technische universität
dortmund

Intrinsic topologies on H-contraction groups with applications to semistability

Wilfried Hazod

Preprint 2012-01 Januar 2012

Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

INTRINSIC TOPOLOGIES ON H-CONTRACTION GROUPS WITH APPLICATIONS TO SEMISTABILITY

WILFRIED HAZOD

Abstract. Semistable continuous convolution semigroups on Lie groups with non-trivial idempotent are characterized by semistable continuous convolution semigroups with trivial idempotent on a contractible, hence homogeneous Lie group. (Cf., e.g. [9], [10], III, theorem 3.5.4.) In fact, this homogeneous group is obtained by a re-topologization of the contractible subgroup on which the original semistable laws are concentrated. In [26] E. Siebert investigated such *intrinsic topologies* for contractible subgroups of Polish groups, generalizing partially the before mentioned situation of Lie groups. Here we use these ideas to obtain intrinsic topologies for H-contractible subgroups of Polish groups, where H denotes a compact subgroup. This allows, under additional assumptions (which are satisfied in the Lie group case) to obtain similar characterization of semistable laws with non-trivial idempotents.

Introduction

If $\mathbb{G}$ is a Lie group and $\{\mu_t\}_{t\geq 0}$ a (τ,α)-semistable continuous convolution semigroup with idempotent ω_H, ω_H denoting the Haar measure on a compact subgroup $H \subseteq \mathbb{G}$, $\tau \in \mathrm{Aut}(\mathbb{G})$ and $0 < \alpha < 1$, then it is well known that $\{\mu_t\}$ is concentrated on the H-contraction group $C_H(\tau) := \{x \in \mathbb{G} : \tau^n(x) \to e \bmod H\}$. Furthermore, Lie groups have the decomposition property $C_H(\tau) = C(\tau)\cdot H$, where $C(\tau) = C_{\{e\}}(\tau) = \{x : \tau^n(x) \to e\}$. There exists an *intrinsic* or *natural* topology on $C(\tau)$ (defined by the Lie algebra) turning $C(\tau)$ into a contractible Lie group $\widetilde{C}(\tau)$ (hence in particular simply connected and nilpotent). Moreover, $\{\mu_t\}$ is uniquely determined by a continuous convolution semigroup $\left\{\nu_t^\dagger\right\}$ on a quotient of $\widetilde{C}(\tau)$ with trivial idempotent. (Cf., e.g., [9], [10], III, theorem 3.5.4). In [26] E. Siebert generalized the concept of intrinsic topologies of contractible subgroups to the class of Polish (not necessary locally compact) groups.

In the following we investigate intrinsic topologies of H-contractible subgroups which are of the form $C(\tau)\cdot H \subseteq C_H(\tau)$ and generalize (partially) the above mentioned characterizations of semistable laws with non-trivial idempotents from the Lie group case to more general classes of groups. The results are far from being as complete and satisfactory as in the Lie group case, as additional conditions are needed, which are automatically fulfilled for Lie groups (and, in general, the intrinsic topology on $\widetilde{C}(\tau)$ need no longer be locally compact.)

Date: 12.2.2012

Keywords: Contractible groups; Polish groups; contracting automorphism; semistable laws; intrinsic topology.

MSC 2010: 22D05; 43A05; 60B15; 60E07.

1. Intrinsic topology for $C(\tau)$

Let $\mathbb{G}, \mathcal{G}, \Lambda, \Gamma \dots$ denote topological Hausdorff groups, mostly second countable. In interesting examples $\mathbb{G}$ is assumed to be a locally compact and second countable group or a completely metrizable topological vector space, hence in particular, a Polish group. $H, K, L, D, \dots$ denote τ-invariant compact subgroups, τ denoting an automorphism. Recall that for $\tau \in \operatorname{Aut}(\mathbb{G})$ the τ-contractible subgroup is defined as $C(\tau) := \{x : \tau^n(x) \to e\}$, and for a τ-invariant compact subgroup H, $C_H(\tau) = \{x : \tau^n(x) \to e \bmod H\}$. W.l.o.g. we assume $C_H(\tau)^- = \mathbb{G}$. Note that $C_H(\tau)$ will in general not be closed.

Let $\mathfrak{U}_{\mathbb{G}}$ denote the neighbourhood filter of e, $\mathfrak{U}^b_{\mathbb{G}}, \mathfrak{U}^o_{\mathbb{G}}, \mathfrak{U}^c_{\mathbb{G}} \dots$ the bases consisting of Borel, resp. open resp. closed sets. We repeat shortly the construction of the intrinsic topology for $C(\tau)$: Let $\Gamma := \mathbb{C}(\tau)^-$. For $U \in \mathfrak{U}^o_\Gamma$ put for $n \in \mathbb{Z}$, $U_{(n)} := \bigcap_{-\infty < k \leq n} \tau^k(U)$ and $U_n := U_{(n)} \cap C(\tau)$. Then the filter basis $\widetilde{\mathfrak{U}}^o_{C(\tau)} := \{U_n : n \in \mathbb{Z}, U \in \mathfrak{U}^o_\Gamma\}$ generates a Hausdorff topology on $C(\tau)$ (inherited by the topology of uniform convergence of $\{\tau^k(x)\}_{k\in\mathbb{N}} \in \mathbb{G}^{\mathbb{N}}$, cf. [26], lemma 4.) Denote by $\widetilde{C}(\tau)$ the group $C(\tau)$ endowed with this topology, called *intrinsic topology.* $\widetilde{C}(\tau)$ is a topological Hausdorff group, the identity $\varphi : \widetilde{C}(\tau) \to C(\tau) \subseteq \Gamma$ is a continuous bijective homomorphism, called *canonical homomorphism.* ($C(\tau)$ will always denote the group endowed with the subspace topology inherited by Γ resp. $\mathbb{G}$). $\widetilde{\tau}$, defined by the the restriction of τ to $C(\tau)$, is a automorphism of $\widetilde{C}(\tau)$, such that $\varphi \circ \widetilde{\tau} = \tau \circ \varphi$. $\widetilde{\tau}$ is contracting, in particular, $\widetilde{C}(\tau) = C(\widetilde{\tau})$.

Remark 1.1. *As well known, if $\mathcal{G}$ is locally compact or Polish and $\tau \in \operatorname{Aut}(\mathcal{G})$, and H a compact τ-invariant subgroup, such that $\mathcal{G} = C_H(\tau)$, then τ is compactly contracting, i.e., for all open $U \supseteq H$ and all compact subsets $F \subseteq \mathcal{G}$ we have: There exists a $N \in \mathbb{N}$ such that $\tau^n(F) \subseteq U$ for all $n \geq N$. ⟦Cf. e.g., [26], lemma 1, [10], lemma 3.1.3 resp. lemma 3.1.3* in § 3.3.IV for $H = \{e\}$. The proof for $H \neq \{e\}$ is analogous.⟧ $\mathcal{G}$ is supposed to be Polish or locally compact in order to apply Baire's category theorem. In the above situation, if $\mathbb{G}$ is Polish, so is $\widetilde{C}(\tau)$ (cf., Remark 1.5 below). Hence $\widetilde{\tau}$ is also compactly contracting.*

Note that the filter basis $\mathfrak{U}^o_\Gamma$ may be replaced by any filter basis without changing the intrinsic topology.

Recall that a topological $\mathcal{G}$ group is called Polish (in analogy to vector spaces) if $\mathcal{G}$ is metrizable and complete w.r.t. a (left) invariant metric generating the topology of $\mathcal{G}$. Obviously we have:

Lemma 1.2. *Let $F \subseteq \operatorname{Aut}(\Gamma)$ be a subgroup such that $\tau F = F\tau$. Let $U \in \mathfrak{U}^o_\Gamma$ be F-invariant. Then $U_{(n)}$ and U_n are F-invariant.*

⟦ Note that in that case, $\sigma(C(\tau)) = C(\tau)$ for all $\sigma \in F$. ⟧

For a compact τ-invariant subgroup $H \subseteq \mathbb{G}$ let $\beta : H \to \operatorname{Aut}(\Gamma) : H \ni \kappa \mapsto \beta(\kappa)$ be defined as restriction of the inner automorphisms $\beta(\kappa)(x) = i_\kappa(x) := \kappa x \kappa^{-1}$ for $x \in \Gamma$, resp. its restriction to $C(\tau)$. ⟦In fact, we always have $C(\tau) \lhd C_H(\tau)$), hence H normalizes $C(\tau)$ and thus Γ.⟧ In particular, it follows that $C(\tau) \cdot H$ is a subgroup of $C_H(\tau)$.

Applying Lemma 1.2 for $F := \beta(H)$ we obtain:

Corollary 1.3. *Let $\mathbb{G}$ be a topological group, let, as before, $\tau \in \mathrm{Aut}(\mathbb{G})$ and $\Gamma := C(\tau)^-$, and let H be a compact τ-invariant subgroup of $\mathbb{G}$. Then there exists a basis $\mathfrak{U}_\Gamma^H$ of the neighbourhood filter consisting of $\beta(H)$-invariant subsets. Thus, by Lemma 1.2, $\{U_{(n)} : U \in \mathfrak{U}_\Gamma^H\}$ and $\{U_n : U \in \mathfrak{U}_\Gamma^H\}$ are $\beta(H)$-invariant for all $n \in \mathbb{Z}$.*

Consequently, $\widetilde{C}(\tau)$ possesses a a filter basis of $\widetilde{\beta}(H)$-invariant subsets, $\widetilde{\beta}$ denoting the homomorphism $H \to \mathrm{Aut}(\widetilde{C}(\tau))$ corresponding to β, $\varphi(\widetilde{\beta}(\kappa)(x)) = \beta(\kappa)(\varphi(x)), x \in \widetilde{C}(\tau), \kappa \in H$. (Cf. Proposition 1.4 below.)

Proof. If U is $\beta(H)$-invariant then $U_{(n)}$ and hence U_n share this property (Lemma 1.2). Moreover, $\tau \circ \beta(\kappa) = \beta(\tau(\kappa)) \circ \tau$ shows that $\tau\beta(H) = \beta(H)\tau$. (Recall that $\tau(H) = H$.)

There exists a $\beta(H)$-invariant filter basis $\mathfrak{U}_\Gamma^H$ of $\mathfrak{U}_\Gamma$. This follows e.g. by [11], theorem 4.9, saying that $\forall U \in \mathfrak{U}_\Gamma$, $\bigcap_{\kappa\in H} \beta(\kappa)(U) \in \mathfrak{U}_\Gamma$. Hence there exists a $\widetilde{\beta}(H)$-invariant filter basis of $\widetilde{C}(\tau)$. ☐

Proposition 1.4. *The mappings $(x,\kappa) \mapsto \beta(\kappa)(x)$: $C(\tau) \times H \to C(\tau)$ and $(x,\kappa) \mapsto \widetilde{\beta}(\kappa)(x)$: $\widetilde{C}(\tau) \times H \to \widetilde{C}(\tau)$ are (simultaneously) continuous.*

Proof. (1) As mentioned above, there exist $\beta(H)$-invariant filter-bases $\mathfrak{U}_\Gamma^H$ of $\mathfrak{U}_\Gamma$ and $\mathfrak{U}_{\widetilde{C}(\tau)}^H$ of $\mathfrak{U}_{\widetilde{C}(\tau)}$.

(2) $(x,\kappa) \mapsto \beta(\kappa)(x)$: $C(\tau) \times H \to C(\tau)$ is continuous. ⟦In fact, $(x,\kappa) \mapsto \beta(\kappa)(x)$: $\mathbb{G} \times H \to \mathbb{G}$ is continuous. ⟧

(3) For all $\kappa \in H$, $x \mapsto \widetilde{\beta}(\kappa)(x)$ is continuous, $\widetilde{C}\tau) \to \widetilde{C}(\tau)$.

⟦Let $\{x_n, x\} \subseteq \widetilde{C}(\tau)$ such that $x_n \overset{\widetilde{C}(\tau)}{\to} x$. I.e., for all $U \in \mathfrak{U}_\Gamma^H$, for all $N \in \mathbb{Z}$ there exist $M \in \mathbb{Z}_+$ satisfying: $x_n \in U_N \cdot x$ for all $n \geq M$. Hence $\beta(\kappa)(x_n) \in \beta(\kappa)(U_N) \cdot \beta(\kappa)(x) = U_N \cdot \beta(\kappa)(x)$ for $n \geq M$. Whence the assertion follows. ⟧

(4) For all $x \in \widetilde{C}(\tau)$, $H \ni \kappa \mapsto \widetilde{\beta}(\kappa)(x) \in \widetilde{C}(\tau)$ is continuous.

⟦Let $\kappa_n \overset{H}{\to} \kappa$ and $x \in \widetilde{C}(\tau)$ (fixed). Then $\beta(\kappa_n)(x) \overset{\Gamma}{\to} \beta(\kappa)(x)$ and, for all $k \in \mathbb{Z}$, we have $\tau^k(\beta(\kappa_n)(x)) \overset{\Gamma}{\to} \tau^k(\beta(\kappa)(x))$. We see that this is equivalent to $\beta(\tau^k(\kappa_n))(\tau^k(x)) \overset{\Gamma}{\to} \beta(\tau^k(\kappa))(\tau^k(x))$, using that $\tau^k \circ \beta(\chi) = \beta(\tau^k(\chi)) \circ \tau^k$, for all $\chi \in H$.

Let $W, U \in \mathfrak{U}_\Gamma^H$, symmetric, with $W^2 \subseteq U$. Fix $N \in \mathbb{Z}$.

For all $k \in \mathbb{Z}$ there exist $N_k \in \mathbb{Z}_+$ such that for all $n \geq N_k$ we have: $\tau^k(\beta(\kappa_n)(x)) \in U \cdot \tau^k(\beta(\kappa)(x))$.

τ acts contracting on x, hence there exists $K \in \mathbb{Z}_+$, w.l.o.g. $K > N$, such that $\tau^k(x) \in W$ for all $k \geq K$. Let $N^* := \max_{N\leq k\leq K} N_k$.

- For $N \leq k \leq K$ and $n \geq N^*$: $\tau^k(\beta(\kappa_n)(x)) \in U \cdot \tau^k(\beta(\kappa)(x))$.
- For $k > K$ and all $n \in \mathbb{Z}_+$: $\tau^k(\beta(\kappa_n)(x)) = \beta(\tau^k(\kappa_n))(\tau^k(x)) \in \beta(\tau^k(\kappa))(W) = W$ and $\tau^k(\beta(\kappa)(x)) = \beta(\tau^k(\kappa))(\tau^k(x)) \in \beta(\tau^k(\kappa))(W) = W$, hence $\tau^k(\beta(\kappa_n)(x)) \in W^2 \cdot \beta(\tau^k(\kappa))(\tau^k(x)) \subseteq U \cdot \tau^k(\beta(\kappa)(x))$.

Hence, according to the definition of U_N it follows: For all $U \in \mathfrak{U}_\Gamma^H$, for all $N \in \mathbb{Z}$, there exists $N^* \in \mathbb{Z}_+$ such that for all $n \geq N^*$ we have: $\beta(\kappa_n)(x) \in U_N \cdot \beta(\kappa)(x)$. ⟧

(5) $(x,\kappa) \mapsto \widetilde{\beta}(\kappa)(x)$ is simultaneously continuous $\widetilde{C}(\tau) \times H \to \widetilde{C}(\tau)$.

⟦Choose as before, in (4), $W, U \in \mathfrak{U}_\Gamma^H$ symmetric such that $W^2 \subseteq U$, hence $W_N^2 \subseteq U_N$ for all N, and fix $N \in \mathbb{Z}$. Assume $\kappa_m \overset{H}{\to} \kappa$ and $x_n \overset{\widetilde{C}(\tau)}{\to} x$.

There exists $N_* = N_*(W, N) \in \mathbb{Z}_+$ such that $x_n \in W_N \cdot x$ for $n \geq N_*$, hence (according to (3)) $\beta(\kappa_m)(x_n) \in W_N \cdot \beta(\kappa_m)(x)$ for all $m \in \mathbb{Z}_+$.

There exist $M_* = M_*(N, W)$ such that (according to (4)) we have: $\beta(\kappa_m)(x) \in W_N \cdot \beta(\kappa)(x)$ for all $m \geq M_*$.
Hence $\beta(\kappa_m)(x_n) \in W_N^2 \cdot \beta(\kappa)(x) \subseteq U_N \cdot \beta(\kappa)(x) \forall n \geq N_*, m \geq M_*$.⟧ □

We list properties of the intrinsic topology:

Remarks 1.5. *Let $\mathbb{G}$ be a topological Hausdorff group with $\tau \in \mathrm{Aut}(\mathbb{G})$.*

a) If $\mathbb{G}$ is metrizable then $\widetilde{C}(\tau)$ is metrizable too.

b) If $\mathbb{G}$ is Polish then $\widetilde{C}(\tau)$ is Polish too.

c) $C(\tau)$ and $C_H(\tau)$ are Borel measurable and $\varphi^{-1} : C(\tau) \to \widetilde{C}(\tau)$ is a Borel isomorphism.

d) If $\mathbb{G}$ is Polish, then the intrinsic topology of $\widetilde{C}(\tau)$ is characterized in the following way: Let C be a topological Hausdorff group with the properties (i) there exists a continuous bijective homomorphism $\xi : C \to C(\tau)$, (ii) C is Polish and (iii) $\tau' = \xi \circ \tau \circ \xi^{-1}$ acts contracting on C. Then there exists a continuous bijective homomorphism $\widetilde{C}(\tau) \to C$. (I.e., the intrinsic topology is the weakest topology with the properties (i),(ii), (iii).)

e) If $\mathbb{G}$ is Polish and $C(\tau)$ is closed, i.e., $C(\tau) = \Gamma$, then for all $U \in \mathfrak{U}_\Gamma$ and all n we have: $U_{(n)} = U_n \in \mathfrak{U}_\Gamma$. Hence $\widetilde{C}(\tau) = C(\tau)(= \Gamma)$.

f) If $\mathbb{G}$ is a Lie group then $C(\tau)$ and hence Γ are nilpotent, $\widetilde{C}(\tau)$ is a simply connected contractible (hence nilpotent Lie) group. Therefore, in particular, if C is a simply connected Lie group with a continuous bijective homomorphism $\xi : C \to C(\tau)$ then $C \cong \widetilde{C}(\tau)$.

g) If $\mathbb{G}$, hence $C(\tau)$, is totally disconnected then $\widetilde{C}(\tau)$ is totally disconnected too and if $\widetilde{C}(\tau)$ is connected then $C(\tau)$ and hence Γ are connected.

h) $C(\tau)$ is arc-wise connected iff $\widetilde{C}(\tau)$ shares this property. (In fact, it is only needed that φ^{-1} is measurable.)

⟦ *Cf.* [26], *or* [10], *§ 3.3.IV; there the assertions a)–f) are proved, g) being obvious. Assertion h), concerning arc-wise connectedness, follows by Corollary 3.3. For measurability of φ^{-1} cf. Lemma 2.2 and Proposition 2.4.* ⟧

However we have: if $\mathbb{G}$ is locally compact, $\widetilde{C}(\tau)$ will in general not be locally compact. And conversely, if $\widetilde{C}(\tau)$ is locally compact, $\mathbb{G}$ need not share this property. (Cf., [26], containing criteria for local compactness of the intrinsic topology.) Furthermore, there exist examples of compact, connected groups $\mathbb{G}$ and $\tau \in \mathrm{Aut}(\mathbb{G})$, with $\mathbb{G} = C(\tau)^- = C(\tau^{-1})^-$, furthermore, with dense arc-wise connected subgroup $\mathbb{G}_a$, such that $\widetilde{C}(\tau)$ is locally compact and totally disconnected. In contrast, $C(\tau^{-1}) = \mathbb{G}_a$, i.e. arc-wise connected, and $\widetilde{C}(\tau^{-1}) \cong \mathbb{R}$. And in addition, $C(\tau) \cap C(\tau^{-1}) = \{e\}$. ⟦[26], section 3, theorem. ⟧

Definition 1.6. *Let, as before, $\mathbb{G}$ be a topological Hausdorff group, $\tau \in \mathrm{Aut}(\mathbb{G})$ with $\mathbb{G} = C_H(\tau)^-$. Let again $\Gamma := C(\tau)^-$. Then $\Lambda := C(\tau) \cdot H$ is a subgroup of $\mathbb{G}$, $\Lambda \subseteq C_H(\tau)$.*

The triple $(\mathbb{G}, \tau, H)$ has the decomposition property if $C_H(\tau) = \Lambda = C(\tau) \cdot H$. And $(\mathbb{G}, \tau, H)$ has the strong decomposition property if in addition, $C(\tau) \cap H = \{e\}$.

The pair $(\mathbb{G}, \tau)$ has the (strong) decomposition property if for all compact τ-invariant subgroups H, $(\mathbb{G}, \tau, H)$ has the (strong) decomposition property.

Remarks 1.7. *a) If $\mathbb{G}$ is a Lie group, then for all automorphisms $\tau \in \mathrm{Aut}(\mathbb{G})$, $(\mathbb{G}, \tau)$ has the decomposition property.* $[\![$*Cf. e.g.,* [9], [10], *III, theorem 3.2.13.* $]\!]$

b) If $\mathbb{G}$ is a locally compact totally disconnected group then again for all τ, $(\mathbb{G}, \tau)$ has the decomposition property. $[\![$*Cf.* [6], [16], *Theorem 1, and the literature mentioned there.* $]\!]$

c) [17]*, section 10, contains a characterization of all compact groups $\mathbb{G}$ and $\tau \in \mathrm{Aut}(\mathbb{G})$ such that $(\mathbb{G}, \tau)$ has the decomposition property. Moreover, in* [20] *the authors generalize this result and obtain sufficient conditions for $(\mathbb{G}, \tau)$ to have the (strong) decomposition property. See in particular proposition 4.5, theorem 4.4 and corollary 4.11.*

d) If $\mathbb{G}$ is locally compact or Polish, if $(\mathbb{G}, \tau, H)$ has the decomposition property and $C(\tau)$ is closed, then $(\mathbb{G}, \tau, H)$ has the strong decomposition property. $[\![$ *In fact, then $C(\tau)$ is locally compact or Polish, hence $\tau|_{C(\tau)}$ is compactly contracting and $C(\tau) \cap H$ is compact. Whence $C(\tau) \cap H = \{e\}$ follows.* $]\!]$

e) Thus, if $\mathbb{G}$ is locally compact and totally disconnected and if τ is a tidy *automorphism (cf. e.g.,* [29]*), then $(\mathbb{G}, \tau)$ has the strong decomposition property.* $[\![$ *If $\mathbb{G}$ is a p-adic Lie group then $C(\tau)$ is closed, (*[28]*); for the general case see* [6]*, lemma 1,* [16]. $]\!]$

f) In [17]*, example 4.1, it is shown that e.g., for $\mathbb{G} = \bigotimes_{\mathbb{Z}} \mathbb{T}$, $\mathbb{T}$ denoting the one-dimensional torus, and τ denoting the shift, there exists a closed subgroup $H \cong \mathbb{T}$, such that $(\mathbb{G}, \tau, H)$ fails to have the decomposition property.*

We have in mind this example in the sequel and describe it – and some related examples – in details in Section 5.

2. Intrinsic topologies for $\Lambda = C(\tau) \cdot H$

In the following we generalize the concept of intrinsic topologies to groups of the form $\Lambda := C(\tau) \cdot H$ ($= C_H(\tau)$ if $(\mathbb{G}, \tau, H)$ has the decomposition property). We use the notations introduced in Section 1. Let $\mathbb{G}$ be a Polish topological group. We assume now w.l.o.g. $\mathbb{G} = (C(\tau) \cdot H)^-$ and put again $\Gamma := C(\tau)^-$. $C(\tau)$ and Λ are always endowed with the subspace topology of $\mathbb{G}$, and, as before, $\widetilde{C}(\tau)$ denotes $C(\tau)$ endowed with the intrinsic topology. Furthermore, $\varphi : \widetilde{C}(\tau) \to C(\tau)$ and $\widetilde{\tau} = \varphi^{-1}\tau\varphi$ are defined as before. Our aim is to construct a Polish group $\widetilde{C(\tau) \cdot H}$ containing monomorphic images $\widetilde{C}$ and $\widetilde{H}$ of $\widetilde{C}(\tau)$ and H, and a canonical homomorphism $\Psi : \widetilde{C(\tau) \cdot H} = \widetilde{C} \cdot \widetilde{H} \to C(\tau) \cdot H = \Lambda$, such that the restriction $\Psi|_{\widetilde{H}}$ is an isomorphism and φ factorizes, $\varphi = \Psi|_{\widetilde{H}} \circ g$, g denoting a continuous bijective homomorphism $\widetilde{C}(\tau) \to \widetilde{C}$. $[\![$Indeed, it turns out that in general, g can not be an isomorphism.$]\!]$ Furthermore, there exists an automorphism $\rho_2 \in \mathrm{Aut}(\widetilde{C(\tau) \cdot H})$ such that $\Psi \circ \rho_2 = \tau \circ \Psi$ and $\widetilde{C} = C(\rho_2)$. $\mathcal{A} := C(\tau) \rtimes_\beta H$ and $\mathcal{B} := \widetilde{C}(\tau) \rtimes_{\widetilde{\beta}} H$ are well-defined topological groups (β resp. $\widetilde{\beta}$ denoting the homomorphisms $H \to \mathrm{Aut}(C(\tau))$ and $H \to \mathrm{Aut}(\widetilde{C}(\tau))$ respectively. $[\![$Indeed, from Proposition 1.4 it follows that the semi-direct products define topological groups, cf. [11], (6.20). Note that, as immediately verified, $\mathcal{A}$ and $\mathcal{B}$ are semi-topological groups, i.e., the group operation is separately continuous. In the locally compact case then simultaneous continuity follows by Ellis' theorem on transformation groups [5]. For the general case, I could not find a reference, hence we had to prove Proposition 1.4. ⟧

By definition of $\varphi, \beta, \widetilde{\beta}$ we have: $\varphi(\widetilde{\beta}(\kappa)(x)) = \beta(\kappa)(\varphi(x))$. The canonical homomorphism φ defines a continuous bijective homomorphism $\phi = \varphi \otimes \mathrm{id}_H \colon \mathcal{B} \to \mathcal{A}$, $(x, \kappa) \mapsto (\varphi(x), \kappa)$. We define furthermore automorphisms of $\mathcal{A}$ resp. $\mathcal{B}$, $\rho := \tau|_{C(\tau)} \otimes \tau|_H \in \mathrm{Aut}(\mathcal{A})$: $(x, \kappa) \mapsto (\tau(x), \tau(\kappa))$ and $\widetilde{\rho} \in \mathrm{Aut}(\mathcal{B})$, $\widetilde{\rho} := \widetilde{\tau} \otimes \tau|_H$: $(x, \kappa) \mapsto (\widetilde{\tau}(x), \tau(\kappa))$. ⟦Note that by definition of $\beta(\cdot)$ we have: $\tau(\beta(\kappa)(y)) = \beta(\tau(\kappa))(\tau(y))$.⟧ H as well as $C(\tau)$ resp. $\widetilde{C}(\tau)$ are considered as a compact resp. invariant subgroups of Λ, $\mathcal{A}$ and $\mathcal{B}$ respectively, i.e., if no confusion is possible, we identify tacitly H with $\{e\} \otimes H$ and $C(\tau)$ with $C(\tau) \otimes \{e\}$ resp. $\widetilde{C}(\tau)$ with $\widetilde{C}(\tau) \otimes \{e\}$. The product function $f : \mathcal{A} \to \Lambda$, $(x, \kappa) \mapsto x \cdot \kappa$ is a continuous homomorphism. f is bijective iff $C(\tau) \cap H = \{e\}$.

Obviously, we have:

Proposition 2.1. *ρ is contractive* mod H *on $\mathcal{A}$ and $\widetilde{\rho}$ is contractive* mod H *on $\mathcal{B}$. Furthermore we have: $\rho \circ \phi = \phi \circ \widetilde{\rho}$.*

Polish groups have the following nice properties:

Lemma 2.2. *Let A, B be second countable topological groups, let A be Polish and let $g : A \to B$ be a continuous injective homomorphism.*

a) Then $C := g(A)$ is a measurable subgroup and $g^{-1} : C \to A$ is a bijective Borel-measurable homomorphism.

b) In addition, for any compact subgroup $K \subseteq C$ the image $\widetilde{K} := g^{-1}(K)$ is a compact subgroup of A, and $g|_{\widetilde{K}} : \widetilde{K} \to K$ is continuous, hence a topological (and algebraic) isomorphism. ⟦Indeed for this result it is sufficient to suppose that A, B are metrizable and g^{-1} is measurable. ⟧

c) If $D \subseteq A$ is closed then D is Polish, and if $D \triangleleft A$, $g : A \to A/D$, then $B := A/D$ is Polish.

Proof. a) The first assertions, already mentioned, follow by Kuratowski's theorem, cf. e.g., [19], I, theorem 3.9, corollary 3.3.

b) Let ω_K be the Haar measure on the a compact group K, hence an idempotent probability. Therefore, $\lambda := g^{-1}(\omega_K)$ is an idempotent probability in $\mathcal{M}^1(A)$. Thus (cf. [19], III, theorem 3.1), $\lambda = \omega_{\widetilde{K}}$ for a compact subgroup $\widetilde{K} \subseteq A$. Obviously, $\widetilde{K} = g^{-1}(K)$. The last assertion follows since $\widetilde{K}$ is compact and $g : \widetilde{K} \to K$ is continuous and bijective.

c) Obviously, a closed subset D is complete, hence Polish. Completeness of the quotient group is surely well-known. It relies on metrizability. (Note that e.g., quotients of complete non-metrizable vector spaces need not be complete.) We adapt the standard proof for vector spaces, cf. e.g., [21], I, § 6, 6.3:

Let d denote a left invariant metric on A such that (A, d) is complete, and define the metric on the quotient group by $d_0(\overline{x}, \overline{y}) := \inf\{d(x, y) : x \in \overline{x}, y \in \overline{y}\}$. Let $\{\overline{x}_n\}$ be a Cauchy sequence in A/D. Then there exists a subsequence (n_k) with $d_0(\overline{x}_{n_k}, \overline{x}_{n_{k+1}}) < 1/2^{k+1}$, equivalently, $d_0(\overline{x}_{n_k}^{-1}\overline{x}_{n_{k+1}}, \overline{e}) < 1/2^{k+1}$. Choose $y_k \in \overline{x}_{n_k}^{-1}\overline{x}_{n_{k+1}}$ with $d(y_{k+1}, e) < 1/2^k$. Fix $x_1 \in \overline{x}_{n_1}$ and put $z_K := x_1 \cdot y_1 \cdots y_K$ $(K \geq 2)$, hence $z_K \in \overline{x}_{n_{K+1}}$.

Hence $\{z_k\}$ is a Cauchy sequence in A, thus $\lim z_k =: z_\infty \in A$ exists, and by continuity, $\lim \overline{x}_{n_k} \in B$ exists. Since $\{\overline{x}_{n_k}\}$ is Cauchy, $\lim \overline{x}_n$ exists. □

If $E = C(\tau) \cap H = \{e\}$ then $\mathcal{B}, \mathcal{A}, \varphi, \phi, f$ and Φ are sufficient to characterize semistable laws with idempotent ω_H (Theorem 4.1).

For the general case, $E \neq \{e\}$, we need some more preparations:

Notations 2.3. We are interested in the subgroup $\Lambda = C(\tau) \cdot H \subseteq C_H(\tau)$. Hence, if $E := C(\tau) \cap H \neq \{e\}$, we put $\Delta := \{(c, c^{-1}) : c \in E\}$. It is easily verified that Δ resp. $\widetilde{\Delta}$ are closed normal subgroups of $\mathcal{A}$ and of $\mathcal{B}$ respectively, and moreover, E resp. $\widetilde{E}$ are closed normal τ- resp. $\widetilde{\tau}$-invariant subgroups of $C(\tau)$ and $\widetilde{C}(\tau)$ respectively. (We write $\widetilde{E}$ if E is considered as subgroup of $\widetilde{C}(\tau)$. Hence $\widetilde{E} = \varphi^{-1}(E)$. Analogously, we write $\widetilde{\Delta}$ if the group is considered as subgroup of $\mathcal{B}$, $\widetilde{\Delta} = \phi^{-1}(\Delta)$). Let π_Δ resp. $\pi_{\widetilde{\Delta}}$ denote the quotient homomorphisms $\mathcal{A} \to \overline{\mathcal{A}} := \mathcal{A}/\Delta$ resp. $\mathcal{B} \to \overline{\mathcal{B}} := \mathcal{B}/\widetilde{\Delta}$. Since $\mathbb{G}$ is Polish, $\widetilde{C}(\tau)$, hence $\mathcal{B}$ and (by Lemma 2.2 c)) $\overline{\mathcal{B}}$ are Polish.

As mentioned afore, $C(\tau)$ and $\Lambda = C(\tau) \cdot H$ are always endowed with the subspace topology.

1. $\overline{\mathcal{A}} := \mathcal{A}/\Delta \cong C(\tau) \cdot H$ algebraically and $f : \mathcal{A} \to C(\tau) \cdot H$: $(x, \kappa) \mapsto x \cdot \kappa \in \Lambda$ is a continuous surjective homomorphism with kernel Δ. Hence there exists a bijective continuous homomorphism $\theta : \overline{\mathcal{A}} \to \Lambda$, with $f = \theta \circ \pi_\Delta$. If f is open then $\overline{\mathcal{A}} \cong \Lambda$. ⟦[11], theorem 5.27.⟧ But in general, θ need not be an isomorphism, even in case $C(\tau) \cap H = \{e\}$: the topology of the semi-direct product may be different from the subspace topology of Λ. (Cf., e.g., Example 5.8.)

Put $C := \theta^{-1}(C(\tau)) = \pi_\Delta(C(\tau) \otimes \{e\})$, and $H' := \theta^{-1}(H) = \pi_\Delta(\{e\} \otimes H)$. Obviously, $C \cdot H' = \overline{\mathcal{A}}$.

2. The afore defined homomorphism $\phi : \mathcal{B} \to \mathcal{A}$ and $\theta : \mathcal{A}/\Delta \to C(\tau) \cdot H$ induce continuous bijective homomorphisms $\Psi : \overline{\mathcal{B}} \to \overline{\mathcal{A}} \to \Lambda$, $\Psi = \theta \circ \psi$, where $\psi \circ \pi_{\widetilde{\Delta}} = \pi_\Delta \circ \phi$, $\psi : \overline{\mathcal{B}} \ni \overline{(x, \kappa)} := \pi_{\widetilde{\Delta}}(x, \kappa) \overset{\psi}{\mapsto} \pi_\Delta(\phi(x, \kappa)) = \overline{\phi((x, \kappa)} \in \overline{\mathcal{A}}$.

$f^{-1}(H) =: H_1$ is a closed subgroup of $\mathcal{A}$, $H_1 = E \otimes H$. Hence $H' := \pi_\Delta(H_1) \subseteq \overline{\mathcal{A}}$, $H' \cong H$. Analogously, $\widetilde{H}_1 := \phi^{-1}(H_1)$, $\widetilde{H}_1 = \widetilde{E} \otimes H$, is closed in $\mathcal{B}$ and $\widetilde{H} := \pi_{\widetilde{\Delta}}(\widetilde{H}_1)$ is a compact subgroup of $\mathcal{B}/\Delta$. ⟦Note that $\widetilde{H} = \psi^{-1}(H') = \Psi^{-1}(H)$, hence $\widetilde{H} \cong H$. Cf. Lemma 2.2 b).⟧

If $E = \{e\}$, then obviously $\phi = \psi$ and $\psi|_{\widetilde{C}(\tau)} : \widetilde{C}(\tau) \to C(\tau) \subseteq \mathcal{A}$ coincides with φ. (We identify $C(\tau) \otimes \{e\} \subseteq \mathcal{A}$ with $C(\tau) \subseteq \Lambda$, since $f|_{C(\tau) \otimes \{e\}}$ is a topological isomorphism. And also $\widetilde{C}(\tau)$ is identified with $\widetilde{C}(\tau) \otimes \{e\} \subseteq \mathcal{B}$, hence $\Psi|_{\widetilde{C}(\tau)}$ coincides with φ.)

If $E \neq \{e\}$ we have: $\widetilde{C} := \Psi^{-1}(C(\tau))$ is a Borel subgroup of $\overline{\mathcal{B}}$ such that $\widetilde{C} \cdot \widetilde{H} = \overline{\mathcal{B}}$ and $\widetilde{C} \cap \widetilde{H} = \Psi^{-1}(E)$. ($\widetilde{C}$ need not be closed in $\overline{\mathcal{B}}$ since $C(\tau)$ need not be closed in Λ.) As easily verified, $\widetilde{C} = \pi_{\widetilde{\Delta}}(\widetilde{C}(\tau) \otimes \{e\})$, and $C = \pi_\Delta(C(\tau) \otimes \{e\}) \subseteq \overline{\mathcal{A}}$. Furthermore, $\psi^{-1} = \pi_{\widetilde{\Delta}} \phi^{-1} \pi_\Delta^{-1}$. ψ and Ψ are continuous bijective, hence the restriction $\psi|_{\widetilde{C}} : \widetilde{C} \to C(\tau)$ shares this property. And on the other hand, $\pi_{\widetilde{\Delta}}|_{\widetilde{C}(\tau)} : \widetilde{C}(\tau) \to \widetilde{C}$ is continuous and bijective. Furthermore, $\varphi = f \circ \phi|_{\widetilde{C}(\tau)}$ factorizes, $\varphi = \Psi \circ \pi_{\widetilde{\Delta}}|_{\widetilde{C}(\tau)}$, and we have $C = \psi(\widetilde{C})$. Thus there exists a chain of continuous bijective homomorphisms $\widetilde{C}(\tau) \to \widetilde{C} \to C \to C(\tau) \subseteq \Lambda$.

3. Δ resp. $\widetilde{\Delta}$ are invariant w.r.t. the afore defined automorphisms ρ and $\widetilde{\rho}$, respectively, hence there exist automorphisms $\rho_1 \in \mathrm{Aut}(\overline{\mathcal{A}})$, $\rho_2 \in \mathrm{Aut}(\overline{\mathcal{B}})$, such that $\pi_\Delta \rho = \rho_1 \pi_\Delta$, $\pi_{\widetilde{\Delta}} \widetilde{\rho} = \rho_2 \pi_{\widetilde{\Delta}}$, and furthermore, $f\rho_1 = \tau|_\Lambda f$.

If $E = \{e\}$, equivalently, $\Delta = \{e\}$, we have $\phi = \psi$ and $\rho_2 = \widetilde{\rho} = = \phi^{-1}\rho_1\phi = \psi^{-1}\tau|_\Lambda\psi$, and if $E \neq \{e\}$, $\psi^{-1}\rho_1\psi = \rho_2$, and $\rho_1 = \theta^{-1}\tau|_\Lambda\theta$.

ρ_1 and ρ_2 are contracting mod H' and mod $\widetilde{H}$ respectively. Furthermore, it is again easily verified that $\widetilde{C} = C(\rho_2)$ and $C = C(\rho_1)$.

Claim: Define $g := \pi_\Delta|_{C(\tau)}$ and $h := \pi_{\widetilde{\Delta}}|_{\widetilde{C}(\tau)}$. Then $g : C(\tau) \to C$ and $h : \widetilde{C}(\tau) \to \widetilde{C}$ are continuous bijective with continuous resp. measurable inverses g^{-1} and h^{-1} respectively. In particular, $C(\tau) \cong C$. $[\![$ Let $U_1, W_1 \in \mathfrak{U}^o_{\mathbb{G}}$ such that $U_1^2 \subseteq W_1$. Thus $U := U_1 \cap C(\tau), W := W_1 \cap C(\tau) \in \mathfrak{U}^o_{C(\tau)}$ and $V := U_1 \cap H \in \mathfrak{U}^o_H$ such that $U \cdot V \subseteq W$, hence also $U \cdot (V \cap E) \subseteq W$. Hence $\mathfrak{B} := \{U \cdot (V \cap E) : U \in \mathfrak{U}^o_{C(\tau)}, V \in \mathfrak{U}^o_H\}$ is a basis for the filter $\mathfrak{U}_{C(\tau)}$.

Since $\Delta \cap (C(\tau) \otimes \{e\}) = \{e\}$, bijectivity of g follows. Continuity is obvious. To prove continuity of g^{-1} it suffices to show that g maps open sets of $C(\tau) \otimes \{e\}$ to open sets of $C (\subseteq \overline{\mathcal{A}})$. It suffices to show that $g(U \cdot (V \cap E) \otimes \{e\})$ is open in C. Indeed, this follows since $g(U \cdot (V \cap E) \otimes \{e\}) = \pi_\Delta(U \otimes V) \cap C$. $[\![\overline{(u,v)\Delta} = \overline{(u,v)} = \overline{(x,e)}$ (with $x \in C(\tau), u \in U, v \in V$) yields $u \cdot v = x$, hence $v \in V \cap E$. Thus $\overline{(u,v)} = \overline{(u \cdot v, e)}$ follows. Whence $g(U \cdot (V \cap E) \otimes \{e\}) \supseteq \pi_\Delta(U \otimes V) \cap C$. Conversely, if $u \in U, v \in V \cap E$, then $\overline{(u \cdot v, e)} = (u,v)\Delta = \overline{(u,v)}$, whence $g(U \cdot (V \cap E) \otimes \{e\}) \subseteq \pi_\Delta(U \otimes V) \cap C.]\!]$

As before, $h : \widetilde{C}(\tau) \to \widetilde{C}$ is bijective and continuous. But note that h is in general not an isomorphism. (See e.g., Example 5.7.) Indeed, we have: h is an isomorphism iff $E = \{e\}$, hence iff $\mathcal{B} = \overline{\mathcal{B}}$. $[\![$Assume $\widetilde{C}(\tau) \cong \widetilde{C}$. $\widetilde{C}(\tau)$ is complete (as closed subgroup of $\mathcal{B}$), hence, $\widetilde{C}$ is complete and hence closed (in $\overline{\mathcal{B}}$). Thus $\widetilde{C} \cap \widetilde{H}$ is compact. But ρ_2 acts contracting on the Polish group $\widetilde{C}$, hence compactly contracting. Thus $\widetilde{C} \cap \widetilde{H} = \{e\}$. Hence $E = \{e\}$. The converse is obvious.$]\!]$

The construction of $\overline{\mathcal{B}}$ (and $\overline{\mathcal{A}}$) is sufficient to define the intrinsic topology of $C(\tau) \cdot H$. (Definition 2.5 below). However, for a characterization of semistable laws we have to pass to a quotient group. At the first glance, the following way seems to be natural:

4. $F := E^-$ is a compact subgroup of H, hence of $C(\tau) \cdot H$, and hence $D := f^{-1}(F)$ and $\widetilde{D} := \psi^{-1}(D)$ are compact subgroups of $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ (according to Lemma 2.2), and $\widetilde{D} \subseteq \widetilde{H} = \Psi^{-1}(H)$. If E is a *normal* subgroup of $\mathbb{G}$, (as in Assumption (2), cf. (4.1)), then F, D and $\widetilde{D}$ are compact normal subgroups of $C(\tau) \cdot H$ (hence of $\mathbb{G}$), of $\overline{\mathcal{A}}$ and of $\overline{\mathcal{B}}$ respectively. Let π, π_D and $\pi_{\widetilde{D}}$ denote the quotient homomorphisms $\mathbb{G} \to \mathbb{G}/F =: \overline{\mathbb{G}}$, $\overline{\mathcal{A}} \to \overline{\mathcal{A}}/D =: \mathcal{A}_1$ and $\overline{\mathcal{B}} \to \overline{\mathcal{B}}/\widetilde{D} =: \mathcal{B}_1$ respectively. Put $K := \pi_{\widetilde{D}}(\widetilde{H})$, $\overline{H} := \pi_D(H')$, $H^* := \pi(H) \subseteq \overline{\mathbb{G}}$ and $\overline{C} := \pi_D(C(\tau))$, $C^* := \pi_{\widetilde{D}}(\widetilde{C})$, $C_1 := \pi(C(\tau))$ (in $\overline{\mathbb{G}}$). K, $\overline{H}$ and H^* are compact isomorphic subgroups in $\mathcal{B}$, $\overline{\mathcal{A}}$ and $\overline{\mathbb{G}}$ respectively. Furthermore, σ, ξ and $\overline{\tau}$ denote the automorphisms in $\mathrm{Aut}(\mathcal{B}_1)$, $\mathrm{Aut}(\mathcal{A}_1)$ and $\mathrm{Aut}(\overline{\mathbb{G}})$ respectively, which are induced by by $\rho_2 \in \mathrm{Aut}(\overline{\mathcal{B}})$, $\rho_1 \in \mathrm{Aut}(\overline{\mathcal{A}})$ resp. $\tau \in \mathrm{Aut}(\mathbb{G})$.

We have $\mathcal{B}_1 = C^* \cdot K$, $\mathcal{A}_1 = \overline{C} \cdot \overline{H}$ and $\overline{\Lambda} := \Lambda/F = C_1 \cdot H^*$. Furthermore, $\pi(C_F(\tau)) = C(\overline{\tau})$ and $C_F(\tau) = \pi^{-1}(C(\overline{\tau}))$ $[\![$cf., e.g., [10], 3.2.3 with $K = N = F$⟧ , hence, if we assume $C_F(\tau) = C(\tau) \cdot F$ (as we shall do in Assumption (3), cf. (4.2)), we obtain $C(\overline{\tau}) = \pi(C(\tau)) = C_1$. And, applying Ψ resp. θ, $C(\xi) = \pi_D(C(\rho_1))$ and $C(\sigma) = \pi_{\widetilde{D}}(C(\rho_2))$ follow. Thus $C^* = C(\sigma) = \pi_{\widetilde{D}}(\widetilde{C})$ and $\overline{\Psi}|_{C^*} : C^* \to C_1$ is bijective and continuous.

Furthermore, C^* is closed in $\mathcal{B}_1$, hence also Polish. (Note, if $E \neq \{e\}$, $\widetilde{C}$ is not closed in $\overline{\mathcal{B}}$, but $C^* = \pi_{\widetilde{D}}(\widetilde{C})$ is closed). We have $C^* = \pi_{\widetilde{D}}\pi_{\widetilde{\Delta}}(\widetilde{C}(\tau) \otimes \{e\}) = \pi_{\widetilde{D}}\pi_{\widetilde{\Delta}}(\widetilde{C}(\tau) \otimes \{e\} \cdot S)$ with $S := \pi_{\widetilde{\Delta}}^{-1}(\widetilde{D})$. We observe $S = \widetilde{E} \otimes F$. ⟦Indeed, $S = \{(x, \kappa) \in \mathcal{B} : \Phi(x, \kappa) \in F\}$, hence, putting $(x', \kappa') := \phi(x, \kappa)$, we obtain $(x, \kappa) \in S$ iff $x' \cdot \kappa' \in F$. Since $x' \in C(\tau)$ and $\kappa' \in H$, it follows $x' \in E$ and thus $\kappa' \in F$. I.e., $(x, \kappa) \in \widetilde{E} \otimes F$. Thus $S \subseteq \widetilde{E} \otimes F$. Conversely, if $(x, \kappa) \in \widetilde{E} \otimes F$, we have $(x', \kappa') \in E \otimes F$ and hence $x' \cdot \kappa' \in F$. Hence $\widetilde{E} \otimes F \subseteq S$. ⟧

$(\widetilde{C}(\tau) \otimes \{e\}) \cdot S$ is closed in $\mathcal{B}$, therefore, according to Lemma 2.2 c), C^* is complete, hence closed in $\mathcal{B}_1$. ⟦Indeed, $(\widetilde{C}(\tau) \otimes \{e\}) \cdot S = (\widetilde{C}(\tau) \otimes \{e\}) \cdot (\widetilde{E} \otimes F) = \widetilde{C}(\tau) \otimes F$ is closed since F is closed in H. ⟧

According to Remark 1.5 d), there exists a continuous bijective homomorphism $\delta : \widetilde{C}(\tau) \to C^*$, hence also such a homomorphism $\alpha = \delta \otimes \mathrm{id}_{H^*} : \mathcal{B}_2 \to \mathcal{B}_1$. But we cannot prove that topology of C^* is the intrinsic one, i.e. that $C^* \cong \widetilde{C}(\overline{\tau})$.

5. Therefore we have to replace step 4. by a repetition of the first step:

Let $\mathcal{A}_2 := C(\overline{\tau}) \rtimes_\gamma H^*$, $\gamma : H^* \to \mathrm{Aut}(C(\overline{\tau}))$ defined in obvious way (note that F is τ-invariant), and $\mathcal{B}_2 := \widetilde{C}(\overline{\tau}) \rtimes_{\widetilde{\gamma}} H^* = \widetilde{C(\overline{\tau}) \cdot H^*}$ ($\widetilde{\gamma}$ defined according to Proposition 1.4). Denote furthermore by φ^*, ϕ^* the corresponding canonical homomorphisms, and $\Phi^* := f^* \circ \phi^* : \mathcal{B}_2 \to \overline{\Lambda}$, $f^* : \mathcal{A}_2 \to \overline{\Lambda} : (\overline{x}, \overline{\kappa}) \mapsto \overline{x} \cdot \overline{\kappa}$, $\phi^* : \mathcal{B}_2 \to \mathcal{A}_2$. Note that $\widetilde{C}(\overline{\tau}) = C(\widetilde{\overline{\tau}})$.

Finally we have $C(\overline{\tau}) \cap H^* = \{e\}$. ⟦Indeed, assume $\overline{x} = \pi(x) \in C_1 \cap H^* = \pi_F(C(\tau) \cap H)$. Then there exist $x \in E$ with $\pi(x) = \overline{x} = x \cdot F$. But $E \subseteq F$, hence $\overline{x} = \overline{e}$. ⟧ Moreover, $\overline{\mathbb{G}}$ is Polish (Lemma 2.2).

The properties of $\mathcal{B}_2$ will enable us to reduce the investigation of semistable laws on Λ – even in the case $E \neq \{e\}$– to the case of semi-direct products (cf. Theorem 4.2).

Note that the role of the groups $\mathcal{A}, \overline{\mathcal{A}}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ is not essential; these groups were introduced to construct $\mathcal{B}, \overline{\mathcal{B}}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, and to link these groups via projections and canonical homomorphisms to Λ and $\Lambda/F = \overline{\Lambda}$.

In the following diagrams we collect the above introduced notation:

$$
\begin{array}{lllll}
\Phi : & \mathcal{B} = \widetilde{C}(\tau) \rtimes_{\widetilde{\beta}} H & \xrightarrow{\phi} \mathcal{A} = C(\tau) \rtimes_\beta H & \xrightarrow{f} & \Lambda = C(\tau) \cdot H \\
& \downarrow \pi_{\widetilde{\Delta}} & \quad \downarrow \pi_\Delta & & || \\
\Psi : & \overline{\mathcal{B}} = \widetilde{C} \cdot \widetilde{H} & \xrightarrow{\psi} \overline{\mathcal{A}} = C \cdot H' & \xrightarrow{\theta} & \Lambda \\
& \downarrow \pi_{\widetilde{D}} & \quad \downarrow \pi_D & & \downarrow \pi \\
\overline{\Psi} : & \mathcal{B}_1 = C^* \cdot K & \xrightarrow{\overline{\psi}} \mathcal{A}_1 = \overline{C} \cdot \overline{H} & \xrightarrow{\overline{\theta}} & \overline{\Lambda} = C(\overline{\tau}) \cdot H^* \\
& \uparrow \alpha & & & || \\
\Psi^* : & \mathcal{B}_2 = \widetilde{C}(\overline{\tau}) \rtimes_{\widetilde{\gamma}} H^* & \xrightarrow{\phi^*} \mathcal{A}_2 = C(\overline{\tau}) \rtimes_\gamma H^* & \xrightarrow{f^*} & \overline{\Lambda}
\end{array}
$$

$$
\begin{array}{ccccc}
\widetilde{C}(\tau)\otimes\{e\} = C(\widetilde{\rho}) & \overset{\varphi}{\longrightarrow} & C(\tau)\otimes\{e\} = C(\rho) & \overset{f|_{C(\tau)}}{\longleftrightarrow} & C(\tau)\subseteq\Lambda \\
\downarrow h & & \updownarrow g & & || \\
\widetilde{C} = C(\rho_2) & \overset{\psi|_{\widetilde{C}}}{\longleftrightarrow} & C = C(\rho_1) & \overset{\theta|_{C}}{\longrightarrow} & C(\tau) \\
\downarrow \pi_{\widetilde{D}} & & \downarrow \pi_D & & \downarrow \pi \\
C^* = C(\sigma) & \overset{\overline{\psi}|_{C^*}}{\longrightarrow} & \overline{C} = C(\xi) & \overset{\overline{\theta}|_{\overline{C}}}{\longrightarrow} & C_1 = C(\overline{\tau}) \\
\uparrow \delta & & & & || \\
\widetilde{C}(\overline{\tau}) & \overset{\varphi^*}{\longrightarrow} & C(\overline{\tau}) & \overset{f^*|_{C(\overline{\tau})}}{\longleftrightarrow} & C(\overline{\tau})\subseteq\overline{\Lambda}
\end{array}
$$

The existence of measurable inverses of the canonical homomorphisms follows by Kuratowski's theorem (Lemma 2.2 a)). It seems remarkable that the special construction of the intrinsic topology implies measurability without completeness assumption:

Proposition 2.4. *Let $\mathbb{G}$ be a second countable topological Hausdorff group (hence metrizable but not necessarily complete), let $\tau \in \mathrm{Aut}(\mathbb{G})$ and let H be a compact τ-invariant subgroup. With the afore introduced notations we obtain:*

a) $C(\tau), C_H(\tau), \Lambda$ are Borel subsets of $\mathbb{G}$.

b) The bijective continuous homomorphisms in the above diagrams have measurable inverses. In particular, this is true for for:

b1) $\varphi : \widetilde{C}(\tau) \to C(\tau)$, $\quad \varphi^* : \widetilde{C}(\overline{\tau}) \to C(\overline{\tau})$,

b2) $\phi : \mathcal{B} \to \mathcal{A}$, $\quad \phi^* : \mathcal{B}_2 \to \mathcal{A}_2$,

b3) $\theta : \overline{\mathcal{A}} \to \Lambda$, $\quad \overline{\theta} : \mathcal{A}_1 \to \overline{\Lambda}$, $\quad f^* = \theta^* : \mathcal{A}_2 \to \overline{\Lambda}$,

b4) $\psi : \overline{\mathcal{B}} \to \overline{\mathcal{A}}$, $\quad \Psi : \overline{\mathcal{B}} \to \Lambda$, $\quad \overline{\Psi} : \mathcal{B}_1 \to \overline{\Lambda}$, $\quad \Psi^* : \mathcal{B}_2 \to \overline{\Lambda}$

Proof. a) is already used. It follows by the representation $C(\tau) = \bigcap_{n\geq 1}\bigcup_{m\geq 1}\bigcap_{k\geq m}\tau^{-k}(V_n)$, where $\{V_n\}$ denotes a (countable) basis of $\mathfrak{U}_\Gamma$, and by analogous representations for $C_H(\tau)$.

b1) φ^{-1} is measurable: We show: For all $U \in \mathfrak{U}^o_\Gamma$ hence all $U_n \in \mathfrak{U}^o_{\widetilde{C}(\tau)}$, $\varphi(U_n)$ is a G_δ-set. Indeed, $U_n = \varphi(U)_n = \bigcap_{k\geq n}\tau^{-k}(U\cap C(\tau))$, with $\tau^{-k}((U\cap C(\tau)) \in \mathfrak{U}^o_{C(\tau)}$. The proof for φ^* is analogous.

b2) follows immediately by $\phi = \varphi\otimes \mathrm{id}_H$ etc.

b3) θ^{-1} is measurable: We have $\theta = f\circ\pi_\Delta^{-1}$ (with $\ker f = \Delta$). We show that for $U \in \left\{U = \pi_\Delta(V_1\times V_2) : V_1 \in \mathfrak{U}^b_{C(\tau)}, V_2 \in \mathfrak{U}^b_H\right\}$, $\theta(U)$ is a Borel set: Indeed, let V_1 be open, V_2 closed (hence compact in $\mathbb{G}$), then there exist open W_n in $\mathbb{G}$, such that $V_2 = \bigcap_n W_n$. Therefore, $\theta(U) = f(V_1\times V_2) = V_1\cdot V_2 = \bigcap_n V_1\cdot W_n$, is a G_δ set, since $V_1\cdot W_n$ are open in $\mathbb{G}$. Measurability of $\overline{\theta}^{-1}$ and f^{*-1} is proved analogously.

b4) ψ^{-1} is measurable: Let $\mathcal{O}_\mathcal{A}$, $\mathcal{O}_\mathcal{B}$ etc. denote the open sets of $\mathcal{A}, \mathcal{B}$ etc. and let $\mathcal{O}^\Delta_\mathcal{A}$ etc. denote the Δ-invariant open sets (i.e., open sets W with $W\cdot x = W$ for all $x \in \Delta$). Then $\pi_\Delta(\mathcal{O}^\Delta_\mathcal{A}) = \mathcal{O}_{\overline{\mathcal{A}}}$ and $\pi_\Delta^{-1}(\mathcal{O}_{\overline{\mathcal{A}}}) = \mathcal{O}^\Delta_\mathcal{A}$. Hence π_Δ defines also a bijection between the σ-algebras generated by these sets, i.e., between the Δ-invariant Borel sets in $\mathcal{A}$ and Borel sets in $\overline{\mathcal{A}}$ respectively. An analogous assertion holds true for $\mathcal{O}^{\widetilde{\Delta}}_\mathcal{B}$ and $\mathcal{O}_{\overline{\mathcal{B}}}$.

We have $\psi = \pi_\Delta\circ\phi\circ\pi_{\widetilde{\Delta}}^{-1}$ (with $\phi(\widetilde{\Delta}) = \Delta$). Let $U \in \mathcal{O}_{\overline{\mathcal{B}}}$, hence $\psi(U) = \pi_\Delta\phi\pi_{\widetilde{\Delta}}^{-1}(U) = \pi_\Delta(\phi(U'))$, $U' \in \mathcal{O}^{\widetilde{\Delta}}_\mathcal{B}$. ϕ^{-1} is measurable and U'

is $\widetilde{\Delta}$-invariant, hence $\phi(U')$ is a Δ-invariant Borel set in $\mathcal{A}$, and thus $\pi_\Delta(\phi(U'))$ is a Borel set in $\overline{\mathcal{A}}$.

Since compositions of measurable mappings are measurable, measurability of the resting homomorphisms follows immediately. $\square$

Definition 2.5. *a) If $E = C(\tau) \cap H = \{e\}$ we call the topology of $\mathcal{B}$ the intrinsic topology of $C(\tau) \cdot H$. (Recall that in this case, $f : \mathcal{A} = C(\tau) \rtimes_\beta H \to C(\tau) \cdot H$ is bijective and continuous).*

b) If $\Delta \neq \{e\}$, equivalently, $E \neq \{e\}$, then we call the quotient topology of $\overline{\mathcal{B}} = \mathcal{B}/\widetilde{\Delta}$ intrinsic topology of $\widetilde{C(\tau) \cdot H} = \Lambda$.

c) In either case, we use the notation $\widetilde{C(\tau) \cdot H} =: \mathcal{B}$ (in case a)) resp. $=: \mathcal{B}/\widetilde{\Delta} = \overline{\mathcal{B}}$ (in case b)). The continuous bijective mapping $\widetilde{C(\tau) \cdot H} \to C(\tau) \cdot H$, defined by $\Phi = f \circ \phi : (x, \kappa) \mapsto x \cdot \kappa$ (in case a)) resp. $\Psi = \theta \circ \psi : \pi_{\widetilde{\Delta}}(x, \kappa) \mapsto x \cdot \kappa$ (in case b)) is called canonical homomorphism.

(In case a), we have $\mathcal{B} = \overline{\mathcal{B}}$, hence then we may identify Φ with Ψ.)

Putting things together we obtain

Theorem 2.6. *Let $\mathbb{G}$ be a Polish group, $\tau \in \mathrm{Aut}(\mathbb{G})$ and let H be a τ-invariant compact subgroup. Then $C(\tau)$ and hence Λ are Borel sets.*

a) The afore defined group $\widetilde{C(\tau) \cdot H}$ is a Polish group, the canonical homomorphism $\Psi : \widetilde{C(\tau) \cdot H} \to C(\tau) \cdot H = \Lambda \subseteq \mathbb{G}$ is continuous and bijective, with measurable inverse, where the subgroup $C(\tau) \cdot H$ is always endowed with the subspace topology inherited by $\mathbb{G}$.

b) $\widetilde{H} := \Psi^{-1}(H)$ is a compact subgroup of $\widetilde{C(\tau) \cdot H}$, $\widetilde{H} \cong H$, and $\widetilde{C} := \Psi^{-1}(C(\tau))$ is a Borel subgroup, such that $\widetilde{C(\tau) \cdot H} = \widetilde{C} \cdot \widetilde{H}$. Furthermore, $\Psi^{-1}|_H$ is a topological (and algebraic) isomorphism, and $\Psi|_{\widetilde{C}} : \widetilde{C} \to C(\tau)$ is bijective, continuous and there exists a bijective continuous homomorphism $\widetilde{C}(\tau) \to \widetilde{C}$.

c) $\rho_2 := \Psi\tau\Psi^{-1} \in \mathrm{Aut}(\widetilde{C(\tau) \cdot H})$ is contracting $\mathrm{mod}\ \widetilde{H}$. (Compactly contracting since $\widetilde{C(\tau) \cdot H}$ is Polish, cf. d1) below.)

d) The intrinsic topology $\widetilde{C(\tau) \cdot H}$ has the following properties:

d1) $\mathbb{G}$ is Polish, so are $\mathcal{B}$ and $\overline{\mathcal{B}}$, hence $\widetilde{C(\tau) \cdot H}$.

d2) If $C(\tau)$ is closed in $\mathbb{G}$ then $\mathcal{A} = \mathcal{B} = \overline{\mathcal{B}} = \widetilde{C(\tau) \cdot H} \cong \Lambda$.

d3) If $\mathbb{G}$, hence $C(\tau)$, is totally disconnected, so is $\widetilde{C(\tau) \cdot H}$.

d4) If $\widetilde{C(\tau) \cdot H}$ is connected, so is $C(\tau) \cdot H$ and hence $(C(\tau) \cdot H)^-$.

d5) $C(\tau) \cdot H$ is arc-wise connected iff $\widetilde{C(\tau) \cdot H}$ is.

d6) $\mathcal{B}$ is locally compact iff $\widetilde{C}(\tau)$ is locally compact. (Cf. the criteria for locally compactness of the intrinsic topology of $\widetilde{C}(\tau)$ in [26]*.) In that case, $\overline{\mathcal{B}}$ and $\mathcal{B}_1$ are locally compact too.*

e) It is not surprising that we also obtain a characterization of the intrinsic topology $\widetilde{C(\tau) \cdot H}$ analogous to Remark 1.5 d):

Let $\mathcal{G}$ be a Polish group, representable as $\mathcal{G} = A \cdot B$, with $\zeta \in \mathrm{Aut}(\mathcal{G})$, such that $A = C(\zeta)$, $\zeta(B) = B$, a compact subgroup, and a continuous bijective homomorphism $F : \mathcal{G} \to \Lambda = C(\tau) \cdot H$, with $F(B) = H$ and $F \circ \zeta = \tau \circ F$. Then there exists a continuous bijective homomorphism $\Xi : \widetilde{C(\tau) \cdot H} \to \mathcal{G}$.

Proof. a) follows by definition of $\widetilde{C(\tau)\cdot H}$ and Ψ (Notations 2.3) and by Proposition 2.4.

b) Cf. Notations 2.3, 2. Compactness of $\widetilde{H}$ follows by Lemma 2.2.

c) is easily verified and

d) follows by Remark 1.5 (see also [25], or [10], lemma 3.3.17), and the product representation $\mathcal{B} = \widetilde{C}(\tau) \rtimes H$ resp. $\overline{\mathcal{B}} = \pi_{\widetilde{\Delta}}(\mathcal{B})$. Note that $\mathcal{B}$ is Polish resp. (arc-wise) connected resp. totally disconnected iff $\widetilde{C}(\tau)$ and H are Polish resp. (arc-wise)connected resp. totally disconnected.

e) We identify B with H via $F|_B$ and consider the intrinsic topology and canonical homomorphisms $\widetilde{C}(\zeta) = C(\widetilde{\zeta}) \stackrel{\widehat{\varphi}}{\to} C(\zeta) = A$, $\widehat{\phi} : \widehat{\mathcal{B}} :\to \widehat{\mathcal{A}}$, $\widehat{\phi} = \widehat{\varphi} \otimes \mathrm{id}_H$, $\widehat{\mathcal{B}} := \widetilde{C}(\zeta) \rtimes H$, $\widehat{f} : \widehat{\mathcal{A}} \to \mathcal{G}$. Let $\widehat{E} := A \cap B$, $\widehat{\Delta} := \{(c, c^{-1}) : c \in \widehat{E}\}$, $\widehat{\widehat{\Delta}} := \widehat{\Phi}^{-1}(\widehat{\Delta})$.

$\widetilde{C}(\zeta)$ is Polish and $Z := F \circ \widehat{\varphi} : \widetilde{C}(\zeta) = C(\widetilde{\zeta}) \to C(\tau)$ is continuous, bijective and $Z \circ \widetilde{\zeta} = \tau \circ Z$. Hence (Remark 1.5 d)) there exists a continuous bijective homomorphism $\widehat{\delta} : \widetilde{C}(\tau) \to \widetilde{C}(\zeta)$, hence also $\widehat{\alpha} := \widehat{\delta} \otimes \mathrm{id}_H : \mathcal{B} \to \widehat{\mathcal{B}}$. $\widehat{\Phi} := \widehat{f} \circ \widehat{\phi} : \widehat{\mathcal{B}} \to \mathcal{G}$ is continuous with kernel $\widehat{\widehat{\Delta}}$, hence $T := \widehat{\Phi} \circ \widehat{\alpha} : \mathcal{B} \to \mathcal{G}$ is continuous with kernel $\ker(T) = \widetilde{\Delta} (= \widehat{\alpha}^{-1}(\widehat{\widehat{\Delta}}))$. Therefore, there exists a factorization $T = \Xi \circ \pi_{\widetilde{\Delta}}$ with continuous bijective $\Xi : \overline{\mathcal{B}} \to \mathcal{G}$. □

3. Liftings, retracts and disintegrations of continuous convolution semigroups

Definition 3.1. *a) Let $\mathcal{G}$ be a topological Hausdorff group, let H, K be compact subgroups. Let $\{\mu_t\}$ and $\{\nu_t\}$ be continuous convolution semigroups with idempotents $\mu_0 = \omega_H$ and $\nu_0 = \omega_K$. Then $\{\nu_t\}$ is a retract of $\{\mu_t\}$ (equivalently, $\{\mu_t\}$ is a projection of $\{\nu_t\}$) if $\mu_t = \omega_H \star \nu_t = \nu_t \star \omega_H$ for $t \geq 0$. (Hence $K \subseteq H$.)*

b) Let $\mathcal{G}$ and Γ be topological Hausdorff groups, $\Lambda \subseteq \mathcal{G}$ a measurable subgroup and $\phi : \Gamma \to \Lambda \subseteq \mathcal{G}$ a continuous bijective homomorphism with measurable inverse ϕ^{-1}. Let $\{\mu_t\}$ and $\{\nu_t\}$ be continuous convolution semigroups in $\mathcal{M}^1(\mathcal{G})$ and $\mathcal{M}^1(\Gamma)$ respectively, such that $\mu_t(\Lambda) = 1$ for all $t \geq 0$. (For short: $\{\mu_t\}$ is concentrated on Λ.*) $\{\nu_t\}$ is a ϕ-lifting of $\{\mu_t\}$ if $\phi(\nu_t) = \mu_t$ for all t.*

c) Let $\mathcal{G}, \Gamma, \Lambda, \phi$ as in b). Let H be a compact subgroup of $\mathcal{G}$, normalizing Λ and put $\Lambda_1 = \Lambda \cdot H$. Let $\{\mu_t\}$ be a continuous convolution semigroup concentrated on Λ_1 with idempotent $\mu_0 = \omega_H$. A continuous convolution semigroup $\{\nu_t\} \subseteq \mathcal{M}^1(\Gamma)$ such that $\mu_t = \phi(\nu_t) \star \omega_H = \omega_H \star \phi(\nu_t)$ is called a ϕ-disintegration of $\{\mu_t\}$.

d) If in a), for $\tau \in \mathrm{Aut}(\mathcal{G})$, with $\tau(H) = H$, $\{\mu_t\}$ and $\{\nu_t\}$ are (τ, α)-semistable, then $\{\nu_t\}$ is called semistable retract, and analogously, if, in b), c) $\tau \in \mathrm{Aut}(\mathcal{G})$ and $\tau' \in \mathrm{Aut}(\Gamma)$ such that $\phi\tau' = \tau\phi$, and $\{\mu_t\}$ and $\{\nu_t\}$ are (τ, α)- and (τ', α)-semistable, we call $\{\nu_t\}$ a semistable lifting resp. disintegration.

Proposition 3.2. *Let A, B be Polish groups and $\phi : B \to C \subseteq A$ a continuous bijective homomorphism. Then C is measurable. And in that case, any continuous convolution semigroup $\{\mu_t\}$ in $\mathcal{M}^1(A)$ concentrated on C has a ϕ-lifting $\{\nu_t\}$ in $\mathcal{M}^1(B)$. The lifting is (τ', α)-semistable, if $\{\mu_t\}$ is (τ, α)-semistable (where $\tau \in \mathrm{Aut}(A)$ and $\tau' \in \mathrm{Aut}(B)$ and we assume again $\tau\phi = \phi\tau'$).*

Indeed, if we suppose that C is measurable and ϕ^{-1} is measurable then it is sufficient to assume A and B to be second countable (hence metrizable), not necessarily complete.

This result is trivial in the case of Lie groups for $B := \widetilde{C}(\tau)$ and $\phi := \varphi$, the canonical homomorphism $\widetilde{C}(\tau) \to C = C(\tau) \subseteq \Gamma$, since in that case $\widetilde{C}(\tau)$ is a simply connected nilpotent Lie group, hence $\varphi^{-1}(\mu_1)$ is continuously embeddable. For the general result cf. [26], appendix 1, [10], proposition 3.4.11. However, in the first reference the proof of continuity is missing, in the second that proof is not complete. Therefore we give a new proof.

Proof. Let C be a Borel set and let ϕ^{-1} be measurable (e.g., A, B Polish, in view of Lemma 2.2.) Then $\{\nu_t := \phi^{-1}(\mu_t)\}$ is a convolution semigroup in $\mathcal{M}^1(B)$. We have to prove continuity. It is sufficient to prove continuity (at 0) of $t \mapsto \langle \nu_t, f\rangle$ for $f \in \mathcal{U}^b(B), \mathcal{U}^b(B)$ denoting the space of bounded right uniformly continuous functions (cf. e.g., [19], theorem 6.1). Furthermore, it suffices to prove this for f belonging to a $||\cdot||_\infty$-dense subset of $\mathcal{U}^b(B)$.

For this purpose we use a method invented in [2], [3]; see also [27]: Let $\rho := \int_0^\infty \mathrm{e}^{-t}\mu_t \mathrm{d}t \in \mathcal{M}^1(A)$ be the resolvent measure. Obviously, $\rho(C) = 1$. The convolution operators $T_{\mu_t} : f \mapsto \int_C f(x\cdot)\mathrm{d}\mu_t(x)$ form a continuous operator semigroup on the Banach space $\mathcal{L}^1(C, \rho)$ with $||T_{\mu_t}|| \le \mathrm{e}^t$. Let $\overline{\rho} := \phi^{-1}(\rho) \in \mathcal{M}^1(B)$. Since $\mathcal{L}^1(B, \overline{\rho}) = \{f \circ \phi : f \in \mathcal{L}^1(C, \rho)\}$, the convolution operators T_{ν_t} form a continuous operator semigroup on $\mathcal{L}^1(B, \overline{\rho})$, again with $||T_{\nu_t}|| \le \mathrm{e}^t$. In particular, this semigroup is weakly continuous, hence for $f \in \mathcal{L}^1(B, \overline{\rho})$, in particular, $f \in \mathcal{U}^b$, $g \in \mathcal{L}^\infty(B, \overline{\rho})$ we observe:

$$\begin{aligned} t \;\mapsto\; \langle T_{\nu_t} f, g\rangle &= \int_B \int_B f(xy)\mathrm{d}\nu_t(x) g(y)\mathrm{d}\overline{\rho}(y) \qquad (3.1)\\ &= \int_B \left(\int_B f(xy)g(y)\mathrm{d}\overline{\rho}(y)\right) \mathrm{d}\nu_t(x) =: \int_B h_{f,g}(x)\mathrm{d}\nu_t(x) \end{aligned}$$

is continuous. We choose for g an approximative unit: Let, for $\delta > 0$, $g_\delta = g := c \cdot 1_{B_\delta}$, where B_δ denotes the ball $\{x : d(x, e) < \delta\}$, d denoting a left invariant metric, and $c := 1/\overline{\rho}(B_\delta)$. If $\epsilon > 0$ and $\delta > 0$ such that $|f(xy) - f(x)| < \epsilon$ for all x and all $y \in B_\delta$, then $||f - h_{f,g}||_\infty < \epsilon$. Hence $\{h_{f,g}\}$ is $||\cdot||$-dense in $\mathcal{U}^b(B)$.

Above, in (3.1), we have proved continuity of $t \mapsto \langle \nu_t, h_{f,g}\rangle$ for all $h_{f,g}$, therefore $\{\nu_t\}$ is a continuous convolution semigroup as asserted.

The last assertion, concerning semistability, is obvious since $\nu_t = \phi^{-1}(\mu_t)$ and $\tau\phi = \phi\tau'$. □

Corollary 3.3. *a) Let A, B be Polish groups, $\phi : A \to B$ a continuous injective homomorphism. Or assume that A and B are metrizable, that $\phi(A)$ is a Borel set and ϕ^{-1} is measurable.*

Then for all continuous one-parameter groups $(x(t))_{t\in\mathbb{R}} \subseteq \phi(A)$, $(y(t) := \phi^{-1}(x(t)))_{t\in\mathbb{R}}$ is a continuous one-parameter group in A.

b) Applied to the situation in Sections 1 and 2, in particular Theorem 2.6, we observe: $C(\tau)$ resp. $C(\tau)\cdot H$ is arc-wise connected iff $\widetilde{C}(\tau)$ resp. $\widetilde{C(\tau)\cdot H}$ shares this property.

⟦ Apply the above Proposition 3.2 to $\{\mu_t = \varepsilon_{x(\pm t)}\}$ and φ resp. ψ. ⟧

For later use we note:

Proposition 3.4. *Let A be a locally compact or Polish group. Let H, K be compact subgroups, $K \subseteq H, K \lhd A$. Let $\{\mu_t\}$ be a continuous convolution semigroup in $\mathcal{M}^1(A)$ with idempotent $\mu_0 = \omega_H$. Put $\nu_t := \pi_K(\mu_t)$, π_K denoting the quotient homomorphism $A \to A/K$. Then $\{\mu_t\}$ is uniquely determined by $\{\nu_t\}$, and (trivially) vice versa.*

$[\![$ π_K defines a bijection between $\{\lambda \in \mathcal{M}^1(A) : \lambda = \lambda \star \omega_K = \omega_K \star \lambda\}$ and $\mathcal{M}^1(A/K)$. Since $K \subseteq H$ and $\mu_t \star \omega_H = \omega_H \star \mu_t = \mu_t$ it follows $\pi_K^{-1}\{\pi_K(\mu_t)\} = \{\mu_t\}$. (See [12], theorem 1.12.15 for locally compact groups. A proof for Polish groups is analogous.) $]\!]$

3.1. **Application to semistable laws with non-trivial idempotents.** Let throughout $\mathbb{G}$ be Polish, $\tau \in \mathrm{Aut}(\mathbb{G})$ and $0 < \alpha < 1$. Recall that (τ, α)-semistable continuous convolution semigroups with trivial idempotents are concentrated on $C(\tau)$. And for semistable continuous convolution semigroups with non-trivial idempotent ω_H we have: H is τ-invariant and the measures are concentrated on $C_H(\tau)$. $[\![$See e.g., [10], proposition 3.4.4 for locally compact groups. The proof for non locally compact groups is analogous. It relies on the existence of Lévy measures, which is guaranteed e.g., by [24], section 2, remark. $]\!]$

In the following we assume throughout for $\{\mu_t\}$

$$\textbf{Assumption } (1) \quad \mu_t(\Lambda) = \mu_t(C(\tau) \cdot H) = 1 \ \forall\, t \geq 0. \tag{3.2}$$

For (τ, α)-semistable $\{\mu_t\}$, and also for non-dissipating random walks or continuous convolution semigroups, and for semi-self decomposable continuous convolution semigroups with idempotent ω_H, this condition is satisfied if $(\mathbb{G}, \tau, H)$ has the decomposition property; in particular for Lie groups and for totally disconnected locally compact groups.

Thus we obtain the following corollary to Proposition 3.2.

Corollary 3.5. *Let $\mathbb{G}, \tau, H$ be as in Theorem 2.6. Let $\{\mu_t\}$ be (τ, α)-semistable in $\mathcal{M}^1(\mathbb{G})$ with idempotent $\mu_0 = \omega_H$. By Assumption (1) (cf. (3.2)) we have: $\mu_t(C(\tau) \cdot H) = 1$ for all $t \geq 0$. Then there exists a Ψ-lifting to a (ρ_2, α)-semistable continuous convolution semigroup $\{\nu_t = \Psi^{-1}(\mu_t)\}$ in $\mathcal{M}^1(\widetilde{C(\tau) \cdot H})$.*

And analogously, for semi-self decomposable or non-dissipating continuous convolution semigroups there exist Ψ-liftings to $\mathcal{M}^1(\widetilde{C(\tau) \cdot H})$.

The following observations will illustrate Assumption (1), (3.2):

Proposition 3.6. *Let $\mathcal{G}$ be Polish. Let $\{\mu_t\}$ be a continuous convolution semigroup in $\mathcal{M}^1(\mathcal{G})$ with $\mu_0 = \omega_H$, let $\{\nu_t\}$ be a a retract, i.e. $\mu_t = \nu_t \star \omega_H = \omega_H \star \nu_t$.*

a) $\nu_t(C(\tau)) = 1$ for all $t \geq 0 \Longrightarrow \mu_t(C(\tau) \cdot H) = 1$ for all $t \geq 0$.

b) Assume $\nu_0 = \varepsilon_e$. Let $\widetilde{\eta}$ denote the Lévy measure of $\{\nu_t\}$. Then we have: $\nu_t(C(\tau)) = 1$ for all $t \geq 0$ iff $\widetilde{\eta}(\complement C(\tau)) = 0$.

c) Assume $H \lhd \mathcal{G}$. Put $\{\overline{\mu}_t := \pi_H(\mu_t)\}$ (π_H denoting the quotient homomorphism $\mathcal{G} \to \mathcal{G}/H$). Let η denote the Lévy measure of $\{\mu_t\}$. Then we have: $\mu_t(C(\tau) \cdot H) = 1$ for all $t \geq 0$ iff $\overline{\mu}_t(\pi_H(C(\tau))) = 1$. This is the case iff $\eta(\complement(C(\tau) \cdot H)) = 0$.

d) Let $\{\mu_t\}$ be (τ, α)-semistable (hence concentrated on $C_H(\tau)$). If the retract $\{\nu_t\}$ is (τ, α)-semistable with idempotent $\nu_0 = \varepsilon_e$ then $\{\mu_t\}$ is concentrated on $C(\tau) \cdot H$.

Proof. a) is obvious since $\mu_t = \omega_H \star \nu_t = \nu_t \star \omega_H$.

b) follows by [27], corollary 4.5. $[\![C(\tau)$ is a measurable subgroup.$]\!]$

c) The investigations in [27] are concerned with continuous convolution semigroups with trivial idempotents only. Hence in order to apply these results we have to assume $H \triangleleft \mathcal{G}$ and pass to the quotient group $\mathcal{G}/H$. $\mathcal{G}/H$ is Polish.) Note that $\overline{\eta}(\complement\pi_H(C(\tau)\cdot H)) = \overline{\eta}(\complement\pi_H(C(\tau))) = \eta(\complement C(\tau)\cdot H)$, $\overline{\eta}$ denoting the Lévy measure of $\{\overline{\mu}_t\}$. (By Assumption (1), $C_H(\tau) = C(\tau)\cdot H$, hence $\pi_H(C_H(\tau)) = \pi_H(C(\tau))$.

d) $\{\nu_t\}$ is concentrated on $C(\tau)$, hence $\{\mu_t = \nu_t \star \omega_H\}$ is concentrated on $C(\tau)\cdot H$. □

Proposition 3.7. *Let C, H, be Polish groups, H compact. Let $\beta : H \to \mathrm{Aut}(C)$ be a homomorphism such that $C\times H \ni (x,\kappa) \mapsto \beta(\kappa)(x) \in C$ is continuous. Define $\mathcal{G} := C \rtimes_\beta H$. Let $\{\mu_t\} \subseteq \mathcal{M}^1(\mathcal{G})$ be a continuous convolution semigroup with idempotent ω_H.*

a) If there exist measures $\left\{\mu_t^\dagger\right\} \subseteq \mathcal{M}^1(C)$ such that $\mu_t = \mu_t^\dagger \otimes \omega_H$ for all $t \geq 0$, then $\left\{\mu_t^\dagger\right\}$ is a $\beta(H)$-invariant continuous convolution semigroup.

Conversely, if $\left\{\mu_t^\dagger\right\} \subseteq \mathcal{M}^1(C)$ is a $\beta(H)$-invariant continuous convolution semigroup, then $\left\{\mu_t = \mu_t^\dagger \otimes \omega_H\right\}$ is a continuous convolution semigroup in $\mathcal{M}^1(\mathcal{G})$ with idempotent $\mu_0 = \omega_H$.

In that case, $\left\{\mu_t^\dagger\right\}$ is uniquely determined by $\{\mu_t\}$. In fact, $\mu_t^\dagger = q(\mu_t)$, where q denotes the projection $(x,\kappa) \mapsto x$. And furthermore, we have $\mu_0^\dagger = \varepsilon_e$.

b) If $\rho \in \mathrm{Aut}(\mathcal{G})$ such that H and C are ρ-invariant, then $\{\mu_t\}$ is (ρ,α)-semistable iff $\left\{\mu_t^\dagger\right\}$ is $(\rho|_C,\alpha)$-semistable.

If in a) or b), C and H are considered as subgroups of $\mathcal{G}$, then the $\beta(H)$-invariant continuous convolution semigroup $\left\{\mu_t^\dagger\right\} \subseteq \mathcal{M}^1(\mathcal{G})$ is concentrated on C and $\mu_t^\dagger \otimes \omega_H$ is identified with $\mu_t = \mu_t^\dagger \star \omega_H$.

Proof. a) As immediately verified (and well known), in case of semidirect products we have: Let $\lambda \in \mathcal{M}^1(\mathcal{G})$. Then $\lambda = \lambda^\dagger \otimes \omega_H$ with $\beta(H)$-invariant $\lambda^\dagger \in \mathcal{M}^1(C)$ iff $\lambda = \lambda \star \omega_H = \omega_H \star \lambda$. (Cf. e.g., [10] Lemma 3.5.1.) And for such $\lambda_i = \lambda_i^\dagger \otimes \omega_H$, $i = 1,2$, we have:

$$\lambda_1 \star \lambda_2 = (\lambda_1^\dagger * \lambda_2^\dagger) \otimes \omega_H \tag{3.3}$$

($\star$ and $*$ denoting convolution on $\mathcal{G}$ and C respectively.) Whence the first assertion follows.

Let $q : (x,\kappa) \mapsto x$ denote the projection onto the homogeneous space $C \rtimes_\beta H/H$ (topologically isomorphic to C), put $q(\mu_t) =: \mu_t^\dagger$. μ_t is H-bi-invariant, i.e., $\varepsilon_{(e,\kappa)} \star \mu_t = \mu_t = \mu_t \star \varepsilon_{(e,\kappa)}$ for all $\kappa \in H$, whence $\beta(\kappa)(\mu_t^\dagger) = \mu_t^\dagger$ follows, and vice versa. As easily seen, $t \mapsto \mu_t$ is weakly continuous iff $t \mapsto \mu_t^\dagger$ is. Furthermore, by (3.3) it follows that $\left\{\mu_t^\dagger\right\}$ is a continuous convolution semigroup in $\mathcal{M}^1(C)$.

b) Let $\rho_1 := \rho|_C$ and $\rho_2 = \rho|_H$. For all $\kappa \in H$ we have

$$\beta(\rho_2(\kappa)) \circ \rho_1 = \rho_1 \circ \beta(\kappa) \tag{3.4}$$

and conversely, for any $\rho_1 \in \mathrm{Aut}(C)$, $\rho_2 \in \mathrm{Aut}(H)$ satisfying (3.4) it follows $\rho := \rho_1 \otimes \rho_2 \in \mathrm{Aut}(C \rtimes_\beta H)$.

Furthermore, $\rho_2(H) = H$ yields: $\mu_{\alpha t}^\dagger \otimes \omega_H = \mu_{\alpha t} = \rho(\mu_t) = \rho_1(\mu_t^\dagger) \otimes \rho_2(\omega_H) = \rho_1(\mu_t^\dagger) \otimes \omega_H$ iff $\rho_1(\mu_t^\dagger) = \mu_{\alpha t}^\dagger$.

If $\mu_t = \mu_t^\dagger \otimes \omega_H$ for all t, in particular, $\mu_0 = \omega_H = \mu_0^\dagger \otimes \omega_H$, then it follows easily that $\mu_0^\dagger = \varepsilon_e$. (Note that, if C and H are considered as subgroups of $\mathcal{G}$, we have $H \cap C = \{e\}$.) □

4. Main results

4.1. **The case of semidirect products,** $E = C(\tau) \cap H = \{e\}$**.** In that case we obtain in analogy to the Lie group case ([10], proposition 3.5.2) that (τ, α)-semistable laws on $\mathbb{G}$ with (non-trivial) idempotent ω_H are uniquely determined by $(\widetilde{\tau}, \alpha)$-semistable laws (with trivial idempotent) on the Polish contractible group $\widetilde{C}(\tau)$. Precisely, we have:

Theorem 4.1. *Let, as before,* $\mathbb{G}$ *be a Polish topological group,* $\tau \in \mathrm{Aut}(\mathbb{G})$*,* H *a compact* τ*-invariant subgroup. Assume* $E := C(\tau) \cap H = \{e\}$*. Hence* $\widetilde{C(\tau) \cdot H} = \widetilde{C}(\tau) \rtimes_{\widetilde{\beta}} H =: \mathcal{B}$ *is Polish. Let* $\Psi = \Phi = f \circ \phi : \widetilde{C}(\tau) \cdot H = \widetilde{C}(\tau) \rtimes_{\widetilde{\beta}} H \to C(\tau) \cdot H$ *and* $\varphi : \widetilde{C}(\tau) \to C(\tau)$ *denote the canonical homomorphisms of the Polish groups* $\mathcal{B}$ *resp.* $\widetilde{C}(\tau)$ *onto the measurable subgroups* $\Lambda = C(\tau) \cdot H \subseteq \mathbb{G}$ *resp. onto* $C(\tau)$*. Let* $q : \mathcal{B} = \widetilde{C}(\tau) \rtimes_{\widetilde{\beta}} H \to \widetilde{C}(\tau)$ *denote the projection onto the homogeneous space:* $(x, \kappa) \mapsto x$*. (According to Assumption (1) we restrict the considerations to* Λ^-*, hence we may assume w.l.o.g. that* $\Lambda = C(\tau) \cdot H$ *is dense in* $\mathbb{G}$*.)*
a) (i) Let $\{\mu_t\}$ *be a continuous convolution semigroup in* $\mathcal{M}^1(\mathbb{G})$ *with idempotent* $\mu_0 = \omega_H$*, which is concentrated on* $C(\tau) \cdot H$*. And let* $\{\nu_t := \Psi^{-1}(\mu_t)\}$ *denote the continuous convolution semigroup on the Polish group* $\mathcal{B}$*. Then* $\left\{\nu_t^\dagger := q(\nu_t)\right\}$ *is a* $\widetilde{\beta}(H)$*-invariant continuous convolution semigroup in* $\mathcal{M}^1(\widetilde{C}(\tau))$ *with trivial idempotent* $\nu_0 = \varepsilon_e$*. And* $\mu_t = \Psi\left(\nu_t^\dagger \otimes \omega_H\right) = \varphi(\nu_t^\dagger) \otimes \omega_H$*,* $t \geq 0$*.*

(ii) Conversely, let $\left\{\nu_t^\dagger\right\}$ *be a* $\widetilde{\beta}(H)$*-invariant continuous convolution semigroup in* $\mathcal{M}^1(\widetilde{C}(\tau))$ *with idempotent* $\mu_0^\dagger = \varepsilon_e$*. Then* $\left\{\mu_t := \Psi(\nu_t^\dagger \otimes \omega_H) = \varphi(\nu_t^\dagger) \otimes \omega_H\right\}$ *is a continuous convolution semigroup in* $\mathcal{M}^1(\mathbb{G})$ *concentrated on* $C(\tau) \cdot H$ *with idempotent* $\nu_0 = \omega_H$*.*

In either case, $\left\{\nu_t^\dagger\right\}$ *is uniquely determined by* $\{\mu_t\}$ *and vice versa.*
b) Moreover, $\{\mu_t\}$ *is* (τ, α)*-semistable on* $\mathbb{G}$ *(and concentrated on* $C(\tau) \cdot H$*, Assumption (1)), iff* $\left\{\nu_t^\dagger\right\}$ *is* $(\widetilde{\tau}, \alpha)$*-semistable on* $\widetilde{C}(\tau) \subseteq \mathcal{B}$*.*

Proof. Recall (Proposition 3.7): If $\nu_t = \nu_t^\dagger \otimes \omega_H$ for all t, in particular, $\nu_0 = \omega_H = \nu_0^\dagger \otimes \omega_H$, since $H \cap \widetilde{C}(\tau) = \{e\}$, it follows $\nu_0^\dagger = \varepsilon_e$.

(ii) According to Proposition 3.7, $\left\{\nu_t^\dagger \otimes \omega_H = \nu_t\right\}$ is a continuous convolution semigroup in $\mathcal{M}^1(\widetilde{C(\tau) \cdot H})$ and Ψ is a continuous bijective homomorphism onto $C(\tau) \cdot H$. Whence the assertion follows.

(i) Let $\{\mu_t\}$ be a continuous convolution semigroup concentrated on the measurable subgroup $C(\tau) \cdot H$ with idempotent ω_H. Then (Proposition 3.2) $\{\nu_t = \Psi^{-1}(\mu_t)\}$ is a continuous convolution semigroup in $\mathcal{M}^1(\widetilde{C(\tau) \cdot H})$ with idempotent ω_H. By assumption, $\widetilde{C(\tau) \cdot H} = \widetilde{C}(\tau) \rtimes_{\widetilde{\beta}} H$. Let $\nu_t^\dagger$ denote the projection of ν_t onto $\widetilde{C}(\tau)$. According to Proposition 3.7 a), $\nu_t^\dagger$ is uniquely determined, $\widetilde{\beta}(H)$-invariant and we have $\nu_t = \nu_t^\dagger \otimes \omega_H$, for $t \geq 0$.

As already mentioned, continuity of $\{\nu_t\}$, hence, according to Proposition 3.2, continuity of $\{\mu_t\}$, is equivalent with continuity of $\left\{\nu_t^\dagger\right\}$.

The last assertion b) follows by Proposition 3.7 b). □

Recall that the assumption $E = \{e\}$ holds if $(\mathbb{G}, \tau, H)$ has the strong decomposition property (cf. Remark 1.7). However, in contrast to the following Theorem 4.2, in the proof of Theorem 4.1 the decomposition property is not needed.

4.2. **The case $E = C(\tau) \cap H \neq \{e\}$.** As already shown in the case of Lie groups, the situation is less simple if $E := C(\tau) \cap H \neq \{e\}$ (cf. e.g., [10], theorem 3.5.4). In addition, our results in the general case are less complete as two further additional conditions are necessary (which are also satisfied in the Lie group case). These additional assumptions allow to reduce the problem without loss of information to a quotient group, for which Theorem 4.1 applies: Again, semistable laws with nontrivial idempotents are characterized by semistable laws with trivial idempotents on a Polish contractible group.

To formulate our main results, we need some preparations. Recall that in 2.3 we introduced notations which will be used in the sequel: We assume for the rest of this section Assumption (1), (3.2), in addition $E \neq \{e\}$, equivalently, $\Delta \neq \{e\}$, moreover

$$\textbf{Assumption } (2) \qquad E \lhd \mathbb{G}, \tag{4.1}$$

thus F, D and $\widetilde{D}$ are compact normal subgroups of $C(\tau) \cdot H$ (hence of $\mathbb{G}$), of $\overline{\mathcal{A}}$ and of $\overline{\mathcal{B}}$ respectively, and furthermore,

$$\textbf{Assumption } (3) \qquad C_F(\tau) = C(\tau) \cdot F. \tag{4.2}$$

I.e., Assumption (3) is satisfied iff $(\mathbb{G}, \tau, F)$ has the decomposition property. This is equivalent with $C_D(\rho_1) = C(\rho_1) \cdot D$, resp. $C_{\widetilde{D}}(\rho_2) = C(\rho_2) \cdot \widetilde{D} = \widetilde{C} \cdot \widetilde{D}$. ⟦In fact, $x \in C_{\widetilde{D}}(\rho_2)$ iff $\mathrm{LIM}\{{\rho_2}^k(x)\} \subseteq \widetilde{D}$, equivalently, iff $\Psi(x) \in C_F(\tau)$. (LIM denoting the set of accumulation points.) According to Assumption (3), then we have $\Psi(x) \in C(\tau) \cdot F$, whence $x \in \widetilde{C} \cdot \widetilde{D}$. ⟧

We have $C(\sigma) = \pi_{\widetilde{D}}(C_{\widetilde{D}}(\rho_2)) \supseteq \pi_{\widetilde{D}}(\widetilde{C} \cdot \widetilde{D}) = \pi_{\widetilde{D}}(\widetilde{C}) = C^*$.

1. Hence, by Assumption (3) (cf. (4.2)), we have $C(\sigma) = C^*$ and analogously, $\overline{C} = \pi(C_F(\tau)) = C(\overline{\tau})$, as mentioned in Notations 2.3 4.

2. Under the above assumptions we have: $C(\overline{\tau}) \cap H^* = \{\overline{e}\}$ (cf. Notations 2.3, 5.). This allows to reduce the investigations to Theorem 4.1. We obtain, again in analogy to the case of Lie groups (cf. [10], theorem 3.5.3):

Theorem 4.2. *Let $\mathbb{G}$ be a Polish topological group, with $\tau \in \mathrm{Aut}(\mathbb{G})$, and compact, τ-invariant subgroup H. Assume $E = C(\tau) \cap H \neq \{e\}$. With the above introduced notations and Assumptions (1), (2), (3) – again w.l.o.g. $\mathbb{G} = \Lambda^-$ – we have:*

a) Let $\{\mu_t\}$ be a continuous convolution semigroup in $\mathcal{M}^1(\mathbb{G})$ with idempotent $\mu_0 = \omega_H$ concentrated on $C(\tau) \cdot H$ (Assumption (1)). Then $\{\mu_t\}$ is uniquely determined by $\{\overline{\mu}_t := \pi(\mu_t)\}$, thus by $\left\{\overline{\lambda}_t = \Psi^{*-1}(\overline{\mu}_t)\right\}$ $\subseteq \mathcal{M}^1(\mathcal{B}_2)$. We have $C(\overline{\tau}) \cap H^* = \{\overline{e}\}$, hence $\mathcal{B}_2 = \widetilde{C}(\overline{\tau}) \rtimes_\gamma H^*$ $= C(\widetilde{\overline{\tau}}) \cdot H^*$, and therefore Theorem 4.1 applies: Thus $\{\mu_t\}$ is uniquely determined by the projection $\left\{\overline{q}(\overline{\lambda}_t) := \overline{\lambda}_t^\dagger\right\}$ onto $\widetilde{C}(\overline{\tau})$, and $\left\{\overline{\lambda}_t^\dagger\right\}$ is a

$\widetilde{\gamma}(H^*)$-invariant continuous convolution semigroup in $\mathcal{M}^1(\widetilde{C}(\overline{\tau}))$ with trivial idempotent.

b) And conversely, a $\widetilde{\gamma}(H^*)$-invariant continuous convolution semigroup $\left\{\overline{\lambda}_t^\dagger\right\}$ in $\mathcal{M}^1(\widetilde{C}(\overline{\tau}))$ with trivial idempotent defines a continuous convolution semigroup $\{\mu_t\}$ in $\mathcal{M}^1(\mathbb{G})$ with idempotent ω_H which is concentrated on $C(\tau)\cdot H$.

c) In case a) or b) we have: $\left\{\overline{\lambda}_t^\dagger\right\}$ is $(\widetilde{\overline{\tau}},\alpha)$-semistable iff $\{\mu_t\}$ is (τ,α)-semistable.

Proof. The proof runs along the following steps:
1. As mentioned in Proposition 3.4, $\pi:\mathbb{G}\to\mathbb{G}/F=\overline{\mathbb{G}}$ induces an isomorphism between $\{\lambda\in\mathcal{M}^1(\mathbb{G}):\ \omega_F\star\lambda=\lambda\star\omega_F=\lambda\}$ and $\mathcal{M}^1(\overline{\mathbb{G}})$.
2. As above mentioned, $\pi(C(\tau))=\pi(C_F(\tau))=C(\overline{\tau})$, and therefore $\pi(C(\tau)\cdot H)=\pi(C(\tau))\cdot\pi(H)=\pi(C_F(\tau))\cdot\pi(H)=C(\overline{\tau})\cdot H^*$.
3. Again mentioned above, $C(\overline{\tau})\cap H^*=\{\overline{e}\}$.

Therefore we have the means to finish the proof: According to step 1., $\{\mu_t\}$ is uniquely determined by $\{\overline{\mu}_t:=\pi(\mu_t)\}$, a continuous convolution semigroup in $\mathcal{M}^1(\overline{\mathbb{G}})$, and step 2. yields that that this continuous convolution semigroup is concentrated on $C(\overline{\tau})\cdot H^*$, thus uniquely determined by $\left\{\overline{\lambda}_t=\Psi^{*-1}(\overline{\mu}_t)\right\}\subseteq\mathcal{M}^1(\widetilde{C}(\overline{\tau})\rtimes_{\widetilde{\gamma}}H^*)$.

Applying Theorem 4.1, we obtain that $\overline{\lambda}_t=\overline{\lambda}_t^\dagger\otimes\omega_{H^*}$, where $\left\{\overline{\lambda}_t^\dagger\right\}$ is a uniquely determined $\widetilde{\gamma}(H^*)$-invariant continuous convolution semigroup in $\mathcal{M}^1(\widetilde{C}(\overline{\tau}))$.

The proof of the assertions b) and c) runs along the same lines in view of Proposition 3.7. □

Remarks 4.3. *a) In view of Propositions 2.4, 3.2 and Corollary 3.3 there also exist θ- and Ψ- liftings of $\{\mu_t\}$ to $\mathcal{M}^1(\overline{\mathcal{A}})$ and $\mathcal{M}^1(\overline{\mathcal{B}})$ and their projections resp. $\overline{\theta}$- and $\overline{\Psi}$- liftings of $\{\overline{\mu}_t\}$ to $\mathcal{M}^1(\mathcal{A}_1)$ and $\mathcal{M}^1(\mathcal{B}_1)$ respectively. And again, $\mathcal{B}_1=C(\sigma)\cdot K$ with $C(\sigma)\cap K=\{e\}$. But it could not be shown that $C(\sigma)\cong\widetilde{C}(\overline{\tau})$, hence we had to use the representation as lifting in $\mathcal{M}^1(\mathcal{B}_2)$. But $C(\sigma)$, $C(\overline{\tau})$ and $\widetilde{C}(\overline{\tau})$ are Borel isomorphic, and thus the existence of a disintegration $\left\{\overline{\lambda}_t=\overline{\lambda}_t^\dagger\otimes\omega_{H^*}\right\}$ in $\mathcal{M}^1(\mathcal{B}_2)$ implies an analogous disintegration $\left\{\overline{\nu}_t^\dagger\otimes\omega_K\right\}$ in $\mathcal{M}^1(\mathcal{B}_1)$. (Since $C(\sigma)=C^*$ is complete, according to Remark 1.5, there exists a continuous bijective homomorphism $\delta:\widetilde{C}(\overline{\tau})\to C(\sigma)$, hence $\overline{\nu}_t^\dagger=\delta(\overline{\lambda}_t^\dagger)$.)*

b) Assumptions (1)–(3), (cf. (3.2), (4.1) and (4.2)), are satisfied for Lie groups $\mathbb{G}$. Indeed, (3.2) and (4.2) follow since then $(\mathbb{G},\tau)$ has the decomposition property for all $\tau\in\mathrm{Aut}(\mathbb{G})$. And (4.1) is proved e.g., in [10], *3.2.20. Analogously, for totally disconnected locally compact groups $\mathbb{G}$, Assumptions (1) and (3) hold true. (Cf. Remark 1.7.)*

c) If $\mathbb{G}$ is a Lie group the situation is considerably simple ([10], *§ 3.2 II, 3.2 III, § 3.5.): As mentioned (Remark 1.5), $C(\tau)$, Γ and $\widetilde{C}(\tau)$ are nilpotent, connected. (It is easy to reduce the situation to the case of connected groups,* [10], *3.2.8). D, E and $\widetilde{E}$ are connected, $E\subseteq D\subseteq\mathrm{Cent}(\Gamma_0)$ (*[10], *3.2.20) and thus $\widetilde{E}$ is a connected subgroup of the centre of $\widetilde{C}(\tau)$, hence isomorphic to a vector space.*

d) As mentioned in Remark 1.5 g), in the Lie group case, the existence of a continuous bijective homomorphism yields that the Lie algebras and the simply connected covering groups coincide. Since C^* is closed in $\mathcal{B}_1$ (cf. Notations 2.3, 4.), $\delta : \widetilde{C}(\overline{\tau}) \to C^*$ is a bijective Lie group homomorphism and $\widetilde{C}(\overline{\tau})$ is simply connected. Thus $\widetilde{C}(\overline{\tau}) \cong C^*$.

Furthermore we note: Let p denote the continuous homomorphism $\Lambda \supseteq C(\tau) \to C(\overline{\tau}) \subseteq \overline{\Lambda}$ defined by the restriction $p := \pi|_{C(\tau)}$. Then p defines a homomorphism of the underlying Lie algebras and hence of the covering groups, $\widetilde{p} : \widetilde{C}(\tau) \to \widetilde{C}(\overline{\tau})$. We have $\ker p = E, \ker \widetilde{p} = \widetilde{E}$.

e) More generally, if we assume $F \lhd \mathbb{G}$ (Assumption (2)) and if $\mathbb{G}$ is locally compact and connected, then $F \subseteq \mathrm{Cent}(\mathbb{G})$. ⟦[11], *(26.10).* ⟧

Note that, if $F \subseteq \mathrm{Cent}(\mathbb{G})$, then $\beta(\kappa)|_F$ is trivial for all $\kappa \in H$. This is in particular the case for connected Lie groups.

The following results enable us to enlarge slightly the class of examples satisfying our assumptions. Throughout let $\mathbb{G}$ be a locally compact second countable group, $\tau \in \mathrm{Aut}(\mathbb{G})$ and let $H, L_i, i \geq 1$, be compact τ-invariant subgroups, such that $L_i \lhd \mathbb{G}$ and $L_i \downarrow \{e\}$. Assume that $C_H(\tau)$ is dense in $\mathbb{G}$, and denote by $\pi_i : \mathbb{G} \to \mathbb{G}_i := \mathbb{G}/L_i$ the quotient homomorphisms and by $\tau_i \in \mathrm{Aut}(\mathbb{G}_i)$ the induced automorphisms. Finally put $H_i := \pi_i(H)$.

Proposition 4.4. *a)* $C_H(\tau) = \bigcap_i \pi_i^{-1}(C_{H_i}(\tau_i))$, *in particular,* $C(\tau) = \bigcap_i \pi_i^{-1}(C(\tau_i))$.

b) $C_H(\tau) = C(\tau) \cdot H$ *iff for all i,* $C_{H_i}(\tau_i) = C(\tau_i) \cdot H_i$.

c) $E = C(\tau) \cap H \lhd \mathbb{G}$ *iff for all i,* $E_i = C(\tau_i) \cap H_i \lhd \mathbb{G}_i$.

⟦ a) The proof is analogous to [10], 3.1.22, 3.1.22*.

b) resp. c) follow immediately by a) and the (obvious) representation $H = \bigcap_i \pi_i^{-1}(H_i)$. ⟧

Corollary 4.5. *Let $\mathbb{G}, \tau, H, L_i$ be as above. Assume moreover that $\mathbb{G}/L_i$ are Lie groups, $i \in \mathbb{N}$. Then Assumption (2) is satisfied and, as $(\mathbb{G}, \tau)$ has the decomposition property, also assumption (3). (A particular case of* [20]*, corollary 4.11.)*

Finally we note an induction principle for groups with decomposition property:

Proposition 4.6. *Let $\mathbb{G}$ be a locally compact group, $\tau \in \mathrm{Aut}(\mathbb{G})$, and let H, N be compact τ-invariant subgroups with $N \subseteq H$ and $N \lhd \mathbb{G}$. Let $\pi : \mathbb{G} \to \mathbb{G}/N$ resp. $\overline{\tau} \in \mathrm{Aut}(\mathbb{G}/N)$ denote the quotient homomorphism resp. the induced automorphism. Then we have:*

If $(\mathbb{G}, \tau, N)$ and $(\mathbb{G}/N, \overline{\tau}, H/N)$ have the decomposition property then $(\mathbb{G}, \tau, H)$ shares this property.

⟦ According to [10], 3.2.3 and by the second assumption we have $C_H(\tau) = \pi^{-1}(C_{H/N}(\overline{\tau})) = \pi^{-1}(C(\overline{\tau}) \cdot H/N)$. Since $\pi^{-1}(C(\overline{\tau})) = C_N(\tau)$, this equals $C_N(\tau) \cdot H$. By the first assumption, $C_N(\tau) \cdot H = C(\tau) \cdot N \cdot H = C(\tau) \cdot H$. ⟧

5. Examples

The following examples will illustrate our results:

Example 5.1. *An example $(\mathbb{G}, \tau, H)$ without decomposition property. (Cf.* [17]*, example 4.1.)*

Let $\mathbb{T}$ denote the one-dimensional torus, let $\mathbb{G} = \bigotimes_{\mathbb{Z}} \mathbb{T}$ endowed with *the product topology, hence a compact Abelian group.* $\vec{x} \in \mathbb{G}$ *is represented as function* $\mathbb{Z} \ni k \mapsto x(k) = \mathrm{e}^{2\pi i \xi(k)}$, $-1/2 \le \xi(k) < 1/2 \pmod 1$ *with unit* $\vec{e} : k \mapsto 1$. *Obviously we have* $C(\tau) = \{\vec{x} : \lim_{n\to\infty} x(n) = 1\}$. $H := \{\vec{x} : x(n) \equiv x(0)\} \cong \mathbb{T}$ *is a compact,* τ*-invariant subgroup, and we have* $C_H(\tau) = \{\vec{x} : \mathrm{LIM}\,\{\tau^n(\vec{x})\} \subseteq H\}$ *(*LIM *denotes the set of accumulation points), and* $C_H(\tau) = \{\vec{x} : \exists \vec{y}_n \in H : \tau^n(\vec{x})\vec{y}_n^{\,-1} \to \vec{e}\}$. *Furthermore,* $C(\tau)\cdot H = \{\vec{x} : \lim x(n) =: x(\infty)$ *exists*$\}$. *Hence* $\vec{x} \in C(\tau)\cdot H$ *is canonically decomposable as* $\vec{x} = \vec{y} \cdot \vec{h}$ *with* $\vec{y} \in C(\tau)$, $y(n) = x(n) \cdot x(\infty)^{-1}$ *with* $y(n) \to 1$ *and* $\vec{h} : n \mapsto x(\infty)$ $(\in H)$.

In [17] *it is shown that, e.g., for* $f : n \mapsto \sum_1^n 1/j$, *the element* $\vec{x} : n \mapsto x(n) := \mathrm{e}^{2\pi i f(n)}$ *for* $n \ge 1$ *and* $x(n) = 1$ *for* $n \le 0$, *belongs to* $C_H(\tau)\backslash C(\tau)\cdot H$. *(As easily seen, we may replace the function* f *by* $n \mapsto \log n$, $n \ge 1$.*)*

Example 5.2. *The (intrinsic) topology of* $\widetilde{C}(\tau)$. *We continue Example 5.1:*

Let e.g., $\epsilon > 0$, $U = U^\epsilon := \{\vec{x} : |\xi(0)| < \epsilon\}$. *Then we have:*
$U_{(n)} = \bigcap_{k\le n} \tau^k(U) = \{\vec{x} : |\xi(-k)| < \epsilon, k \le n\} = \{\vec{x} : |\xi(k)| < \epsilon, k \ge -n\}$,
and $U_n = U_n^\epsilon = \{\vec{x} : \xi(k) \to 0$ *and* $|\xi(k)| < \epsilon, k \ge -n\}$.

The finite intersections of $\{U_n^\epsilon\}$ *form a neighbourhood basis of the unit in* $\widetilde{C}(\tau)$. *With other words,* $\vec{x}_n \to \vec{x}$ *in* $\widetilde{C}(\tau)$ *iff (i) for all* n, $\xi_n(k) \overset{k\to\infty}{\to} 0$ *and (ii)* $\xi(k) \overset{k\to\infty}{\to} 0$, *(i.e.,* $\vec{x}, \vec{x}_n \in C(\tau)$*) and moreover, (iii) for all* $N \in \mathbb{Z}$, $\sup_{k\ge N} |x_n(k) - x(k)| \overset{n\to\infty}{\to} 0$, *equivalently,* $\sup_{k\ge N} |\xi_n(k) - \xi(k)| \overset{n\to\infty}{\to} 0$.

Example 5.3. *The (intrinsic) topology of* $\widetilde{C(\tau)\cdot H}$.
Note first that in example 5.1, $C(\tau) \cap H = \{e\}$, *hence* $\mathcal{A} = C(\tau) \rtimes_\beta H$ $= \mathbb{C}(\tau) \otimes H$ *and also* $\mathcal{B} = \widetilde{C(\tau)\cdot H} = \widetilde{C}(\tau) \otimes H$ *(since* β *and* $\widetilde{\beta}$ *are trivial.)*

Thus the topology of $\mathcal{A}$ *is characterized by:* $\vec{x}_m \to \vec{x}$ *iff (i) for all* m, $x_m(k) \overset{k\to\infty}{\to} x_m(\infty)$, *(ii)* $x(k) \overset{k\to\infty}{\to} x(\infty)$, *(iii) for all* k, $x_m(k) \overset{m\to\infty}{\to} x(k)$ *and (iv)* $x_m(\infty) \overset{m\to\infty}{\to} x(\infty)$. ⟦*In fact, with the representations* $\vec{x}_n = \vec{y}_n\cdot\vec{h}_n$, $\vec{x} = \vec{y}\cdot\vec{h}$, *with* $\vec{y}_n, \vec{y} \in C(\tau)$ *and* $\vec{h}_n, \vec{h} \in H$ *we have:* $\vec{x}_n \to \vec{x}$ *iff* $\vec{y}_n \to \vec{y}$ *in* $C(\tau)$ *and* $\vec{h}_m \to \vec{h}$ *in* H, *equivalently,* $x_m(\infty) \to x(\infty)$. ⟧

And analogously, $\vec{x}_m \to \vec{x}$ *in* $\widetilde{C(\tau)\cdot H}$ *iff* $\vec{y}_m \to \vec{y}$ *in* $\widetilde{C}(\tau)$ *(cf. Example 5.2) and* $\vec{h}_m \to \vec{h}$ *(in* H*), i.e.,* $x_m(\infty) \to x(\infty)$.

Example 5.4. *Existence of semistable laws concentrated on* $C(\tau)$.
For all $0 < \alpha < 1$ *there exist* (τ, α)*-semistable continuous convolution semigroups in* $\mathcal{M}^1(\mathbb{G})$ *with trivial idempotent, hence concentrated on* $C(\tau)$. *(And therefore, there exists a semistable lifting to* $\widetilde{C}(\tau)$.*)*

a) Let $\vec{x}_0 \in C(\tau)$ *be defined by* $x_0(n) = \mathrm{e}^{2\pi i \xi_0(n)}$ *with* $|\xi_0(n)| \le \beta^n$ *for some* $0 < \beta < \sqrt{\alpha}$, *and* $n \mapsto |\xi_0(n)|$ *decreasing. Put, for* $N \in \mathbb{N}$, $\eta_N := \sum_{k=-\infty}^{N} \alpha^{-k}\tau^k((\varepsilon_{\vec{x}_0} + \varepsilon_{\vec{x}_0^{-1}})/2)$ *and* $c_N := ||\eta_N|| = \sum_{-\infty}^N \alpha^{-k}$. *The measures* η_N *converge vaguely to a (unbounded) positive measure* η *on* $\mathbb{G}\backslash\{\vec{e}\}$, *which will turn out to be the Lévy measure of a continuous convolution semigroup. Indeed, we show that the Fourier-transforms of* $\lambda_N := \eta_N - c_N \cdot \varepsilon_{\vec{e}}$ *converge to a function* $f : \mathbb{Z} \to \mathbb{R}$: ⟦*Let* $\vec{k} = (k_\ell) \in \widehat{\mathbb{G}} = \bigotimes^* \mathbb{Z}$ *(i.e.,* $k_\ell \in \mathbb{Z}\backslash\{0\}$ *finitely often), and* $\langle \vec{k}, \vec{x}\rangle = \mathrm{e}^{2\pi i \sum_\ell k_\ell \xi(\ell)}$. *Then* $\widehat{\lambda}_N(\vec{k}) = \sum_{-\infty}^N \alpha^{-k}\left(\cos(2\pi \sum_\ell k_\ell \xi(k + k_\ell)) - 1\right)$ *is* *absolutely convergent since* $|\cos(2\pi\sum_\ell k_\ell\xi(k+k_\ell))-1| < C_1\beta^{2k}$ *for sufficiently large* k, *where* C_1 *is a constant depending on* $\vec{k}$.$\rrbracket$ *Thus* $\mathrm{e}^{t\widehat{\lambda}_N} \to \mathrm{e}^{tf}$. *And therefore there exists a continuous convolution semigroup* $\{\mu_t\}$ *with trivial idempotent and Lévy measure* η *such that* $\mu_t^{(N)} \to \mu_t$, *where* $\mu_t^{(N)} := \exp t\lambda_N$.

Furthermore, $\tau(\eta_N) = \alpha\cdot\eta_{N+1}$, *for all* N, *yields* $\tau(\mu_t) = \mu_{\alpha t}$, $t \geq 0$.

b) A further example proving existence of semistable laws is constructed as follows: Let $\sigma \in \mathcal{M}^1(\mathbb{T})$ *be symmetric.* σ *is considered as probability supported by* $\mathbb{T}_0 \subseteq \bigotimes_k \mathbb{T}_k$ *(with* $\mathbb{T}_k \cong \mathbb{T}$ *for all* k*). Let, for* $0 < \alpha < 1$, $\eta := \sum_{-\infty}^{\infty}\alpha^{-k}\tau^k(\sigma)$, *and again as in a),* $\lambda_N := \sum_{-\infty}^{N}\alpha^{-j}\left(\tau^j(\sigma)-\varepsilon_{\vec{e}}\right)$. *For* $\vec{k} \in \widehat{\mathbb{G}}$, *put* $\tau^*(\vec{k}) : \ell \mapsto k_{\ell+1}$. *Then* $\widehat{\lambda}_N(\vec{k}) = \sum_{-\infty}^{N}\alpha^{-j}\left(\widehat{\sigma}(\tau^{*j}(\vec{k}))-1\right)$. *Obviously the Fourier-transforms of* λ_N *converge for all* $\vec{k} \in \widehat{\mathbb{G}}$ *(since we have only a finite number of non-zero summands). Hence the limit generates a continuous convolution semigroup* $\{\mu_t\}$. *Note that in this example, the measures* μ_t *are infinite convolution products,* $\mu_t = \mathop{\bigstar}_{k\in\mathbb{Z}} \exp t\alpha^{-k}\left(\tau^k(\sigma)-\varepsilon_e\right)$ *with factors concentrated on* $\mathbb{T}_k$.

As above, it follows easily that $\tau(\lambda_N) = \alpha\cdot\lambda_{N+1}$, *hence* $\{\mu_t\}$ *is semistable.*

Example 5.5. *Existence of semistable laws with idempotent* ω_H *concentrated on* $C(\tau)\cdot H$.

Let $\{\mu_t\}$ *be defined as in Example 5.4 (with trivial idempotent). Then* $\{\nu_t := \mu_t \star \omega_H\}$ *is a suitable* (τ,α)*-semistable continuous convolution semigroup concentrated on* $C(\tau)\cdot H$. *(Cf. Proposition 3.6 a)).*

The next example shows that the situation $E \neq \{e\}$ may appear, and moreover, it shows that the results of Theorem 4.2 can not essentially be improved: The construction there is rather complicated, at the first glance one could expect to find a characterization of (τ,α)-semistable continuous convolution semigroups $\{\mu_t\}$ with non-trivial idempotents by semistable laws on $\widetilde{C}(\tau)$ (avoiding the projections π_D and $\pi_{\widetilde{D}}$). I.e., to find a $(\widetilde{\tau},\alpha)$-semistable continuous convolution semigroup $\left\{\mathring{\nu}_t\right\} \subseteq \mathcal{M}^1(\widetilde{C}(\tau))$ with $\mu_t = \varphi(\mathring{\nu}_t)\star\omega_H$. If so, we had a bijective mapping $T : \left\{\mathring{\nu}_t\right\} \mapsto \{\mu_t\} \mapsto \left\{\overline{\nu}_t^\dagger\right\}$, to a $(\widetilde{\overline{\tau}},\alpha)$-semistable continuous convolution semigroup $\left\{\overline{\nu}_t^\dagger\right\}$ in $\mathcal{M}^1(\widetilde{C}(\overline{\tau}))$. The mapping $T : (\left\{\mathring{\nu}_t\right\}) \mapsto \left\{\overline{\nu}_t^\dagger\right\}$, is induced by a homomorphism $\widetilde{p} : \widetilde{C}(\tau) \to \widetilde{C}(\overline{\tau})$. By construction, $\widetilde{C}(\tau)$ and $\widetilde{C}(\overline{\tau})$ are the covering groups of $C(\tau)$ and $C(\overline{\tau})$:

Example 5.6. *We sketch a counter-example, the details are found (with slightly different notations) e.g., in* [9], [10], *example 3.5.6.*

Let $\mathbb{G} = \mathbb{T}^4$ *(a Lie group, hence satisfying Assumptions (1), (2) and (3)). Define* $\tau : (z_1,z_2,z_3,z_4) \mapsto (z_2z_3, z_1z_2z_4, z_4, z_3z_4)$ *and* $H := \mathbb{T}^2\times\{1\}\times\{1\} \cong \mathbb{T}^2$. *Then we have:* $C(\tau) = \{(\mathrm{e}^{is},\mathrm{e}^{ias},\mathrm{e}^{it},\mathrm{e}^{iat}) : s,t\in\mathbb{R}\}$ *with* $a = (1-\sqrt{5})/2$. *Furthermore,* $C_H(\tau) = \mathbb{T}^2\times\{(\mathrm{e}^{is},\mathrm{e}^{ias}) : s\in\mathbb{R}\}$ $= C(\tau)\cdot H$. *Hence* $E = H\cap C(\tau) = \{(\mathrm{e}^{is},\mathrm{e}^{ias}) : s\in\mathbb{R}\}\times\{1\}\times\{1\}$ *and thus* $F = D = H$.

We have $\widetilde{C}(\tau) \cong \mathbb{R}^2$, $\varphi(s,t) = (\mathrm{e}^{is},\mathrm{e}^{ias},\mathrm{e}^{it},\mathrm{e}^{iat})$, *and it can be shown that* $\widetilde{\tau}$ *is the linear operator* $(s,t) \mapsto (s+at, at)$.

Applying π ($=\pi_D=\pi_H$ in our case) we obtain: $\overline{\mathbb{G}}\cong\mathbb{T}^2$, and $\overline{\tau}:(z_1,z_2)\mapsto(z_2,z_1z_2)$. Therefore, $\widetilde{C}(\overline{\tau})\cong\mathbb{R}$, $\widetilde{\overline{\tau}}$ is the homothetical transformation $t\mapsto a\cdot t$, and we have $\varphi^*=\Psi^*|_{\widetilde{C}(\overline{\tau})}:t\mapsto(\mathrm{e}^{it},\mathrm{e}^{iat})\in\overline{\mathbb{G}}$.

Furthermore, $\widetilde{E}=\varphi^{-1}(E)=\varphi^{-1}(\{(\mathrm{e}^{is},\mathrm{e}^{ias},1,1):s\in\mathbb{R}\})=\{(s,0)\}$, $\widetilde{E}\cong\mathbb{R}$. The projection $\widetilde{p}:\widetilde{C}(\tau)\to\widetilde{C}(\overline{\tau})$, is given by $(s,t)\mapsto(0,t)$, hence $\ker\widetilde{p}=\widetilde{E}$ and $\varphi^*\widetilde{p}=\pi\varphi$. (Cf. Remark 4.3, d).)

(1) It is not surprising that the afore defined mapping T is not injective: If $\left\{\overset{\circ}{\nu}_t\right\}$ is a continuous convolution semigroup concentrated on the $\widetilde{\tau}$-invariant subspace $\widetilde{E}=\ker\widetilde{p}$, then we obtain $\overline{\nu}_t^\dagger\equiv\varepsilon_e$ and thus $\overline{\nu}_t^\dagger\equiv\varepsilon_e$.

(2) T is not surjective (for semistable laws): Let $\left\{\overline{\nu}_t^\dagger\right\}$ be Gaussian on $\mathbb{R}$, hence $(\widetilde{\overline{\tau}},\alpha)$-semistable. Assume that $\left\{\overset{\circ}{\nu}_t\right\}$ is $(\widetilde{\tau},\alpha)$-semistable on $\mathbb{R}^2$ and $\widetilde{p}(\overset{\circ}{\nu}_t)=\overline{\nu}_t^\dagger$. Then $\overset{\circ}{\nu}_t$ has a non-trivial Gaussian part. Since the only $\widetilde{\tau}$-invariant subspaces are $\widetilde{E}=\ker\widetilde{p}$ and $\mathbb{R}^2$, $\overset{\circ}{\nu}_t$ must be full Gaussian. But a full Gaussian law on $\mathbb{R}^2$ cannot be $(\widetilde{\tau},\alpha)$-semistable for the particular operator $\widetilde{\tau}$, a contradiction.

The next example shows that in general $\widetilde{C}(\tau)\not\cong\widetilde{C}$:

Example 5.7. *Let $\mathbb{G}=\mathbb{T}^4$ and τ as in Example 5.6. We have $\mathcal{A}=C(\tau)\otimes H=\{(\mathrm{e}^{is},\mathrm{e}^{ias},\mathrm{e}^{it},\mathrm{e}^{iat},u,v):s,t\in\mathbb{R},u,v\in\mathbb{T}\}$, and $\mathcal{B}=\widetilde{C}(\tau)\otimes H=\{(s,t,u,v):s,t\in\mathbb{R},u,v\in\mathbb{T}\}\cong\mathbb{R}^2\otimes\mathbb{T}^2$. Furthermore, $\Delta=\{(\mathrm{e}^{is},\mathrm{e}^{ias},1,1,\mathrm{e}^{-is},\mathrm{e}^{-ias}):s\in\mathbb{R}\}$ (closed in $\mathcal{A}$) and $\widetilde{\Delta}=\{(s,0,\mathrm{e}^{-is},\mathrm{e}^{-ias}):s\in\mathbb{R}\}$ (closed in $\mathcal{B}$). Hence $\overline{\mathcal{A}}=\mathcal{A}/\Delta=\{(1,1,\mathrm{e}^{it},\mathrm{e}^{iat},x,y):t\in\mathbb{R},x,y\in\mathbb{T}\}$ and $\overline{\mathcal{B}}=\mathcal{B}/\overline{\Delta}=\{(0,t,x,y):t\in\mathbb{R},x,y\in\mathbb{T}\}=\mathbb{R}\otimes\mathbb{T}^2$, a Polish group. Finally, $\widetilde{C}=\pi_{\widetilde{\Delta}}(\widetilde{C}(\tau)\otimes\{e\})=\{(0,t,\mathrm{e}^{is},\mathrm{e}^{ias}):s,t\in\mathbb{R}\}$ (not closed in $\overline{\mathcal{B}}$). (Note that $(\widetilde{C}(\tau)\otimes\{e\})\cdot\widetilde{\Delta}=\left\{(s,t,1,1)(s',0,\mathrm{e}^{-is'},\mathrm{e}^{-ias'})\right\}=\left\{(s+s',t,\mathrm{e}^{-is'},\mathrm{e}^{-ias'}):s,t,s'\in\mathbb{R}\right\}$ is not closed in $\mathcal{B}$.)*

Hence $h:\mathbb{R}^2\cong\widetilde{C}(\tau)\to\widetilde{C}$ is not an isomorphism.

The construction of the intrinsic topologies could be simplified if the groups Λ and $\mathcal{A}$ (in case $E=\{e\}$) resp. $\overline{\mathcal{A}}$ (in case $E\neq\{e\}$) were identified, and f resp. θ were replaced by *id*. However, even for semi-direct products, the product map $C\rtimes H\to C\cdot H$ is bijective, continuous, but in general not open, hence the inverse is not continuous:

Example 5.8. *Choose $\mathbb{G}=\mathbb{T}^4$ and $C:=\{(\mathrm{e}^{it},\mathrm{e}^{iat},\mathrm{e}^{is},\mathrm{e}^{ias}):s,t\in\mathbb{R}\}$ as in Example 5.6. The subgroup $K:=\{(u,u,v,v):u,v\in\mathbb{T}\}$ ($\cong\mathbb{T}^2$) is compact, $C\cap K=\{e\}$ and $C\cdot K=\mathbb{T}^4$. But the topology of $\mathbb{T}^4$ and that of $C\otimes K$ are not identical since C is not closed in $\mathbb{T}^4$.*

Conjectures and problems

1.) Do there exist examples of semistable laws on groups, hence concentrated on $C_H(\tau)$, but $\mu_t(C(\tau)\cdot H)\neq 1$? ⟦The methods of Example 5.4 a) do not work in that case: Let $\xi_n=\log(n)(\mathrm{mod}\ 1)$, $\vec{x}_1:n\mapsto\mathrm{e}^{2\pi i\xi(n)}$ and define η_N,λ_N as before, in Example 5.4 a), replacing $\vec{x}_0$ by $\vec{x}_1$ (with $\vec{x}_1\in C_H(\tau)\backslash C(\tau)\cdot H$). However, then $\widehat{\lambda}_N(\vec{k})$ is not convergent for all $\vec{k}\in\widehat{\mathbb{G}/H}$, i.e., for $\vec{k}$ with $\sum k_\ell=0$. ⟧

2.) Is $C^*\cong\widetilde{C}(\overline{\tau})$ (and hence $\mathcal{B}_2\cong\mathcal{B}_1$) true also for non-Lie groups? ⟦For Lie groups cf. Remarks 4.3. E.g., with the notations of Example

5.6, 5.7, we obtain: $D \cong H \cong \widetilde{D} = \{(0,0,x,y) : x,y \in \mathbb{T}\}$. Hence $C^* = \pi_{\widetilde{D}}(\widetilde{C}) = \{(0,t,\mathrm{e}^{-is},\mathrm{e}^{-ias}) : t,s \in \mathbb{R}\}/\widetilde{D} \cong \mathbb{R} \cong \widetilde{C}(\overline{\tau})$.]]

3.) Do we have $E \lhd \mathbb{G}$ also for non-Lie groups?

4.) Do there exist characterizations of the intrinsic topology $\widetilde{C(\tau)\cdot H}$ similar to Theorem 2.6 e) (but less complicated)?

5.) Find examples of locally compact totally disconnected groups $\mathbb{G}$ with $C_H(\tau) = C(\tau)\cdot H$ and $C(\tau)\cap H \neq \{e\}$ (in particular, $C(\tau)$ not closed).

6.) Find characterizations of (non-compact) locally compact groups $\mathbb{G} = C_H(\tau)^-$ (hence $\mathbb{G} = C_K(\tau)$ for some $K \supseteq H$) such that $(\mathbb{G},\tau)$ fails to have the decomposition property. [[As mentioned in Remark 1.7, [20], proposition 4.5, theorem 4.4 and corollary 4.11 contain sufficient conditions for $(\mathbb{G},\tau)$ to have the decomposition property. This is the case e.g., if $\mathbb{G}$ is locally compact, and possesses compact τ-invariant normal subgroups L_i with $\bigcap L_i = \{e\}$, such that $L_i \subseteq \mathbb{G}_0$ and G_0/L_i is a Lie group. But in example 4.10 it is shown that these conditions are not necessary.]]

References

[1] **Baumgartner, U., Willis, G.A.:** *Contraction groups and scales of automorphisms of totally disconnected locally compact groups.* Israel J. Math. **142**, 221–248 (2004)

[2] **Byczkovski,T., Hulanicki, A.:** *Gaussian measure of normal subgroups.* Ann. Probab. **11** (1983), 685–691

[3] **Byczkovski,T., Żak, T.:** *Decomposition of convolution semigroups of probability measures on groups.* In: Probability Measures on Groups VII, Proceedings Oberwolfach (1983). H. Heyer, ed. Lecture Notes Math. **1064**, 23–35 (1984)

[4] **Ellis, R.:** *Locally compact transformation groups.* Duke Math. J. **24**, 119–126 (1957)

[5] **Dani, S.G., Shah, R.:** *Contraction subgroups and semistable measures on p-adic Lie groups.* Math. Proc. Camb. Phil. Soc. **110**, 299–306 (1991)

[6] **Glöckner, H.:** *Contraction groups for tidy automorphisms of totally disconnected groups.* Glasgow Math. J. **47**, 329–333 (2005)

[7] **Hazod, W., Scheffler, H-P.:** *Strongly τ-decomposable and selfdecomposable laws on simply connected nilpotent groups.* Mh. Math. **128**, 269–282 (1999)

[8] **Hazod, W., Siebert, E.:** *Continuous automorphism groups on a locally compact group contracting modulo a compact subgroup and applications to stable convolution semigroups.* Semigroup Forum **33**, 111-143 (1986)

[9] **Hazod, W., Siebert, E.:** *Automorphisms of a Lie group contracting modulo a compact subgroup and applications to semistable convolution semigroups.* J. Theoret. Probab. **1**, 211-226 (1988)

[10] **Hazod, W., Siebert, E.:** *Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups. Structural Properties and Limit Theorems.* Mathematics and its Applications vol. **531**. Kluwer A. P. (2001)

[11] **Hewitt, E. Ross, K.A..:** *Abstract Harmonic Analysis I.* Berlin-Heidelberg-New York. Springer (1977)

[12] **Heyer, H.:** *Probability Measures on Locally Compact Groups.* Berlin-Heidelberg-New York. Springer (1963)

[13] **Jaworski, W., Rosenblatt, J., Willis, G.:** *Concentration functions in locally compact groups.* Math. Ann. **305**, 673–691 (1996)

[14] **Jaworski, W.:** *Contractive automorphisms of locally compact groups and the concentration function problem.* J. Theoret. Probab. **10**, 967–989 (1997)

[15] **Jaworski, W.:** *On shifted convolution powers and concentration functions in locally compact groups.* Proceedings Gainsville (1999), Contemp. Math. **261**, 23–41 (2000)

[16] **Jaworski, W.:** *On contraction groups of automorphisms of totally disconnected locally compact groups.* Israel J. Math. **172**, 1–8 (2009)

[17] **Jaworski, W.:** *Contraction groups, ergodicity, and distal properties of automorphisms of compact groups* Manuscript (2011)

[18] **Lashof, R.:** *Lie algebras of locally compact groups.* Pacific J. Math. **7**, 1145–1162 (1957)

[19] **Parthasarathy, K.R.:** *Probability measures on metric spaces.* Academic Press (1967)

[20] **Raja, C.R.E., Shah, R.:** *Some properties of distal actions on locally compact groups.* Manuscript (2012)

[21] **Schaefer, H. H.:** *Topological Vector Spaces.* 4^{th} edition, Springer: Berlin, Heidelberg-New York (1980)

[22] **Shah, R.:** *Semistable measures and limit theorems on real and p-adic Lie groups.* Mh. Math. **115**, 191–213 (1993)

[23] **Shah, R.:** *Selfdecomposable measures on simply connected nilpotent Lie groups.* J. Theoret. Probab. **13**, 65–83 (2000)

[24] **Siebert, E.:** *Jumps of stochastic processes with values in a topological group.* Prob. Math. Stat. **5**, 197–209 (1985)

[25] **Siebert, E.:** *Contractive automorphisms on locally compact groups.* Math. Z. **191**, 73–90 (1986)

[26] **Siebert, E.:** *Semistable convolution semigroups and the topology of contraction groups.* In: Probability Measures on Groups IX, Proceedings Oberwolfach (1988). H. Heyer, ed. Lecture Notes Math. **1379**, 325–343 (1989)

[27] **Siebert, E.:** *Decomposition of convolution semigroups on Polish groups and zero-one laws.* Hokkaido Math. J. **16** (1987), 235–255

[28] **Wang, J.S.P.:** *The Mautner pheneomen on p-adic Lie groups.* Math. Z. **185**, 403–412 (1984)

[29] **Willis, G.:** *The structure of totally disconnected, locally compact groups.* Math. Ann. **300**, 341–363 (1994)

Faculty of Mathematics, Technische Universität Dortmund, Vogelpothsweg 81, D-44227 Dortmund, Germany

E-mail address: wilfried.hazod@mathematik.tu-dortmund.de

Preprints ab 2009/15

2012-01 **Wilfried Hazod**
Intrinsic topologies on H-contraction groups with applications to semistability

2011-14 **Guy Bouchitté and Ben Schweizer**
Plasmonic waves allow perfect transmission through sub-wavelength metallic gratings

2011-13 **Waldemar Grundmann**
Moment functions and Central Limit Theorem for Jacobi hypergroups on $[0, \infty[$

2011-12 **J. Koch, A. Rätz, and B. Schweizer**
Two-phase flow equations with a dynamic capillary pressure

2011-11 **Michael Voit**
Central limit theorems for hyperbolic spaces and Jacobi processes on $[0, \infty[$

2011-10 **Ben Schweizer**
The Richards equation with hysteresis and degenerate capillary pressure

2011-09 **Andreas Rätz and Matthias Röger**
Turing instabilities in a mathematical model for signaling networks

2011-08 **Matthias Röger and Reiner Schätzle**
Control of the isoperimetric deficit by the Willmore deficit

2011-07 **Frank Klinker**
Generalized duality for k-forms

2011-06 **Sebastian Aland, Andreas Rätz, Matthias Röger, and Axel Voigt**
Buckling instability of viral capsides - a continuum approach

2011-05 **Wilfried Hazod**
The concentration function problem for locally compact groups revisited: Non-dissipating space-time random walks, τ-decomposable laws and their continuous time analogues

2011-04 **Wilfried Hazod, Katrin Kosfeld**
Multiple decomposability of probabilities on contractible locally compact groups

2011-03 **Alexandra Monzner* and Frol Zapolsky†**
A comparison of symplectic homogenization and Calabi quasi-states

2011-02 **Stefan Jäschke, Karl Friedrich Siburg and Pavel A. Stoimenov**
Modelling dependence of extreme events in energy markets using tail copulas

2011-01 **Ben Schweizer and Marco Veneroni**
The needle problem approach to non-periodic homogenization

2010-16 **Sebastian Engelke and Jeannette H.C. Woerner**
A unifying approach to fractional Lévy processes

2010-15 **Alexander Schnurr and Jeannette H.C. Woerner**
Well-balanced Lévy Driven Ornstein-Uhlenbeck Processes

2010-14 **Lorenz J. Schwachhöfer**
On the Solvability of the Transvection group of Extrinsic Symplectic Symmetric Spaces

2010-13 **Marco Veneroni**
Stochastic homogenization of subdifferential inclusions via scale integration

2010-12 **Agnes Lamacz, Andreas Rätz, and Ben Schweizer**
A well-posed hysteresis model for flows in porous media and applications to fingering effects

2010-11 **Luca Lussardi and Annibale Magni**
Γ-limits of convolution functionals

2010-10 **Patrick W. Dondl, Luca Mugnai, and Matthias Röger**
Confined elastic curves

2010-09 **Matthias Röger and Hendrik Weber**
Tightness for a stochastic Allen–Cahn equation

2010-08 **Michael Voit**
Multidimensional Heisenberg convolutions and product formulas for multivariate Laguerre polynomials

2010-07 **Ben Schweizer**
Instability of gravity wetting fronts for Richards equations with hysteresis

2010-06 **Lorenz J. Schwachhöfer**
Holonomy Groups and Algebras

2010-05 **Agnes Lamacz**
Dispersive effective models for waves in heterogeneous media

2010-04 **Ben Schweizer and Marco Veneroni**
Periodic homogenization of Prandtl-Reuss plasticity equations in arbitrary dimension

2010-03 **Holger Dette and Karl Friedrich Siburg and Pavel A. Stoimenov**
A copula-based nonparametric measure of regression dependence

2010-02 **René L. Schilling and Alexander Schnurr**
The Symbol Associated with the Solution of a Stochastic Differential Equation

2010-01 **Henryk Zähle**
Rates of almost sure convergence of plug-in estimates for distortion risk measures

2009-16 **Lorenz J. Schwachhöfer**
Nonnegative curvature on disk bundles

2009-15 **Iuliu Pop and Ben Schweizer**
Regularization schemes for degenerate Richards equations and outflow conditions