t) = 0,\\ W^{\pm}(x,0) = c_0^{\pm}(x)\end{aligned}$$

for $(x,t) \in \mathbb{R} \times (0,T)$ and $c_0^{\pm}$ as in (3.1)*.*

The existence and uniqueness as well as energy estimates of a solution are a direct consequence of the results in Ref.[17], see Section 3 for more details. In fact, the weakly dispersive problem and the lKdV problem have strong solutions, which can be explicitly constructed by means of Fourier analysis.

We are now able to state the main one-dimensional results. The first shows that the weakly dispersive problem of Definition 2.3, which is also known as linear Boussinesq equation, provides a good approximation of the original problem.

Theorem 2.5. *Let $c_0, d_0 \in C_c^{\infty}\big((-R,R)\big)$ and $\int_{-R}^{R} d_0(x)\,dx = 0$. Consider u_ε from the time-scaled homogenization problem of Definition 2.2 and the weakly dispersive solution v_ε of Definition 2.3. Then there exists an ε-independent constant C such that*

$$\|u_\varepsilon(x,t) - v_\varepsilon(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C\varepsilon. \tag{2.2}$$

We note that in (4.6), (4.7) a slightly stronger convergence is derived. We further remark that in the original time scale, Theorem 2.5 reads as

Remark 2.6. *Consider $\bar{u}_\varepsilon$ as in Definition 2.1 and $\bar{v}_\varepsilon$ the unique solution to the rescaled weakly dispersive problem*

$$\begin{aligned}\partial_\tau^2 \bar{v}_\varepsilon(x,\tau) - a^*\partial_x^2 \bar{v}_\varepsilon(x,\tau) - \varepsilon^2\frac{a_2^*}{a^*}\partial_\tau^2\partial_x^2 \bar{v}_\varepsilon(x,\tau) = 0,\\ \bar{v}_\varepsilon(x,0) = c_0(x), \qquad \partial_\tau \bar{v}_\varepsilon(x,0) = d_0(x).\end{aligned}$$

Then

$$\|\bar{u}_\varepsilon(x,\tau) - \bar{v}_\varepsilon(x,\tau)\|_{L^\infty(0,T/\varepsilon^2;L^\infty(\mathbb{R}))} \leq C\varepsilon.$$

The weakly dispersive solution and the lKdV-solution can be compared as follows.

Theorem 2.7. *Consider the weakly dispersive solution v_ε of Definition 2.3 and the shifts $w_\varepsilon^{\pm}$ of the lKdV-solutions of Definition 2.4*

$$w_\varepsilon^+(x,t) := W^+\left(x - \frac{c^*}{\varepsilon^2}t, t\right), \tag{2.3}$$

$$w_\varepsilon^-(x,t) := W^-\left(x + \frac{c^*}{\varepsilon^2}t, t\right). \tag{2.4}$$

Then there exists an ε-independent constant C such that

$$\begin{aligned}\|\partial_x v_\varepsilon(x,t) - \partial_x\big(w_\varepsilon^+ + w_\varepsilon^-\big)(x,t)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C\varepsilon^2,\\ \|\partial_t v_\varepsilon(x,t) - \partial_t\big(w_\varepsilon^+ + w_\varepsilon^-\big)(x,t)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C.\end{aligned} \tag{2.5}$$

Moreover,

$$\|v_\varepsilon(x,t) - \big(w_\varepsilon^+ + w_\varepsilon^-\big)(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C\varepsilon. \tag{2.6}$$

Theorem 2.7 provides a comparison between the solution of the weakly dispersive problem and the lKdV-solutions. This result is not surprising, see Ref.[29] in the nonlinear case. Similarly, articles such as [7, 30, 6] establish a link between Boussinesq and KdV-equations. Nevertheless, they do not explicitly recover the setting of Theorem 2.7, which will be obtained in Section 3 with a direct argument. We include the theorem for the sake of self-containedness and to give precise estimates. By contrast, the proof of Theorem 2.5 requires more computationally intensive methods and is performed in Section 4.

By applying the triangle inequality to (2.2) and (2.6) one directly obtains the following result.

Corollary 2.8. *Consider u_ε from the time-scaled homogenization problem and the lKdV-solutions $W^\pm$. Then there exists an ε-independent constant C such that*

$$\left\| u_\varepsilon(x,t) - W^+\Big(x - \frac{c^*}{\varepsilon^2}t, t\Big) - W^-\Big(x + \frac{c^*}{\varepsilon^2}t, t\Big) \right\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C\varepsilon. \tag{2.7}$$

The corollary suggests that the solution to the time-scaled homogenization problem of Definition 2.2 is approximatively equal to two waves propagating with speed c^*/ε^2 in opposite directions. Moreover, it shows that the shape of the right going wave is well described by the solution W^+ to the lKdV-problem, those of the left going wave by W^-. Similarly to Remark 2.6, this result provides also an approximation of $\bar{u}_\varepsilon$ in the original time scale.

Remark 2.9. *Consider $\bar{u}_\varepsilon$ as in Definition 2.1 and the lKdV-solutions $W^\pm$. Then there exists an ε-independent constant C such that*

$$\left\| \bar{u}_\varepsilon(x,\tau) - W^+\Big(x - c^*\tau, \varepsilon^2\tau\Big) - W^-\Big(x + c^*\tau, \varepsilon^2\tau\Big) \right\|_{L^\infty(0,T/\varepsilon^2;L^\infty(\mathbb{R}))} \leq C\varepsilon.$$

2.2 Main results in an abstract framework and the multi-dimensional case

Theorem 2.5 and Corollary 2.8 contain a complete description of the long time behavior of waves in the one-dimensional case. In the multi-dimensional setting, $x \in \mathbb{R}^N$, the situation is much more complicated and it seems that, up to now, comparable results or even an effective equation are out of reach.

In the last part of this thesis, Sections 6-8, we will therefore study the multi-dimensional homogenization problem in a more abstract framework. More precisely, we define an energy density $E_\varepsilon(\xi,t)$ and show that a weak star limit $\mu \in L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ exists. Based on the one-dimensional results of Sections 3-5 we prove that for $N=1$ the energy measure μ is in fact a Dirac measure. For $N \geq 2$ the identification of μ remains an open problem. In Section 8 we derive at least restrictions on the support of μ.

Let us now state the N-dimensional long time homogenization problem.

Definition 2.10 (N-dimensional homogenization problem)**.** *Let $N \in \mathbb{N}$, $N \geq 2$ be arbitrary. Let $T, R > 0$ and let $A(\cdot) \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$ be $(0,1)^N$-periodic. Moreover, let $A(y) \in M(\alpha,\beta)$ be symmetric and admissible in the sense of Definition*

1.1 for every $y \in \mathbb{R}^N$. Let $c_0, d_0 \in C_c^\infty(B_R(0))$. We call $\bar{u}_\varepsilon(x,\tau)$ a solution to the N-dimensional long time homogenization problem if $\bar{u}_\varepsilon \in L^\infty(0,T/\varepsilon^2; H^2(\mathbb{R}^N))$, $\partial_\tau \bar{u}_\varepsilon \in L^\infty(0,T/\varepsilon^2; H^1(\mathbb{R}^N))$ and

$$\begin{aligned} \partial_\tau^2 \bar{u}_\varepsilon(x,\tau) &= \nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon(x,\tau)\Big)\,, \\ \bar{u}_\varepsilon(x,0) &= c_0(x), \\ \partial_t \bar{u}_\varepsilon(x,0) &= d_0(x). \end{aligned} \tag{2.8}$$

Analogous to the one-dimensional case, see Definition 2.2, the time-scaled homogenization problem in N space dimensions reads as follows.

Definition 2.11 (Time-scaled homogenization problem in N space dimensions)**.** *Consider the setting of Definition 2.10. We denote by $u_\varepsilon(x,t)$ the unique solution to*

$$\begin{aligned} \varepsilon^4 \partial_t^2 u_\varepsilon(x,t) &= \nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big) \nabla u_\varepsilon(x,t)\Big)\,, \\ u_\varepsilon(x,0) &= c_0(x), \\ \partial_t u_\varepsilon(x,0) &= \frac{1}{\varepsilon^2} d_0(x) \end{aligned}$$

for $(x,t) \in \mathbb{R}^N \times (0,T)$.

We are now in the position to introduce the N-dimensional energy measure μ and to state the main results in the abstract framework.

Definition 2.12 (Energy density and energy measure)**.** *Let $N \in \mathbb{N}$ be arbitrary.*

For $N = 1$ let u_ε be the solution to the one-dimensional time-scaled homogenization problem of Definition 2.2. For $N \geq 2$ let u_ε be the solution to the multi-dimensional time-scaled homogenization problem of Definition 2.11.

1. *We define the N-dimensional energy density $E_\varepsilon(\xi,t)$ by*

$$E_\varepsilon(\xi,t) := \frac{1}{\varepsilon^{2N}} \left[\left(A\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon(\cdot,t)\right)^T A\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon(\cdot,t) \right] \left(\frac{\xi}{\varepsilon^2}\right). \tag{2.9}$$

2. *We call $\mu \in L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ an energy measure if there exists a subsequence E_{ε_k} of E_ε such that $E_{\varepsilon_k} \overset{*}{\rightharpoonup} \mu$ in $L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$.*

The existence of a weak star limit μ is shown in Section 6. It is a direct consequence of the energy estimate for the time-scaled wave equation, which can be found in the appendix.

Remark 2.13. *In the one-dimensional case, $N = 1$, the energy density E_ε of Definition 2.12 is defined through a homogenization problem with well adapted initial data. In this setting, we will use the fine results of Subsection 2.1 to identify the one-dimensional energy measure μ, see* (2.10) *below.*

For non-adapted initial data we can only prove restrictions on the support of μ. More precisely, the result of Theorem 2.15 below is also valid for the one-dimensional homogenization problem with non-adapted initial data.

In Subsection 6.1 we prove that for $N=1$ the energy measure μ of Definition 2.12 is in fact a Dirac measure,

$$\mu(\xi,t)=\left[(a^*)^2\|\partial_x c_0^+\|_{L^2(\mathbb{R})}^2\right]\delta_{c^*t}(\xi)+\left[(a^*)^2\|\partial_x c_0^-\|_{L^2(\mathbb{R})}^2\right]\delta_{-c^*t}(\xi), \tag{2.10}$$

where c_0^+ is the right going part of the initial data and c_0^- is the left going part, respectively. Eq. (2.10) reflects the fact that the solution to the one-dimensional long time homogenization problem is approximatively equal to two waves propagating with speed c^* in opposite directions. We observe that, unlike Theorem 2.5 and Corollary 2.8, the energy measure μ doesn't contain any information about the fine properties of the solution.

In the multi-dimensional case, $N\geq 2$, a rigorous identification of μ seems to be out of reach. In Subsection 7.3 we introduce the notion of effective speeds. More precisely, we define the *energetic effective speed* $\hat{c}$ through the slope of the smallest cone in space-time which contains the support of any energy measure μ. We set

$$\hat{c}:=\inf\left\{c>0\,|\,\operatorname{supp}\mu\subseteq C(c)\text{ for every energy measure}\,\mu\right\}, \tag{2.11}$$

where $C(c)$ denotes the cone with slope $\frac{1}{c}$, $C(c):=\{(\xi,t)\in\mathbb{R}^N\times(0,T)\,|\,|\xi|\leq ct\}$.

Our main result in the multi-dimensional setting is Theorem 2.15 below. It provides an upper bound for the energetic effective speed $\hat{c}$ in terms of another homogenization problem. To be more precise, we exploit a connection between u_ε, solution to the multi-dimensional homogenization problem of Definition 2.11, and the Riemannian distance q_ε according to $A\left(\frac{\cdot}{\varepsilon}\right)$.

Definition 2.14 (Riemannian distance)**.** *Let $N\in\mathbb{N}$ be arbitrary. Let $A(\cdot)$ be as in Definition 2.10 and let $x_0\in\mathbb{R}^N$ be fixed. We define $q_\varepsilon(x)$ as the unique viscosity solution to the Hamilton-Jacobi equation*

$$\begin{gathered}(\nabla q_\varepsilon)^T(x)A\Big(\frac{x}{\varepsilon}\Big)\nabla q_\varepsilon(x)=1,\;\; q_\varepsilon(x)>0\;\text{ in }\mathbb{R}^N\setminus\{x_0\},\\ q_\varepsilon(x_0)=0.\end{gathered}\tag{2.12}$$

In what follows we assume that the matrix $A\left(\frac{\cdot}{\varepsilon}\right)$ is in fact a scalar function, $A\left(\frac{\cdot}{\varepsilon}\right)=a\left(\frac{\cdot}{\varepsilon}\right)\,Id_N$ with Id_N denoting the $N\times N$ unit matrix. Due to Proposition 7.7 of Subsection 7.3 the Riemannian distance q_ε converges uniformly on $\mathbb{R}^N$ to the effective distance $\bar{q}$,

$$\bar{q}(x)=|x-x_0|\bar{b}\left(\frac{x-x_0}{|x-x_0|}\right)$$

with an effective cost function $\bar{b}$. We define the *geometric effective speed* $\bar{c}$ as the maximal value of $1/\bar{b}$,

$$\bar{c}:=\left(\min_{|x|=1}\bar{b}(x)\right)^{-1}. \tag{2.13}$$

Let us now state the main result in the N-dimensional abstract setting. The proof of Theorem 2.15 is given in Section 8.

Theorem 2.15 (Upper bound for the energetic effective speed)**.** *Let* $N \in \mathbb{N}$ *be arbitrary. Let* $\hat{c}$ *be the energetic effective speed of* (2.11) *and let* $\bar{c}$ *be the geometric effective speed of* (2.13) *Then the following inequality holds*

$$\hat{c} \leq \bar{c}. \tag{2.14}$$

The theorem provides an upper bound for the energetic effective speed $\hat{c}$ in terms of the effective distance $\bar{q}$. However, this bound is not optimal. The non-optimality is shown in Subsection 7.4 using the fine results of the one-dimensional case.

Part II
The one-dimensional case

3 Weakly dispersive equation and the lKdV-problems

This section is devoted to the proof of Theorem 2.7. We will discuss the well-posedness of the lKdV-problem in Subsection 3.1. After some preliminaries in Subsection 3.2, the proof of Theorem 2.7 is given in Subsection 3.3.

3.1 The lKdV-problems and their shifts

Let us start with some preliminaries concerning the *left going* and the *right going* part of the initial data. In fact, due to classical theory, each solution to the one-dimensional wave equation

$$\partial_t^2 u(x,t) - (c^*)^2 \partial_x^2 u(x,t) = 0$$

is given by

$$u(x,t) = f(x - c^* t) + g(x + c^* t).$$

In particular,

$$u(x,0) = f(x) + g(x) \qquad \text{and} \qquad \partial_t u(x,0) = -c^* \partial_x f(x) + c^* \partial_x g(x).$$

Consequently, the following definition is useful.

Definition 3.1 (Decomposition of the initial data)**.** *Let $c^* > 0$. We define $P^+_{c^*}(c_0, d_0)$ and $P^-_{c^*}(c_0, d_0)$ as solutions of*

$$\begin{aligned} \big(P^+_{c^*}(c_0,d_0)\big)(x) + \big(P^-_{c^*}(c_0,d_0)\big)(x) &= c_0(x), \\ -c^*\partial_x\big(P^+_{c^*}(c_0,d_0)\big)(x) + c^*\partial_x\big(P^-_{c^*}(c_0,d_0)\big)(x) &= d_0(x). \end{aligned}$$

In the following, we will use the abbreviations $c_0^+ := P^+_{c^*}(c_0,d_0)$ and $c_0^- := P^-_{c^*}(c_0,d_0)$.

Remark 3.2. *The decomposition of Definition 3.1 satisfies the following properties.*

1. *The projections c_0^+, c_0^- are uniquely defined up to an additive constant.*
2. *In the case of smooth initial data, $c_0, d_0 \in C^\infty\big((-R,R)\big)$, also their projections are smooth, i.e. c_0^+, $c_0^- \in C^\infty\big((-R,R)\big)$.*
3. *In the case that $c_0, d_0 \in C_c^\infty((-R,R))$ with $\int_{-R}^R d_0(x)\,dx = 0$, there exist unique compactly supported projections, $c_0^+, c_0^- \in C_c^\infty((-R,R))$.*

We will always work with 3. and have unique projections. In this case

$$\begin{aligned} c_0^+(x) &= \frac{1}{2} c_0(x) + \frac{1}{2c^*}\int_x^R d_0(\xi)\,d\xi, \\ c_0^-(x) &= \frac{1}{2} c_0(x) - \frac{1}{2c^*}\int_x^R d_0(\xi)\,d\xi. \end{aligned} \tag{3.1}$$

Moreover, we remark that in our case the wave speed as well as the initial data are rescaled, i.e. we consider the wave speed c^*/ε^2 and initial data $(c_0, d_0/\varepsilon^2)$. Nevertheless, a simple calculation shows that $P^{\pm}_{c^*/\varepsilon^2}(c_0, d_0/\varepsilon^2) = P^{\pm}_{c^*}(c_0, d_0)$.

Next, let us discuss existence and properties of the lKdV-solution $W^+(x,t)$. We omit the analysis of left moving initial data $c_0^- = P^-_{c^*}(c_0, d_0)$ for convenience, since it can be handled in exactly the same way.

Proposition 3.3 (Existence of the lKdV-solution and its shift on $\mathbb{R}$)**.** *Let $c_0^+ \in C_c^\infty((-R,R))$. Consider the right going lKdV-problem of Definition 2.4. Then,*

1. *there exists a unique solution W^+ and a constant C such that*

$$\begin{aligned}
&\|\partial_t\partial_x^3 W^+(x,t)\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\|c_0^+(x)\|^2_{H^6((-R,R))},\\
&\|\partial_t^2\partial_x^2 W^+(x,t)\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\|c_0^+(x)\|^2_{H^9((-R,R))}.
\end{aligned}\tag{3.2}$$

2. *The shifted function w_ε^+ of* (2.3) *satisfies $w_\varepsilon^+, \partial_t w_\varepsilon^+, \partial_t^2 w_\varepsilon^+ \in L^\infty\big(0,T;H^2(\mathbb{R})\big)$.*

Here, $H^n((-R,R))$ as usually denotes the Sobolev-space $H^n((-R,R)) := \{u(x) \in L^2((-R,R)) \textit{ with } \partial_x^i u(x) \in L^2((-R,R)) \textit{ for } i = 1,...,n\}$ equipped with the norm $\|u\|_{H^n((-R,R))} := \sum_{i=0}^n \|\partial_x^i u\|_{L^2((-R,R))}$. Moreover, $H^0((-R,R)) := L^2((-R,R))$.

Proof. **First step.** In the first step we show the existence of a solution W^+ on bounded intervals $\Omega = (-L, L)$.

Claim 1. On a fixed bounded domain $\Omega = (-L,L)$ with $L > R$, the lKdV-problem with boundary condition

$$W^+(-L,t) = W^+(L,t) = \partial_x W^+(L,t) = 0$$

has a unique classical solution W^+. Moreover, there exists an Ω-independent constant C, such that

$$\|\partial_t\partial_x^3 W^+(x,t)\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\|c_0^+(x)\|^2_{H^6(\Omega)}, \tag{3.3}$$

$$\|\partial_t^2\partial_x^2 W^+(x,t)\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\|c_0^+(x)\|^2_{H^9(\Omega)}. \tag{3.4}$$

A proof of existence and uniqueness of a classical solution can be found in Ref.[17]. Let us merely remark that fixing the spatial derivative of W^+ at the right endpoint of the interval is necessary to ensure the well-posedness of the boundary value problem.

We have to show the energy estimates in (3.3) and (3.4). Actually, multiplying the lKdV-equation by $W^+(x,t)$, integrating over Ω and applying integration by parts leads to

$$\begin{aligned}
0 &= \frac{1}{2}\frac{d}{dt}\|W^+(\cdot,t)\|^2_{L^2(\Omega)} - \frac{a_2^*}{2c^*}\int_\Omega \frac{1}{2}\frac{d}{dx}(\partial_x W^+)^2(x,t)\,dx\\
&= \frac{1}{2}\frac{d}{dt}\|W^+(\cdot,t)\|^2_{L^2(\Omega)} - \frac{a_2^*}{4c^*}\Big((\partial_x W^+)^2(L,t) - (\partial_x W^+)^2(-L,t)\Big)\\
&= \frac{1}{2}\frac{d}{dt}\|W^+(\cdot,t)\|^2_{L^2(\Omega)} + \frac{a_2^*}{4c^*}(\partial_x W^+)^2(-L,t)\\
&\ge \frac{1}{2}\frac{d}{dt}\|W^+(\cdot,t)\|^2_{L^2(\Omega)}
\end{aligned}$$

and thus for each $t \in (0, T)$

$$\|W^+(\cdot, t)\|^2_{L^2(\Omega)} \le \|W^+(\cdot, 0)\|^2_{L^2(\Omega)} = \|c_0^+(x)\|^2_{L^2(\Omega)}. \tag{3.5}$$

Next, we differentiate the lKdV-equation twice with respect to t arriving at

$$\begin{aligned} \partial_t(\partial_t^2 W^+(x,t)) + \frac{a_2^*}{2c^*}\partial_x^3(\partial_t^2 W^+(x,t)) &= 0, \\ \partial_t^2 W^+(x,0) &= \frac{(a_2^*)^2}{4(c^*)^2}\partial_x^6 c_0^+(x). \end{aligned}$$

In exactly the same way as above we conclude

$$\|\partial_t^2 W^+(\cdot, t)\|^2_{L^2(\Omega)} \le C\|c_0^+(x)\|^2_{H^6(\Omega)}$$

and thus, taking into account $\partial_t^2 W^+ = -\frac{a_2^*}{2c^*}\partial_t\partial_x^3 W^+$, estimate (3.3) follows. To obtain (3.4) we differentiate once more with respect to t to find

$$\|\partial_t^2\partial_x^3 W^+(x,t)\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\|c_0^+(x)\|^2_{H^9(\Omega)}.$$

An interpolation argument provides the control of $\|\partial_t^2\partial_x^2 W^+(x,t)\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}$. This proves Claim 1.

Second step. Since the estimates in (3.3),(3.4) are independent of $\Omega = (-L, L)$, one obtains a solution W^+ to the lKdV-problem on $\mathbb{R}$ by considering $L \to \infty$. Then W^+ satisfies the same bounds as its approximations.

The regularity of the shifted function $w_\varepsilon^+(x,t) = W^+\left(x - \frac{c^*}{\varepsilon^2}t, t\right)$ follows directly from the regularity of W^+. □

3.2 Equation for v_ε in the moving frame

In what follows, we use that v_ε from Definition 2.3 has some regularity; a proof of existence and regularity can be found in Subsection 4.1. To motivate the statement of Theorem 2.7, let us firstly decompose the weakly dispersive solution $v_\varepsilon = v_\varepsilon^+ + v_\varepsilon^-$ as in Definition 3.1. To be more precise, we solve the weakly dispersive problem of Definition 2.3 with initial data that belong to a right going wave,

$$v_\varepsilon^+(x,0) = c_0^+(x) \quad \text{and} \quad \partial_t v_\varepsilon^+(x,0) = -\frac{c^*}{\varepsilon^2}\partial_x c_0^+(x). \tag{3.6}$$

We remark that the analysis of the left going part v_ε^- is analogous.

Observation 1 (Equation in the moving frame)**.** *Let v_ε^+ be the weakly dispersive solution of Definition 2.3 with initial data* (3.6)*. Then, the shift*

$$V_\varepsilon^+(x,t) := v_\varepsilon^+\left(x + \frac{c^* t}{\varepsilon^2}, t\right)$$

satisfies the equation

$$-2c^*\partial_x\left(\partial_t V_\varepsilon^+ + \frac{a_2^*}{2c^*}\partial_x^3 V_\varepsilon^+\right) = -\varepsilon^2\partial_t\left(\partial_t V_\varepsilon^+ + 2\frac{a_2^*}{c^*}\partial_x^3 V_\varepsilon^+\right) + \varepsilon^4\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 V_\varepsilon^+. \tag{3.7}$$

This is a direct consequence of the chain rule.

Loosely speaking, Observation 1 suggests that the shift V_ε^+ satisfies the lKdV-equation

$$\partial_t V_\varepsilon^+(x,t) + \frac{a_2^*}{2c^*}\partial_x^3 V_\varepsilon^+(x,t) = O(\varepsilon^2)$$

and can therefore be compared to leading order with $W^+(x,t)$. In order to make the connection precise, we now start from the lKdV-solution $W^+(x,t)$ and derive a corresponding equation for $w_\varepsilon^+(x,t)$.

Lemma 3.4. *Consider the lKdV-solution $W^+(x,t)$ of Definition 2.4 and its shift $w_\varepsilon^+(x,t) = W^+(x - \frac{c^*t}{\varepsilon^2}, t)$. Then $w_\varepsilon^+(x,t)$ solves the following problem*

$$\begin{aligned}
\varepsilon^4\partial_t^2 w_\varepsilon^+(x,t) - a^*\partial_x^2 w_\varepsilon^+(x,t) - \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 w_\varepsilon^+(x,t)& \\
= -\varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 W^+\left(x - \frac{c^*}{\varepsilon^2}t, t\right) &+ \varepsilon^4\frac{3a_2^*}{2c^*}\partial_x^3\partial_t W^+\left(x - \frac{c^*}{\varepsilon^2}t, t\right),\\
w_\varepsilon^+(x,0) = c_0^+(x),&\\
\partial_t w_\varepsilon^+(x,0) = -\frac{c^*}{\varepsilon^2}\partial_x c_0^+(x) - \frac{a_2^*}{2c^*}\partial_x^3 c_0^+(x).&
\end{aligned}$$

Proof. Analogous to Observation 1, the chain rule yields

$$0 = \partial_t W^+ + \frac{a_2^*}{2c^*}\partial_x^3 W^+ = \frac{c^*}{\varepsilon^2}\partial_x w_\varepsilon^+ + \partial_t w_\varepsilon^+ + \frac{a_2^*}{2c^*}\partial_x^3 w_\varepsilon^+. \tag{3.8}$$

Now, we apply the partial differential operator $\varepsilon^4\partial_t - \varepsilon^2 c^*\partial_x$ to (3.8). Inserting the term $\varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 w_\varepsilon^+$, one arrives at

$$\begin{aligned}
0 =&\varepsilon^4\partial_t^2 w_\varepsilon^+ - a^*\partial_x^2 w_\varepsilon^+ - \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 w_\varepsilon^+ + \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 w_\varepsilon^+ + \varepsilon^2\frac{a_2^*}{2c^*}\left(\varepsilon^2\partial_t\partial_x^3 w_\varepsilon^+ - c^*\partial_x^4 w_\varepsilon^+\right)\\
=&\varepsilon^4\partial_t^2 w_\varepsilon^+ - a^*\partial_x^2 w_\varepsilon^+ - \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 w_\varepsilon^+ + \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 W^+ - \varepsilon^4\frac{3a_2^*}{2c^*}\partial_x^3\partial_t W^+,
\end{aligned}$$

which is the claimed result. The initial data are again a direct consequence of the chain rule. ☐

3.3 Proof of Theorem 2.7

We are now in the position to prove Theorem 2.7. The main estimate is contained in Proposition 3.5 below for right moving initial data. We will observe later that Theorem 2.7 can easily be derived from that proposition. In what follows, C denotes different ε-independent constants.

Proposition 3.5. *Let v_ε^+ be the solution to the weakly dispersive problem of Definition 2.3 with initial data* (3.6). *Let W^+ be the solution to the right going lKdV-problem of Definition 2.4 and let w_ε^+ be its shift, $w_\varepsilon^+(x,t) = W^+(x - \frac{c^*t}{\varepsilon^2}, t)$. Then there exists an ε-independent constant C such that*

$$\|\partial_x v_\varepsilon^+ - \partial_x w_\varepsilon^+\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\varepsilon^2, \tag{3.9}$$

$$\|v_\varepsilon^+ - w_\varepsilon^+\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_t v_\varepsilon^+ - \partial_t w_\varepsilon^+\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C. \tag{3.10}$$

Proof. Since v_ε^+ solves the weakly dispersive equation, Lemma 3.4 yields that the difference $w_\varepsilon^+ - v_\varepsilon^+$ satisfies

$$\begin{aligned}
\varepsilon^4\partial_t^2\big(w_\varepsilon^+ - v_\varepsilon^+\big)(x,t) - a^*\partial_x^2\big(w_\varepsilon^+ - v_\varepsilon^+\big)(x,t) - \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2\big(w_\varepsilon^+ - v_\varepsilon^+\big)(x,t)\\
= -\varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 W^+\left(x-\frac{c^*}{\varepsilon^2}t,t\right) + \varepsilon^4\frac{3a_2^*}{2c^*}\partial_x^3\partial_t W^+\left(x-\frac{c^*}{\varepsilon^2}t,t\right). \qquad (3.11)
\end{aligned}$$

Now, we multiply (3.11) by $\partial_t\big(w_\varepsilon^+ - v_\varepsilon^+\big)$ and integrate over $\mathbb{R}$. We then apply integration by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\varepsilon^4\|\partial_t w_\varepsilon^+(\cdot,t) - \partial_t v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2 + a^*\|\partial_x w_\varepsilon^+(\cdot,t) - \partial_x v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2\\
&\quad + \varepsilon^6\frac{a_2^*}{a^*}\|\partial_t\partial_x w_\varepsilon^+(\cdot,t) - \partial_t\partial_x v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2\Big)\\
&= -\int_{\mathbb{R}}\left(\varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2W^+ + \varepsilon^4\frac{3a_2^*}{2c^*}\partial_x^3\partial_tW^+\right)\left(x-\frac{c^*}{\varepsilon^2}t,t\right)\left(\partial_t w_\varepsilon^+ - \partial_t v_\varepsilon^+\right)(x,t)\,dx\\
&\le \frac{4}{\varepsilon^4}\left(\left\|\varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2W^+\left(\cdot-\frac{c^*}{\varepsilon^2}t,t\right)\right\|_{L^2(\mathbb{R})}^2 + \left\|\varepsilon^4\frac{3a_2^*}{2c^*}\partial_x^3\partial_tW^+\left(\cdot-\frac{c^*}{\varepsilon^2}t,t\right)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad + \frac{\varepsilon^4}{2}\|\partial_t w_\varepsilon^+(\cdot,t) - \partial_t v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2\\
&= 4\left(\varepsilon^8\left\|\frac{a_2^*}{a^*}\partial_t^2\partial_x^2W^+\left(\cdot-\frac{c^*}{\varepsilon^2}t,t\right)\right\|_{L^2(\mathbb{R})}^2 + \varepsilon^4\left\|\frac{3a_2^*}{2c^*}\partial_x^3\partial_tW^+\left(\cdot-\frac{c^*}{\varepsilon^2}t,t\right)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad + \frac{\varepsilon^4}{2}\|\partial_t w_\varepsilon^+(\cdot,t) - \partial_t v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2
\end{aligned}$$

for almost every $t \in [0,T]$. Next, we define

$$\begin{aligned}
A(t) :=&\frac{1}{2}\Big(\varepsilon^4\|\partial_t w_\varepsilon^+(\cdot,t) - \partial_t v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2 + a^*\|\partial_x w_\varepsilon^+(\cdot,t) - \partial_x v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2\\
&+ \varepsilon^6\frac{a_2^*}{a^*}\|\partial_t\partial_x w_\varepsilon^+(\cdot,t) - \partial_t\partial_x v_\varepsilon^+(\cdot,t)\|_{L^2(\mathbb{R})}^2\Big).
\end{aligned}$$

Taking into account

$$\big(w_\varepsilon^+ - v_\varepsilon^+\big)(x,0) = 0 \qquad \text{and} \qquad \partial_t\big(w_\varepsilon^+ - v_\varepsilon^+\big)(x,0) = -\frac{a_2^*}{2c^*}\partial_x^3 c_0^+(x)$$

and applying the Gronwall Lemma finally leads to

$$\begin{aligned}
A(t) \le& C\left(\frac{\varepsilon^4}{2}\left\|\frac{a_2^*}{2c^*}\partial_x^3c_0^+(x)\right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^6 a_2^*}{2a^*}\left\|\frac{a_2^*}{2c^*}\partial_x^4c_0^+(x)\right\|_{L^2(\mathbb{R})}^2\right.\\
&+ \varepsilon^8\left\|\frac{a_2^*}{a^*}\partial_t^2\partial_x^2W^+\left(x-\frac{c^*}{\varepsilon^2}t,t\right)\right\|_{L^2(0,T;L^2(\mathbb{R}))}^2\\
&\left.+\varepsilon^4\left\|\frac{3a_2^*}{2c^*}\partial_x^3\partial_tW^+\left(x-\frac{c^*}{\varepsilon^2}t,t\right)\right\|_{L^2(0,T;L^2(\mathbb{R}))}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\varepsilon^4\|c_0^+(x)\|^2_{H^4((-R,R))}+\varepsilon^8\left\|\partial_t^2\partial_x^2W^+\left(x-\frac{c^*}{\varepsilon^2}t,t\right)\right\|^2_{L^2(0,T;L^2(\mathbb{R}))}\right.\\
&\quad\left.+\,\varepsilon^4\left\|\partial_x^3\partial_tW^+\left(x-\frac{c^*}{\varepsilon^2}t,t\right)\right\|^2_{L^2(0,T;L^2(\mathbb{R}))}\right)\\
&\overset{(1)}{\leq} C\varepsilon^4\|c_0^+(x)\|^2_{H^9((-R,R))}\leq C\varepsilon^4.
\end{aligned}
$$

for almost every $t\in[0,T]$. Let us remark that inequality (1) is a consequence of Proposition 3.3. This shows (3.9) and that

$$
\|\partial_t w_\varepsilon^+-\partial_t v_\varepsilon^+\|_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C. \tag{3.12}
$$

Finally, to prove inequality (3.10), one simply writes

$$
\begin{aligned}
(w_\varepsilon^+-v_\varepsilon^+)(x,t)&=(w_\varepsilon^+-v_\varepsilon^+)(x,0)+\int_0^t(\partial_t w_\varepsilon^+-\partial_t v_\varepsilon^+)(x,\tau)\,d\tau\\
&=\int_0^t(\partial_t w_\varepsilon^+-\partial_t v_\varepsilon^+)(x,\tau)\,d\tau
\end{aligned}
$$

and applies (3.12). □

In exactly the same way the analogous result can be stated for the left going weakly dispersive solution v_ε^- and its shift $w_\varepsilon^-(x,t):=W^-(x+\frac{c^*t}{\varepsilon^2},t)$. Due to $v_\varepsilon=v_\varepsilon^++v_\varepsilon^-$ and the linearity of the weakly dispersive problem one obtains the following result.

Proposition 3.6. *Let v_ε be the weakly dispersive solution of Definition 2.3 and let $w_\varepsilon^+,w_\varepsilon^-$ be as in Theorem 2.7. Then the following estimates are valid*

$$
\|\partial_x v_\varepsilon-\partial_x\big(w_\varepsilon^++w_\varepsilon^-\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C\varepsilon^2, \tag{3.13}
$$

$$
\|v_\varepsilon-\big(w_\varepsilon^++w_\varepsilon^-\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))}+\|\partial_t v_\varepsilon-\partial_t\big(w_\varepsilon^++w_\varepsilon^-\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C. \tag{3.14}
$$

We are now in the position to prove Theorem 2.7.

Proof of Theorem 2.7. Ineq. (2.5) appears in Proposition 3.6.

We have to prove (2.6), that

$$
\|v_\varepsilon(x,t)-\big(w_\varepsilon^++w_\varepsilon^-\big)(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}\leq C\varepsilon. \tag{3.15}
$$

Considering the difference $z_\varepsilon(x,t):=\Big(v_\varepsilon-\big(w_\varepsilon^++w_\varepsilon^-\big)\Big)(x,t)$, we firstly claim that for every fixed $y_0\in\mathbb{R}$

$$
\sup_{t\in[0,T]}\ \sup_{|x-y_0|<\frac{1}{\varepsilon^2}}|z_\varepsilon(x,t)-z_\varepsilon(y_0,t)|\leq C\varepsilon \tag{3.16}
$$

with C independent of ε and y_0.

Indeed, due to Jensen's inequality and Proposition 3.6 one obtains

$$
\begin{aligned}
&\sup_{t\in[0,T]}\ \sup_{|x-y_0|<\frac{1}{\varepsilon^2}}|z_\varepsilon(x,t)-z_\varepsilon(y_0,t)|^2=\sup_{t\in[0,T]}\ \sup_{|x-y_0|<\frac{1}{\varepsilon^2}}\left|\left(\int_{y_0}^x\partial_x z_\varepsilon(\xi,t)\,d\xi\right)\right|^2\\
&\leq\sup_{t\in[0,T]}\ \sup_{|x-y_0|<\frac{1}{\varepsilon^2}}|x-y_0|\left|\int_{y_0}^x(\partial_x z_\varepsilon)^2(\xi,t)\,d\xi\right|\\
&\leq\frac{1}{\varepsilon^2}\|\partial_x z_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C\varepsilon^2,
\end{aligned}
$$

which shows (3.16).

We are now in the position to prove estimate (3.15). Let $t \in (0, T)$ be arbitrary and fixed. We divide $\mathbb{R}$ into countably many intervals $I_k := (\frac{k}{\varepsilon^2}, \frac{k+1}{\varepsilon^2})$ with $k \in \mathbb{Z}$. Due to Ineq. (3.14) there exists some constant C, independent of ε, k and t, such that the following holds:

$$\forall \varepsilon > 0,\ \forall k \in \mathbb{Z}\ \exists\, y_k(\varepsilon) \in I_k \text{ such that } z_\varepsilon(y_k(\varepsilon), t) \leq C\varepsilon. \tag{3.17}$$

Indeed, in the opposite case,

$$\int_{I_k} |z_\varepsilon(x,t)|^2\, dx \geq |I_k| C^2 \varepsilon^2 = C^2.$$

Choosing C appropriately shows (3.17). Estimate (3.16) then implies

$$\|z_\varepsilon(\cdot, t)\|_{L^\infty(I_k)} \leq C_1 \varepsilon$$

with C_1 independent of ε, k and t, which was the claim (3.15).

□

4 The original homogenization problem and the weakly dispersive equation

This section is devoted to the proof of Theorem 2.5. We will study properties of the weakly dispersive problem in Subsection 4.1. In Subsection 4.2 we introduce and discuss the adaption operator $\mathcal{A}_\varepsilon$, which is an essential tool in our analysis. Finally, in Subsection 4.3 the proof of Theorem 2.5 is given.

4.1 The weakly dispersive problem

In this subsection, we prove existence and uniqueness of a solution to the weakly dispersive problem of Definition 2.3. Besides, energy estimates and regularity properties are discussed. As for the lKdV-problem, we firstly construct solutions on bounded intervals $\Omega = (-L, L)$ and show that the corresponding energy estimates are Ω-independent.

Proposition 4.1 (Existence on $\mathbb{R}$ and energy estimates). *Let $k \in \mathbb{N}$ and let $c_0, d_0 \in C_c^\infty\big((-R,R)\big)$. Consider the weakly dispersive problem of Definition 2.3. Then there exists a unique solution v_ε such that $v_\varepsilon \in L^\infty(0,T;H^k(\mathbb{R})), \partial_t v_\varepsilon \in L^\infty(0,T;H^{k-1}(\mathbb{R}))$, $\partial_t^2 v_\varepsilon \in L^2(0,T;H^{k-1}(\mathbb{R}))$. Moreover, the following energy estimate is valid*

$$\begin{aligned}\|\partial_x v_\varepsilon\|_{L^\infty(0,T;H^{k-1}(\mathbb{R}))} &+ \varepsilon^2 \|\partial_t v_\varepsilon\|_{L^\infty(0,T;H^{k-1}(\mathbb{R}))} \\ &+ \varepsilon^5 \|\partial_t^2 v_\varepsilon\|_{L^2(0,T;H^{k-1}(\mathbb{R}))} \le C\, K_k(c_0, d_0),\end{aligned} \tag{4.1}$$

where the constant C is independent of ε and
$K_k(c_0,d_0) := \Big(\|\partial_x c_0\|^2_{H^{k-1}((-R,R))} + \|d_0\|^2_{H^{k-1}((-R,R))} + \varepsilon^2 \|\partial_x d_0\|^2_{H^{k-1}((-R,R))}\Big)^{1/2}.$

Proof. Our aim is to apply the Rothe time discretization method. More precisely, we discretize the time variable by introducing a finite number of time-steps t_j and replace the time-derivative by a difference quotient $\partial_t v_\varepsilon(t_j) := \frac{v_\varepsilon(t_j) - v_\varepsilon(t_{j-1})}{\Delta t}$. In each time-step t_j we then solve an ordinary differential equation on $\Omega = (-L, L)$ and derive a priori estimates which are independent of the domain Ω. Considering $L \to \infty$ provides a solution on $\mathbb{R}$ in each time-step. Finally, we consider the limit $\Delta t \to 0$ in the discretization scheme. In the following we will merely give the proof of the corresponding a priori estimates for $k = 1$ and omit the details of the method. Actually, one derives estimates for higher order spatial derivatives by differentiating the weakly dispersive equation with respect to x.

First energy estimates. Multiplying the weakly dispersive equation of Definition 2.3 by $\partial_t v_\varepsilon$ and integrating over Ω, one obtains

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon^4 \|\partial_t v_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + a^* \|\partial_x v_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \varepsilon^6 \frac{a_2^*}{a^*} \|\partial_t \partial_x v_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}\right) = 0.$$

Next, we integrate over $[0,t]$ arriving at

$$\begin{aligned}
\varepsilon^4\|\partial_t v_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} &+ a^*\|\partial_x v_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \varepsilon^6\frac{a_2^*}{a^*}\|\partial_t\partial_x v_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} \\
&= \varepsilon^4\left\|\frac{1}{\varepsilon^2}d_0(x)\right\|^2_{L^2(\Omega)} + a^*\|\partial_x c_0(x)\|^2_{L^2(\Omega)} + \varepsilon^6\frac{a_2^*}{a^*}\left\|\frac{1}{\varepsilon^2}\partial_x d_0(x)\right\|^2_{L^2(\Omega)} \\
&= \|d_0(x)\|^2_{L^2((-R,R)} + a^*\|\partial_x c_0(x)\|^2_{L^2((-R,R))} + \varepsilon^2\frac{a_2^*}{a^*}\|\partial_x d_0(x)\|^2_{L^2((-R,R))} \\
&\le C\big(K_1(c_0,d_0)\big)^2
\end{aligned}$$

for each $t\in[0,T]$, which proves the estimates for the first two terms in (4.1).

Second energy estimates. It remains to verify the estimate for $\partial_t^2 v_\varepsilon(x,t)$. Actually,

$$\varepsilon^4\partial_t^2 v_\varepsilon(x,t) - \varepsilon^6\frac{a_2^*}{a^*}\partial_x^2\partial_t^2 v_\varepsilon(x,t) - a^*\partial_x^2 v_\varepsilon(x,t) = 0. \tag{4.2}$$

Multiplying Eq. (4.2) by $\partial_t^2 v_\varepsilon(x,t)$, integrating over $\Omega\times(0,T)$ and applying integration by parts leads to

$$\begin{aligned}
\varepsilon^4\|\partial_t^2 v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} &+ \varepsilon^6\frac{a_2^*}{a^*}\|\partial_t^2\partial_x v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \\
&= -\int_0^T\int_\Omega a^*\partial_x v_\varepsilon(x,t)\,\partial_t^2\partial_x v_\varepsilon(x,t)\,dx\,dt \\
&\le \|a^*\partial_x v_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\,\|\partial_t^2\partial_x v_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\le \varepsilon^6\gamma\|\partial_x\partial_t^2 v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{\varepsilon^6\gamma}\|a^*\partial_x v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

where we choose $\gamma\in(0,\frac{a_2^*}{a^*})$. Hence,

$$\begin{aligned}
\varepsilon^4\|\partial_t^2 v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} + \varepsilon^6\|\partial_t^2\partial_x v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} &\le \frac{C}{\varepsilon^6}\|\partial_x v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \\
&\le \frac{C}{\varepsilon^6}\left(K_1(c_0,d_0)\right)^2
\end{aligned}$$

due to the first energy estimates. Consequently,

$$\|\partial_t^2 v_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le \frac{1}{\varepsilon^5}C\,K_1(c_0,d_0),$$

which is the claimed result. □

4.2 The adaption operator

The aim of this subsection is to adapt a function $f(x,t)$ to the micro-structure of the material. More precisely, our aim is to construct an operator $\mathcal{A}_\varepsilon$ such that $\partial_x\Big(a\big(\frac{x}{\varepsilon}\big)\partial_x\big(\mathcal{A}_\varepsilon(f)\big)\Big)$ can be expanded in derivatives of f, i.e.

$$\partial_x\Big(a\Big(\frac{x}{\varepsilon}\Big)\partial_x\big(\mathcal{A}_\varepsilon(f)\big)\Big) - \mathcal{A}_\varepsilon\Big(\sum_{i=0}^m \varepsilon^i a_i^*\partial_x^{2+i}f\Big) = \rho, \tag{4.3}$$

where ρ is a small error, $m \in \mathbb{N}$ and $a_i^* \in \mathbb{R}$. The concept of $\mathcal{A}_\varepsilon$ is related to the classical asymptotic expansion approach, see for instance Ref.[23]. If (4.3) is satisfied, we have a way to replace the elliptic operator with oscillating coefficients by a finite sum of differential operators with constant coefficients.

Let us start with some preliminaries. Considering $Y = (0,1)$ as in the introduction, we denote by $H^1_{per}(Y)$ the closure of $\{u(\cdot) \in C^\infty(\mathbb{R}) \mid u \text{ } is \text{ } Y\text{-}periodic\}$ with respect to the Sobolev-norm $\| \, \|_{H^1(Y)}$. Moreover, let

$$\langle u(\cdot) \rangle_Y := \frac{1}{|Y|} \int_Y u(y)\, dy = \int_Y u(y)\, dy$$

be the mean value of $u(\cdot)$ and let $C^2_{per}(Y) := \{L(\cdot) \in C^2(\mathbb{R}) \mid L(\cdot) \text{ } is \text{ } Y\text{-}periodic\}$. Then, formal series expansions in powers of ε, see Ref.[23], and numerical results (see for instance Ref.[12, 13]) suggest that the solution to the time-scaled homogenization problem of Definition 2.2 can be approximated by functions which obey the following general structure. We cut off the expansion after the fifth expansion term.

Definition 4.2 (Adaption operator $\mathcal{A}_\varepsilon$). *Let $f(\cdot,t) \in H^7(\mathbb{R})$ for each t and let $L_1(\cdot), ..., L_5(\cdot) \in C^2_{per}(Y)$ be the smooth functions introduced in Definition 5.1. We define the adapted function $\mathcal{A}_\varepsilon(f)(x,t)$ by*

$$\mathcal{A}_\varepsilon(f)(x,t) := f(x,t) + \sum_{i=1}^{5} \varepsilon^i \, L_i\left(\frac{x}{\varepsilon}\right) \, \partial_x^i f(x,t).$$

We now discuss the regularity of an adapted function $\mathcal{A}_\varepsilon(f)$.

Claim 2. Let $f(\cdot,t) \in H^7(\mathbb{R})$ for each t. Then

$$\mathcal{A}_\varepsilon(f)(\cdot,t) \in H^2(\mathbb{R}).$$

Indeed, since $L_i(\cdot) \in C^2_{per}(Y)$, there exists an ε-independent constant C such that

$$\|\mathcal{A}_\varepsilon(f)(\cdot,t)\|_{H^2(\mathbb{R})} = \left\| f(\cdot,t) + \sum_{i=1}^{5} \varepsilon^i \, L_i\left(\frac{\cdot}{\varepsilon}\right) \, \partial_x^i f(\cdot,t) \right\|_{H^2(\mathbb{R})} \leq \frac{C}{\varepsilon} \|f(\cdot,t)\|_{H^7(\mathbb{R})}.$$

Our aim is to construct the auxiliary functions $L_i(y)$ such that (4.3) is satisfied. This is possible as we show in Section 5. Actually, if the error $\rho = O(\varepsilon^4)$ and $m = 3$, the functions $L_i(y)$ and the coefficients a_i^* are uniquely determined, see Section 5 for details. Let us point out that, except for $i = 1, 2$, the functions L_i do not correspond to the classical auxiliary functions from elliptic homogenization theory.

Next, let us make the implications of characterization (4.3) more precise. Setting $a^\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$, one immediately discovers

Observation 2. *Let $\mathcal{A}_\varepsilon$ be the adaption operator of Definition 4.2. Let $\mathcal{A}_\varepsilon$ satisfy characterization* (4.3) *with $m = 3$ and let $f(x,t)$ solve the constant coefficient problem*

$$\varepsilon^4 \partial_t^2 f(x,t) - \sum_{i=0}^{3} \varepsilon^i a_i^* \partial_x^{2+i} f(x,t) = 0.$$

Then the adaption of f is an approximative solution of the time-scaled homogenization problem of Definition 2.2 in the sense that

$$\begin{aligned}
&\varepsilon^4\partial_t^2\Big(\mathcal{A}_\varepsilon(f)(x,t)\Big) - \partial_x\Big(a^\varepsilon(x)\partial_x\big(\mathcal{A}_\varepsilon(f)\big)(x,t)\Big)\\
&= \mathcal{A}_\varepsilon\Big(\varepsilon^4\partial_t^2 f(x,t) - \sum_{i=0}^{3}\varepsilon^i a_i^*\partial_x^{2+i} f(x,t)\Big) - \rho = \mathcal{A}_\varepsilon(0) - \rho = -\rho
\end{aligned}$$

is small.

In the light of classical homogenization theory, we can expect for the first coefficient a_0^* that $a_0^* = a^* = (c^*)^2$, where c^* denotes the homogenized wave speed. This is also shown in Section 5. In fact, it is a common procedure to modify the classical homogenization limit by adding higher order corrector terms in order to improve the quality of the approximation, see for instance Ref.[2].

Finally, we make the connection between the adaption operator $\mathcal{A}_\varepsilon$ of Definition 4.2 and formula (4.3) more precise. Following the construction of Section 5 we insert $\mathcal{A}_\varepsilon(f)$ with appropriate $L_i(\cdot)$ in the left hand side of characterization (4.3). Claim 2 suggests that the application of the differential operator $\partial_x\Big(a^\varepsilon(x)\partial_x\Big)$ to $\mathcal{A}_\varepsilon(f)(x,t)$ produces a $L^2(\mathbb{R})$-function.

Lemma 4.3 (Algebraical Lemma)**.** *Let $f(x,t) \in H^7(\mathbb{R})$ for each t and let $\mathcal{A}_\varepsilon$ be the adaption operator of Definition 4.2. Then*

$$\begin{aligned}
&\partial_x\Big(a^\varepsilon(x)\partial_x\mathcal{A}_\varepsilon(f)(x,t)\Big)\\
&=a^*\partial_x^2 f(x,t) + \varepsilon^2 a_2^*\partial_x^4 f(x,t)\\
&\quad + \varepsilon L_1\big(\frac{x}{\varepsilon}\big)\partial_x\Big(a^*\partial_x^2 f(x,t) + \varepsilon^2 a_2^*\partial_x^4 f(x,t)\Big)\\
&\quad + \varepsilon^2 L_2\big(\frac{x}{\varepsilon}\big)\partial_x^2\Big(a^*\partial_x^2 f(x,t)\Big)\\
&\quad + \varepsilon^3 L_3\big(\frac{x}{\varepsilon}\big)\partial_x^3\Big(a^*\partial_x^2 f(x,t)\Big) + \tilde{\rho}(x,t)
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned}
\tilde{\rho}(x,t) =&\varepsilon^4\partial_x^6 f(x,t)\,\Big(\partial_y\Big(a(.)L_5(.)\Big)\Big(\frac{x}{\varepsilon}\Big) + a^\varepsilon(x)\,\Big(L_4\Big(\frac{x}{\varepsilon}\Big) + \partial_y L_5\Big(\frac{x}{\varepsilon}\Big)\Big)\Big)\\
&+ \varepsilon^5\partial_x^7 f(x,t) a^\varepsilon(x) L_5\Big(\frac{x}{\varepsilon}\Big).
\end{aligned}$$

Proof. The lemma follows from formula (5.2) and Lemmas 5.2 and 5.3 of the subsequent section. □

Regarding the characterization (4.3), the Algebraical Lemma yields that

$$\partial_x\Big(a^\varepsilon(x)\partial_x\mathcal{A}_\varepsilon(f)(x,t)\Big) = \mathcal{A}_\varepsilon\big(a^*\partial_x^2 f(x,t) + \varepsilon^2 a_2^*\partial_x^4 f(x,t)\big) + O(\varepsilon^4)$$

in $L^2(\mathbb{R})$ independent of t. Observation 2, exploiting $a_1^* = 0, a_3^* = 0$, then suggests to consider the problem

$$\varepsilon^4\partial_t^2 f(x,t) - a^*\partial_x^2 f(x,t) - \varepsilon^2 a_2^*\partial_x^4 f(x,t) = 0. \tag{4.5}$$

Remark 4.4. *Keeping in mind that $a_2^* > 0$, see Lemma 5.4, we observe that Problem (4.5) is ill-posed. However, to lowest order, one formally can replace $\partial_x^4 f$ by $\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^2 f$ arriving at the weakly dispersive equation of Definition 2.3.*

4.3 Proof of Theorem 2.5

We are now in the position to prove Theorem 2.5. In what follows, u_ε denotes the solution of the time-scaled homogenization problem of Definition 2.2 and v_ε the weakly dispersive solution of Definition 2.3. Actually, in our first step we show that the difference between u_ε and $\mathcal{A}_\varepsilon(v_\varepsilon)$ is small. In the second step the adaption $\mathcal{A}_\varepsilon$ is omitted. Our first result is analogous to Proposition 3.6.

Proof of Theorem 2.5.
First step. The crucial point is to show

Claim 3 There exists an ε-independent constant C such that

$$\|\partial_x u_\varepsilon - \partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\varepsilon^2, \tag{4.6}$$

$$\|u_\varepsilon - \big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_t u_\varepsilon - \partial_t\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C. \tag{4.7}$$

The idea is to apply the partial differential operator $\varepsilon^4\partial_t^2 - \partial_x(a^\varepsilon(x)\partial_x)$ to $\mathcal{A}_\varepsilon(v_\varepsilon)$. The difference between u_ε and $\mathcal{A}_\varepsilon(v_\varepsilon)$ is then controlled by standard energy estimates.

In the subsequent calculation, equality (1) holds due to the Algebraical Lemma 4.3. Using in (2) that v_ε solves the weakly dispersive problem of Definition 2.3, $\varepsilon^4\partial_t^2 v_\varepsilon - a^*\partial_x^2 v_\varepsilon = \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^2 v_\varepsilon$, we obtain

$$\begin{aligned}
&\varepsilon^4\partial_t^2\left(\mathcal{A}_\varepsilon v_\varepsilon\right)(x,t) - \partial_x\Big(a^\varepsilon(x)\partial_x\left(\mathcal{A}_\varepsilon v_\varepsilon\right)(x,t)\Big)\\
&\overset{(1)}{=}\varepsilon^4\partial_t^2 v_\varepsilon(x,t) - a^*\partial_x^2 v_\varepsilon(x,t) - \varepsilon^2 a_2^*\partial_x^4 v_\varepsilon(x,t)\\
&\quad + \varepsilon L_1\big(\frac{x}{\varepsilon}\big)\partial_x\left(\varepsilon^4\partial_t^2 v_\varepsilon(x,t) - a^*\partial_x^2 v_\varepsilon(x,t) - \varepsilon^2 a_2^*\partial_x^4 v_\varepsilon(x,t)\right)\\
&\quad + \varepsilon^2 L_2\big(\frac{x}{\varepsilon}\big)\partial_x^2\left(\varepsilon^4\partial_t^2 v_\varepsilon(x,t) - a^*\partial_x^2 v_\varepsilon(x,t)\right)\\
&\quad + \varepsilon^3 L_3\big(\frac{x}{\varepsilon}\big)\partial_x^3\left(\varepsilon^4\partial_t^2 v_\varepsilon(x,t) - a^*\partial_x^2 v_\varepsilon(x,t)\right)\\
&\quad + \varepsilon^8 L_4\big(\frac{x}{\varepsilon}\big)\partial_x^4\partial_t^2 v_\varepsilon(x,t) + \varepsilon^9 L_5\big(\frac{x}{\varepsilon}\big)\partial_x^5\partial_t^2 v_\varepsilon(x,t)\\
&\quad - \varepsilon^4\partial_x^6 v_\varepsilon(x,t)\left(\partial_y\Big(a(.)L_5(.)\Big)\Big(\frac{x}{\varepsilon}\Big) + a^\varepsilon(x)\left(L_4\Big(\frac{x}{\varepsilon}\Big) - \partial_y L_5\Big(\frac{x}{\varepsilon}\Big)\right)\right)\\
&\quad - \varepsilon^5\partial_x^7 v_\varepsilon(x,t)a^\varepsilon(x)L_5\Big(\frac{x}{\varepsilon}\Big)\\
&\overset{(2)}{=}a_2^*\,\varepsilon^2\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^2 v_\varepsilon - \partial_x^4 v_\varepsilon\right)(x,t) + a_2^*\,\varepsilon^3 L_1\big(\frac{x}{\varepsilon}\big)\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^3 v_\varepsilon - \partial_x^5 v_\varepsilon\right)(x,t)\\
&\quad - \varepsilon^4\partial_x^6 v_\varepsilon(x,t)\Big(\partial_y\big(a(.)L_5(.)\big)\left(\frac{x}{\varepsilon}\right) + a^\varepsilon(x)\big(L_4(.) + \partial_y L_5(.)\big)\left(\frac{x}{\varepsilon}\right)\Big)\\
&\quad - \varepsilon^5 a^\varepsilon(x)L_5\left(\frac{x}{\varepsilon}\right)\partial_x^7 v_\varepsilon(x,t) + \varepsilon^8\partial_t^2\partial_x^4 v_\varepsilon(x,t)\left(\frac{a_2^*}{a^*}L_2\left(\frac{x}{\varepsilon}\right) + L_4\left(\frac{x}{\varepsilon}\right)\right)\\
&\quad + \varepsilon^9\partial_t^2\partial_x^5 v_\varepsilon(x,t)\left(\frac{a_2^*}{a^*}L_3\left(\frac{x}{\varepsilon}\right) + L_5\left(\frac{x}{\varepsilon}\right)\right).
\end{aligned} \tag{4.8}$$

To sum up, the effect of the wave operator on $\mathcal{A}_\varepsilon(v_\varepsilon)$ is characterized by

$$\varepsilon^4\partial_t^2\left(\mathcal{A}_\varepsilon v_\varepsilon\right)(x,t) - \partial_x\Big(a^\varepsilon(x)\partial_x\left(\mathcal{A}_\varepsilon v_\varepsilon\right)(x,t)\Big) = f_\varepsilon(x,t). \tag{4.9}$$

The estimate for f_ε is obtained in a quite elementary way and can be found in Lemma 5.5 in the next section. It provides

$$\|f_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\varepsilon^4.$$

Now, we consider the energy estimate for the time-scaled homogenization problem of Definition 2.2 and apply it to $u_\varepsilon - \mathcal{A}_\varepsilon(v_\varepsilon)$. We refer to the appendix for details. Let us remark that due to classical regularity theory, Proposition 4.1 and Claim 2, u_ε and $\mathcal{A}_\varepsilon(v_\varepsilon)$ are sufficiently regular.

Exploiting estimates for the initial data of Lemma 5.6,

$$\|\partial_x u_\varepsilon(x,0) - \partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0)\|_{L^2(\mathbb{R})}^2 \le C\varepsilon^4,$$
$$\varepsilon^4\|\partial_t u_\varepsilon(x,0) - \partial_t\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0)\|_{L^2(\mathbb{R})}^2 \le C\varepsilon^4$$

and taking into account that u_ε solves the time-scaled homogenization problem, one arrives at

$$\begin{aligned}
&\|\partial_x u_\varepsilon - \partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 + \varepsilon^4\|\partial_t\big(u_\varepsilon\big) - \partial_t\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 \\
\le& C\Big(\|\partial_x u_\varepsilon(x,0) - \partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0)\|_{L^2(\mathbb{R})}^2 \\
&+ \varepsilon^4\|\partial_t u_\varepsilon(x,0) - \partial_t\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0)\|_{L^2(\mathbb{R})}^2 + \frac{1}{\varepsilon^4}\|f_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}))}^2\Big) \\
\le& C\varepsilon^4.
\end{aligned} \tag{4.10}$$

Now (4.6) is a direct consequence of (4.10). Moreover, one obtains

$$\|\partial_t u_\varepsilon - \partial_t\mathcal{A}_\varepsilon(v_\varepsilon)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C$$

and thus, writing

$$\big(u_\varepsilon - \mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,t) = \big(u_\varepsilon - \mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0) + \int_0^t \partial_t\big(u_\varepsilon - \mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,\tau)\,d\tau,$$

inequality (4.7) follows directly.

Second step. We would like to show that

$$\|u_\varepsilon(x,t) - v_\varepsilon(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \le C\varepsilon.$$

Arguing in exactly the same way as in the proof of Theorem 2.7, Ineq. (4.6), (4.7) yield

$$\|u_\varepsilon(x,t) - \mathcal{A}_\varepsilon(v_\varepsilon)(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \le C\varepsilon.$$

It remains to avoid the adaption operator $\mathcal{A}_\varepsilon$. Actually, in the subsequent calculation we use in (1) that $\big\|L_1\big(\frac{x}{\varepsilon}\big)\big\|_{L^\infty(\mathbb{R})}, \ldots, \big\|L_5\big(\frac{x}{\varepsilon}\big)\big\|_{L^\infty(\mathbb{R})}$ are uniformly bounded in ε. Inequality (2) is a consequence of Proposition 4.1 using once more the fundamental theorem of calculus,

$$\begin{aligned}
&\|\mathcal{A}_\varepsilon(v_\varepsilon)(x,t) - v_\varepsilon(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \\
=& \Big\|\varepsilon L_1\big(\frac{x}{\varepsilon}\big)\partial_x v_\varepsilon(x,t) + \ldots + \varepsilon^5 L_5\big(\frac{x}{\varepsilon}\big)\partial_x^5 v_\varepsilon(x,t)\Big\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \\
\overset{(1)}{\le}& C\varepsilon\left(\|\partial_x v_\varepsilon(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} + \ldots + \|\partial_x^5 v_\varepsilon(x,t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}\right) \\
\overset{(2)}{\le}& C\varepsilon.
\end{aligned}$$

This proves Theorem 2.5. □

Let us point out that estimates as in Claim 3 can be obtained also on an intermediate time scale, namely time intervals of order $1/\varepsilon$ instead of $1/\varepsilon^2$. To be more precise, the following estimates, which are not available with standard homogenization theory, hold.

Proposition 4.5. *Let $\tilde{u}_\varepsilon(x,\tilde{t}) := \bar{u}_\varepsilon(x,\frac{\tilde{t}}{\varepsilon})$ be the time-scaled version of the original homogenization problem. Let $\tilde{v}_\varepsilon(x,\tilde{t}) := \bar{v}_\varepsilon(x,\frac{\tilde{t}}{\varepsilon})$ with $\bar{v}_\varepsilon$ as in Remark 2.6 and let $\tilde{T} > 0$ be fixed. Then, there exists an ε-independent constant C such that*

$$\|\partial_x \tilde{u}_\varepsilon - \partial_x \left(\mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\right)\|_{L^\infty(0,\tilde{T};L^2(\mathbb{R}))} \leq C\varepsilon^2, \tag{4.11}$$

$$\|\tilde{u}_\varepsilon - \mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\|_{L^\infty(0,\tilde{T};L^2(\mathbb{R}))} + \|\partial_{\tilde{t}} \tilde{u}_\varepsilon - \partial_{\tilde{t}} \left(\mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\right)\|_{L^\infty(0,\tilde{T};L^2(\mathbb{R}))} \leq C\varepsilon. \tag{4.12}$$

Proof. The proof is straightforward but lengthy, it follows the first step in the proof of Theorem 2.5. For the sake of brevity, we skip the calculations and just state the essential steps. Indeed, in the setting above, relation (4.8) reads as

$$\varepsilon^2 \partial_{\tilde{t}}^2 \left(\mathcal{A}_\varepsilon \tilde{v}_\varepsilon\right)(x,\tilde{t}) - \partial_x \left(a^\varepsilon(x) \partial_x \left(\mathcal{A}_\varepsilon \tilde{v}_\varepsilon\right)(x,\tilde{t})\right) = \tilde{f}_\varepsilon(x,\tilde{t})$$

with $\|\tilde{f}_\varepsilon\|_{L^2(0,\tilde{T};L^2(\mathbb{R}))} \leq C\varepsilon^4$. Then,

$$\begin{aligned}
&\|\partial_x \tilde{u}_\varepsilon - \partial_x \left(\mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\right)\|^2_{L^\infty(0,\tilde{T};L^2(\mathbb{R}))} + \varepsilon^2 \|\partial_{\tilde{t}} \tilde{u}_\varepsilon - \partial_{\tilde{t}} \left(\mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\right)\|^2_{L^\infty(0,\tilde{T};L^2(\mathbb{R}))} \\
\leq & C \Big(\|\partial_x \tilde{u}_\varepsilon(x,0) - \partial_x \left(\mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\right)(x,0)\|^2_{L^2(\mathbb{R})} \\
&+ \varepsilon^2 \|\partial_{\tilde{t}} \tilde{u}_\varepsilon(x,0) - \partial_{\tilde{t}} \left(\mathcal{A}_\varepsilon(\tilde{v}_\varepsilon)\right)(x,0)\|^2_{L^2(\mathbb{R})} + \frac{1}{\varepsilon^2} \|f_\varepsilon\|^2_{L^2(0,\tilde{T};L^2(\mathbb{R}))} \Big) \\
\leq & C(\varepsilon^4 + \varepsilon^4 + \varepsilon^6) \leq C\varepsilon^4.
\end{aligned}$$

This is the desired estimate. □

Let us give some remarks on the higher dimensional case, $x \in \mathbb{R}^N$ with $N > 1$. In principle, the adaption operator approach is not restricted to the one-dimensional case. Under appropriate assumptions on the coefficient $A\left(\frac{x}{\varepsilon}\right) \in M^{N\times N}$, namely assuming that the material is macroscopically orthotropic, one again formally derives a linear Boussinesq equation in multi-dimensions, see Ref.[14]. Nevertheless, rigorous results in higher dimensions are not yet available.

5 Construction of $\mathcal{A}_\varepsilon$ and algebraical properties

This section is devoted to the construction of $\mathcal{A}_\varepsilon$ and the derivation of algebraical properties. The key point of the previous section, the Algebraical Lemma 4.3, is a direct consequence of the subsequent results.

5.1 Construction of the auxiliary problems

According to characterization (4.3), in this subsection we derive the periodic boundary problems which determine the auxiliary functions $L_1(y), ..., L_5(y)$.

Our aim is to construct the adaption operator $\mathcal{A}_\varepsilon$ such that

$$\partial_x\Big(a^\varepsilon(x)\partial_x\mathcal{A}_\varepsilon(f)(x,t)\Big) = \mathcal{A}_\varepsilon\Big(\sum_{i=0}^{3}\varepsilon^i a_i^*\partial_x^{i+2}f(x,t)\Big) + O(\varepsilon^4) \tag{5.1}$$

for every fixed smooth function f. We calculate the left hand side of (5.1) as

$$\begin{aligned}
&\partial_x\Big(a^\varepsilon(x)\partial_x\big(\mathcal{A}_\varepsilon(f)\big)(x,t)\big)\Big)\\
=&\partial_x\left(a^\varepsilon(x)\left(1+\partial_y L_1\left(\frac{x}{\varepsilon}\right)\right)\partial_x f(x,t) + \sum_{i=2}^{5}\varepsilon^{i-1}a^\varepsilon(x)\Big(L_{i-1}(.)+\partial_y L_i(.)\Big)\left(\frac{x}{\varepsilon}\right)\partial_x^i f(x,t)\right)\\
&+\partial_x\left(\varepsilon^5 a^\varepsilon(x)L_5\left(\frac{x}{\varepsilon}\right)\partial_x^6 f(x,t)\right)\\
=&\frac{1}{\varepsilon}\partial_x f(x,t)\partial_y\Big(a(.)\big(1+\partial_y L_1(.)\big)\Big)\left(\frac{x}{\varepsilon}\right) + \partial_x^2 f(x,t)a^\varepsilon(x)\left(1+\partial_y L_1\left(\frac{x}{\varepsilon}\right)\right)\\
&+\sum_{i=1}^{4}\varepsilon^{i-1}\partial_y\Big(a(.)\big(L_i(.)+\partial_y L_{i+1}(.)\big)\Big)\left(\frac{x}{\varepsilon}\right)\partial_x^{i+1}f(x,t)\\
&+\sum_{i=2}^{5}\varepsilon^{i-1}a^\varepsilon(x)\Big(L_{i-1}(.)+\partial_y L_i(.)\Big)\left(\frac{x}{\varepsilon}\right)\partial_x^{i+1}f(x,t)\\
&+\varepsilon^4\partial_y\Big(a(.)L_5(.)\Big)\left(\frac{x}{\varepsilon}\right)\partial_x^6 f(x,t) + \varepsilon^5 a^\varepsilon(x)L_5\left(\frac{x}{\varepsilon}\right)\partial_x^7 f(x,t).
\end{aligned} \tag{5.2}$$

Considering terms of order $(1/\varepsilon)$ in (5.1), one immediately derives the following equation for $L_1(y)$

$$\partial_y\Big(a(y)\big(1+\partial_y L_1(y)\big)\Big) = 0. \tag{5.3}$$

The additional claim $\langle L_1(y)\rangle_Y = 0$ makes the periodic boundary problem (5.3) well-posed. Hence, a unique solution $L_1(y) \in H^1_{per}(Y)$ exists, see Ref.[10] for details.

At order (1), as a pre-factor of $\partial_x^2 f$, the following equation arises

$$\partial_y\Big(a(y)\big(L_1(y)+\partial_y L_2(y)\big)\Big) = a_0^* - a(y)\left(1+\partial_y L_1(y)\right). \tag{5.4}$$

Considering the mean value

$$\begin{aligned}
0 = \Big\langle\partial_y\Big(a(y)\big(L_1(y)+\partial_y L_2(y)\big)\Big)\Big\rangle_Y &= \langle a_0^* - a(y)\left(1+\partial_y L_1(y)\right)\rangle_Y\\
&= a_0^* - \langle a(y)\left(1+\partial_y L_1(y)\right)\rangle_Y,
\end{aligned}$$

one directly concludes $a_0^* = \langle a(y)\,(1 + \partial_y L_1(y))\rangle_Y$. Demanding $\langle L_2(y)\rangle_Y = 0$, problem (5.4) determines $L_2(y)$ uniquely.

Next, considering terms of order (ε), one finds as pre-factor of $\partial_x^3 f$

$$\partial_y\Big(a(y)\big(L_2(y) + \partial_y L_3(y)\big)\Big) = a_1^* + a_0^* L_1(y) - a(y)\,(L_1(y) + \partial_y L_2(y))\,, \tag{5.5}$$

where $a_1^* = \langle a(y)\,(L_1(y) + \partial_y L_2(y))\rangle_Y$, since

$$\begin{aligned} 0 = \Big\langle \partial_y\Big(a(y)\big(L_2(y) + \partial_y L_3(y)\big)\Big)\Big\rangle_Y &= \langle a_1^* + a_0^* L_1(y) - a(y)\,(L_1(y) + \partial_y L_2(y))\rangle_Y \\ &= a_1^* - \langle a(y)\,(L_1(y) + \partial_y L_2(y))\rangle_Y. \end{aligned}$$

Then a unique solution to problem (5.5), with $\langle L_3(y)\rangle_Y = 0$, exists. Let us point out that the equation for L_3 does not correspond to the classical auxiliary problem, in which the term $a_0^* L_1(y)$ on the right hand side of (5.5) does not appear. In an analogous manner, the equation at order (ε^2) is given by

$$\begin{aligned} &\partial_y\Big(a(y)\big(L_3(y) + \partial_y L_4(y)\big)\Big) \\ &\qquad = a_2^* + a_1^* L_1(y) + a_0^* L_2(y) - a(y)\,(L_2(y) + \partial_y L_3(y))\,. \end{aligned} \tag{5.6}$$

Again, the mean value yields

$$\begin{aligned} 0 &= \Big\langle \partial_y\Big(a(y)\big(L_2(y) + \partial_y L_3(y)\big)\Big)\Big\rangle_Y \\ &= \langle a_2^* + a_1^* L_1(y) + a_0^* L_2(y) - a(y)\,(L_2(y) + \partial_y L_3(y))\rangle_Y \\ &= a_2^* - \langle a(y)\,(L_2(y) + \partial_y L_3(y))\rangle_Y \end{aligned}$$

and thus $a_2^* = \langle a(y)\,(L_2(y) + \partial_y L_3(y))\rangle_Y$. Problem (5.6) uniquely determines $L_4(y)$ with $\langle L_4(y)\rangle_Y = 0$. Following this procedure at order (ε^3), one at least arrives at the following definition. By smoothness of $a(\cdot)$, the auxiliary functions are smooth, $L_1(y), ..., L_5(y) \in C^2_{per}(Y)$.

Definition 5.1 (Auxiliary problems). *Denote by $L_1(y), ..., L_5(y)$ the unique solution to the following problem*

$$\begin{aligned} \partial_y\Big(a(y)\big(1 + \partial_y L_1(y)\big)\Big) =& 0, \\ \partial_y\Big(a(y)\big(L_1(y) + \partial_y L_2(y)\big)\Big) =& a_0^* - a(y)\,(1 + \partial_y L_1(y))\,, \\ \partial_y\Big(a(y)\big(L_2(y) + \partial_y L_3(y)\big)\Big) =& a_1^* + a_0^* L_1(y) - a(y)\,(L_1(y) + \partial_y L_2(y))\,, \\ \partial_y\Big(a(y)\big(L_3(y) + \partial_y L_4(y)\big)\Big) =& a_2^* + a_1^* L_1(y) + a_0^* L_2(y) - a(y)\big(L_2(y) + \partial_y L_3(y)\big), \\ \partial_y\Big(a(y)\big(L_4(y) + \partial_y L_5(y)\big)\Big) =& a_3^* + a_2^* L_1(y) + a_1^* L_2(y) + a_0^* L_3(y) \\ & - a(y)\big(L_3(y) + \partial_y L_4(y)\big) \end{aligned}$$

and

$$\langle L_1(y)\rangle_Y = \langle L_2(y)\rangle_Y = \langle L_3(y)\rangle_Y = \langle L_4(y)\rangle_Y = \langle L_5(y)\rangle_Y = 0,$$

where

$$\begin{aligned} a_0^* :&= \langle a(y)\,(1 + \partial_y L_1(y))\rangle_Y, \\ a_i^* :&= \langle a(y)\,(L_i(y) + \partial_y L_{i+1}(y))\rangle_Y \end{aligned}$$

for $1 \le i \le 3$.

5.2 Algebraical simplifications

In this subsection we derive some algebraical results which considerably simplify Definition 5.1. Moreover, we show that the coefficient a_2^* in Definition 5.1 is strictly positive.

Lemma 5.2. *The auxiliary functions $L_1(y), L_2(y) \in C^2_{per}(Y)$ satisfy*

$$a(y)\big(1 + \partial_y L_1(y)\big) = a_0^* = a^*, \tag{5.7}$$

$$a(y)\big(L_1(y) + \partial_y L_2(y)\big) = a_1^* = 0, \tag{5.8}$$

where a^* is the effective coefficient of (2.1).

Proof. Since $a_0^* = \langle a(y)\,(1 + \partial_y L_1(y))\rangle_Y$, one obtains

$$\partial_y\Big(a(y)\big(1 + \partial_y L_1(y)\big)\Big) = 0 \qquad \Rightarrow \qquad a(y)\big(1 + \partial_y L_1(y)\big) = a_0^*$$
$$\Rightarrow 1 + \partial_y L_1(y) = \frac{a_0^*}{a(y)} \quad \Rightarrow \quad \langle 1 + \partial_y L_1(y)\rangle_Y = a_0^* \left\langle \frac{1}{a(y)} \right\rangle_Y .$$

Due to $L_1(y) \in C^2_{per}(Y)$ and thus $\langle \partial_y L_1(y)\rangle_Y = 0$, we conclude

$$1 = a_0^* \left\langle \frac{1}{a(y)} \right\rangle_Y = \frac{a_0^*}{a^*} \qquad \Rightarrow \qquad a_0^* = a^*.$$

Inserting (5.7) in the equation for L_2, it follows that $a(y)\big(L_1(y) + \partial_y L_2(y)\big) = 0$. This provides $a_1^* = 0$ in the next equation. □

Next, we show that the coefficient a_3^* vanishes.

Lemma 5.3 (The mean value a_3^*)**.** *Consider the mean value a_3^* of Definition 5.1. Then in fact*

$$a_3^* = 0.$$

Proof. In what follows, equalities (1) and (3) are direct consequences of Definition 5.1 and Lemma 5.2. Equality (2) is valid, since $a(y)\big(L_1(y) + \partial_y L_2(y)\big) = 0$ and thus $\partial_y L_2(y) = -L_1(y)$. We calculate

$$\begin{aligned}
&\partial_y\Big(\partial_y\Big(a(y)\big(L_3(y) + \partial_y L_4(y)\big)\Big)\Big) \\
&= \partial_y\Big(a_2^* + a^* L_2(y) - a(y)\big(L_2(y) + \partial_y L_3(y)\big)\Big) \overset{(1)}{=} a^* \partial_y L_2(y) - a^* L_1(y) \\
&\overset{(2)}{=} -2a^* L_1(y) \overset{(3)}{=} -2\partial_y\Big(a(y)\big(L_2(y) + \partial_y L_3(y)\big)\Big).
\end{aligned}$$

Consequently,

$$\partial_y\Big(a(y)\big(L_3(y) + \partial_y L_4(y)\big)\Big) = -2a(y)\big(L_2(y) + \partial_y L_3(y)\big) + C$$

for a constant C. Considering the mean value one discovers

$$0 = -2a_2^* + C$$

and thus

$$\begin{aligned}\partial_y\Big(a(y)\big(L_3(y)+\partial_y L_4(y)\big)\Big) &= -2a(y)\big(L_2(y)+\partial_y L_3(y)\big)+2a_2^* \\ &= a_2^* + a^* L_2(y) - a(y)\big(L_2(y)+\partial_y L_3(y)\big).\end{aligned}$$

The last equality is valid due to Definition 5.1. Consequently,

$$-a(y)\big(L_2(y)+\partial_y L_3(y)\big)+a_2^* = a^* L_2(y)$$

and thus

$$\partial_y\Big(a(y)\big(L_3(y)+\partial_y L_4(y)\big)\Big) = 2a^* L_2(y). \tag{5.9}$$

Next, multiplying Eq. (5.9) by $L_1(y)$, integrating over Y and applying integration by parts yields

$$\begin{aligned}-\int_Y \partial_y L_1(y) a(y)\big(L_3(y)+\partial_y L_4(y)\big)\,dy &= 2a^*\int_Y L_2(y)L_1(y)\,dy \\ &\overset{(1)}{=} -2a^*\int_Y L_2(y)\partial_y L_2(y)\,dy = -a^*\int_Y \partial_y (L_2(y))^2\,dy = 0,\end{aligned}$$

where (1) again uses the fact that $\partial_y L_2(y) = -L_1(y)$. Moreover, taking into account $\partial_y L_1(y) = \frac{a^*}{a(y)} - 1$, see Lemma 5.2, one concludes

$$0 = \int_Y a(y)(L_3+\partial_y L_4)(y)\,dy - \int_Y a^*(L_3+\partial_y L_4)(y)\,dy = \int_Y a(y)(L_3+\partial_y L_4)(y)\,dy,$$

which is the claimed result. □

Finally, we prove that the coefficient a_2^* is strictly positive. We note that the positivity of a_2^* ensures the well-posedness of the weakly dispersive problem of Definition 2.3.

Lemma 5.4 (The mean value a_2^*)**.** *Consider the mean value a_2^* of Definition 5.1. Then*

$$a_2^* > 0.$$

Proof. As in the proof of Lemma 5.3, one multiplies the equation

$$\partial_y\Big(a(y)\big(L_2(y)+\partial_y L_3(y)\big)\Big) = a^* L_1(y)$$

by $L_1(y)$, integrates over Y and applies integration by parts arriving at

$$-\int_Y \partial_y L_1(y)\Big(a(y)\big(L_2(y)+\partial_y L_3(y)\big)\Big)\,dy = a^*\langle L_1(y)L_1(y)\rangle_Y > 0.$$

Now, we again use that $\partial_y L_1(y) = \frac{a^*}{a(y)} - 1$. Consequently,

$$\begin{aligned}&-\int_Y \partial_y L_1(y)\Big(a(y)\big(L_2(y)+\partial_y L_3(y)\big)\Big)\,dy \\ &\qquad= \int_Y (a(y)\big(L_2(y)+\partial_y L_3(y)\big)\,dy - \int_Y a^*\big(L_2(y)+\partial_y L_3(y)\big)\,dy \\ &\qquad= \int_Y (a(y)\big(L_2(y)+\partial_y L_3(y)\big)\,dy = a_2^*\end{aligned}$$

and thus the positivity of a_2^*. □

The Algebraical Lemma 4.3 directly follows by considering Definition 5.1 and inserting in (5.2) the results of Lemma 5.2 and Lemma 5.3.

5.3 Effect of the wave operator on $\mathcal{A}_\varepsilon(v_\varepsilon)$

The aim of this subsection is to estimate the error generated by the application of the wave operator $\varepsilon^4\partial_t^2 - \partial_x\left(a^\varepsilon(x)\partial_x\right)$ to $\mathcal{A}_\varepsilon(v_\varepsilon)$.

Lemma 5.5 (Effect of the wave operator on $\mathcal{A}_\varepsilon(v_\varepsilon)$)**.** *Let*

$$f_\varepsilon := \varepsilon^4\partial_t^2\left(\mathcal{A}_\varepsilon v_\varepsilon\right)(x,t) - \partial_x\Big(a^\varepsilon(x)\partial_x\left(\mathcal{A}_\varepsilon v_\varepsilon\right)(x,t)\Big).$$

Then there exists an ε-independent constant C such that

$$\|f_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C\varepsilon^4. \tag{5.10}$$

Proof. Due to (4.8) we have to estimate the following expressions

1. $a_2^*\,\varepsilon^2\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^2 v_\varepsilon - \partial_x^4 v_\varepsilon\right)(x,t)$,
2. $a_2^*\,\varepsilon^3 L_1\left(\frac{x}{\varepsilon}\right)\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^3 v_\varepsilon - \partial_x^5 v_\varepsilon\right)(x,t)$,
3. $\varepsilon^4\partial_x^6 v_\varepsilon(x,t)\Big(a^\varepsilon(x)\big(L_4(.) + \partial_y L_5(.)\big)\left(\frac{x}{\varepsilon}\right) + \partial_y\big(a(.)L_5(.)\big)\left(\frac{x}{\varepsilon}\right)\Big)$,
4. $\varepsilon^5 L_5\left(\frac{x}{\varepsilon}\right)a^\varepsilon(x)\partial_x^7 v_\varepsilon(x,t)$,
5. $\varepsilon^8\Big(\frac{a_2^*}{a^*}L_2\left(\frac{x}{\varepsilon}\right) + L_4\left(\frac{x}{\varepsilon}\right)\Big)\partial_t^2\partial_x^4 v_\varepsilon(x,t)$,
6. $\varepsilon^9\Big(\frac{a_2^*}{a^*}L_3\left(\frac{x}{\varepsilon}\right) + L_5\left(\frac{x}{\varepsilon}\right)\Big)\partial_t^2\partial_x^5 v_\varepsilon(x,t)$.

In what follows, we exploit that due to $L_i(y) \in C^2_{per}(Y)$ there exists an ε- independent constant $K > 0$ such that

$$\left\|L_1\left(\frac{x}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R})}, ..., \left\|L_5\left(\frac{x}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R})} \leq K,$$
$$\left\|\partial_y L_1\left(\frac{x}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R})}, ..., \left\|\partial_y L_5\left(\frac{x}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R})} \leq K.$$

Concerning expression 1: Due to

$$\begin{aligned}\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^2 v_\varepsilon - \partial_x^4 v_\varepsilon\right)(x,t) &= \partial_x^2\left(\frac{\varepsilon^4}{a^*}\partial_t^2 v_\varepsilon - \partial_x^2 v_\varepsilon\right)(x,t) = \frac{\varepsilon^6 a_2^*}{(a^*)^2}\partial_t^2\partial_x^4 v_\varepsilon(x,t)\\ &= \frac{\varepsilon^2 a_2^*}{(a^*)^2}\partial_x^4\left(\varepsilon^4\partial_t^2 v_\varepsilon(x,t)\right) = \frac{\varepsilon^2 a_2^*}{(a^*)^2}\left(a^*\partial_x^6 v_\varepsilon(x,t) + \frac{a_2^*}{a^*}\varepsilon^6\partial_t^2\partial_x^6 v_\varepsilon(x,t)\right),\end{aligned}$$

one obtains

$$\begin{aligned}&\left\|a_2^*\varepsilon^2\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^2 v_\varepsilon - \partial_x^4 v_\varepsilon\right)(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))}\\ &\leq \frac{\varepsilon^4 (a_2^*)^2}{(a^*)^2}\left(\|a^*\partial_x^6 v_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} + \varepsilon^6\left\|\frac{a_2^*}{a^*}\partial_t^2\partial_x^6 v_\varepsilon(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))}\right)\\ &\leq C\varepsilon^4,\end{aligned}$$

where the last inequality is a consequence of Proposition 4.1.

Concerning expression 2: We observe that

$$\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^3 v_\varepsilon - \partial_x^5 v_\varepsilon\right)(x,t) = \varepsilon^6 \frac{a_2^*}{(a^*)^2}\partial_t^2\partial_x^5 v_\varepsilon(x,t)$$

and thus

$$\begin{aligned}
&\left\| a_2^*\varepsilon^3 L_1\left(\frac{x}{\varepsilon}\right)\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^3 v_\varepsilon - \partial_x^5 v_\varepsilon\right)(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))} \\
&\quad \le a_2^*\varepsilon^3 \left\| L_1\left(\frac{x}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R})} \left\|\left(\frac{\varepsilon^4}{a^*}\partial_t^2\partial_x^3 v_\varepsilon - \partial_x^5 v_\varepsilon\right)(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))} \\
&\quad \le K\varepsilon^9 \frac{(a_2^*)^2}{(a^*)^2}\|\partial_t^2\partial_x^5 v_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} \\
&\quad \le C\varepsilon^4
\end{aligned}$$

due to Proposition 4.1.

Concerning expressions 3. and 4: Taking into account $a\left(\frac{x}{\varepsilon}\right) \le \beta$ and applying Proposition 4.1, one arrives at

$$\left\|\varepsilon^5 L_5\left(\frac{x}{\varepsilon}\right) a^\varepsilon(x)\partial_x^7 v_\varepsilon(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))} \le \varepsilon^5 K\beta\|\partial_x^7 v_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\varepsilon^5.$$

In the same manner, exploiting that $\partial_y a(\cdot)$ is bounded, we obtain

$$\left\|\varepsilon^4\partial_x^6 v_\varepsilon(x,t)\Big(a^\varepsilon(x)\big(L_4(.) + \partial_y L_5(.)\big)\left(\frac{x}{\varepsilon}\right) + \partial_y\big(a(.)L_5(.)\big)\left(\frac{x}{\varepsilon}\right)\Big)\right\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\varepsilon^4.$$

Concerning expressions 5. and 6: Application of Proposition 4.1 leads to

$$\begin{aligned}
&\left\|\varepsilon^8\Big(\frac{a_2^*}{a^*}L_2\left(\frac{x}{\varepsilon}\right) + L_4\left(\frac{x}{\varepsilon}\right)\Big)\partial_t^2\partial_x^4 v_\varepsilon(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\varepsilon^4 K\|\varepsilon^4\partial_t^2\partial_x^4 v_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} \\
&= C\varepsilon^4 K\|a^*\partial_x^6 v_\varepsilon(x,t) + \varepsilon^6\frac{a_2^*}{a^*}\partial_t^2\partial_x^6 v_\varepsilon(x,t)\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\varepsilon^4
\end{aligned}$$

and

$$\left\|\varepsilon^9\Big(\frac{a_2^*}{a^*}L_3\left(\frac{x}{\varepsilon}\right) + L_5\left(\frac{x}{\varepsilon}\right)\Big)\partial_t^2\partial_x^5 v_\varepsilon(x,t)\right\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\varepsilon^4.$$

□

We conclude this section by proving an estimate for the initial data of $u_\varepsilon - \mathcal{A}_\varepsilon(v_\varepsilon)$. The stimate has been used in the proof of Theorem 2.5.

Lemma 5.6 (Estimate for the initial data)**.** *There holds*

$$\|\partial_x u_\varepsilon(x,0) - \partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0)\|_{L^2(\mathbb{R})}^2 \le C\varepsilon^4, \tag{5.11}$$

$$\varepsilon^4\|\partial_t u_\varepsilon(x,0) - \partial_t\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)(x,0)\|_{L^2(\mathbb{R})}^2 \le C\varepsilon^4. \tag{5.12}$$

Proof. We calculate

$$\begin{aligned}&\Big(\partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon)\big)-\partial_x u_\varepsilon\Big)(x,0)\\&\quad=\varepsilon^2\partial_y L_3\left(\frac{x}{\varepsilon}\right)\partial_x^3 c_0(x)+\varepsilon^3\Big(L_3\left(\frac{x}{\varepsilon}\right)+\partial_y L_4\left(\frac{x}{\varepsilon}\right)\Big)\partial_x^4 c_0(x)\\&\qquad+\varepsilon^4\Big(L_4\left(\frac{x}{\varepsilon}\right)+\partial_y L_5\left(\frac{x}{\varepsilon}\right)\Big)\partial_x^5 c_0(x)+\varepsilon^5 L_5\left(\frac{x}{\varepsilon}\right)\partial_x^6 c_0(x).\end{aligned}$$

Since $c_0(x)\in C_c^\infty\big((-R,R)\big)$, one directly obtains

$$\Big\|\varepsilon^2\partial_y L_3\Big(\frac{x}{\varepsilon}\Big)\partial_x^3 c_0(x)\Big\|^2_{L^2(\mathbb{R})}\leq\varepsilon^4K^2\|\partial_x^3 c_0(x)\|^2_{L^2(\mathbb{R})}\leq C\varepsilon^4$$

estimating the remaining terms in an analogous way. Similarly, we derive (5.12). $\square$

Part III
Abstract framework and the multi-dimensional case

6 The energy measure

Our previous considerations concerning the long time behavior of waves in heterogeneous media have been made rigorous only in the one-dimensional case. In this section we want to introduce a more abstract framework, which can be used to describe the behavior of waves in the multi-dimensional setting. In what follows, we define an energy density $E_\varepsilon(\xi, t)$ in Definition 6.1. We show that a weak star limit μ in $L^\infty(0, T; \mathcal{M}(\mathbb{R}^N))$ exists for $\varepsilon \to 0$.

In one space dimension, based on the results of Sections 3-5, the measure μ turns out to be a weighted Dirac measure. For $N > 1$, the identification of μ remains an open problem. We can only prove restrictions on the support of the multi-dimensional energy measure, see Section 8 for details.

Definition 6.1 (Energy density and energy measure)**.** *Let $N \in \mathbb{N}$ be arbitrary.*

For $N = 1$ let u_ε be the solution to the one-dimensional time-scaled homogenization problem of Definition 2.2. For $N \geq 2$ let u_ε be the solution to the multi-dimensional time-scaled homogenization problem of Definition 2.11.

1. *We define the N-dimensional energy density $E_\varepsilon(\xi, t)$ by*
$$E_\varepsilon(\xi, t) := \frac{1}{\varepsilon^{2N}} \left[\left(A\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon(\cdot, t) \right)^T A\left(\frac{\cdot}{\varepsilon}\right) \nabla u_\varepsilon(\cdot, t) \right] \left(\frac{\xi}{\varepsilon^2}\right). \tag{6.1}$$

2. *The energy density E_ε is uniformly bounded in $L^\infty(0, T; L^1(\mathbb{R}^N))$. We call $\mu \in L^\infty(0, T; \mathcal{M}(\mathbb{R}^N))$ an energy measure if there exists a subsequence E_{ε_k} of E_ε such that $E_{\varepsilon_k} \overset{*}{\rightharpoonup} \mu$ in $L^\infty(0, T; \mathcal{M}(\mathbb{R}^N))$.*

We have to prove the uniform boundedness of $E_\varepsilon(\cdot, t) \in L^1(\mathbb{R}^N)$.

Proof. The energy estimate for u_ε, see Lemma A.1 in the appendix, provides
$$\|\nabla u_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + \varepsilon^4 \|\partial_t u_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C. \tag{6.2}$$

We can therefore calculate

$$\begin{aligned}
&\|E_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^N))} \\
&= \left\| \int_{\mathbb{R}^N} \frac{1}{\varepsilon^{2N}} \left[\left(A\left(\frac{\xi}{\varepsilon^3}\right) \nabla u_\varepsilon\left(\frac{\xi}{\varepsilon^2}, \cdot\right) \right)^T A\left(\frac{\xi}{\varepsilon^3}\right) \nabla u_\varepsilon\left(\frac{\xi}{\varepsilon^2}, \cdot\right) \right] d\xi \right\|_{L^\infty(0,T)} \\
&= \left\| \int_{\mathbb{R}^N} \left(A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x, \cdot) \right)^T A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x, \cdot)\, dx \right\|_{L^\infty(0,T)} \\
&\leq \beta^2 \|\nabla u_\varepsilon\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C,
\end{aligned}$$

which is the claimed result. □

6.1 Identification of μ for $N = 1$

Our aim is to translate the one-dimensional results from Sections 3-5 into the abstract framework of energy measures. In what follows we will only investigate u_ε^+, the right going wave in the one-dimensional homogenization problem. It corresponds to right going initial data, see Definition 3.1. The treatment of the left going part u_ε^- is analogous.

We show that in the one-dimensional right going case the measure μ is in fact a Dirac measure, $\mu(\xi,t) = E^+\,\delta_{c^*t}(\xi)$, where the density E^+ is determined as $E^+ = (a^*)^2\|\partial_x c_0^+\|^2_{L^2(\mathbb{R})}$.

Let us first of all recall some one-dimensional results from Sections 3-5.

Lemma 6.2 (One-dimensional results from Sections 3-5)**.** *Let u_ε^+ be the right going wave in the homogenization problem of Definition 2.2. Let W^+ be the lKdV-solution of Definition 2.4 and let v_ε^+ be the right going part of the weakly dispersive solution of Definition 2.3.*

1. *There exists an ε-independent constant C such that*

$$\|\partial_x u_\varepsilon^+(x,t) - \partial_x\big(\mathcal{A}_\varepsilon(v_\varepsilon^+)\big)(x,t)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\varepsilon^2. \tag{6.3}$$

2. *There exists an ε-independent constant C such that*

$$\|\partial_x v_\varepsilon^+(x,t) - \partial_x W^+\left(x - \frac{c^*}{\varepsilon^2}t, t\right)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\varepsilon^2. \tag{6.4}$$

Using Lemma 6.2 we derive the following result.

Lemma 6.3. *Let u_ε^+ and W^+ be as in Lemma 6.2 . There exists an ε-independent constant C such that*

$$\left\|\partial_x u_\varepsilon^+(x,t) - \partial_x\left(\mathcal{A}_\varepsilon W^+\left(\cdot - \frac{c^*}{\varepsilon^2}t, t\right)\right)(x)\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C\varepsilon^2.$$

Proof. The lemma is a consequence of Lemma 6.2 and the fact that estimate (6.4) is also valid for higher order space derivatives. □

We are now in the position to identify the energy measure μ for $N = 1$.

Proposition 6.4 (Energy measure in one space dimension.)**.** *Let u_ε be the solution to the one-dimensional homogenization problem with right going initial data, $c_0 = c_0^+$. Let μ be an energy measure as in Definition 6.1 for $N = 1$. Then*

$$\mu(\xi,t) = E^+(t)\delta_{c^*t}(\xi),$$

where the density $E^+(t)$ is given through

$$E^+(t) = (a^*)^2\|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})}. \tag{6.5}$$

Proof. Our aim is to show that for $E^+(t)$ as in (6.5) and for every test function $\Phi \in L^1(0,T;C_0(\mathbb{R}))$ there holds

$$\int_0^T \int_{\mathbb{R}} \Phi(\xi,t) E_\varepsilon(\xi,t)\, d\xi\, dt \to \int_0^T E^+(t)\Phi(c^* t,t)\, dt \qquad \text{for } \varepsilon \to 0. \tag{6.6}$$

We introduce

$$g_\varepsilon(x,t) := \partial_x \left(\mathcal{A}_\varepsilon W^+ \left(\cdot - \frac{c^*}{\varepsilon^2} t, t \right) \right)(x).$$

With this function the left hand side of (6.6) reads

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \Phi(\xi,t) E_\varepsilon(\xi,t)\, d\xi\, dt \\
=& \int_0^T \int_{\mathbb{R}} \Phi(\xi,t) a^2 \left(\frac{\xi}{\varepsilon^3}\right) \frac{1}{\varepsilon^2} \left((\partial_x u_\varepsilon)^2 \left(\frac{\xi}{\varepsilon^2}, t\right) - g_\varepsilon^2 \left(\frac{\xi}{\varepsilon^2}, t\right) + g_\varepsilon^2 \left(\frac{\xi}{\varepsilon^2}, t\right) \right)\, d\xi\, dt \\
=& \int_0^T \int_{\mathbb{R}} \Phi(\xi,t)\, a^2 \left(\frac{\xi}{\varepsilon^3}\right) \frac{1}{\varepsilon^2} \left(\partial_x u_\varepsilon \left(\frac{\xi}{\varepsilon^2}, t\right) - g_\varepsilon \left(\frac{\xi}{\varepsilon^2}, t\right) \right) \\
&\left(\partial_x u_\varepsilon \left(\frac{\xi}{\varepsilon^2}, t\right) + g_\varepsilon \left(\frac{\xi}{\varepsilon^2}, t\right) \right)\, d\xi\, dt + \int_0^T \int_{\mathbb{R}} \Phi(\xi,t)\, a^2 \left(\frac{\xi}{\varepsilon^3}\right) \frac{1}{\varepsilon^2}\, g_\varepsilon^2 \left(\frac{\xi}{\varepsilon^2}, t\right)\, d\xi\ dt \\
=:& A_\varepsilon + B_\varepsilon
\end{aligned} \tag{6.7}$$

We will show that the first term on the right hand side of (6.7) converges to zero. The second term produces in the limit the desired measure, $B_\varepsilon \to \int_0^T E^+(t)\Phi(c^* t,t)\, dt$.

Concerning A_ε: We substitute $x := \frac{\xi}{\varepsilon^2}$ and apply Lemma 6.3 to arrive at

$$\begin{aligned}
|A_\varepsilon| &= \left| \int_0^T \int_{\mathbb{R}} \Phi(x\varepsilon^2,t)\, a^2 \left(\frac{x}{\varepsilon}\right) \left(\partial_x u_\varepsilon\,(x,t) - g_\varepsilon\,(x,t)\right) \left(\partial_x u_\varepsilon\,(x,t) + g_\varepsilon\,(x,t)\right)\, dx\, dt \right| \\
&\le C \|\Phi\|_{L^1(0,T;C_0(\mathbb{R}))} \|\partial_x u_\varepsilon - g_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} \|\partial_x u_\varepsilon + g_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} \\
&\le C\varepsilon^2 \to 0.
\end{aligned}$$

In the last last line we have used that $\|g_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}$ and $\|\partial_x u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}$ are uniformly bounded.

Concerning B_ε: We calculate

$$\begin{aligned}
a\left(\frac{\xi}{\varepsilon^3}\right) g_\varepsilon \left(\frac{\xi}{\varepsilon^2}, t\right) = a\left(\frac{\xi}{\varepsilon^3}\right) \Bigg[& \left(1 + \partial_y L_1 \left(\frac{\xi}{\varepsilon^3}\right)\right) \partial_x W^+ \left(\frac{\xi - c^* t}{\varepsilon^2}, t\right) \\
&+ \varepsilon \left(L_1(\cdot) + \partial_y L_2(\cdot)\right) \left(\frac{\xi}{\varepsilon^3}\right) \partial_x^2 W^+ \left(\frac{\xi - c^* t}{\varepsilon^2}, t\right) \\
&+ \ldots + \varepsilon^4 \left(L_4(\cdot) + \partial_y L_5(\cdot)\right) \left(\frac{\xi}{\varepsilon^3}\right) \partial_x^5 W^+ \left(\frac{\xi - c^* t}{\varepsilon^2}, t\right) \\
&+ \varepsilon^5 L_5 \left(\frac{\xi}{\varepsilon^3}\right) \partial_x^6 W^+ \left(\frac{\xi - c^* t}{\varepsilon^2}, t\right) \Bigg] \\
&= a^* \partial_x W^+ \left(\frac{\xi - c^* t}{\varepsilon^2}, t\right) + O(\varepsilon),
\end{aligned} \tag{6.8}$$

where we have applied Lemma 5.2 in the last line and $O(\varepsilon)$ is used in the sense of the $L^\infty(0,T;L^2(\mathbb{R}))$-norm. Consequently,

$$
\begin{aligned}
\lim_{\varepsilon\to 0} B_\varepsilon &= \lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}} \Phi(\xi,t)\frac{(a^*)^2}{\varepsilon^2}\left(\partial_x W^+\left(\frac{\xi - c^*t}{\varepsilon^2}, t\right)\right)^2 \, d\xi\, dt \\
&= \lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}} \Phi(\varepsilon^2 x + c^*t, t)(a^*)^2 \left(\partial_x W^+(x,t)\right)^2 \, dx\, dt \\
&= \int_0^T \int_{\mathbb{R}} \Phi(c^*t, t)(a^*)^2 \left(\partial_x W^+(x,t)\right)^2 \, dx\, dt \\
&= \int_0^T \Phi(c^*t,t)(a^*)^2 \|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})} \, dt \\
&= \int_0^T E^+(t)\Phi(c^*t,t)\, dt,
\end{aligned}
$$

which is the claimed result. □

We remark that the treatment of u_ε^-, the left going wave in the one-dimensional homogenization problem, is analogous. Hence, we directly obtain the following result.

Corollary 6.5 (Energy measure in one space dimension.)**.** *Let μ be the energy measure of Definition 6.1 for $N = 1$. Then*

$$\mu(\xi,t) = E^+(t)\delta_{c^*t}(\xi) + E^-(t)\delta_{-c^*t}(\xi),$$

where the densities $E^+(t)$, $E^-(t)$ are given through

$$
\begin{aligned}
E^+(t) &= (a^*)^2\|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})}, \\
E^-(t) &= (a^*)^2\|\partial_x W^-(\cdot,t)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

We remark that the densities E^+ and E^- are in fact independent of t.

Remark 6.6. *Let E^+ and E^- be as in Corollary 6.5. Then there holds*

$$E^+(t) = (a^*)^2\|\partial_x c_0^+\|^2_{L^2(\mathbb{R})} \quad \text{and} \quad E^-(t) = (a^*)^2\|\partial_x c_0^-\|^2_{L^2(\mathbb{R})}$$

for all $t \in (0,T)$.

Proof. Let W^+ be the right going lKdV-solution of Definition 2.4. Multiplying the right going lKdV-equation by $\partial_x^2 W^+$, integrating over $\mathbb{R}$ and applying integration by parts leads to

$$
\begin{aligned}
0 &= -\frac{1}{2}\frac{d}{dt}\|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{a_2^*}{2c^*}\int_{\mathbb{R}} \partial_x^2 W^+(x,t)\partial_x^3 W^+(x,t)\, dx \\
&= -\frac{1}{2}\frac{d}{dt}\|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})} + \frac{a_2^*}{2c^*}\int_{\mathbb{R}} \frac{1}{2}\partial_x\left(\partial_x^2 W^+\right)^2(x,t)\, dx \\
&= -\frac{1}{2}\frac{d}{dt}\|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently,

$$\|\partial_x W^+(\cdot,t)\|^2_{L^2(\mathbb{R})} = \|\partial_x W^+(\cdot,0)\|^2_{L^2(\mathbb{R})} = \|\partial_x c_0^+\|^2_{L^2(\mathbb{R})}$$

for every $t \in (0,T)$, which is the claimed result. We remark that the analysis of W^- is analogous. □

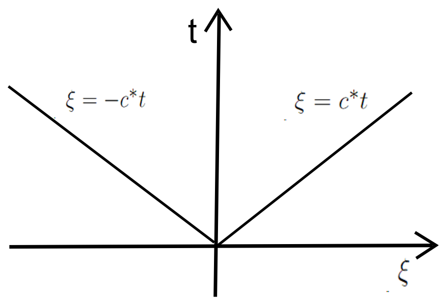

Figure 3: Support of the one-dimensional energy measure μ.

6.2 Other energy densities

We have discovered that in the one-dimensional case the energy density E_ε converges to the sum of two Dirac measures, $E_\varepsilon \overset{*}{\rightharpoonup} E^+\delta_{c^*t}(\xi) + E^-\delta_{-c^*t}(\xi)$. However, E_ε of Definition 6.1 does not correspond to the classical energy density, in which the term $\nabla u_\varepsilon(\cdot,t)^T A\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon(\cdot,t)$ appears instead of $\left(A\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon(\cdot,t)\right)^T\left(A\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon(\cdot,t)\right)$, see Eq. (1.21).

In this subsection, we will therefore investigate a slight modification of E_ε, which in fact is more natural.

Definition 6.7 (Natural energy density). *Let $N \in \mathbb{N}$ be arbitrary. Let u_ε be as in Definition 6.1.*

1. *We define the natural energy density $\tilde{E}_\varepsilon$ by*

$$\tilde{E}_\varepsilon(\xi,t) := \frac{1}{\varepsilon^{2N}}\left(\nabla u_\varepsilon(\cdot,t)^T A\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon(\cdot,t)\right)\left(\frac{\xi}{\varepsilon^2}\right). \tag{6.9}$$

2. *We call $\tilde{\mu} \in L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ a natural energy measure if there exists a subsequence $\tilde{E}_{\varepsilon_k}$ of $\tilde{E}_\varepsilon$ such that $\tilde{E}_{\varepsilon_k} \overset{*}{\rightharpoonup} \tilde{\mu}$ in $L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$.*

The existence of an energy measure $\tilde{\mu}$ is straightforward. It follows from the fact that $|\tilde{E}_\varepsilon| \le C|E_\varepsilon|$. Analogous to Proposition 6.4 we can derive a corresponding result for the one-dimensional natural energy measure $\tilde{\mu}$. However, we cannot give an exact formula for the corresponding density $\tilde{E}^+(t)$.

Corollary 6.8 (Natural energy measure in one space dimension). *Let u_ε be the solution to the one-dimensional homogenization problem with right going initial data, $c_0 = c_0^+$. Let $\tilde{\mu}$ be a natural energy measure as in Definition 6.7 for $N = 1$. Then*

$$\tilde{\mu}(\xi,t) = \tilde{E}^+(t)\delta_{c^*t}(\xi)$$

for some density $\tilde{E}^+(t) \in L^\infty(0,T)$.

Proof. We consider a test function $\Phi \in L^1(0,T;C_0(\mathbb{R}))$ with $\Phi(\xi,t) = 0$ if $\xi = c^*t$. Then

$$\int_0^T\int_{\mathbb{R}} |\Phi(\xi,t)| E_\varepsilon(\xi,t)\, d\xi\, dt \to 0 \tag{6.10}$$

due to Proposition 6.4. Consequently,

$$\left|\int_0^T \int_{\mathbb{R}} \Phi(\xi,t)\tilde{E}_\varepsilon(\xi,t)\, d\xi\, dt\right| = \left|\int_0^T \int_{\mathbb{R}} \Phi(\xi,t) a\left(\frac{\xi}{\varepsilon^3}\right)^{-1} E_\varepsilon(\xi,t)\, d\xi\, dt\right|$$
$$\leq \int_0^T \int_{\mathbb{R}} |\Phi(\xi,t)|\, a\left(\frac{\xi}{\varepsilon^3}\right)^{-1} E_\varepsilon(\xi,t)\, d\xi\, dt$$
$$\leq \frac{1}{\alpha}\int_0^T \int_{\mathbb{R}} |\Phi(\xi,t)|\, E_\varepsilon(\xi,t)\, d\xi\, dt \to 0$$

due to (6.10). In the second line we have used that E_ε is nonnegative. This is the claimed result. □

6.3 Fine properties of solutions for $N = 1$

We conclude the analysis of the one-dimensional case with an abstract description of the fine properties of the solution.

According to the fact that for right going initial data $E_\varepsilon(\xi,t) \overset{*}{\rightharpoonup} E^+\delta_{c^*t}(\xi)$ and $\tilde{E}_\varepsilon(\xi,t) \overset{*}{\rightharpoonup} \tilde{E}^+(t)\delta_{c^*t}(\xi)$, respectively, we follow the right going wave u_ε^+ along the ray $\xi = c^*t$ and consider u_ε^+ in the neighborhood of $x = \frac{\xi}{\varepsilon^2} = \frac{c^*t}{\varepsilon^2}$.

Definition 6.9 (Shifted gradient)**.** *Let u_ε^+ be the right going wave in the homogenization problem of Definition 2.2. We define the shifted gradient $U_\varepsilon^+(x,t)$ by*

$$U_\varepsilon^+(x,t) := \left(a\left(\frac{\cdot}{\varepsilon}\right)\partial_x u_\varepsilon^+(\cdot,t)\right)\left(x + \frac{c^*t}{\varepsilon^2}\right).$$

Due to the energy estimates for u_ε^+ we obtain that $\|U_\varepsilon^+\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C$. Hence, after passing to a subsequence, a weak star limit $U^+ \in L^\infty(0,T;L^2(\mathbb{R}))$ exists. In the subsequent proposition we identify the limit function $U^+(x,t)$. Moreover, we obtain that the weak star convergence is in fact a strong convergence.

Proposition 6.10. *Let U_ε^+ be as in Definition 6.9 and let W^+ be the right going lKdV-solution of Definition 2.4. Then there holds*

$$\|U_\varepsilon^+(x,t) - a^*\partial_x W^+(x,t)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \to 0 \quad \textit{for } \varepsilon \to 0.$$

Proof. We recall from Lemma 6.3 the estimate

$$\left\|\partial_x u_\varepsilon^+\left(x + \frac{c^*t}{\varepsilon^2}, t\right) - \partial_x\left(\mathcal{A}_\varepsilon W^+\left(\cdot - \frac{c^*}{\varepsilon^2}t, t\right)\right)\left(x + \frac{c^*t}{\varepsilon^2}\right)\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C\varepsilon^2. \tag{6.11}$$

A direct calculation similar to that in (6.8) yields

$$
\begin{aligned}
&a\left(\frac{x+c^*t/\varepsilon^2}{\varepsilon}\right)\left[\partial_x u_\varepsilon^+\left(x+\frac{c^*t}{\varepsilon^2},t\right)-\partial_x\left(\mathcal{A}_\varepsilon W^+\left(\cdot-\frac{c^*}{\varepsilon^2}t,t\right)\right)\left(x+\frac{c^*t}{\varepsilon^2}\right)\right]\\
=&U_\varepsilon^+(x,t)-\left[\left(a\left(\frac{\cdot}{\varepsilon}\right)\left(1+\partial_y L_1\left(\frac{\cdot}{\varepsilon}\right)\right)\right)\left(x+\frac{c^*t}{\varepsilon^2}\right)\partial_x W^+(x,t)\right.\\
&+\varepsilon\left(a\left(\frac{\cdot}{\varepsilon}\right)\left(L_1\left(\frac{\cdot}{\varepsilon}\right)+\partial_y L_2\left(\frac{\cdot}{\varepsilon}\right)\right)\right)\left(x+\frac{c^*t}{\varepsilon^2}\right)\partial_x^2 W^+(x,t)\\
&\left.+\ldots+\varepsilon^5\left(a\left(\frac{\cdot}{\varepsilon}\right)\partial_y L_5\left(\frac{\cdot}{\varepsilon}\right)\right)\left(x+\frac{c^*t}{\varepsilon^2}\right)\partial_x^6 W^+(x,t)\right]\\
=&U_\varepsilon^+(x,t)-a^*\partial_x W^+(x,t)+O(\varepsilon),
\end{aligned}
$$

where $O(\varepsilon)$ is used in the sense of $L^\infty(0,T;L^2(\mathbb{R}))$. Taking into account (6.11), we arrive at

$$
\|U_\varepsilon^+(x,t)-a^*\partial_x W^+(x,t)\|_{L^\infty(0,T;L^2(\mathbb{R}))}\le C\varepsilon^2+O(\varepsilon)\to 0,
$$

which is the claimed result. □

Proposition 6.4 and Proposition 6.10 contain a complete description of the long-time behavior of the one-dimensional wave equation.

In the multi-dimensional case the situation is much more complicated. In fact, even a rigorous identification of the energy measure μ seems not to be available, much less a statement about the fine properties of the solution.

7 Effective speeds and Riemannian distance

In this section we want to present a very general approach, the concept of domains of dependence and propagation speeds. With this approach we will be able to prove at least restrictions on the support of the N-dimensional energy measure μ in Section 8.

7.1 Domains of dependence

In this subsection we introduce and discuss the concept of domains of dependence.

Suppose that $L(t, x, u(x,t), \partial_t u(x,t), D_x u(x,t), D_x^2 u(x,t)) = 0$ is a homogeneous system of linear partial differential equations. Suppose that $c_0(x) := u(x, 0)$ is a given initial value. Now, we select an arbitrary point $p_0 = (x_0, t_0) \in \mathbb{R}^N \times (0, \infty)$. Our aim is to investigate how the value of u at the point p_0 is influenced by the initial datum c_0. In particular, we want to decide whether the point p_0 lies in the support of u or not.

It turns out that an appropriate tool to study this problem is the notion of domains of dependence. The following definition is due to [20].

Definition 7.1 (Domain of dependence). *Let $p_0 = (x_0, t_0) \in \mathbb{R}^N \times (0, \infty)$ and $p_1 = (x_1, t_1) \in \mathbb{R}^N \times [0, \infty)$ be two points in space-time.*

1. *The point $p_1 = (x_1, t_1)$ is said to influence the point p_0 if, given any spatial neighbourhood U of x_1 and any space-time neighbourhood D of p_0, there exists a solution u to $L(t, x, u(x,t), \partial_t u(x,t), D_x u(x,t), D_x^2 u(x,t)) = 0$ such that the following holds. At time t_1 the solution u is zero in $\mathbb{R}^n \setminus U$ but u is nonzero somewhere in D.*

2. *The domain of dependence $W(p_0, u)$ of the point p_0 is the set of all point p_1 influencing it.*

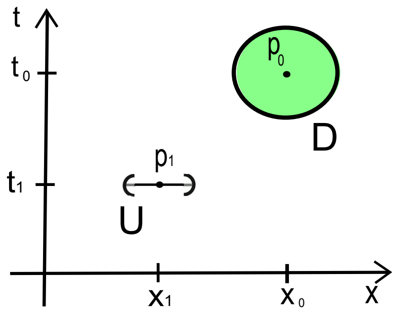

Figure 4: Suppose that the point p_1 influences the point p_0. Then there exists a solution u such that at time t_1 the solution u is zero outside U but is nonzero somewhere in D.

Let us discuss a prominent example, for which the domain of dependence can be directly determined. For the homogeneous wave equation we obtain the following result, see [20] for more details.

Example 1 (Domain of dependence for homogeneous wave equations). *Let $c > 0$. Consider the homogeneous wave equation $\partial_t^2 u(x,t) - (c^2)\Delta u(x,t) = 0$.*

1. *In the one-dimensional case, $x \in \mathbb{R}$, the domain of dependence of the point $p_0 = (x_0, t_0) \in \mathbb{R} \times (0, \infty)$ is*

$$W(p_0, u) = \{(x,t) \in \mathbb{R} \times [0,\infty) \,|\, |x - x_0| = c(t_0 - t)\}\,. \tag{7.1}$$

2. *In the two-dimensional case, $x \in \mathbb{R}^2$, the domain of dependence of the point $p_0 = (x_0, t_0) \in \mathbb{R}^2 \times (0, \infty)$ is*

$$W(p_0, u) = \big\{(x,t) \in \mathbb{R}^2 \times [0,\infty) \,|\, |x - x_0| \le c(t_0 - t)\big\}\,. \tag{7.2}$$

We observe that in one space dimension the domain of dependence is just the boundary of a cone in space-time with vertex $p_0 = (x_0, t_0)$. In the case of two space dimensions the domain of dependence also includes the interior of the cone. In particular, $W(p_0, u)$ does not only depend on the respective differential equation but also on the dimension of the space. This fact already suggests that an exact identification of domains of dependence is in general a difficult problem. Subsection 7.2 is devoted to the analysis of $W(p_0, u)$ in the case of the N-dimensional homogenization problem of Definition 2.10.

With the notion of domains of dependence we are now in the position to give restrictions on the support of a solution u. In what follows we assume that the initial value c_0 is compactly supported, $\operatorname{supp} c_0 \subseteq B_R(0) \subset \mathbb{R}^N$.

Corollary 7.2 (Support of u)**.** *Consider the setting of Definition 7.1. Let c_0 be compactly supported,* $\operatorname{supp} c_0 \subseteq B_R(0) \subset \mathbb{R}^N$*. Then $u(x_0, t_0) = 0$ for all $p_0 = (x_0, t_0) \in \mathbb{R}^N \times (0, \infty)$ satisfying the following condition*

$$(x, 0) \notin W(p_0, u) \quad \textit{for all} \;\; x \in B_R(0) \subset \mathbb{R}^N. \tag{7.3}$$

Proof. Let $p_0 \in \mathbb{R}^N \times (0, \infty)$ be such that condition (7.3) is satisfied. We have to show that no point $p_1 = (x_1, 0)$ with $x_1 \in \operatorname{supp} c_0$ influences the point p_0. Indeed, $\operatorname{supp} c_0 \subseteq B_R(0)$ and thus $p_1 \notin W(p_0, u)$ due to condition (7.3). □

In Subsection 7.2 we will use Corollary 7.2 to give restrictions on the support of $\bar{u}_\varepsilon$, solution to the N-dimensional homogenization problem of Definition 2.10.

7.2 Riemannian distance and Hamilton-Jacobi equations

In this subsection, we introduce the Riemannian distance q_ε corresponding to $A\left(\frac{\cdot}{\varepsilon}\right)$ and show that it localizes the domain of dependence for the N-dimensional homogenization problem of Definition 2.10. We remark that this approach is motivated by geometric optics, see [11, 20] for more details.

Definition 7.3 (Riemannian distance)**.** *Let $A(\cdot)$ be as in Definition 2.10, but not necessarily $(0,1)^N$-periodic, and let $x_0 \in \mathbb{R}^N$ be fixed. We define $q_\varepsilon(x)$ as the unique viscosity solution to the Hamilton-Jacobi equation*

$$\begin{gathered}(\nabla q_\varepsilon)^T(x) A\Big(\frac{x}{\varepsilon}\Big) \nabla q_\varepsilon(x) = 1, \;\; q_\varepsilon(x) > 0 \;\; \textit{in } \mathbb{R}^N \setminus \{x_0\}, \\ q_\varepsilon(x_0) = 0.\end{gathered} \tag{7.4}$$

A proof of existence and uniqeness of q_ε can be found in [11].

The subsequent proposition shows that the Riemannian distance q_ε determines a set, which at least contains the domain of dependence of $\bar{u}_\varepsilon$. We emphasize that this set need not be optimal.

Proposition 7.4 (Domain of dependence)**.** *Let $\tau_0 > 0$ and let $\tau_1 \in [0, \tau_0)$. Consider $A(\cdot)$, x_0 and q_ε as in Definition 7.3. We define the Riemannian cone*

$$C^\varepsilon := \{(x, \tau) \in \mathbb{R}^N \times (\tau_1, \tau_0) \,|\, q_\varepsilon(x) < \tau_0 - \tau\}$$

and the cross section

$$C^\varepsilon_\tau := \{x \in \mathbb{R}^N \,|\, q_\varepsilon(x) < \tau_0 - \tau\}.$$

Assume that q_ε is Lipschitz continuous and that $\bar{u}_\varepsilon$ is a smooth solution to the N-dimensional ε-problem (2.8) *without periodicity assumption. Then there holds the following statement on domains of dependence:*

If $\bar{u}_\varepsilon = \partial_\tau \bar{u}_\varepsilon = 0$ on $C^\varepsilon_{\tau_1}$, then $\bar{u}_\varepsilon = 0$ within the cone C^ε.

The following calculation is adapted from a calculation performed in the proof of Theorem 8 in Chapter 7.2 of Ref.[11].

Proof. We define the energy

$$e(\tau) := \frac{1}{2} \int_{C^\varepsilon_\tau} (\partial_\tau \bar{u}_\varepsilon)^2 + (\nabla \bar{u}_\varepsilon)^T A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon \, dx$$

for $\tau_1 \leq \tau \leq \tau_0$. Our aim is to show that $e(\tau) \equiv 0$. Since $e(\tau_1) = 0$ holds by assumption, it suffices to compute $\frac{d}{d\tau} e(\tau)$. Due to the symmetry of $A(y)$ we obtain

$$\begin{aligned}
\frac{d}{d\tau} e(\tau) &= \int_{C^\varepsilon_\tau} \partial_\tau \bar{u}_\varepsilon \partial_\tau^2 \bar{u}_\varepsilon + (\partial_\tau \nabla \bar{u}_\varepsilon)^T A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon \, dx \\
&\quad - \frac{1}{2} \int_{\partial C^\varepsilon_\tau} \Big((\partial_\tau \bar{u}_\varepsilon)^2 + (\nabla \bar{u}_\varepsilon)^T A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon \Big) \frac{1}{|\nabla q_\varepsilon|} \, dS \\
&=: L - M.
\end{aligned}$$

In this formula we have used the Coarea-formula, which implies for every continuous function f the relation

$$\frac{d}{d\tau} \Big(\int_{C^\varepsilon_\tau} f(x) \, dx \Big) = - \int_{\partial C^\varepsilon_\tau} \frac{f(x)}{|\nabla q_\varepsilon(x)|} \, dS. \tag{7.5}$$

Note that due to $A(y) \in M(\alpha, \beta)$ for ever $y \in \mathbb{R}^N$ and the Hamilton-Jacobi equation, the Riemannian distance q_ε satisfies $|\nabla q_\varepsilon| \geq \frac{1}{\sqrt{\beta}}$.

In order to evaluate L, we integrate by parts to find

$$\begin{aligned}
L &= \int_{C^\varepsilon_\tau} \partial_\tau \bar{u}_\varepsilon \Big(\partial_\tau^2 \bar{u}_\varepsilon - \nabla \cdot \Big(A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon \Big) \Big) \, dx + \int_{\partial C^\varepsilon_\tau} \partial_\tau \bar{u}_\varepsilon \; A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon \cdot \boldsymbol{\nu} \, dS \\
&= \int_{\partial C^\varepsilon_\tau} \partial_\tau \bar{u}_\varepsilon \; A\Big(\frac{x}{\varepsilon}\Big) \nabla \bar{u}_\varepsilon \cdot \boldsymbol{\nu} \, dS,
\end{aligned}$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\tau)$ denotes the outer unit normal of $\partial C^\varepsilon_\tau$. Taking into account the symmetry of $A(y)$ and the fact that $A(y)$ is nonnegative, we can apply the Cauchy-Schwarz inequality to obtain

$$\left|A\Big(\frac{x}{\varepsilon}\Big)\nabla\bar{u}_\varepsilon\cdot\boldsymbol{\nu}\right| \le \Big((\nabla\bar{u}_\varepsilon)^T A\Big(\frac{x}{\varepsilon}\Big)\nabla\bar{u}_\varepsilon\Big)^{1/2}\Big(\boldsymbol{\nu}^T A\Big(\frac{x}{\varepsilon}\Big)\boldsymbol{\nu}\Big)^{1/2}. \tag{7.6}$$

Due to the definition of C^ε_τ, we have $\boldsymbol{\nu} = \frac{\nabla q_\varepsilon}{|\nabla q_\varepsilon|}$ and thus

$$\boldsymbol{\nu}^T A\Big(\frac{x}{\varepsilon}\Big)\boldsymbol{\nu} = \frac{(\nabla q_\varepsilon)^T A\big(\frac{x}{\varepsilon}\big)\nabla q_\varepsilon}{|\nabla q_\varepsilon|^2} = \frac{1}{|\nabla q_\varepsilon|^2}, \tag{7.7}$$

where we used the Hamilton-Jacobi equation for q_ε. We conclude

$$\begin{aligned}
|L| &\le \int_{\partial C^\varepsilon_\tau} |\partial_\tau\bar{u}_\varepsilon|\left|A\Big(\frac{x}{\varepsilon}\Big)\nabla\bar{u}_\varepsilon\cdot\boldsymbol{\nu}\right| dS\\
&\le \int_{\partial C^\varepsilon_\tau} |\partial_\tau\bar{u}_\varepsilon|\Big((\nabla\bar{u}_\varepsilon)^T A\Big(\frac{x}{\varepsilon}\Big)\nabla\bar{u}_\varepsilon\Big)^{1/2}\frac{1}{|\nabla q_\varepsilon|}\,dS\\
&\le \frac{1}{2}\int_{\partial C^\varepsilon_\tau}\Big((\partial_\tau\bar{u}_\varepsilon)^2 + (\nabla\bar{u}_\varepsilon)^T A\Big(\frac{x}{\varepsilon}\Big)\nabla\bar{u}_\varepsilon\Big)\frac{1}{|\nabla q_\varepsilon|}\,dS = M.
\end{aligned}$$

Consequently,

$$\frac{d}{d\tau}e(\tau) \le 0.$$

Taking into account that $e(\tau_1) = 0$ we deduce that $e(\tau) = 0$ for each $\tau_1 \le \tau \le \tau_0$, which is the claimed result. □

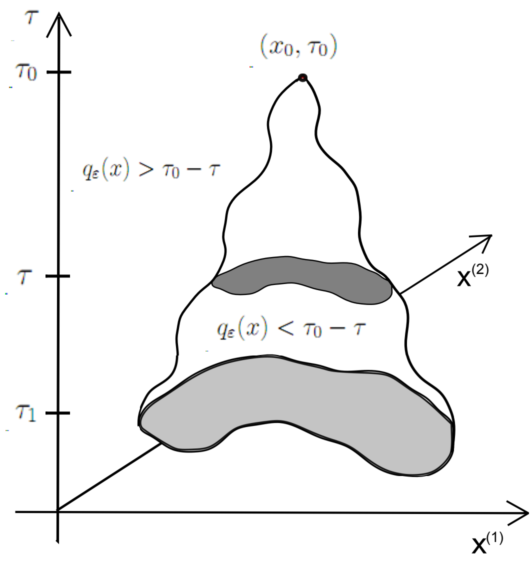

Figure 5: Riemannian cone C^ε with cross sections $C^\varepsilon_{\tau_1}$ and C^ε_τ in 2 space dimensions.

Next, we introduce the geometric cone of dependence $C^\varepsilon(p_0, \bar{u}_\varepsilon)$ of $\bar{u}_\varepsilon$ for a point $p_0 = (x_0, \tau_0) \in \mathbb{R}^N \times (0, T/\varepsilon^2)$.

Definition 7.5 (Geometric cone of dependence of $\bar{u}_\varepsilon$)**.** *Let $\bar{u}_\varepsilon$ be the solution to the N-dimensional homogenization problem of Definition 2.10. Let $p_0 = (x_0, \tau_0) \in \mathbb{R}^N \times (0, T/\varepsilon^2)$ be an arbitrary point in space time and let q_ε be the corresponding Riemannian distance. We define the geometric cone of dependence $C^\varepsilon(p_0, \bar{u}_\varepsilon)$ of $\bar{u}_\varepsilon$ for the point p_0 as follows*

$$C^\varepsilon(p_0, \bar{u}_\varepsilon) := \left\{(x, \tau) \in \mathbb{R}^N \times [0, \tau_0] \,|\, q_\varepsilon(x) \leq \tau_0 - \tau\right\}.$$

With the notion of cones of dependence we can now give restrictions on the domain of dependence of $\bar{u}_\varepsilon$.

Observation 3. *Proposition 7.4 yields the following relation for the domain of dependence of $\bar{u}_\varepsilon$ in the point $p_0 = (x_0, \tau_0) \in \mathbb{R}^N \times (0, T/\varepsilon^2)$*

$$W(p_0, \bar{u}_\varepsilon) \subseteq C^\varepsilon(p_0, \bar{u}_\varepsilon). \tag{7.8}$$

The problem of optimality in (7.8) is difficult and is treated in articles such as [24, 25]. In [24] the authors characterize domains of dependence of $\bar{u}_\varepsilon$ by means of two other approaches, namely the description via influence curves and the method of spacelike deformations. They show that all approaches are linked by the Riemannian distance. Furthermore, they show that the set $C^\varepsilon(p_0, \bar{u}_\varepsilon)$ is in fact optimal.

In view of Corollary 7.2, Observation 3 in particular gives restrictions on the support of $\bar{u}_\varepsilon$ in terms of the Riemannian distance q_ε.

Corollary 7.6 (Suport of $\bar{u}_\varepsilon$)**.** *Let $\operatorname{supp} c_0, \operatorname{supp} d_0 \subseteq B_R(0) \subset \mathbb{R}^N$ and let $\tau_0 > 0$ and $x_0 \in \mathbb{R}^N$ be fixed. Let $\bar{u}_\varepsilon$ be the solution to the N-dimensional homogenization problem of Definition 2.10. Moreover, let $p_0 = (x_0, \tau_0)$ satisfy the following condition:*

$$q_\varepsilon(x) > \tau_0 \quad \textit{for all } x \in B_R(0), \tag{7.9}$$

where q_ε denotes the Riemannian distance corresponding to x_0 and $A\left(\frac{\cdot}{\varepsilon}\right)$. Then

$$\bar{u}_\varepsilon(x_0, \tau_0) = 0.$$

Proof. Corollary 7.2 yields $\bar{u}_\varepsilon(x_0, \tau_0) = 0$ provided that $(x, 0) \notin W(p_0, \bar{u}_\varepsilon)$ for all $x \in B_R(0)$. On the other hand, when condition (7.9) is satisfied, we obtain that $(x, 0) \notin C^\varepsilon(p_0, \bar{u}_\varepsilon)$ for all $x \in B_R(0)$ and thus, using (7.8), $(x, 0) \notin W(p_0, \bar{u}_\varepsilon)$ for all $x \in B_R(0)$. Hence the claimed result follows. $\square$

We conclude this subsection by a brief discussion on the homogeneous wave equation, $A(\cdot) = c^2 \cdot Id_N$. What we expect in view of Example 1 is that $q_\varepsilon(x) = \frac{1}{c}|x - x_0|$. Indeed, the following holds.

Example 2 (The case of a constant A)**.** *Let $N \in \mathbb{N}$ be arbitrary. Let $A(\cdot) = c^2 \cdot Id_N$ for some $c > 0$ with Id_N denoting the $N \times N$ unit matrix. Then*

1. *the Riemannian distance q_ε of Definition 7.3 satisfies*
$$|\nabla q_\varepsilon(x)|^2 = \frac{1}{c^2}.$$
2. *the unique viscosity solution is $q_\varepsilon(x) = \frac{1}{c}|x - x_0|$.*
3. *the geometric cone of dependence of Definition 7.5 is*
$$C^\varepsilon(q, \bar{u}_\varepsilon) = \left\{(x, \tau) \in \mathbb{R}^N \times [0, \tau_0] \,|\, \frac{1}{c}|x - x_0| \leq \tau_0 - \tau\right\}.$$

7.3 Geometric effective cone of dependence and effective speeds

Corollary 7.6 provides a way to localize the support of $\bar{u}_\varepsilon$, solution to the N-dimensional homogenization problem of Definition 2.10. In particular, it gives restrictions on the support of the N-dimensional energy density E_ε of Definition 6.1.

For $\varepsilon \to 0$ the N-dimensional energy measure μ is the weak star limit of E_ε. Using that the Riemannian distance q_ε converges uniformly to some limit function $\bar{q}$, see Proposition 7.7 below, in Section 8 we derive restrictions on the support of μ in terms of $\bar{q}$.

In what follows we assume that the matrix $A(\cdot)$ in the N-dimensional homogenization problem of Definition 2.10 is in fact a scalar function, $A(\cdot) = a(\cdot) Id_N$. The following result on periodic homogenization of Hamilton-Jacobi equations is due to [22, 9]. It is a consequence of rewriting q_ε in the variational formulation

$$q_\varepsilon(x) = \inf_{x(\cdot)\in W^{1,1}((0,t);\mathbb{R}^N)} \left\{ \int_0^t b_\varepsilon(x(s), \dot{x}(s))\, ds \,|\, x(0) = x_0,\, x(t) = x \right\}$$

with cost function $b_\varepsilon(x,p) := \frac{|p|}{\sqrt{a\left(\frac{x}{\varepsilon}\right)}}$. In other words, $q_\varepsilon(x)$ corresponds to the distance of x and x_0 in the Riemannian metric induced by $a\left(\frac{\cdot}{\varepsilon}\right)$.

Proposition 7.7 (Homogenization of Hamilton-Jacobi equations)**.** *Let $x_0 \in \mathbb{R}^N$ be fixed. Let q_ε be the Riemannian distance of Definition 7.3 corresponding to $A\left(\frac{\cdot}{\varepsilon}\right) = a\left(\frac{\cdot}{\varepsilon}\right) \cdot Id_N$ and x_0. Then q_ε converges uniformly on $\mathbb{R}^N$ to a 1-homogeneous limit $\bar{q}$,*

$$\bar{q}(x) = |x - x_0| \bar{b}\left(\frac{x - x_0}{|x - x_0|}\right).$$

The effective cost function $\bar{b}$ can be characterized with a variational problem,

$$\bar{b}(p) = \lim_{\varepsilon\to 0} \inf_{x(\cdot)} \left\{ \int_0^t b_\varepsilon(x(s), \dot{x}(s))\, ds \,|\, x(0) = 0,\, x(t) = p \right\} \tag{7.10}$$

for $p \in \mathbb{R}^N$, $|p| = 1$.

Analogous to Definition 7.5 we define the geometric effective cone of dependence $\bar{C}(p_0)$ for the point p_0 as follows.

Definition 7.8 (Geometric effective cone of dependence and geometric effective speed)**.** *Let $N \in \mathbb{N}$ be arbitrary. Let $p_0 = (x_0, \tau_0) \in \mathbb{R}^N \times (0, T/\varepsilon^2)$ be an arbitrary point in space time. Let $q_\varepsilon(\cdot)$ and $\bar{q}(\cdot)$ be as in Proposition 7.7.*

1. *We define the geometric effective cone of dependence $\bar{C}(p_0)$ for the point p_0 through*

$$\bar{C}(p_0) := \left\{ (x,\tau) \in \mathbb{R}^N \times [0,\tau_0] \,|\, \bar{q}(x) = |x - x_0| \bar{b}\left(\frac{x - x_0}{|x - x_0|}\right) \le \tau_0 - \tau \right\}.$$

2. *We define the geometric effective speed $\bar{c}$ through*

$$\bar{c} = \left(\min_{|x|=1} \bar{b}(x) \right)^{-1}. \tag{7.11}$$

We remark that the geometric effective speed $\bar{c}$ of (7.11) is well defined. Indeed, the following holds.

Remark 7.9. *Let $\bar{b}$ and $\bar{c}$ be as in Definition 7.8. Then*

1. *the minimal value $\min_{|x|=1} \bar{b}(x)$ exists.*
2. *the geometric effective speed $\bar{c}$ is finite, $\bar{c} \leq \sqrt{\beta}$.*

Proof. Concerning 1: The effective metric $\bar{q} = |x - x_0| \bar{b}\left(\frac{x-x_0}{|x-x_0|}\right)$ is continuous, since it is the uniform limit of continuous functions. Hence, the minimal value $\min_{|x-x_0|=1} \bar{q}(x) = \min_{|x|=1} \bar{b}(x)$ exists.

Concerning 2: Due to $\alpha \leq a(\cdot) \leq \beta$, we obtain

$$|b_\varepsilon(x,p)| = \frac{|p|}{\sqrt{a\left(\frac{x}{\varepsilon}\right)}} \geq |p| \frac{1}{\sqrt{\beta}}.$$

Consequently, the effective cost function $\bar{b}$ satisfies

$$\begin{aligned}
\bar{b}(p) &= \liminf_{\varepsilon \to 0} \inf_{x(\cdot)} \left\{ \int_0^t b_\varepsilon(x(s), \dot{x}(s))\, ds \,|\, x(0) = 0,\ x(t) = p \right\} \\
&\geq \liminf_{\varepsilon \to 0} \inf_{x(\cdot)} \left\{ \int_0^t |\dot{x}(s)| \frac{1}{\sqrt{\beta}}\, ds \,|\, x(0) = 0,\ x(t) = p \right\} = |p| \frac{1}{\sqrt{\beta}}
\end{aligned}$$

and thus $\bar{c} \leq \sqrt{\beta}$. □

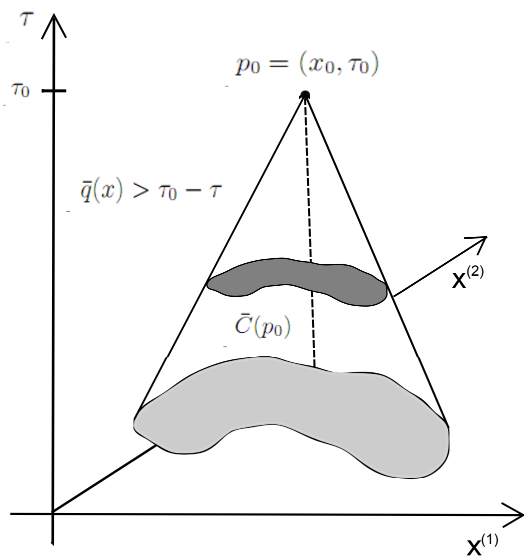

Figure 6: Geometric effective cone of dependence $\bar{C}(p_0)$ in 2 space dimensions.

We conclude this subsection by introducing another notion of a speed of propagation. More precisely, we introduce the *energetic effective speed* $\hat{c}$, which is defined through the support of the energy measure μ of Definition 6.1.

Definition 7.10 (Energetic effective speed $\hat{c}$)**.** *Let $N \in \mathbb{N}$ be arbitrary. We define the energetic effective speed $\hat{c}$ by*

$$\hat{c} := \inf\{c > 0 \,|\, \operatorname{supp}\mu \subseteq C(c) \textit{ for every energy measure } \mu\},$$

where $C(c)$ is a cone in space time with slope $\frac{1}{c}$,

$$C(c) := \{(\xi, t) \in \mathbb{R}^N \times (0, T) \,|\, |\xi| \le ct\}.$$

Let us give a comment on the definition of $\hat{c}$. It might be confusing at first sight that the apexes of the enveloping cones $C(c)$ of Definition 7.10 lie in the origin. In fact, the particular scaling of the space variable $\xi = x\varepsilon^2$ effects that, in the limit as $\varepsilon \to 0$, the support of the initial data $c_0, d_0 \in C_c^\infty(B_R(0))$ concentrates in $\xi = 0$.

Due to Corollary 6.5 the one-dimensional energy measure μ is a Dirac measure, $\mu(\xi, t) = E^+\delta_{c^*t}(\xi) + E^-\delta_{-c^*t}(\xi)$. Consequently, the energetic effective speed $\hat{c}$ in one space dimension, $N = 1$, is $\hat{c} = c^* = \left(\sqrt{\int_Y \frac{1}{a(y)}\,dy}\right)^{-1}$.

In the case of $N \ge 2$ space dimensions a rigorous identification of $\hat{c}$ seems to be out of reach. However, in Section 8 we show that $\hat{c} \le \bar{c}$. In other words, the geometric effective speed $\bar{c}$ provides at least an upper bound for the effective speed $\hat{c}$.

7.4 Geometric effective speed in one space dimension

In this subsection, we determine the geometric effective speed $\bar{c}$ of Definition 7.8 in the case of one space dimension, $N = 1$.

It turns out that $\bar{c}$ and $\hat{c}$ do not coincide, $\bar{c} > \hat{c}$. In other words, the geometric effective speed provided by the Riemannian distance approach is not optimal in the sense that it overestimates the energetic effective speed $\hat{c}$. In Subsection 7.5 we explain this phenomenon with a simple example.

Proposition 7.11 (Homogenization of the Riemannian distance for $N = 1$)**.** *Consider q_ε as in Definition 7.3 for $N = 1$. Then $q_\varepsilon \to \bar{q}$ uniformly on $\mathbb{R}$, where the limit function $\bar{q}$ is given by*

$$\bar{q}(x) = |x - x_0| \left\langle \frac{1}{\sqrt{a(\cdot)}} \right\rangle_Y. \tag{7.12}$$

Proof. Due to a direct calculation one obtains

$$(\partial_x q_\varepsilon(x))^2 = \frac{1}{a\left(\frac{x}{\varepsilon}\right)} \Leftrightarrow \partial_x q_\varepsilon(x) = \pm\frac{1}{\sqrt{a\left(\frac{x}{\varepsilon}\right)}}$$

and thus, using $q_\varepsilon(x) \ge 0$,

$$q_\varepsilon(x) = \begin{cases} \int_{x_0}^x \frac{1}{\sqrt{a\left(\frac{\xi}{\varepsilon}\right)}}\,d\xi & \text{for } x \ge x_0, \\ -\int_{x_0}^x \frac{1}{\sqrt{a\left(\frac{\xi}{\varepsilon}\right)}}\,d\xi & \text{for } x < x_0. \end{cases}$$

The periodicity of $a(\cdot)$ yields

$$\begin{aligned}\sup_{x\ge x_0}|q_\varepsilon(x)-\bar q(x)| &= \sup_{x\ge x_0}\left|\int_{x_0}^{x}\frac{1}{\sqrt{a\left(\frac{\xi}{\varepsilon}\right)}}\,d\xi-(x-x_0)\left\langle\frac{1}{\sqrt{a(\cdot)}}\right\rangle_Y\right|\\ &= \sup_{x\ge x_0}\left|\int_{x_0}^{x}\frac{1}{\sqrt{a\left(\frac{\xi}{\varepsilon}\right)}}-\left\langle\frac{1}{\sqrt{a(\cdot)}}\right\rangle_Y d\xi\right|\le C\varepsilon,\end{aligned}$$

where the constant C is independent of ε. An analogous calculation yields the same result for $x<x_0$, hence

$$\sup_{x\in\mathbb{R}}|q_\varepsilon(x)-\bar q(x)|\le C\varepsilon,$$

which is the claimed result. □

The proposition directly provides the geometric effective cone of dependence $\bar C(p_0)$ and the geometric effective speed $\bar c$ in the case of one space dimension.

Remark 7.12 (Geometric effective cone of dependence and geometric effective speed $\bar c$ for $N=1$)**.** *Let $N=1$. Let $p_0=(x_0,\tau_0)\in\mathbb{R}\times(0,T/\varepsilon^2)$ be an arbitrary point in space time. Then*

1. *the geometric effective cone of dependence $\bar C(q)$ of Definition 7.8 is*

$$\bar C(p_0)=\left\{(x,\tau)\in\mathbb{R}\times[0,\tau_0]\,|\,\bar q(x)=|x-x_0|\left\langle\frac{1}{\sqrt{a(\cdot)}}\right\rangle_Y\le\tau_0-\tau\right\}.$$

2. *the geometric effective speed $\bar c$ of Definition 7.8 is*

$$\bar c=\left\langle\frac{1}{\sqrt{a(\cdot)}}\right\rangle_Y^{-1}. \tag{7.13}$$

However, Corollary 6.5 suggests that the energetic effective speed $\hat c$ of Definition 7.10 is $\hat c=\left(\sqrt{\int_Y\frac{1}{a(y)}\,dy}\right)^{-1}$. A comparison between the two effective speeds yields the following relation.

Corollary 7.13. *Let $N=1$. For $a(\cdot)\neq const$ the two effective wave speeds $\bar c$ and $\hat c$ do not coincide,*

$$\bar c=\left(\int_Y\frac{1}{\sqrt{a(y)}}\,dy\right)^{-1}>\left(\sqrt{\int_Y\frac{1}{a(y)}\,dy}\right)^{-1}=\hat c. \tag{7.14}$$

Proof. The relation is a direct consequence of Jensen's inequality. □

In the next subsection we will explain this phenomenon with a simple example. The example is well known and can for instance be found in [23]. The crucial point is that the Riemannian distance approach does not account for the amplitude of the propagating wave.

7.5 Analysis of $\bar{c}$ by means of waves in stratified media

We want to analyze $\bar{c}$ by presenting a simple one-dimensional example.

Example 3 (Waves across an interface)**.** *The one-dimensional wave equation across one interface is*

$$\partial_\tau^2 u(x,\tau) - \partial_x \left(a(x)\partial_x u(x,\tau)\right) = 0$$

with

$$a(x) := \begin{cases} (c_1)^2 & \text{for } x < 0, \\ (c_2)^2 & \text{for } x > 0, \end{cases} \tag{7.15}$$

where $c_1, c_2 > 0$ *and* $c_1 \neq c_2$*.*

Suppose that a wave $\varphi(\cdot) \in C^1(\mathbb{R},\mathbb{R})$ reaches the interface of the two media ($x = 0$) from the side of negative x. We decompose the solution into an incoming, reflected and transmitted wave as

$$u(x,\tau) = \begin{cases} \varphi_2(x - c_2\tau) & \text{for } x \geq 0, \\ \varphi_1(x + c_1\tau) + \varphi(x - c_1\tau) & \text{for } x \leq 0. \end{cases}$$

Since $u(.,\tau)$ and $a(.)\partial_x u(.,\tau)$ have to be continuous across the interface $x = 0$, one obtains the following system of equations

$$\begin{aligned} \varphi(-c_1\tau) + \varphi_1(c_1\tau) &= \varphi_2(-c_2\tau), \\ (c_1)^2\Big(\varphi'(-c_1\tau) + \varphi_1'(c_1\tau)\Big) &= (c_2)^2\varphi_2'(-c_2\tau). \end{aligned} \tag{7.16}$$

This system can be solved by a direct calculation providing the amplitude of the transmitted wave and the reflected wave, respectively.

Lemma 7.14 (Reflection and Transmission coefficients)**.** *Let* $\varphi, \varphi_1, \varphi_2 \in C^1(\mathbb{R},\mathbb{R})$ *be as in* (7.16)*. Then*

$$\begin{aligned} \varphi_1(y) &= \frac{c_1 - c_2}{c_1 + c_2}\varphi(-y), \\ \varphi_2(y) &= \frac{2c_1}{c_1 + c_2}\varphi\left(\frac{c_1}{c_2}y\right). \end{aligned}$$

In our next step, we investigate the case of ε-periodic interfaces.

Example 4 (Waves in stratified media)**.** *The one-dimensional wave equation in an* ε*-stratified medium is*

$$\partial_\tau^2 \bar{u}_\varepsilon(x,\tau) - \partial_x \left(a\left(\frac{x}{\varepsilon}\right)\partial_x \bar{u}_\varepsilon(x,\tau)\right) = 0,$$

where $a(\cdot)$ *is* 1*-periodic with*

$$a(y) = \begin{cases} (c_1)^2 & \text{for } -1/2 < x < 0, \\ (c_2)^2 & \text{for } 0 < x < 1/2. \end{cases}$$

We recall that in the case of stratified media the geometric effective speed $\bar{c}$ from (7.13) is given by

$$\bar{c} = \left(\int_Y \frac{1}{\sqrt{a(y)}}\, dy \right)^{-1} = \left(\frac{1}{2c_1} + \frac{1}{2c_2} \right)^{-1} = \frac{2c_1c_2}{c_1 + c_2}.$$

We recover $\bar{c}$ by investigating the speed of the leading wave front, i.e. the wave which is transmitted at each interface.

Observation 4 (Propagation speed of the leading wave front)**.** *The leading wave front travels with average speed $\bar{c}$. Indeed, the front covers the distance ε during the time*

$$\tau_\varepsilon = \frac{\frac{\varepsilon}{2}}{c_1} + \frac{\frac{\varepsilon}{2}}{c_2} = \frac{\varepsilon(c_1 + c_2)}{2c_1c_2}$$

and thus the average speed equals $\frac{\varepsilon}{\tau_\varepsilon} = \frac{2c_1c_2}{c_1+c_2} = \bar{c}$. In other words, $\bar{c}$ is obtained by adding propagation times.

Observation 4 suggests that there actually exists a wave traveling with asymptotic speed $\bar{c} > \hat{c}$. However, we did not account for the amplitude of the leading wave front.

Suppose again that a wave $\varphi(\cdot)$ reaches the interface of the two media ($x = 0$) from the side of negative x. Lemma 7.14 suggests that the amplitude of the transmitted wave corresponds to the amplitude of φ multiplied by $\frac{2c_1}{c_1+c_2}$. When the transmitted wave crosses the next interface, $x = \varepsilon/2$, its amplitude is multiplied by $\frac{2c_2}{c_1+c_2}$. To be more precise, $u(x,\tau) = \theta\, \varphi(x - c_1\tau)$ for $\frac{\varepsilon}{2} < x < \varepsilon$. The transmission coefficient θ is determined as $\theta := \frac{4c_1c_2}{(c_1+c_2)^2} = 1 - \left(\frac{c_1-c_2}{c_1+c_2}\right)^2 < 1$. Consequently, after crossing $n = \Theta(1/\varepsilon)$ layers, the leading wave front amplitude is of order θ^n and thus vanishes in the limit $\varepsilon \to 0$.

The above calculations suggest that a wave of small amplitude propagates in front of the principal wave which travels with speed $\hat{c} = c^*$. The principal wave $\bar{u}$, solution to the homogenized equation, is generated by many interacting reflections and transmissions within the heterogeneous material.

8 Estimate for the N-dimensional energy measure

In this section we prove that in the multi-dimensional case the geometric effective speed $\bar{c}$ of Definition 7.8 provides at least an upper bound for the energetic effective speed $\hat{c}$ of Definition 7.10, $\hat{c} \le \bar{c}$. We recall that in the case of one space dimension, $N = 1$, this bound is not optimal, $\hat{c} < \bar{c}$.

In what follows, we assume that the matrix $A(\cdot)$ in the homogenization problem of Definition 2.10 is a scalar function, $A(y) = a(y) \cdot Id_N$.

We are now in the position to prove the main result of this section: The geometric effective speed $\bar{c}$ provides an upper bound for the energetic effective speed $\hat{c}$.

Theorem 8.1 (Upper bound for the energetic effective speed $\hat{c}$)**.** *Let $N \in \mathbb{N}$ be arbitrary. Let $\hat{c}$ be the energetic effective speed of Definition 7.10 and let $\bar{c}$ be the geometric effective speed of Definition 7.8. Then there holds*

$$\hat{c} \le \bar{c}. \tag{8.1}$$

Proof. We have to show that $\operatorname{supp} \mu \subseteq \left\{(\xi, t) \in \mathbb{R}^N \times (0,T) \,|\, |\xi| \le \bar{c}t\right\}$. More precisely, we prove that for the energy density E_ε of Definition 6.1 and for every test function $\Phi \in L^1(0,T; C_0(\mathbb{R}^N))$ with $\operatorname{supp} \Phi \subseteq \left\{(\xi, t) \in \mathbb{R}^N \times (0,T) \,|\, |\xi| > \bar{c}t\right\}$ there holds

$$\int_0^T \int_{\mathbb{R}^N} \Phi(\xi,t) E_\varepsilon(\xi,t)\, d\xi\, dt \to 0 \qquad \text{as } \varepsilon \to 0. \tag{8.2}$$

First step. In our first step we give restrictions on the support of $\bar{u}_\varepsilon$, solution to the N-dimensional homogenization problem of Definition 2.10 in terms of the effective distance $\bar{q}$.

We apply Corollary 7.6 to obtain $\bar{u}_\varepsilon(x_0, \tau_0) = 0$ for all points $(x_0, \tau_0) \in \mathbb{R}^N \times (0, T/\varepsilon^2)$ such that the following condition is satisfied

$$q_\varepsilon(x) > \tau_0 \quad \text{for all } x \in B_R(0) \subset \mathbb{R}^N. \tag{8.3}$$

We now use that $q_\varepsilon(\cdot)$ converges uniformly to $\bar{q}(\cdot) = |\cdot - x_0| \bar{b}\left(\frac{\cdot - x_0}{|\cdot - x_0|}\right)$. This implies that there exists some positive constant $\delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for all $x \in \mathbb{R}^N$ there holds

$$q_\varepsilon(x) > \bar{q}(x) - \delta(\varepsilon). \tag{8.4}$$

Due to $\bar{c} = \left(\min_{|x|=1} \bar{b}(x)\right)^{-1}$ we obtain for a point $|x| \le R$

$$\begin{aligned} \bar{q}(x) - \delta(\varepsilon) &= |x - x_0| \bar{b}\left(\frac{x - x_0}{|x - x_0|}\right) - \delta(\varepsilon) \\ &\ge \frac{1}{\bar{c}} |x - x_0| - \delta(\varepsilon) \\ &\ge \frac{1}{\bar{c}} (|x_0| - R) - \delta(\varepsilon). \end{aligned} \tag{8.5}$$

In particular, condition (8.3) is satisfied if

$$\frac{1}{\bar{c}} (|x_0| - R) - \delta(\varepsilon) > \tau_0. \tag{8.6}$$

Consequently, $\bar{u}_\varepsilon(x_0,\tau_0)=0$ holds for all (x_0,τ_0) satisfying (8.6).

Second step. We use condition (8.6) to derive restrictions on the support of the energy density E_ε. Taking into account the definition of E_ε,

$$E_\varepsilon(\xi,t)=\frac{1}{\varepsilon^{2N}}\,a^2\left(\frac{\xi}{\varepsilon^3}\right)\left|\nabla\bar{u}_\varepsilon\left(\frac{\xi}{\varepsilon^2},\frac{t}{\varepsilon^2}\right)\right|^2,$$

we obtain that $E_\varepsilon(\xi_0,t_0)=0$ for all $(\xi_0,t_0)\in\mathbb{R}^N\times(0,T)$ satisfying

$$\frac{1}{\bar{c}}\left(\left|\frac{\xi_0}{\varepsilon^2}\right|-R\right)-\delta(\varepsilon)>\frac{t_0}{\varepsilon^2}$$

or, equivalently,

$$\frac{1}{\bar{c}}\left(|\xi_0|-\varepsilon^2R\right)-\varepsilon^2\delta(\varepsilon)>t_0. \tag{8.7}$$

Third step. We are now in the position to prove the convergence (8.2). We define the set A_ε by

$$A_\varepsilon:=\left\{(\xi_0,t_0)\in\mathbb{R}^N\times(0,T)\,|\,t_0\geq\frac{1}{\bar{c}}\left(|\xi_0|-\varepsilon^2R\right)-\varepsilon^2\delta(\varepsilon)\right\} \tag{8.8}$$

and rewrite the left hand side of (8.2). Since E_ε vanishes outside A_ε,

$$\left|\int_0^T\int_{\mathbb{R}^N}\Phi(\xi,t)E_\varepsilon(\xi,t)\;d\xi\,dt\right|=\left|\int_0^T\int_{\mathbb{R}^N}\Phi(\xi,t)E_\varepsilon(\xi,t)\chi_{A_\varepsilon}(\xi,t)\;d\xi\,dt\right|.$$

Since $\operatorname{supp}\Phi\subseteq\left\{(\xi_0,t_0)\in\mathbb{R}^N\times(0,T)\,|\,t_0<\frac{1}{\bar{c}}|\xi_0|\right\}$, there exists some $\varepsilon_0>0$ such that for all $\varepsilon<\varepsilon_0$ and all $(\xi,t)\in A_\varepsilon$ there holds $\Phi(\xi,t)=0$. Consequently,

$$\int_0^T\int_{\mathbb{R}^N}\Phi(\xi,t)E_\varepsilon(\xi,t)=0$$

for $\varepsilon<\varepsilon_0$. This proves Theorem 8.1. □

Let us give some remarks on $\bar{c}$ and $\hat{c}$. In fact, the two effective speeds are just maximal values, where the maximum is taken over all directions of propagation. We can perform a more exact analysis on the effective speed of waves in heterogeneous media by introducing direction-dependent quantities.

Definition 8.2 (Direction-dependent effective speeds)**.** *Let $\bar{b}$ be the effective cost function of Proposition 7.7. Let $\vartheta\in\mathbb{R}^N$ with $|\vartheta|=1$. We define the direction-dependent effective speeds $\bar{c}(\vartheta)$ and $\hat{c}(\vartheta)$ by*

$$\bar{c}(\vartheta):=\left(\bar{b}(\vartheta)\right)^{-1},$$
$$\hat{c}(\vartheta):=\inf\left\{c>0\,|\,\operatorname{supp}\mu\cap\{t\vartheta\,|\,t\in\mathbb{R}\}\subseteq C(c)\text{ for every energy measure }\mu\right\},$$

where $C(c)$ is a cone in space time with slope $\frac{1}{c}$,

$$C(c):=\{(\xi,t)\in\mathbb{R}^N\times(0,T)\,|\,|\xi|\leq ct\}.$$

Analogous to Theorem 8.1 one can show that for every direction $\vartheta\in\mathbb{R}^N$, $|\vartheta|=1$, there holds

$$\hat{c}(\vartheta)\leq\bar{c}(\vartheta). \tag{8.9}$$

The proof of (8.9) follows the proof of Theorem 8.1.

Appendix

A Energy estimate for the time-scaled wave equation

Lemma A.1. *Let $u \in L^\infty(0,T;H^2(\mathbb{R}^N))$ with $\partial_t u \in L^\infty(0,T;H^1(\mathbb{R}^N)), \partial_t^2 u \in L^2(0,T;L^2(\mathbb{R}^N))$ be the unique solution to the time-scaled wave equation*

$$\begin{aligned} \varepsilon^4 \partial_t^2 u(x,t) - \nabla \cdot \Big(A\left(\frac{x}{\varepsilon}\right) \nabla u(x,t) \Big) &= f(x,t), \\ u(x,0) &= a_0(x), \\ \partial_t u(x,0) &= b_0(x) \end{aligned} \tag{1.1}$$

with $f \in L^2(0,T;L^2(\mathbb{R}^N))$, $a_0 \in H^2(\mathbb{R}^N)$ and $b_0 \in H^1(\mathbb{R}^N)$. Then there exists an ε-independent constant C such that

$$\begin{aligned} &\|\nabla u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \varepsilon^4 \|\partial_t u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^N))} \\ &\quad \le C\Big(\|a_0\|^2_{H^1(\mathbb{R}^N)} + \varepsilon^4 \|b_0\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\varepsilon^4} \|f\|^2_{L^2(0,T;L^2(\mathbb{R}^N))} \Big). \end{aligned}$$

Proof. We multiply the time-scaled wave Eq. (1.1) by $\partial_t u(x,t)$, integrate over $\mathbb{R}^N$ and apply integration by parts. Then

$$\begin{aligned} \varepsilon^4 \frac{d}{dt} \|\partial_t u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + \frac{d}{dt} \int_{\mathbb{R}^N} \nabla u(x,t) \cdot \Big(A\left(\frac{x}{\varepsilon}\right) \nabla u(x,t) \Big)\, dx &= 2 \int_{\mathbb{R}^N} f(x,t) \partial_t u(x,t)\, dx \\ &\le \Big(\frac{4}{\varepsilon^4} \|f(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + \varepsilon^4 \|\partial_t u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \Big) \end{aligned} \tag{1.2}$$

for almost every $t \in (0,T)$. Taking into account that $A\left(\frac{x}{\varepsilon}\right) \in M(\alpha,\beta)$ and integrating over $(0,t)$, one arrives at

$$\begin{aligned} &\varepsilon^4 \|\partial_t u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + \alpha \|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \\ &\le \varepsilon^4 \|b_0(x)\|^2_{L^2(\mathbb{R}^N)} + \beta \|a_0(x)\|^2_{H^1(\mathbb{R}^N)} + \frac{4}{\varepsilon^4} \|f(x,t)\|^2_{L^2(0,T;L^2(\mathbb{R}^N))} \\ &\quad + \int_0^t \varepsilon^4 \|\partial_t u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^N)}\, d\tau \\ &\le \varepsilon^4 \|b_0(x)\|^2_{L^2(\mathbb{R}^N)} + \beta \|a_0(x)\|^2_{H^1(\mathbb{R}^N)} + \frac{4}{\varepsilon^4} \|f(x,t)\|^2_{L^2(0,T;L^2(\mathbb{R}^N))} \\ &\quad + \int_0^t \varepsilon^4 \|\partial_t u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^N)} + \alpha \|\nabla u(\cdot,\tau)\|^2_{L^2(\mathbb{R}^N)}\, d\tau. \end{aligned}$$

Application of the Gronwall lemma finally leads to the claimed result. □

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