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Moment functions and Central Limit Theorem for Jacobi hypergroups on $[0,\infty[$

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Abstract

In this paper we derive sharp estimations and asymptotic results for moment functions on Jacobi-type hypergroups. Moreover, we use these estimations to prove a Central Limit Theorem for random walks on Jacobi hypergroups with growing parameters $\alpha,\beta\to\infty$. As a special case we obtain a CLT for random walks on the hyperbolic spaces $H_d(\mathbb{F})$ with growing dimensions d over the fields $\mathbb{F}=\mathbb{R},\ \mathbb{C}$ or the quaternions $\mathbb{H}$.

1 Introduction

Let $(K,*)$ be a hypergroup in the sense of Jewett [7]. The convolution $*$ allows the notion of random walks on $(K,*)$ by saying that a (time-homogeneous) Markov chain $(S_n)_{n\geq 0}$ is a random walk on $(K,*)$ with law $\nu\in\mathcal{M}^1(K)$ if

$$\mathbb{P}(S_{n+1}\in A|S_n=x)=(\delta_x*\nu)(A) \tag{1.1}$$

for all $n\geq 0$, $x\in X$ and Borel sets $A\subset K$. A lot of research was carried out in this setting, such as, for example, on recurrence, laws of large numbers, large deviation principle and central limit theorems. But there are also many issues where the presentation of S_n as sums of i.i.d. random variables would be useful. Typical examples are laws of large numbers, where truncature methods are used, see [22] and Section 7.3 of [3].

On a hypergroup, there is in general no deterministic operation corresponding to the convolution of measures. Consequently, sums of hypergroup-valued random variables cannot be defined directly. This obstacle was overcome in a particular case by Kingman [10] by applying the concept of randomized addition, in such a way that, as in the classical case, the distribution of the sum of two independent K-valued random variables equals the convolution of the distributions of the summands. Later, this construction was generalized by Zeuner [22] under the name of concretisation.

In studying of central limit theorems, the modified moments of a random variable, which are adapted to the hypergroup operation, were introduced to formulate the conditions under which a particular limit theorem holds, and to calculate the actual value of the limit. The notions of moments of the first and second order and of dispersion were introduced for special cases by Tutubalin [15]. Later, the idea of dispersion appeared in the work of Faraut [5] and Trimèche [14]. The modified moment functions on Sturm-Liouville hypergroups and polynomial hypergroups were studied by Zeuner [22] and Voit [16] respectively. A systematic study of this subject on an arbitrary hypergroup has been carried out by Zeuner [23].

Today, there are various CLT's for random walks on hypergroups available. For an overview on results we refer to the monograph [3] and references cited there. Random walks $(S_n)_{n\geq 1}$ on hypergroups $(K,*)$, where the convolution $* = *_n$ is in a certain way coupled with the number of steps n, were investigated by M. Voit (see [18], [19] and [20]) and is motivated by the following problem: For $\nu \in \mathcal{M}^1([0,\infty[)$ fix, and dimension $d \in \mathbb{N}$ there is a unique rotation-invariant probability measure $\nu_d \in \mathcal{M}^1(\mathbb{R}^d)$ with $\varphi_d(\nu_d) = \nu$, where $\varphi_d(x) = \|x\|_2$ is the norm mapping. For each $d \in \mathbb{N}$ consider i.i.d. $\mathbb{R}^d$-valued random variables X_k^d, $k \in \mathbb{N}$, with law ν_d as well as the associated radial random walks $\left(S_n^d := \sum_{k=1}^n X_k^d\right)_{n\geq 0}$ on $\mathbb{R}^d$. The aim is to find limit theorems for the $[0,\infty[$-valued random variables $\left\|S_n^d\right\|_2$ for $n, d \to \infty$ coupled in a suitable way. Two associated CLT's under disjointed growth conditions for $d = d_n$ were presented in [18] and in a more general setting in [19]. The first CLT for $d_n << n$ was thereby a consequence of Berry-Esseen estimates of $\mathbb{R}^d$ with explicit constants, depending on the dimension d, which are due to Bentkus [1]. A version of this result without the strong restriction $\frac{n}{d_n^3} \to \infty$ and $\frac{n^2}{d_n} \to 0$ but with the assumption of the existence of moments $\int x^j d\nu(x) < \infty$ for $j \in \mathbb{N}$ was recently derived in [6] using different methods, as in the references above.

The aim of this paper is to derive a CLT for random walks $(S_n)_{n\geq 0}$ on Jacobi-type hypergroup $([0,\infty[, *_{\alpha,\beta})$ for α, $n \to \infty$ as in [19] without any restriction on the growth of the "dimension" $\alpha = \alpha(n)$. We shall prove this here by using algebraic transformation and inequalities for moment functions as well as relations between moment function of the first and second order. A part of these results seems to be interesting in general, therefore we study them in a general setting of an arbitrary non commutative hypergroup in Section 2. Furthermore, special results for the moment functions on Jacobi-type hypergroups $([0,\infty[, *_{\alpha,\beta})$, in particular asymptotic behaviour of these as $\alpha \to \infty$, will be presented in the forth section.

The content of the paper is as follows. In the second section, after recalling some basic facts about hypergroups, concretisation and randomized sums, which are fundamental for the construction of random walks on hypergroups, we recapitulate the concept of moment functions for arbitrary hypergroups and derive relationships between them. These are essential for the proof of the central limit theorem 5.1. In the third section we collect the necessary background information on hyperbolic spaces and Jacobi hypergroups and indicate the connection between them. In the fourth section we derive estimations and asymptotic results for moment functions on Jacobi-type hypergroups. In the fifth section we use the results of the previous sections to prove the CLT for Jacobi hypergroups and its corollary. The last section is devoted to a weak law of large numbers for the sum of many "small" random variables on Jacobi hypergroups.

2 Random walks and moment functions on hypergroups

2.1 Hypergroups. The dual of a commutative hypergroup

Let $(K,*)$ be a *hypergroup* in the sense of Jewett [7]; this means that K is a locally compact space with an associative convolution $(x,y) \mapsto \delta_x * \delta_y \in \mathcal{M}^1(K)$ such that there exists a neutral element $e \in K$ and an inversion $x \mapsto \check{x}$ satisfying certain conditions. For a list of examples we refer to [3] and [23]. We call a hypergroup $(K,*)$ Hermitian, if $\check{x} = x$ for all $x \in K$, in particular this implies the commutativity of $(K,*)$.

The *dual* $\hat{K}$ of a Hermitian hypergroup $(K,*)$ is the space of all real-valued multiplicative functions φ on K with $\varphi(e) = \|\varphi\|_\infty = 1$ [7, 6.3]. For every probability measure P on K the

Fourier transform $\mathcal{F}P$ is the continuous real-valued function $\varphi \mapsto \mathcal{F}P(\varphi) := \int_K \varphi(x)dP(x)$ on $\hat{K}$. It is a well known fact that the uniqueness theorem and the continuity theorem for the Fourier transform are valid for commutative hypergroups (see [2, 7]). In the following, let $(K,*)$ be a hypergroup (not necessarily Hermitian).

2.2 Concretization of hypergroups, randomized sums

The forming of *sums* of K-valued random variables is not directly possible, as there is no deterministic operation on K in general. It is clear, that the "sum" of two independent random variables X and Y should be $P_X * P_Y$. In this section we recapitulate the construction of the *randomized sum* of K-valued random variables using the concept of the *concretization of hypergroups* (see [3, Chapter 7]). For this, we need the following definition:

Definition 2.1. Let $(K,*)$ be a hypergroup, μ a probability measure on a compact set M, and let $\Phi : K \times K \times M \to K$ be Borel-measurable. The triple (M,μ,Φ) is called a concretization of $(K,*)$ if

$$\mu\{\Phi(x,y,\cdot) \in A\} = (\delta_x * \delta_y)(A) \quad \text{for} \quad x,\ y \in K,\ A \in \mathcal{B}(K).$$

($\mathcal{B}(K)$ denotes the Borel σ-field of K.) Since $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$, we obviously have

$$\Phi(e,x,\cdot) = \Phi(x,e,\cdot) = x \quad \mu - a.s. \tag{2.1}$$

For a list of examples of concretisation we refer to [22]. It has been proven by Zeuner [23], that for every countable hypergroup $(K,*)$ there exists a measurable mapping Φ from $K \times K \times [0,1]$ in K such that $([0,1], \lambda_{[0,1]}, \Phi)$ is a concretization of K.

Assumption 2.2. In the sequel let (M,μ,Φ) be a concretization of the hypergroup $(K,*)$ and $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space. Moreover let $X, Y, X_0, X_1, X_2 \ldots$ be a sequence of K-valued random variables with $X_0 :\equiv e$, and $\Lambda, \Lambda_0, \Lambda_1, \Lambda_2 \ldots$ be a sequence of M-valued random variables such that all random variables $X, \Lambda, Y, X_0, \Lambda_0, X_1, \Lambda_1, X_2, \Lambda_2, \ldots$ are independent. We set $\nu_n := \mathbb{P}_{X_n}$ (so $\nu_0 = \delta_e$) and $\mu_n := \mathbb{P}_{\Lambda_n}$ $(n \geq 0)$. For a set $B \subset K$ we will denote by χ_B the map

$$\chi_B : K \to K, \qquad \chi_B(x) = \begin{cases} x & \text{for } x \in B, \\ e & \text{for } x \notin B. \end{cases}$$

Construction of randomized sums: Let X, Y and Λ be random variables as in assumption 2.2. We define

$$X \overset{\Lambda}{+} Y := \Phi(X,Y,\Lambda).$$

This is a K-valued random variable.

More generally, let $(X_n)_{n\geq 0}$ and $(\Lambda_n)_{n\geq 0}$ be sequences as in 2.2. Then we define the *randomized sum* $S_{j,n}$ recursively by

$$S_{j,n} := \begin{cases} X_j & \text{for } n = j, \\ \Phi(S_{j,n-1}, X_n, \Lambda_{n-1}) & \text{for } j < n. \end{cases}$$

for $j,\ n \in \mathbb{N}_0$ with $j \leq n$. We write S_n instead of $S_{0,n}$, i.e. we have

$$S_n = S_{n-1} \overset{\Lambda_{n-1}}{+} X_n = \big(S_{n-1} \overset{\Lambda_{n-2}}{+} X_{n-1}\big) \overset{\Lambda_{n-1}}{+} X_n.$$

If the random variables $X_1, X_2, \ldots$ are identically distributed with $P_{X_i} = \nu$ $(i \in \mathbb{N})$ then we will call $(S_n)_{n\in\mathbb{N}}$ a *random walk* on $(K,*)$ with law ν. In fact, direct computation shows that the sequence $\left(S_n\right)_{n\in\mathbb{N}}$ is a (non-homogeneous) Markov chain, with the transition kernel

$$\mathbb{P}\left(S_n \in B \middle| S_{n-1} = x\right) = \left(\nu_n * \delta_x\right)(B) \quad \mathbb{P} - \text{a.s.}$$

for all $x \in K$, $B \in \mathcal{B}(K)$. We will denote by $S_n(j,Z)$ $(j,n \in \mathbb{N},\ j \leq n)$ the *randomized sum* S_n, where the j-th term X_j is replaced by a K-valued random variable Z, i.e. $S_n(j,Z)$ can be recursively defined as follows

$$S_n(j,Z) := \begin{cases} \Phi(S_{n-1}(j,Z), X_n, \Lambda_{n-1}) & \text{for } j < n, \\ \Phi(S_{j-1}, Z, \Lambda_{j-1}) & \text{for } j = n. \end{cases}$$

For instance, if $n = 5$, $j = 3$ then

$$S_5(3,Z) = S_4(3,Z) \overset{\Lambda_4}{+} X_5 = \ldots = \Big(\big((X_1 \overset{\Lambda_1}{+} X_2) \overset{\Lambda_2}{+} Z\big) \overset{\Lambda_3}{+} X_4\Big) \overset{\Lambda_4}{+} X_5.$$

Clearly $S_n(j,e)$ coincides $\mathbb{P}$-a.s. with the randomized sum S_n, where the j-th term is omitted.

Remark 2.3. Forming randomized sums is generally non an associative operation although convolution of distributions obviously is. Indeed, it is easy to check that

$$\begin{aligned} P_{S_{j,n}} &= \nu_j * \nu_{j+1} * \ldots * \nu_n, \\ P_{S_n(j,Z)} &= \nu_1 * \ldots * \nu_{j-1} * \mathbb{P}_Z * \nu_{j+1} * \ldots * \nu_n. \end{aligned}$$

For $\nu \in M^1(K)$ and $n \in \mathbb{N}$ we denote the n-fold convolution power of ν w.r.t. the convolution $*$ by ν^n. If $X_1, X_2, \ldots$ are identically distributed, say $\mathbb{P}_{X_n} = \nu$ for $n \geq 1$ then

$$\mathbb{P}_{S_{j,n}} = \nu^{n-j+1} \quad \text{and} \quad \mathbb{P}_{S_n(j,Z)} = \nu^{j-1} * \mathbb{P}_Z * \nu^{n-j} \quad (j,n \in \mathbb{N},\ j \leq n).$$

2.3 Moments on hypergroups

We recapitulate the concept of moment function introduced by Zeuner; see [22, 23] and [3, Section 7.2].

Definition 2.4. Define the function $m_0(x) = 1$ for $x \in K$. A finite sequence $(m_i)_{i=1,\ldots,n}$ of measurable and locally-bounded functions $m_j : K \to \mathbb{C}$ $(j = 1, \ldots, n)$ is called a sequence of moment functions of length $n \in \mathbb{N}$ if

$$\int_K m_i(z)\ d\delta_x * \delta_y(z) = \sum_{j=0}^{i} \binom{i}{j} m_j(x) m_{i-j}(y) \quad (i = 1, \ldots, n; \quad x, y \in K). \tag{2.2}$$

Moreover, the function m_j $(j = 1, \ldots, n)$ is called a moment function of j-th order (associated with the sequence $(m_i)_{i=1,\ldots,n}$). By induction, we conclude from (2.2) that $m_j(e) = 0$ for a moment function of j-th order.

Here and subsequently, let $(m_i)_{i=1,\ldots,n}$ be a sequence of moment functions of length $n \in \mathbb{N}$. For every K-valued random variable X and $k \in \{1, \ldots, n\}$ such that $m_k(X)$ is integrable,

$$\mathbb{E}_k^*(X) := \mathbb{E}(m_k(X)) = \int m_k(X) d\mathbb{P}$$

will be called the *k-th modified moment* of X (with respect to the moment function m_k). We write $\mathbb{E}_*(X)$ for a modified moment of first order and refer to a *modified expectation.* The modified $*$-variance $\mathbb{V}_*(X)$ of X is defined as

$$\mathbb{V}_*(X) := \int m_2(X)d\mathbb{P} - \mathbb{E}_*(X)^2$$

if $\int m_2(X)d\mathbb{P}$ is finite and $\mathbb{V}_*(X) := \infty$ if not (c.f. [22, Sections 5-6]).

Let X, Y and Λ be random variables as in 2.2 such that $m_k(X)$ and $m_k(Y)$ are integrable for all $k \le n$. Then, we recapitulate from [22] that $m_n(X \overset{\Lambda}{+} Y)$ is integrable and

$$\mathbb{E}_n^*(X \overset{\Lambda}{+} Y) = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}_k^*(X)\mathbb{E}_{n-k}^*(Y) = \mathbb{E}_n^*(Y \overset{\Lambda}{+} X). \tag{2.3}$$

In particular, for $\mu_i \in \mathcal{M}^1(K)$, $(i = 1, 2, 3)$ we obtain following commutativity property

$$\int_K m_k(x)d(\mu_1 * \mu_2 * \mu_3)(x) = \int_K m_k(x)d(\mu_{\sigma(1)} * d\mu_{\sigma(2)} * d\mu_{\sigma(3)})(x), \tag{2.4}$$

where σ is a permutation of $\{1, 2, 3\}$.

Lemma. 2.5. *Let $X_1, \dots, X_n$ be K-valued and $\Lambda_1, \dots, \Lambda_{n-1}$ be M-valued independent random variables, such that $\mathbb{P}_{X_i} = \nu$ and $\mathbb{P}_{\Lambda_i} = \mu$ for $i = 1, \dots, n$ with corresponding random walk $(S_n)_{n\ge1}$ (with law ν). Then*

$$\mathbb{E}\big(m_k(S_n)|X_j\big) = \sum_{l=0}^{k} \binom{k}{l} m_l(X_j)\mathbb{E}_{k-l}^*\big(S_n(j, e)\big) \quad \mathbb{P} - a.s.$$

for all $k \in \mathbb{N}_0$, $j \in \{1, \dots, n\}$. In particular, for $k = 1$ we have

$$\mathbb{E}\big(m_1(S_n)|X_j\big) = m_1(X_j) + \mathbb{E}_*\big(S_n(j, e)\big) \quad \mathbb{P} - a.s.$$

Proof. Let $B \in \mathcal{B}(K)$, $k \in \mathbb{N}_0$ and $j \in \{1, \dots, n\}$. Consider the random variable $S_n(j, \chi_B(X_j))$. Obviously, by the definition of χ_B we have

$$S_n(j, \chi_B(X_j))(\omega) = \begin{cases} S_n(\omega) & \text{for } \omega \in \{X_j \in B\} \\ S_n(j, e)(\omega) & \text{for } \omega \notin \{X_j \in B\}. \end{cases}$$

This and the independence of X_j and $S_n(j, e)$ clearly forces

$$\begin{aligned} \mathbb{E}\big(1_{\{X_j\in B\}} m_k(S_n)\big) &= \mathbb{E}_k^*\big(S_n(j, \chi_B(X_j))\big) - \mathbb{E}\big(1_{\{X_j\notin B\}} \cdot m_k(S_n(j, e))\big) \\ &= E_k^*\big(S_n(j, \chi_B(X_j))\big) - \mathbb{P}(X_j \notin B)\mathbb{E}_k^*\big(S_n(j, e)\big). \end{aligned} \tag{2.5}$$

On the other side, by Remark 2.3, Eq. (2.3) and (2.4) it follows that

$$\begin{aligned} \mathbb{E}_k^*\big(S_n(j, \chi_B(X_j))\big) &= \int_\Omega m_k(x)d(\mathbb{P}_{\chi_B(X_j)} * \nu^{n-1})(x) \\ &= \sum_{\alpha=0}^{k} \binom{k}{\alpha} \mathbb{E}_\alpha^*\big(\chi_B(X_j)\big)\mathbb{E}_{k-\alpha}^*\big(S_n(j, e)\big) \end{aligned} \tag{2.6}$$

Since $1_{\{X_j\in B\}} m_l(X_j) = m_l(\chi_B(X_j))$ ($l \in \mathbb{N}$), taking (2.5) and (2.6) in account we obtain

$$\begin{aligned}
\mathbb{E}_k^*\big(\chi_B(S_n)\big) &= \sum_{\alpha=0}^{k} \binom{k}{\alpha} \mathbb{E}\big(1_{\{X_j\in B\}} m_\alpha(X_j)\big) \mathbb{E}_{k-\alpha}^*\big(S_n(j,e)\big) - \mathbb{P}(X_j \notin B)\mathbb{E}_k^*\big(S_n(j,e)\big) \\
&= \sum_{\alpha=1}^{k} \binom{k}{\alpha} \mathbb{E}\big(1_{\{X_j\in B\}} m_\alpha(X_j)\big) \mathbb{E}_{k-\alpha}^*\big(S_n(j,e)\big) + \mathbb{P}(X_j \in B)\mathbb{E}_k^*\big(S_n(j,e)\big) \\
&= \sum_{\alpha=0}^{k} \binom{k}{\alpha} \mathbb{E}\big(1_{\{X_j\in B\}} m_\alpha(X_j)\big) \mathbb{E}_{k-\alpha}^*\big(S_n(j,e)\big).
\end{aligned}$$

□

Theorem 2.6. *Let (m_1, m_2) be a sequence of moment functions such that*

$$m_2(x) \geq m_1(x)^2 \quad \textit{for all } x \in K. \tag{2.7}$$

Suppose that assumptions of Lemma 2.5 hold. Then

$$\mathbb{E}\Big(\Big\{m_1(S_n) - \sum_{j=1}^{n} m_1(X_j)\Big\}^2\Big) \leq n\Big(\mathbb{E}\big(m_2(X_1)\big) - \mathbb{E}\big(m_1(X_1)^2\big)\Big) \tag{2.8}$$

Proof. We define $Z_n := m_1(S_n) - \sum_{j=1}^{n} m_1(X_j)$ and calculate

$$Z_n^2 = m_1(S_n)^2 - 2m_1(S_n) \cdot \sum_{j=1}^{n} m_1(X_j) + \Big\{\sum_{j=1}^{n} m_1(X_j)\Big\}^2.$$

Since the random variables $X_1, \dots, X_n$ are i.i.d., Assumption (2.7) yields

$$\mathbb{E}\big(Z_n^2\big) \leq \mathbb{E}\big(m_2(S_n)\big) - 2\sum_{j=1}^{n} \mathbb{E}\big(m_1(S_n) m_1(X_j)\big) + n(n-1)\mathbb{E}\big(m_1(X_1)\big)^2 + n\mathbb{E}\big(m_1(X_1)^2\big). \tag{2.9}$$

For $j \in \{1, \dots, n\}$, using Lemma 2.5 and Eq. (2.3) we obtain

$$\begin{aligned}
\mathbb{E}\big(m_1(S_n) m_1(X_j)\big) &= \mathbb{E}\Big(m_1(X_j)\mathbb{E}\big(m_1(S_n)\big|X_j\big)\Big) = \mathbb{E}\Big(m_1(X_j)\big\{m_1(X_j) + \mathbb{E}_*\big(S_n(j,e)\big)\big\}\Big) \\
&= \mathbb{E}\big(m_1(X_1)^2\big) + (n-1)\mathbb{E}\big(m_1(X_1)\big)^2.
\end{aligned}$$

Iterative application of (2.3) to $\mathbb{E}\big(m_2(S_j) \overset{\Lambda}{+} X_{j+1}\big)$ $(j = 1, \dots, n-1)$ leads to

$$\begin{aligned}
\mathbb{E}_2^*\big(S_n\big) &= \mathbb{E}_2^*\big(S_{n-1} \overset{\Lambda_{n-1}}{+} X_n\big) = \mathbb{E}_2^*\big(S_{n-1}\big) + 2(n-1)\mathbb{E}_*\big(X_1\big)^2 + \mathbb{E}_2^*\big(X_1\big) \\
&= \mathbb{E}_2^*\big(S_{n-2}\big) + 2(n-2)\mathbb{E}_*\big(X_1\big)^2 + \mathbb{E}_2^*\big(X_1\big) + 2(n-1)\mathbb{E}_*\big(X_1\big)^2 + \mathbb{E}_2^*\big(X_1\big) \\
&= \ldots = n(n-1)\mathbb{E}\big(m_1(X_1)\big)^2 + n\mathbb{E}_2^*\big(X_1\big)
\end{aligned}$$

Therefore, we obtain from (2.9)

$$\mathbb{E}\big(Z_n^2\big) \leq n\Big(\mathbb{E}\big(m_2(X_1)\big) - \mathbb{E}\big(m_1(X_1)^2\big)\Big).$$

□

Remark 2.7. While the randomized sum S_n clearly depends on the particular choice of the underlying concretization on K the estimation in (2.8) does not.

3 Hyperbolic spaces and Jacobi hypergroups

3.1 Hyperbolic spaces

Let $d \geq 2$, and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or the skew field of the quaternions $\mathbb{H}$. We denote with $U(d, \mathbb{F})$ the orthogonal, unitary or symplectic group respectively. Moreover, we consider

$$U(d,1,\mathbb{F}) := \left\{A \in GL(d+1,\mathbb{F}) : A^* I_{1,d} A = I_{1,d}\right\},$$

where $I_{1,d}$ is the diagonal matrix of the form $diag(-1,1,\ldots,1)$. The *hyperbolic space* $H_d(\mathbb{F})$ of dimension d over $\mathbb{F}$ may be regarded as the symmetric space

$$H_d(\mathbb{F}) := G_d/V_d,$$

where $G_d := U(d,1,\mathbb{F})$ and $V_d := U(1,\mathbb{F}) \times U(d,\mathbb{F})$. In all cases, the double coset $G_d//V_d$ can be regarded as the interval $[0,\infty[$ by identifying $t \geq 0$ with the double coset

$$V_d a_t V_d \quad \text{with} \quad a_t = \begin{pmatrix} \text{ch}\,(t)\; 0 & \ldots & 0\; \text{sh}\,(t) \\ 0 & & 0 \\ \vdots & I_{d-1} & \vdots \\ 0 & & 0 \\ \text{sh}\,(t)\; 0 & \ldots & 0\; \text{ch}\,(t) \end{pmatrix};$$

(see [5] and [11, Ch. 3]). We define the *hyperbolic distance* on $H_d(\mathbb{F})$ by

$$dist(xV_d, yV_d) := \varphi_d(V_d y^{-1} x V_d), \quad \text{for } x,\ y\ \in\ G_d,$$

where $\varphi_d : V_d a_t V_d \mapsto t$ is the homeomorphism between $G_d//V_d$ and $[0,\infty[$. For a fixed probability measure $\nu \in \mathcal{M}^1([0,\infty[)$ there exists a unique radial (i.e. V_d-invariant) measure $\nu_d \in \mathcal{M}^1(H_d(\mathbb{F}))$ with $\varphi_d(\nu_d) = \nu$ (see in a more general context [19] and references cited there). In this way, we introduce the time-homogeneous radial (i.e. V_d-invariant) random walks $\left(S_n^d\right)_{n\geq 0}$ associated with the ν_d by $S_0^d := V_d \in H_d(\mathbb{F})$ and

$$\mathbb{P}\left(S_{n+1}^d \in A \cdot V_d \mid S_n^d = x \cdot V_d\right) = \nu_d\left(x^{-1} A V_d\right) = \nu\left(V_d x^{-1} A V_d\right)$$

for $n \geq 0$, $x \in G_d$, and $A \subset G_d$ a Borel set (see [7, 19] for details). Among other results, we shall derive the following central limit theorem for the random walk $\left(S_n^d\right)_{n\geq 0}$ on $H_d(\mathbb{F})$ where for a fixed field $\mathbb{F}$, the dimension d and the number of steps n tends to infinity.

Theorem 3.1. *Let $(d_n)_{n\geq 1} \subset \mathbb{N}$ be an increasing sequence of dimensions with $\lim_{n\to\infty} d_n = 0$ and fix $\mathbb{F}$ as above. Let $\nu \in \mathcal{M}^1\big([0,\infty[\big)$ with a finite second moment. For each dimension $d \geq 2$ consider the V_d-invariant time-homogeneous random walk $\left(S_n^d\right)$ on $H_d(\mathbb{F})$ such that for all n, d, the random variables $dist\left(S_n^{d+1}, S_n^d\right)$ have distribution ν. Then, $r_j := \int_0^\infty (\ln\,(\text{ch}\, x))^j d\nu(x) < \infty$ exist for $j = 1, 2$, and*

$$\frac{1}{\sqrt{n}}\Big(dist\big(S_n^{d_n}, S_0^{d_n}\big) - n r_1\Big)$$

tends in distribution for $n \to \infty$ to the normal distribution $\mathcal{N}(0, r_2 - r_1^2)$.

The above theorem will be proved by considering the moments of the distributions of $\big(dist(S_n^d, S_0^d)\big)_{n\geq 0}$ on the spaces $G//H \simeq [0,\infty[$ equipped with the associated double coset convolutions. These convolutions may be regarded as special cases of the so-called *Jacobi-convolution* on $[0,\infty[$ (c.f. Section 3.2). The same result, but with some restrictions on the growth of $d = d(n)$ in dependence of n was derived by M. Voit in [19, 20] by using different methods.

3.2 Jacobi-functions and Jacobi-hypergroups

For fixed parameters $\alpha \geq \beta \geq -\frac{1}{2}$ let

$$a_{\alpha,\beta}(x) := \text{sh}\,(x)^{2\alpha+1}\text{ch}\,(x)^{2\beta+1} \quad \text{and} \quad \rho := \frac{1}{2}\lim_{x\to\infty}\frac{a'_{\alpha,\beta}(x)}{a_{\alpha,\beta}(x)}.$$

This gives $\frac{a'_{\alpha,\beta}(x)}{a_{\alpha,\beta}(x)} = (2\alpha+1)\coth(x) + (2\beta+1)\tanh(x)$ and $\rho = \alpha+\beta+1$. Moreover, let $\mathcal{L} := \mathcal{L}_{(\alpha,\beta)}$ be the differential operator on $[0,\infty[$, defined by

$$\mathcal{L}f(x) = -f''(x) - \frac{a'_{\alpha,\beta}(x)}{a_{\alpha,\beta}(x)}f'(x) \tag{3.1}$$

for $x > 0$ and $f \in \mathcal{C}^2([0,\infty[)$ with $f'(0) = 0$. The *Jacobi-functions* $\varphi_\lambda := \varphi_\lambda^{(\alpha,\beta)}$ may be introduced as the unique solutions to the Sturm-Liouville problem

$$\mathcal{L}\varphi_\lambda(x) = (\rho^2+\lambda^2)\varphi_\lambda, \quad \varphi_\lambda(0) = 1, \quad \varphi'_\lambda(0) = 0. \tag{3.2}$$

It is well-known (see e.g. [4, 14, 22]) that there is a unique hypergroup operation $* := *_{\alpha,\beta}$ on $K := [0,\infty[$ such that

$$\int_K \varphi_\lambda(t)d(\delta_x * \delta_y)(t) = \varphi_\lambda(x)\varphi_\lambda(y) \quad \text{for all} \quad x,\ y \in \mathbb{R}_+, \quad \lambda \in \mathbb{C}. \tag{3.3}$$

We denote $([0,\infty[, *_{\alpha,\beta})$ and $*_{\alpha,\beta}$ as a *Jacobi-hypergroup* and a *Jacobi-convolution* on $[0,\infty[$ with parameters (α,β) respectively. The neutral element of this hypergroup is 0 and the inversion is the identity mapping. According to (3.3), the Jacobi functions are multiplicative functions w.r.t. the operation $* := *_{\alpha,\beta}$ on K. Furthermore, the dual $\hat{K}$ of K satisfies

$$\hat{K} = \{\varphi_\lambda : \quad \lambda \in \mathbb{R}_+ \cup i[0,\rho]\}.$$

The Plancherel measure π_K of $K := (\mathbb{R}_+, *)$ associated with the Haar measure $\omega_K := a_{\alpha,\beta}\lambda^1_{\mathbb{R}_+}$ ($\lambda^1_{\mathbb{R}_+}$ is Lebesgue measure on $\mathbb{R}_+$) and is given by

$$d\pi_K(\lambda) := \frac{1}{|c(\lambda)|^2}d\lambda^1_{\mathbb{R}_+} \text{ with } c(\lambda) := \frac{\sqrt{2\pi}2^{-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma((\rho+i\lambda)/2)\Gamma((\rho+i\lambda)/2-\beta)}$$

for all $\lambda \in \mathbb{R}_+$. The proof of the preceding results can be found in [3, 4].
The *(Fourier-) Jacobi transform* $f \mapsto \mathcal{F}f$ or $\mu \mapsto \mathcal{F}\mu$ is defined by

$$\mathcal{F}f(\lambda) := \int_0^\infty f(t)\varphi_\lambda(t)d\omega_K(t), \quad \mathcal{F}\mu(\lambda) := \int_0^\infty \varphi_\lambda(t)d\mu(t)$$

for all functions f and Borel measures μ on $\mathbb{R}_+$ respectively, and $\lambda \in \mathbb{C}$, for which the right-hand side is well-defined. With the notations above, the hyperbolic spaces $H_d(\mathbb{F})$ and their associated double coset convolutions are related to the Jacobi-convolution $*_{\alpha,\beta}$ by

$$\alpha = \frac{dim_{\mathbb{R}}(\mathbb{F}) \cdot d}{2} - 1, \qquad \beta = \frac{dim_{\mathbb{R}}(\mathbb{F})}{2} - 1.$$

An important technical tool will be the Laplace representation for the multiplicative functions φ_λ ($\lambda \in \mathbb{C}$) proved in [4, Proposition I-IV]: For every $x \in \mathbb{R}_+$ there exists a probability measure ν_x on $[-x, x]$ such that

$$\varphi_\lambda(x) = \int e^{-t(\rho+i\lambda)} d\nu_x(t) \quad \text{for } x \in \mathbb{R}_+,\ \lambda \in \mathbb{C}. \tag{3.4}$$

Furthermore, the measure $\tau_x(t) := e^{-\rho t} d\nu_x(t)$ is a symmetric subprobability measure on $\mathbb{R}$ which depends continuously on x in the weak topology on $\mathcal{M}^1(\mathbb{R})$.

4 Moment functions on Jacobi hypergroups on $[0, \infty[$

From now on let $(K, *)$ be a Jacobi-hypergroup on $\mathbb{R}_+$ and φ_λ a Jacobi-function for parameters $\alpha \geq \beta \geq -1/2$. It is well known that $\varphi_\lambda(x)$ is an analytic function of λ for all $x \in \mathbb{R}_+$ (see [11]). The derivations of $\varphi_\lambda(x)$ with respect to λ were established as the most important tool for defining (modified) moments for each probability measure on $\mathbb{R}_+$ in a way, which is consistent with the convolution structure (c.f. in a general context of Sturm-Liouville hypergroup [3, Section 7.2]).

Definition 4.1. For every $x \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ let

$$m_k(x) := m_k^{(\alpha,\beta)}(x) := \left(\frac{\partial}{\partial \mu}\right)^k \varphi_{i(\rho+\mu)}(x)|_{\mu=0} \quad .$$

We recapitulate from [3, Section 7.2] some facts about m_k. For $k = 0$ we have $\varphi_{i\rho} \equiv 1$ and thus $m_0 \equiv 1$. It is easily verified that for any $n \in \mathbb{N}$ the tuple $(m_k)_{k=1,\dots,n}$ is a sequence of moment functions of the length n in the sense of Definition 2.4. The cases $n = 1$ and $n = 2$ are proven in [23, Section 5 and 6]. By differentiating the equation (3.2) with respect to λ we obtain

$$\mathcal{L} m_k = -2k\rho m_{k-1} - k(k-1)m_{k-2}, \quad m_k(0) = m_k'(0) = 0 \quad \text{for} \quad k \geq 1. \tag{4.1}$$

It follows from the Laplace representation (3.4) that

$$m_k(x) = \int_{-x}^{x} t^k d\nu_x(t) = \int_0^x t^k \left(e^{t\rho} + (-1)^k e^{-t\rho}\right) d\nu_x(t) \tag{4.2}$$

for $x \in \mathbb{R}_+$, $\lambda \in \mathbb{C}$ and $k \geq 1$. In particular, m_k is non-negative and in the case $\rho = 0$ (i.e. $\alpha = \beta = -\frac{1}{2}$) it is clear that $m_k = 0$ if k is odd.
Next we prove a series of statements about the moments m_k, which are needed in the next section.

Lemma. 4.2. *For all $k \in \mathbb{N}$ the functions $m_k = m_k^{(\alpha,\beta)}$ are recursively given by*

$$m_k(x) = \int_0^x \int_0^y \frac{a_{\alpha,\beta}(z)}{a_{\alpha,\beta}(y)} \left(2k\rho m_{k-1}(z) + k(k-1)m_{k-2}(z)\right) dz dy. \tag{4.3}$$

Proof. Let $w := m_k'$. The initial value problem (4.1) is equivalent to

$$w' = -\frac{a'_{\alpha,\beta}}{a_{\alpha,\beta}} w + b, \quad w(0) = 0,$$

where $b(x) := 2k\rho m_{k-1}(x) + k(k-1)m_{k-2}(x)$. With variation of constants we obtain

$$w(x) = \exp\big(F(x)\big) \int_0^x b(t) \exp\big(-F(t)\big) dt,$$

where $F(x) := -\int_0^x \frac{a'_{\alpha,\beta}(t)}{a_{\alpha,\beta}(t)} dt$. By integrating the equation above, one obtains the asserted recursion formula for m_k. □

Remark 4.3. By using the Recursion formula (4.3) we obtain for the moment function $m_1^{(\alpha,\beta)}$

$$m_1^{(\alpha,\beta)}(x) = 2\rho \int_0^x \int_0^y \frac{a_{\alpha,\beta}(z)}{a_{\alpha,\beta}(y)} dz dy \quad (-1/2 \le \beta \le \alpha, \quad x \in [0, \infty[). \tag{4.4}$$

We set

$$A_{\alpha,\beta}(y) := \int_0^y a_{\alpha,\beta}(z) dz, \quad (-1/2 \le \beta \le \alpha, \quad y \in [0, \infty[). \tag{4.5}$$

For $\beta = 0$ we obtain $A_{\alpha,0}(y) = \frac{1}{2(\alpha+1)} \text{sh}\,(y)^{2(\alpha+1)}$ and therefore

$$m_1^{(\alpha,0)}(x) = 2\rho \int_0^x \frac{A_{\alpha,0}(y)}{a_{\alpha,0}(y)} dy = \ln(\text{ch}\, x), \quad (x \in [0, \infty[). \tag{4.6}$$

Lemma. 4.4. *For all $k,\ l \in \mathbb{N}$ with $l > 1$ we have*

$$m_k(x)^l \le m_{kl}(x) \le x^{kl} \quad \textit{for every} \quad x \ge 0.$$

In particular, m_1 and m_2 satisfy the growth condition (2.7) of Theorem 2.6.

Proof. Since the function $t \mapsto t^l$ is convex on $\mathbb{R}_+$, the first inequality follows from Jensen's inequality and (4.2). The second inequality is a consequence of the fact that the measure ν_x in (4.2) is supported by $[-x, x]$. □

Lemma. 4.5. *Let $\alpha \ge \beta \ge -\frac{1}{2}$ with $(\alpha, \beta) \ne (-\frac{1}{2}, -\frac{1}{2})$. Then*

$$\big(1 - \frac{|\beta|}{\alpha+1}\big)\ln(\text{ch}\, x) \le m_1^{(\alpha,\beta)}(x) \le \begin{cases} \big(1 + \frac{\beta}{\alpha+1}\big)\ln(\text{ch}\, x) & \textit{for } \beta \ge 0 \\ \big(1 + \frac{1}{2\alpha+1}\big)\ln(\text{ch}\, x) & \textit{for } \beta \in [-\frac{1}{2}, 0]. \end{cases}$$

Proof. Let $x \in [0, \infty[$. Firstly, we consider the case $0 \le \beta \le \alpha$. By the monotonicity of ch and Formula (4.6) we obtain

$$m_1^{(\alpha,\beta)}(x) \le \big(1 + \frac{\beta}{\alpha+1}\big) m_1^{(\alpha,0)}(x) = \big(1 + \frac{\beta}{\alpha+1}\big)\ln(\text{ch}\, x).$$

On the other hand, by means of partial integration, using $\frac{e^z}{e^y} \le \frac{\text{ch}\, z}{\text{ch}\, y}$ $(0 \le z \le y)$ we have

$$\begin{aligned} m_1^{(\alpha,\beta)}(x) &\ge 2(\alpha+\beta+1) \int_0^x \int_0^y \frac{a_{\alpha,0}(z)e^{2\beta z}}{a_{\alpha,0}(y)e^{2\beta y}} dz dy \\ &= 2(\alpha+\beta+1) \int_0^x \frac{A_{\alpha,0}(y)}{a_{\alpha,0}(y)} dy - 2(\alpha+\beta+1) \int_0^x \int_0^y \frac{A_{\alpha,0}(z) 2\beta e^{2\beta z}}{a_{\alpha,0}(y)e^{2\beta y}} dz dy \\ &\ge \frac{\alpha+\beta+1}{\alpha+1} \ln(\text{ch}\, x) - \frac{(\alpha+\beta+1)\beta}{(\alpha+1)^2} \ln(\text{ch}\, x) \ge \big(1 - \frac{\beta}{\alpha+1}\big)\ln(\text{ch}\, x). \end{aligned}$$

We now turn to the case $\beta \geq -1/2$. Since $\frac{\operatorname{sh} z}{\operatorname{sh} y} \leq \frac{\operatorname{ch} z}{\operatorname{ch} y}$ for $0 \leq z \leq y$, we conclude

$$m_1^{(\alpha,\beta)}(x) \leq \frac{\alpha+\beta+1}{\alpha+1/2} m_1^{(\alpha-1/2,0)}(x) \leq \big(1 + \frac{1}{2\alpha+1}\big) \ln(\operatorname{ch} x).$$

For $0 \leq z \leq y$ we get $\big(\frac{\operatorname{ch} z}{\operatorname{ch} y}\big)^{2\beta} \geq 1$, hence that

$$m_1^{(\alpha,\beta)}(x) \geq \frac{\alpha+\beta+1}{\alpha+1} m_1^{(\alpha,0)}(x) = \big(1 - \frac{|\beta|}{\alpha+1}\big) \ln(\operatorname{ch} x).$$

It is clear that for $\alpha = \beta = -\frac{1}{2}$ the first moment $m_1^{(\alpha,\beta)}$ vanishes. □

Lemma. 4.6. *There is a constant C, such that for all $\alpha \geq \beta \geq -\frac{1}{2}$, $x \in [0, \infty[$,*

$$\Big| x - m_1^{(\alpha,\beta)}(x) \Big| \leq C + \frac{|\beta|}{\alpha+1} x \quad \text{and} \quad \Big| \Big(m_1^{(\alpha,\beta)}\Big)^{-1}(x) - x \Big| \leq C + \frac{|\beta|}{\alpha+1} x. \qquad (4.7)$$

Proof. By Lemmas 4.4 and 4.5 using the inequality

$$0 \leq x - \ln(\operatorname{ch} x) \leq \ln(2),$$

we obtain

$$\Big| x - m_1^{(\alpha,\beta)}(x) \Big| \leq x - \ln(\operatorname{ch} x) + \frac{|\beta|}{\alpha+1} \ln(\operatorname{ch} x) \leq \ln(2) + \frac{|\beta|}{\alpha+1} x.$$

Since the graph of $\big(m_1^{(\alpha,\beta)}\big)^{-1}$ is obtained by reflecting the graph of $m_1^{(\alpha,\beta)}$ across the line $y = x$, the second inequality follows immediately from the first one. □

Lemma. 4.7. *Let $\alpha \geq \beta \geq -\frac{1}{2}$ and $(\alpha, \beta) \neq (-\frac{1}{2}, -\frac{1}{2})$ then*

$$m_1(x)^2 \leq m_2(x) \leq m_1(x)^2 + \frac{1}{\rho} m_1(x), \quad (x \in \mathbb{R}_+). \qquad (4.8)$$

Proof. Because of Lemma 4.4 we have only to verify the second inequality. From (4.4) and (4.5) we obtain $m_1'(x) = 2\rho \frac{A_{\alpha,\beta}(x)}{a_{\alpha,\beta}(x)}$. Hence, by Lemma 4.2 and partial integration we observe

$$\begin{aligned} m_2(x) &= 4\rho \int_0^x \int_0^y \frac{a_{\alpha,\beta}(z)}{a_{\alpha,\beta}(y)} m_1(z) dz dy + \frac{1}{\rho} m_1(x) \\ &= 4\rho \int_0^x \frac{A_{\alpha,\beta}(y)}{a_{\alpha,\beta}(y)} m_1(y) dy - 4\rho \int_0^x \int_0^y \frac{A_{\alpha,\beta}(z)}{a_{\alpha,\beta}(y)} m_1'(z) dz dy + \frac{1}{\rho} m_1(x) \\ &\leq 2 \int_0^x m_1'(y) m_1(y) dy + \frac{1}{\rho} m_1(x) = m_1(x)^2 + \frac{1}{\rho} m_1(x). \end{aligned}$$

For $\alpha = \beta = -\frac{1}{2}$ we conclude from (4.3) that $m_2(x) = x^2$. □

For $j \in \mathbb{N}_0$, $-1/2 \leq \beta \leq \alpha$ and $\nu \in \mathcal{M}^1([0, \infty[)$ we define

$$\begin{aligned} r_j &:= \int_0^\infty \ln(\operatorname{ch} x)^j d\nu(x), & \hat{r}_j(\alpha) &:= \int_0^\infty m_j^{(\alpha,\beta)}(x) d\nu(x), \\ \tilde{r}_j &:= \int_0^\infty x^j d\nu(x), & \check{r}_j(\alpha) &:= \int_0^\infty m_1^{(\alpha,\beta)}(x)^j d\nu(x). \end{aligned}$$

Remark 4.8. From Lemmas 4.5 and 4.7 we have

$$\lim_{\alpha\to\infty} \hat{r}_k(\alpha) = r_k = \lim_{\alpha\to\infty} \check{r}_k(\alpha) \quad (k = 1, 2). \tag{4.9}$$

Lemma. 4.9. *Let $k \in \mathbb{N}_0$ and $\alpha \geq -\frac{1}{2}$. Then*

$$m_k^{(\alpha,\alpha)}(x) = 2^{-k} m_k^{(\alpha,-\frac{1}{2})}(2x) \quad (x \in \mathbb{R}_+). \tag{4.10}$$

Proof. The idea of the following proof goes back to Koornwinder (see Section 5.3 of [11]). For $\alpha \geq \beta \geq -\frac{1}{2}$ let $\mathcal{L}_{(\alpha,\beta)}$ be the differential operator as in (3.1). For a function $g \in \mathcal{C}^2(\mathbb{R}_+)$ with $g'(0) = 0$ we define a function $\tilde{g}$ by $\tilde{g}(t) := g(2t)$, $(t \in \mathbb{R}_+)$. By a straightforward calculation one obtains

$$\big(\mathcal{L}_{(\alpha,\alpha)}\tilde{g}\big)(t) = 4\big(\mathcal{L}_{(\alpha,-\frac{1}{2})}g\big)(2t). \tag{4.11}$$

For $k = 0$ the Formula (4.10) is obviously true. Let $k > 0$; we set $f(t) := m_k^{(\alpha,\alpha)}(t)$ and $h(t) := 2^{-k} m_k^{(\alpha,-\frac{1}{2})}(2t)$. Since (4.1) we have

$$\big(\mathcal{L}_{(\alpha,\alpha)}f\big)(t) = -2k(2\alpha+1)m_{k-1}^{(\alpha,\alpha)}(t) - k(k-1)m_{k-2}^{(\alpha,\alpha)}(t).$$

On the other side we calculate

$$\begin{aligned}
\big(\mathcal{L}_{(\alpha,\alpha)}h\big)(t) &= 4\big(\mathcal{L}_{(\alpha,-\frac{1}{2})}\tilde{h}\big)(2t) = 4\cdot 2^{-k}\big(\mathcal{L}_{(\alpha,-\frac{1}{2})}m_k^{(\alpha,-\frac{1}{2})}\big)(2t) \\
&= 4\cdot 2^{-k}\Big(-2k(\alpha+\frac{1}{2})m_{k-1}^{(\alpha,-\frac{1}{2})}(2t) - k(k-1)m_{k-2}^{(\alpha,-\frac{1}{2})}(2t)\Big) \\
&= 4\cdot 2^{-k}\Big(-2k(\alpha+\frac{1}{2})2^{k-1}m_{k-1}^{(\alpha,\alpha)}(t) - k(k-1)2^{k-2}m_{k-2}^{(\alpha,\alpha)}(t)\Big) \\
&= -2k(2\alpha+1)m_{k-1}^{(\alpha,\alpha)}(t) - k(k-1)m_{k-2}^{(\alpha,\alpha)}(t).
\end{aligned}$$

By the uniqueness of the solution of the underlying initial value problem we finally conclude that $f \equiv h$. □

5 A central limit theorem with growing dimensions

Let $\big(S_n^{(\alpha,\beta)}\big)_{n\geq 0}$ be the time-homogeneous random walk on $([0,\infty[, *_{\alpha,\beta})$ with law ν as defined in Section 2.2. In this section we study of the asymptotic behaviour of $\big(S_n^{(\alpha,\beta)}\big)_{n\geq 0}$ for increasing parameters α and β. From now on we will suppose that the variables $X_1,\ X_2,\ \ldots$ are i.i.d. with finite usual second moment $\tilde{r}_2 = \tilde{r}(\nu) < \infty$. We already know [21, 4.2] that

$$\frac{1}{\sqrt{n}}\left\{S_n^{(\alpha,\beta)} - \left(m_1^{(\alpha,\beta)}\right)^{-1}(n\hat{r}_1(\alpha))\right\}$$

converges in distribution for every fixed index (α,β) $(-\frac{1}{2} \leq \beta \leq \alpha)$. It is an interesting fact that the limit distribution is some normal law on $\mathbb{R}$, independent of which hypergroup $(\mathbb{R}_+, *_{\alpha,\beta})$ has been considered. It is also known that for a fixed parameter $\beta \geq -\frac{1}{2}$ in the case of the finite second moment $\tilde{r}_2 < \infty$ under strong requirements on the growth of the sequence $(\alpha_n)_{n\in\mathbb{N}} \subset [\beta,\infty[$, namely $\frac{n}{\sqrt{\alpha_n}} \to 0$,

$$\frac{1}{\sqrt{n}}\left\{S_n^{(\alpha_n,\beta)} - nr_1\right\}$$

tends in distribution for $n \to \infty$ to the normal distribution $\mathcal{N}(0, r_2 - r_1^2)$ (see [19, Theorem 4.2]). In analogy with the radial limit theorems on $\mathbb{R}^{\alpha_n}$ for $\alpha_n \to \infty$ (see [18, 19]) one might suspect that, also in our situation, the case $n >> \alpha_n$ would establish another limit distribution as in the case $n << \alpha_n$. However, we shall prove:

Theorem 5.1. *Let $\beta \geq -\frac{1}{2}$ and let $(\alpha_n)_{n\in\mathbb{N}} \subset [\beta, \infty[$ be an arbitrary increasing sequence with $\lim_{n\to\infty} \alpha_n = \infty$. Let $\nu \in \mathcal{M}^1([0,\infty[)$ with a finite second moment $\tilde{r_2} < \infty$. Then, r_1 and r_2 exist and for the random walks $\left(S_n^{(\alpha_n,\beta)}\right)_{n\geq 0}$ on $[0,\infty[$ with law ν,*

$$Y_n := \frac{1}{\sqrt{n}} \left(m_1^{(\alpha_n,\beta)} \left(S_n^{(\alpha_n,\beta)} \right) - n\hat{r}_1(\alpha_n) \right)$$

tends in distribution for $n \to \infty$ to $\mathcal{N}(0, r_2 - r_1^2)$.

The proof is essentially based on the asymptotic behaviour of modified moments $m_k^{(\alpha,\beta)}$, $k \in \mathbb{N}$ for $\alpha \to \infty$.

Proof. In the first step we show that the random variables

$$\frac{1}{\sqrt{n}} m_1^{(\alpha_n,\beta)} \big(S_n^{(\alpha_n,\beta)}\big) \quad \text{and} \quad \frac{1}{\sqrt{n}} \sum_{j=1}^{n} m_1^{(\alpha_n,\beta)} (X_j)$$

are "asymptotically uncorrelated". More precisely, we check that the random variables

$$Z_n := \frac{1}{\sqrt{n}} \Big(m_1^{(\alpha_n,\beta)} \big(S_n^{(\alpha_n,\beta)}\big) - \sum_{j=1}^{n} m_1^{(\alpha_n,\beta)} (X_j) \Big)$$

converge to zero in the L^2-sence. For this, we conclude from Theorem 2.6 that

$$\mathbb{E}\big(Z_n^2\big) \leq \mathbb{E}\big(m_2(X_1)\big) - \mathbb{E}\big(m_1(X_1)^2\big) = \hat{r}_2(\alpha_n) - \check{r}_2(\alpha_n).$$

As by Remark 4.8,

$$\hat{r}_2(\alpha_n) - \check{r}_2(\alpha_n) \longrightarrow 0 \qquad (n \to \infty),$$

the claimed convergence follows. We define

$$U_{n,\alpha} := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} V_j^{(n,\alpha)} \quad \text{with} \quad V_j^{(n,\alpha)} := m_1^{\alpha,\beta} (X_j) - \hat{r}_1(\alpha)$$

and denote the distribution of $U_{n,\alpha}$ by $\mu_{n,\alpha}$. Now we will prove that U_{n,α_n} tends in distribution for $n \to \infty$ to $\mathcal{N}(0, r_2 - r_1^2)$. Let $f \in \mathcal{C}_b(\mathbb{R})$ be a bounded continuous function on $\mathbb{R}$, $\alpha \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. We have

$$\Big| \int f d\mu_{n,\alpha_n} - \int f d\mathcal{N}(0, r_2 - r_1^2) \Big| \leq \Big| \int f d\mu_{n,\alpha_n} - \int f d\mu_{n,\alpha} \Big| +$$
$$+ \Big| \int f d\mu_{n,\alpha} - \int f d\mathcal{N}(0, \check{r}_2(\alpha) - \hat{r}_1(\alpha)^2) \Big| + \Big| \int f d\mathcal{N}(0, \check{r}_2(\alpha) - \hat{r}_1(\alpha)^2) - \int f d\mathcal{N}(0, r_2 - r_1^2) \Big|.$$

Since the random variables $V_j^{(n,\alpha)}$, $j = 1, \ldots, n$ are i.i.d. and

$$\mathbb{E}\big(V_j^{(n,\alpha)}\big) = 0, \quad \mathbb{V}(V_j^{(n,\alpha)}) = \check{r}_2(\alpha) - \hat{r}_1(\alpha)^2$$

we conclude with Markov's inequality and the estimation of m_1 in Lemma 4.5 that

$$\begin{aligned}
0 &\le \mathbb{E}\big((U_{n,\alpha_n} - U_{n,\alpha_0})^2\big) = \mathbb{E}\big(U_{n,\alpha_n}^2 - 2U_{n,\alpha_n}U_{n,\alpha_0} + U_{n,\alpha_0}^2\big) \\
&= \frac{1}{n}\Big\{n\mathbb{E}\big((V_1^{(n,\alpha_n)})^2\big) - 2n\mathbb{E}\big(V_1^{(n,\alpha_n)}V_1^{(n,\alpha_0)}\big) + n\mathbb{E}\big((V_1^{(n,\alpha_0)})^2\big)\Big\} \\
&= \check{r}_2(\alpha_n) - \hat{r}_1(\alpha_n)^2 - 2\mathbb{E}\big(m_1^{(\alpha_n,\beta)}(X_1)m_1^{(\alpha_0,\beta)}(X_1)\big) + 2\hat{r}_1(\alpha_n)\hat{r}_1(\alpha_0) + \check{r}_2(\alpha_0) - \hat{r}_1(\alpha_0)^2 \\
&\le \check{r}_2(\alpha_n) - \hat{r}_1(\alpha_n)^2 - 2r_2 + \frac{c_\beta r_2}{\min(\alpha_n,\alpha_0)} + 2\hat{r}_1(\alpha_n)\hat{r}_1(\alpha_0) + \check{r}_2(\alpha_0) - \hat{r}_1(\alpha_0)^2,
\end{aligned}$$

where c_β is a positive constant dependent only on β.

Let $\varepsilon > 0$, $A_\delta := \{|U_{n,\alpha_n} - U_{n,\alpha}| \le \delta\}$ $(\delta > 0)$ and $f \in \mathcal{C}_b^u(\mathbb{R})$ be a bounded uniformly continuous function on $\mathbb{R}$ satisfying $f \not\equiv 0$. It follows that

$$\exists\, \delta > 0 : \quad \int_{A_\delta} |f(U_{n,\alpha_n}) - f(U_{n,\alpha})|\, d\mathbb{P} \le \frac{\varepsilon}{6}$$

By (4.9) we observe that for $\tilde{\varepsilon} := \min\left(\frac{\varepsilon}{6}, \frac{\delta^2\varepsilon}{12\|f\|_\infty}\right)$

$$\exists\, n_0,\ \alpha_0 : \quad \mathbb{E}\Big(\big(U_{n,\alpha_n} - U_{n,\alpha}\big)^2\Big) \le \tilde{\varepsilon} \quad \forall n \ge n_0,\ \alpha \ge \alpha_0.$$

By Chebyshev's inequality follows for α and n large enough

$$\begin{aligned}
\left|\int f(U_{n,\alpha_n}) - f(U_{n,\alpha})d\mathbb{P}\right| &\le \int_{A_\delta} |f(U_{n,\alpha_n}) - f(U_{n,\alpha})|\, d\mathbb{P} + \int_{\Omega\setminus A_\delta} |f(U_{n,\alpha_n}) - f(U_{n,\alpha})|\, d\mathbb{P} \\
&\le \frac{\varepsilon}{6} + 2\|f\|_\infty \mathbb{P}(|U_{n,\alpha_n} - U_{n,\alpha}| > \delta) \le \frac{\varepsilon}{6} + 2\|f\|_\infty \frac{\tilde{\varepsilon}}{\delta^2} \le \frac{\varepsilon}{3}.
\end{aligned}$$

In summary, we get

$$\exists\, n_0,\ \alpha_0 : \quad \left|\int f d\mu_{n,\alpha_n} - \int f d\mu_{n,\alpha}\right| \le \frac{\varepsilon}{3} \quad \forall n \ge n_0,\ \alpha \ge \alpha_0.$$

From the classical central limit theorem we deduce

$$\forall\, \alpha\ \exists\, n_1 : \quad \left|\int f d\mu_{n,\alpha} - \int f d\mathcal{N}(0, \check{r}_2(\alpha) - \hat{r}_1^2(\alpha))\right| \le \frac{\varepsilon}{3} \quad \forall\, n \ge n_1.$$

As the sequence of measures $\big(\mathcal{N}(0, \check{r}_2(\alpha) - \hat{r}_1^2(\alpha))\big)_\alpha$ converges weakly to $\mathcal{N}(0, r_2 - r_1^2)$, we have

$$\exists\, \alpha_1 : \quad \left|\int f d\mathcal{N}(0, \check{r}_2(\alpha) - \hat{r}_1^2(\alpha)) - \int f d\mathcal{N}(0, r_2 - r_1^2)\right| \le \frac{\varepsilon}{3} \quad \forall\, \alpha \ge \alpha_1.$$

Hence, U_{n,α_n} and therefore, finally Y_n, converges to the normal distribution $\mathcal{N}(0, r_2 - r_1^2)$. □

Corollary. 5.2. *In the situation as in the theorem above,*

$$L_n := \frac{1}{\sqrt{n}}\left\{S_n^{(\alpha_n,\beta)} - \left(m_1^{(\alpha_n,\beta)}\right)^{-1}(n\hat{r}_1(\alpha_n))\right\}$$

tends in distribution for $n \to \infty$ to $\mathcal{N}(0, r_2 - r_1^2)$.

Proof. Let $x_n := m_1^{(\alpha_n,\beta)}\big(S_n^{(\alpha_n,\beta)}\big)$ and $y_n := n\hat{r}_1(\alpha_n)$. Adapted from the mean value theorem there is a ξ between x_n and y_n such that

$$\big|(x_n - y_n) - (m_1^{-1}(x_n) - m_1^{-1}(y_n))\big| = \big|x_n - y_n\big| \cdot \big|1 - (m_1^{-1})'(\xi)\big|.$$

Since $\big(m_1^{-1}\big)'(x) \searrow 1$ as $x \to \infty$ (see [22, proof of Lemma 5.7]) we obtain

$$\big|(x_n - y_n) - (m_1^{-1}(x_n) - m_1^{-1}(y_n))\big| \leq \Big(\big(m_1^{-1}\big)'\big(\min\{x_n, y_n\}\big) - 1\Big) \cdot \big|x_n - y_n\big|.$$

Therefore, by the preceding theorem, L_n tends in distribution for $n \to \infty$ to the normal distribution $\mathcal{N}(0, r_2 - r_1^2)$. □

Corollary. 5.3. *Let $(\alpha_n)_{n\geq 1} \subset [0,\infty[$ be an increasing sequence with $\lim_{n\to\infty} \frac{n}{\alpha_n^2} = 0$. In the situation as in Corollary 5.2,*

$$\frac{1}{\sqrt{n}}\Big(S_n^{(\alpha_n,\beta)} - n\hat{r_1}^{(\alpha_n,\beta)}\Big) \tag{5.1}$$

tends in distribution for $n \to \infty$ to the normal distribution $\mathcal{N}(0, r_2 - r_1^2)$.

Proof. We have

$$L_n = \frac{1}{\sqrt{n}}\Big(S_n^{(\alpha_n,\beta)} - n\hat{r_1}^{(\alpha_n,\beta)}\Big) + \frac{1}{\sqrt{n}}\Big(n\hat{r_1}^{(\alpha_n,\beta)} - \Big(m_1^{(\alpha_n,\beta)}\Big)^{-1}(n\hat{r}_1(\alpha_n))\Big).$$

In particular, the growth condition on α_n and the second inequality in (4.7) implies the corollary. □

In the central limit theorem above β is fixed and α_n tends to infty. It is natural to think of variants of theorem 5.1 for α_n and $\beta_n \to \infty$ in certain coupled ways (see [19]). Usually, such kinds of CLT no longer have a geometric interpretation. Nevertheless, we present here a CLT for the case $\beta_n \to \infty$ and $\alpha_n = \beta_n + c$ for some constant $c \geq 0$:

Theorem 5.4. *Let $c \geq 0$ be a constant, and let $(\beta_n)_{n\in\mathbb{N}} \subset [-\frac{1}{2}, \infty[$ be an arbitrary, increasing sequence of indices. Let ν be a probability measure on $[0,\infty[$ with second moment $\int_{\mathbb{R}_+} x^2 d\nu(x) < \infty$. Then, $\rho := \int_{\mathbb{R}_+} \ln(\mathrm{ch}(2x)) d\nu(x) < \infty$ and $\sigma^2 := \int_{\mathbb{R}_+} \big(\ln(\mathrm{ch}(2x))\big)^2 d\nu(x) < \infty$ exist, and*

$$\frac{1}{\sqrt{n}}\Big(m_1^{(\beta_n+c,\beta_n)}\big(S_n^{(\beta_n+c,\beta_n)}\big) - n\rho\Big)$$

tends in distribution for $n \to \infty$ to $\mathcal{N}(0, \sigma^2 - \rho^2)$.

Proof. By (4.4), monotonicity of sh and Lemma 4.9 we obtain

$$m_1^{(\beta_n+c,\beta_n)}(x) \leq \frac{2\beta_n + c + 1}{2\beta_n + 1} m_1^{(\beta_n,\beta_n)}(x) = \frac{2\beta_n + c + 1}{2(2\beta_n + 1)} m_1^{(\beta_n,-\frac{1}{2})}(2x)$$

and

$$m_1^{(\beta_n+c,\beta_n)}(x) \geq \frac{2\beta_n + c + 1}{2\beta_n + 1}\,\frac{\beta_n + 1}{\beta_n + c + 1} m_1^{(\beta_n,\beta_n)}(x) = \frac{2\beta_n + c + 1}{2(2\beta_n + 1)}\,\frac{\beta_n + 1}{\beta_n + 1 + c} m_1^{(\beta_n,-\frac{1}{2})}(2x)$$

respectively. From Lemmas 4.5 and 4.7 it follows that

$$\lim_{n\to\infty} m_j^{(\beta_n+c,\beta_n)}(x) = \Big(\frac{\ln\,(\mathrm{ch}\,((2x))}{2}\Big)^j \quad \text{for } j = 1, 2 \text{ and } x \in \mathbb{R}_+.$$

The proof of Theorem 5.1 can now be transferred word by word to the setting above, which then leads to the proof of the assertion. □

6 The addition of small random variables with growing dimensions

Let $(K, *)$ be a hypergroup with $K := [0, \infty[$ and (M, μ, Φ) a fixed concretization of K (c.f. [3, Section 7.1]). Moreover, let $(X_n)_{n\geq 0}$ and $(\Lambda_n)_{n\geq 0}$ be sequences of random variables as in the Assumption 2.2, such that $\nu = P_{X_i}$ and $\mu = P_{\Lambda_i}$ for all $i \in \mathbb{N}$. It is well known that in contrast to the situation on $(\mathbb{R}, +)$ with the ordinary addition $+$, the distributive law on $(\mathbb{R}_+, *)$ does not hold for the addition $\hat{+}$, i.e.

$$a \cdot (X \hat{+} Y) \neq a \cdot X \hat{+} a \cdot Y.$$

This important difference is also reflected in central limit theorems on $(\mathbb{R}_+, *)$. To describe this, we consider for an $r \geq 0$ the randomized sum $S_n^{(r)}$ with initial compression n^{-r}, recursively defined by

$$S_n^{(r)} := \begin{cases} n^{-r} X_1 & \text{for } n = 1 \\ \Phi(S_{n-1}^{(r)}, n^{-r} X_n, \Lambda_{n-1}) & \text{for } n > 1 \end{cases}$$

as well as the associated compressed random walk $(S_n^{(r)})_{n\geq 1}$ with law ν on $[0, \infty[$. The most classical case appears for $r = 1/2$, which states that the sum of many small random variables $S_n^{(1/2)}$ has approximately a so called Gaussian distribution, has been proved for the hyperbolic plane and space in [9], [15]; (see also Section 3.2 of [13]) and in a more general setting by Trimèche in [14]. On Chébli- Trimèche hypergroups, Zeuner [21] has shown that $\frac{1}{\sqrt{n}}\left(S_n^{(0)} - \mathbb{E}(S_n^{(0)})\right)$ is asymptotically normal and $\frac{1}{\sqrt{n}} S_n^{(0)}$ approaches $+\infty$.

On Jacobi hypergroup $([0, \infty[, *_{\alpha,\beta})$ it has been proven by M. Voit (see [17]) that for $r > \frac{1}{2}$ the random variables $n^{r-1/2} S_n^{(r)}$ after an suitable normalization tend in distribution to the Rayleigh distribution ρ_α.

6.1 The Gaussian distributions on Jacobi-hypergroup $\left([0, \infty[, *_{(\alpha,\beta)}\right)$

Definition 6.1. The Gaussian distribution on Jacobi-hypergroup $\left([0, \infty[, *_{(\alpha,\beta)}\right)$ with parameter t is the unique probability measure μ_t on $[0, \infty[$ with

$$\mathcal{F}\mu_t(\lambda) = e^{-\frac{t}{2}(\rho^2+\lambda^2)} \quad \text{for} \quad \lambda \in \mathbb{R}_+ \cup i[0, \rho].$$

The existence of μ_t is a consequence of [21, Theorem 5.5.]. Although μ_t is uniquely determined for every given hypergroup $(\mathbb{R}_+, *_{\alpha,\beta})$, a different hypergroup will in general have different Gaussian measures. The family of Gaussian measures $(\mu_t : \ t \geq 0)$ forms a convolution semigroup. By forward calculation we obtain for μ_t-distributed random variable X_t and for fixed indices (α, β)

$$\mathbb{E}_* (X_t) = \rho \cdot t \quad \text{and} \quad \mathbb{V}_* (X_t) = t.$$

By the inversion formula for (Fourier-) Jacobi transform (see [11, Section 2] and [8]), the density h_t of a Gaussian distribution μ_t with respect to the Haar measure $\omega_K := a_{\alpha,\beta} \lambda^1_{\mathbb{R}_+}$ is given by

$$h_t(x) := \int_0^\infty e^{-\frac{t}{2}(\rho^2+\lambda^2)} \varphi_\lambda(x) d\pi_K(\lambda).$$

Let $\mathbb{E}(\mu_t)$, $\mathbb{V}(\mu_t)$ be the usual expectation and variance of μ_t respectively. We conjecture that the density $\tilde{h}_t$ of $T(\mu_t)$, where T is the linear transformation

$$T(x) := \frac{1}{\mathbb{V}(\mu_t)}\big(x - \mathbb{E}(\mu_t)\big),$$

converges to the density $n_{0,1}$ of the normal distribution on $\mathbb{R}$. This would imply a CLT for $S_n^{(1/2)}$ as $n,\ \alpha \to \infty$. For this, one needs a "good" short-time asymptotic of h_t for $t \to 0$. Nevertheless, we present here a weak LLN for the sum of many small random variables as $\alpha \to \infty$.

Theorem 6.2. *Let $\beta \geq -\frac{1}{2}$ and let $(\alpha_n)_{n\in\mathbb{N}} \subset [\beta, \infty[$ be an arbitrary increasing sequence with $\lim_{n\to\infty} \alpha_n = \infty$. Let $\nu \in \mathcal{M}^1([0,\infty[)$ with a finite second moment $\tilde{r_2} < \infty$ and*

$$\big(Y_n := S_n^{(\alpha_n,\beta),(1/2)}\big)_{n\geq 1}$$

the associated random walk on $([0,\infty[, *_{\alpha_n,\beta})$. Then Y_n converges in L^2-norm to $\frac{1}{2}\mathbb{E}\big(X_1^2\big)$.

Proof. By Lemma 4.4 we have

$$\mathbb{E}\Big(\big(m_1^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big) - \mathbb{E}m_1^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big)\big)^2\Big) \leq \mathbb{E}\Big(m_2^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big)\Big) - \mathbb{E}\Big(m_1^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big)\Big)^2.$$

For all $x \in \mathbb{R}_+$ we obtain by a straightforward calculation

$$\lim_{n\to\infty} n \ln\big(\mathrm{ch}\,\frac{x}{\sqrt{n}}\big) = \frac{1}{2}x^2.$$

Therefore, by dominated convergence, Eq. (2.3) and Lemma 4.5, follows

$$\lim_{n\to\infty} \mathbb{E}\Big(m_1^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big)\Big) = \frac{1}{2}\mathbb{E}\big(X_1^2\big).$$

Hence, by means of the Identity (2.3) (see also the proof of Theorem 5.1)

$$\mathbb{E}\Big(m_2^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big)\Big) = n(n-1)\mathbb{E}\Big(m_1^{(\alpha_n,\beta)}\big(X_1/\sqrt{n}\big)\Big)^2 + n\mathbb{E}\Big(m_2^{(\alpha_n,\beta)}\big(X_1/\sqrt{n}\big)\Big)$$

and 4.7 we conclude

$$\lim_{n\to\infty} \mathbb{E}\Big(m_2^{(\alpha_n,\beta)}\big(S_n^{(1/2)}\big)\Big) = \frac{1}{4}\mathbb{E}\big(X_1^2\big)^2.$$

□

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